Diffusive random process approximation in certain nonstationary statistical problems of physics

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The review considers, on the basis of a unified approach, the problem of Brownian motion in nonlinear dynamic systems, including a linear oscillator acted upon by random forces, parametric resonance in an oscillating system with random parameters, turbulent diffusion of particles in a random-velocity field, and diffusion of rays in a medium with random inhomogeneities of the refractive index. The same method is used to consider also more complicated problems such as equilibrium hydrodynamic fluctuations in an ideal gas, description of hydrodynamic turbulence by the method of random forces, and propagation of light in a medium with random inhomogeneities. The method used to treat these problems consists of constructing equations for the probability density of the system or for its statistical moments, using as the small parameter the ratio of the characteristic time of the random actions to the time constant of the system (in many problems, the role of the time is played by one of the spatial coordinates). The first-order approximation of the method is equivalent to replacement of the real correlation function of the action by a δ function; this yields equations for the characteristics in closed form. The method makes it possible to determine also higher approximations in terms of the aforementioned first-order small parameter.

CONTENTS

1. INTRODUCTION

Statistical problems occupy at present a prominent position in various branches of physics. Even if we disregard problems that belong traditionally to statistical physics, there are many questions in which we encounter the need of taking fluctuation effects into account. Although the causes of the fluctuations are entirely different in different problems (these may be thermal noise, instabilities, turbulence, etc.), the methods used for their theoretical analysis are frequently very similar.

A powerful tool that makes it possible at present to solve rather complicated statistical problems is the theory of processes of diffusive type, which was developed on the basis of the theory of Brownian motion, and the theory of Markov random processes. The purely mathematical aspects of this theory is treated in an extensive literature (see, e.g.,[6-14]) and these questions will not be touched upon in this review.

The purpose of the present article is to illustrate how different physical problems can be solved on the basis of a generalized Brownian-motion theory. We reveal here interesting analogies between quite different physical problems. The examples considered below are borrowed mainly from statistical radiophysics and hydrodynamics, this being the branches in which the authors work. Analogous problems and methods of their solutions arise, however, also in plasma physics, in solid-state theory, in magnetohydrodynamics, etc.

The method used below is a theory based on the expansion of the solutions in a small parameter, namely the ratio of the correlation time of random actions to the observation time and other characteristic temporal scales of the problem (in a number of cases these are not temporal but spatial scales). In the theory of Brownian motion this approximation corresponds to neglect of the time between random collisions in comparison with all other temporal scales.

As applied to the problems of dynamic systems whose motion satisfies ordinary differential stochastic equations, the employed method leads to the Markov random-process approximation. In more complicated problems, which are described by partial differential equations (among the examples considered below, these include problems of hydrodynamic fluctuations in an ideal gas, the functional description of hydrodynamic turbulence, and the propagation of light in a medium with random inhomogeneities), this method leads to a generalized equation of the Einstein-Fokker type, in which connection it can be called the diffusive random-process approximation.

In many recent papers in which the Einstein-Fokker equation (EFE) is used, this equation is formulated on the basis of intuitive considerations, and the dynamic equations are used only to calculate the coefficients that enter in the EFE. This approach is somewhat inconsistent. Indeed, the statistical problem is fully defined by the dynamic equations and by the assumptions made in these equations concerning the random actions. The EFE, as an approximate solution of the problem, should in this case be a logical consequence of the dynamic equations and of certain assumptions concerning the character of the random actions. It is clear that the solution of the problem reduces to the EFE in by far not all cases. An example where the EFE was obtained under very general assumptions from dynamic equations may be Bogolyubov's work[15].

The method used below for obtaining the EFE was proposed by Novikov[16] in turbulence theory. This method makes it possible to obtain the EFE by starting directly from the dynamic equation of the problem, and also to investigate the corrections necessitated by the finite correlation time of the random actions. At the same time, the limitations under which the EFE is valid are determined. We note that in some problems this leads to limitations of the "intensity" of the random actions. In many cases it is possible to construct also more accurate equations that take into account the finite correlation time of the random actions. An example illustrating this more accurate method can be found in the problem of propagation of light in an active medium.

2. DIFFUSIVE RANDOM PROCESS APPROXIMATION FOR NONLINEAR DYNAMIC SYSTEMS OF GENERAL FORM

a) The Einstein-Fokker equation for a system of differential equations. Let a certain quantity $\xi(t) = \{\xi_1(t), \ldots, \xi_n(t)\}$ satisfy the system of dynamic equations

$$d\xi_i(t)/dt = v_i(\xi, t) + f_i(\xi, t), \quad \xi(0) = \xi_0, \tag{2.1}$$

where $v_i(\boldsymbol{\xi}, t)$ are determined functions, and $f_i(\mathbf{x}, t)$ are random functions of n + 1 variables, having the following properties: (a) $f_i(x, t)$ is a random Gaussian field in an (n + 1)-dimensional space; (b) $\langle f_i(x, t)\rangle = 0$. We assume for completeness that t is the temporal coordinate and $\mathbf{x}$ stands for the spatial coordinates.

The statistical characteristics of the field $f_i(\mathbf{x}, t)$ are completely described by specifying its correlation tensor

$$B_{ij}(\mathbf{x}, t; \mathbf{x}', t') = \langle f_i(\mathbf{x}, t) f_j(\mathbf{x}', t')\rangle,$$

where the angle brackets denote averaging over the ensemble of the realizations of f.

Since the system (2.1) is of first order in t, the initial conditions for which are stipulated at t = 0, the functions $\xi_i(t)$ will depend functionally only on values of $f_j(\mathbf{x}, t')$ which come earlier in terms of t in the interval $0 \le t' \le t$. It follows therefore that $\xi_i(t)$ does not vary when the functions $f_j(\mathbf{x}, t')$ are varied outside this interval, i.e., in the sections $t' < 0$ and $t' > t$. Consequently, the variational derivatives $\delta\xi_i/\delta f_j$ satisfies the condition (at (fixed $\xi_i(0)$)

$$\delta\xi_i(t)/\delta f_j(\mathbf{x}, t') = 0, \quad \text{if} \quad t' < 0, \quad t' > t, \tag{2.2}$$

which we shall call the causality condition.

In what follows we shall need the value $\delta\xi_i(t)/\delta f_j(\mathbf{x}, t')$ at $t' = t$. This quantity can be obtained in the following manner. Integrating (2.1) with respect to t, we obtain

$$\xi_i(t) = \xi_i(0) + \int_0^t d\tau\, [v_i(\xi(\tau), \tau) + f_i(\xi(\tau), \tau)]$$

$$= \xi_i(0) + \int_0^t d\tau \int d\boldsymbol{\eta}\, [v_i(\boldsymbol{\eta}, \tau) + f_i(\boldsymbol{\eta}, \tau)]\, \delta(\boldsymbol{\eta} - \xi(\tau)).$$

We apply to this formula the operator $\delta/\delta f_j(\mathbf{x}, t')$ and recognize that, in accordance with the definition of the variational derivative we have $\delta f_i(\boldsymbol{\eta}, \tau)/\delta f_j(\mathbf{x}, t') = \delta_{ij}\delta(\boldsymbol{\eta} - \mathbf{x})\delta(\tau - t')$. Since $\xi_i(0)$ does not depend on $\mathbf{f}$, we obtain by differentiating the product under the integral sign

$$\frac{\delta\xi_i(t)}{\delta f_j(\mathbf{x}, t')} = \int_0^t d\tau \int d\boldsymbol{\eta}\, \Big\{\delta_{ij}\delta(\boldsymbol{\eta} - \mathbf{x})\,\delta(\tau - t')\,\delta(\boldsymbol{\eta} - \xi(\tau))$$
$$+ [v_i(\boldsymbol{\eta}, \tau) + f_i(\boldsymbol{\eta}, \tau)]\frac{\delta}{\delta f_j(\mathbf{x}, t')}\delta(\boldsymbol{\eta} - \xi(\tau))\Big\} = \delta_{ij}\delta(\mathbf{x} - \xi(t'))$$
$$+ \int_{t'}^t d\tau \int d\boldsymbol{\eta}\, [v_i(\boldsymbol{\eta}, \tau) + f_i(\boldsymbol{\eta}, \tau)]\, \frac{\delta}{\delta f_j(\mathbf{x}, t')}\delta(\boldsymbol{\eta} - \xi(\tau)).$$

The lower limit of integration in the second term is replaced by t′, since, according to (2.2), the variational derivative under the integral sign is equal to zero if $t' > \tau$. Putting now $t' = t$, we cause the second term, which can be shown not to contain singularities at $\tau = t$, to vanish, as a result of which we obtain the formula

$$\delta\xi_i(t)/\delta f_j(\mathbf{x}, t) = \delta_{ij}\delta(\mathbf{x} - \xi(t)). \tag{2.3}$$

As indicated above, the functions $\xi_i(t)$ depend functionally only on the preceding values $f_j(\mathbf{x}, t')$. A statistical connection may exist, however, between $\xi_i(t)$ and succeeding values $f_j(\mathbf{x}, t'')$, since the values of $f_j(\mathbf{x}, t'')$ at $t'' > t$ are correlated with the values of $f_j(\mathbf{x}, t')$ at $t' < t$. It is clear that the correlation of the functions $\xi_i(t)$ with the subsequent values of $f_j(\mathbf{x}, t'')$ is noticeable only at $t'' - t \lesssim \tau_0$, where τ_0 is the correlation radius of the field $f_j(x, t)$ with respect to the variable t. On the other hand, if the characteristic correlation radius $\xi_i(t)$ is of the order of $T \gg \tau_0$ (this holds true for a rather large class of real physical processes), then there exists in the considered problem a small parameter τ_0/T, which can be used to construct an approximate solution.

In first-order approximation in this small parameter, we can put $\tau_0 = 0$. In this case the values of $\xi_i(t')$ at $t' < t$ will be independent not only functionally but also statistically of the values of $f_j(\mathbf{x}, t'')$ at $t'' > t$. This circumstance is equivalent to replacing the correlation function B_{ij} by the effective correlation

$$B_{ij}^{\text{eff}}(\mathbf{x}, t; \mathbf{x}', t') = 2\delta(t - t')\, F_{ij}(\mathbf{x}, \mathbf{x}', t), \tag{2.4}$$

with the quantity F_{ij} determined from the condition of equality of the integrals of B_{ij} and B_{ij}^{eff} with respect to t′:

$$F_{ij}(\mathbf{x}, \mathbf{x}', t) = (1/2)\int_{-\infty}^{\infty} dt' B_{ij}(\mathbf{x}, t; \mathbf{x}', t'). \tag{2.5}$$

We introduce the probability density for the solution $\boldsymbol{\xi}(t)$ of the system (2.1)

$$P_t(\mathbf{x}) = \langle\delta(\mathbf{x} - \xi(t))\rangle, \tag{2.6}$$

where $\boldsymbol{\xi}(t)$ is the solution of the system (2.1) corresponding to a definite realization of $\mathbf{f}(\mathbf{x}, t)$, and the averaging is carried out over the set of all the realizations.

Differentiating (2.6) with respect to t, we obtain, taking (2.1) into account, the equation

$$\frac{\partial P_t(\mathbf{x})}{\partial t} = \Big\langle\frac{\partial}{\partial t}\delta(\mathbf{x} - \xi(t))\Big\rangle = -\Big\langle\frac{d\xi_k}{dt}\frac{\partial}{\partial x_k}\delta(\mathbf{x} - \xi(t))\Big\rangle$$
$$= -\frac{\partial}{\partial x_k}\langle\delta(\mathbf{x} - \xi(t))\,[v_k(\xi(t), t) + f_k(\xi(t), t)]\rangle. \tag{2.7}$$

Using the identity $\delta(\mathbf{x} - \boldsymbol{\xi}(t))\, v_k(\boldsymbol{\xi}(t), t)$

$\equiv \delta(\mathbf{x} - \boldsymbol{\xi}(t))\, v_k(\mathbf{x}, t)$, taking the non-random factor $v_k(\mathbf{x}, t)$ outside the averaging symbol, and again using (2.6), we obtain

$$\partial P_t(\mathbf{x})/\partial t = -(\partial/\partial x_k)\,[v_k(\mathbf{x}, t)\, P_t(\mathbf{x}) + \langle f_k(\mathbf{x}, t)\, \delta(\mathbf{x} - \boldsymbol{\xi}(t))\rangle]. \qquad (2.8)$$

To calculate the correlation in the right-hand side of (2.8), we use a formula obtained by Furutsu[17] and Novikov[16] (see also[18])

$$\langle f(\mathbf{r})\, R[f]\rangle = \int d\mathbf{r}'\, \langle f(\mathbf{r})\, f(\mathbf{r}')\rangle\, \langle \delta R[f]/\delta f(\mathbf{r}')\rangle, \qquad (2.9)$$

which makes it possible to calculate the correlation of a Gaussian random field $\mathbf{f}(\mathbf{r})$ ($\langle \mathbf{f}(\mathbf{r})\rangle = 0$) with a functional $R[f]$ of the field ($\mathbf{r}$ stands for all the arguments of the function f, including the indices over which it is necessary to sum). Formula (2.9) can be proved, for example, by expanding $R[f]$ in a functional Taylor series, and represents a generalization of the well known formula for the correlation of a Gaussian random quantity with a function of this quantity $\langle z f(z)\rangle = \langle z^2\rangle\langle f'(z)\rangle$ ($\langle z\rangle = 0$) to include the case of continuous fields.

If we use expression (2.4) for the correlation function of the field $f_j(\mathbf{x}, t)$, then we obtain terms connected with the values $\delta\xi_i(t)/\delta f_j(\mathbf{x}, t')$ at coinciding arguments $t' = t$, which, in accordance with (2.3), are expressed in terms of the functions $\xi_i(t)$ themselves. We then arrive at the EFE

$$\frac{\partial P_t(\mathbf{x})}{\partial t} + \frac{\partial}{\partial x_k}\{[v_k(\mathbf{x}, t) + A_k(\mathbf{x}, t)]\, P_t(\mathbf{x})\} - \frac{\partial^2}{\partial x_k\, \partial x_l}[F_{kl}(\mathbf{x}, \mathbf{x}, t)\, P_t(\mathbf{x})] = 0, \qquad (2.10)$$

where

$$A_k(\mathbf{x}, t) = \frac{\partial}{\partial x_l} F_{kl}(\mathbf{x}, \mathbf{x}', t)\Big|_{\mathbf{x}'=\mathbf{x}}.$$

Equation (2.10) should be solved with the initial condition

$$P_0(\mathbf{x}) = \delta(\mathbf{x} - \boldsymbol{\xi}_0) \qquad (2.11)$$

or else with an initial condition of more general form $P_0(\mathbf{x}) = W(\mathbf{x})$.

It is similarly easy to obtain also an equation for the probability density

$$\mathcal{P}_m(\mathbf{x}_1, t_1; \ldots; \mathbf{x}_m, t_m) = \langle \delta(\boldsymbol{\xi}(t_1) - \mathbf{x}_1) \ldots \delta(\boldsymbol{\xi}(t_m) - \mathbf{x}_m)\rangle, \qquad (2.12)$$

which pertains to m different instants of time.

