Electromagnetic properties of pions at low energies

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The theoretical structure of the pion-photon interaction amplitudes at low energies and the possibility of studying them experimentally are reviewed. It is pointed out that it is possible to measure the $\pi\pi$ scattering lengths in photon-pion transitions. For processes involving an odd number of pions, a discussion is given of the fundamental role of the $\pi^0 \to 2\gamma$ decay amplitude, which determines the form of all more complex amplitudes. This fact is connected with the anomalies in the axial-vector current that appears in the weak interaction. In processes involving an even number of pions, the radius and polarizability of the π^- meson appear as basic parameters. The polarizability is related to the weak decay amplitude.

1. INTRODUCTION

This review is concerned with the properties of the pion-photon interaction amplitudes. Some of the simplest amplitudes of this type are shown in Figs. 1–7. In these figures a wavy line denotes a photon, and a dashed line denotes a pion.

The experimental and theoretical investigation of the electromagnetic vertex of the pion (Fig. 1) and the decay $\pi^0 \to 2\gamma$ (Fig. 2) has a long history. Interest in more complex processes has grown in recent years as a result of the development of colliding beam techniques. It is obvious that there exists a fundamental possibility of studying the amplitudes for $\gamma \to (n)\pi$ $(n = 2, 3, \ldots)$ in accelerator experiments with colliding beams ($e^+e^- \to \gamma \to (n)\pi$). There have also been many discussions of the possibility of studying the processes $\gamma\gamma \to (n)\pi$ $(n = 1, 2, \ldots)$ in the reactions $e^+e^- \to e^+e^-(n)\pi$ in those cases in which the two-photon production mechanism corresponding to the diagram in Fig. 8 is definitive. This mechanism was first considered in the early work[1] (see also[2], p. 471). Various aspects of the two-photon production mechanism in colliding beams have been considered in recent years in numerous works (see, e.g.,[3-13]).

In connection with the possibility of studying the pion-photon interaction experimentally, it should also be noted that there are genuine prospects for achieving intense intermediate-energy meson beams ("pion factories") in the near future. When the exploitation of "meson factories" begins, it will become possible to study with a high accuracy a number of the amplitudes for the Coulomb scattering of pions by nucleons and atomic nuclei which are considered here (Fig. 9). The experimental investigation of the processes of Coulomb production on nuclei is evidently sufficiently practicable even at the present time.

The amplitudes in Figs. 2, 4, 5 and 7 can also be studied in the reactions of pion photoproduction in the Coulomb field of a nucleus. Thus, the Primakoff effect[14] —the photoproduction of a single pion in a Coulomb field —is currently a principal method of measuring the lifetime of the π^0 meson (see[15]).

The processes in Figs. 1–7 represent a rich and interesting object of study from the theoretical point of view. Gauge invariance and the low-energy pion technique make it possible to establish relations among the amplitudes under consideration and to make a number of statements which have very high accuracy. Certain features of these amplitudes reflect the deep dynamical properties of the strong interactions.

The above-mentioned circumstances have led to a voluminous literature. The results of many authors overlap to a great extent. Unfortunately, in this torrent

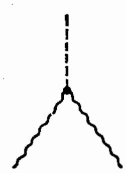
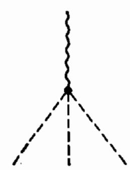


FIG. 1


FIG. 2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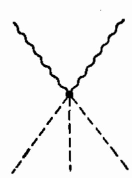
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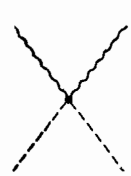
FIG. 5

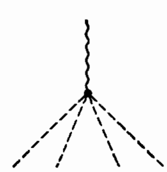
FIG. 6

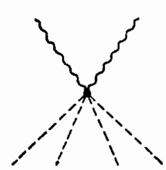
FIG. 7

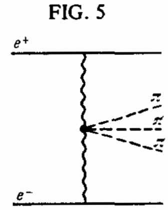

FIG. 8

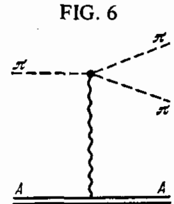

FIG. 9

there are also a large number of works which are in error. In short, we have the usual situation that follows from crowds and haste, although in this sense the situation is not as strained as in certain other, more "fashionable" fields.

