Vortices in type-II superconductors

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The review is devoted to an investigation of the behavior of vortex filaments in hard superconductors. The analysis is carried out in the region of weak magnetic fields, where linear electrodynamics is applicable. This permits a simple physical interpretation of both the properties of isolated vortices and of interactions of vortices with one another and with the surface of the superconductor. A number of examples of the influence of the superconductor outer boundary on the vortex structure are analyzed. It follows from these examples that the vortex lattice is stable relative to a transverse transport current even in an ideally homogeneous sample. Estimates are presented of the maximum transport current at which the vortex structure still remains stable. The role played by the internal surfaces in a superconductor is also illustrated with the interaction of a vortex with a cylindrical cavity or with an interface between two different superconductors as an example.

CONTENTS

1. INTRODUCTION

If a type-II superconductor is placed in a sufficiently strong magnetic field, superconducting vortices (or vortex filaments) are produced in it. Such a vortex comprises a thin non-superconducting core around which an undamped superconducting current circulates.

The existence of vortices gives rise to many interesting and important (from the applied point of view) properties of type-II superconductors. The desire to be able to have at ones fingertip the facts that take place with vortices is therefore natural. This turns out to be possible. In moderate fields, when the London approximation of superconductivity theory can be used, a linear electrodynamic situation arises wherein the principle of superposition of the currents and fields can be extensively used. The vortices can then be regarded as independent objects and their interaction with one another and with the surfaces of the superconductor can be taken into account. This makes it possible to study the magnetization curves of type-II superconductors and their dependence on the shape of the sample, and to understand how the vortices interact and are pinned by inhomogeneities of the material.

We consider the last item to be particularly important, since vortex pinning by inhomogeneities is the cause of irreversible effects in the magnetization of superconductors, because of the residual magnetic flux, and finally, the cause of the existence of a sufficiently large critical current in inhomogeneous type-II superconductors (hard superconductors).

In this review we consider first a single vortex, its magnetic flux, its interaction with the boundary of the superconductor and with the extraneous current flowing around it, and the pinning of this vortex by an internal cavity and a superconductor. We then investigate a system of vortices that interact with one another, with the boundaries of the superconductor, and with certain defects. This enables us to investigate the mixed state in bounded type-II superconductors. Moreover, it explains how even an ideally homogeneous type-II superconductor, when in the mixed state, still retains the ability of passing a finite transport current in a direction perpendicular to the magnetic field.

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Vortices in superconductors are investigated by the method of the theory of Ginzburg and Landau (GL)[1]. Although this theory has been detailed in a number of reviews and monographs (see, e.g.,[2,3,11]), we consider it appropriate to mention here its fundamental premises.

It is known from experiment that at a critical temperature T_c there occurs in a superconductor a transition from a normal state to a superconducting state, and that this transition is a second-order phase transition. The order parameter, which is equal to zero at $T > T_c$ and is small at $T_c - T \ll T_c$, is the so-called effective wave function of the superconducting electrons $\psi(\mathbf{r})$. The quantity $|\psi|^2$ can be treated as the density of the superconducting electrons, i.e., the number of metal electrons per unit volume in the condensate (contained in the Cooper pairs). If the Landau theory of second-order phase transitions is applied to a superconductor T_c, then the expression for the density of the free energy of the superconductor in the magnetic field takes the form

$$F_s = F_n + \frac{H^2}{8\pi} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}\left|\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\psi\right|^2;$$

here F_n is the density of the free energy of the normal phase, $\mathbf{H}$ is the intensity of the magnetic field inside the superconductor, $\alpha(T)$ and β are constants that depend on the material, m is the electron mass, and e is the electron charge. The last term is the density of the kinetic energy of the superconducting electrons in the presence of a magnetic field whose vector potential is $\mathbf{A}$.

Integrating F_s over the volume of the superconductor and equating to zero the first variations of the free energy with respect to ψ^* and $\mathbf{A}$, we obtain the well known GL equations, to which Maxwell's equations should be added:

$$\frac{1}{2m}\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0, \tag{1}$$

$$\mathbf{j} = -\frac{ie\hbar}{m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{4e^2}{mc}|\psi|^2\mathbf{A}, \tag{2}$$

$$\mathbf{j}=\frac{c}{4\pi}\operatorname{rot}\mathbf{H}=\frac{c}{4\pi}\operatorname{rot}\operatorname{rot}\mathbf{A}. \qquad (3)$$

It follows from these equations that the magnetic field and the current can vary significantly in a superconductor over distances on the order of

$$\lambda=\sqrt{\frac{mc^2\beta}{16\pi e^2|\alpha|}}.$$

This quantity is called the depth of penetration of the weak magnetic field.

The distance over which the order parameter ψ can vary significantly (as follows from (1)) is equal to the so-called coherence length

$$\xi=\frac{\hbar}{\sqrt{2m|\alpha|}}.$$

The parameter κ of the GL theory is equal to

$$\varkappa=\frac{\lambda}{\xi}.$$

In the absence of a magnetic field, the equilibrium value of $|\psi|^2$, as follows from (1), is $|\psi_0|^2=-\alpha/\beta$. Derivation of the GL equations from the microscopic theory[5] has made it possible to express the coefficients α and β (and consequently λ and ξ) in terms of the electronic characteristics of the material. We present the final results.

For a pure superconductor

$$\lambda(T)=\frac{1}{\sqrt{2}}\lambda_L(0)\sqrt{\frac{T_c}{T_c-T}},\quad \xi(T)=0.74\xi_0\left(\frac{T_c}{T_c-T}\right)^{1/2}.$$

For a dirty superconductor

$$\lambda(T)=0.615\lambda_L(0)\sqrt{\frac{\xi_0}{l}}\sqrt{\frac{T_c}{T_c-T}},\quad \xi(T)=0.85\sqrt{\xi_0 l}\sqrt{\frac{T_c}{T_c-T}};$$

here $\lambda_L^2(0)=3c^2/(8\pi e^2 v_F^2 N(0))$ is the London depth of penetration, $\xi_0=0.18\hbar v_F/(k_B T_c)$, v_F is the electron velocity on the Fermi surface, N(0) is the density of states at the Fermi level, and l is the electron mean free path. A superconductor is regarded as dirty if $l\ll\xi_0$.

If $\kappa>1/\sqrt{2}$, then the energy of the boundary between the normal and superconducting phases (the n—s boundary) is negative. The n—s boundary can be in the equilibrium state only when the superconductor is in an external magnetic field and the n and s phases are in equilibrium. The magnetic field then passes through the n phase. Let $\lambda\gg\xi$. This means that in the region of the n—s boundary there is located a layer (of thickness $\sim\lambda$) in which both a magnetic field and a superconducting condensate are present. It is clear that the free-energy density in this field is smaller than the free-energy density of the n phase, by an amount released upon condensation (pairing) of the electrons of this layer. This means that the energy of the n—s boundary is negative in this case. Superconductors with $\kappa>1/\sqrt{2}$ are called type-II superconductors.

Superconductors with $\kappa<1/\sqrt{2}$ have a positive n—s boundary energy and are called type I superconductors.

On the basis of the GL theory, Abrikosov[6] developed the theory of type-II superconductors. Let a type-II superconductor be in so strong a magnetic field that it has certainly gone over into the normal state. We start to decrease the external field. It turns out that at a certain field H_{c2} (the second critical field) the so-called mixed state sets in, wherein the homogeneous normal sample becomes laminated into alternating regions of normal and superconducting phases. Here

$$H_{c2}=\sqrt{2}\varkappa H_{c,m},$$

where $H_{c,m}$ is the thermodynamic critical field and is given by the relation

$$\frac{H_{c,m}^2}{8\pi}=F_n-F_s.$$

This lamination is energywise favored, since the n—s boundary has a negative surface energy in our superconductor. Abrikosov[6] has shown that this lamination should be realized in the form of a vortex lattice, i.e., a two-dimensional periodic structure of vortex filaments parallel to the external magnetic field and permeating through the entire body of the superconductor.

With decreasing external field, the period of the vortex lattice increases and the number of vortices inside the superconductor decreases. Finally, in a field H_{c1} (the first critical field) the stay of the vortices inside the superconductor becomes energywise unprofitable and the last vortex leaves the sample.

2. MAGNETIC FLUX OF VORTEX, ITS MAGNETIC FIELD AND FREE ENERGY

We now explain what the nature of a single vortex is. This can be done conveniently by assuming that $\lambda\gg\xi$. Then Eqs. (1) and (2) are easy to analyze. Let the center of the vortex be at the origin. Then $\psi(0)=0$. It is easy to show[6] that $|\psi(r)|^2\sim(r/\xi)^2$ at $r\ll\xi$. At $r\gg\xi$, on the other hand, we have $r\gg\xi$ $|\psi|^2=\psi_0^2=-\alpha/\beta$, since ξ is the distance over which the order parameter changes appreciably; the growth of the order parameter $|\psi|^2$ from zero to ψ_0^2 occurs over a length $\sim\xi$.

A microscopic analysis[7,8], has shown that the spectrum of the elementary excitations in this region of the vortex differs little from the spectrum of the elementary excitations of a nonsuperconducting metal. Therefore at $\lambda\gg\xi$ one frequently employs the following model of the vortex: the core of the vortex, of radius ξ, is assumed to be a normal metal, and all the remaining space is assumed to be superconducting. The vortical superconducting current producing the magnetic field of the vortex flows around the normal core and occupies a region of radius λ. At distances $r\gg\lambda$ the vortex current attenuates exponentially. Figure 1 shows schematically $\psi(r)$ and the filament field H(r) at $\lambda\gg\xi$.

We now determine the magnetic field of a single vortex situated in an unbounded superconductor. The center of the vortex will be assumed as before to be at the origin. We draw a circle of radius $R\gg\lambda$ in a plane perpendicular to the filament axis. The flux density of the vortex on this circle is equal to zero, and the modulus of the ordering parameter has long ago passed to the level ψ_0. Representing the order parameter in the form $\psi=\psi_0 e^{i\theta}$, where θ is the phase of the GL wave function, we rewrite (2) in the form

$$\mathbf{j}=\frac{2e\hbar}{m}\psi_0^2\nabla\theta-\frac{4e^2}{mc}\psi_0^2\mathbf{A}. \qquad (4)$$

We now integrate this equation along the contour of our circle of radius R. We consider first $\oint_R \nabla\theta\, dl$. This

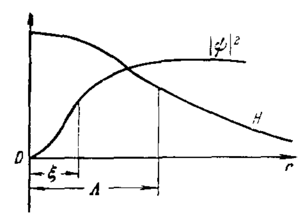

FIG. 1. Dependence of $|\psi|^2$ and of H on the distance r to the isolated vortex.

integral is obviously equal to the total change of the phase of the wave function after going around the closed contour. We stipulate the natural requirement that the wave function be unique. Then the total change of the phase on going around the closed contour can be only a multiple of 2π. If the contour surrounds one vortex filament, then the multiplicity is minimal[1)] and

$$\oint_R \nabla\theta\, dl = 2\pi.$$

Recognizing now that $\oint_R \mathbf{j}\, d\mathbf{l} = 0$ and that the total magnetic flux of the vortex is equal to $\Phi_0 = \oint_R \mathbf{A}\, d\mathbf{l}$, we readily obtain from (4)

$$\Phi_0 = \frac{ch}{2e} = \frac{\pi c\hbar}{e} \approx 2\cdot 10^{-7}\ \text{G-cm}^2.$$

This quantity is called the magnetic-flux quantum.

We emphasize right away that the vortex filament carries a magnetic-flux quantum Φ_0 only if it is far from the surface of the superconductor. If the filament is close to the superconducting surface, then its magnetic flux is smaller than Φ_0. We shall return to this question later on.

We now proceed to calculate the magnetic field of the filament.

Taking the foregoing into account, we assume the core of the vortex, of radius ξ, to be a normal metal. What is important here is that for the calculations it suffices to know only the order of magnitude of the radius of the core, since it will be shown later on that the radius of the core enters under the logarithm sign in the expressions for the field of the filament and for its energy.