Let $t_1 < t_2 < \ldots < t_{m-1} < t_m$. Differentiating (2.12) with respect to t_m, and then using the dynamic equation (2.1), formula (2.9), the causality condition (2.2), and relation (2.3), we can obtain the equation

$$\frac{\partial \mathcal{P}_m}{\partial t_m}(\mathbf{x}_1, t_1; \ldots; \mathbf{x}_m, t_m) + \sum_{i=1}^{n} \frac{\partial}{\partial x_{mi}}\{[v_i(\mathbf{x}_m, t_m) + A_i(\mathbf{x}_m, t_m)]\, \mathcal{P}_m\} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_{mi}\, \partial x_{mj}}[F_{ij}(\mathbf{x}_m, \mathbf{x}_m, t_m)\, \mathcal{P}_m] \qquad (2.13)$$

(we do not sum here over the index m). The initial condition of (2.13) can be obtained from (2.12). Putting $t_m = t_{m-1}$ in (2.12) and noting that

$$\delta(\boldsymbol{\xi}(t_{m-1}) - \mathbf{x}_{m-1})\, \delta(\boldsymbol{\xi}(t_{m-1}) - \mathbf{x}_m) = \delta(\mathbf{x}_m - \mathbf{x}_{m-1})\, \delta(\boldsymbol{\xi}(t_{m-1}) - \mathbf{x}_{m-1}),$$

we obtain

$$\mathcal{P}_m(\mathbf{x}_1, t_1; \ldots; \mathbf{x}_{m-1}, t_{m-1}; \mathbf{x}_m, t_{m-1}) = \delta(\mathbf{x}_m - \mathbf{x}_{m-1})\, \mathcal{P}_{m-1}(\mathbf{x}_1, t_1; \ldots; \mathbf{x}_{m-1}, t_{m-1}). \qquad (2.14)$$

We can seek the solution of (2.13) in the form

$$\mathcal{P}_m(\mathbf{x}_1, t_1; \ldots; \mathbf{x}_{m-1}, t_{m-1}; \mathbf{x}_m, t_m) = p(\mathbf{x}_m, t_m \mid \mathbf{x}_{m-1}, t_{m-1})\, \mathcal{P}_{m-1}(\mathbf{x}_1, t_1; \ldots; \mathbf{x}_{m-1}, t_{m-1}). \qquad (2.15)$$

Since all the differential operations in (2.13) pertain to $\mathbf{x}_m$ and t_m it follows that, substituting (2.15) in (2.13) and (2.14), we obtain an equation for the probability density of the transition (we denote $\mathbf{x}_m$ and t_m by $\mathbf{x}$ and t, and $\mathbf{x}_{m-1}$ and t_{m-1} by $\mathbf{x}_0$ and t_0)

$$\frac{\partial}{\partial t} p(\mathbf{x}, t \mid \mathbf{x}_0, t_0) + \frac{\partial}{\partial x_i}\{[v_i(\mathbf{x}, t) + A_i(\mathbf{x}, t)]\, p\} = \frac{\partial^2}{\partial x_i\, \partial x_j}[F_{ij}(\mathbf{x}, \mathbf{x}, t)\, p(\mathbf{x}, t \mid \mathbf{x}_0, t_0)] \qquad (2.16)$$

with the initial condition

$$p(\mathbf{x}, t_0 \mid \mathbf{x}_0, t_0) = \delta(\mathbf{x} - \mathbf{x}_0). \qquad (2.17)$$

By a second application of formula (2.15), we obtain the expression

$$\mathcal{P}_m(\mathbf{x}_1, t_1; \ldots; \mathbf{x}_m, t_m) = p(\mathbf{x}_m, t_m \mid \mathbf{x}_{m-1}, t_{m-1})\, p(\mathbf{x}_{m-1}, t_{m-1} \mid \mathbf{x}_{m-2}, t_{m-2}) \ldots p(\mathbf{x}_2, t_2 \mid \mathbf{x}_1, t_1)\, P_{t_1}(\mathbf{x}_1)\ (t_m > t_{m-1} > \ldots > t_2 > t_1), \qquad (2.18)$$

where $P_{t_1}(\mathbf{x}_1)$ is the probability density defined by (2.10) and (2.11), and pertains to one instant of time. Expression (2.18) for the many-time probability density in terms of a product of the transition probability densities p denotes that $\boldsymbol{\xi}(t)$ is a random Markov process.

Thus, we can make the following statement.

If the nonlinear dynamic system is described by Eq. (2.1), in which the random "force" $\mathbf{f}(\mathbf{x}, t)$ satisfies the conditions (a) and (b), and is δ-correlated in time (i.e., its correlation tensor takes the form specified by the right-hand side of (2.4)), then the random process $\boldsymbol{\xi}(t)$ is a Markov process, and is described by the EFE (2.10) or by relations (2.16)–(2.18).

An important role is played here by the causality condition (2.2), which follows from Eq. (2.1) itself and from its initial conditions.

These facts are well known (see, for example, the monographs[19,20]). A derivation of the EFE by the method described above is contained in[21].

In a number of physical problems one deals not with a finite-dimensional system of ordinary differential equations, but with a system of partial differential equations ($n = \infty$). In this case the concept of the probability density is not always meaningful, and it is necessary to consider a characteristic functional for the corresponding fields. The equation for the characteristic functional is in this case a functional equation with variational derivatives and constitutes an infinite-dimensional analog of the EFE, so that the described approximation can be called the diffusive random-process approximation.

Let us examine in greater detail the quantities contained in (2.10). The terms of Eq. (2.10) with A_k and F_{kl} are due to fluctuations of the field $f_j(\mathbf{x}, t)$. If the field $\mathbf{f}$ is stationary in time, then the quantities A_k and F_{kl} do not depend on t. If furthermore in all the spatial coordinates the field $\mathbf{f}$ is homogeneous and isotropic, then the quantity $F_{kl}(\mathbf{x}, \mathbf{x}, t)$ is a constant corresponding to a constant diffusion coefficient, and $A_k(\mathbf{x}, t) = 0$.[1)] Let now the field $\mathbf{f}$ not be spatially homogeneous. If the intensity of the fluctuations of $\mathbf{f}$ is low enough, then the fluctuations of $\xi(t)$, due to the fluctuations of $\mathbf{f}$, are also sufficiently small in a number of cases. In this case we can expand $f_i(\mathbf{x}, t)$ in the right-hand side of (2.1) in a series in the spatial variables about the point $\mathbf{x} = \boldsymbol{\xi}_0$:

$$f_i(\mathbf{x}, t) = a_i(t) + b_{ij}(t)(x_j - \xi_{0j}), \tag{2.19}$$

where the quantities $a_i(t)$ and $b_{ij}(t)$ are Gaussian random quantities that are stationary in time. The first term in (2.19) leads to a constant diffusion coefficient, and the second leads to a diffusion coefficient that is quadratic in the spatial variables. These diffusion coefficients possess that property that the terms corresponding to them in (2.10) are "homogeneous" in the sense that if we write down with the aid of (2.10) equations for the moments of the function $\boldsymbol{\xi}(t)$, then the diffusive terms generate moments of $\boldsymbol{\xi}$ of order not higher than the considered one. If furthermore the determined functions $v_i(\mathbf{x}, t)$ are also linear functions of the spatial arguments, then we obtain for the moments of any order of the function $\boldsymbol{\xi}(t)$ equations that in many cases are more convenient for analysis than Eq. (2.10) itself.

b) Conditions for the applicability of the Einstein-Fokker equation. To derive the conditions for the applicability of the EFE it is necessary to take into account the finite correlation radius τ_0 of the field $f_j(\mathbf{x}, t)$ with respect to the temporal coordinate. In this case we obtain in place of (2.10)

$$\hat{E}P_t(\mathbf{x}) = -\partial S'_k(\mathbf{x}, t)/\partial x_k, \tag{2.20}$$

where $\hat{E}$ is the operator in the left-hand side of (2.10), in which $F_{kl}(\mathbf{x}, \mathbf{x}, t)$ is replaced by

$$\widetilde{F}_{kl}(\mathbf{x}, \mathbf{x}', t) = \int_0^t dt' B_{kl}(\mathbf{x}, t; \mathbf{x}', t'), \tag{2.21}$$

and $S'_k(\mathbf{x}, t)$ are corrections to the probability flux density vector to account for the finite value of τ_0. We note that as $\tau_0 \to 0$ the right-hand side of (2.20) tends to zero, and we return to (2.10). In the general case, Eq. (2.20) differs from the EFE in two respects: F_{kl} is replaced by $\widetilde{F}_{kl}$, and (2.20) acquires a right-hand side. The difference between $\widetilde{F}_{kl}$ and F_{kl} is noticeable only in the region $t \lesssim \tau_0$. Allowance for S'_k imposes limitations, generally speaking, on the intensity of the fluctuations of the field $\mathbf{f}$. Expressions corresponding to the system of Eqs. (2.1) are given for S'_k in [21]. In the general case these expressions are rather cumbersome, and each different physical problem requires its own condition for the applicability of the EFE, and consequently for the applicability of the diffusive random-process approximation.

Thus, the condition that the parameter τ_0/T be small is necessary but, generally speaking, insufficient to be able to describe the statistical characteristics of the solution of the system (2.1) on the basis of the diffusive random-process approximation (EFE), and for each particular problem it is necessary to carry out more detailed investigations.

We proceed now to consider particular physical problems. The simplest problems of this type are those of Brownian motion of various dynamic systems.

3. BROWNIAN MOTION OF DYNAMIC SYSTEMS

When considering Brownian motion of dynamic systems, we start with the Langevin approach, which is based on the study of the statistical characteristics of the solutions of phenomenological stochastic equations. We shall not dwell on the justification of this approach, on its connection with other methods of describing Brownian motion, and on the conditions of its applicability. These questions have been adequately investigated and their exposition can be found in many papers, both original and surveys [1-5,22-27,95], which a complete bibliography of these problems can also be found.

a) Brownian motion of a system of interacting particles. We consider a system of n particles with mass m, interacting with one another with a potential $V(\mathbf{r}_1, \ldots, \mathbf{r}_n)$, and placed in a thermostat consisting of a gas of lighter particles. If the time of interaction between the thermostat particles and the singled-out particles is short in comparison with all the other characteristic times, then we can use for the particle motion the Langevin equation

$$m\ddot{\mathbf{r}}_i + \mu\dot{\mathbf{r}}_i + \partial U/\partial \mathbf{r}_i = \mathbf{f}_i(t) \quad (i = 1, 2, \ldots, n), \tag{3.1}$$

where μ is the effective friction coefficient, and the random forces $f_i(t)$, the mean value of which is zero, are described by the function

$$\langle f_i^\alpha(t) f_j^\beta(t')\rangle = 2D\delta_{ij}\delta_{\alpha\beta}\delta(t-t') \tag{3.2}$$

(α and β are vector indices).

Equation (3.1) can be written in the standard form (2.1), by introducing the momenta $\mathbf{p}_i = m\dot{\mathbf{r}}_i$ and the Hamiltonian $H = (p_i^2/2m) + V(\mathbf{r}_1, \ldots, \mathbf{r}_n)$:

$$\dot{\mathbf{r}}_i = \partial H/\partial \mathbf{p}_i, \quad \dot{\mathbf{p}}_i = -\partial H/\partial \mathbf{r}_i - (\mu/m)\mathbf{p}_i + \mathbf{f}_i(t). \tag{3.3}$$

Then the EFE corresponding to (3.3) take the form

$$\partial P_t(r, p)/\partial t + \sum_{k=1}^n \{P_t H\}_{(k)} - (\mu/m) \sum_{k=1}^n \partial\{\mathbf{p}_k P_t\}/\partial \mathbf{p}_k = D \sum_{k=1}^n \partial^2 P_t/\partial \mathbf{p}_k^2, \tag{3.4}$$

where

$$\{\varphi, \psi\}_{(k)} = (\partial\varphi/\partial\mathbf{p}_k)\,\partial\psi/\partial\mathbf{r}_k - (\partial\varphi/\partial\mathbf{r}_k)\,\partial\psi/\partial\mathbf{p}_k$$

is the Poisson bracket for the k-th particle.

It is easy to verify that the stationary solution of (3.4) takes the form of a Gibbs canonical distribution

$$P_\infty(\mathbf{r}, \mathbf{p}) = \text{const}\cdot\exp(-\beta H) \quad (\beta = 1/kT), \tag{3.5}$$

and in order that (3.5) be a solution of (3.4) it is necessary to satisfy the relation

$$D = \mu/\beta = \mu kT, \tag{3.6}$$

where k is the Boltzmann constant and T is an equilibrium temperature; relation (3.6) is Einstein's formula for the diffusion coefficient[2)].

Integrating (3.5) over all $\mathbf{r}_i$, we obtain a Maxwellian velocity distribution describing the velocity fluctuations of the Brownian particles. The case when U is a positive quadratic form of its arguments describes the Brownian motion of a system of coupled oscillators. To demonstrate the general procedure given in the second part of the paper, we shall dwell on this case in greater detail.

b) Brownian motion of an oscillator. One-dimensional Brownian motion of an oscillator, according to (3.1) is described by the stochastic equation

$$\begin{gathered}\ddot{x} + \mu\dot{x} + \omega^2 x = f(t) \quad (m = 1),\\ \langle f(t) f(t')\rangle = 2\mu kT\delta(t-t'), \quad \langle f(t)\rangle = 0.\end{gathered} \tag{3.1'}$$

We rewrite Eq. (3.1′) in the form of the system of equations

$$\dot{x} = y, \quad \dot{y} = -\mu y - \omega^2 x + f(t). \tag{3.7}$$

For the initial conditions for the system (3.7) we assume x = 0 and y = 0 at t = 0.

Averaging (3.7) over the ensemble f, we obtain $\langle x\rangle = \langle y\rangle = 0$ for $t > 0$.

We now consider the equal-time correlations of the functions x(t) and y(t). In the derivation of the equations for them, there arise terms due to the correlations of f(t) with x and y, taken at the same instant of time. To calculate these correlations, we use formula (2.9) and take into account the relations $\delta x(t)/\delta f(t) = 0$, $\delta y(t)/\delta f(t) = 1$. Then the system of equations for the equal-time correlations becomes

$$d\langle x^2\rangle/dt = 2\langle xy\rangle,\quad d\langle xy\rangle/dt = \langle y^2\rangle - \mu\langle xy\rangle - \omega^2\langle x^2\rangle, \tag{3.8}$$

and

$$d\langle y^2\rangle/dt = -2\mu\langle y^2\rangle - 2\omega^2\langle xy\rangle + 2\mu kT, \tag{3.9}$$

and the initial conditions for it are the conditions

$$\langle x^2\rangle = \langle xy\rangle = \langle y^2\rangle = 0 \text{ and } t = 0. \tag{3.10}$$

The stationary solution of the system (3.8) and (3.9) is

$$\langle xy\rangle = 0, \tag{3.11}$$

$$\langle y^2\rangle = \omega^2\langle x^2\rangle = kT \tag{3.12}$$

in accordance with the law of equipartition of the energy over the degrees of freedom.

We consider now the temporal correlations of the functions x and y. It is easily seen that they are determined by the dynamics of the system (3.1′) in the absence of random forces. In fact, assume for the sake of argument that $t > t'$. Then, multiplying (3.7) by $x(t')$ and averaging, we obtain the system of equations

$$(d/dt)\langle x(t)x(t')\rangle = \langle y(t)x(t')\rangle, \tag{3.13}$$

$$(d/dt)\langle y(t)x(t')\rangle = -\mu\langle y(t)x(t')\rangle - \omega^2\langle x(t)x(t')\rangle, \tag{3.14}$$

the boundary conditions for which are expressed in terms of the equal-time correlations at $t = t'$. In the derivation of (3.11)–(3.14) we use the fact that $\langle f(t)x(t')\rangle = 0$ at $t > t'$, since the function f is δ-correlated.

A similar system of equations can be obtained also for other temporal correlations, both in the case $t > t'$ and for $t < t'$. The solutions of these systems can be easily obtained and we shall not stop to discuss them here.

c) **Brownian motion of hydrodynamic-type systems.** We now discuss a different class of phenomena, which are closely related to Brownian motion, namely equilibrium thermal fluctuations in a continuous medium.

The equilibrium thermal fluctuations can be described within the framework of macroscopic variables (for a discussion of this question see, e.g., [28]), by including in the corresponding macroscopic equations the "extraneous forces"—Gaussian random fields, which are δ-correlated in time (this has a direct bearing on the fluctuation-dissipation theorem; see, e.g., [29], Appendix 1). The correlation theories of equilibrium thermal fluctuations constructed on this basis in electrodynamics, hydrodynamics, and in elastic-viscous media are described in[29-32].