By now the problem is apparently settled to a great extent, and it is possible to quote theoretical results concerning each of the processes in Figs. 1—7. This, as well as a qualitative discussion of the prospects of making experimental measurements in a number of cases, is the main content of the present review.

As many of the processes in question are related to the $\pi\pi$ scattering amplitude, we also consider (very briefly) the current theoretical and experimental information on the $\pi\pi$ interaction.

It should be stressed that photon-pion interactions at low energies constitute, in a sense, a self-contained area of physics. All the amplitudes in question can be calculated by making use of a limited set of initial hypotheses. These hypotheses are not arbitrary. They are supported by a whole series of theoretical arguments and experimental facts.

The theoretical results that will be discussed below consist mainly of the derivation of formulas which express all the pion-photon amplitudes in terms of a small number of initial parameters: the radius of the pion, its polarizability, the $\pi^0 \to 2\gamma$ decay constant, and, in a number of cases, parameters which characterize $\pi\pi$ scattering. All these input parameters can be measured independently in other processes. Thus, the low-energy physics of pions and photons is very rigidly determined. Consequently, the experimental data in this area should have an unambiguous interpretation. Unfortunately, practically no data exist at the present time, but it seems that we can expect an accumulation of a large amount of experimental information here in the near future.

The following notation is employed in this review. The system of units is $\hbar = c = 1$, $e^2 = 4\pi\alpha = 4\pi/137$. The metric is given by $a_\nu b_\nu = a_0 b_0 - \mathbf{a}\cdot\mathbf{b}$. The normalization of the states corresponds to choosing the phase space of a particle in the form $(2\pi)^{-3} d^3p/2p_0$. The following matrices are used:

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\gamma_0\gamma_1\gamma_2\gamma_3, \qquad \gamma_0^\dagger = \gamma_0, \ \gamma_j^\dagger = -\gamma_j \qquad (j = 1, 2, 3).$$

The matrix element for each process, T_{fi}, is defined by the relation

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta(p_f - p_i) T_{fi}, \tag{1.1}$$

where S is the scattering matrix.

If the process contains a photon in the initial or final state, we consider the vector T_ν (or the tensor $T_{\nu\mu}$ for the case of two photons) defined by the relation

$$T_{fi} = \xi_\nu T_\nu, \tag{1.2}$$

where either ξ_ν is the polarization 4-vector of the photon or $\xi_\nu = A_\nu(q)/(2\pi)^4$, where $A_\nu(q)$ is the external field.

2. BASIC HYPOTHESES

a) We shall discuss mainly the consequences of gauge invariance and the following assumptions:

A. Conservation of the vector current:

$$\partial_\nu v_\nu^k(x) = 0. \tag{2.1}$$

B. Partial conservation of the axial-vector current (PCAC):

$$\partial_\nu a_\nu^k(x) = \mu^2 F_\pi \varphi^k(x). \tag{2.2}$$

In (2.1) and (2.2), v_ν^k and a_ν^k are the vector and axial-vector hadronic currents, k = 1, 2, 3 are isotopic indices, φ^k is the pion field, μ is the pion mass, and F_π is the $\pi \to e\nu$ decay constant.