We now define the region of applicability of our calculations. It is necessary to stipulate here, besides the inequality $\lambda \gg \xi$, that the modulus of the order parameter $|\psi|$ be constant outside the core. This requirement is satisfied when the cores of the vortices are far from one another, i.e., when the external field H_0 satisfies the inequality $H_{c1} \leq H_0 \ll H_{c2}$. The last condition means that the center of the vortex should be far enough (in comparison with ξ) from the surface of the superconductor, for otherwise the variation of $|\psi|$ cannot be neglected. To derive the equation for the field of the vortex filament, we start from the GL equation (4), which is valid everywhere outside the core of the vortex. We now impose the boundary conditions. The magnetic field of the filament $H(r) \to 0$ as $r \to \infty$. This is the first boundary condition. The second condition can be obtained from the requirement that the phase of the wave function ψ change by an amount 2π on going around the center of the vortex filament. We draw a circle of radius ξ around the center of the filament. We integrate (4) along this circle. Neglecting the magnetic flux surrounded by the integration contour, i.e., the magnetic flux of the core, we obtain

$$\oint \mathbf{j}\, d\mathbf{l} = \frac{c}{4\pi}\frac{\Phi_0}{\lambda^2},$$

where

$$|\text{rot}\,\mathbf{H}|_{r=\xi} = \frac{\Phi_0}{2\pi\lambda^2\xi}. \tag{5}$$

This is our second boundary condition. In the derivation of (5), we used Maxwell's equation curl $\mathbf{H} = (4\pi/c)\mathbf{j}$, which connects the field of the vortex with its current. The condition (5) can be satisfied automatically by taking the curl of both sides of (4) and rewriting the equation for $\mathbf{H}$ in the form

$$\mathbf{H} + \lambda^2\,\text{rot rot}\,\mathbf{H} = \Phi_0\mathbf{e}\delta(\mathbf{r} - \mathbf{r}_0); \tag{6}$$

here $\mathbf{r}_0$ is the radius vector of the center of the vortex filament, specified on a plane perpendicular to the filament, and $\mathbf{e}$ is a unit vector directed along the filament. Let us in fact integrate (6) over the area of a circle of radius ξ with center on the vortex axis. Again neglecting the flux $\int \mathbf{H}d\mathbf{S}$ through the core, we obtain immediately the condition (5). For a single vortex filament, Eq. (6) has the following solution that falls off at infinity:

$$H = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{|\mathbf{r}-\mathbf{r}_0|}{\lambda}\right), \tag{7}$$

where $K_0(0)$ is a Hankel function of zero order and imaginary argument. We recall the asymptotic forms of this function: $K_0(z) \approx \ln(2/\gamma z)$ at $z \ll 1$, where $\gamma = e^C \approx 1.78$, and $K_0(z) \approx \sqrt{\pi/2z}\,e^{-z}$ at $z \gg 1$.

Thus, the magnetic field of the vortex filament decreases exponentially at $|\mathbf{r} - \mathbf{r}_0| \gg \lambda$, and increases logarithmically on approaching the core of the filament. To find the field at the center of the filament, we substitute $|\mathbf{r} - \mathbf{r}_0| = \xi$ in (7), i.e., we cut off the logarithmic divergence at the radius ξ. Then

$$H(\mathbf{r}_0) = \frac{\Phi_0}{2\pi\lambda^2}\ln\varkappa. \tag{8}$$

We now find the free energy of a single filament. Inasmuch as $\lambda \gg \xi$ by assumption, we neglect the energy change due to the change of the condensation energy in the normal core of the vortex. The free energy of the isolated filament then takes the form

$$\mathcal{F} = \frac{1}{8\pi}\int [\mathbf{H}^2 + \lambda^2(\text{rot}\,\mathbf{H})^2]\, dV. \tag{9}$$

The integral is taken over the entire volume of the sample. It would be more correct to integrate only over the volume of the superconductor without the region occupied by the core of the vortex. However, since $\xi \ll \lambda$ and since the calculations are carried out with logarithmic accuracy, the integration in (9) can be extended over the entire volume of the sample. The first term in (9) yields the energy of the magnetic field of the filament, and the second term yields the kinetic energy of the vortex currents.

The free energy per unit length of the vortex ϵ_0 is obtained by integrating (9) over a plane-parallel layer of unit thickness, located perpendicular to the filament. Substituting in (9) $(\text{curl}\,\mathbf{H})^2 = \mathbf{H}\,\text{curl curl}\,\mathbf{H} + \text{div}[\mathbf{H}\times\text{curl}\,\mathbf{H}]$, and recognizing that

$$\int \text{div}\,[\mathbf{H}\times\text{rot}\,\mathbf{H}]\, dV = \oint [\mathbf{H}\times\text{rot}\,\mathbf{H}]\, d\mathbf{S} = 0$$

(S is the outer surface of the superconductor, which is located at infinity), and using (6), we get

$$\varepsilon_0 = \left(\frac{\Phi_0}{4\pi\lambda}\right)^2\ln\varkappa. \tag{10}$$

This agrees with logarithmic accuracy with the result $\epsilon_0 = (\Phi_0/4\pi\lambda)^2(\ln\kappa - 0.18)$ of Abrikosov's numerical calculation[6]. Thus, if a single vortex is located in a type-II superconductor, then the free energy of the superconductor (per unit length of the vortex) increases by an amount ϵ_0. Why do vortices appear then in a type-II superconductor? The point is that if the superconductor is an external homogeneous magnetic field $\mathbf{H}_0$, then the energy that is minimal in the equilibrium state is not the free energy $\mathcal{F}$, but the Gibbs free energy[12], the density G of which is given by

$$G = F - \frac{\mathbf{H}_0\mathbf{H}}{4\pi}, \tag{11}$$

where the density of the free energy is

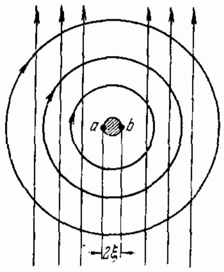

FIG. 2. Vortex filament in transport current. The vortex current is added to the transport current at point a and is subtracted at point b.

$F = (1/8\pi)[\mathbf{H}^2 + \lambda^2(\text{curl } \mathbf{H})^2]$; here $\mathbf{H}$ is the local magnetic field in the superconductor. The origin is assumed to be the energy of the superconductor without the external field and without the vortex filament.

We integrate (11) over the entire volume of a superconductor containing one vortex filament. Then we obtain per unit filament length

$$\int G\,dV = \varepsilon_0 - \frac{\Phi_0 H_0}{4\pi}.$$

Equating the right-hand side of this equation to zero, we obtain immediately the external field H_0 at which the penetration of the first vortex filament in the superconductor becomes energy wise favored, i.e., the first critical field

$$H_{c1} = \frac{4\pi\varepsilon_0}{\Phi_0} = \frac{\Phi_0}{4\pi\lambda^2}\ln\varkappa. \tag{12}$$

Comparison of (8) and (12) shows that the field at the center of the filament is approximately double the first critical field.

3. FORCE OF INTERACTION BETWEEN THE VORTEX AND THE CURRENT

We now can proceed to the study of a very important practical problem, that of the interaction force between a vortex and the transport current flowing around it, i.e., the current produced by some external source. This force is usually called the Lorentz force. We consider first the qualitative picture. Figure 2 shows a vortex with transport current flowing around it. It is easy to see that the superfluid velocities of the transport current and of the vortex are added together on the left of the core and oppose each other on the right. This means that, according to Bernoulli's law, the pressure is lower to the left of the core than to the right. Consequently a Lorentz force acts on the core of the vortex, as shown in the figure.

We emphasize that this force is applied to the boundary of the vortex core, and therefore the very name "Lorentz force" seems to us inappropriate. Indeed, the Lorentz force is taken to mean the force exerted by the magnetic-field source on a charge moving in this field. Yet in the volume of the superconductor all the forces acting on the charges are balanced. To explain this, we turn to the book by F. London[13]. The general expression for the density of the volume force acting on the moving charges is

$$\mathbf{f} = \frac{1}{c}[\mathbf{j}, \mathbf{H}], \tag{13}^*$$

where $\mathbf{j}$ is the current density. This is the density of the force customarily called the Lorentz force. This force can be represented in the form of a divergence of the stress tensor of the electromagnetic field $(-\hat{T})$, which in the stationary case (when there is no electric field) is given by

$$T_{ik} = -\frac{1}{4\pi}\left(H_i H_k - \delta_{ik}\frac{H^2}{2}\right), \tag{14}$$

i.e.,

$$f_i = -\sum_k \frac{\partial T_{ik}}{\partial x_k},$$

or

$$\mathbf{f} = -\operatorname{Div}\hat{T}(\mathbf{H}). \tag{15}$$

This can be easily verified by substituting in (13) Maxwell's equation $\mathbf{j} = (c/4\pi)\text{curl }\mathbf{H}$ and, recognizing that $\operatorname{div}\mathbf{H} = 0$, represent $\mathbf{f}$ in the form

$$\mathbf{f} = \frac{1}{4\pi}(\mathbf{H}\operatorname{div}\mathbf{H} - [\mathbf{H}, \operatorname{rot}\mathbf{H}]).$$

Elementary calculations show that the right-hand side of this formula can be written in the form $-\operatorname{Div}\hat{T}(\mathbf{H})$, where the tensor $T_{ik}(\mathbf{H})$ is defined by (14). So far we have made no use of the fact that our arguments apply to a superconductor. We now take this circumstance into account.

For a superconductor we have the London equation

$$\operatorname{rot}(\Lambda\mathbf{j}) = -\frac{1}{c}\mathbf{H}, \qquad \Lambda = \frac{4\pi}{c^2}\lambda^2. \tag{16}$$

This equation is easily obtained from (6) by using Maxwell's equation for $\mathbf{j}$.

We now substitute (16) in (13). The density of the force $\mathbf{f}$ can now be expressed already as the divergence of the tensor $\Lambda\hat{S}$:

$$\mathbf{f} = \operatorname{Div}(\Lambda\hat{S}), \qquad S_{ik} = \left(j_i j_k - \delta_{ik}\frac{j^2}{2}\right). \tag{17}$$

This can be easily verified if (recognizing that $\operatorname{div}\mathbf{j} = 0$ in the stationary case) we represent $\mathbf{f}$ in the form

$$\mathbf{f} = \Lambda(\mathbf{j}\operatorname{div}\mathbf{j} - [\mathbf{j}, \operatorname{rot}\mathbf{j}]).$$

Combining (15) and (17), we get the following condition for equilibrium and stationary flow of current in a superconductor:

$$\operatorname{Div}(\hat{T}(\mathbf{H}) + \Lambda\hat{S}(\mathbf{j})) = 0. \tag{18}$$

This means that the forces acting in the superconductor on a current-carrying volume element are balanced and the resultant force is equal to zero. In other words, the true Lorentz force acting on the volume element of the superconductor with current $(-\operatorname{Div}\hat{T})$ is exactly balanced by the inertia forces $(-\operatorname{Div}\Lambda\hat{S})$. Equation (18) can be regarded as the law for the conservation of the momentum in a superconductor in the stationary state.

However, there is one extremely important difference between the tensors $\hat{T}(\mathbf{H})$ and $\hat{S}(\mathbf{j})$. On the boundary between a superconductor and a dielectric, $\hat{T}(\mathbf{H})$ remains continuous while $\hat{S}(\mathbf{j})$ experiences a discontinuity, inasmuch as $\hat{S} = 0$ in the dielectric. The balance (18) is therefore violated on the boundary and the surface of the superconductor is acted upon by a force usually interpreted as magnetic pressure. Let us explain the foregoing with a simple example. We consider a superconductor occupying the half-space $x > 0$. The plane $x = 0$ is its boundary with vacuum. A magnetic field H_0 along the Oz axis is specified on this surface. It is well known that in a layer $\sim\lambda$ next to the surface there is produced a Meissner current in the direction of the Oy axis, $j_y = (c/4\pi\lambda)H_0 e^{-x/\lambda}$. The magnetic field intensity in the same layer is $H_z = H_0 e^{-x/\lambda}$. Thus in the layer $\sim\lambda$ next to the surface there flows a current j_y in a field

H_z. This means that this current is acted upon by a true Lorentz force $(1/c)\mathbf{j}\times\mathbf{H}$, which in our case is directed along the Ox axis and is equal to $(H_0^2/4\pi\lambda)\,e^{-2x/\lambda}$. Why does the current flow directly along the Ox axis? Why is not the motion of the superconducting electrons distorted by this force? The answer is that the distribution of the superfluid velocities in the considered layer λ is such that the superconducting electrons are acted upon by a Bernoulli force in the negative Ox direction, and this force exactly balances the force $(1/c)\mathbf{j}\times\mathbf{H}$. This can be easily verified by calculating $\mathrm{Div}(-\Lambda\hat{S})$ for our case.

