Cyclotron oscillations of plasma in an inhomogeneous magnetic field

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Chapter I considers cyclotron oscillations of an equilibrium plasma (the cyclotron heating problem). It is devoted mainly to the analysis of processes that lead to energy exchange between oscillations and a plasma in cyclotron resonance in an inhomogeneous magnetic field. According to present-day concepts, such an exchange is effected by excitation of modulated beams. The oscillations due to individual beams are analogous to the known Van Kampen waves. In the presence of a thermal spread of the plasma-particle velocity, the interference quenches these oscillations. As a result, the energy drawn by the plasma from the cyclotron wave takes on a thermal (random) form. An appendix (at the end of the article) traces the analogy with Cerenkov resonance in an inhomogeneous plasma and with surface absorption of the oscillations incident on an abrupt plasma boundary (anomalous skin effect). Chapter II summarizes earlier results on the analysis of the stability of a plasma not in thermal equilibrium situated in the inhomogeneous field of a magnetic trap. It was shown in the earlier papers that, depending on the conditions on the plasma boundaries, an increase in the inhomogeneity of the magnetic field can lead to either a lowering of the increment of the unstable natural oscillations (oscillation-reflecting boundaries), or to their total stabilization (absorbing boundaries). Two approaches to the investigation of stability are compared in the review, viz., analysis of the natural oscillations and an investigation of the evolution of the initial perturbations with time.

CONTENTS

INTRODUCTION

If electromagnetic oscillations propagate in a plasma situated in a non-uniform magnetic field, then at some (resonant) point their frequency ω can become equal to the cyclotron-rotation frequency of the charged particles ω_j ($j = e, i$). This gives rise to a number of characteristic phenomena, the investigation of which is the purpose of the present review. This investigation is carried out by using as examples two problems that have stimulated the development of the theory of cyclotron oscillations. We have in mind the theory of cyclotron heating of a plasma with equilibrium charged-particle velocity distribution, and the problem of the stability of a nonequilibrium plasma in a magnetic trap.

Cyclotron resonance produces conditions for energy exchange between the oscillations and the rotational degree of freedom of the charged particles in the magnetic field. This has naturally suggested the idea of using cyclotron oscillations for plasma heating[1,2]. The dynamics of the processes that lead to heating in cyclotron resonance in a non-uniform magnetic field was investigated in a number of papers[3–8]. The general picture of the phenomenon can by now be regarded as already clarified. To present it lucidly, it is useful to make use of the beam model of the plasma, i.e., to assume that the plasma consists of a set of beams of charged particles moving along the magnetic field. As it passes through the resonance region, each beam is modulated by the oscillations, and charges and currents are induced in it. If the oscillation frequency is ω and the beam moves with velocity v, then the spatial period of the modulation is $2\pi v/(\omega - \omega_j)$. Individual beams produce in the plasma oscillations analogous to the well-known Van Kampen waves. If the beam velocities have a continuous distribution, these oscillations are shifted in phase with increasing distance from the resonance point, and are ultimately attenuated through interference. Since there is no visible order in the particle motion in the final state, one can say in a certain sense that the oscillation energy goes over into thermal energy. The formation of modulated beams and their subsequent interference is apparently typical of any process that leads to energy exchange between the wave and the particles, if the action of this process is confined to a small region of space. In the Appendix we consider from this point of view resonant Cerenkov interaction in an inhomogeneous plasma[9], and also the interaction that occurs when the charged-particle velocity changes abruptly[10]. The latter, in particular, leads to absorption of oscillations incident on an abrupt plasma boundary (the problem of the anomalous skin effect[11]). It should be noted that in the case of interference of modulated beams, the information on the phase relations is preserved in hidden form, a fact that can lead to phenomena of the echo type (see[9,12] and also[8]).

If the charged-particle velocity distribution is not in thermodynamic equilibrium, then energy can be transferred from the plasma to the oscillations at cyclotron resonance. Nonequilibrium distributions are established in many systems intended for the containment of hot plasma. Thus, for example, only particles that move at sufficiently large angles to the magnetic field are contained in magnetic bottles. Particles that fall in the so-called loss cone leave the bottle. As a result, there are fewer particles having low velocities transverse to the magnetic field.

There are two possible approaches to the study of plasma stability: determination of the normal modes, and investigation of the time evolution of the initial perturbations. In the former case one determines the asymptotic form of the perturbation as $t \to \infty$, and in the latter one investigates the initial stage of the evolution until the normal modes are established. The normal modes as a rule, take the form of standing waves, and for these to exist it is necessary that the system contain reflection points. In an inhomogeneous magnetic field, the oscillations passing through the resonance point not only experience a change of amplitude (they are amplified in a nonequilibrium plasma and absorbed in a equilibrium plasma), but are also partly reflected in part (in a non-equilibrium plasma the reflection is accompanied by amplification, so that the reflection coefficient can be larger than unity). Normal modes "trapped" between the resonance points can therefore exist in a nonmonotonic magnetic field[13,14]. This possibility is particularly important if the oscillations are absorbed at the plasma boundaries. If, however, the oscillations are reflected from the boundaries, then the additional reflection from the resonance points can be disregarded.

The evolution of unstable cyclotron oscillations in a non-uniform magnetic field was investigated in[15]. It was shown that the perturbations grow for a finite time, after which their amplitude becomes constant, and it turned out that the ratio of the final amplitude to the initial one is equal to the gain of the oscillations on passing through cyclotron resonance. The perturbations can be regarded as practically stable if the gain is close to unity. It is interesting to note that under dissipative boundary conditions the plasma is stable in this case with respect to the normal modes. Thus, both approaches to the investigation of plasma stability are mutually complementary and are in agreement.

I. EQUILIBRIUM PLASMA

1. Dynamics of resonant cyclotron interaction.
a) Homogeneous magnetic field. We consider the motion of an electron situated in a constant uniform magnetic field and acted upon by an alternating electric field. We assume that the electric-field vector is perpendicular to the magnetic field and rotates with the same frequency and in the same direction as the electrons in the magnetic field. From the equation of motion we get

$$dv/dt = -(e/m_e)\,E; \tag{1.1}$$

here v is the modulus of the velocity in the direction transverse to the magnetic field, $\mathbf{v}_\perp = \mathrm{Re}(v;\ iv)\exp(i\omega_e t)$, the quantity E is defined by the relation $\mathbf{E} = \mathrm{Re}(E;\ iE;\ 0) \times \exp(-i\omega t)$, and $\omega_e = eH/m_e c$ is the electron-cyclotron frequency, $\omega = \omega_e$; a Cartesian coordinate system is used with Oz axis along the magnetic field.

With the aid of (1.1) we obtain

$$dw_e/dt = -eEv_{x0} + (e^2E^2/m_e)\,t; \tag{1.2}$$

here w_e is the electron kinetic energy and v_{x0} is the initial value of the velocity projection on Ox. For an assembly of electrons with uniform distribution with respect to the phase shifts of the Larmor rotation, the first term in the right-hand side of (1.2) vanishes after averaging. It is interesting to note that if the electric field is not maintained by external sources, then it attenuates with a decrement of the same order as the electron plasma frequency $\omega_{pe} = (4\pi e^2 n_0/m_e)^{1/2}$. Indeed, it follows from (1.2) that after a time $t \approx \omega_{pe}^{-1}$ the electrons acquire an energy comparable with the field energy $E^2/8\pi$. It is assumed in these estimates that the electrons are uniformly distributed in space with a density n_0.

b) Cyclotron absorption in an inhomogeneous magnetic field. Assume now that the magnetic field varies in the longitudinal direction, $H = H(z)$, and that the electrons move along the field. In what follows we need to know not only the amplitude but also the phase shift of the Larmor rotation. We therefore introduce $v_- = v_x - iv_y$, for which the equation of motion takes the form

$$dv_-/dt + i\omega_e(z(t))\,v_- = -(eE/m_e)\,e^{-i\omega t}. \tag{1.3}$$

From (1.3) we get

$$v_-(t) = \Big[v_-(0) - \frac{eE}{m_e}\int_0^t \exp\Big(-i\omega t' + i\int_0^{t'} \omega_e(t'')\,dt''\Big)\,dt'\Big]\exp\Big(-i\int_0^t \omega_e(t')\,dt'\Big). \tag{1.4}$$

We assume that the cyclotron frequency at some (resonance) point coincides with the wave frequency. We consider large time intervals $t \gg \delta\omega_e^{-1}$, where $\delta\omega_e$ is the change in the electron-cyclotron frequency along the electron trajectory. In this case, we can use asymptotic methods to calculate the integral with respect to dt′ in (1.4). If the electron has not yet reached the resonance point $t < t_s$ $(z(t_x) = z_s)$, then the asymptotic form of $v_-(t)$ is

$$v_-(t) \approx v_-(0)\exp\Big(-i\int_0^t \omega_e(t')\,dt'\Big) - \frac{ieE}{m_e}\frac{e^{-i\omega t}}{\omega - \omega_e(z(t))}; \tag{1.5}$$

only the contribution of the upper integration limit $(t' = t)$ was taken into account here, since it was assumed that the electron had time to "forget" the instant when the field was turned on.

At $t > t_s$ it is necessary to take additional account of the resonance-point contribution. The electron is in phase with the wave in the vicinity of the resonance point, so that the integrand of (1.4) has at $t' = t_s$ a stationary-phase point[4,6,8]:

$$v_-(t) \approx \Big\{v_-(0) - \frac{eE}{m_e}\Big[\frac{2\pi i}{v_{\parallel}(d\omega_e/dz)_{z_s}}\Big]^{1/2}\exp\Big[-i\omega t_s + i\int_0^{t_s}\omega_e(z(t'))\,dt'\Big]\Big\} \times \exp\Big[-i\int_0^t \omega_e(z(t'))\,dt'\Big] - \frac{ieE}{m_e}\frac{e^{-i\omega t}}{\omega - \omega_e(z(t))}. \tag{1.6}$$

It follows from a comparison of (1.5) and (1.6) that the amplitude and phase of the cyclotron rotation change after passing through the resonance point. The electron energy increases then by an amount

$$\delta w_e = (\pi e^2E^2/m_e)\,[v_{\parallel}(d\omega_e/dz)_{z_s}]^{-1}.$$

(We have averaged here over the initial phase of the Larmor rotation.) This increment can be obtained from the following considerations:

An electron moving in a non-uniform magnetic field remains in the resonance state for a finite time that can be obtained from the condition $|\delta\Phi| < \pi/2$, where

$$\delta\Phi = -\omega(t - t_s) + \int_{t_s}^t dt'\,\omega_e(z(t'))$$

is the difference between the phase of the electron Larmor rotation and the phase of the wave. Substituting in this expression $\omega_e(z) = \omega + (z - z_s)(d\omega_e/dz)_{z_s}$, $z - z_s = v_{\parallel}(t - t_s)$, we determine the duration $\delta t \approx [2\pi v_{\parallel}(d\omega_e/dz)_{z_s}]^{-1/2}$ of the resonance interaction. At

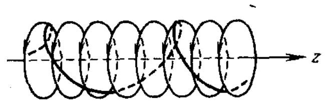

FIG. 1. Modulation of electron beam at the resonance point. The thick line shows the position of the group of electrons whose velocity modulus has changed by the same amount $\delta v_\perp$. Groups with other values of $\delta v_\perp$ form analogous helices shifted along the z axis and covering the entire surface of the cylinder made up of the Larmor circles (thin lines).