In Chap. 8 below we shall dwell in greater detail on equilibrium hydrodynamic fluctuations. Here we note only the following circumstance. It is well known that there are many hydrodynamic phenomena described by finite numbers of degrees of freedom (for example, thermal convection, the flow of liquid between rotating cylinders). To describe such phenomena, Obukhov[33-35] has introduced the concept of mechanical systems of the hydrodynamic type (HTS) S_n, defined by finite sets of quantities $\{v_n\}$, the dynamic equations for which have the same properties as the equations of hydrodynamics of an ideal incompressible liquid (quadratic nonlinearity, energy conservation, regularity[3)]). We note that these systems are generally speaking not Hamiltonian. The simples such system (S_3) is analogous to the Euler equations in the dynamics of a rigid body.

Hydrodynamic-type systems, with allowance for linear friction, can be described by the dynamic equations

$$dv_i/dt = F_i(\mathbf{v}) - \lambda^{(i)}v_i \quad (i = 1, \ldots, N), \tag{3.15}$$

where $\lambda^{(i)}$ is the friction coefficient of the i-th component of the N-dimensional vector[4)] $\mathbf{v}$, and $F_i(\mathbf{v})$ is a function quadratic in v and has the following properties:

a) $V_i F_i(\mathbf{v}) = 0$ (at $\lambda^{(i)} = 0$ we get the energy conservation law $dE/dt = 0$, $E = v_i^2/2$);

b) $\partial F_i/\partial v_i = 0$ (at $\lambda^{(i)} = 0$ the conditions of the Liouville theorem are satisfied, and the equality b) is the incompressibility equation $\partial\dot{v}_i/\partial v_i = 0$ in phase space).

Equilibrium thermal fluctuations are described in HTS by Eq. (3.13), in which we include the "extraneous forces" $f_i(t)$:

$$dv_i/dt = F_i(\mathbf{v}) - \lambda^{(i)}v_i + f_i(t), \tag{3.16}$$

which are Gaussian random fluctuations, δ-correlated in time, with a correlation tensor

$$\langle f_i(t)f_j(t')\rangle = 2\mu_{(i)}^2\delta_{ij}\delta(t-t')\ (\langle \mathbf{f}\rangle = 0) \tag{3.17}$$

We note that the system (3.16) can be regarded as a system of equations describing the Brownian motion of the HTS. In this case the coefficients $\lambda^{(i)}$ are certain effective friction coefficients. Thus, in the case N = 3, the system (3.16) describes rotational Brownian motion of a rigid body in a medium (in velocity space), and the quantities $\lambda^{(i)}v_i$ are determined as the corresponding forces.

The probability density for the solution v(t) of Eqs. (3.16), i.e., the function $P_t(\mathbf{v}) = \langle\delta(\mathbf{v}(t) - \mathbf{v})\rangle$, satisfies in accordance with (2.10) the EFE

$$\partial P_t(\mathbf{v})/\partial t = -(\partial/\partial v_i)\{F_i(\mathbf{v})P_t\} + \lambda^{(i)}(\partial/\partial v_i)\{v_i P_t\} + \mu_{(i)}^2\partial^2 P_t/\partial v_i^2. \tag{3.18}$$

A stationary solution of Eq. (3.18), independent of the initial data, should have the character of a Maxwellian distribution corresponding to the law of equipartition of the energy over the degrees of freedom:

$$P_\infty(\mathbf{v}) = \text{const}\cdot\exp(-v_i^2/2kT). \tag{3.19}$$

Substituting (3.19) in (3.18) we obtain for $\mu_{(i)}^2$ the relations

$$\mu_{(i)}^2 = \lambda^{(i)}kT \text{ (Einstein's formula).} \tag{3.20}$$

By virtue of conditions (a) and (b), we have here $(\partial/\partial v_i)\{F_i(\mathbf{v})P_\infty(\mathbf{v})\} \equiv 0$. Relations (3.17) and (3.20) thus determine completely the statistics of the "extraneous forces," and the nonlinear term in (3.16) plays no role in the equal-time fluctuations of v(t) (in analogy with the case of Hamiltonian systems).

We note that Eqs. (3.16) with the expression for the correlations (3.17) can be used not only to describe equilibrium thermal fluctuations in SHT, but also the action of small-scale motions (for example, microturbulence) on large-scale motions. If the HTS can at the same time be described by the system of phenomenological equations (3.16) with $\lambda^{(i)} \equiv \lambda$ and $\mu_{(i)}^2 \equiv \mu^2$, then the stationary distribution of the probabilities for $\mathbf{v}$ has a form similar to the distribution (3.19):

$$P_\infty(\mathbf{v}) = \text{const}\cdot\exp[-(\lambda/2\mu^2)v_1^2]. \tag{3.21}$$

Assume now that the HTS is acted upon by a regular force $\mathbf{g}$. In this case the thermal fluctuations are not in equilibrium. From physical considerations, however, it is natural to assume that they can be described by the equation

$$dv_i/dt = F_i(\mathbf{v}) - \lambda^{(i)}v_i + f_i(t) + g_i \tag{3.22}$$

with the same "extraneous forces" as in (3.16).[5)]

In the absence of fluctuations, the system (3.22) has stationary solutions described by the solution of the system of algebraic equations

$$F_i(\mathbf{v}_{st}) + g_i = \lambda^{(i)}v_{i\,st}. \tag{3.23}$$

The solution of the system (3.23) is not unique, and when choosing definite solutions one must take into account their stability with respect to infinitesimally small perturbations.

Let us consider by way of example the simplest system (S_3), which we write in dimensionless form:

$$\begin{aligned} &\dot{v}_0 = v_2^2 - v_1^2 - v_0 + R + f_0(t), \quad \dot{v}_1 = v_0v_1 - v_1 + f_1(t), \\ &\dot{v}_2 = -v_0v_2 - v_2 + f_2(t), \quad \langle f_i(t+\tau)f_j(t)\rangle = 2\delta_{ij}\sigma^2\delta(\tau). \end{aligned} \tag{3.24}$$

In the absence of "extraneous forces," this system is equivalent to the dynamic description of the motion of a gyroscope with isotropic friction, excited by a constant external torque, relative to an unstable axis, and can be realized in the well known problem of the motion of a liquid in an ellipsoidal cavity[35,36]. The stationary solution of this system is determined by the parameter $R = \mathrm{Re}$ (by the Reynolds number). The critical value is in this case $\mathrm{Re}_{cr} = 1$. If $\mathrm{Re} > 1$, this regime becomes unstable with respect to infinitesimally small perturbations, and a new regime $v_{0\,st} = 1$, $v_{1\,st} = \pm(\mathrm{Re} - 1)^{1/2}$, $v_{2\,st} = 0$ is established. A randomness element is present here, namely, the quantity $v_{1\,st}$ can be either positive or negative, depending on the sign of the amplitudes of the infinitesimally small perturbations.

The system (3.24) was simulated numerically in[37]. Figures 1 and 2 show the solutions of this system, corresponding to two realizations of the field $f(t)$ at $\mathrm{Re} = 6$ and $\sigma^2 = 0.01$. The function $v_2(t)$, which is not shown in these figures, fluctuates little about the position $v_2 \sim 0$.

The stationary distribution of the probabilities for v_1 at $\mathrm{Re} > 1$ has a form shown schematically in Fig. 3, with two maxima in the vicinity of $v_1 = \pm(\mathrm{Re} - 1)^{1/2}$, corresponding to stable stationary states, and with a minimum at $v_1 = 0$, corresponding to an unstable state. We note that this probability distribution corresponds to averaging over the ensemble of realizations of the field $\mathbf{f}(t)$. On the other hand, if there is one realization, then the system will arrive, with probability 1/2, at one of the positions corresponding to the maximum of the distribution. In this case there will be formed a probability distribution (with averaging over the time) in the vicinity of the positions of the maximum (Fig. 4). However, owing to the existence of values of sufficiently large functions $f(t)$, the system will be transferred, after the lapse of a certain time T (which is larger the smaller σ^2 and the larger Re), from one state into another (for details see Chap. 4). Figure 5 shows the solution of the system (3.24) for one realization of $f(t)$ at $\mathrm{Re} = 6$ and $\sigma^2 = 0.1$,

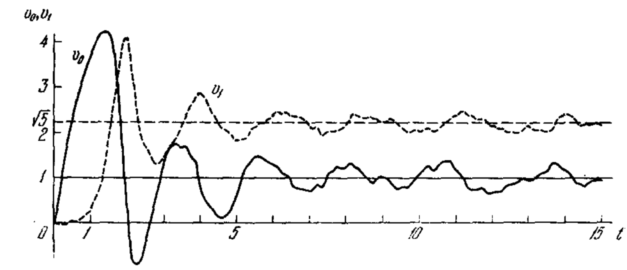

FIG. 1. Numerical solution of the system (3.24) for one realization of the random forces at $\mathbf{Re} = 6$ and $\sigma^2 = 0.01$. The solid line shows the component v_0, and the dashed line v_1.

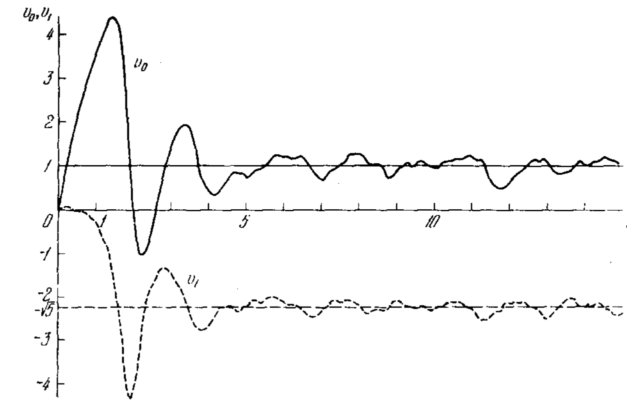

FIG. 2. Numerical solution of the system (3.24) for another realization of the random forces at $\mathbf{Re} = 6$ and $\sigma^2 = 0.01$. The solid line shows the component v_0, and the dashed line shows v_1.

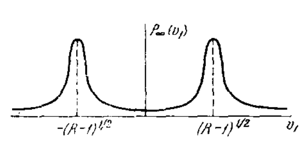

FIG. 3

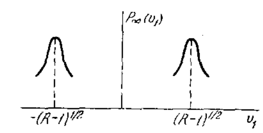

FIG. 4

FIG. 3. Stationary probability distribution density at $\mathbf{Re} > 1$ for the component v_1 of a dynamic system (3.24) (averaging over the ensemble of random forces).

FIG. 4. Probability distribution densities for the component v_1 of the system (3.24), obtained for one realization of the random forces (averaging over the time).

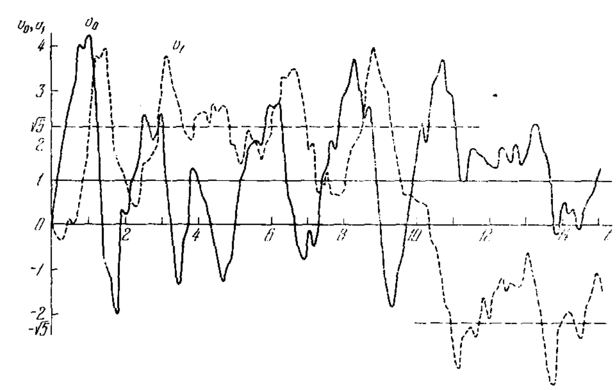

FIG. 5. Numerical solution of the system (3.24) for one realization of the random forces at $\mathbf{Re} = 6$ and $\sigma^2 = 0.1$, illustrating the "transfer" phenomenon. The solid line shows the component v_0, and the dashed line shows v_1.

where such a transfer was realized. Thus, the probability distribution shown in Fig. 3 will be formed only within a time $t \gg T$ (ergodicity).

4. NONLINEAR OSCILLATOR UNDER THE ACTION OF RANDOM FORCES

By way of example of a nonlinear dynamic problem that admits of an exact solution, we consider the equation

$$\ddot{x}+\alpha\dot{x}+\omega_0^2x+\beta x^3=f(t)\quad(\alpha,\ \beta>0),\quad\langle f(t)f(t')\rangle=2D\delta(t-t'),\tag{4.1}$$

which describes the motion of the nonlinear oscillator with linear friction, excited by a random Gaussian force that is δ-correlated in t.

In the case $\alpha = 0$ and $f = 0$, this equation is called the Duffing equation.

Equation (4.1) can be rewritten in standard form for the functions $x(t)$ and $y(t) = \dot{x}(t)$:

$$\dot{x}=\partial H/\partial y,\quad \dot{y}=-\partial H/\partial x-\alpha y+f(t),\tag{4.2}$$

where $H = (y^2/2) + (\omega_0^2x^2/2) + \beta(x^4/4)$.

The EFE for the joint probability density of x and y, is given according to (2.10) by

$$\partial P_t(x,\ y)/\partial t+\{P_t,\ H\}-\alpha\,\partial\{yP_t\}/\partial y=D\,\partial^2P_t/\partial y^2,\tag{4.3}$$

where $\{P_t,\ H\} = (\partial P_t/\partial x)\partial H/\partial y - (\partial P_t/\partial y)\partial H/\partial x$ is a Poisson bracket.

The stationary solution of (4.3) takes a form analogous to the Gibbs distribution (3.5):

$$P_\infty(x,\ y)=\mathrm{const}\cdot\exp(-\alpha H/D).\tag{4.4}$$

It is seen from the solution (4.4) that the stationary distribution of the probability for y(t) is a Gaussian distribution, and the probability distribution for the quantity x(t) itself is not Gaussian, and they do not correlate with each other.

Integrating (4.4) with respect to y, we obtain the probability distribution for x

$$P_\infty(x)=\mathrm{const}\cdot\exp\{-(\alpha/D)[(\omega_0^2x^2/2)+(\beta x^4/4)]\}.\tag{4.5}$$

This distribution has a maximum at the point $x = 0$, corresponding to a stable equilibrium position. Equation (4.5) can be rewritten in the form

$$\begin{gathered}P_\infty(\tilde{x})=\mathrm{const}\cdot\exp[-(\tilde{x}^2+\tilde{x}^4)/\mu],\\(\tilde{x}=(\beta/2\omega_0^2)^{1/2}x,\quad\mu=\beta D/\alpha\omega_0^4).\end{gathered}\tag{4.6}$$

We note that at a specified ratio of the correlations (4.1), expression (4.5) is an exact solution of the problem. Eq. (4.1) was simulated in[38] with an analog device, a white-noise generator being used as the source exciting the motion. The results of a comparison of the quantities $\langle\tilde{x}^2\rangle$ and $\langle\tilde{x}^4\rangle/\langle\tilde{x}^2\rangle^2$, obtained by numerical integration with the distribution (4.6), and of the corresponding quantities measured with the analog device, are shown in Fig. 6.

We consider now the equation

$$\ddot{x}+\alpha\dot{x}-\omega_0^2x+\beta x^3=f(t)\quad(\alpha,\ \beta>0).\tag{4.7}$$

In this case the EFE is of the same form as in (4.3), where

$$H=(y^2/2)+U(x),\quad U(x)=-(\omega_0^2x^2/2)+(\beta x^4/4).\tag{4.8}$$

The stationary distribution of the probabilities for the quantity x(t) is given by

$$P_\infty(x)=\mathrm{const}\cdot\exp(-\alpha U(x)/D).\tag{4.9}$$

The distribution (4.9) is shown in Fig. 7 and has maxima at the points $x = \pm(\omega_0^2/\beta)^{1/2}$ and a minimum at the point $x = 0$, corresponding to the positions of the stable and

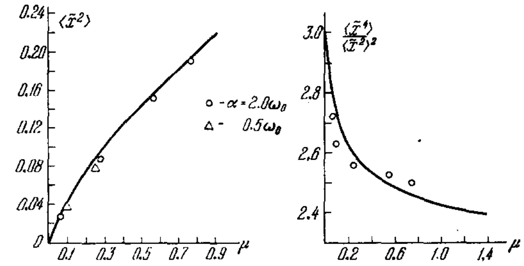

FIG. 6. The solid line corresponds to calculation in accordance with the distribution (4.6), and the experimental points correspond to quantities measured with an analog device.