C. Current algebra*):

$$[v_0^k(x),\ v_\nu^n(x')]\,\delta(x_0 - x_0') = i\varepsilon_{knm} v_\nu^m(x)\,\delta^{(4)}(x - x'), \tag{2.3}$$

$$[a_0^k(x),\ a_\nu^n(x')]\,\delta(x_0' - x_0) = i\varepsilon_{knm} v_\nu^m(x)\,\delta^{(4)}(x - x'), \tag{2.4}$$

$$[a_0^k(x),\ v_\nu^n(x')]\,\delta(x_0 - x_0') = i\varepsilon_{knm} a_\nu^m(x)\,\delta^{(4)}(x - x'), \tag{2.5}$$

$$[v_0^k(x),\ a_\nu^n(x')]\,\delta(x_0 - x_0') = i\varepsilon_{knm} a_\nu^m(x)\,\delta^{(4)}(x - x'). \tag{2.6}$$

D. The structure of the electromagnetic current $j_\mu(x)$:

$$j_\mu(x) = e(v_\mu^0 + v_\mu^3), \tag{2.7}$$

where v_μ^0 is the isoscalar current (in the framework of SU(3) symmetry, the eighth component of the octet). The currents v_μ^0 and v_μ^3 have opposite G-parity ($Gv_\mu^0G^{-1} = -v_\mu^0$, $Gv_\mu^3G^{-1} = v_\mu^3$; we note also that $Ga_\mu^kG^{-1} = -a_\mu^k$), and the current v_μ^0 commutes with the isovector currents v_μ^k and a_μ^k.

We turn now to a brief discussion of the nature of the hypotheses formulated above.

Equation (2.1) is a consequence of the isotopic invariance of the strong interactions.

The relation (2.2) is an identity on the mass shell of the pion (here any operator is proportional to the pion field). Off the mass shell, it actually serves as a definition of the field $\varphi^k(x)$ and is significant only if the non-pole contributions to the matrix elements from the operator $\partial_\nu a_\nu^k(x)$ are slowly varying functions of the momenta and allow a series expansion for small momenta ($\sim \mu$). As a rule, the matrix elements of the operator $\partial_\nu a_\nu^k(x) \cdot \exp(ipx)$ can be calculated for the momentum $p \to 0$ by using current algebra. The hypothesis that a series expansion is possible allows the result to be continued to the point $p^2 = \mu^2$, where in accordance with (2.2), the matrix element in question is related to the amplitude for the emission of a pion. Thus, by calculating the matrix elements of the divergence of the axial-vector current for $p \to 0$, we obtain information on the amplitudes for the emission of a pion.

As a result, we can say that Eq. (2.2) is a conventional representation of the hypothesis that certain hadronic amplitudes are slowly varying over a scale of the order of the pion mass. For a more detailed discussion of the physical content of the hypothesis (2.2), see[16,17].

Current algebra (2.3)—(2.6) was proposed by Gell-Mann. It is realized in the theory of free fermion fields (where $v_\nu^k = (1/2)\bar{\psi}\gamma_\nu\tau^k\psi$ and $a_\nu^k = (1/2)\bar{\psi}\gamma_\nu\gamma_5\tau^k\psi$), in the quark model, and in the σ model[18]. Its applicability in the actual theory of the strong interactions can only be substantiated by the agreement between the experimental data and the predictions based on its use.

Equations (2.3)–(2.6) imply simple commutation relations for the charge operators $Q_j = \int d^3x v_0^j(\mathbf{x}, 0)$ and $Q_{5j} = \int d^3x a_0^j(\mathbf{x}, 0)$. The "left" ($L_j = Q_j - Q_{5j}$) and "right" ($R_j = Q_j + Q_{5j}$) combinations of charges generate two independent (left and right) SU(2) algebras. The axial-vector current a_μ^j is conserved when $\mu^2 = 0$, so that Q_{5j}, like Q_j, is time-independent and the quantities L_j and R_j are the generators of the "left" and "right" SU(2) groups, respectively. Consequently, the hypotheses A, B and C amount to the assumption that the strong interactions possess SU(2) × SU(2) symmetry at $\mu^2 = 0$. It is convenient to represent the strong interaction hamiltonian in the form $\mathscr{H} = \mathscr{H}_0 + \epsilon\mathscr{H}_1$, where $\mathscr{H}_0$ is invariant under the group SU(2) × SU(2), $\mathscr{H}_1$ is the symmetry breaking, and ϵ is a numerical parameter characterizing the scale of the symmetry breaking. It is then readily shown that $\int d^3x \partial a_\mu^j(x)/\partial x_\mu = -i\epsilon[Q_{5j}(t), \mathscr{H}_1]$.