$|t - t_s| < \delta t$, the magnetic field can be regarded as homogeneous, so that we get in accordance with (1.2) $\delta w_e = (e^2E^2/2m_e)(\delta t)^2$.

c) Modulated beams. We have shown that electrons moving along an inhomogeneous magnetic field draw a finite amount of energy from the wave. Let us consider another important consequence connected with the motion of the electrons in a non-uniform magnetic field. We assume that a stationary and spatially homogeneous electron beam moves with constant velocity $v_\parallel$. In accordance with (1.5) and (1.6), the velocity of the Larmor rotation of the electrons is modulated by the electric field at the resonance point. The velocity change depends here both on the initial position of the electron on the Larmor circle, characterized by $v_-(0)$, and on the instant t_s of passage through the resonance point. We assume that the magnetic field falls off along the Oz axis. In this field, the electrons passing through the resonant point lag the electric field in phase. It is easy to see that those electrons whose velocity modulus has changed by the same amount form a helical line in space (Fig. 1[1)]). This line moves along the z axis with velocity $v_\parallel$ and rotates about this axis at the local Larmor frequency. Thus, electric currents with circular polarization appear behind the resonance point. Since the process is stationary, the fields excited by these currents should have the same frequency ω as the primary field. The dependence of the phase of the secondary wave on the coordinate z and on the velocity $v_\parallel$ is obtained from (1.6):

$$\Phi(z, v_\parallel) = -[(z - z_s)^2/2v_\parallel]\, d\omega_e/dz\,|_{z_s}. \qquad (1.7)$$

The similarity between the secondary wave and the Van Kampen waves is easily seen[16]. The latter, as is well known are oscillations due to charged-particle bea beams in which the density is modulated in the direction of motion. Brambilla[7] succeeded in finding the asymptotic solution of the self-consistent wave equation of the cyclotron oscillations; this solution can be regarded as the analog of the Van Kampen wave.

2. **Field of modulated beam.** a) Adiabatic wave equation. In the preceding sections we have assumed the electric field of the oscillations to be constant in space. Of real interest is the problem of resonance interaction between electromagnetic waves. The solution must then be self-consistent, i.e., allowance must be made for the change introduced into the wave fields by the currents produced by the wave.

We assume that the magnetic field varies with distance linearly, $dH/dz = \text{const} < 0$, and that the electrons move with the same constant velocity $v_\parallel > 0$. Under these conditions, the wave equation takes the form:

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2} E + i\frac{\omega\omega_{pe}^2}{v_\parallel c^2}\int_{-\infty}^{z} dz' E(z') \exp\left[-\frac{i}{2v_\parallel}\frac{d\omega_e}{dz}(z^2 - z'^2)\right] = 0; \qquad (1.8)$$

The origin is chosen here at the resonance point $\omega_e(0) = \omega$.

It is easy to establish a correspondence between (1.8) and (1.4). Indeed, (1.4) determines the electron-velocity perturbation δv due to the electric field. Connected with the velocity perturbation is a current $\delta j = -en_0\delta v$. It is the influence of this current on the oscillations which is accounted for by the last terms in (1.8), where, unlike in (1.4), the integration with respect to time is replaced by integration with respect to the coordinate on the electron trajectory $dz'/dt' = v_\parallel$, the initial instant is referred to $-\infty$, and the electric field is assumed to be a function of coordinate and is therefore left under the integral sign. If we calculate in (1.8) the asymptotic form of the integral with respect to dz', assuming E(z') to be a slowly varying function and taking into account only the contribution of the upper integration limit, then we obtain

$$d^2E/dz^2 + (\omega^2/c^2)\{1 + [\omega_{pe}^2/\omega(\omega_e(z) - \omega)]\} E = 0; \qquad (1.9)$$

We have made here the substitution $z\, d\omega_e/dz = \omega_e(z) - \omega$.

Equation (1.9), which is frequently used in the analysis of cyclotron oscillations in an inhomogeneous magnetic field, can be obtained by describing the motion of electrons in oscillations in the hydrodynamic approximation (see, e.g.,[17-19]). It differs from the wave equation in a homogeneous magnetic field only in that the field is assumed to be a slowly varying parameter. Therefore (1.9) can be called the adiabatic wave equation[8]. The adiabatic equation is asymptotic and does not take into account the influence of the resonance interaction (the contribution of the stationary-phase point to the integral with respect to dz' at $z > 0$). It is therefore not surprising that it becomes meaningless at the resonance point $z = 0$ (it has a singularity). To continue the solution through the singular point, it is customary to introduce the Landau bypass rule. Namely, an analytic continuation of (1.9) to the complex z plane is considered and it is assumed that oscillations with $\mathrm{Im}\,\omega = 0$ are the limit of growing oscillations with $\mathrm{Im}\,\omega > 0$. If $\mathrm{Im}\,\omega > 0$, then the resonance (singular) point z_s shifts downward from the real axis at $d\omega_e/dz|_{z_s} < 0$ and upward at $d\omega_e/dz|_{z_s} > 0$ ($\delta z = i\,\mathrm{Im}\,\omega\,[(d\omega_e/dz)_{z_s}]^{-1}$). Accordingly, the singular point should be bypassed from above when $d\omega_e/dz|_{z_s} < 0$ and from below when $d\omega_e/dz|_{z_s} > 0$. In some papers (see, for example,[17-19]) it is indicated that to justify the Landau bypass rules it suffices to take the electron-ion collisions into account. Indeed, in the presence of collisions with frequency ν, the resonant denominator in (1.9) takes the form $1/(\omega_e(z) - \omega - i\nu)$, which is equivalent to a changeover to growing oscillations. As the result of the collisions, the oscillation energy is transformed in the vicinity of the resonance point into Joule heat. This process predominates at $\nu \gg (v_\parallel |d\omega_e/dz|)^{1/2}$ (see below). Under the experimental conditions, however, the opposite inequality is frequently satisfied. In this case the energy exchange between the wave and the electrons proceeds without the collisions taking part (see the preceding sections) and consequently a justification for the Landau bypass rules must be obtained by another method.

b) The Brambilla solution. We now turn to wave equation (1.8) which, of course, is regular at all values of z. We multiply (1.8) by $\exp[(i/2v_\parallel)(d\omega_e/dz)z^2]$, differentiate the result with respect to dz, and then multiply by $\exp[(-i/2v_\parallel)(d\omega_e/dz)z^2]$. We obtain

$$E'''_{\zeta^3} - 2ip^2\zeta E''_{\zeta^2} + E'_\zeta + i(A - 2p^2\zeta)E = 0; \qquad (1.10)$$

we have introduced here the notation $A = (\omega_{pe}^2/\omega^2)c/v_\parallel$,

$p^2 = -(1/2v_{\|})(d\omega_e/dz)c^2/\omega^2$, $\zeta = \omega z/c$. The last equation has terms (first and third derivative) not contained in (1.9). The addition of the third derivative eliminates the singularity at the origin (at the resonance point). At $|\zeta| \to \infty$, Eq. (1.10) has solutions of two types, large-scale and small-scale. It can be assumed approximately that the large-scale solutions satisfy the adiabatic equation (1.9), which does not contain the third derivative. For the small-scale solutions, we can leave out the last two terms of (1.10). We then obtain $E \approx C(\zeta)\exp(ip^2\zeta^2)$, where $C(\zeta)$ is a slowly varying function. The phase of this solution coincides with (1.7), and the solution itself describes the field of a modulated beam going off from the resonance point. In accordance with the arguments given in Sec. 1c, such beams should, for example, be excited when oscillations with larger wavelength are incident on the resonance point. The change of the spatial scale of the oscillations is customarily called transformation. It is known (see, for example, the review[20]) that this effect is indeed described by equations with a small parameter preceding the highest-order derivative. If the electron velocity and the magnetic-field gradient are not too large, then the small parameter in our case is the quantity $1/p^2$.

Equation (1.10) was solved in[7] by using the Laplace-transform method. The following results were obtained: the solution representing at $z < 0$ a wave incident on the resonance point,

$$E(\zeta) \underset{\zeta\to-\infty}{\approx} |\zeta|^{-i\beta/2\pi} e^{i\zeta}, \tag{1.11}$$

assumes at $z > 0$ the form

$$E(\zeta) \approx |\zeta|^{-i\beta/2\pi} e^{i\zeta-(\beta/2)} - \frac{i^{(\beta/2\pi)+(3/2)}\pi^{1/2}\cdot 2^{\beta/2\pi}}{p^3\zeta^2\Gamma(i\beta/2\pi)} e^{ip^2\zeta^2}, \tag{1.12}$$

where $\beta = \pi\omega_{pe}^2 L/\omega c$.

It follows from (1.12) that the large-scale oscillations are indeed transformed into small-scale oscillations. The amplitude of the large-scale oscillations decreases by a factor

$$T = e^{-\beta/2}, \tag{1.13}$$

and consequently the transformation coefficient (the fraction of the transferred energy) is

$$\eta = 1 - e^{-\beta}. \tag{1.14}$$

This result could be obtained by using the simplified adiabatic wave equation (1.9) supplemented by the Landau bypass rule. Indeed, at $\zeta < 0$ we take the solution in the form $E(\zeta) = \zeta^{-i\beta/2\pi}e^{i\zeta}$. We continue it into the region $\zeta > 0$, bypassing the point $\zeta = 0$ in the upper half-plane. Then $\arg\zeta$ acquires an increment $-\pi$, and accordingly the amplitude $E(\zeta)$ is decreased by a factor $e^{-\beta/2}$.

The present analysis does not take into account relativistic effects. It is shown in[7] that allowance for these effects leads to the substitution $\beta \to \beta[1 - (v_{\|}/c)]$.

3. Effect of thermal spread. Landau bypass rule. We now assume that the electron distribution with respect to the longitudinal velocities has a certain thermal spread $\delta v_{\|}$. We can then assume that an assembly of electron beams with a continuous velocity distribution travels towards the resonance point. At the resonance point, each of these beams is modulated by the oscillations. In accordance with (1.7), the phases of the individual beams are shifted with increasing distance from the resonant point. At $z - z_s \gg v_{\|}(\delta v_{\|} d\omega_e/dz)^{-1/2}$, the interference of the fields connected with the individual beams leads to their mutual annihilation[2)]. The same result would be obtained if the phases of the individual beams were randomly distributed, so that we can state in a certain sense that the ordered energy of the oscillations goes over into thermal energy. The phase information is retained in latent form, and in principle can become manifest in effects of the echo type (see[9,12]).