We turn now to a vortex around which a transverse transport current flows. We replace the core of the vortex in our model by a nonsuperconducting cylinder of radius ξ. The superfluid velocities on the boundary with the core at the points a and b (see Fig. 2) are different, and consequently the pressure on the boundary from the normal core at the point a will be larger than at the point b. Their difference produces the resultant force of interaction between the vortex and the transport current, which is usually also called the Lorentz force. We shall use this term frequently, always taking it to mean the force of interaction between the vortex filament and the current.

We now find a quantitative expression for this force. This is easiest to do by considering the interaction of two parallel vortex filaments. Let the coordinates of the centers of the filaments be $\mathbf{r}_1$ and $\mathbf{r}_2$. We calculate the free energy (9) of this system. The same transformations as used in the derivation of (10) yield in this case

$$\mathcal{F}=\frac{\Phi_0}{8\pi}[H(\mathbf{r}_1)+H(\mathbf{r}_2)]; \tag{19}$$

here $H(\mathbf{r}_1)$ and $H(\mathbf{r}_2)$ are the total fields at the centers of the first and second vortices. We consider the first vortex. The total field at its center consists of its proper field (formula (8)) induced at the point $\mathbf{r}_1$, and the field $H_{21}(\mathbf{r}_1)$ induced by the second vortex at the center of the first.

Since we have to calculate the force of interaction between vortices, we are interested in this calculation only in that part of $\mathcal{F}$ which depends on the relative distance between the vortices $r = |\mathbf{r}_1-\mathbf{r}_2|$. Leaving out the part that does not depend on r, we get

$$\mathcal{F}=\frac{\Phi_0}{4\pi}H_{12}(r),$$

where H_{12} is the field produced by the first vortex at the center of the second. Recognizing that this field is directed along the Oz axis, we can easily show that the vortex current $j(\mathbf{r})$ produced by the first vortex at the point corresponding to the center of the second vortex is equal to

$$\mathbf{j}(\mathbf{r})=\frac{c}{4\pi}\,\mathrm{rot}\,\mathbf{H}_{12}=-\frac{c}{4\pi}[\mathbf{e},\nabla H_{12}].$$

This yields immediately

$$\nabla H_{12}=-\frac{4\pi}{c}[\mathbf{j},\mathbf{e}]. \tag{20}$$

On the other hand, the force acting on the vortex is, by definition,

$$\mathbf{f}=-\nabla\mathcal{F}=-\frac{\Phi_0}{4\pi}\nabla H_{12}(r).$$

Substituting here formula (20) we obtain finally the force on a unit length of the second vortex:

$$\mathbf{f}=\frac{1}{c}[\mathbf{j},\mathbf{\Phi}_0], \tag{21}$$

where $\mathbf{\Phi}_0=\Phi_0\mathbf{e}$ and $\mathbf{j}$ is the current produced by the first vortex at the center of the second vortex. This is the formula for the Lorentz force, or the force of interaction between the vortex and the extraneous current flowing around it.

We have considered a particular case when the current flowing around the vortex is produced by a second vortex in the vicinity of the first. It can be shown[14,15] that formula (21) is valid in the general case. We shall make extensive use of this formula, since the concept of the Lorentz force as the force of interaction between a vortex and an extraneous current flowing around it is very fruitful and makes it easy to understand the physics of the situation.

4. VORTEX NEAR A SUPERCONDUCTOR BOUNDARY

So far we have considered vortices in an unbounded superconductor. A much more realistic and interesting case is a vortex in a bounded superconductor.

In the general case a vortex filament located near the surface of the superconductor produces a magnetic field outside the superconductor. To calculate the field distribution it is necessary to solve simultaneously Eq. (6) for the region inside the superconductor, and the equation curl H = 0 for the external space. The last equation is simply the condition under which there are no currents in the external space. For an arbitrary sample geometry this is a complicated problem. We confine ourselves therefore to the case when the vortex filament is parallel to the generator of an infinitely long superconducting cylinder of arbitrary cross section. In this case the vortex does not produce a magnetic field in the external space and the field of the vortex on the surface of the conductor is equal to zero. Let us prove this.

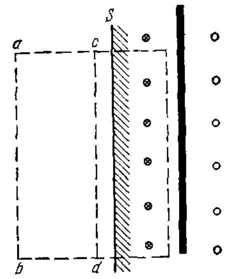

FIG. 3. For use in the proof that the field of the vortex is zero on the surface S. The thick line shows the core of the vortex. The circles are the directions of the vortex currents.

Figure 3 shows a vortex filament placed near the surface of a superconductor and parallel to the surface, as well as the contour abdc, the segments ab and cd being parallel to the superconductor surface. It follows from the symmetry of the problem that

$$\oint_{abdc}\mathbf{H}_v\,d\mathbf{l}=\int_{ab}\mathbf{H}_v\,d\mathbf{l}+\int_{dc}\mathbf{H}_v\,d\mathbf{l}.$$

On the other hand, this integral is equal to zero, since no electric current flows through the area abcd.

Hence

$$\int_{ab}\mathbf{H}_v\,d\mathbf{l}=\int_{cd}\mathbf{H}_v\,d\mathbf{l}; \tag{22}$$

here and throughout $\mathbf{H}_V$ is the magnetic field produced by the vortex filament. It is clear from physical considerations that if the section is moved very far from the superconductor surface then we have there $\mathbf{H}_V = 0$ and $\int_{ab}\mathbf{H}_V\,dl = 0$. Then, by virtue of (22) we have $\int_{cd}\mathbf{H}_V\,dl = 0$, and, taking the symmetry of the problem into account, we get $\mathbf{H}_V|_{cd} = 0$. Placing the segment cd on the surface

S of the superconductor, we obtain ultimately

$$\mathbf{H}_v |_S = 0. \tag{23}$$

a) **Magnetic flux of a vortex situated near the boundary of the superconductor.** In Chap. 2 we have established that in an infinite superconductor the vortex carries a magnetic flux equal to the quantum Φ_0. We now find the magnetic flux of a single vortex, when the vortex is near the outer boundary of superconducting space and is parallel to this boundary. Let the vortex filament be in a given superconducting cylinder of arbitrary cross section, let it be parallel to the cylinder generator, and let its center pass through the point $\mathbf{r}_0$. The magnetic flux Φ_V carried by such a filament is defined by

$$\Phi_v = \int \mathbf{H}_v \, d\mathbf{S}, \tag{24}$$

where the integration is carried out over the cross section area of the cylinder. It is shown in [16] that this expression can be represented in the form

$$\Phi_v = \Phi_0 (1 - \mathbf{eh}(\mathbf{r}_0)), \tag{25}$$

where $\mathbf{h}(\mathbf{r})$ is the field that would be present inside the cylinder if there were no vortex at all in the cylinder, and if the cylinder were placed in a unit magnetic field parallel to its generator. In other words $\mathbf{h}(\mathbf{r})$ satisfies the equation $\mathbf{h} + \lambda^2 \operatorname{curl} \operatorname{curl} \mathbf{h} = 0$ with boundary condition $\mathbf{h}|_\sigma = \mathbf{e}$ on the surface σ of the cylinder; $\mathbf{e}$ is, as before a unit vector directed along the vortex.

It is easily seen that when the center of the vortex approaches the surface of the cylinder we have $\mathbf{h}(\mathbf{r}_0) \to \mathbf{e}$ and $\Phi_V \to 0$. To the contrary, when the vortex moves away into the interior of the superconductor we have $\mathbf{h}(\mathbf{r}_0) \to 0$ and $\Phi_V \to \Phi_0$.

It is frequently more convenient to calculate the flux in accordance with formula (25) than by direct integration of (24). For example, for a round cylinder of radius R with a vortex parallel to its axis we have

$$\Phi_v = \Phi_0 \left(1 - \frac{I_0(r_0/\lambda)}{I_0(R/\lambda)}\right),$$

where r_0 is the distance from the center of the vortex to the cylinder axis and $I_0(z)$ is a Bessel function of imaginary argument. For a plate of thickness d with a filament parallel to its surface, we have

$$\Phi_v = \Phi_0 \left\{1 - \frac{\operatorname{ch}[(2x_0 - d)/2\lambda]}{\operatorname{ch}(d/2\lambda)}\right\}, \tag{26}$$

where x_0 is the distance from the center of the vortex to the surface of the plate. For the superconducting half-space $x > 0$, the boundary of which is the plane $x = 0$, we have

$$\Phi_v = \Phi_0 (1 - e^{-x_0/\lambda}), \tag{27}$$

where x_0 is the coordinate of the center of the vortex.

This variation of the magnetic flux of the vortex can be explained by using the method of images. This method is easiest to apply and is most lucid in the case of a superconducting half-space. Inasmuch as the vortex field $\mathbf{H}_V$ in the superconductor is determined by the solution of Eq. (6) with the boundary condition $\mathbf{H}_V|_{x=0} = 0$, a field of precisely the same type is produced by virtue of the linearity of Eq. (6), in the region $x > 0$ in an unbounded superconductor in which there are two vortices parallel to the plane $x = 0$ and passing through the points $x = x_0$ and $x = -x_0$, with the directions of both vortices opposite to each other. Indeed, by virtue of the symmetry of the problem, the magnetic field of the vortices at the plane $x = 0$ is equal to zero. The vortex at the point $x = -x_0$ will be called the image. Since it is of opposite sign, its magnetic flux in the region $x > 0$ cancels out part of the magnetic flux of the first vortex. When the vortex emerges to the surface of the superconductor, its image emerges to the same place. Total compensation sets in and $\Phi_V = 0$.

b) **Free energy of a vortex in a bounded superconductor.** We now find the free energy of the vortex if it is located inside a superconducting cylinder and is parallel to the generator:

$$\mathcal{F} = \frac{1}{8\pi} \int [\mathbf{H}_v^2 + \lambda^2 (\operatorname{rot} \mathbf{H}_v)^2] \, dV. \tag{28}$$

The integration is over the volume of the cylinder and is referred to a unit of the cylinder length. The field $\mathbf{H}_V(\mathbf{r})$ satisfies Eq. (6) with the boundary condition $\mathbf{H}_V|_\sigma = 0$. Therefore, using the identity $\mathbf{a} \operatorname{curl} \mathbf{b} = \mathbf{b} \operatorname{curl} \mathbf{a} - \operatorname{div}[\mathbf{a} \times \mathbf{b}]$ and the Gauss theorem, we easily obtain from (28) an expression for F [16]:

$$\mathcal{F} = \frac{\Phi_0 \mathbf{H}_v(\mathbf{r}_0)}{8\pi}. \tag{29}$$

Strictly speaking, the representation of the filament energy in the form (29) is incorrect, inasmuch as $H_V \to \infty$ as $\mathbf{r} \to \mathbf{r}_0$, but this logarithmic divergence can be cut off at a distance $|\mathbf{r} - \mathbf{r}_0| \sim \xi$. It is in this sense that $\mathbf{H}_V(\mathbf{r}_0)$ should be understood. It is easily seen that formula (29) goes over into the well known expression for the energy of a single filament in an unbounded superconductor if the vortex is moved into the interior of the cylinder to a distance much larger than λ from the surface. On the other hand, if the vortex approaches the surface, then $\mathcal{F} \to 0$, since $\mathbf{H}_V|_\sigma = 0$. All this can be easily explained in the language of the Lorentz force, and we shall do so when we consider the surface barrier.

We now take into account the interaction of our vortex in the cylinder with the external magnetic field, i.e., we find the Gibbs free energy G.

In this section we confine ourselves to the case of a long cylinder in a longitudinal homogeneous field $\mathbf{H}_0$. The case of such a geometry makes it possible to avoid complications connected with introduction of a demagnetizing factor.