The above-mentioned possibility of continuing the matrix elements in the momenta over intervals of the order of μ implies that the pion mass is small with respect to the scale of the strong interactions. This, in turn, means that the violation of SU(2) × SU(2) symmetry is small ($\epsilon \ll 1$). A detailed discussion of the relation of the hypotheses A–C to the problem of SU(2) × SU(2) symmetry can be found in[19].

The hypothesis (2.7) about the structure of the electromagnetic current, as formulated in D, contains, in essence, two statements. First, the electromagnetic current contains no isotensor component; second, and more exacting, the current has an octet structure. Most of the results which we propose to discuss are based on the first, weaker assumption.

Unfortunately, both assumptions, while being extremely plausible from the theoretical point of view, have few direct experimental confirmations (see, e.g.,[20]).

The hypotheses A–D together constitute the basis of the low-energy pion technique and lead to a large number of results that are in good agreement with experiment (see[16,17]).

For the constant F_π in (2.2), we shall use the value

$$F_\pi = \frac{0.83\mu}{\sqrt{2}}, \tag{2.8}$$

which follows from the Goldberger-Treiman relation[16,17]

$$F_\pi = \frac{G_A m_N}{G_V g_{\pi N}}; \tag{2.9}$$

here G_V and G_A are the constants associated with the vector and axial-vector currents in β decay, m_N is the nucleon mass, and $g_{\pi N}$ is the πN coupling constant. The value (2.8) differs slightly (~ 10%) from the value $F_\pi^{\exp} \approx 0.93\mu/\sqrt{2}$ which can be obtained from the data on the $\pi \to e\nu$ decay probability. The choice (2.8) can be justified by the fact that Eq. (2.2) is usually applied not at the pion pole ($p^2 = \mu^2$), where it is exact if $F_\pi = F_\pi^{\exp}$, but at the point $p = 0$, where it is approximate. It is at $p = 0$ that Eq. (2.9) follows from (2.2), but it is satisfied exactly for the choice (2.8). We may therefore expect that the choice (2.8) also "rectifies" Eq. (2.2) at the point $p = 0$ for other processes. However, from a "high-level" theoretical point of view, we should not distinguish between (2.8) and the value $F_\pi = F_\pi^{\exp}$. This difference lies within the limits of the accuracy of the low-energy pion technique. A possible ~ 10% uncertainty in the choice of the value of the constant F_π in (2.2) should be borne in mind.

b) The highly successful application of the low-energy energy pion technique seems rather odd at first sight, in view of the fact that the hypothesis (2.2) is incompatible with the occurrence of the decay $\pi^0 \to 2\gamma$. (It follows from (2.2) that the decay $\pi^0 \to 2\gamma$ should be strongly suppressed, and this is not observed experimentally).

There should be some theoretical grounds for supposing that Eq. (2.2) cannot be applied to the amplitude for $\pi^0 \to 2\gamma$.

In this connection, it was shown in[21] that in the presence of an electromagnetic field the condition (2.2) for the neutral components (k = 3) must be rewritten in the form

$$\partial_\nu a_\nu^3(x) = \mu^2 F_\pi \varphi^3(x) + c\frac{\alpha}{8\pi}\varepsilon_{\nu\mu\alpha\beta}F_{\nu\mu}(x)F_{\alpha\beta}(x), \tag{2.10}$$

where $F_{\nu\mu}(x)$ is the electromagnetic field tensor, and c is an arbitrary constant. This modification of the PCAC condition does not show up in all the results of the low-energy technique discussed in[16,17], but removes the contradiction in the decay $\pi^0 \to 2\gamma$.