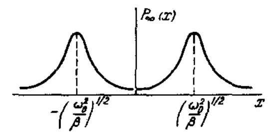

FIG. 7.

unstable equilibrium in the problem of oscillations described by the nonlinear equation (4.7) at $\alpha = 0$ and $f = 0$ (see, e.g.,[39]). The transition from the initial state to the stationary state occurs in this case in two stages: a Maxwellian velocity distribution is first established rapidly, followed by a much slower establishment of the distribution (4.9) over the coordinates. The second stage is described by the Einstein-Smoluchowski equation (see, e.g.,[28])

$$\partial P_t(x)/\partial t=(1/\alpha)(\partial/\partial x)[(\partial U/\partial x)P_t)]+(D/\alpha^2)\,\partial^2P_t/\partial x^2,\tag{4.10}$$

which is the EFE for a stochastic dynamic system described by the equation

$$\alpha\dot{x}=\omega_0^2x-\beta x^3+f(t)\quad(\alpha,\ \beta>0).\tag{4.11}$$

Introducing the dimensionless coordinates $x \to (\omega_0^2/\beta)^{1/2}x$ and time $t \to (\alpha/\omega_0^2)t$, we can rewrite (4.10) in the form

$$\frac{\partial P_t}{\partial t}=\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}P_t\right)+\mu\frac{\partial^2P_t}{\partial x^2}\quad\left(\mu=\frac{\beta D}{\alpha\omega_0^4},\ U(x)=-\frac{x^2}{2}+\frac{x^4}{4}\right).\tag{4.12}$$

Since the stationary distribution (4.9) is similar to the distribution for v_1 (see Fig. 3), it follows that everything said above concerning the behavior of v_1 applies to the present problem. Let us estimate the time in which the system goes over from one most probable state ($x = -1$) to the other ($x = 1$), following the classical paper[40] (see also[41]).

Assume that at instant $t = 0$ the system described by Eq. (4.10) is at the point ξ, i.e., $P_0(x) = \delta(x - \xi)$. Then the probability that a random point situated at the initial instant at the point ξ inside a segment (a, b) will leave this segment after a time t, going (at least once) through either the point a or the point b, satisfies the equation

$$\partial\varphi(t,\ \xi)/\partial t=-(\partial U(\xi)/\partial\xi)\,\partial\varphi/\partial\xi+\mu\,\partial^2\varphi/\partial\xi^2,\tag{4.13}$$

which must be solved with the initial and boundary conditions

$$\varphi(0,\ \xi)=0,\quad\varphi(t,\ a)=\varphi(t,\ b)=1,\quad\varphi(\infty,\ \xi)=1.\tag{4.14}$$

The average time required for the random point to leave the segment (a, b) is a, b $T(\xi) = \int_0^\infty t[\partial\varphi(t,\xi)/\partial t]dt$ and satisfies, as can be readily seen, the equation

$$\mu\,d^2T/d\xi^2-(\partial U(\xi)/d\xi)\,dT/d\xi=-1,\quad T(a)=T(b)=0.\tag{4.15}$$

Equation (4.15) is easy to solve, and we obtain for the average time of the transition of the system from the state $x = -1$ to the state $x = 1$ the expression

$$T = \frac{1}{\mu}\int_{-1}^{1} d\xi \int_{-\infty}^{\xi} d\eta \exp\left\{\frac{1}{\mu}[U(\xi)-U(\eta)]\right\} = \frac{C(\mu)}{\mu}\int_{0}^{1} d\xi \exp\left[\frac{1}{\mu}U(\xi)\right], \quad C(\mu)=\int_{-\infty}^{\infty} d\xi \exp\left[-\frac{1}{\mu}U(\xi)\right]. \tag{4.16}$$

$T \approx \sqrt{2}\pi \exp(1/4\mu)$ at $\mu \ll 1$, i.e., the average transition time increases exponentially with decreasing intensity of the forced fluctuations.

5. PARAMETRIC RESONANCE IN AN OSCILLATING SYSTEM WITH RANDOM PARAMETERS

We consider now the statistical description of the oscillator, described by a linear equation but with fluctuating coefficients[6]:

$$d^2x/dt^2 + [1 + f(t)]\, x = 0, \tag{5.1}$$

where f is a random Gaussian function characterized by its correlation function $\langle f(t) f(t+\tau)\rangle = F(\tau)$. Such equations arise for example, in the problem of parametric excitation of oscillations in electric circuits with fluctuating circuit parameters[42], in the problem of tsunami propagation over a statistically rough sea floor[43], the one-dimensional problem of propagation of waves in a medium with fluctuating parameters[44-47], etc.

We consider Eq. (5.1) with initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. We rewrite (5.1) in the form of the system of equations

$$\dot{x} = y, \quad \dot{y} = -x - xf(t). \tag{5.2}$$

In the diffusive random process approximation, the probability densities of the functions x(t) and y(t) satisfy, in accord with (2.10), the EFE

$$\partial P_t/\partial t = -y\,\partial P_t/\partial x + x\,\partial P_t/\partial y + \mu^2 x^2 \partial^2 P_t/\partial y^2 \quad \left(\mu^2 = (1/2)\int_{-\infty}^{\infty} d\tau F(\tau)\right) \tag{5.3}$$

with initial conditions $P_0(x, y) = \delta(x)\delta(y-1)$.

Equation (5.3) corresponds to an EFE with a quadratic diffusion coefficient, and cannot be solved in the general case[7]. However, as indicated in Chap. 2, in this case we can obtain closed systems of equations for moments of any order. Thus, for the mean values of x(t) and y(t) we obtain the system of equations

$$\langle \dot{x}\rangle = \langle y\rangle, \quad \langle \dot{y}\rangle = -\langle x\rangle, \tag{5.4}$$

The solution of which with the initial conditions $\langle x(0)\rangle = 0$, $\langle y(0)\rangle = 1$ is

$$\langle x(t)\rangle = \sin t, \quad \langle y(t)\rangle = \cos t \tag{5.5}$$

and coincides with the solution of (5.1) in the absence of fluctuations. The second moments of the quantities x(t) and y(t) are described by a system of equations

$$d\langle x^2\rangle/dt = 2\langle xy\rangle, \quad d\langle xy\rangle/dt = \langle y^2\rangle - \langle x^2\rangle, \quad d\langle y^2\rangle/dt = -2\langle xy\rangle + 2\mu^2\langle x^2\rangle \tag{5.6}$$

with initial conditions

$$\langle x^2\rangle = \langle xy\rangle = 0, \quad \langle y^2\rangle = 1 \quad \text{for} \quad t = 0. \tag{5.7}$$

We note that the solution of the system (5.6) contains terms that increase with time, corresponding to statistical parametric buildup of the dynamic system (5.2) as a result of fluctuations of parameters. In the case of weak fluctuations, the growth increment is $\sim\mu^2$ ($\mu^2 \ll 1$). In the presence of weak friction in the system (described by a term $-\gamma\dot{x}$ in the right-hand side of (5.1), ($\gamma \ll 1$), parametric excitation takes place if the condition[8] $\mu^2 > \gamma$ is satisfied.

For our problem, Eqs. (5.6) can be obtained also directly by starting from the system (5.2), in an analogy with the procedure used in Sec. b) of Chap. 3. In this case one uses formula (2.9) and the relations that follow from the character of the dynamic system (5.4):

$$\delta x(t)/\delta f(t) = 0, \quad \delta y(t)/\delta f(t) = -x(t). \tag{5.8}$$

As to the temporal correlations of the functions x(t) and y(t), it is easily seen that the temporal structure is described by dynamic equations of the type (5.2) in the absence of fluctuations and that the initial conditions to these equations will contain equal-time correlations defined by the system (5.6).

We now stop to discuss the conditions for the applicability of the diffusive random process approximation to the problem in question. In principle it would be necessary to consider an equation of the type (2.13) for the probability density. If, however, we are interested only in the moments of the corresponding distribution, then it suffices to determine the conditions for the applicability of formulas (5.5) to the mean values, the applicability of (5.6) to the correlations, etc.

By way of illustration, we consider the derivation of the conditions for the applicability of formulas (5.5) to mean values, following[49].

Let us average the system (5.2). To calculate the correlation $\langle f(t)x(t)\rangle$ we use formula (2.9) and relation (5.2). Then the system of equations for the mean values is

$$\langle\dot{x}\rangle = \langle y\rangle, \quad \langle\dot{y}\rangle = -\langle x\rangle - \int_0^t d\tau F(t-\tau)\langle\delta x(t)/\delta f(\tau)\rangle. \tag{5.9}$$

The system (5.9) is not closed, since it contains a new unknown function $\delta x(t)/\delta f(\tau)$. We obtain an equation for this function by varying (5.2) with respect to $f(\tau)$ at $\tau < t$:

$$\frac{\partial}{\partial t}\frac{\delta x(t)}{\delta f(\tau)} = \frac{\delta y(t)}{\delta f(\tau)}, \quad \frac{\partial}{\partial t}\frac{\delta y(t)}{\delta f(\tau)} = -\frac{\delta x(t)}{\delta f(\tau)} - f(t)\frac{\delta x(t)}{\delta f(\tau)}. \tag{5.10}$$

The boundary conditions for the system (5.10) are the expressions (5.8). When (5.10) is averaged, a new unknown function $\delta^2 x(t)/\delta f(\tau_1)\delta f(\tau_2)$ appears, etc. If the δ-correlation condition $(F(\tau) \to 2\mu^2\delta(\tau))$ is used in (5.9), then we arrive at the system (5.4), which corresponds to the EFE (5.3). The δ-correlation condition can be used in (5.10). We then obtain a closed system of higher-order equations, which is already more accurate (this will be discussed in greater detail in Chap. 10). In this approximation we have

$$\langle\delta x(t)/\delta f(\tau)\rangle = -\langle x(\tau)\rangle \sin(t-\tau) \tag{5.11}$$

and Eq. (5.9) for $\langle x\rangle$ takes the form

$$\langle\ddot{x}\rangle + \langle x\rangle = \int_0^t d\tau F(\tau)\sin\tau\,\langle x(t-\tau)\rangle. \tag{5.12}$$

The characteristic time of variation of the quantity $\langle x\rangle$ is $t \sim 1$, and the quantity $F(\tau)$ is characterized by a correlation radius τ_0. If $\tau_0 \ll 1$, then we can neglect the change in the quantity $\langle x(t-\tau)\rangle$ and confine ourselves to the first term of the expansion of $\sin\tau$ in τ. For times $t \gg \tau_0$, the integration limit in (5.12) can be replaced by infinity, and consequently, (5.12) under the condition $\mu^2\tau_0 \ll 1$ goes over into an equation corresponding to the EFE. Thus, the conditions for the applicability of the solution (5.5) are the conditions

$$\tau_0 \ll 1, \quad t \gg \tau_0, \quad \mu^2\tau_0 \ll 1, \tag{5.13}$$

which, generally speaking, impose a weak limitation on μ^2.

The problem considered above concerning the statistical parametric buildup of a dynamic system due to fluctuations of the parameters could be described in the diffusive approximation because the initial conditions were specified at one point. On the other hand, if the boundary conditions are specified at different points, then the causality conditions (2.2) will not be satisfied for the corresponding problem, and it is consequently impossible to go over to the diffusive random process approximation. In a number of cases, however, it is possible to introduce certain auxiliary equations satisfying the causality condition, and consequently admitting of a transition to the diffusive approximation; the statistical characteristics of the solutions of the auxiliary equation determine the statistical properties of the formulated problem. By way of such an example we can indicate the one-dimensional problem of the propagation of a wave through a layer of a medium with fluctuating parameters[44-47].

Outside this layer, the wave is described by the equation

$$d^2u/dx^2 + u = 0 \quad (-\infty < x < 0, \quad L < x < \infty). \tag{5.14}$$

Inside the layer, the field of the wave satisfies the equation

$$d^2u/dx^2 + [1 + \varepsilon(x)]\, u = 0 \quad (0 < x < L). \tag{5.15}$$

The boundary conditions for the considered problem are the conditions for the continuity of the functions u and du/dx at the boundaries of the layer. It is clear, that the considered problem does not satisfy the causality conditions (2.2) and cannot be directly described in the diffusive approximation.

Assume that a wave exp(ix) is incident on this fluctuating medium; then the solutions of the problem in the corresponding regions are

$$\begin{aligned} u(x) &= \exp(ix) + R\exp(-ix) \quad (-\infty < x \leqslant 0),\\ u(x) &= T\exp\{i(x-L)\} \quad (L \leqslant x < \infty), \end{aligned} \tag{5.16}$$

where R(L) and T(L) determine respectively the complex amplitudes of the reflected and transmitted waves. Inside the layer, on the other hand, the wave is described by Eq. (5.15) with the boundary conditions

$$\begin{aligned} u(0) &= 1 + R, \quad du/dx\,|_{x=0} = i(1-R),\\ u(L) &= T, \quad du/dx\,|_{x=L} = iT. \end{aligned} \tag{5.17}$$

We note that the boundary conditions (5.17) are random and the amplitudes R and T must themselves be determined.

To find the statistical properties of the wave transmission and reflection coefficients we can write for them first-order equations in x with initial conditions at a single point. These equations already possess the causality property and therefore can be described in the diffusive approximation. We note that for the complex amplitude R(L) of the reflected wave itself it is possible to obtain a nonlinear differential equation of first order in L with a condition at L = 0.[46]

6. PARTICLE DIFFUSION IN A RANDOM-VELOCITY FIELD

The study of the diffusion of particles, both liquid and impurity, in a turbulent flow has been the subject of many papers a detailed exposition of which is contained in the monograph[50]. We note only the classical work by Taylor[51] who apparently was the first to pose and solve the problem of the diffusion of a single particle, and the work by Obukhov[52], where it was proposed to describe the diffusion of an impurity in a turbulent stream as a Markov random process described by the Einstein-Fokker equation.

We consider the diffusion, in a three-dimensional field of random velocities, of a particle with coordinates described by the dynamic equation[9)]

$$d\mathbf{r}(t)/dt = \mathbf{V}(t), \quad \mathbf{V}(t) = \mathbf{u}(\mathbf{r}(t), t), \quad \mathbf{r}(0) = 0. \tag{6.1}$$

The velocity field $\mathbf{u}(\mathbf{r}, t)$ will be regarded as a Gaussian random field with zero mean value, homogeneous and isotropic in space and stationary in time. It is completely characterized by its correlation tensor

$$B_{ij}(\mathbf{x}, \tau) = \langle u_i(\mathbf{r}+\mathbf{x}, t+\tau)\, u_j(\mathbf{r}, t)\rangle \tag{6.2}$$

or by the corresponding spatial spectral tensor

$$\Phi_{ij}(\varkappa, \tau) = (2\pi)^{-3}\int d\mathbf{x}\, B_{ij}(\mathbf{x}, \tau)\exp(-i\varkappa\mathbf{x}). \tag{6.3}$$

According to the second part of the paper, in the diffusive random process approximation, the probability density for the particle position, i.e., the function $P_t(\mathbf{x}) = \langle\delta(\mathbf{r}(t) - \mathbf{x})\rangle$ is described by the EFE

$$\partial P_t(\mathbf{x})/\partial t = \partial^2 [F_{kl} P_t(\mathbf{x})]/\partial x_k\, \partial x_l, \tag{6.4}$$

where F_{kl} determines the tensor of the diffusion coefficients and

$$F_{kl} = \frac{1}{2}\int_{-\infty}^{\infty} d\tau B_{kl}(0, \tau) = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \int d\varkappa \Phi_{kl}(\varkappa, \tau) \tag{6.5}$$

and for a statistically isotropic velocity field it takes the form $F_{kl} = F\delta_{kl}/3$.