To describe the interference phenomenon quantitatively, let us derive an expression for the current produced by the wave past the resonance point. Using (1.6), we get

$$\delta j(z) = -en_0 \int dv_{\|} f_{0e}(v_{\|})\,\delta v \approx \frac{e^2 n_0}{m_e}\left[\frac{2\pi i}{(d\omega_e/dz)_{z_s}}\right]^{1/2} E(z_s)\int \frac{dv_{\|}}{v_{\|}^{1/2}} f_{0e}(v_{\|})\, e^{i\Phi(z, v_{\|})} + i\frac{\omega_{pe}^2}{4\pi}\frac{1}{\omega - \omega_e(z)} E(z). \tag{1.15}$$

In the case of a Maxwellian electron velocity distribution, $f_{0e}(v_{\|}) = (1/\pi^{1/2}\delta v_{\|})e^{-(v_{\|}/\delta v_{\|})^2}$, the integral with respect to the velocities in (1.15) can be easily calculated by the saddle-point method:

$$\int \frac{dv_{\|}}{v_{\|}^{1/2}} f_{0e}(v_{\|})\, e^{i\Phi(z, v_{\|})} \approx \left(6\left|\frac{d\omega_e}{dz}\right|\right)^{-1/6} (3\delta v_{\|}(z - z_s))^{-1/3} \times \exp\left[\frac{i\pi}{12} - \frac{3}{2^{3/4}} e^{-i\pi/3} \frac{(z - z_s)^{4/3}|d\omega_e/dz|^{2/3}}{(\delta v_{\|})^{2/3}}\right]. \tag{1.16}$$

It follows from (1.16) that the characteristic damping length of the fields of the modulated beams is of the order of $(\partial v_{\|}/|d\omega_e/dz|)^{1/2}$.

The expressions obtained by us are valid on the real axis. For an analytic continuation to the complex z plane it is necessary to use a wave equation that takes full account of the resonance effects[8]. It can be obtained by averaging in (1.8) over $v_{\|}$. An analysis of the complete wave equation shows that effects connected with resonance interaction (modulated beams) must be taken into account not only in the vicinity of the resonance point, ($|z - z_s| \lesssim (\delta v_{\|}/|d\omega_e/dz|)^{1/2}$), but also at $|z - z_s| \to \infty$ in the sectors $-3\pi/4 < \arg(z - z_s) < -5\pi/8$, $-3\pi/8 < \arg(z - z_s) < -\pi/4$. These sectors are shown shaded in Fig. 2. In the unshaded region we have $\delta j \approx (i\omega_{pe}^2/4\pi)E(z)/(\omega - \omega_e(z))$, and accordingly the complete wave equation reduces to the adiabatic equation (1.9)[3)]. Thus, if we bypass the resonance point from above at a sufficiently large distance $|z - z_s| \gg (\delta v_{\|}/|d\omega_e/dz|)^{1/2}$, then we can confine ourselves to the use of the simplified adiabatic wave equation (1.9). Since we assume in the present analysis that $(d\omega_e/dz)|_{z_s} < 0$, this bypassing rule coincides with the Landau rule. When the thermal spread is decreased, and we go from a Maxwellian velocity distribution to distributions of the type

$$f_0(v_{\|}) = \frac{1}{\sqrt{\pi}\,\delta v_{\|}} \exp\left[-\left(\frac{v_{\|} - v_{\|0}}{\delta v_{\|}}\right)^2\right],$$

the bypass raidus increases, $|z - z_s|$

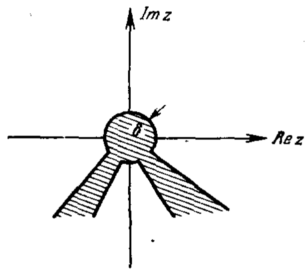

FIG. 2. Complex-variable plane in the vicinity of the resonance point. The shaded region is where the simplified adiabatic wave equation (1.9) is not valid. The distribution of the electrons with respect to the longitudinal velocities is assumed to be Maxwellian: $=(1/\pi^{1/2}\delta v_{\|})\,e^{-(v_{\|}/\delta v_{\|})^2}$, $\delta = (\delta v_{\|}/|d\omega_e/dz|)^{1/2}$.

$\gtrsim v_{\parallel 0}(\delta v_{\parallel}\,|d\omega_e/dz|)^{1/2}$. In the limit as $dv_{\parallel} \to 0$, there is no interference and the bypass radius becomes infinite. This was precisely the case considered in the preceding section.

If the condition

$$\max\left(\frac{\omega^2}{c^2}\frac{v_{\parallel}}{|d\omega_e/dz|},\ \frac{\omega_{pe}^2\omega v_{\parallel}^{1/2}}{c^2\,|d\omega_e/dz|^{3/2}}\right) \gg 1$$

is satisfied, then the resonance region is broadened by the Doppler effect. If the change $\Delta\omega_e$ of the cyclotron frequency within the limits of the system is sufficiently small,

$$\frac{\Delta\omega_e}{\omega_e} \ll \max\left[\frac{v_{\parallel}}{c},\ \left(\frac{\omega_{pe}}{\omega}\frac{v_{\parallel}}{c}\right)^{2/3}\right],$$

then the resonance region overlaps the entire system. In this case the inhomogeneity of the magnetic field can be disregarded completely.

It is of interest to compare the considered oscillations with the electron Langmuir oscillations of a hot plasma. In the problem of electron Langmuir oscillations, the contribution of the resonant particles to the expression for the perturbed particle velocity distribution function led to the singularity $f_1(v) \sim [v - (\omega/k)]^{-1}$. To eliminate this singularity it was necessary to take into account the nonlinear effect of the change of the resonant-particle velocity under the influence of the wave field, which imposed a limit on the resonance-interaction time (see, for example, the review[21]). In the present case, however, allowance for the nonlinear effect would not lead to the desired result, since the cyclotron frequency is independent of the energy in the nonrelativistic limit. At the same time, in an inhomogeneous magnetic field, the resonance condition for a separate electron may be violated, if the electron, moving along the field, leaves the resonance zone. This was the effect taken into account by us.

In both problems it is possible to use under certain conditions singular expressions supplemented by the Landau bypass rule. When electron Langmuir oscillations are considered, such an approach is valid if modulated beams (Van Kampen waves) do not exist at the initial instant of time, and only so long as the wave does not transfer an appreciable fraction of its energy to these beams. In the problem of cyclotron oscillations, the adiabatic wave equation (1.9) is valid in a region free of the fields due to the modulated beams.

4. Quantitative characteristics of resonant interaction. We assume that the electron velocity distribution along the magnetic field has a thermal scatter. In this case, at a sufficient distance from the resonant point, we can use the adiabatic wave equation supplemented by the Landau circuiting rule. It takes the form of a Schrödinger equation with a "potential" U(z) $= \omega_{pe}^2\omega/c^2\ (\omega - \omega_e(z))$ and an "energy" $W = \omega^2/c^2$ (Fig. 3). It follows from the figure that the region where the oscillations propagate (the transparency regions) are separated by a potential barrier ($z_s < z < z_0$). Here z_0 is the usual turning point, at which the "kinetic energy" ($W = U(z_0)$) vanishes, and z_s is the singular turning point at which the "potential energy" is infinite.

We consider oscillations propagating from the larger values of the magnetic field (from left to right). The ratio of the amplitude of the oscillations passing through the non-transparency region to the amplitude of the incident wave is given by the usual quantum-mechanical expression

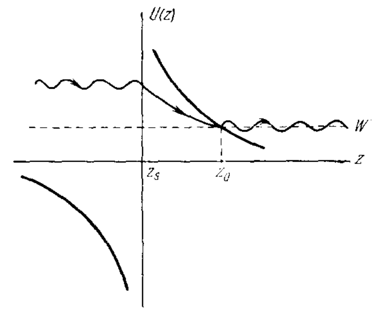

FIG. 3. Absorption of oscillations incident from a strong magnetic field. The oscillation propagation direction is marked by arrows; in the transparency region, where the oscillations take the form of traveling waves, their amplitude is shown by the wavy line; $U(z) = \omega\omega_{pe}^2/c^2(\omega - \omega_e(z))$ is the potential energy in the Schrödinger equation equivalent to (1.9) ; $W = \omega^2/c^2$ is the total energy; z_s is the singular turning point (resonant point); z_0 is the usual turning point.

$$T = \exp\left(i\int_{z_s}^{z_0} k_{\parallel}(z)\,dz\right), \qquad (1.17)$$

where

$$k_{\parallel}(z) = \frac{\omega}{c}\left[1 + \frac{\omega_{pe}^2}{\omega(\omega_e(z) - \omega)}\right]^{1/2};$$

If the magnetic field varies linearly,

$$\omega_e(z) = \omega\left(1 - \frac{z - z_s}{L}\right).$$

the integral in the argument of the exponentical can be easily calculated and turns out to be $\beta/2$ (see (1.13)).

It is noted in Sec. 3 that in the absence of thermal spread we can use for the description of the long-wave part of the oscillations the adiabatic wave equation supplemented by the Landau bypass rule. It is therefore not surprising that, regardless of the value of the spread, the attenuation of the long-wave part of the oscillations is given by (1.13).

Unlike the problems considered in quantum mechanics, there is no reflected wave in the present case (see[14,17] and also[42]). It must therefore be assumed that the difference between the energies of the incident wave and the transmitted wave is absorbed. It was shown in the preceding sections that the oscillation energy is lost to modulation of the beam of charged particles incident on the resonance point. This process, at first glance, is not reflected in any way in the adiabatic wave equation employed by us, and the correspondence between (1.13) and (1.17) may be accidental[4)]. In fact, this is not so. In[8], where the complete wave equation was obtained, it was shown that the interaction between the particle oscillations is described by a single analytic expression, which breaks up into an adiabatic part and a resonant part only asymptotically, at sufficiently large distance from the resonance point. Each of these parts, however, contains information concerning the other part, so that any modification of the resonance-interaction process, for example under the influence of acceleration, the Doppler effect, etc., is reflected in the adiabatic wave equation. Naturally, the results obtained from a consideration of the motion of individual particles are the same as those obtained with the aid of the adiabatic wave equation. Let us demonstrate the validity of this statement, using as the simplest example a low-density plasma ($\omega_{pe}^2 L/\omega c \ll 1$).

It follows from (1.17) that if $\omega_{pe}^2 L/\omega c \ll 1$, then the absorption coefficient (the fraction of the absorbed energy) is equal to

$$\eta = 1 - T^2 \approx \pi\omega_{pe}^2 L/\omega c. \tag{1.18}$$

On the other hand, the energy absorbed by the electrons can be obtained with the aid of (1.6). It is equal to $\delta w_e = (m_e n_0/2)|\delta v|^2$; here δv is the change of the Larmor-rotation velocity (the term linear in δv is eliminated after averaging over the phase of the Larmor rotation). At low density, the electromagnetic energy of the modulated beams can be disregarded, and we can assume for the absorption coefficient the expression $\eta \approx \delta w_e/S$, where $S = c|E|^2/4\pi$ and E is the amplitude of the oscillations incident on the resonance point. Simple calculations lead to (1.18). It is shown in[6] that at higher density, when the condition $c/\omega L \ll (\omega_{pe}/\omega)^2 \ll c^2/\omega^{3/2} L^{3/2} v_{\|}^{1/2}$ is satisfied, the absorption coefficient obtained by both methods is one and the same ($\eta \approx 1$).

The adiabatic wave equation in the form (1.9) is valid, generally speaking, only if the magnetic field varies linearly, and the Doppler effect and the particle acceleration in the inhomogeneous magnetic field are disregarded. The equivalence of the two approaches was verified in[8] by calculating the corresponding corrections to the absorption coefficient (1.18)[5)]:

$$\frac{\delta\eta}{\eta} = -\frac{1}{8c^2}\left\langle \frac{v_\perp^4}{v_{\|}^2} \right\rangle - \langle v_{\|}^2 \rangle \left[\frac{5}{8}\left(\frac{L}{\omega}\right)^6 \left(\frac{d^2\omega_e}{dz^2}\right)^4 + \frac{1}{6}\left(\frac{L}{\omega}\right)^4 \left(\frac{d^3\omega_e}{dz^3}\right)^2 \right]; \tag{1.19}$$

here the first term takes into account the joint action of the Doppler effect and of the particle acceleration in the inhomogeneous magnetic field, and the second is connected with the nonlinear character of the variation of the magnetic field, while the angle brackets denote averaging over the velocities; the plasma density is assumed low ($\beta = \omega_{pe}^2 L/\omega c \ll 1$).