We shall call relations for $\partial_\nu a_\nu^3(x)$ that differ from (2.2) anomalous PCAC conditions. The decay $\pi^0 \to 2\gamma$ is allowed within the framework of the hypothesis (2.10), although the corresponding decay constant f(0) cannot be calculated, since the parameter c in (2.10) cannot be determined without invoking additional model-dependent assumptions. We shall employ the constant f(0) as an "input" phenomenological parameter, which, as we shall see, will determine the amplitudes with an odd number of pions in Figs. 3 and 4. This is equivalent to the assumption that an anomalous PCAC condition exists for certain amplitudes. We shall not make use of the explicit form of (2.10) here. It is only important to suppose that the usual form of PCAC undergoes certain modifications in the presence of an electromagnetic field, since $f(0) \neq 0$. This point is discussed in greater detail in Sec. 5.

c) As we remarked in Sec. 1, some of the results which we shall discuss depend on the $\pi\pi$ scattering amplitude. This quantity cannot be calculated completely with the framework of the hypotheses A, B and C and contains one arbitrary parameter γ (see[17]). The amplitude was nevertheless calculated in[22], and the value $\gamma = 0$ was obtained. This was done, however, by making use of an additional hypothesis about the structure of the commutator:

$$\left[\partial_\nu a_\nu^k(x), \int d^3x' a_0^3(0, \mathbf{x}')\right] \sim \mu^2\delta_{kn}. \tag{2.11}$$

The assumption is in fact that the right-hand side of (2.11) is proportional to δ_{kn}. This is quite natural and can be justified in a number of field-theoretic models.

As we have already noted in considering the content of the hypotheses B and C, the divergence of the axial-vector current is proportional to the term $\epsilon\mathscr{H}_1$ which breaks SU(2) × SU(2) symmetry. Isotopic invariance requires that $\mathscr{H}_1$ is an isoscalar. This means that $\mathscr{H}_1$ must transform according to some representation of the group SU(2) × SU(2) of the type (n, n). By assuming that H_1 belongs to the simplest representation (1/2, 1/2), we can easily reproduce the result (2.11). Certain theoretical arguments for choosing the representation (1/2, 1/2) were given, in particular, in[23] and have been discussed further in a large number of works.

Nevertheless, it should be stressed that Eq. (2.11) lies outside the framework of the hypotheses B and C and determines the isotopic properties of the interaction that breaks $SU(2) \times SU(2)$ symmetry.

We shall discuss the basic results as far as possible without making use of (2.11), but in a number of cases we shall point out simplifications which result if the condition (2.11) is satisfied.

3. ELECTROMAGNETIC VERTEX OF THE PION

a) **Phenomenological structure.** We define the vertex function $\tau_\nu(k_1, k_2)$ of the pion by means of the relation

$$\langle \pi^a(k_1), \pi^b(k_2) \mid j_\nu(0) \mid 0 \rangle = -ie\varepsilon_{3ab}\mathcal{T}_\nu(k_1, k_2), \tag{3.1}$$

where j_ν is the electromagnetic current (2.7). The G-parity selection rules exclude a contribution of v_ν^0 to (3.1). At $k_1^2 = k_2^2 = \mu^2$, the function $\tau_\nu(k_1, k_2)$ has the form

$$\mathcal{T}_\nu(k_1, k_2) = (k_1 - k_2)_\nu F(q^2), \tag{3.2}$$

where $q = k_1 + k_2$, and $F(q^2)$ is the electromagnetic form factor. At small q^2, we have

$$F(q^2) \approx 1 + \frac{\langle r^2 \rangle}{6} q^2, \tag{3.3}$$

where $\langle r^2 \rangle^{1/2}$ is the electromagnetic radius of the pion.