The field inside the cylinder is now determined by the solution of Eq. (6) with the boundary condition $\mathbf{H}|_\sigma = \mathbf{H}_0$. This is understandable, for if there were no vortex in the cylinder, then the field would be determined by the solution of the homogeneous London equation

$$\mathbf{H}_1 + \lambda^2 \operatorname{rot} \operatorname{rot} \mathbf{H}_1 = 0, \quad \mathbf{H}_1|_\sigma = \mathbf{H}_0. \tag{30}$$

The field $\mathbf{H}_V$ of the vortex itself is also known to us and it is determined by Eq. (6) with boundary condition $\mathbf{H}_V|_\sigma = 0$. The total field is

$$\mathbf{H} = \mathbf{H}_v + \mathbf{H}_1. \tag{31}$$

Reckoning the free energy of the cylinder from its value in the absence of a magnetic field, we obtain

$$\mathcal{F} = \frac{1}{8\pi} \int [\mathbf{H}^2 + \lambda^2 (\operatorname{rot} \mathbf{H})^2] \, dV.$$

Substituting here (31) we get

$$\mathcal{F} = \frac{1}{8\pi} \int [\mathbf{H}_v^2 + \lambda^2 (\operatorname{rot} \mathbf{H}_v)^2] \, dV + \frac{1}{8\pi} \int [\mathbf{H}_1^2 + \lambda^2 (\operatorname{rot} \mathbf{H}_1)^2] \, dV + \frac{1}{4\pi} \int [\mathbf{H}_1 \mathbf{H}_v + \lambda^2 \operatorname{rot} \mathbf{H}_1 \operatorname{rot} \mathbf{H}_v] \, dV.$$

It is easy to show that the last integral is equal to zero. Indeed,

$$\int \text{rot}\,\mathbf{H}_1\,\text{rot}\,\mathbf{H}_v\,dV = \int \mathbf{H}_v\,\text{rot}\,\text{rot}\,\mathbf{H}_1\,dV + \int [\mathbf{H}_v,\,\text{rot}\,\mathbf{H}_1]\,d\sigma,$$

but the surface integral is equal to zero, since $\mathbf{H}_V|_\sigma = 0$. From this we get, taking (30) into account,

$$\int (\mathbf{H}_1\mathbf{H}_v + \lambda^2\,\text{rot}\,\mathbf{H}_1\;\text{rot}\,\mathbf{H}_v)\,dV = \int \mathbf{H}_v\,(\mathbf{H}_1 + \lambda^2\,\text{rot}\,\text{rot}\,\mathbf{H}_1)\,dV = 0.$$

Thus, the free energy $\mathcal{F}$ is simply the sum of the free energies of the vortex in the cylinder without the external field and of the cylinder in the field but without the vortex. The interaction of the vortex with the field does not enter in $\mathcal{F}$. It is therefore clear that in the thermodynamic-equilibrium state the quantity going through a minimum will not be $\mathcal{F}$ but the other thermodynamic potential, the Gibbs free energy

$$\mathcal{G} = \mathcal{F} - \frac{1}{4\pi}\int \mathbf{H}\mathbf{H}_0\,dV.$$

The integral is taken over the volume of the cylinder.

Reckoning $\mathcal{G}$ from its value in the absence of the vortex, we obtain

$$\mathcal{G} = \frac{1}{8\pi}\int [\mathbf{H}_v^2 + \lambda^2(\text{rot}\,\mathbf{H}_v)^2]\,dV - \frac{1}{4\pi}\int \mathbf{H}_0\mathbf{H}_v\,dV,$$

i.e., using (29) and (24),

$$\mathcal{G} = \frac{\Phi_0 \mathrm{H}_v(\mathbf{r}_0)}{8\pi} - \frac{\Phi_v(\mathbf{r}_0)\,\mathrm{H}_0}{4\pi}, \qquad (32)$$

where $\boldsymbol{\Phi}_V$ is a vector directed along the vortex; its modulus is equal to the magnetic flux of the vortex.

c) The Bean-Livingston barrier. The physical meaning of formula (32) can be explained by considering a superconducting half-space as an example. Let the superconductor occupy the region $x \geq 0$, let the vortex be located at the point x_0, and let the external field H_0 be directed along the Oz axis. The first term in (32), i.e., F represents the self energy of the free vortex ϵ_0 and the energy of the interaction between the vortex and the current produced by the image of the same vortex. Inasmuch as $H_V|_{x_0 \to 0} \to 0$, the function $\mathcal{F}$ increases monotonically (Fig. 4), and consequently the vortex filament in the absence of an external field H_0 is always attracted to the surface. This can be quite easily explained with the aid of Fig. 5. The distribution of the vortex current is distorted by the surface of the superconductor, and the current lines are denser to the left of the center of the vortex; consequently, the superfluid velocity is larger than on the right of the vortex core. We see again that the force determined by the difference between the Bernoulli pressures on the right and on the left of the core is applied to the vortex core and always draws the vortex to the surface. Differently stated, this is the Lorentz force $\mathbf{f}_\sigma$ exerted on the vortex by its image. The work required to overcome this force is indeed equal to F.

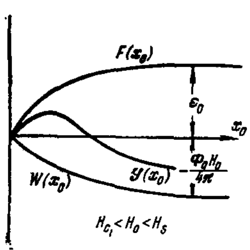

FIG. 4

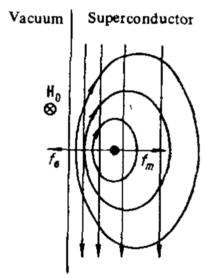

FIG. 5

FIG. 4. Illustrating the theory of the surface barrier. $\mathcal{F}(x_0)$ is the energy of attraction of the vortex to the surface. $W(x_0)$ is the energy of the interaction between the vortex and the Meissner current, and $\mathcal{G}(x_0)$ is the Gibbs free energy near the surface.

FIG. 5. Vortex currents and Meissner current near the surface. f_σ is the force of attraction of the vortex to the surface, due to the asymmetry of the vortex currents near the surface, f_m is the force of the interaction of the vortex with the Meissner current.

We now examine the second term of formula (32), which we designate W. Inasmuch as Φ_V is given by formula (25), we have

$$W = -\frac{\mathrm{H}_0\Phi_v}{4\pi} = \frac{\Phi_0}{4\pi}(\mathrm{H}_1(x_0) - \mathrm{H}_0),$$

where $\mathbf{H}_1$ is the field produced inside the superconductor by the external field $\mathbf{H}_0$, without allowance for the vortex field. It is then easily seen that

$$-\nabla W = \frac{1}{c}[\mathbf{j}_m(\mathbf{r}_0),\,\boldsymbol{\Phi}_0],$$

where

$$\mathbf{j}_m = \frac{c}{4\pi}\,\text{rot}\,\mathbf{H}_1,$$

i.e., $\mathbf{j}_m$ is the Meissner current produced by the external field $\mathbf{H}_0$. Consequently $-\nabla W$ is the Lorentz force $\mathbf{f}_m$ exerted by the Meissner current on a unit length of the vortex filament situated near the surface of the superconductor (Fig. 5). If the vortex and the external field have the same direction, then the force f_m is directed towards the interior of the superconductor. It is clear that if the vortex moves away from the surface under the influence of this force, then its energy will be lower by an amount $W(x_0)$, which becomes equal to $-\mathbf{H}_0\cdot\boldsymbol{\Phi}_0/4\pi$ at $x_0 \gg \lambda$. The Gibbs free energy $\mathcal{G}(x_0)$ is equal to the sum $\mathcal{F}(x_0) + W(x_0)$. As shown in Fig. 4, the quantity $\mathcal{G}(x_0)$ can be a nonmonotonic function of x_0, producing a barrier for the penetration of the vortices. This barrier is frequently called the Bean-Livingston barrier[17]. Obviously, the barrier disappears when the forces f_σ and f_m become equal for the vortex situated on the surface.

So far we have not made use at all of the specific property that our superconductor is a half-space, so that everything stated above is valid for a superconductor cylinder of any shape. We now use the fact that the surface of the superconductor is the plane $x = 0$. Then the force of interaction of the vortex with its image is

$$f_\sigma = \frac{1}{c}\,j_v(x_0)\,\Phi_0,$$

where $j_V(x_0)$ is the current produced at the point x_0 by the image situated at the point $(-x_0)$. According to (7), the free-vortex field is equal to $(\Phi_0/2\pi\lambda^2)\,K_0(r/\lambda)$. Then the vortex current at a distance $r = 2x_0$ is equal to $(c/4\pi)(\Phi_0/2\pi\lambda^3)K_1(2x_0/\lambda)$. At $z \ll 1$ we have $K_1(z) \approx 1/z$. For a vortex close to the surface we therefore have

$$f_\sigma = \frac{\Phi_0^2}{16\pi^2\lambda^2 x_0}.$$

The force f_σ tends to infinity when the vortex approaches the surface. However, the very procedure used for the calculation no longer holds as $x_0 \to 0$, for we can expect here strong changes in the order parameter ψ, and we have neglected these changes throughout. It is therefore clear that our analysis should be confined to the region $x_0 \gg \xi$. The limiting case, in which one can no longer count on the correctness of the numerical coefficients, but one can be sure of the order of magnitude, is $x_0 = \xi$. It is this which we mean when we refer to a "vortex on the surface of a superconductor." The force of interaction of the vortex with the Meissner current is $f_m = (1/c) j_m \Phi_0$, where $j_m = (c/4\pi\lambda) H_0 e^{-x_0/\lambda}$. At $x_0 \ll \lambda$ we have

$$f_m = \frac{\Phi_0}{4\pi\lambda} H_0.$$

Equating now the forces f_σ and f_m on the surface of the superconductor, we obtain the condition for the determination of the field H_s at which the Bean-Livingston barrier vanishes:

$$H_s \approx \frac{\Phi_0}{4\pi\lambda\xi}.$$

Recognizing that according to the GL theory we have $\kappa = \lambda/\xi = 2\sqrt{2}(e/\hbar c)\lambda^2 H_{c,m}$, we readily obtain

$$H_s \approx \frac{1}{\sqrt{2}} H_{c,m}.$$

As already mentioned, this formula can claim to be accurate only in order of magnitude. The field H_s is in essence the maximum superheat field of the Meissner vortex-free state of a type-II superconductor, so that it can be calculated "from the other end," by considering the superheating of the homogeneous superconducting state. This problem was solved by de Gennes[18], who obtained

$$H_s = H_{c,m}. \tag{33}$$

The change of the order parameter is taken into account here, and this formula is already exact.

Thus, it can be stated that to overcome the attraction of the vortex to the surface (to its image) the vortex must be acted upon by a force f_m produced by the Meissner current $j_m = cH_{c,m}/4\pi\lambda$, since it is precisely this current which flows over the surface when the field on the surface is equal to $H_{c,m}$. But this current is equal to the limiting current at which the breaking of the electron pairs begins.

Thus, the penetration of the vortices into a superconductor situated in an external magnetic field becomes energywise favored even in a relatively weak external field $H_0 = H_{c1}$. For many type-II superconductors with $\kappa \sim 100$ we have $H_{c1} \sim 100$ Oe. This penetration is hindered, however, by the barrier, which in the case of an ideal surface vanishes at $H_0 = H_s = H_{c,m}$, i.e., in a field on the order of 10^3 Oe. Actually, the surface is never ideally smooth. The microscopic roughnesses make this barrier locally lower, and unless special care is taken to ensure a high-grade surface, the penetration of the vortices begins at a field close to H_{c1}. The height of the barrier is characterized by the difference $H_s - H_{c1}$. According to the data of[19], in the eutectic alloy Pb – Bi we have $H_s = 220$ Oe, $H_{c1} = 160$ Oe, and $H_{c,m} = 900$ Oe. The theoretical value of the barrier is $H_{c,m} - H_{c1} = 740$ Oe, and the actual barrier is $H_s - H_{c1} = 60$ Oe, i.e., the barrier is in fact lower by one order of magnitude than the theoretical value.

If the surface is thoroughly polished, then the penetration of the vortices into the superconductor is greatly hindered, thus, De Blois and De Sorbo[20] have investigated the magnetization of bulky niobium samples with 0.3 at.% oxygen. The critical parameters were $H_{c,m} = 1360$ Oe, $H_{c1} = 580$ Oe, $H_{c2} = 7000$ Oe (at 4.2°K), and $T_c = 8.8$°K. The sample surface was thoroughly polished electrolytically. It turned out that in this case the vortices begin to penetrate into the sample at $H_s = 1330$ Oe. This agrees well with the theoretical formula (33).