So far we have considered cyclotron oscillations incident on the resonant point from larger values of the magnetic field. We shall show below that oscillations propagating in the opposite direction interact less strongly with the plasma, and are therefore of less interest. It follows from Fig. 4 that such oscillations are first incident on the usual turning point, from which they are partly reflected. The reflected wave can be obtained in the quasiclassical approximation by the Zwaan method (see[14]) or, if the magnetic field has a linear profile, with the aid of an exact solution of (1.9) expressed in terms of Whittaker functions[14,17]. The calculations lead to the following results:

$$\xi = |R|^2 = (1 - e^{-\beta})^2, \tag{1.20}$$

$$\eta = 1 - |R|^2 - |T|^2 = e^{-\beta}(1 - e^{-\beta}); \tag{1.21}$$

here, as before, $\beta = 2i \int_{z_s}^{z_0} k_{\|}\, dz$ (if the magnetic field has a linear profile we have $\beta = \pi\omega_{pe}^2 L/\omega c$, where R is the ratio of the reflected and incident wave amplitudes; the attenuation coefficient T does not depend on the propagation direction of the oscillations and is given by (1.13) and (1.17). We recall that the reflection coefficient for oscillations coming from the direction of the stronger magnetic field is equal to zero.

At $\beta \ll 1$ (low density, large magnetic-field gradients), the plasma interacts in practically the same manner with oscillations arriving at the cyclotron-resonance point from opposite directions. In this case, accurate to quantities of the order of β, we have $\xi \approx 0$ and $\eta \approx \beta$. At the same time, when $\beta \gg 1$, the oscillations coming from the direction of the stronger magnetic field are almost completely absorbed, while those coming from the weaker field are reflected. To transfer energy to the plasma effectively at $\beta \gtrsim 1$, it is therefore necessary that the magnetic field decrease between the radiator and the resonance point[1]. This region was called the magnetic shore "in analogy with dissipation of waves lapping on the inclined shore of the ocean"[18].

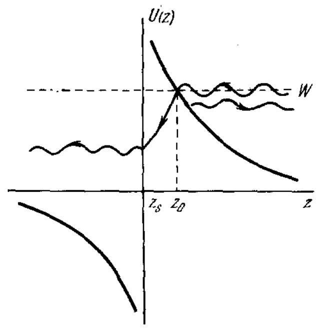

FIG. 4. Absorption of oscillations incident from the weaker part of the magnetic field. The notation is the same as in Fig. 3.

Ion heating in an installation with a magnetic shore was realized in[2,6]. This method was further developed in[22,23] and elsewhere. Cyclotron oscillations were used also to heat the electronic component of a plasma (see, e.g.,[24]), and electron cyclotron resonance was frequently realized, i.e., the energy needed to ionize the gas was supplied by the cyclotron oscillations. We know of no experimental studies aimed at verifying the characteristic relations derived in the theory, for example, the dependence of the absorption coefficient on the plasma density and on the magnetic-field gradient. This is due apparently to difficulty of varying the parameters in real systems. At the same time we note that in[6], in accordance with (1.13), almost complete absorption of the oscillations was observed at sufficiently high plasma density.

In[25] it was proposed to use cyclotron oscillations to "cork" magnetic bottles. The point is that magnetic bottles cannot retain particles traveling at a sufficiently small angle to the actual magnetic field, i.e., those falling in the so-called loss cone. It was assumed that if the cyclotron-resonance points are located in the magnetic mirrors, then the particles leaving the bottle will increase the transverse velocity component as a result of the cyclotron absorption. The velocity vector will then reverse, and the outgoing particles will be recontained. It was shown in[26], however, (see also[27]) that the main result of resonant cyclotron interaction is an enhanced diffusion of the distribution function of the charged particles with respect to the transverse velocities. The diffusion should lead to a filling of the loss cone, and consequently to an increased loss of particles. Indeed, it follows from (1.6) that the change of the transverse velocity component depends on the ratio of the phase of the Larmor rotation of the particles to the phase of the field at the instant of passage through cyclotron resonance. If the particles are uniformly distributed over the Larmor circles, then, in first order in the field amplitude, the energy of the Larmor rotation is just as likely to increase as to decrease. Systematic energy absorption is a weaker quadratic effect in terms of the wave field.

II. NON-EQUILIBRIUM PLASMA

1. Resonant cyclotron interaction in a non-equilibrium plasma. a) Uniform magnetic field. Under real conditions, the plasma particle velocity distribution frequently differs from the equilibrium or Maxwellian distribution. The non-equilibrium character can be due to the presence of beams, to the anisotropy of the distribution function (inequality of the mean energies parallel and perpendicular to the magnetic field), to the absence of low-velocity particles in the distribution with respect to the transverse velocities, etc. As already noted, distributions poor in particles with low transverse velocities are typical of magnetic mirrors. It is these distributions, called conical, which are of interest to us. To separate effects connected with the presence of a loss cone, it is necessary to consider oscillations having a certain spatial structure in the direction perpendicular to the magnetic field. In the simplest case, this is a plane wave with wave vector perpendicular to the magnetic field: $\mathbf{E} = E(\sin(\omega t - kx);\ 0;\ 0)$. Rather cumbersome calculations lead to the following expression for the rate of change of the charged-particle (ion) energy under the influence of the wave field[7]:

$$dW_i/dt = (n/2)\,(J_{n-1}^2(s) - J_{n+1}^2(s))\,(e^2E^2/m_i)\,t; \tag{2.1}$$

Here $s = kr_i$, $r_i = v_\perp/\omega_i$ is the Larmor radius of the ions, and we consider resonance with the n-th cyclotron harmonic, $\omega \approx n\omega_i$.

Expression (2.1) differs from (1.1) by a factor

$$A_n(s) = (n/2)\,(J_{n-1}^2(s) - J_{n+1}^2(s)) = (n^2/s)\,(J_n^2(s))_s'.$$

At $s \ll n$ we have

$$A_n(s) \approx [n^3/2\Gamma^2(n+1)]\,(s/2)^{2(n-1)},$$

and $A_n(s)$ has an oscillatory structure at $s \gg n$, namely[29], $A_n(s) \approx (-1)^n\,(2n^2/\pi s^2)\cos 2s$. If $A_n(s) < 0$, then the kinetic energy of the ions will decrease under the influence of the field, and consequently, by virtue of the energy conservation, the wave will increase. The cyclotron interaction leads in this case to emission rather than absorption of oscillations. Let us explain this result. We expand the plane wave in cylindrical harmonics in accordance with the well-known formula $e^{\pm ikx} = \sum_{n=-\infty}^{\infty} J_n(s)\,e^{\pm in\theta}$; where $x = r_i\sin\theta$, and θ is the azimuthal angle; the origin is chosen at the center of the Larmor circle. By virtue of the equality[29], $\sum_{n=-\infty}^{\infty} J_n^2 = 1$, the weight of the n-th cylindrical harmonic in the expansion is equal to J_n^2. In the coordinate system connected with the particle, the frequencies of the individual cylindrical harmonics are equal to $\omega_n = \omega - n\dot\theta$, where $\dot\theta = \omega_i$. It is obvious that the term $n\dot\theta$ plays the same role as the Doppler shift $k_\parallel v_\parallel$ for a wave that is periodic along the magnetic field. In analogy, we can speak of the normal and anomalous Doppler effects $\omega_n = \omega - n\dot\theta = \pm\omega_i$. Energy is absorbed by the charged particle in the normal effect and emitted in the anomalous effect. Inasmuch as $\dot\theta = \omega_i$, the $(n-1)$-st cylindrical harmonic will be absorbed and the $(n+1)$-st emitted in the case of oscillations with $\omega = n\omega_i$. The emission effect predominates if the weight of the $(n+1)$-st harmonic in the expansion of the plane wave turns out to be larger.

For an aggregate of particles with a certain transverse-velocity distribution $f_{0i}(v_\perp)$, the resultant effect is determined by the mean value of $A_n(s)$:

$$\alpha_n = 2\pi \int_0^\infty dv_\perp v_\perp f_{0i}(v_\perp)\,A_n(s).$$

The quantity α_n can become negative if the distribution is depleted in the region of low velocities $v_\perp \lesssim n\omega_i/k_\perp$, for in this case the region of small values of s, where $A_n(s) > 0$, is cut out of the integral with respect to $dv_\perp$.

b) Magnetic field that varies monotonically in space. In connection with the problem of plasma stability in adiabatic traps, magnetized electron Langmuir oscillations were intensively investigated in the frequency region $\omega \approx n\omega_i$. The adiabatic wave equation describing these oscillations is (see for example,[13,14,30])

$$\frac{d^2\varphi}{dz^2} + k_\perp^2\left(\frac{\omega}{\omega_{pe}}\right)^2\left[1 - \frac{\omega_{pi}^2\alpha_n}{n\omega_i\,(\omega - n\omega_i(z))}\right]\varphi = 0; \tag{2.2}$$

here $\varphi(z)$ is the perturbed potential, $\varphi(\mathbf{r}, t) = \varphi(z)\exp(-i\omega t + i\mathbf{k}_\perp\cdot\mathbf{r}_\perp)$, and the oscillation frequency is assumed close to $n\omega_i$. Therefore only the n-th resonant term is retained in the sum over n, which generally speaking should be present in (2.2), and the dependence on the coordinate z is taken into account only in the resonant denominator. This approximation is valid if $(\omega_{pi}/\omega_i)\,|\alpha_n/n|^{1/2} \ll 1$, i.e., at sufficiently low plasma density or at sufficiently high values of $k_\perp$. (For conical distributions we have

$$\alpha_n \underset{k_\perp r_i \gg n}{\approx} -2^{-3/2}\pi^{-1/2}n^{-2}(k_i r_\perp)^{-3};$$

where r_i is the average Larmor radius of the ions.)