The solution of (6.4) corresponds to a Gaussian probability distribution with a variance tensor

$$\langle r_i(t)\, r_j(t)\rangle = 2F_{ij}t. \tag{6.6}$$

We consider now the temporal correlation of the coordinates of the diffusing particle. Let, for example, $t > t'$. Then, multiplying the i-component of (6.1) by $r_j(t')$ and averaging, we obtain

$$\begin{aligned} (\partial/\partial t)\,\langle r_i(t)\, r_j(t')\rangle &= \langle u_i(\mathbf{r}(t), t)\, r_j(t')\rangle,\\ \langle r_i(t)\, r_j(t')\rangle\,|_{t=t'} &= 2F_{ij}t'. \end{aligned} \tag{6.7}$$

The right-hand side of (6.7), as can be readily seen, vanishes by virtue of the δ-correlation of the field $\mathbf{u}$ with respect to t, and therefore

$$\langle r_i(t)\, r_j(t')\rangle = 2F_{ij}\min\{t, t'\}. \tag{6.8}$$

With the aid of (6.6) or (6.8) we can obtain the correlations of the position of the particle and its velocity, and the correlation between its velocities:

$$\langle r_i(t)\, V_j(t)\rangle = F_{ij}, \quad \langle V_i(t)\, V_j(t')\rangle = 0. \tag{6.9}$$

We consider now the problem of the joint diffusion of two particles. In this case we have the dynamic system

$$d\mathbf{r}_i/dt = \mathbf{u}(\mathbf{r}_i, t), \quad \mathbf{r}_i(0) = r_i^0, \tag{6.10}$$

where i = 1 and 2 numbers the particles. Introducing the six-dimensional vectors $\boldsymbol{\xi} = \{\mathbf{r}_1, \mathbf{r}_2\}$, $\mathbf{f} = \{\mathbf{u}(\mathbf{r}_1, t), \mathbf{u}(\mathbf{r}_2, t)\}$, we can write down this system in the form (2.1), and consequently we can write for the joint probability density the EFE

$$\begin{aligned} P_t(\mathbf{x}, \mathbf{y}) &= \langle\delta(\mathbf{r}_1 - \mathbf{x})\,\delta(\mathbf{r}_2 - \mathbf{y})\rangle,\\ \frac{\partial P_t}{\partial t} &= \left[\frac{\partial^2}{\partial x_i\,\partial x_j} + \frac{\partial^2}{\partial y_i\,\partial y_j}\right] F_{ij}(0)\, P_t + 2\frac{\partial^2}{\partial x_i\,\partial y_j}[F_{ij}(\mathbf{x}-\mathbf{y})\, P_t]. \end{aligned} \tag{6.11}$$

In (6.11) we can introduce new variables $\boldsymbol{\rho} = \mathbf{x} - \mathbf{y}$, $\mathbf{R} = (\mathbf{x} + \mathbf{y})/2$. After changing over to the new variables, we can integrate with respect to $\mathbf{R}$ and obtain an equation for the function $P_t(\boldsymbol{\rho}) = \langle\delta(\mathbf{r}_1 - \mathbf{r}_2 - \boldsymbol{\rho})\rangle$, which describes the relative diffusion of the two particles:

$$\partial P_t/\partial t = \partial^2 [D_{ij}(\rho) P_t]/\partial\rho_i\,\partial\rho_j, \tag{6.12}$$

where the quantity $D_{ij}(\boldsymbol{\rho}) = 2[F_{ij}(0) - F_{ij}(\boldsymbol{\rho})]$ plays the role of the variable in the space of the diffusion-coefficient tensor.

In the case $\rho > l_0$ (l_0 is the spatial correlation radius of the field $\mathbf{u}$), we have $D_{ij}(\boldsymbol{\rho}) = 2F_{ij}(0)$. This relation denotes that if the initial distances between the particles are large in comparison with l_0, their relative diffusion occurs with double the diffusion coefficient of one particle, corresponding to independent diffusion of each particle. The joint probability distribution can be regarded in this case as Gaussian.

In the general case, however, Eq. (6.12) cannot be solved. It is clear only that when the diffusion coefficient is variable the solution is not a Gaussian distribution.

The case $\rho \ll l_0$ admits, however, of a more complete analysis. Expanding $D_{ij}(\boldsymbol{\rho})$ in powers of $\boldsymbol{\rho}$, we find that at $\rho \ll l_0$ we have

$$D_{ij}(\rho) = g_{ij,\alpha\beta}\rho_\alpha\rho_\beta,\quad g_{ij,\alpha\beta} = (1/2)\int_{-\infty}^{\infty} d\tau \int d\varkappa\,\varkappa_\alpha\varkappa_\beta\Phi_{ij}(\varkappa, \tau). \tag{6.13}$$

If Eq. (6.12) with the coefficients (6.13) is multiplied by $\boldsymbol{\rho}^2$ and integrated over all $\boldsymbol{\rho}$, then after integration by parts we obtain a closed equation for $\langle\boldsymbol{\rho}^2\rangle$

$$\frac{d}{dt}\langle\rho^2\rangle = \gamma\langle\rho^2\rangle,\quad \gamma = \frac{1}{3}\int_{-\infty}^{\infty} d\tau \int d\varkappa\,\varkappa^2\Phi_{ii}(\varkappa, \tau). \tag{6.14}$$

Solving (6.14) with the initial condition $\langle\boldsymbol{\rho}^2\rangle|_{t=0} = \rho_0^2$, we obtain the solution

$$\langle\rho^2(t)\rangle = \rho_0^2 \exp(\gamma t). \tag{6.15}$$

For the time interval $\gamma t \ll 1$ it follows from (6.15) that

$$\langle\rho^2(t)\rangle = \rho_0^2(1 + \gamma t). \tag{6.16}$$

It is clear that in this case the probability distribution for $\rho(t)$ can also be regarded as Gaussian.

If there exists an interval of values of t on which $\gamma t \gg 1$, but nevertheless $\rho_0^2 \exp(\gamma t) \ll l_0^2$ (such an interval always exists for sufficiently small ρ_0), then the quantity $\langle\boldsymbol{\rho}^2(t)\rangle$ increases exponentially in this region. Formula (6.15) no longer holds when $\langle\boldsymbol{\rho}^2\rangle$ becomes comparable with l_0^2. At large values of t, the relative diffusion will be described by the formula

$$\langle\rho^2(t)\rangle = 4F_{ii}(0)\,t. \tag{6.17}$$

We note that in the case when there is a constant average transport of particles by the field $\mathbf{u}$ ($\langle\mathbf{u}\rangle = \mathbf{U}$), then the diffusion of an individual particle is described by the probability density $P_t(\mathbf{x} - \mathbf{U}t)$, and the relative diffusion of the two particles does not depend on $\mathbf{U}$.

We now stop to discuss the conditions under which we can describe the diffusion of particles in the EFE approximation. We rewrite (6.1) in the form of an integral equation

$$r_i(t) = \int_0^t d\tau \int d\varkappa\, u_i(\varkappa, \tau) \exp[i\varkappa \times \mathbf{r}(\tau)], \tag{6.18}$$

where

$$u_i(\varkappa, t) = (2\pi)^{-3} \int d\mathbf{x}\, u_i(\mathbf{x}, t) \exp(-i\varkappa\mathbf{x}).$$

The statistical characteristics of the field $\mathbf{u}(\boldsymbol{\kappa}, t)$ are determined completely by the spatial spectral tensor $\Phi_{ij}(\boldsymbol{\kappa}, t)$ (6.3)

$$\langle u_i(\mathbf{q}_1, t + \tau)\, u_j(\mathbf{q}_2, t)\rangle = \delta(\mathbf{q}_1 + \mathbf{q}_2)\,\Phi_{ij}(\mathbf{q}_1, \tau). \tag{6.19}$$

Squaring (6.18) and averaging, we obtain for $\langle\mathbf{r}_i^2(t)\rangle$ an equation consisting of two terms, the first of which is connected with the direct averaging of the product of the fields $\mathbf{u}$, and the second with the correlation of the field $\mathbf{u}$ and the function $\mathbf{r}$. We consider the first term, which can be written in the form

$$\int_0^t\int_0^t d\tau_1\,d\tau_2 \int d\varkappa\,\Phi_{ii}(\varkappa, \tau_1 - \tau_2)\,\langle\exp\{i\varkappa[\mathbf{r}(\tau_1) - \mathbf{r}(\tau_2)]\}\rangle. \tag{6.20}$$

Since we are interested only in the condition for the applicability of formula (6.6), obtained on the basis of the EFE (6.4), to estimate $\langle\mathbf{r}_i^2(t)\rangle$ it suffices to calculate the average value in (6.20) in the diffusion approximation. In this approximation, the quantities $\mathbf{r}(t)$ are Gaussian random functions determined by the correlation (6.8), and therefore the expression (6.20) can be rewritten in the form

$$2\int_0^t d\tau\,(t - \tau)\int d\varkappa\,\Phi_{ii}(\varkappa, \tau)\exp\left(-\frac{1}{3}\varkappa^2 F\tau\right)$$

Let the field $\mathbf{u}$ be characterized by the temporal and spatial scales τ_0 and l_0, respectively. In the case when the inequality $F\tau_0 \ll l_0^2$ is satisfied, the exponential in the integral does not play any role, and can be replaced by unity. If we are interested here by times $t \gg \tau_0$ and take into account the fact that $\Phi_{ii}(\boldsymbol{\kappa}, \tau)$ is even in τ, then we arrive at expression (6.6), but with the indices reversed. Thus, the first term obtained when $\langle\mathbf{r}_i^2\rangle$ is calculated leads to (6.6) under the following conditions

$$F\tau_0 \ll l_0^2,\quad t \gg \tau_0. \tag{6.21}$$

Allowance for the second term does not lead to any new limitations. Thus, the conditions (6.21) are conditions for the applicability of formula (6.6) and by the same token for the applicability of the diffusive approximation. The first inequality in (6.21) can be rewritten in a form that has a lucid physical meaning $\langle\mathbf{r}^2(\tau_0)\rangle \ll l_0^2$, i.e., it reduces to smallness of the displacement of the diffusing particle during the time τ_0, in comparison with the spatial scale of the fluctuations of the field $\mathbf{u}$.

We note that the diffusive approximation is not suitable to describe particle diffusion in a developed turbulent stream, for in this case it is precisely the relation $\langle\mathbf{r}^2(\tau_0)\rangle \sim l_0^2$ which holds true.

7. DIFFUSION OF RAYS IN RANDOMLY-INHOMOGENEOUS MEDIA

A problem closely related to that considered above is that of the diffusion of rays in randomly-inhomogeneous media, which serves in a large number of papers as the basis for describing the propagation of waves in a medium with fluctuating inhomogeneities in the geometrical-optics approximation[53-56].

Usually the variable describing the behavior of rays is the length of the ray l. Taking l to be the independent variable, we can write down the ray equations in the form

$$dr_i(l)/dl = \tau_i(l),\quad d\tau_i(l)/dl = (\delta_{ik} - \tau_i\tau_k)\,\partial\mu(\mathbf{r})/\partial r_k \quad (i = 1, 2, 3;\ |\boldsymbol{\tau}| = 1), \tag{7.1}$$

where $\mu = \ln n$, and n is the refractive index. The ray equations have been written in this form, for example, in[57,58]. We assume that μ is a Gaussian homogeneous isotropic random field with zero mean value. If we introduce the six-vectors $\boldsymbol{\xi} = \{\mathbf{r}, \boldsymbol{\tau}\}$, $\mathbf{v} = \{\boldsymbol{\tau}, 0\}$ and $\mathbf{f} = \{0, \mathbf{a}\}$, where $a_i = (\delta_{ik} - \tau_i\tau_k)\partial\mu(\mathbf{r})/\partial r_k$, then the system (7.1) can be rewritten in the form (2.1). Conditions (a) and (b) are satisfied, and as to the transition to the correlation-function approximation for μ with the aid of the δ function (2.4), we encounter here an unsurmountable difficulty. The point is that $a_i = a_i(\mathbf{r}, \boldsymbol{\tau}) \equiv a_i(\boldsymbol{\xi})$ and does not depend on l (which in this case plays the role of the time). Formally it can be assumed that the function $B_{kl}(\boldsymbol{\xi}, \boldsymbol{\xi}', l, l')$ (which does not depend on l and l' in this case) has an infinite correlation interval with respect to the variable $(l - l')$, inasmuch as this function does not decrease at arbitrary $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$. Thus, it is impossible to write down an EFE corresponding to the dynamic system (7.1).

It is possible, however, to write down ray equations by choosing as the independent variable the coordinate z. If the ray equation is sought in the form $\mathbf{R}_\perp = \mathbf{R}_\perp(z)$, where $\mathbf{R}_\perp = \{x, y\}$ is the transverse displacement, then we obtain in place of (7.1) the dynamic system

$$d\mathbf{R}_\perp(z)/dz = \boldsymbol{\tau}_\perp(z)/(1-\boldsymbol{\tau}_\perp^2)^{1/2}, \quad d\boldsymbol{\tau}_\perp/dz = \mathbf{a}_\perp(\mathbf{R}_\perp, \boldsymbol{\tau}_\perp, z)/(1-\boldsymbol{\tau}_\perp^2)^{1/2}, \tag{7.2}$$

where $\boldsymbol{\tau}_\perp = \{\tau_x, \tau_y\}$, $\mathbf{a}_\perp = \{a_1, a_2\}$. We note, however, that in this form we can use the ray equation only up to the first turning point, where the denominator $(1 - \boldsymbol{\tau}_\perp^2)^{1/2}$ vanishes. It follows therefore that in the statistical problem one can use the equations in (7.2) only in the region where the probability of negative τ_z is small, i.e., in the region of small ray inclination angles ($\tau_z \sim 1$). In this case we can write in place of (7.2) an approximate system of equations for the rays in the small-angle approximation:

$$d\mathbf{R}_\perp(z)/dz = \boldsymbol{\tau}_\perp(z), \quad d\boldsymbol{\tau}_\perp(z)/dz = \nabla_\perp\mu(\mathbf{R}_\perp, z). \tag{7.3}$$

For the system (7.3), an independent variable no longer enters among the arguments of $\mathbf{f}$, and we can therefore go over to the diffusion approximation. The corresponding EFE (2.10) takes the form

$$\partial P_z/\partial z + \boldsymbol{\tau}_\perp \partial P_z/\partial \mathbf{R}_\perp = D\partial^2 P_z/\partial\boldsymbol{\tau}_\perp^2, \tag{7.4}$$

where D is the diffusion coefficient, which arises when F_{kl} is calculated and is given for statistically homogeneous and isotropic fluctuations of μ by

$$D = \pi^2 \int_0^\infty d\varkappa\varkappa^3\Phi(\varkappa), \tag{7.5}$$

$\Phi(\kappa)$ is the three-dimensional spectral density of the correlation function μ.