A very elegant experiment confirming the existence of the surface barrier was performed by Lowell[21]. The experimental conditions were the following: The sample was a plate of very homogeneous Nb + 20 at.%Mo alloy, with an almost reversible magnetization curve. The surface of the plate was chemically polished to a mirror finish. A current was made to flow through the plate, and the external magnetic field ($\sim H_{c2}/2$) was parallel to the surface of the plate and perpendicular to the current. Two types of constantan wire heaters were secured to the surface of the plate: α, located perpendicular to the current, and β, oriented obliquely to the current. The critical current was measured as a function of the power released by the heaters. The experimental results are shown in Fig. 6: the heater perpendicular to the current (position α) is much more effective than the heater in the position β. This is interpreted as follows: The heater destroys the barrier locally. In position α, a lowered-barrier region is produced parallel to the vortices that "wish" to penetrate into the plate. The vortices overcome this lower barrier, tend to move into the "doors" opened by the heater, and a resistive state is produced. The situation is different when the heater occupies position β. Inasmuch as the nascent vortex filament is perpendicular to the current, as before, it senses the lowered barrier only in that section of its length which crosses the heater. Therefore the heater in position β should be much less effective. This is indeed observed in the experiment.

d) **Vortex interaction with a cavity inside a superconductor.** So far we have been interested in the behavior of the vortices near the outer boundary of the superconductor. We now consider how the vortex interacts inside an infinite superconductor with a cylindrical empty channel parallel to the vortex.

As is well known, the flux carried by a cylindrical cavity in a bulky superconductor should be quantized, i.e., consist of an integer number of flux quanta Φ_0. With increasing number of flux quanta in the cavity, the field intensity in the cavity increases and, starting with a certain number n_0 of quanta it may be energywise more profitable for one flux quantum to become released in the form of a vortex filament. In other words, it may turn out that a superconductor with a cavity carrying $n_0 + 1$ flux quanta has a higher energy than in the case of the cavity with n_0 quanta and with an infinitely remote vortex parallel to its axis. However, the presence of a surface barrier prevents the creation of a filament on the boundary of the cavity and its penetration into the superconductor. Therefore the maximum number n_s of flux quanta in the cavity can exceed n_0. The problem of determining n_0 and n_s is solved by investigating the interaction of a cavity carrying n flux quanta with a vortex parallel to its axis[22].

Thus, we consider an infinite superconductor with

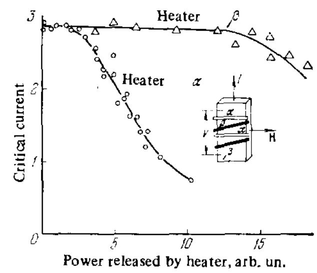

FIG. 6. Lowell's experiment. Plot of the critical current against the power released by heaters situated in positions and (from [21]).

cylindrical cavity of radius r, having n flux quanta, and with a vortex parallel to its axis and located at a distance ρ_0 away from its center. The field in the cavity consists of the field connected with n flux quanta of the cavity and the field produced inside the cavity by the vortex filament. The latter follows from the fact that the filament currents flowing around the cavity produce in the cavity a magnetic field, and it is easy to verify that this field is constant over the entire volume of the cavity. We denote the total field in the cavity by $\mathbf{H}_0$. Then the distribution of the field in the superconductor is described by the equation

$$\mathbf{H}+\lambda^2 \operatorname{rot}\operatorname{rot}\mathbf{H}=\Phi_0\mathbf{e}\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0) \qquad (34)$$

with the boundary condition $\mathbf{H}|_\sigma=\mathbf{H}_0$ on the surface σ of the cavity. In (34), $\mathbf{e}$ is a unit vector along the filament, i.e., along the cavity axis, and $\boldsymbol{\rho}_0$ is the two-dimensional radius vector of the center of the filament. The solution $\mathbf{H}(\boldsymbol{\rho})$ of this equation depends parametrically on $\boldsymbol{\rho}_0$ and $\mathbf{H}_0$, and the field $\mathbf{H}_0$ should be determined in a self-consistent manner. In the limit $\kappa\gg 1$, the second GL equation (4) can be written in the form

$$\lambda^2\operatorname{rot}\mathbf{H}=\frac{\Phi_0}{2\pi}\nabla\theta-\mathbf{A},$$

where θ is the phase of the wave function of the GL theory and $\mathbf{A}$ is the vector potential. Integrating this equation along the circular contour of the cavity, i.e., along a circle of radius r, we obtain

$$r\int_0^{2\pi}\operatorname{rot}_\rho\mathbf{H}\,d\varphi=\Phi_0 n-\pi r^2 H_0.$$

Substituting the solution of (34) in this expression we obtain H_0, which is given in the case $r\ll\lambda$ by

$$H_0=\frac{\Phi_0}{2\pi\lambda^2}K_0\left(\frac{\rho_0}{\lambda}\right)+\frac{\Phi_0}{2\pi\lambda^2}nK_0\left(\frac{r}{\lambda}\right) \qquad (35)$$

This shows clearly the constituents of the field in the cavity. The first term is the field produced in the cavity by the vortex filament. This field is not quantized. The second term is the field determined by the number n of the flux quanta contained in the cavity. This field is quantized and is equal to the field that remains in the cavity if the vortex filament is removed to infinity. It follows from (35) that H_0 increases monotonically as the filament approaches the cavity. At the instant when ρ_0 becomes equal to r, the filament vanishes, but H_0 becomes equal to $(\Phi_0/2\pi\lambda^2)(n+1)K_0(r/\lambda)$, i.e., the cavity acquires one more flux quantum. The free energy of the superconductor per unit length of the vortex filament takes the form

$$\mathscr{F}=\frac{1}{8\pi}\int[\mathbf{H}^2+\lambda^2(\operatorname{rot}\mathbf{H})^2]\,dV+\frac{1}{8\pi}H_0^2\pi r^2.$$

In this expression, the integration is carried out over the volume of a layer of unit thickness perpendicular to the axis of the cavity; the second term gives the energy of the field in the cavity per unit cavity length. This energy is a function of the position ρ_0 of the filament and of the number n of the flux quanta of the cavity. The function $\mathscr{F}_n(\rho_0)$ for various n is given by the formulas

$$\mathscr{F}_n(\rho_0)=\begin{cases}\dfrac{H_{c,m}^2\lambda^2}{\kappa^2}\left[\dfrac{1}{2}\ln\left(1-\dfrac{r^2}{\rho_0^2}\right)+nK_0\left(\dfrac{\rho_0}{\lambda}\right)\right], & \rho_0\ll\lambda;\\[2ex] \dfrac{H_{c,m}^2\lambda^2}{\kappa^2}nK_0\left(\dfrac{\rho_0}{\lambda}\right), & \rho_0\gg\lambda,\ n\neq 0.\end{cases}$$

A plot of $\mathscr{F}_n(\rho_0)$ is shown schematically in Fig. 7. The origin of $\mathscr{F}_n$ is taken to be the energy of the system with a vortex filament removed to infinity. It is seen from the figure that a cavity that is free of flux quanta attracts

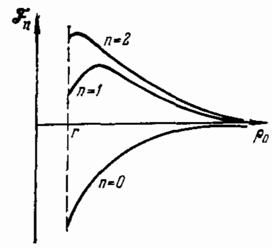

FIG. 7. Free energy of vortex located near a cavity containing n magnetic-flux quanta.

a vortex filament to itself. On reaching the surface of the cavity, the vortex vanishes from the superconductor, and the cavity acquires one flux quantum. This situation continues until the number of flux quanta in the cavity exceeds a certain value n_0, starting with which the capture of n_0+1 quanta by the cavity is no longer energywise favored. Were it not for the surface barrier, n_0 would indeed be the maximum number of flux quanta that the cavity can carry, since it is profitable for a cavity with n_0+1 flux quanta to release one quantum in the form of a vortex filament. The presence of the surface barrier, however, prevents formation of a filament on the surface of the cavity and prevents the filament from penetrating into the superconductor. The flux-quantum number n_s at which the barrier vanishes is the maximum number of quanta that the cavity can retain. Calculation shows[22] that $n_s\sim r/2\xi$, whereas $n_0\sim 1$ at $r\ll\lambda$. Since it is assumed that $r\gg\xi$, it is clear that $n_s\gg n_0$.

The force per unit vortex length that must be applied to the vortex on the cavity surface to detach it from the cavity is obviously

$$f=-\frac{\partial\mathscr{F}}{\partial\rho_0}\Big|_{\rho_0=r}.$$

If we substitute in this formula the function $\mathscr{F}_n(\rho_0)$ determined in[22], we obtain

$$f=\frac{H_{c,m}^2\xi}{2}\left(1-\frac{2n\xi}{r}\right).$$

Thus, if n = 0, the force of attraction of the vortex to the cavity is maximal, does not depend on r (at $r\gg\xi$), and to overcome it it is necessary to have an extraneous current

$$j=\frac{cH_{c,m}}{4\pi\sqrt{2}\lambda},$$

i.e., again a current of the same order as magnitude as the current that leads to the breaking of the electron pairs.

5. MIXED STATE OF TYPE-II SUPERCONDUCTORS

a) Mixed state of unbounded superconductor. We have considered so far one isolated vortex in an unbounded or bounded superconductor. We now consider a system of interacting vortices. We start with the case of unbounded space, i.e., we consider a system of vortices that are parallel to one another and are in equilibrium with one another and with the external magnetic field; we neglect in this section all effects due to the interaction of the vortices with the surface of the sample. We increase the external field H_0. Up to the first critical field H_{c1}, the sample is in a pure Meissner state, i.e., there are no vortices inside the superconductor (they are not favored energywise), and Meissner current flows over the surface. The field inside the sample is equal to zero. The magnetic moment increases linearly with the field:

$$M=\frac{B-H_0}{4\pi},$$

where M is the magnetic moment per unit volume and P is the average magnetic field inside the sample; in the Meissner state we have B = 0. When H_0 reaches the value H_{c1} the penetration of the vortices becomes energywise profitable, and as a result of their interaction they begin to align themselves parallel to one another, forming a regular two-dimensional lattice. The distances between the vortices, however, are still large, so that it suffices to take only the interaction of each given vortex with its nearest neighbors into account. Calculation of $M(H_0)$ in this field region[23] leads to an infinite value of $(dM/dH_0)_{H_0=H_{c1}}$, owing to the exponentially weak interaction of the filaments at distances that are large in comparison with λ. The filaments approach each other abruptly to distances $\sim\lambda$. With further increase of the external field H_0, we arrive at the magnetic-field region $H_{c1} \ll H_0 \ll H_{c2}$, where it is easy to determine the equilibrium structure by using the single-vortex-filament model analyzed above.

In perfect analogy with the derivation of formula (32) for the Gibbs free energy of a single filament, we can easily obtain the density of the Gibbs free energy of our superconductor in the mixed state:

$$G = n\left[\varepsilon_0 + \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \sum_i{}' K_0\left(\frac{|\mathbf{r}_i|}{\lambda}\right)\right] - \frac{n\Phi_0 H_0}{4\pi};$$

here n is the density of the vortex filaments, the summation is carried out over the vortex-lattice points, and the prime at the summation sign means that the term with $|\mathbf{r}_i| = 0$ has been limited.

Recognizing that the average field or induction is equal to

$$B = n\Phi_0,$$

we have

$$G = \frac{B}{4\pi}\left[\frac{4\pi\varepsilon_0}{\Phi_0} - H_0 + \frac{\Phi_0}{4\pi\lambda^2}\sum_i{}' K_0\left(\frac{|\mathbf{r}_i|}{\lambda}\right)\right]. \qquad (36)$$

The equilibrium value of the vortex-structure unit-cell parameter a corresponds to the minimum value of G, so that the equilibrium condition is

$$\frac{\partial G}{\partial a} = 0. \qquad (37)$$

From this equation we determine the equilibrium value of the induction at a given external field H_0. Let us specify a vortex-lattice unit cell of arbitrary type, say an equilateral triangle with side a. Replacing the summation in (36) by integration over the reciprocal lattice, we obtain the following expression for the induction[2]:

$$B = H_0 - H_{c1} - \frac{\Phi_0}{8\pi\lambda^2}\ln\left(\beta\frac{\Phi_0}{\lambda^2 B}\right), \qquad (38)$$

where β is a numerical coefficient that depends on the type of lattice. To calculate it it is necessary to carry out exact summation in formula (36). This was done in[24,25], and the result is $\ln\beta_\Delta = -3.872$ and $\ln\beta_\Box = -3.852$ for a triangular and a quadratic lattice, respectively.