After performing a number of changes in the notation, (2.2) can be reduced to (1.9) and accordingly (2.1) can be reduced to (1.11). Therefore resonant interaction in a non-equilibrium plasma can be described by the expression derived in Sec. 1d. At $\alpha_n < 0$ it is convenient to introduce

$$\beta_1 = -\beta = 2i\int_{z_0}^{z_s} k_\parallel\,dz,$$

where $k_\parallel$ is obtained from (2.2):

$$k_\parallel = k_\perp\frac{\omega}{\omega_{pe}}\left[1 - \frac{\omega_{pi}^2\alpha_n}{n\omega_i\,(\omega - n\omega_i(z))}\right]^{1/2}.$$

In a magnetic field that varies linearly we have

$$\beta_1 = -\pi\,(\alpha_n/n)\,(m_e/m_i)^{1/2}\,(k_\perp L)\,\omega_{pi}/\omega_i.$$

In analogy with (1.17) and (1.20) we get (see[14])

$$T_1 = e^{\beta_1/2}, \tag{2.3}$$

$$\xi_1 = |R_1|^2 = (e^{\beta_1} - 1)^2. \tag{2.4}$$

At $\alpha_n < 0$ the ions emit rather than absorb oscillations, so that the oscillations become amplified on passing through the resonant ration ($T_1 > 1$). If $\beta_1 > \ln 2$, then the reflected wave is also amplified. We shall need in what follows the ratio of the amplitudes of the reflected and incident waves, with allowance for the phase shift. This ratio was determined in[14] by the Zwaan method:

$$R_1 = -i\,(e^{\beta_1} - 1); \tag{2.5}$$

the phase is reckoned here from the resonant point, i.e., the incident and reflected waves are taken in the form

$$\varphi \sim \exp\left(\pm i\int_{z_s}^{z} k_\parallel\,dz\right).$$

Just as in an equilibrium plasma, the reflected oscilla-

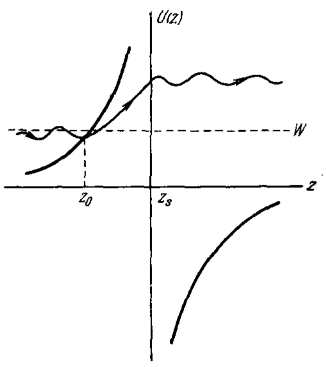

FIG. 5

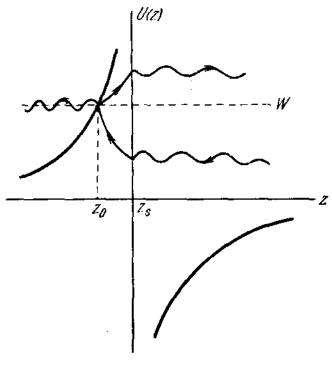

FIG. 6

FIG. 5. Amplification of oscillations incident from the stronger side of the magnetic field. $U(z) = (m_e/m_i)\, n^2\alpha_n\omega/\omega(-n\omega_i(z))$ is the potential energy in the Schrodinger equation equivalent to (2.2); $W = k_\perp^2 \omega^2/\omega_{pe}^2$ is the total energy; the remaining symbols are the same as in Figs. 3 and 4.

FIG. 6. Amplification of oscillations coming from the weaker side of the magnetic field. The notation is the same as in Fig. 5.

tions are those coming from the weaker side of the magnetic field; oscillations propagating in the opposite directions pass through the resonance without reflection. The gain does not depend on the propagation direction of the oscillations.

We note that at $\alpha_n > 0$ the ions absorb the oscillation energy, so that potential oscillations, as well as nonpotential ones, can be used to heat the ionic component of a plasma[31].

It is of interest to trace the processes that lead to the amplification of the transmitted and reflected waves. Figure 5 and 6 show the potential that enters in the Schrödinger equation equivalent to (2.2). From a comparison of these figures with Fig. 3 it follows that in a nonequilibrium plasma ($\alpha_n < 0$) the potential reverses sign (as before, we assume that $dH/dz < 0$). We consider oscillations incident on the resonance point from the stronger side of the magnetic field. Such oscillations first reach the usual turning point z_0 (see Fig. 5)[8]. In the non-transparency region ($z_0 < z < z_s$) their amplitude increases exponentially $\sim k_\parallel^{-1/2} \exp(i \int^z k_\parallel dz)$, and at the point z_s it exceeds the initial one by a factor $\sim \beta_1^{-1/2} \exp(\beta_1/2)$. However, the energy flux remains unchanged up to a point z_s where the oscillations draw energy from the ions. The amount of energy given up by the ions is estimated with the aid of (2.1), by putting $t \approx (v_\parallel d\omega_i/dz)^{-1/2}$. From the obvious relation $T_1^2 \approx n_0 v_{\parallel i}\, \delta W_i/S_1$ we get $T_1 = \exp(\beta_1/2)$; here δW_i is the energy given up by one ion, $S_1 = V_0 E^2/4\pi$ is the energy flux in the region to the left of z_s, E is the amplitude of the oscillations at the point z_s, and $V_0 = \omega_{pe}/k_\perp$ is their group velocity as $|z - z_s| \to \infty$. Figure 6 illustrates the process of reflection of oscillations coming from the weaker side of the magnetic field. The reflection coefficient can be estimated by the method described above.

c) Nonmonotonic magnetic field. With decreasing magnetic-field gradient at the resonant point, the amplification and reflection coefficients of the oscillations increase (see (2.3)–(2.5)). Since the magnetic field in an adiabatic trap has a minimum at the center of the system, the coefficients T_1 and R_1 should be maximal for oscillations with $\omega \approx n\omega_{i\,min}$. However, their calculation entails considerable difficulties in this case. First, the adiabatic wave equation, in the case of a nonmonotonic profile of the magnetic field, does not reduce to a standard equation, and second, as $z_s \to z_{min}$, the very use of this equation is not legitimate. Indeed, in accordance with the Landau bypass rule, in a decreasing magnetic field the resonance point should be bypassed from above in the complex plane and from below in a decreasing magnetic field. Therefore at resonance, near the extremum, when the resonance points are encountered in pairs, we must pass between them. As z_s approaches z_{min}, the damping length of the modulated beams generated at the resonant points may turn out to be larger than the distance between z_s and z_{min}. In this case, when constructing the solution, we must fall into the region occupied by the beams, where the simplified adiabatic wave equation is not valid (see Sec. 1c).

At large (thermonuclear) densities, the Coulomb collisions should broaden the ion velocity distribution function. In such a plasma, the thermal spread of the longitudinal velocities will be quite large ($\delta v_\parallel \approx v_\parallel$), and consequently modulated beams should attenuate over distances on the order of the modulation wavelength (see Sec. 1c). The resonance band δz_s is of the same order of magnitude. At $|z_s - z_{min}| \gg r_i^{1/3}L_1^{2/3}$ we have $\delta z_s \approx L_1(r_i/z_s)^{1/2}$ and at $|z_s - z_{min}| \lesssim r_i^{1/3}L_1^{2/3}$ we have $\delta z_s \approx r_i^{1/3}L_1^{2/3}$; it is assumed here that

$$H(z) = H_{\min}\left[1 + \left(\frac{z - z_{\min}}{L_1}\right)^2\right].$$

We therefore assume that the approach of z_s to z_{min} only broadens the resonance band, making the resonance interaction more intense[9].

Let us estimate the gain of oscillations with $\omega \approx n\omega_{i\,min}$. The energy drawn by the oscillations from the ions is proportional to the square of the amplitude of the oscillations at the resonance point (see (2.1)). The amplitude increases in the non-transparency region like $\sim \exp(\int^z \mathrm{Im}\, k_\parallel dz)$. As a result, the argument of the exponential in the expression for the gain T_1 turns out to be $\beta_1 = \int_{z_0}^{z_s} \mathrm{Im}\, k_\parallel dz$ (see (2.3)). At resonance in the minimum of the magnetic field, approximate estimates give $\beta_1 \approx C(k_\perp L_1)(m_e/m_i)^{1/2}$; here C is a quantity of the order of unity. This approximate expression can be used when $|z_s - z_{min}| \lesssim L_1|\alpha_n|^{1/2}\omega_{pi}/n\omega_i$. If $|z_s - z_{min}| \gg L_1|\alpha_n|^{1/2}\omega_{pi}/n\omega_i$, then the regions in which the oscillations differ from plane waves are separated. In this case, each of the resonances can be regarded separately as a resonance in a monotonically varying magnetic field.

d) Oscillations that grow in time. In a bounded plasma, the amplification in the resonant region leads to establishment of oscillations that grow in time ($\mathrm{Im}\, \omega = \gamma > 0$). It turns out that in the simplest case of a linearly varying magnetic field the gain and reflection coefficients (2.3)–(2.5) do not depend on the increment γ. In spite of this, however, the process of energy transfer from the ions to the oscillations proceeds in a different manner when $\gamma > 0$. First, when $\gamma \gg (\delta v_\parallel d\omega_i/dz)^{1/2}$ no modulated beams are excited. Indeed, the frequency of the growing oscillations cannot be determined with an accuracy $\delta\omega$ exceeding the increment. Therefore, if $\gamma \neq 0$ we can assume that many waves participate simultaneously in the excitation of the beams. It follows from (1.16) that when $\gamma \gg (\delta v_\parallel d\omega_i/dz)^{1/2}$ the interference should suppress the modulated beams. Accordingly, the shaded region in Fig. 2 drops below the real axis, and the adiabatic wave equations can be used on the entire axis, with the inhomogeneity of the magnetic field taken into account parametrically. (Compare with the oscillations in a collision plasma, Chap. I, Sec. 2a.) This result can also be obtained by considering the motion of a single ion. Indeed, at $\gamma \gg (\delta v_{\parallel} d\omega_i/dz)^{1/2}$ the ion does not "feel" the inhomogeneity of the magnetic field, since the amplitude of the oscillations changes before the ion leaves the resonance zone.

When estimating the energy given up by the waves to the growing oscillations, it is necessary to take into account, from the entire prior history, the nearest interval $\delta t \approx \gamma^{-1}$, since the oscillation amplitude is exponentially small at earlier instants of time. By virtue of the uncertainty of the frequency of the growing oscillations, the resonance zone becomes smeared out at a distance of the order of $\delta z_s \approx \gamma (d\omega_i/dz)^{-1}_{z=z_s}$. The number of particles in this zone is $\approx n_0 \delta z_s$, and the energy given up by them per unit time is $\approx n_0 \delta z_s \dot{W}_i \approx n^2 \alpha_n \omega_{pi}^2 (d\omega_i/dz)^{-1} \times \times E^2/4\pi$. This quantity does not depend on the increment (an estimate for the case $\gamma = 0$ was obtained above). If the magnetic field varies nonlinearly in the vicinity of the resonance point, then the resonance interaction becomes weaker with increasing γ. Thus, for example, in the limiting case of resonance at the minimum of the magnetic field, the argument of the exponential β_1 in the expression for the gain, determined in the preceding section, can be used only if $\gamma \ll (\omega_{pi}^2/\omega_i)|\alpha_n/n|$. In the opposite limiting case, it is necessary to make the substitution $\beta_1 \to \beta_1[\omega_{pi}/(\omega_i\gamma)^{1/2}]|\alpha_n/n|^{1/2}$.

2. Stability of cyclotron oscillations in magnetic traps. a) Boundaries reflecting the oscillations. In adiabatic traps, the magnetic field increases from the center towards the edges of the system, thus limiting the motion of the charged particles along the magnetic field. In bounded systems, the evolution of arbitrary perturbations leads ultimately to establishment of natural oscillations. The latter can be regarded as consisting of an incident wave and a reflected wave. If the system is symmetrical about the magnetic field ($z = 0$) and the oscillations are reflected both from the resonance region and from the plasma boundary, then the condition for matching the incident and reflected waves is

$$R_1 \exp\left(2i \int_0^{z_s} k_{\parallel}\, dz\right) + R_b \left(2i \int_0^{z_b} k_{\parallel}\, dz\right) = \pm 1; \tag{2.6}$$

here +1 corresponds to symmetrical modes and −1 to antisymmetrical modes, R_1 is the coefficient of reflection from the resonance region (2.5), and R_b is the coefficient of reflection from the plasma boundary $z = z_b$. If the plasma density varies quite abruptly at the boundary, $ak_{\parallel}(z_b) \ll 1$ ($k_{\parallel}(z_b) \approx k_\perp \omega/\omega_{pe}(z_b)$, where a is the characteristic scale of density variation), and if a metallic wall is placed at a distance b from the boundary, then $R_b = e^{-i\rho}$, where $\rho = \tan^{-1}[2B/(1 - B^2)]$ and $B = [k_{\parallel}(z_b)/k_\perp] \coth(k_\perp b)$. It follows from this expression that reflection from the plasma boundary changes the phase but not the amplitude of the oscillations. Here and below we approximate the field of the magnetic trap by the expression $H(z) = H_0[1 + (z^2/L_1^2)]$.