Equation (7.4) can be easily solved, and its solution, corresponding to the initial condition $P_0(\mathbf{R}_\perp, \boldsymbol{\tau}_\perp) = \delta(\mathbf{R}_\perp)\delta(\boldsymbol{\tau}_\perp)$, takes the form of a Gaussian distribution with moments

$$\langle R_{\perp i}(z) R_{\perp k}(z)\rangle = (2/3) D\delta_{ik}z^3, \quad \langle R_{\perp i}(z)\tau_{\perp k}(z)\rangle = D\delta_{ik}z^2, \quad \langle \tau_{\perp i}(z)\tau_{\perp k}(z)\rangle = 2D\delta_{ik}z. \tag{7.6}$$

On the basis of (7.3) we can easily obtain also the longitudinal correlation function of the ray displacements:

$$\frac{\langle \mathbf{R}_\perp(z)\mathbf{R}_\perp(z')\rangle}{(\langle \mathbf{R}_\perp^2(z)\rangle\langle \mathbf{R}_\perp^2(z')\rangle)^{1/2}} = \left(1+\frac{3}{2}\xi\right)(1+\xi)^{-3/2}, \quad \xi = \frac{|z-z'|}{z_{\min}}. \tag{7.7}$$

We consider now the problem of joint diffusion of two rays. In this case we have the following eighth-order dynamic system:

$$\partial\mathbf{R}_{\perp i}/\partial z = \boldsymbol{\tau}_{\perp i}, \quad d\boldsymbol{\tau}_{\perp i}/dz = \partial\mu(\mathbf{R}_{\perp i}, z)/\partial\mathbf{R}_{\perp i}$$

$$(i = 1, 2 - \text{ray number}), \tag{7.8}$$

and, proceeding in analogy with the case of particle diffusion, we arrive at an EFE describing the relative diffusion of the rays,

$$P_z(\boldsymbol{\rho}, \mathbf{l}) = \langle \delta(\mathbf{R}_{\perp 1} - \mathbf{R}_{\perp 2} - \boldsymbol{\rho})\,\delta(\boldsymbol{\tau}_{\perp 1} - \boldsymbol{\tau}_{\perp 2} - \mathbf{l})\rangle,$$
$$\partial P_z/\partial z + \mathbf{l}\,\partial P_z/\partial\boldsymbol{\rho} = D_{\alpha\beta}(\boldsymbol{\rho})\,\partial^2 P_z/\partial l_\alpha\,\partial l_\beta, \tag{7.9}$$
$$D_{\alpha\beta}(\boldsymbol{\rho}) = 2\pi\int d\boldsymbol{\varkappa}\,[1 - \cos\boldsymbol{\varkappa}\boldsymbol{\rho}]\,\varkappa_\alpha\varkappa_\beta\Phi(\varkappa).$$

If l_0 is the correlation radius for the gradients of the refractive index, then $D_{\alpha\beta}(\boldsymbol{\rho}) = 2D\delta_{\alpha\beta}$ at $\rho \gg l_0$. This relation means that if the initial distances between the rays are large in comparison with l_0, then their relative diffusion takes place with double the diffusion coefficient, corresponding to independent diffusion of each ray. The joint probability distribution can be regarded in this case as Gaussian. In the case $\rho \ll l_0$ we have

$$D_{\alpha\beta}(\boldsymbol{\rho}) = \pi B(\rho^2\delta_{\alpha\beta} + 2\rho_\alpha\rho_\beta), \quad B = (\pi/4)\int_0^\infty d\varkappa\varkappa^5\Phi(\varkappa). \tag{7.10}$$

We note that the quantity B determines in the geometrical-optics approximation the amplitude fluctuations $\sigma_I^2 = \pi B z^3/3$[59] (see Chap. 10). This is natural, since the amplitude fluctuations are connected with the changes in the cross section of the ray tube, i.e., with the relative displacements of the rays. An approach to the wave-intensity fluctuations, based on this fact, was considered in[60,96].

Expression (7.10) can be used only if the mean-squared distance between rays is small in comparison with l_0^2. In this case the equations for the moments form a closed system that can be easily solved[21]. From the solution of this system it follows that if there exists an interval of values of z such that $\alpha z \gg 1$ ($\alpha = (16\pi B)^{1/3}$), but still $l_0^2 \gg \rho_0^2\exp(\alpha z)$ (it always exists for sufficiently small ρ_0^2), then exponential growth of $\langle\boldsymbol{\rho}^2\rangle$, $\langle\boldsymbol{\rho}\cdot\mathbf{l}\rangle$, and $\langle\mathbf{l}^2\rangle$ occurs in this region. We note that the start of the exponential-growth region, determined by the condition $\alpha z \sim 1$, coincides with the start of the region of strong intensity fluctuations, since $\alpha z \sim \sigma_I^{2/3} = \langle[\ln(I/I_0)]^2\rangle^{1/3}$.

We now stop to discuss the conditions for the applicability of the EFE. As already noted, the EFE for ray diffusion can be justified only in the small-angle approximation[10]. This, in accordance with (7.6), leads to the condition

$$\langle\boldsymbol{\tau}_\perp^2\rangle \ll 1 \quad \text{or} \quad Dz \sim \sigma_\mu^2 z/l_0 \ll 1. \tag{7.11}$$

This limitation imposes a very weak condition on the transverse ray displacement $\langle\mathbf{R}_\perp^2(z)\rangle \ll z^2$. As to the corrections connected with the finite character of the longitudinal correlation radius, the requirement that they be small leads to the condition $z \gg l_0$ and again to (7.11).

If the inequality (7.11) is not satisfied, the small-angle approximation no longer holds, and with further distribution of the rays the probability distribution law tends to an isotropic law that is the analog of the Boltzmann distribution in statistical mechanics[61].

With this example we complete the analysis of physical processes based on a system of ordinary differential equations, and proceed to their infinite-dimensional generalization.

8. EQUILIBRIUM HYDRODYNAMIC FLUCTUATIONS IN AN IDEAL GAS

As noted in the third part of the paper, a general correlation theory of hydrodynamic fluctuations was constructed in a paper by Landau and Lifshitz[30] by introducing "extraneous" random terms in the hydrodynamic equation, namely the "extraneous" stress tensor s_{ik} in the Navier-Stokes equation and the vector of "extraneous" thermal flux $\mathbf{g}$ in the heat transfer equation. We shall not stop to derive these stochastic equations (they can be obtained by different methods on the basis of either the Boltzmann equations[62-65] or the Klimontovich kinetic equation[66]), but, following[67], consider the solution of this statistical problem for an ideal gas.

a) Formulation of the problem. Assume that at $t < 0$ the hydrodynamic system was in a state of thermodynamic equilibrium, $\mathbf{v} = 0$, $\rho = \rho_0$, $p = p_0$, $T = T_0$, and $S = S_0$ (the notation is the same as in the book[57]). At the instant of time $t = 0$ the molecular noise characterized by the random fields s_{ik} and $\mathbf{g}$ is turned on. Then the equilibrium hydrodynamic fluctuations in an ideal gas are described in the absence of dispersion by a system of equations linearized with respect to the equilibrium states:

$$\frac{\partial v_i}{\partial t}+c_0^2\frac{\partial P}{\partial x_i}-\nu\Delta v_i-\left(\eta+\frac{\nu}{3}\right)\frac{\partial^2 v_k}{\partial x_i\,\partial x_k}=\frac{1}{\rho_0}\frac{\partial s_{ik}}{\partial x_k},\quad \frac{\partial R}{\partial t}+\nabla\mathbf{v}=0,$$
$$\frac{\partial P}{\partial t}+\nabla\mathbf{v}-\lambda\gamma\Delta P+\lambda\Delta R=-\frac{1}{\rho_0 T_0 c_p}\nabla\mathbf{g},\quad \frac{T}{T_0}=\gamma P-R,\quad S=c_p(P-R); \tag{8.1}$$

here $P = p/\gamma p_0$, $R = \rho/\rho_0$, $c_0^2 = \gamma p_0/\rho_0$ is the square of the speed of sound; $\gamma = c_p/c_v$; ν and η are the first- and second-viscosity kinematic coefficients; λ is the coefficient of diffusivity, and the "extraneous" fields are Gaussian random fields with correlation relations

$$\begin{aligned}\langle s_{ik}(\mathbf{r}_1,t_1)\,s_{lm}(\mathbf{r}_2,t_2)\rangle &=2kT_0\rho_0\left[\nu(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl})+\left(\eta-\frac{2}{3}\nu\right)\delta_{ik}\delta_{lm}\right]\delta(\mathbf{r}_1-\mathbf{r}_2)\,\delta(t_1-t_2),\\ \langle s_{ik}(\mathbf{r}_1,t_1)\,g_l(\mathbf{r}_2,t_2)\rangle=0,\quad \langle g_i(\mathbf{r}_1,t_1)\,g_k(\mathbf{r}_2,t_2)\rangle &=2kT_0^2\rho_0 c_p\lambda\delta_{ik}\delta(\mathbf{r}_1-\mathbf{r}_2)\,\delta(t_1-t_2).\end{aligned} \tag{8.2}$$

The problem consists of finding the stationary statistical characteristics of the solution of the system (8.1).

We note that this problem is an infinite-dimensional generalization of the Langevin equation, and the corresponding diffusion tensor, determined by formulas (8.2), is a corresponding generalization of the Einstein formulas (3.6) and (3.20).

The system (8.1) without right-hand sides (sources) was investigated by many workers (see, e.g.,[50]), who have shown that all the motions of the medium break up into three types that do not interact with one another: vortical, acoustical, and entropy waves. The interaction between them are described by nonlinear terms of the hydrodynamic equations. This circumstance holds also for the system (8.1), which describes generation of waves of the indicated type by the thermal motion of the molecules.

In the case when there are no sources, it is easy to obtain a solution of (8.1). Introducing the Fourier transforms for the hydrodynamic fields

$$\varphi(\mathbf{r},t)=\int d\mathbf{q}\,\tilde{\varphi}(\mathbf{q},t)\exp(i\mathbf{q}\mathbf{r})\quad(\varphi=\{P,\ R,\ D=\nabla\mathbf{v}\})$$

and using the smallness of the parameter $\delta = \nu q/c_0 \sim l/\Lambda \ll 1$, where l is the mean free path in the gas and $\Lambda = 2\pi/q$ is the perturbation wavelength ($\nu \sim \eta \sim \lambda$ and $\delta \ll 1$ are the conditions for the applicability of the hydrodynamic description of the medium), we obtain for $\tilde{\varphi}(\mathbf{q}, t)$

$$\tilde{\varphi}(\mathbf{q},t)=C_1^{\tilde{\varphi}}\exp(-\lambda q^2 t)+C_2^{\tilde{\varphi}}\exp[ic_0qt-(zq^2/2)\,t^2]+C_3^{\tilde{\varphi}}\exp[-ic_0qt-(zq^2/2)\,t], \tag{8.3}$$

where $z = \eta + (4\nu/3) + \lambda(\gamma - 1)$ determines the damping of the acoustic waves.

b) Spatial correlations of hydrodynamic fields. We note first that by virtue of the linearity of the system (8.1), all the hydrodynamic fields will be Gaussian random fields, the statistical properties of which are described completely by their space-time correlation tensors. In view of the spatial homogeneity of the "sources," the hydrodynamic fields are also homogeneous random fields. We note also that since the system (8.1) is of first order in the time, the condition that the "sources" be δ-correlated in t is evidence that the process of generation of hydrodynamic fluctuations is a diffusive random process.

We consider equal-time correlations of the hydrodynamic fields

$$B_{\varphi,\psi}(\mathbf{r},t)=\langle\varphi(\mathbf{r}_1+\mathbf{r},t)\,\psi(\mathbf{r}_1,t)\rangle,\quad\text{where}\quad\{\varphi,\psi\}=\{D,\ R,\ P\}.$$

In the derivation of the dynamic equations for $B_{\varphi,\psi}$ there appear terms due to the correlations of the "sources" $s_{ik}(t)$ and $\mathbf{g}(t)$ with the fields φ, taken at the same instant of time. The Furutsu-Novikov formula (29) makes it possible to calculate such correlations. Using at the same time the expressions for the values of the variational derivatives of the hydrodynamic fields with respect to the sources, taken at the same instant of time

$$\frac{\delta D(\mathbf{r},t)}{\delta s_{ik}(\mathbf{r}',t)}=\frac{1}{\rho_0}\frac{\partial^2}{\partial r_i\,\partial r_k}\delta(\mathbf{r}-\mathbf{r}'),\quad \frac{\delta P(\mathbf{r},t)}{\delta g_i(\mathbf{r}',t)}=-\frac{1}{\rho_0 T_0 c_p}\frac{\partial}{\partial r_i}\delta(\mathbf{r}-\mathbf{r}'),$$
$$\frac{\delta D}{\delta g_i}=\frac{\delta R}{\delta s_{ik}}=\frac{\delta R}{\delta g_j}=\frac{\delta P}{\delta s_{ik}}=0, \tag{8.4}$$

which are the consequence of the dynamic system (8.1), we obtain a closed system of equations for the equal-time correlations. The stationary solution for the spatial spectral functions

$$B_{\varphi,\psi}(\mathbf{q},t)=\frac{1}{(2\pi)^3}\int d\mathbf{r}\,B_{\varphi,\psi}(\mathbf{r},t)\exp(-i\mathbf{q}\mathbf{r})$$

is

$$B_{TR}=B_{RD}=B_{PD}=0,\quad B_{DD}=\frac{kT_0}{\rho_0(2\pi)^3}\mathbf{q}^2,\quad B_{PP}=B_{RP}=\frac{kT_0}{\rho_0(2\pi)^3}\frac{1}{c_0^2},$$
$$B_{RR}=\frac{kT_0}{\rho_0(2\pi)^3}\frac{\gamma}{c_0^2},\quad B_{TT}=\frac{kT_0}{\rho_0 c_v(2\pi)^3},\quad B_{SS}=\frac{kc_p}{\rho_0(2\pi)^3}. \tag{8.5}$$

All these formulas, of course, agree with the results of the theory of the thermodynamic fluctuations in an ideal gas[68].

It is possible to obtain analogously the stationary solution for the spectral function of the vortical part of the velocity and an expression for the stationary spectral tensor of the velocity field, the form of which is

$$B_{ij}(\mathbf{q})=[kT_0/\rho_0(2\pi)^3]\,\delta_{ij}. \tag{8.6}$$

Formulas (8.5) and (8.6) make it possible to determine the characteristic scale of the fluctuations, up to which the macroscopic turbulence theory is applicable. This scale is determined by the intersection of the corresponding macroscopic spectral functions with the functions described by the formulas given above. The corresponding estimates for turbulence in an incompressible liquid are given in[67,69].

c) Space-time correlations of hydrodynamic fields.

We proceed now to find the space-time correlations. We consider the quantities $B_{\varphi,\psi'}(\mathbf{r}, t, t') = \langle\varphi(\mathbf{r}_1+\mathbf{r}, t)\psi(\mathbf{r}_1, t')\rangle$ ($\varphi, \psi = \{D, P, R\}$). The equations for these quantities can be obtained by starting from the general system of equations (8.1). It is easy to see here that the time dependence of these quantities is determined only by the dynamic equations for the acoustic and entropy waves, and does not depend on the "sources." In fact, let us consider, for example, the equation for $B_{DD'}$. Let $t > t'$. We then obtain for $B_{DD'}$ the equation

$$\partial B_{DD'}/\partial t + c_0^2\Delta B_{PD'} - [\eta + (4\nu/3)]\,\Delta B_{DD'} = 0, \qquad (8.7)$$

since $\langle s_{\alpha\beta}(\mathbf{r}_1, t)D(\mathbf{r}_2, t')\rangle = 0$ at $t > t'$ by virtue of the fact that $s_{\alpha\beta}$ is δ-correlated with respect to t. The initial condition for (8.7) is the condition $B_{DD'}|_{t=t'} = B_{D', D'}$. It is clear that a similar picture holds also in the case of $t < t'$ and for arbitrary other correlations of the hydrodynamic fields.

The time dependence of the spectral (spatial) functions of the hydrodynamic fields will therefore be determined by expressions of the type (8.3), with t replaced by $t - t'$ or $t' - t$, depending on the ratio of the times t and t'. The corresponding constants are determined with the aid of the boundary conditions at coinciding values of the times. Thus, for example, for a stationary spectral function $B_{DD'}(\mathbf{q}, \tau)$, where $\tau = t - t'$, we obtain the expression

$$B_{DD'}(\mathbf{q}, \tau) = B_{DD}(\mathbf{q}) \exp[-(zq^2/2)\,|\,\tau\,|] \cos c_0 q\tau, \qquad (8.8)$$

where $B_{DD}(\mathbf{q})$ is described by formula (8.5).

We note in conclusion that the method described above for solving the problem of equilibrium hydrodynamic fluctuations in an ideal gas can be easily generalized to more complicated cases, such as a nonideal gas, the presence of temporal dispersion and dissipative coefficients, the presence of spatial boundaries and corresponding boundary conditions, etc.

9. RANDOM FORCES IN HYDRODYNAMIC THEORY OF TURBULENCE

We consider now a model of turbulent motion, wherein the liquid is in the field of external forces $\mathbf{f}(\mathbf{x}, t)$, which we shall regard as Gaussian homogeneous and stationary random forces with zero mean value. Such a model, of course, is fictitious, since the forces $\mathbf{f}$ have no real analogs. However, if it is assumed that the forces $\mathbf{f}$ ensure a noticeable average influx of energy only to the large-scale components of the velocity, then one can expect, as a result of the premises of the theory of locally isotropic turbulence, that the fictitious character of the field $\mathbf{f}$ does not affect the statistical properties of the small-scale components of the turbulence[50]. Therefore the small-scale properties of the turbulence can be correctly described on the basis of such a model.