The general structure of the vertex function $\tau_\nu(k_1, k_2)$ off the mass shell ($k_1^2 \neq \mu^2$, $k_2^2 \neq \mu^2$) is of the form

$$(k_1 - k_2)_\nu F_1(q^2, k_1^2, k_2^2) + q_\nu F_2(q^2, k_1^2, k_2^2). \tag{3.4}$$

Exploiting conservation of the vector current, we can use standard methods (see, e.g.,[18]) to derive the Ward identity for $\tau_\nu(k_1, k_2)$:

$$q_\nu \mathcal{T}_\nu(k_1, k_2) = (k_1^2 - \mu^2)(k_2^2 - \mu^2)[\Delta(k_2) - \Delta(k_1)]; \tag{3.5}$$

here $\Delta(k)$ is the renormalized Green's function of the pion, with a spectral representation in the form

$$\Delta(k) = \frac{1}{k^2 - \mu^2} + \int_{9\mu^2}^{\infty} \frac{\rho(\varkappa^2)\, d\varkappa^2}{k^2 - \varkappa^2}, \quad \rho > 0. \tag{3.6}$$

If $k^2 \sim \mu^2$, the integral term in (3.6) can be expanded in a series in the parameter $k^2/\kappa_{\text{eff}}^2 \ll 1$. In this case, it follows from (3.6) that

$$(k_1^2 - \mu^2)(k_2^2 - \mu^2)[\Delta(k_2) - \Delta(k_1)] \approx k_1^2 - k_2^2 + O(k^6/\varkappa_{\text{eff}}^4). \tag{3.7}$$

It is significant that the terms $\sim k^4$ in (3.7) cancel. This circumstance makes it possible to determine the functions F_1 and F_2 in (3.4) with an accuracy up to second-order terms in the momenta. It turns out in this case that, as before, the vertex function off the mass shell is determined by the single parameter $\langle r^2 \rangle$ at small momenta and is of the form

$$\mathcal{T}_\nu(k_1, k_2) = (k_1 - k_2)_\nu \left(1 + \frac{\langle r^2 \rangle}{6} q^2\right) + q_\nu \frac{\langle r^2 \rangle}{6}(k_2^2 - k_1^2). \tag{3.8}$$

We note that the vector dominance model yields $\langle r^2 \rangle / 6 = 1/m_\rho^2$, where m_ρ is the mass of the ρ meson. In this case, $\langle r^2 \rangle^{1/2} \approx 0.63$ F. This value is in rather good agreement with the experimental data. This means that the characteristic scale of variation of the form factor $F(q^2)$ (beyond the resonance region) is evidently m_ρ and that the expansion in (3.8) is actually made in the parameter p^2/m_ρ^2, where p is one of the momenta q, k_1 or k_2.

b) **Possible methods of determining** $\langle r^2 \rangle$. The parameter $\langle r^2 \rangle$, as well as the entire from factor $F(q^2)$, are fundamental quantities, the measurement of which is of great interest.

The function $F(q^2)$ can be studied most directly (in the region $q^2 > 4\mu^2$) in the colliding-beam reaction $e^+e^- \to \pi^+\pi^-$. So far, measurements have been performed only in the resonance region at $q \sim m_\rho^2$ [24,25] and in the region $q^2 \gtrsim 1$ GeV^2 [26,27].

The most direct method in the region $q^2 < 0$ is to study pion-electron scattering. In this case, at accessible energies, we can only discuss the region of small q^2 at the present time. So far, the appropriate experiments[28,29] have a poor accuracy.

A number of experiments (see, e.g.,[30,31]) have been performed with the aim of determining $\langle r^2 \rangle$ from the scattering of pions by He. Such a possibility was pointed out in[32]. The interpretation of the data on πHe scattering encounters considerable theoretical uncertainties and, on the whole, the method does not appear to be reliable.

So far, attempts to measure the form factor in the region $0 < -q^2 \lesssim 0.5$ GeV^2 in pion-nucleon electroproduction experiments[33-35] have had the greatest success. The theoretical feasibility of doing this was pointed out in[36,37]. The problem lies in the need to extract from the full amplitude the contribution of the diagram in Fig. 10, which contains the electromagnetic vertex of the pion.

In the theoretical interpretation of the electroproduction data, it is customary to employ the results of[38-40], in which the amplitude for photoproduction of a pion by a virtual photon is reconstructed from dispersion relations. By applying such a procedure, the form factor has been found in the form (see, e.g.,[34])

$$F(q^2) = \left(1 - \frac{q^2}{0.56 \pm 0.08}\right)^{-1} \tag{3.9}$$

(q^2 is measured in units GeV^2/c^2), for which $\langle r^2 \rangle^{1/2} = 0.65$ F.