From (38) we get a final expression for the magnetic-moment density:

$$M(H_0) = -\frac{H_{c1}}{4\pi} - \frac{\Phi_0}{32\pi^2\lambda^2}\ln\left(\beta_\Delta\frac{\Phi_0}{\lambda^2 B(H_0)}\right). \qquad (39)$$

Thus, in the first-order approximation ($B = H_0$) the magnetic moment depends logarithmically on the field H_0 at $H_{c1} \ll H_0 \ll H_{c2}$. At $H_0 \sim H_{c2}$ the magnetic moment

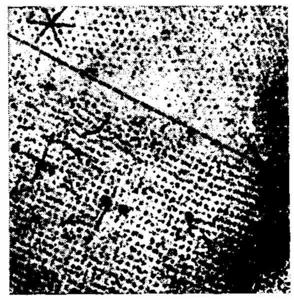

FIG. 8. Electron-microscope photograph of the mixed state, obtained by Obst [28b]. A triangular vortex structure is realized in one of the grains of the polycrystalline material, and a quadratic one in another grain.

of a superconductor with triangular lattice filaments is determined by the formula[6,26a]

$$M(H_0) = \frac{H_0 - H_{c2}}{4\pi\beta(2\varkappa^2-1)}, \quad \varkappa > \frac{1}{\sqrt{2}}, \beta = 1.16. \qquad (40)$$

A comparison of G for triangular and quadratic lattices shows that the triangular lattice is energywise more favored. On the other hand, calculations on the basis of the GL equations have shown[26a] that at arbitrary $\kappa > 1/\sqrt{2}$ near H_{c2}, the triangular lattice is likewise more favored energywise. It is therefore reasonable to assume that at $\kappa \gg 1$ a triangular lattice is realized in the entire range of fields from H_{c1} to H_{c2}. However, the energy gain of the triangular lattice in comparison with the quadratic one is very small, and sometimes, owing to anisotropy of the crystal, the quadratic lattice may be favored. Figure 8 shows a very effective electron-microscope photograph of the structure of the mixed state in two neighboring grains of a polycrystalline sample obtained by Obst[26b]. It is clearly seen that a triangular lattice was formed in one grain and a quadratic one in the other.

b) Mixed state near the boundary of a superconductor. Up to now, when speaking of the structure of the mixed states in a type-II superconductor, it was assumed throughout that the superconductor was infinite. What occurs near the boundary? This is not a trivial question. The vortices interact strongly with the boundary of the sample (in other words, with their own images), as well as with the Meissner currents, and in the region $\sim\lambda$ near the boundary one should expect strong changes in the structure of the mixed state. The calculation result is therefore all the more unexpected, namely the mixed-state structure remains unchanged even in the immediate vicinity of the edge of the superconductor.

Let us consider for simplicity a plane surface of a superconductor occupying the half-space $x > 0$. The constant magnetic field H_0 is directed along the Oz axis, $H_{c1} \ll H_0 \ll H_{c2}$. A mixed state is produced in the superconductor, i.e., a vortex lattice appears. We assume that the vortices are arranged in rows and that the l-th row is located in the plane $x = x_l$. All the vortices in the row are parallel to the external field and are located at a distance a from one another. The distances between the rows, however, are not identical. Our problem consists precisely of finding the equilibrium positions of these rows.

To this end we write down the expression for the Gibbs free energy of our sample:

$$\mathcal{G} = \frac{\Phi_0}{8\pi}\sum_{l,m} H_v(\mathbf{r}_{lm}) - \frac{H_0}{4\pi}\sum_{l,m}\Phi_v(\mathbf{r}_{lm}); \qquad (41)$$

here $\mathbf{r}_{lm}$ is the radius vector of the center of the m-th vortex in the l-th vortex row, H_v is the field produced by all the vortices without exception (and only by vortices) at the center of the vortex (l, m), and $\Phi_v(\mathbf{r}_{lm})$ is the magnetic flux produced by the vortex (l, m). Formula (41) is obtained in exactly the same manner as formula (32). To calculate the energy $\mathcal{G}$ it is convenient to use the method of images. It is necessary, however, to determine the field produced by one vortex row at a point located at a distance x from this row.

In other words, the question is the following: Assume that the centers of the vortices occupy the points (0, $\pm$ma), m = 0, 1, 2, ..., in an unbounded superconducting space. What field do they produce at the point (0, x)? This field $H_v(x)$ can be written down immediately by using the formula (7):

$$H_v(x)=\frac{\Phi_0}{2\pi\lambda^2}\sum_m K_0\left(\sqrt{\frac{x^2+(ma)^2}{\lambda^2}}\right).$$

This expression is summed in[16]. The final result is very simple:

$$H_v(x)=\frac{\Phi_0}{2a\lambda}e^{-x/\lambda}. \tag{42}$$

This formula is valid accurate to terms $\sim a[\exp(-2\pi x/a)]/\lambda$. Since formula (42) will be used with $x \gtrsim a$, it is clear that the omitted terms are negligibly small.

Using now (42) and (27), we write down the energy (41) for a superconductor strip of unit width along Oy and of unit height along Oz:

$$\mathcal{G}=\frac{\Phi_0^2}{16\pi\lambda a^2}\sum_{l,\,l'}[e^{-|x_l-x_{l'}|/\lambda}-e^{-|x_l+x_{l'}|/\lambda}]+\frac{\Phi_0 H_0}{4\pi a}\sum_l e^{-x_l/\lambda}+\text{const.} \tag{43}$$

The constant in this expression stands for all the terms that do not depend on x_l. The physical meaning of this formula is quite clear: the first exponential in the double sum yields the energy of the repulsion of the vortices from one another, and the second exponential gives the energy of attraction to the system of images; finally, the second sum is the energy of interaction of the vortices with the Meissner current.

The equilibrium position of all the vortex rows is obtained from the condition that $\mathcal{G}$ be a minimum

$$\frac{\partial\mathcal{G}}{\partial x_l}=0,$$

or, using (43),

$$\sum_{\mu>l}e^{-(x_\mu-x_l)/\lambda}-\sum_{\mu<l}e^{-(x_l-x_\mu)/\lambda}+\sum_\mu e^{-(x_l+x_\mu)/\lambda}=\frac{2a\lambda H_0}{\Phi_0}e^{-x_l/\lambda}. \tag{44}$$

The exact solution of the system (44) is

$$x_l=bl+x_0,\ l=0,\ 1,\ 2,\ \ldots \tag{45}$$

This can be easily verified by substituting (45) in (44). We obtain in this case an equation that enables us to determine $x_0(H_0)$. It turns out that x_0 is also of the order of b and is practically independent of H_0.

We have thus found that the vortex rows in a semi-infinite superconductor in the mixed state are equally spaced relative to one another even in the immediate vicinity of the edge of the superconductor. The distance b between the vortex rows obviously corresponds to the equilibrium state and is uniquely connected with the induction B in the interior of the superconductor, namely $B = \Phi_0/ab$, with $b/a = \sqrt{3}/2$ for a triangular lattice.

c) Mixed state of film. We now turn to the case when the superconductor is a film of thickness d and the external magnetic field H_0 is parallel to its plane. Let $\xi \ll d \lesssim \lambda$ and let us see what vortex-filament structure is produced in this film. This was determined by Abrikosov[27]. It turns out that the penetration of the vortices into the film becomes favored not at the field H_{c1}, as for an infinite superconductor, but at a field $H_{c1}(d)$, which goes over into H_{c1} as $d \to \infty$. Near this field, the vortices align themselves along the center of the film in a linear chain parallel to the field, and the distances between them are initially very large (much larger than λ). Then, with increasing H_0, the vortices start to come close to one another and splitting of the chain in two, misalignment, etc. set in[28a]. The usual mixed state is produced.

It might seem that the proximity of the edges would strongly alter the mixed-state structure. This is not so, however. A calculation similar to that in the preceding section[16] again shows that the distance b between the vortex rows is the same for any location in the film.

Thus, at $d \ll \lambda$ the surface of the film exerts an appreciable influence on the vortex structure when $H_0 \sim H_{c1}(d)$, and has no effect on the structure at $H_{c1}(d) \ll H_0 \ll H_{c2}$. It turns out that in a plate of arbitrary thickness (provided that $d \gg \xi$) a regular triangular lattice is realized in this range of fields[25]. The magnetization curve in this field range is described as before by expression (39). This "insensitivity" to the existence of the film surface, as shown by calculation[25], is attributed to the fact that in relatively strong fields the interaction between the vortices and the film surfaces is completely offset by the interaction between the vortices and the Meissner currents due to the external field. This makes the lattice parameters independent of the thickness even in the limit $d \ll \lambda$, and the magnetic moment per unit volume of the plate coincides with its value in an unbounded sample.

We turn now to fields close to $H_{c1}(d)$ and examine the onset of the mixed state in a film. We have already mentioned that the equilibrium structure is a linear vortex chain aligned along the center of the film. Abrikosov's calculation[27] shows that near $H_{c1}(d)$ the vortices in the chain are very far from each other and their interaction can be neglected. Let us therefore consider only one vortex.

The dependence of the Gibbs free energy of a film with a vortex on the position of this vortex in the film was investigated in[28b]. This energy is given by formula (32), which can be rewritten, taking (25) into account, in the form

$$\mathcal{G}=\frac{\Phi_0}{8\pi}H_v(x_0)+\frac{\Phi_0}{4\pi}(H_1(x_0)-H_0); \tag{46}$$

here $H_v(\mathbf{r})$ and $H_1(\mathbf{r})$ are the solutions of Eqs. (6) and (30), respectively, with boundary conditions $H_v|_\sigma = 0$ and $H_1|_\sigma = H_0$. The vortex is located at the point (x_0, 0), and the film surfaces coincide with the surfaces x = 0 and x = d. The external magnetic field is directed along the Oz axis.

We can verify by simple substitution that the sought solutions are

$$H_v(x,\,y)=\frac{2\Phi_0}{\lambda^2 d}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{iky}\frac{\sin(\pi nx/d)\sin(\pi nx_0/d)}{k^2+(\pi n/d)^2+1/\lambda^2},$$

$$H_1(x)=H_0\frac{\operatorname{ch}\left(\frac{2x-d}{2\lambda}\right)}{\operatorname{ch}(d/2\lambda)}.$$

After substituting these solutions in (46), we can easily

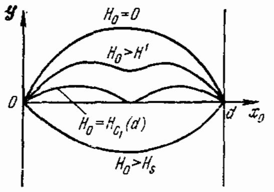

FIG. 9. Gibbs free energy of a vortex in a film at different values of the external field.

calculate $\mathcal{G}(x_0)$, the value of which in the limit $d \ll \lambda$ is

$$\mathcal{G}(x_0)=\varepsilon_0+\left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \ln\left[\frac{\gamma d}{\pi\lambda}\sin\frac{\pi x_0}{d}\right]+\frac{\Phi_0 H_0}{4\pi}\left\{\frac{\operatorname{ch}[(2x_0-d)/2\lambda]}{\operatorname{ch}(d/2\lambda)}-1\right\}, \quad (47)$$

where ϵ_0 is the energy per unit length of the isolated vortex, and $\gamma \approx 1.78$.

A plot of $\mathcal{G}(x_0)$ is shown in Fig. 9. At $H_0 < H'$, the vortices in the film are absolutely unstable. Starting with a certain field H', a potential well for the vortices is produced at the center of the film, and the vortices can exist there in stable fashion, but the penetration of the vortices into the film becomes energywise favored only starting with the field $H_{c1}(d)$. All this is illustrated in Fig. 9.

The field H' is obtained from the condition $(\partial^2\mathcal{G}/\partial x_0^2)|_{x_0=d/2} = 0$:

$$H'=\frac{\pi\Phi_0}{4d^2}, \quad d\ll\lambda. \quad (48)$$

It is natural to call this field the minimum supercooling field of the mixed state. The field $H_{c1}(d)$ is determined by the condition $\mathcal{G}(x_0)|_{x_0=d/2} = 0$. Using (47), we obtain directly Abrikosov's result[27]

$$H_{c1}(d)=\frac{2\Phi_0}{\pi d^2}\ln\left(\frac{\gamma d}{\pi\xi}\right), \quad \varkappa\gg 1, \quad d\ll\lambda.$$

All this is very easily understood. When there is no external field, the vortex is attracted to the film surfaces and is therefore absolutely unstable. This can be explained also in the following manner: If the vortex is not at the center of the film, then its vortex currents are asymmetrical. If $x_0 < d/2$ (see Fig. 10), then this asymmetry is such that the current density to the left of the center of the vortex is larger than to the right. Consequently, the difference between the Bernoulli forces acts on the vortex from right to left, i.e., the vortex is attracted to the surface. Application of an external field H_0 leads to the appearance of a Meissner current which is so directed that, by interacting with the vortex, it produces a Lorentz force that tends to move the vortex to the center of the film.