The motion induced in an incompressible liquid by external forces is described by the Navier-Stokes equation

$$\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu\Delta u_i + f_i(\mathbf{x}, t),$$
$$\operatorname{div}\mathbf{u} = 0 \quad (\operatorname{div}\mathbf{f} = 0).$$

Here ρ_0 is the density of the liquid, and the pressure p is expressed in terms of the velocity field at the same instant of time with the aid of the relation

$$p(\mathbf{x}, t) = -\rho_0\Delta^{-1}(\mathbf{x}, \mathbf{x}')\cdot\partial^2[u_\alpha(\mathbf{x}', t)u_\beta(\mathbf{x}', t)]/\partial x'_\alpha\,\partial x'_\beta, \qquad (9.2)$$

where Δ^{-1} is an integral operator, the inverse of the Laplace operator, and summation over repeated indices is implied.

The Fourier transform of the velocity (with respect to the spatial coordinates)

$$u_i(\mathbf{k}, t) = \int d\mathbf{x}\, u_i(\mathbf{x}, t)\exp(-i\mathbf{k}\mathbf{x}),$$

in the presence of an external-force field, after eliminating the pressure with the aid of formula (9.2), satisfies the equation

$$\frac{\partial u_i(\mathbf{k}, t)}{\partial t} = \frac{i}{2}\int d\mathbf{k}_1\,d\mathbf{k}_2\Lambda_{i,\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k})\,u_\alpha(\mathbf{k}_1, t)\,u_\beta(\mathbf{k}_2, t) - \nu k^2 u_i(\mathbf{k}, t) + f_i(\mathbf{k}, t) \quad (u_i^*(\mathbf{k}) = u_i(-\mathbf{k})), \qquad (9.3)$$

where

$$\Lambda_{i,\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}) = (2\pi)^{-3}\{k_\alpha\Delta_{i\beta}(\mathbf{k}) + k_\beta\Delta_{i\alpha}(\mathbf{k})\}\,\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}),$$
$$\Delta_{ij}(\mathbf{k}) = \delta_{ij} - (k_ik_j/k^2) \quad (i, \alpha, \beta = 1, 2, 3),$$

and $f_i(\mathbf{k}, t)$ is the Fourier transform of the external forces $(\mathbf{k}\cdot\mathbf{f}(\mathbf{k}, t) = 0)$.

Since the field of the external forces $f_i(\mathbf{k}, t)$ is Gaussian, homogeneous, and stationary, the different statistical characteristics of the velocity field are determined by the quantity

$$\langle f_i(\mathbf{k}_1, t+\tau)\,f_j(\mathbf{k}_2, t)\rangle = (1/2)\,\delta(\mathbf{k}_1 + \mathbf{k}_2)\,F_{ij}(\mathbf{k}_1, \tau). \qquad (9.4)$$

Following[16], we assume $F_{ij}(\mathbf{k}, \tau) = F_{ij}(\mathbf{k})\delta(\tau)$. Since (9.3) is an equation of first order in the time, the excitation of the turbulence in this model is a diffusive random process, since the forces are δ-correlated with respect to t.

The most compact complete description of the spatial statistical characteristics of the field $\mathbf{u}(\mathbf{k}, t)$ consists of specifying its characteristic functional

$$\Phi_t[z(\mathbf{k})] = \langle\exp\{i\int d\mathbf{k}\, z_\alpha(\mathbf{k})\,u_\alpha(\mathbf{k}, t)\}\rangle, \qquad (9.5)$$

with the limitation $z_\alpha^*(\mathbf{k}) = z_\alpha(-\mathbf{k})$ imposed on $z_\alpha(\mathbf{k})$.

An equation for this functional can be obtained by differentiating (9.5) with respect to t. Using in this case Eq. (9.3), formula (2.2), and the relation

$$\delta u_i(\mathbf{k}, t)/\delta f_j(\mathbf{k}', t) = \delta_{ij}\delta(\mathbf{k} - \mathbf{k}'), \qquad (9.6)$$

which is the consequence of (9.3), we can write down an equation for Φ_t in the form[16]

$$\frac{\partial\Phi_t}{\partial t} = \int d\mathbf{k}z_i(\mathbf{k})\Big[-\frac{1}{2}\int d\mathbf{k}_1\,d\mathbf{k}_2\Lambda_{i,\alpha\beta}(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k})\frac{\delta^2\Phi_t}{\delta z_\alpha(\mathbf{k}_1)\,\delta z_\beta(\mathbf{k}_2)} - \nu k^2\frac{\delta\Phi_t}{\delta z_i(\mathbf{k})} - \frac{1}{4}F_{ij}(\mathbf{k})\,z_j(-\mathbf{k})\,\Phi_t\Big]. \qquad (9.7)$$

An equation for Φ_t in the absence of external forces was obtained in[70].

Equation (9.7) plays the role of the EFE for the considered problem. It differs from the usual equation of this type in that it is written down for the characteristic functional and not for the probability density which is Fourier-conjugate with this functional. Therefore Eq. (9.7) itself is a functional Fourier transform of the EFE.[11)] Another difference is that (9.7) corresponds to the diffusion equation in infinite-dimensional space, and is therefore an equation with variational derivatives. The role of the diffusion coefficient, which differs for different wave components of the velocity field, is played by the spectral tensor of the forces $F_{ij}(\mathbf{k})$.

We note that allowance for the finite character of the temporal correlation radius of the field of the external forces leads to a system of two functional equations[72], from which it is possible to determine in principle the conditions for the applicability of Eq. (9.7), i.e., the diffusion approximation for the considered problem.

10. PROPAGATION OF LIGHT IN A RANDOMLY-INHOMOGENEOUS MEDIUM

When electromagnetic waves propagate in a medium with random large-scale inhomogeneities (compared to the wavelength), multiple forward scattering causes the fluctuations of the wave field to increase rapidly with increasing distance. Starting with a certain distance, calculations by perturbation theory in one form or another are no longer suitable. This effect was observed experimentally by Gracheva and Gurvich in experiments on the propagation of light in the turbulent atmosphere[73], and was subsequently investigated in greater detail in[74-76].

The general status of the theory of light propagation in randomly-inhomogeneous media is described in the review papers[77,78]. Following[78], we consider henceforth the description of the propagation of light in a randomly-inhomogeneous medium in the diffusive random process approximation.

a) **Diffusive approximation.** The propagation of monochromatic light in a medium with large-scale inhomogeneities, when the depolarization is small, can be described with sufficient accuracy by a scalar wave equation[59]

$$\Delta\psi + k^2 [1 + \tilde{\varepsilon}(\mathbf{r})]\psi = 0; \tag{10.1}$$

where ψ is connected with the component of the electric field E by the relation $E = \psi \exp(-i\omega t)$, $k^2 = (\omega^2/c^2)\langle\epsilon\rangle$ where $\tilde{\epsilon}(\mathbf{r}) = [\epsilon(\mathbf{r}) - \langle\epsilon\rangle]/\langle\epsilon\rangle$ is the fluctuating part of the dielectric constant. If we neglect large-angle scattering, then we can use in place of (10.1) a parabolic equation for the function u, which is connected with ψ by the relation

$$\begin{gathered}\psi = u \exp(ikx),\\ 2ik\,\partial u/\partial x + \Delta_\perp u + k^2\tilde{\varepsilon}(x, \rho)\, u(x, \rho) = 0,\\ \rho = \{x, y\},\quad \Delta_\perp = \partial^2/\partial y^2 + \partial^2/\partial z^2,\end{gathered} \tag{10.2}$$

where the x axis is chosen in the direction of the initial propagation of the wave.

On going from (10.1) to (10.2), we discard the term $\partial^2 u/\partial x^2$.

The initial condition for (10.2) is

$$u(0, \rho) = u_0(\rho). \tag{10.3}$$

We assume the field $\tilde{\epsilon}(x, \boldsymbol{\rho})$ to be a Gaussian homogeneous random field with a correlation function $B_\epsilon(x, \boldsymbol{\rho}) = \langle\tilde{\epsilon}(x_1 + x, \boldsymbol{\rho}_1 + \boldsymbol{\rho})\,\tilde{\epsilon}(x_1, \boldsymbol{\rho}_1)\rangle$.

We note that the light-wave propagation process described by (10.2) with initial condition (10.3) is a generalization, to include the infinite-dimensional case, of the equation considered in Chap. 5 and describing parametric resonance in vibrational systems with random parameters.

Since Eq. (10.2) is of first order in x with an initial condition at x = 0, we can use the diffusive random process approximation (the role of the time t is played by the longitudinal coordinate x). As indicated in Chap. 2, in this approximation the correlation function of the field $\tilde{\epsilon}$ is replaced by the effective function

$$B_\varepsilon^{\mathrm{eff}}(x, \rho) = \delta(x) A(\rho),\quad A(\rho) = \int_{-\infty}^{\infty} dx B_\varepsilon(x, \rho). \tag{10.4}$$

A complete statistical description of the field $u(x, \boldsymbol{\rho})$ in the plane x = const is contained in its characteristic functional

$$\Psi_x[v, v^*] = \langle\exp\{i\int d\rho\,[u(x, \rho)\,v(\rho) + u^*(x, \rho)\,v^*(\rho)]\}\rangle, \tag{10.5}$$

where asterisks denote complex conjugate quantities.

Differentiating (10.5) with respect to x, using the dynamic equation (10.2), the Furutsu-Novikov formula (2.9), and the relation

$$\delta u(x, \rho)/\delta\tilde{\varepsilon}(x, \rho') = i(k/2)\,\delta(\rho - \rho')\,u(x, \rho), \tag{10.6}$$

we obtain a dynamic equation for Ψ_x[79]

$$\begin{aligned}\frac{\partial\Psi_x}{\partial x} = \frac{i}{2k}\int d\rho\Big[v(\rho)\,\Delta_\perp\frac{\delta\Psi_x}{\delta v(\rho)} - v^*(\rho)\,\Delta_\perp\frac{\delta\Psi_x}{\delta v^*(\rho)}\Big]\\ -\frac{k^2}{8}\int d\rho\, d\rho'\, A(\rho - \rho')\,\hat{M}(\rho)\,\hat{M}(\rho')\,\Psi_x,\end{aligned} \tag{10.7}$$

where the Hermitian operator is given by

$$\hat{M}(\rho) = v(\rho)\frac{\delta}{\delta v(\rho)} - v^*(\rho)\frac{\delta}{\delta v^*(\rho)}.$$

Equation (10.7) plays the role of the EFE for the considered problem (with a variable—quadratic—diffusion coefficient).

From (10.7) we can obtain, by variational differentiation, equations in closed form for the moments of the field u in the plane x = const:

$$\begin{gathered}M_{n,m}(x; \{\rho_i\}, \{\rho_j'\}) = \langle u(x, \rho_1)\ldots u(x, \rho_n)\,u^*(x, \rho_1')\ldots u^*(x, \rho_m')\rangle,\\ \frac{\partial}{\partial x}M_{n,m}\\ = \frac{i}{2k}[\Delta_1 + \ldots + \Delta_n - \Delta_1' - \ldots - \Delta_m']\,M_{n,m} - \frac{k^2}{8}Q(\rho_1, \ldots, \rho_m')\,M_{n,m},\end{gathered} \tag{10.8}$$

where

$$Q_{n,m} = \sum_{i=1}^{n}\sum_{j=1}^{n}A(\rho_i - \rho_j) - 2\sum_{i=1}^{n}\sum_{k=1}^{m}A(\rho_i - \rho_k') + \sum_{k=1}^{m}\sum_{l=1}^{m}A(\rho_k' - \rho_l'). \tag{10.9}$$

Equations (10.8) were obtained by different procedures in[80,81], and various particular cases of these equations were obtained in[82-84].

We note that the diffusive approximation can be developed also in the case of non-Gaussian fluctuations of the field $\tilde{\epsilon}$. In this case the moments of the field u in the plane x = const also satisfy closed-form equations which, however, are integro-differential equations and generally speaking cannot be reduced to differential equations[80].

We write down in explicit form equations for the average field $\langle u\rangle$ and for the second-order coherence function $\Gamma_2(x, \mathbf{R}, \boldsymbol{\rho}) = \langle u(x, \mathbf{R} + (1/2)\boldsymbol{\rho}) \times u^*(x, \mathbf{R} - (1/2)\boldsymbol{\rho})\rangle$, which follows from (10.8) at n = 1, m = 0 and n = 1, m = 1:

$$\partial\langle u\rangle/\partial x = (i/2k)\,\Delta_\perp\langle u\rangle - (k^2/8)\,A(0)\,\langle u\rangle,\quad \langle u(0, \rho)\rangle = u_0(\rho), \tag{10.10}$$

$$\begin{gathered}\partial\Gamma_2/\partial x = (i/k)\,\partial^2\Gamma_2/\partial\mathbf{R}\,\partial\rho - (k^2/4)\,D(\rho)\Gamma_2,\quad D(\rho) = A(0) - A(\rho),\\ \Gamma_2(0, \mathbf{R}, \rho) = u_0(\mathbf{R} + (1/2)\rho)\,u_0^*(\mathbf{R} - (1/2)\rho).\end{gathered} \tag{10.11}$$

Equation (10.11) is equivalent to the so-called small-angle approximation of the radiation transport equation[82], and describes the average distribution of the intensity inside a random medium. The equation for $\Gamma_4 = M_{2,2}$ describes already intensity fluctuations.

In addition for the equations for the average values of the product of the fields $u(x, \boldsymbol{\rho}_1) \ldots u^*(x, \boldsymbol{\rho}_m')$, where all the longitudinal coordinates are the same, we can obtain equations also for the functions $\langle u(x_1, \boldsymbol{\rho}_1) \ldots u^*(x_m', \boldsymbol{\rho}_m')\rangle$ at unequal values of x.[85] The initial conditions for these equations contain the functions $M_{n,m}$ at coinciding values of the longitudinal arguments.

Equations (10.10) and (10.11) can be solved in general form:

$$\langle u(x, \rho)\rangle = u_0(x, \rho)\exp(-\gamma x/2), \tag{10.12}$$

where $u_0(x, \rho)$ is the solution in the case of the absence of fluctuations, $\gamma = k^2A(0)/4$ is the extinction coefficient;

$$\Gamma_2(x, \mathbf{R}, \rho) = \frac{k^2}{4\pi^2x^2}\int d\mathbf{R}'\,d\rho'\,\Gamma_2(0, \mathbf{R}-\mathbf{R}', \rho-\rho') \times \exp\left[\frac{ik\rho'\mathbf{R}'}{x} - \frac{k^2}{4}\int_0^x d\xi\, D\left(\rho - \rho'\frac{\xi}{x}\right)\right], \tag{10.13}$$

where $\Gamma_2(0, \mathbf{R}, \rho)$ is the initial value of the coherence function at $x = 0$. For the particular case of a plane incident wave, when $u_0(\rho) \equiv u_0$, we have

$$\langle u\rangle = u_0\exp(-\gamma x/2),\quad \Gamma_2(x, \mathbf{R}, \rho) = |u_0|^2\exp(-k^2D(\rho)x/4). \tag{10.14}$$

If we consider the fluctuations of $\tilde{\epsilon}$ (the dielectric constant) due to turbulent fluctuations of the temperature, then the three-dimensional spectral density $\Phi_\epsilon(\kappa)$ has, in a considerable wave-number interval, the form $\Phi_\epsilon(\kappa) = AC_\epsilon^2\kappa^{-11/3}$ ($\kappa_{min} \ll \kappa \ll \kappa_{max}$), where $A = 0.033$ is a numerical constant and C_ϵ^2 is a structural characteristic of the fluctuations of $\tilde{\epsilon}$ and depends on the external parameters of the flux[59]. In this case we can calculate both the function $\Gamma_2(x, \mathbf{R}, \rho)$ and the average wave intensity $\langle I(x, \mathbf{R})\rangle = \Gamma_2(x, \mathbf{R}, 0)$. Figures 8 and 9 show the spatial spectrum of the coherence function and the average intensity of the wave, respectively, obtained in[86,87]. The same figures show the theoretical plots based on (10.13).