It is indicated in[33] that when the density varies smoothly at the boundary, $ak_{\parallel} \ll 1$, the oscillations incident on the boundary should be absorbed without reflection ($R_b = 0$). Indeed, far from the resonance region we can disregard in the adiabatic wave equation (2.2) the contribution of the ions, in which case the oscillations go over into magnetized electron Langmuir oscillations with $k_{\parallel}(z) \approx k_\perp \omega/\omega_{pe}(z)$. A wave packet made up of such oscillations, when moving towards the boundary, i.e., in the direction in which the plasma density decreases, causes the phase (group) velocity $\omega/k_{\parallel} \approx \omega_{pe}/k_\perp$ to decrease. When this velocity becomes of the same order as the thermal velocity of the electrons, the oscillations are absorbed by the resonant electrons.

On the other hand, in not too long systems ($z_b < L_1$) with an abrupt plasma boundary, we can disregard the reflections from the resonance region. The point is, as shown in[15], that oscillations reflected from the resonance point are delayed by a time $t \approx L_s/V_0$, where $L_s = (H^{-1}dH/dz)^{-1}_{z=z_s}$, $V_0 = \omega_{pe}/k_\perp$. If the condition $\gamma L_s/V_0 \ll 1$ is satisfied, then the oscillations reflected from the boundaries become amplified to a larger degree during this time.

We consider first oscillations of a plasma with an abrupt boundary, when the reflection from the resonance region (the first term in (2.6) can be disregarded. From (2.6) we get

$$2\int_0^{z_b} k_{\parallel}\, dz = p\pi + i \ln R_l; \tag{2.7}$$

here p is an integer. If the variation of the magnetic field within the limits of the system is small enough, $\delta H/H = (z_b/L_1)^2 \ll (\omega_{pi}/\omega_i)|\alpha_n/n|^{1/2}$, then the magnetic field can be regarded as homogeneous. We then obtain from (2.7) $\mathrm{Im}\,\omega = \gamma_0 \approx \omega_{pi}|\alpha_n/2n|^{1/2}$. In the opposite limiting case $(z_b/L_1)^2 \gg (\omega_{pi}/\omega_i)|\alpha_n/n|^{1/2}$, the resonance region occupies a small fraction of the system $\delta z_s/z_b \approx (\gamma_0/z_b)(d\omega_i/dz)^{-1}_{z=z_s}$ (see the preceding section). Since the energy transfer from the ions to the oscillations proceeds in the resonance region at the same intensity as in a homogeneous field, the oscillation increment should decrease in comparison with γ_0 in the ratio $\delta z_s/z_b$. Indeed, at $(z_b/L_1)^2 \gg (\omega_{pi}/\omega_i)|\alpha_n/n|^{1/2}$ we obtain from (2.7) (see[30,34])

$$\gamma \approx (\pi/4)(\alpha_n/n)\{L_1^2/(z_s - z_b)\}\,\omega_{pi}^2/\omega_i.$$

This expression for the increment can be obtained also by another method. The oscillations in question are close to the magnetized electron Langmuir oscillations, whose group velocity is $V_0 = \omega_{pe}/k_\perp$. In one passage between the boundaries, i.e., in the time $t_b = 2z_b/V_0$, their amplitude increases in accordance with (2.3) by a factor $T_1 = \exp(\beta_1)$ (in a parabolic magnetic field, the oscillations pass through the resonance twice). The increment is obtained from the condition $\gamma t_b = \beta_1$. It increases with decreasing v_s. The limiting value can be obtained by taking

$$\beta_1 \approx C\,(k_\perp L_1)\,[(m_e/m_i)(\omega_{pi}^2/\omega_i\gamma)\,|\alpha_n/n|]^{1/2}$$

(see Sec. 1d). We then obtain

$$\gamma \approx (\omega_{pi}^{4/3}/\omega_i^{1/3})\,(L_1/z_b)^{2/3}\,|\alpha_n/n|^{2/3}.$$

b) Boundaries that absorb the oscillations. In a plasma with a gradual boundary ($R_b = 0$), the natural modes are established as a result of the reflection from the resonance points. An approximate form of the oscillations is shown in Fig. 7. Between the resonance points there are two waves of decreasing amplitude, propagat-

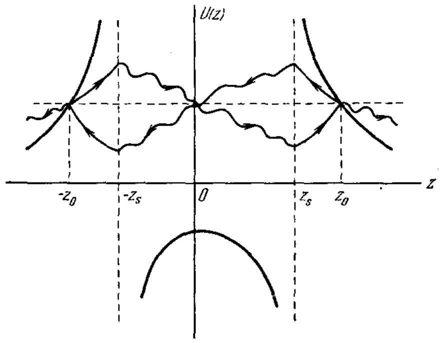

FIG. 7. Growing natural oscillations in the field of a magnetic bottle under dissipative boundary conditions. The notation is the same as in FIGS. 5 and 6.

ing towards each other, and beyond the resonance points there are outgoing waves. The decrease of the amplitude in the region $|z| < z_s$ is due to the fact that the natural modes considered by us increase with time. If the condition $\mathrm{Im}\, k_{\parallel} = (\gamma/\omega)\, \mathrm{Re}\, k_{\parallel}$ is satisfied, then in the region $|z| < z_s$, where the ions do not interact with the oscillations, the amplitude of the oscillations is not altered during the propagation, since the temporal growth is compensated for by the spatial decrease. Since the waves traveling towards the boundaries carry energy away and cause the oscillations to grow, the oscillations are said to have negative energy[35,36].

If $\beta_1 \gg 1$, then

$$R_1 \approx -i \exp\left(2i \int_{z_s}^{z_0} k_{\parallel}\, dz\right),$$

and (2.6) can be reduced to the form

$$2 \int_0^{z_0} k_{\parallel}\, dz = (p + 1/2)\, \pi. \qquad (2.8)$$

We find therefore that the oscillations are unstable[14]:

$$\omega = n\omega_{i0} - \frac{\omega_{pi}^2}{n\omega_{i0}} \frac{\alpha_n}{n} + i\, \frac{2p+1}{k_\perp L_1} \frac{\omega_{pi}^2}{n\omega_{i0}} \left(\frac{m_i}{m_e} |\alpha_n|\right)^{1/2}. \qquad (2.9)$$

In the unstable state, the plasma parameters are such that $\gamma \lesssim \omega_{pi}^2/\omega_{i0}$, whereas $\gamma \gtrsim \omega_{pi}^2/\omega_{i0}$ when the oscillations are reflected from the boundary. With decreasing L_1 (with increasing inhomogeneity), the increment first increases, because of the shortening of the oscillation-localization region. At the same time, however, the reflection coefficient also decreases, since $\beta_1 \lesssim C\, (m_e/m_i)^{1/2} L_1 k_\perp$ (see Sec. 1c). If $L_1 \ll k_\perp^{-1} (m_i/m_e)^{1/2}$, then $\beta_1 \ll 1$; then the oscillations go through the resonance region with practically no reflection (and no amplification), making the existence of natural modes impossible. We find therefore that the instability exists only when $L_1 \gtrsim k_\perp^{-1} (m_i/m_e)^{1/2}$. This estimate of the critical value agrees with the results of an investigation of (2.6)[14]:

$$L_{1\,cr} = (n/k_\perp)\, [(p + 1/2) \ln 2 \cdot |\alpha_n|^{-1} (m_i/m_e)]^{1/2}; \qquad (2.10)$$

Here, as before, $k_\perp \gtrsim r_i^{-1}$ and $|\alpha_n| \lesssim 1$. The oscillation frequency at $L_1 = L_{1\,cr}$ is $\omega = n\omega_{i0} + (p + 1/2) \times (\pi^2/4) \ln 2\, |\alpha_n|\, \omega_{pi}^2/n\omega_{i0}$.

These results pertain to a low-density plasma with $\omega_{pi}/\omega_i \lesssim 1$. Numerical calculations show that the condition for the stabilization of cyclotron oscillations changes little up to "thermonuclear" densities ($(\omega_{pi}/\omega_i)^2 \approx 10^3$)[14]. At $\omega_{pi}/\omega_i \gtrsim 1$, in addition to cyclotron oscillations, there appear oscillations whose development is not effected at all by the magnetic field, since their increment is $\gamma > \omega_i$[33]. Although the cyclotron oscillations have a smaller increment, they are in some respects more dangerous because, for example, they are more difficult to stabilize by addition of cold plasma[37]. We note in conclusion that the stabilization condition $L_1/r_i \lesssim (m_i/m_e)^{1/2}$ ($k_\perp \gtrsim r_i^{-1}$) given above may be difficult to fulfill in real systems, since nonconservation of the magnetic moment gives rise to particle losses from the bottle even starting with $L_1/r_i \approx 25$ (see, for example, [38]).

c) Time evolution of perturbations. Let us examine the evolution of a perturbation having the form of a δ-function at $t = 0$. Such a perturbation will be called elementary. In accordance with[15], the asymptotic form of the short-wave quasiclassical part of the perturbation is[10)]

$$\varphi_{z_0}(z, t) = C(z, z_0, t) \exp\left[-i\omega t + i\, \mathrm{sgn}\,(z - z_0) \int_{z_0}^{z} dz' k_{\parallel}(z', \omega)\right]; \qquad (2.11)$$

Here $C(z, z_0, t)$ is a slowly varying pre-exponential factor, and the frequency ω is defined by the equation

$$t = \mathrm{sgn}\,(z - z_0) \int_{z_0}^{z} dz' \left[\partial k_{\parallel}(z', \omega)/\partial\omega\right], \qquad (2.12)$$

being a function of the elapsed time t, of the observation point z, and of the point z_0 where the elementary perturbation is produced. Since the quantity $\partial\omega/\partial k$ determines the propagation velocity of quasiclassical perturbations, the physical meaning of (2.12) is obvious.

We assume for simplicity a linear variation of the magnetic field, $H(z) = H_0[1 - (z/L)]$. With the aid of (2.11) and (2.12) we obtain the frequency and the complex phase of the oscillations:

$$\omega = n\omega_{i0}\left(1 - \frac{z - z_0}{2L}\right) + \left[(n\omega_{i0})^2 \left(\frac{z - z_0}{2L}\right)^2 - \frac{\alpha_n}{2} \omega_{pi}^2 \frac{(z - z_0)}{V_0 t - (z - z_0)}\right]^{1/2}, \qquad (2.13)$$

$$\Phi(z, z_0, t) = -\omega t + \frac{\omega}{V_0}(z - z_0) + \frac{\beta_1}{2\pi} \ln \frac{\omega - n\omega_i(z_0)}{\omega - n\omega_i(z)}. \qquad (2.14)$$

We consider here that part of the elementary excitation which propagates to the right ($z > z_0$). In the region $z < z_0$, the perturbation evolves in similar fashion.