However, even at $H_0 = H_{c1}(d)$ the vortices cannot penetrate unobstructed into the film—this is prevented by the surface barrier (see Fig. 9). This barrier decreases with increasing field H_0 and vanishes at a field H_S given by the condition $(\partial\mathcal{G}/\partial x_0)|_{x_0=0} = 0$. Our calculations can yield only an estimate of the order of H_S:

$$H_s\sim\frac{\Phi_0}{2\pi\xi d}, \quad d\ll\lambda.$$

It is natural to call the field H_S the maximum superheat field of the Meissner state. It is easy to see that in the case of a thin film the barrier is overcome by the vortex only when the Meissner-current density on the film surface reaches a value $j_S = cH_{c,m}/4\pi\sqrt{2}\lambda$, i.e., a value at which breaking of the electron pairs sets in.

6. TRANSPORT CURRENT IN MIXED STATE

We now proceed to a very interesting and very important practical question: how does the transport electric

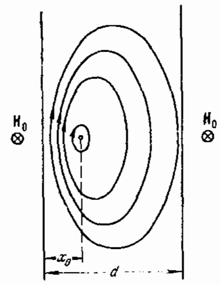

FIG. 10. Configuration of vortex current in film.

current flow in a superconductor in the mixed state? We assume here that the direction of the transport current is perpendicular to the vortices.

It is very frequently stated in the literature that in an ideally homogeneous type-II superconductor, the critical current is equal to zero, in this case, i.e., a nondissipative transport current is impossible. The reasoning is very simple: interaction of the transport current with the vortices generates a Lorentz force, which causes vortex motion, which in turn is accompanied by energy dissipation[29-32]. In real type-II superconductors, however, the vortices are pinned by different imperfections of the material (dislocations, inclusions of another phase, pores, grain boundaries, etc.), and a nondissipative transport current can flow through the superconductor.

We shall show now that, generally speaking, even an ideally homogeneous type-II superconductor in the mixed state is still capable of carrying a nondissipative transport current.

a) Transport current in a film. We turn to the case considered in the preceding section, of a thin film ($d \ll \lambda$) located parallel to the external magnetic field H_0. Let $H_0 \gtrsim H_{c1}(d)$, i.e., a mixed state has set in in the form of a linear chain of almost noninteracting vortices aligned along the center of the film.

What happens if a certain transport current is now made to flow through the film perpendicular to the magnetic field? Let us see what forces act on one of the vortices of the chain. We know these forces well. These are the force of attraction to the edges of the film, the force of interaction with the Meissner currents produced by the field H_0, and now also the force of interaction with the transport current. The vortex is in equilibrium when the sum of all three forces is equal to zero. The sum of the first two forces was investigated in Sec. c of Chap. 5. It is equal to the derivative $(-\partial\mathcal{G}/\partial x_0)$, where $\mathcal{G}(x_0)$ is given by (47) and x_0 is the distance between the vortex chain and one of the film surfaces. This force can be regarded as a restoring force acting on the vortex located in the potential well (see Fig. 9). The displacing force, which draws the vortex out of the minimum of $\mathcal{G}$ at the center in the film, is the Lorentz force f_L, i.e., the force of interaction between the vortex and the transport current. Each value of the transport current corresponds to a certain equilibrium displacement of the vortex chains away from the center of the film, a displacement that can be obtained from the equilibrium condition

$$\frac{\partial\mathcal{G}}{\partial x_0}=f_L.$$

We thus encounter a case in which both a nondissipative transport current and a mixed state exist.

Obviously, the equilibrium will exist until the restoring force reaches the maximum value. This occurs when

x_0 coincides with the abscissa of the inflection point of the function $\mathscr{G}(x_0)$. This is the maximum opposition that the Meissner current can provide for the Lorentz transport current. It is natural to define this transport current as the critical current, i.e., the current at which vortex instability sets in.

A calculation in[28b] has shown that the critical transport-current density determined in this manner is

$$j_c = \frac{c}{4\pi^2}\frac{d}{\lambda_L^2} H_0 \left[\arccos\sqrt{\frac{H'}{H_0}} - \sqrt{\frac{H'}{H_0}\left(1 - \frac{H'}{H_0}\right)}\right], \tag{49}$$

where H' is the minimum supercooling field (48) of the mixed state.

From $j_c(H_0)$ as given by (49) we get $dj_c/dH_0 > 0$, i.e., in the considered field range $H_0 - H_{c1}(d) \ll H_0$ the critical current should increase with increasing external field. This means that somewhere in the field interval from $H_{c1}(d)$ to H_{c2} there should exist a maximum of the critical current. The so-called peak effect, i.e., the existence of a peak in the $j_c(H_0)$ plot, was observed for very many inhomogeneous materials and was due in most cases, of course, to entirely different causes. It seems remarkable to us, however, that in such an ideal object as a perfectly homogeneous film the peak effect comes about quite naturally, without any additional assumptions.

The physical explanation of this phenomenon is very simple. The depth of the potential well $\mathscr{G}(x_0)$ increases with increasing magnetic field H_0 (see Fig. 9); consequently, a larger transport current is needed to upset the stability of the vortex chain.

b) **Transport current in a plate.** We examine now the flow of transport current in a plate of thickness d in the mixed state. The current is perpendicular to the magnetic field H_0, which is parallel to the surfaces of the plate, $H_{c1} \ll H_0 \ll H_{c2}$ and $d \gg \lambda$. The facts we already know concerning the mixed state of a semiinfinite superconductor (see Sec. B of Chap. 5) enables us to consider, at least qualitatively, the mechanism whereby the transport current flows through the plate[16]. Indeed, knowing nothing for the time being concerning the current distribution in the plate, we stipulate the existence of the transport current by requiring the magnetic field on one surface of the plate to be equal to $H_0 + H_I$, and on the other $H_0 - H_I$, where H_I is the field produced on the plate surface by the transport current. Since $d \gg \lambda$, we can apply the results of Sec. b of Chap. 5 to each of the two plate surfaces. We know that the vortex lattice has the same thickness at any point of the superconductor. Consequently only a homogeneous vortex lattice can satisfy the initial equations in a current-carrying plate, i.e., there can be no vortex-density gradient in this case. This pertains, of course, only to an ideal homogeneous superconductor without centers to pin the vortices inside the material. On the other hand, the distance x_0 from the first vortex row to the surface with the plate [Eq. (45)] remains a free parameter determined from the condition for the minimum of the total Gibbs free energy, which includes also a term for the interaction of the vortices with a transport current. Thus, the equilibrium value of x_0 is a function of H_I. This means that when the current is made to flow through the plate the entire vortex lattice is displaced as a unit by a certain distance $\Delta(H_I)$. We call attention to the fact that the absence of a vortex density gradient means the absence of a vortex density gradient means the absence of the transport current over the section of the plate. The

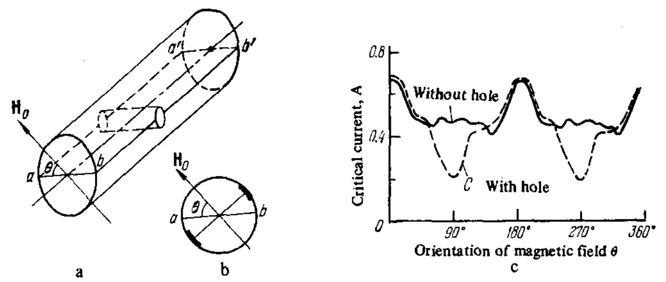

FIG. 11. Illustrating the experiment of Jones and Rose-Innes. a) Schematic diagram of the sample; a hole perpendicular to the cylinder axis was drilled in the sample; b) transport current flows through surface sections but are almost parallel to the external field (thickened parts of the circle in the figure); c) plot of the critical current against the angle between the direction of the filed and the hole axis.

transport current flows over the surfaces of the plate in a layer on the order of the penetration depth. This follows directly from the solution of the London equation for the current under the boundary condition $H_I|_\sigma = \pm H_I$. This result was confirmed by a direct experiment performed by Jones and Rose-Innes[33]. A cylindrical sample was made from a very pure Nb + 50 at.% Ta alloy, with an almost fully reversible magnetization curve. Its surface was electrically polished. The sample was placed in a transverse field H_0, which transform it to the mixed state. The critical current was investigated as a function of the field direction (Fig. 11). A transverse hole was then drilled in the sample and the dependence of the critical current on the field direction was again investigated. The results are shown in Fig. 11. The authors interpret their results as follows: If a homogeneous cylindrical type-II superconductor is located in a transverse field H_0 and has gone into the mixed state, then the transport current flows along the cylinder through two strips oriented along the cylinder on diametrically opposite sections of its surface. These strips are always so oriented that the field H_0 in them is always parallel to the surface of the cylinder. The dependence of the critical current on the field direction when there is no hole in the cylinder is determined by different random inhomogeneities of the cylinder surface. On the other hand, if a transverse hole is drilled through the sample, then at an angle $\theta = 90°$ (see Fig. 11) the current-carrying strips on the surface of the cylinder assume the directions aa′ and bb′. But it is precisely in this case that a part of the current-carrying surface is absent (owing to the transverse hole), and the critical current reaches a minimum value. This is precisely what is observed in the experiment.

We have thus established that the transport current flows over the surfaces of the plate when the plate is in a mixed state, and the entire vortex structure is displaced in this case by a distance $\Delta(H_I)$. This raises the natural question: how far can the vortex lattice be displaced without upsetting its stability? This is in essence the question of the critical current. A detailed analysis of this model for an ideal surface was carried out by Ternovskiĭ and Shekhata[34].

The real surface of the plate is always rough. It is precisely because of this, unless special care is taken with the surface quality, that experiment does not reveal the hysteresis phenomena that would be expected from the existence of a surface barrier.

We shall attempt to take the surface roughness into account in the following manner: If the plate is in an ex-

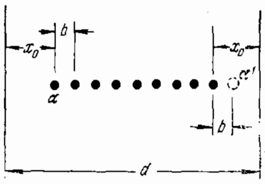

FIG. 12. Transfer of a vortex from a position α to a position α' is equivalent to a rigid-body displacement of the entire vortex structure by a distance b.

ternal field H_0, and the average field (induction) deep in the interior of the plate is equal to B, then a current $I_M = (H_0 - B)c/4\pi$ flows over the surface of the plate in a layer on the order of λ. This current is referred to a unit height of the plate along the Oz axis. In the equilibrium position, when there is no transport current, the entire vortex lattice is symmetrically arranged and the distances of the outermost vortex rows to the surface of the plate are equal to x_0. In Sec. b of Chap. 5 it was shown that the equilibrium value of x_0 is of the order of the distance b between the vortex rows. We already know that when transport current flows the entire vortex lattice as a whole is displaced in the direction of the Lorentz force. When is the equilibrium disturbed? It is natural to assume (taking into account the roughness of the plate surface) that the equilibrium is disturbed when the lattice is displaced by an amount on the order of b, i.e., when the extreme right row of the vortices emerges from the plate and when the extreme left row of the vortices leaves room for a new row. By how much is the free energy of the entire plate increased following such a maximal displacement of all the vortices by an amount b? It is easy to see that this will be precisely the energy needed to remove the extreme left row from the position α and for the appearance of a new row in the position α' (see Fig. 12), with all the remaining vortex rows remaining in the same position. To this end, obviously, it is necessary to perform the work

$$\Delta\mathcal{G} = fb/a,$$

where f is the force of interaction of one vortex with the current I_M and a is the distance between the vortices in the row. The energy $\Delta\mathcal{G}$ pertains to the vortices located on the unit length of the row. Since the density of the current I_M is I_M/λ, we have

$$f = \frac{1}{c}\Phi_0 I_M/\lambda.$$

Recognizing, in addition, that $I_M = c|M|$, where $M = (B - H_0)/4\pi$ is the reversible magnetic moment per unit volume of the superconductor, we have the following condition for the critical transport current:

$$\frac{|M|\Phi_0}{a} = \frac{1}{c} I_c B. \tag{50}$$

Indeed, on the left-hand side we have the restoring force $\Delta\mathcal{G}/b$ acting on all the vortices located in a unit section of the plate along the Oy axis per unit height of the plate along the Oz axis. In other words, this is the maximum vortex-pinning force in the considered section of the plate. On the right-hand side we have the total Lorentz force that the transport current exerts on the vortices. Indeed, we have already established that this current flows in a surface layer of thickness λ. The number of vortices per unit length of this layer is (B/Φ_0), so that the Lorentz force acting on these vortices is

$$\frac{1}{c} j_c\Phi_0\left(\frac{B}{\Phi_0}\right)\lambda = \frac{1}{c} I_c B.$$