As regards the equation for Γ_4 and the equations describing higher moments of the wave intensity, it is impossible to solve them analytically. Results of a numerical solution of the equation for Γ_4 are presented in[88] for a correlation function of $\tilde{\epsilon}$ in the form of a Gaussian curve. The saturation of the intensity fluctuations obtained in that reference agrees qualitatively with the results of[73–76].

We note that the solution of (10.2) with initial condition (10.3) can be written, using a method proposed by Fradkin[89–91] in quantum field theory, in operator form or else in the form of a Feynman continual integral[92]. This form of the solution, using the assumption that $\tilde{\epsilon}$ has a Gaussian distribution and is δ-correlated in the longitudinal coordinate, enables us to calculate $\langle u\rangle$ and Γ_2. In the same cases it is possible to solve also Eq. (10.8). In those cases when it is impossible to obtain the solution of Eqs. (10.8), the representation of the solution in the form of a continual integral also does not make it possible to find an explicit expression for $M_{n,m}$. It is more convenient, however, for the investigation of the asymptotic behavior of the moments of the wave intensity.

b) Successive-approximation method and conditions for the applicability of the diffusive approximation. We now dwell on the conditions for the applicability of the diffusive approximation. We construct a successive-approximation theory that refines the functional dependence of the statistical characteristics of the wave on the field $\tilde{\epsilon}$.[49] The diffusive approximation is the first step in this theory; the next approximations take into account the finite character of the longitudinal correlation radius of the field $\tilde{\epsilon}$ and lead to a system of closed integro-differential equations for the moments. The successive-approximation method is constructed in the following manner: First one writes down an infinite

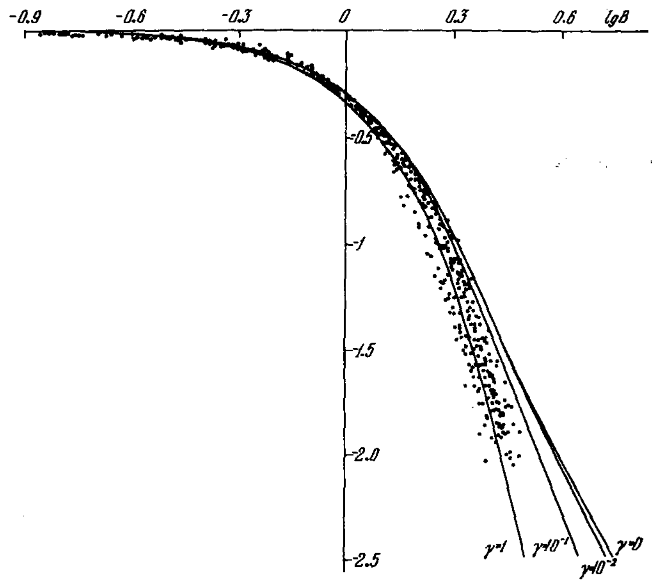

FIG. 8. Spatial spectrum of the coherence function. Solid curve—theoretical plots based on (10.13), symbols—experimental data obtained in [87] (where the symbols are explained).

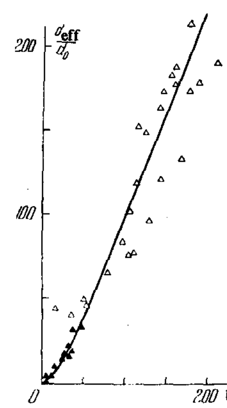

FIG. 9. Effective width of the average intensity distribution of a light beam in a turbulent medium (the solid curve corresponds to the theoretical relation, and the symbols represent the experimental data of [86]).

coupled system of equations for some moment, using the assumption that $\tilde{\epsilon}$ has a Gaussian distribution and using the Furutsu-Novikov formula, but without using the δ-correlation assumption. Each of these equations contains the correlation function $B_\epsilon(x, \rho)$. If we use the δ-correlation assumption (10.4) in the first of these equations, then we obtain the above-described diffusive approximation, and the remaining equations of the system become unnecessary. On the other hand, if we retain in the first $n - 1$ equations the exact value of $B_\epsilon(x, \rho)$ and use in the n-th equation the approximation (10.4), then we arrive at a closed system of n equations for the moment of interest to us. We shall not dwell in greater detail on this method (it is illustrated in part in the parametric-resonance problem). We note only that the second approximation is already much more exact than the diffusive approximation and describes correctly the behavior of the moments both for $x \gg l$ and for $x \ll l$ (l is the longitudinal correlation radius of the field $\tilde{\epsilon}$), and makes it possible to establish the conditions for the applicability of the first (diffusive) approximation. In the general case it is quite difficult to demonstrate this, but this question can be investigated with the aid of an example that admits of an exact solution. By way of example we consider an equation that differs from (10.2) in the absence of the term $\Delta_\perp u$:

$$du/dx = i\,(k/2)\,\tilde{\varepsilon}\,(x)\,u\,(x),\quad u\,(0) = u_0,\quad B_\varepsilon\,(x) = \langle \tilde{\varepsilon}\,(x_1 + x)\,\tilde{\varepsilon}\,(x_1)\rangle.$$

The exact solution for $\bar{u}$ is

$$\bar{u}\,(x) = u_0 \exp\Big[-(k^2/4)\int\limits_0^x d\xi\,(x-\xi)\,B_\varepsilon\,(\xi)\Big].$$

The solution of the problem in the first (diffusive) approximation is

$$\bar{u}_1\,(x) = u_0 \exp\Big[-(k^2 x/4)\int\limits_0^\infty d\xi B_\varepsilon\,(\xi)\Big].$$

Solving the system of second-approximation equations we can reduce the problem to the following integro-differential equation:

$$d\bar{u}_2/dx = -(k^2/4)\int\limits_0^x d\xi \bar{u}_2\,(\xi)\,B_\varepsilon\,(x-\xi)\exp\Big[-(k^2/4)\,(x-\xi)\int\limits_0^\infty d\eta B_\varepsilon\,(\eta)\Big],$$

which can be easily solved by means of a Laplace transformation. If, for example, $B_\epsilon(\xi) = \sigma_\epsilon^2 \exp(-\alpha|\xi|)$, then the exact solution and the first and second approximation are given respectively by

$$\bar{u}\,(x) = u_0 \exp\{-\mu\,[\tau - 1 + e^{-\tau}]\},\tag{10.15}$$

$$\bar{u}_1\,(x) = u_0 \exp(-\mu\tau),\tag{10.16}$$

$$\bar{u}_2\,(x) = [u_0/(1-\mu)]\,[\exp(-\mu\tau) - \mu\exp(-\tau)],\tag{10.17}$$

where $\tau = \alpha x$, $\mu = k^2\sigma_\epsilon^2/4\alpha^2 = k^2 l^2\sigma_\epsilon^2/4$, and $l = 1/\alpha$ is the correlation radius of $\tilde{\epsilon}$. Comparing (10.15), (10.16), and (10.17) at $\tau \gg 1$, we easily establish that the approximate solutions (10.16) and (10.17) can serve as a good approximation of the exact solution only at $\mu \ll 1$. In addition, the function (10.15), unlike (10.16), has at zero the same form as the exact solution. Figure 10 shows the functions (10.15)—(10.17) at $\mu = 0.2$. The difference between the second approximation and the exact solution does not exceed 2.5% in this case. We note also that the conditions for the applicability of the first approximation, namely $\mu \ll 1$ and $\tau \gg 1$, can be obtained not only by comparing $\bar{u}_1$ with $\bar{u}$, but also by comparing $\bar{u}_1$ with u_2.

Although the considered example cannot serve as proof of convergence of the developed method of successive approximation, it does give grounds for hoping that this proof will be obtained in the future.

The second-approximation equations can be solved for $\langle u\rangle$ and Γ_2. A comparison of the first and second approximations makes it possible to find the conditions for the applicability of the diffusive approximation. It turns out here that the applicability conditions for $\langle u\rangle$ are

$$\gamma l \ll 1,\quad x \gg l,\quad kl \gg 1,\tag{10.18}$$

where $\gamma = k^2 A(0)/4$ is the extinction coefficient and l is the correlation radius of the field $\tilde{\epsilon}$. At the same time, the conditions for applicability of the diffusive approximation for $\Gamma_2(x, \mathbf{R}, \rho)$ are

$$\rho \ll x,\quad kx\,|\nabla_\perp A\,(\rho)| \ll 1.\tag{10.19}$$

It is important to emphasize that conditions (10.18) and (10.19) are practically independent, since they impose limitations on different parameters. In particular, it may turn out that the conditions (10.19) are satisfied in the case when the condition $\gamma l \ll 1$ is violated.

As to the conditions for the applicability of the parabolic equation (10.2) itself, they can be obtained either by comparing the solutions (10.1) and (10.2), expressed in operator form[92], or by taking the term $\partial^2 u/\partial x^2$ in (10.1) into account by perturbation theory. It turns out here that the conditions for the applicability of the parabolic equation are satisfied in the entire region of applicability of the diffusive approximation.

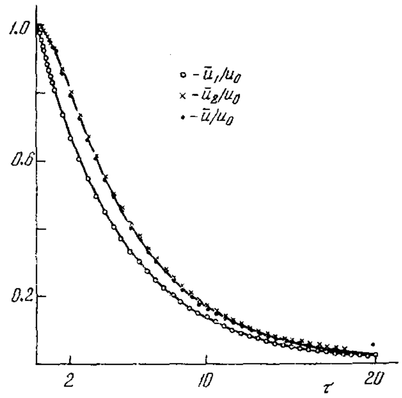

FIG. 10. Comparison of the exact solution of the problem (10.14) with the first (diffusive) and second approximations.

c) **Amplitude-phase fluctuations.** We now stop to discuss amplitude-phase fluctuations of a wave.

The equation describing the amplitude-phase fluctuations can be obtained from (10.2) by introducing a new sought function $\varphi = \ln u = \chi + iS$ ($\chi = \ln A/A_0$):

$$2ik\frac{\partial\varphi}{\partial x} + \Delta_\perp\varphi + (\nabla_\perp\varphi)^2 + k^2\tilde{\varepsilon}(x, \rho) = 0.\tag{10.20}$$

The exact solutions (10.2) and (10.20) are equivalent. The statistical properties of the solution (10.20) can also be described in the diffusive approximation. However, by virtue of the nonlinearity, the equations for the moments of the function φ are not closed.

In the first approximation of Rytov's continuous perturbation method (CPM) one omits from (10.20) the term $(\nabla_\perp\varphi)^2$. The solution in this case is a Gaussian random field and the statistical properties of the amplitude fluctuations are described by the parameter $\sigma_0^2 = \langle\chi^2(x)\rangle$.[59] It turns out here that the condition for the applicability of the first CPM approximation is $\sigma_0^2 \ll 1$. In the region $\sigma_0^2 \gtrsim 1$ (which is called the region of strong fluctuations), it is necessary to study the complete equation.

We note that the diffusive approximation for Eq. (10.2) imposes no limitations on the amplitude fluctuations, and consequently the equations for the moments of the field u (10.8) are valid also in the region of strong amplitude fluctuations.

If the geometrical-optics approximation is valid ($x/kl^2 \ll 1$), then the equation for the phase of the wave can be split off, and both the field $\chi(x, \boldsymbol{\rho})$ itself, and its statistical characteristics are determined by the functional dependences of the phase of the wave on $\tilde{\epsilon}$. The statistical characteristics χ and S are determined by the statistical properties of the rays, described by the stochastic equation[93]

$$d^2\mathbf{R}\,(x')/dx'^2 = (1/2)\,\nabla_\perp\,\tilde{\varepsilon}\,(x',\,\mathbf{R}\,(x'))\tag{10.21}$$

with boundary conditions $\mathbf{R}(x) = \boldsymbol{\rho}$, $d\mathbf{R}(x')/dx'|_{x'=0} = 0$. Since the boundary conditions of (10.21) are stipulated at different points, the causality conditions described in Chap. 2 will not be satisfied for this equation, and consequently, these rays will not be described by the diffusive approximation. Generally speaking, the probability density for these rays will be the conditional probability density of the entire aggregate of rays considered in Chap. 7.

We note that the conditions for the applicability of the diffusive approximation for the description of amplitude-phase fluctuations in the geometrical approximation coincide with the conditions for the applicability of the diffusive approximation for the description of rays in randomly-inhomogeneous media, which were considered in Chap. 7.

11. CONCLUSION

The considered examples thus demonstrate the very wide scope for employing the generalized theory of Brownian motion for highly diverse physical problems. It turns out here frequently that the method under consideration yields a more powerful tool for the solution of a particular problem than other statistical methods (an example is the problem of propagation of light in a medium with random inhomogeneities, where it is difficult to obtain a description of the strong fluctuations of the field by other methods).

Carrying out a formal classification, we can consider by this method nonlinear dynamic systems under the influence of random actions, described by systems of ordinary differential equations, and also problems that reduce to linear partial differential equations, in which the random parameters can enter either in the coefficients of the equations or in the right-hand sides. In all these cases it is possible to obtain a differential equation for the probability distribution, and in some cases also equations in closed form for the mean values, the correlation functions, and higher moments. It is this circumstance which constitutes the advantage of the considered approach, since methods based on the use of perturbation theory usually lead to infinite systems of equations for the moments, and their approximate solutions are frequently valid only under very stringent limitations. One of the advantages of the considered method is also the possibility of simultaneously investigating its applicability limits, and in many cases also the possibility of constructing more accurate approximations.

The developed method can be generalized to include the case when the random actions are not Gaussian but more complicated random processes. To obtain equations in closed form it is necessary in this case that not only the correlation functions but also the cumulant functions of higher order are products of the δ functions (an example of a dynamic system, in which a generalized Poisson process of this type is considered, is contained in[94]. Equations for the description of the propagation of light in a medium with δ-correlated non-Gaussian fluctuations of the refractive index have been constructed in[80]). In this more general case one obtains, generally speaking, integro-differential equations rather than differential equations for the moments and the distribution functions.

From the physical point of view, the theory of the generalized Brownian motion is suitable for those cases when the considered system has already experienced a sufficient number of "independent" random actions, and we can satisfy ourselves with an average description of its behavior over times longer than the time of action of a "single jolt."

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1) We note, however, that the dependence of F_{kl} and A_k on x can be connected also with the use of curvilinear coordinates.

2) The quantity D is connected with the usual diffusion coefficient K by the relation $KD = (kT)^2$.

3) Phase space is conserved in the motion of regular systems (the Liouville theorem).

4) We confine ourselves to the simplest case. In the general case the dissipative term takes the form $\lambda_{ik}v_k$.

5) The inclusion of potential mass forces in the hydrodynamic equation does not change the expressions for the total entropy, and it can therefore be assumed that the results of [30] remain in force also in this case. Generally speaking, Eq. (3.22) should be derived on the basis of non-equilibrium kinetic theory. In the case when f(t) describes the influence of small-scale motions on motions of larger scales, Eqs. (3.22) can be used as the basis of the description.

6) We can consider also a more general problem, with allowance for friction; this problem reduces by a standard procedure to Eq. (5.1).

7) We note that it is possible to find in analytic form the probability distribution for quantities averaged over the period of the oscillations (see, e.g., [48]).

8) A more complete investigation of the problem shows that the conditions of parametric excitation of the system are different for different moments. Thus, the excitation conditions for the fourth moments are weaker than for the second moments.

9) We note that Eq. (6.1) is an equation for the characteristics of the equation $dc/dt = \partial c/\partial t + u\,(5, t)\,\partial c/\partial r = 0$, which describes the mixing of impurity concentration by a random velocity field.

10) We note in this connection that in the previously cited papers [53-55] the EFE was used without justification to describe large-angle scattering of light.

11) Edwards [71] uses, instead of the characteristic functional, the "probability density of the individual realization of a random velocity field in functional space," which can be regarded as a functional Fourier transformation of the characteristic functional. The equation for this probability density is precisely a generalization of the EFE to the considered case.

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