It follows from (2.14) that the oscillations increase if the condition $(z - z_0)(V_0 t - (z - z_0)) < l_r^2$, where $l_r = L(\omega_{pi}/\omega_i)\, |2\alpha_n/n|^{1/2}$, is satisfied. When

$$t = t_r = \frac{1}{V_0}\left[\frac{l_r^2}{z - z_0} + (z - z_0)\right]$$

the growth stops, and two real frequencies appear in place of one complex frequency with $\gamma > 0$. At $t \ll t_r$, one of these frequencies tends to the local cyclotron frequency $\omega \approx n\omega_i(z) + (n\beta_1/2\pi t)$, and the other to the cyclotron frequency $\omega \approx n\omega_i(z_0) - (n\beta_1/2\pi t)$ at the point where the perturbation was produced. The trajectory of the frequency on the complex variable plane is shown in Fig. 8. The general picture of the evolution of the elementary perturbation is shown in Fig. 9.

During the growth time of the oscillations, their amplitude increases by a factor $T_1 = \exp(\beta_1/2)$. Indeed, by the instant of time $t = t_r$, the argument of the logarithm in (2.4) acquires an increment $\approx -\pi$, after which it remains constant.

Thus, the temporal gain of the initial perturbation turns out to be equal to the spatial gain of the periodic perturbation when the perturbation goes through the cyclotron-resonance point. This equality is not a coin-

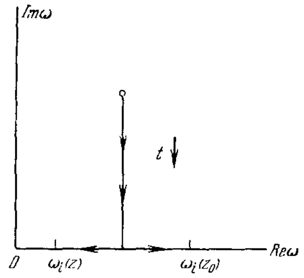

FIG. 8

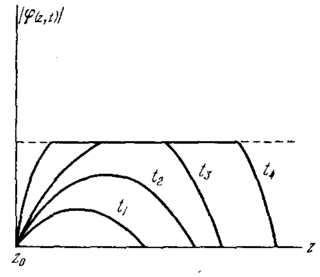

FIG. 9

FIG. 8. Trajectory of the frequency $\omega(t)$ of a perturbation having a δ-function form at the initial instant. The coordinate $z(z > z_0)$ is fixed. The magnetic field has a linear variation, $H(z) = H_0[1 - (z/L)]$.

FIG. 9. Evolution of perturbation having a δ-function form at the initial instant. The magnetic field has a linear variation.

cidence. Indeed, at $t > t_r$ the oscillation frequency at the point z lies in the interval $n\omega_i(z_0) > \omega > n\omega_i(z)$. Such oscillations, moving from z_0 to z, must pass through the cyclotron-resonance point.

The plasma can be regarded as practically stable if the temporal gain is close to unity. On the other hand, it was shown that when the oscillations are absorbed at the plasma boundary this condition ensures elimination of unstable natural modes.

The expression used by us for $\varphi_{z_0}(z, t)$ takes into account waves that move away from the point z_0 (see (2.11)). If the oscillations are reflected from the boundaries, then reflected waves should appear at $t \approx t_b \approx z_b/V_0$. It is shown in[15] that such waves, reflected from the cyclotron-resonance point, appear at $t \gtrsim L/V_0 > t_b$ (the characteristic scale L of variation of the magnetic field is usually larger than z_b). In fact, natural oscillations have time to set in after several reflections. It is necessary in this case to use the results of the preceding sections. Thus, both approaches to the investigation of plasma stability (examination of the time evolution of the perturbations and analysis of the natural oscillations) are in agreement and complement each other.

A more detailed investigation of the evolution of the initial perturbations was carried out in[15]. It was found, in particular, that the temporal and spatial gains coincide also in a nonmonotonic magnetic field. The evolution of the spatially-periodic perturbations and of perturbations in the form of wave packets was traced. It was shown that a wave packet can break up in unstable inhomogeneous media in to formations whose subsequent evolution cannot be described in terms of wave packets.

d) Cyclotron oscillations of a plasma with hot electrons. In the preceding analysis, the electrons were assumed to be cold ($v_e/v_i \ll (\omega_{pe}/\omega_i)/k_\perp r_i$, where $k_\perp \gtrsim r_i^{-1}$). To get an idea of the character of the instability in a plasma with hot electrons, it is useful to consider first the simpler case of oscillations in a homogeneous magnetic field. In homogeneous media, the oscillations take the form of plane waves $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$. The $\omega(\mathbf{k})$ dependence is determined by the dispersion equation, which takes the following form for cyclotron oscillations in a plasma with hot electrons (see, for example,[36])

$$\varepsilon(\omega, \mathbf{k}) = 1 - \left(\frac{k_\perp}{k}\right)^2 \frac{\omega_{pi}^2 \alpha_n}{n\omega_i(\omega - n\omega_i)} + \frac{\omega_{pe}^2}{k^2 v_e^2}\left(1 + i\sqrt{\pi}\frac{\omega}{|k_\parallel| v_e}\right) = 0; \quad (2.15)$$

here $\epsilon(\omega, \mathbf{k})$ is the dielectric constant of the plasma.

In the zeroth approximation in $|\mathrm{Im}\,\epsilon/\mathrm{Re}\,\epsilon| \ll 1$ we obtain from (2.15)

$$\omega = n\omega_i + \frac{\omega_{pi}^2}{n\omega_i}\alpha_n\left[\left(\frac{k}{k_\perp}\right)^2 + \left(\frac{\omega_{pe}}{k_\perp v_e}\right)^2\right]^{-1}.$$

If $\alpha_n < 0$, then the energy of these oscillations, determined from the formula $W = (1/8\pi)\omega(\partial\epsilon/\partial\omega)|E|^2$ is negative (see, for example,[36]). The electrons whose thermal velocity is equal to the phase velocity of the wave absorb energy (collisionless Landau absorption). This effect, which is taken into account by the small imaginary part of ϵ, leads to a buildup of oscillations[36,40]. In essence, the considered instability differs from the instability with dissipative boundary conditions (see Sec. b) only in that the surface dissipation at the plasma boundary is replaced by volume dissipation.

If the magnetic field is inhomogeneous and its variation within the region occupied by the plasma is large enough,

$$\Delta\omega_i \gg \frac{\omega_{pi}^2}{n\omega_i}|\alpha_n|\left[1 + \left(\frac{\omega_{pe}^2 r_i}{v_e}\right)^2\right]^{-1},$$

then the oscillations in question will be characterized by resonances with the cyclotron rotation of the ions. At cyclotron resonance in a nonequilibrium plasma ($\alpha_n < 0$), the energy is radiated, and this should lead to attenuation of oscillations with negative energy. Consequently the wave packet made up of oscillations with negative energy exists only until it reaches the cyclotron-resonance point. Thus, one should expect no unstable natural modes in a plasma with hot electrons (see[30]), and the instability itself is manifest in the form of bursts that are not correlated with one another. In a magnetic field that varies parabolically $H(z) = H_0[1 + (z^2/L_1^2)]$, the lifetime of the burst is

$$\mathrm{Im}\,\varepsilon = \frac{\pi}{|k_\parallel|}\left(\frac{\omega_{pi}}{k}\right)^2\left(\frac{k_\perp^2}{n\omega_i}\alpha_n F_0(v_\parallel) - \frac{\omega - n\omega_i}{v_\parallel}\alpha_{1\,n}F_0'(v_\parallel)\right)$$
$$v_\parallel = (\omega - n\omega_i)/k_\parallel;$$

We note that, generally speaking, unstable natural modes with $\omega < n\omega_{i\,\mathrm{min}}$ are possible, for which the cyclotron-resonance condition is not satisfied, in the vicinity of the minimum of the magnetic field. However, a sufficiently strong inhomogeneity of the magnetic field, $L_1 \lesssim r_i(\omega_i/\omega_{pi})|\alpha_n|/n^{-1/2}$, eliminates these natural oscillations, too. From the mathematical point of view, the problem of oscillations of a plasma with hot electrons in an inhomogeneous magnetic field is close to the problem of oscillations in the flow of a continuous medium moving with a velocity that is variable in the direction transverse to the flow. A detailed exposition of this question can be found in the review[41] (see also[30,42]).

e) Anisotropic instability. Magnetic traps with low plasma density ($\omega_{pi} \lesssim \omega_i$) are characterized by anisotropic ion velocity distributions, at which the average energy of the thermal motion of the ions in the direction across the magnetic field greatly exceeds the longitudinal energy. The anisotropy of the distribution, as well as its conical shape, can be the reason for the buildup of cyclotron oscillations. A feature of anisotropic instability is that even at large anisotropy only oscillations with $\omega < n\omega_i$ are unstable, while oscillations with $\omega > n\omega_i$ are damped. This premise is illustrated by the form of the imaginary part of the dielectric constant (see, for example,[36]):

$$\alpha_{1\,n} = 2\pi\int_0^\infty dv_\perp v_\perp J_n^2(k_\perp v_\perp/\omega_i)F_0(v_\perp), \quad (2.16)$$

Here $F_0(v_{\parallel})$ is the ion distribution function with respect to the longitudinal velocity, which is taken at $v_{\parallel} = (\omega - n\omega_i)/k_{\parallel}$,

$$\Delta t \leqslant \omega_{pi}^{-1} \frac{L_1}{r_i}\left[1+\left(\frac{\omega_{pe} r_i}{v_e}\right)^2\right]^{1/2}.$$

and the magnetic field, as in (2.15), is assumed to be homogeneous.

The sign of Im ϵ characterizes the energy balance in the oscillations. The ions give up energy if Im $\epsilon < 0$. The first term in (2.16) takes into account the transverse ion displacements in the wave field. It can be negative for conical distributions when $\alpha_n < 0$. The second, which takes into account the longitudinal displacements, prevails if the thermal longitudinal-velocity scatter is sufficiently small. If the distribution with respect to $v_{\parallel}$ falls off monotonically, (there are no beams), then this term becomes negative at $\omega < n\omega_i$. In an inhomogeneous magnetic field, a buildup region can exist simultaneously with a damping region. Therefore the instability due to *the anisotropy is possible only if the influence of the* buildup region predominates.