In the case of a triangular lattice, the distance a between the vortices in the row is $(2\Phi_0)^{1/2}3^{-1/4}B^{-1/2}$. Substituting this expression in (50) and introducing the definition of the average density of the critical current $\bar{j}_c = I_c/d$, we have ultimately

$$\bar{j}_c = \frac{3^{1/4}}{\sqrt{2}}\frac{c\sqrt{\Phi_0}\,|M(H_0)|}{\lambda\sqrt{B}}. \tag{51}$$

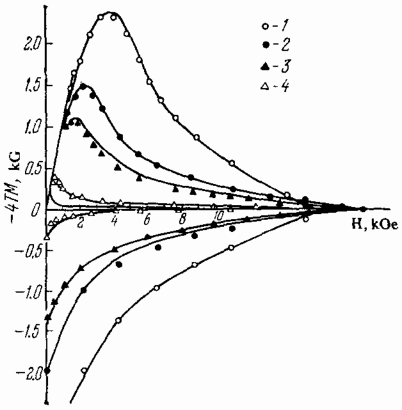

FIG. 13. Dependence of the magnetic moment of eutectic Pb–Bi on the external magnetic field. Symbols—results of experiments for samples subjected to different treatments: 1) one hour, 20°C, 2 mm diameters; 2) 0.5 h, 114°C, 2 mm diameter; 3) 80 min, 114°C, 2 mm diameter, 4) 9 h, 114°C, 0.3 mm diameter. Solid lines—theoretical curve [19].

This is precisely (apart from a numerical coefficient) the formula proposed by Campbell, Evetts, and Dew-Hughes[19]. They checked this formula very carefully. The experiment was performed on a eutectic Pb—Bi alloy. Rather large particles ($> \lambda$) of nonsuperconducting bismuth were uniformly introduced into the superconducting ϵ phase. The authors regarded their sample as an aggregate of thin superconducting plates of thickness assumed to be equal to the average distance between the bismuth particles. They investigated the irreversible magnetic moment of the samples as a function of the external field H_0. The same quantity can be calculated by specifying some dependence of $\bar{j}_c$ on H_0. The authors, performing an independent investigation of the reversible magnetic moment $M(H_0)$ of the ϵ phase, used formula (51) to calculate the irreversible magnetic moment. A comparison of the results of the direct experiment and of the calculation are shown in Fig. 13, where the solid lines are the results of the calculation.

One more confirmation of the previously-advanced point of view concerning the flow of transport current in a plate in a mixed state is a paper by Cardona, Gittleman, and Rosenblum[35], who investigated the real part R of the surface impedance of a flat sample of Pb + 17 at.% In alloy in the mixed state. The measurements were performed in the centimeter band. The constant magnetic field H_0 was applied parallel to the surface of the sample. They observed at $H_{c1} < H_0 < H_{c2}$ the so-called natural hysteresis in the plot of R against H_0. For each value of H_0 there exists a certain field interval ΔH, in which $R(H_0)$ varies reversibly and without hysteresis. The authors offer the following explanation for this phenomenon. The real part R of the surface impedance is determined by the number of normal vortex cores located inside the skin layer. A change (say an increase) of the external field causes the first vortex rows to move away from the surface into the interior of the sample, leaving space for the entry of the next vortex row. But before this new row has entered, the number of vortices in the skin layer is decreased and R has also decreased. Such a reversible elastic displacement of the vortices by the field H_0 should be observed in a certain field interval ΔH. The lower limit of this interval corresponds to the emergence of the extreme vortex row from the sample, and the upper limit to the entry of the new vortex row into the sample.

If we accept this interpretation of [35], then we can compare its result with the concepts developed above. Indeed, the field interval ΔH just referred to is none other than our field H_{I_c} produced by the transport current, inasmuch as it is precisely the field H_{I_c} which causes a new vortex row to enter from that side of the plate where the field is equal to $H_0 + H_{I_c}$ and causes a row to leave where the field is equal to $H_0 - H_{I_c}$. Thus, the initial hypothesis is $H_{I_c} \sim \Delta H$. Then the expression for ΔH takes the form

$$\Delta H = k\frac{\sqrt{3}}{4}\frac{a}{\lambda}(H_0 - B),$$

where a is the side of the triangular unit cell of the vortex lattice, and k is an adjustment coefficient on the order of unity, necessitated by the fact that we do not know exactly the critical displacement of the vortex structure, which we have assumed equal to b. The following parameters of the Pb + 17 at.% In alloy are given in[35]: H_{c2} = 4.8 kOe and H_{c1} = 0.14 kOe. These data suffice to calculate the remaining parameters: κ = 5.3, ξ = 260 Å, λ = 1380 Å. Using now the well known formula $(2\Phi_0)^{1/2}3^{-1/4}B^{-1/2}$ for a, assuming for $H_0 - B$ the logarithmically-accurate relation

$$H_0 - B = H_{c1}\frac{\ln (H_{c2}/B)^{1/2}}{\ln \varkappa},$$

and substituting the values of λ and κ determined above, we obtain ultimately

$$\Delta H\,(\text{kOe}) = k\frac{2\cdot 10^{-2}}{\sqrt{B\,(\text{kOe})}}\ln\frac{4.8}{B\,(\text{kOe})}. \quad (52)$$

Assuming in this formula B = H_0, we can compare this result with the experiment[35] dependence $\Delta H(H_0)$. To determine the adjustment coefficient k, we equate ΔH_{exp} and ΔH_{theor} at the point H_0 = 0.5 kOe. We then obtain k = 1.85. Figure 14 shows the experimental points and the theoretical $\Delta H(H_0)$ curve. The agreement is surprisingly good, when we consider that the adjustment parameter is on the order of unity.

c) **Pinning of vortices on the interface of two superconductors.** In the preceding section we considered the pinning of vortices as a result of their interaction with the surface of the superconductor, and have established that the transport superconductor current flows in this case along this surface, in a layer on the order of λ.

We now consider the case when there is a flat boundary between two different superconductors, and the external magnetic field is parallel to this boundary.

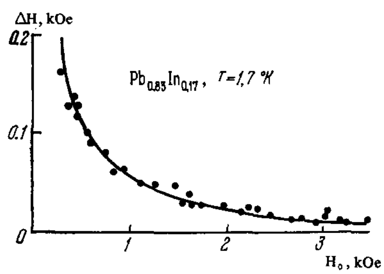

FIG. 14. Dependence of ΔH on the external magnetic field H_0. Points–result of experiment [35]; the solid curve was calculated from formula (52), where k = 1.85.

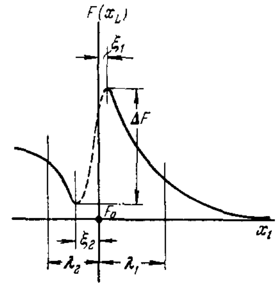

FIG. 15. Dependence of the vortex free energy on its distance to the interface between two superconductors. It is assumed that $\lambda_1 > \lambda_2$ and $\kappa_1 > \kappa_2$.

Let us explain first how one vortex interacts with such a boundary[36]. Let the half-space x > 0 be occupied by a superconductor with a penetration depth λ_1 and a correlation length ξ_1 ($\lambda_1 \gg \xi_1$). At x < 0 we have respectively $\lambda_2 \gg \xi_2$. A vortex filament parallel to the Oz axis passes through the point $\mathbf{r}_L$, with $x_L > 0$. The intensity of the magnetic field produced by the vortex is determined from the solution of the equations

$$\mathbf{H}_1 + \lambda_1^2\,\text{rot rot}\,\mathbf{H}_1 = \Phi_0\delta_2(\mathbf{r} - \mathbf{r}_L),$$
$$\mathbf{H}_2 + \lambda_2^2\,\text{rot rot}\,\mathbf{H}_2 = 0$$

and from the conditions for the matching of the solutions on the x = 0 plane, namely, continuity of the tangential components of the vector potential ($A_{t1} = A_{t2}$) and of the normal components of the current ($j_{n1} = j_{n2}$) on this plane. The first condition means physically that there is no infinite magnetic-field intensity on the separation boundary, and the second means that the current has no divergence (div j = 0). Taken together, these two matching conditions mean that the superconducting current produced by the vortex is refracted by the interface. Indeed, by virtue of the London equation $\mathbf{j} \sim \mathbf{A}/\lambda^2$, the matching conditions are $j_{n1} = j_{n2}$ and $\lambda_1^2 j_{t1} = \lambda_2^2 j_{t2}$. It is precisely this refraction of the vortex superconducting lines which leads to the interaction of the vortex with the interface between two superconductors. A general expression for the energy of this interaction is given in[36]. This energy is shown schematically in Fig. 15. We present here some limiting formulas:

$$F = F_0 + \left(\frac{\Phi_0}{4\pi\lambda_1}\right)^2\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\left(\frac{\pi\lambda_1}{4x_L}\right)^{1/2}e^{-2x_L/\lambda_1},\quad x_L \gg \lambda_1, \quad (53)$$

$$F = F_0 + \left(\frac{\Phi_0}{4\pi\lambda_1}\right)^2 K_0(2x_L/\lambda_1),\quad \lambda_2 \ll \lambda_1,\ x_L, \quad (54)$$

$$F = F_0 - \left(\frac{\Phi_0}{4\pi\lambda_1}\right)^2 K_0(2x_L/\lambda_1),\quad \lambda_2 \gg \lambda_1, \quad (55)$$

$$F = F_0 + \left(\frac{\Phi_0}{4\pi\lambda_1}\right)^2\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2}\ln\frac{\lambda_1}{x_L},\quad x_L \ll \lambda_1, \lambda_2, \quad (56)$$

where $F_0 = (\Phi_0/4\pi\lambda_1)^2 \ln\kappa_1$ is the free energy of the single vortex in the infinite first superconductor, $\kappa_1 = \lambda_1/\xi_1$ and K_0 is a Hankel function of zero order of imaginary argument. Formula (56) makes it possible to estimate with logarithmic accuracy the energy of the vortex at a distance on the order of ξ_1 from the boundary

$$F(\xi_1) \approx \frac{\Phi_0^2}{8\pi^2}\frac{\ln \varkappa_1}{\lambda_1^2 + \lambda_2^2}. \quad (57)$$

The discontinuity of the vortex energy on the boundary (see Fig. 15) is then

$$\Delta F \approx \frac{\Phi_0^2}{8\pi^2}\frac{\ln(\varkappa_1/\varkappa_2)}{\lambda_1^2 + \lambda_2^2}. \quad (58)$$

Thus, it follows from Fig. 15 that at $\lambda_1 > \lambda_2$ and $\kappa_1 > \kappa_2$ there is produced near the interface a potential well for the vortex, i.e., the vortex becomes pinned by such an interface. The expressions given for F enable us to estimate the external force that must be applied to such a vortex to cause it to leave the potential well. This is obviously the maximum value of the derivative $|\partial F/\partial x_L|$. Since the well is asymmetrical (see Fig. 15), it is clear that the pinning force of such a vortex moving to the right differs from that moving to the left.

If both superconductors are in the mixed state, then an array of vortices pinned by the boundary is produced along the interface. These vortices form a wall that prevent the entire vortex system from moving under the influence of the Lorentz force. This force arises if transport current is made to flow along the interface. Since both superconducting half-spaces are assumed to be ideally homogeneous, there can be no pinning of the vortices outside the interface. The transport current can therefore flow only along the interface in a layer of thickness $\sim\lambda_1 + \lambda_2$.

*$[j_1 H] \equiv j \times H$.

[1)]The fact that the multiplicity is minimal for arbitrary fields and κ has not been proved. It can be shown however, [9–11] that at $\kappa \gg 1$ the change of the phase of the order parameter on going around the vortex filament is equal to 2π in the entire mixed-state region.

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