If the magnetic field varies monotonically and its variation within the limits of the system is sufficiently large

$$\frac{\Delta H}{H} \gg \max\left[\frac{\omega_i}{\omega_{pe}}\frac{v_{\parallel i}}{v_{\perp i}},\ \left(\frac{m_e}{m_i}\right)^{1/3}\right],$$

then the gain (attenuation) of the cyclotron oscillations on passing through the resonance region is given by

$$T_1 = \exp\left[\int_{-\infty}^{\infty} dz\,(\gamma/v_0)\right];$$

Here $\gamma = -\mathrm{Im}\,\epsilon/(\partial\,\mathrm{Re}\,\epsilon/\partial\omega)$ is the local increment (temporal gain), $\mathrm{Re}\,\epsilon \approx 1 - [(\omega_{pe}/\omega)k_{\parallel}/k]^2$, $V_0 \approx \omega_{pe}/k_{\perp}$ is *the group velocity of the oscillations; the inhomogeneity* of the magnetic field and of the density is taken into account in these expressions parametrically. Calculations yield $T_1 = T_{11}T_{12}$, where $T_{11} = \exp(\beta_1/2)$ (see (2.3)), and for T_{12} we have

$$T_{12} = \exp\left\{\frac{\pi}{2}\frac{m_e}{m_i}k_{\perp}\frac{n\omega_i^3}{|(\omega_i)'_z|}\left[\left(\frac{\alpha_{1n}}{\omega_{pe}(\omega_i)'_z}\right)'_z\right]_{z=z_s}\right\}. \qquad (2.17)$$

It follows from (2.17) that to obtain amplification of the oscillations through anisotropy it is necessary to satisfy at least one of the inequalities $H'_z H''_{z^2} < 0$, $H'_z(\alpha_{1n})'_z > 0$, $H'_z(n_0)'_z < 0$. *Let us explain these conditions.*

The oscillation amplitude varies within the resonance zone, where Im $\epsilon \neq 0$. It follows from (2.16) that the dimension of the resonance zone is $\approx |k_{\parallel}v_{\parallel}/(\omega_i)'_z|$. If the first inequality is satisfied, then the region where the oscillations are amplified ($\omega < n\omega_i(z)$) is larger than the attenuation region ($\omega > n\omega_i(z)$). The longitudinal motion of the ions in the field of the oscillations is determined by the n-th cylindrical harmonic in the plane-wave expansion (cf. Sec. 1a). Therefore the second inequality ensures a high intensity of the resonant interaction in the gain region (Im $\epsilon \sim \alpha_{1n}$). Finally, the third takes into account the action of several effects, for when the density is increased the number of resonant particles increases like $\sim n_0$, the dimension of the resonance zone decreases like $\sim \Delta z \sim k_{\parallel} \sim n_0^{-1/2}$, the resonance interaction weakens Im $\epsilon \sim (\omega - n\omega_i) \lesssim k_{\parallel}v_{\parallel} \sim n_0^{-1/2}$, and the duration of the interaction increases, since the oscillations slow down, $V_0 \sim n_0^{-1/2}$.

In magnetic bottles, the dimensions of the region occupied by the plasma are as a rule smaller than the characteristic scale L of variation of the magnetic field. In this case the predominant effect is connected with the second derivative of the magnetic field, and since $H'_z H''_{z^2} > 0$, the oscillations weaken as they pass through the *region of the cyclotron resonance.* Exceptions are oscillations with $\omega \approx n\omega_{i\,\min}$, for which $\omega < n\omega_i(z)$ and therefore there is no absorption region at all. Such oscillations will indeed build up under conditions of reflection from the plasma boundaries. If the oscillations are absorbed by the boundaries and the ion distribution, while anisotropic, is monotonic with respect to the transverse energies, then no unstable natural modes can be established. Indeed, under dissipative boundary conditions, the natural modes should be localized between the cyclotron-resonance points (see Sec. b). However, if these points are separated (and are far from the center of the bottle), then the resonance interaction should lead to a weakening of the oscillations. It can also be shown *that unstable natural oscillations are impossible in a* plasma with hot electrons.

f) Drift-cyclotron instability. If the plasma density varies in the direction transverse to the magnetic field, then the electron drift in crossed fields (the electric field of the wave and the constant magnetic field) leads to a separation of the electronic and ionic charges. This effect can be taken into account by introducing in (2.2) an additional term $k_{\perp}\kappa(\omega/\omega_e)\varphi$, where $\kappa = (\mathbf{k}_{\perp}\cdot\nabla n_0)/k_{\perp}n_0$. An analysis of the modified equation shows that unstable oscillations that are strongly elongated along the magnetic field appear even at a small inhomogeneity $\kappa r_i \approx 10^{2}-10^{-3}$ (drift-cyclotron instability with $k_{\parallel} \to 0$)[43]. Such oscillations cannot be absorbed at the plasma boundaries, since any boundary acts like a sharp boundary for them, $ak_{\parallel} \to 0$ (see Sec. b). In analogy with the results obtained in this section, it must be concluded that the inhomogeneity of the magnetic field can only decrease the increment of the drift-cyclotron instability, without leading to full stabilization. Under real conditions, the change of the increment should be small[44].

The main result of Sec. 2 can be formulated in the following manner: An increase of the inhomogeneity of the the magnetic field exerts a stabilizing action on the cyclotron instability. Whether this will decrease the unstable oscillations or lead to complete stabilization depends on the concrete conditions.

Turning to the experimental data characterizing the stability of the cyclotron oscillations in magnetic bottles, we note that in view of the complexity of real systems, which leads to the need for simultaneously accounting for a number of factors (electric field, influence of the end faces, complicated form of the force lines), the influence of the magnetic-field inhomogeneity on the plasma stability has not yet been investigated. At the same time, there are distinct indications that plasma stability is improved when the inhomogeneity of the magnetic field is increased (see, e.g.,[45]).

APPENDIX

At cyclotron resonance in an inhomogeneous magnetic field, energy exchange between the oscillations and the charged particles is via excitation of modulated beams. Other processes that lead to exchange of energy between oscillations and the plasma proceed in similar fashion, if the range of action of these processes is confined to a small region of space. Thus, for example, when oscillations interact with a beam of charged particles moving with velocity $\mathbf{v}_0(\mathbf{r})$ that varies in space, modulated beams will be excited at the point $\mathbf{r}_S$ where the phase velocity of the oscillations $\omega/\mathbf{k}$ coincides with the beam velocity $\mathbf{v}_0(\mathbf{r}_S)$. In this case, the Doppler frequency shift $\mathbf{k}\cdot\mathbf{v}_0(\mathbf{r})$ plays the same role as the $n\omega_i(\mathbf{r})$ shift for cyclotron oscillations[9].

Let us examine now the interaction between the oscillations and charged particles in the case of an abrupt (jumplike) change of the particle velocity. We consider a simplified model. Assume that a plane wave $E = E_1 \exp(-i\omega t + ikz)$ propagates along the OZ axis, and a beam of charged particles moves in the same direction with velocity $V_0(z)$. We assume that in the vicinity of a certain point z_S the beam velocity changes quite abruptly by a small amount $\Delta v_0 \ll v_0(z_S)$. From the equation of motion

$$\dot{v}_1 = (eE_1/m)\, e^{-i\omega t + ikz(t)} \tag{A.1}$$

we obtain

$$v_1 = (eE_1/m) \int_{-\infty}^{t} dt' e^{-i\omega t' + ikz(t')}; \tag{A.2}$$

here z(t) is the particle trajectory.

We insert the quantity $1 \equiv [-\omega + kv_0(t')]/[-\omega + kv_0(t')]$ under the integral sign and integrate by parts

$$v_1 = \frac{eE_1}{m}\left[\frac{i}{\omega - kv_0(t)} e^{-i\omega t + ikz(t)} - i \int_{-\infty}^{t} dt' \frac{k\dot{v}_0(t')}{(\omega - kv_0(t'))^2} e^{-i\omega t' + ikz(t')}\right]. \tag{A.3}$$

The second term in (A.3) takes into account the prior history of the particle motion, which is important only in the presence of acceleration. If the time dependence of the acceleration is in the form of a δ-function, then we obtain from (A.3)

$$v_1 = \frac{eE_1}{m}\left[\frac{i}{\omega - kv_0(t)} e^{-i\omega t + ikz(t)} - \frac{ik\Delta v_0(t_s)}{(\omega - kv_0(t_s))^2} e^{-i\omega t_s + ikz(t_s)}\right]; \tag{A.4}$$

Here $\Delta v_0(t_S)$ is the velocity jump at the point $z_S = z(t_S)$.

It follows from (A.4) that at the point z_S the particles acquire a velocity increment (the second term). This increment is then transported with velocity $v_0(z)$. In other words, modulated beams are excited at the point z_S. The energy needed for this purpose is drawn from the oscillations. This is the mechanism of collisionless absorption (anomalous skin effect) considered in[11], where absorption by a plasma boundary, reflection from which causes the particles to change velocity abruptly (specular reflection, diffuse reflection), is considered.

In the presence of thermal spread, the oscillations connected with the modulated beams attenuate in space. In analogy with (1.9), it is easily seen that in the case of a Maxwellian distribution the damping varies like $\sim \exp[-\alpha(z - z_S)^{2/3}]$. This characteristic relation was first obtained in[46], where, in particular, the propagation of oscillations excited on a sharp plasma boundary was considered. It is naturally explained in terms of modulated beams. A more detailed analysis of the phenomena occurring on a sharp plasma boundary can be found in[10].

1)For the sake of clarity it is assumed in Fig. 1 that the centers of the electron Larmor circles lie on one force line. If the system is homogeneous in the direction transverse to the magnetic field, then the azimuthal dependence drops out after averaging over all the electrons that arrive at the given point of space.

2)An indication of this process is contained in [6].

3)If the electron-ion collisions are frequent enough, $\nu \gg (\delta v_{\parallel} \,|\, d\omega_e/dz\,|)^{1/2}$, then the shaded region in Fig. 2 drops below the real axis. In this case it is possible to use the adiabatic wave equation (1.9) on the entire axis.

4)Actually, the energy absorption is taken into account by the Landau bypass rule. The absence of a reflected wave is the consequence of the use of this rule.

5)At high plasma density $\beta \gg 1$, $\eta \gg 1$, the main effects connected with the nonlinearity of the magnetic field can be taken into account within the framework of the adiabatic wave equation (1.9).

6)So far, the problem of resonant cyclotron interaction was considered using as an example circularly-polarized electron cyclotron oscillations propagating along the magnetic field (whistlers). The resonance-interaction mechanism is, however, the same for charges of either sign. Moreover, the subscript e in the adiabatic wave equation (1.0) is replaced by i, then this equation describes Alfven oscillations in the frequency region $\omega \approx \omega_i$ (see, for example, [18]).

7)Unlike (1.2), this expression is not exact, being only the first term of an expansion in powers of t. The conditions for the applicability of (2.1) coincide with the conditions for the applicability of the linear approximation, which is perfectly sufficient for our purposes. Briefly, the procedure of deriving (2.1) is the following: Averaging over the fast Larmor rotation and over its initial phase, assuming uniform distribution, yields $\dot{W}_i = 2m_i(\dot{v}_\perp^2 + v_\perp \ddot{v}_\perp)\,t$; The quantities $v_\perp$ and $v_\perp$ are determined by the averaging method [28].

8)We should not be surprised at the difference from the case of an equilibrium plasma (see Fig. 4), since we are considering the steady-state picture, when the oscillations incident on the point z_0 have time to "feel" the processes occurring at the resonant point z_S. Formally, the absence of a reflected wave is due to the growth of the amplitude in the region from z_0 to z_S. A discussion of the possible rules for the matching together of the quasiclassical solutions on going through turning points can be found, for example, in [32].

9)If the thermal spread were small enough ($\delta v_\parallel \ll v_\parallel$), then the modulated beam excited by one of the points would give up its energy to another at definite distances between the resonance points. As a result, the gain and reflection coefficient would become oscillating functions of $z_S - z_{min}$.

10)In [39], the frequency of the elementary perturbations was determined from the equation $V^{-1}(z, z_0) = \partial k(z,\omega)/\partial\omega$. The trajectories $z(z_0, t)$ were obtained by using the "Huygens principle." The oscillation frequency is not conserved on trajectories chosen in this manner. We note that the equation $V^{-1} = \partial k/\partial\omega$ is obtained by differentiating (2.12) with respect to time only if the frequency is constant.

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