An attempt has recently been made[41] to extract data on the form factor in the time-like region in studying the reaction $\pi^- p \to e^+e^- n$. Here again it is necessary to isolate the contribution of the diagram in Fig. 10, but now with a time-like photon. The results obtained in[41] correspond to the value $\langle r^2 \rangle^{1/2} \sim 0.7$ F, which is consistent with (3.9).

A number of other proposals for measuring the form factor can be found in the literature. More complex experiments would be required to carry them out, although the theoretical interpretation of the experimental data would be more direct. In[42] it was proposed to study πe scattering in final-state interactions in weak decays of mesons and hyperons. This effect can be isolated by measuring the T-odd spin correlations. In[43] it was pointed out that it is possible to measure the form factor

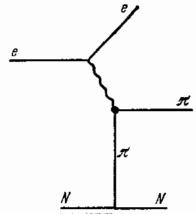

FIG. 10

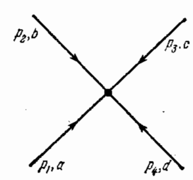

FIG. 11

by studying spin effects in πe scattering. The feasibility of measuring $F(q^2)$ in a pion-nucleus reaction of e^+e^- or $\mu^+\mu^-$ pair production was discussed in[44]. Below (see Sec. 8) we shall discuss the possibility of measuring $\langle r^2\rangle$ in the decay $\pi \to e\nu e^+e^-$.

4. $\pi\pi$ SCATTERING

As we have already pointed out, we shall make use of the $\pi\pi$ scattering amplitude as an "input" parameter in the calculation of the more complex amplitudes in Figs. 3, 4, 6 and 7.

The general structure of the $\pi\pi$ scattering amplitude is of the form (the notation for the momenta and the corresponding isotopic indices is shown in Fig. 11)

$$T_{\pi\pi} = \delta_{ab}\delta_{cd}M(p_1, p_2, p_3, p_4) + \delta_{ac}\delta_{bd}M(p_1, p_3, p_2, p_4) + \delta_{ad}\delta_{bc}M(p_1, p_4, p_3, p_2). \quad (4.1)$$

For small momenta, the function M can be represented in the form[17,22]

$$M(p_1, p_2, p_3, p_4) = \alpha + \beta[(p_1+p_2)^2 - \mu^2] + \gamma(p_1^2 + p_2^2 + p_3^2 + p_4^2 - 3\mu^2). \quad (4.2)$$

In[22] it was found that

$$\alpha = 0, \quad \beta = F_\pi^{-2}, \quad \gamma = 0. \quad (4.3)$$

The equality $\alpha = 0$ follows from the Adler self-consistency condition: $M(0, p_2, p_3, p_4) = 0$ for $p_2^2 = p_3^2 = p_4^2 = \mu^2$. This condition is a direct consequence of Eq. (2.2).

The equality $\beta = F_\pi^{-2}$ follows from (2.2) and the commutation relations (2.5) and (2.4).

The condition $\gamma = 0$ is a consequence of the assumption (2.11) about the structure of the commutator. Considerations in the σ model are the main argument for (2.11). The relation (2.11) does not follow from the basic hypotheses A, B and C in Sec. 2. We shall therefore use, in addition to (4.3), the more general possibility

$$\alpha = 0, \quad \beta = F_\pi^{-2}, \quad \gamma \text{ arbitrary}. \quad (4.4)$$

The $\pi\pi$ scattering lengths enter a large number of processes as parameters (see[45]). Until now, attempts to measure $T_{\pi\pi}$ in peripheral pion-nucleon scattering and in the decays $K \to 3\pi$ and K_{e4} decay have been most effective.

The scattering lengths evaluated using the parameters α, β and γ have the form

$$a_0 = \frac{\mu}{32\pi}(7\beta + 5g), \quad a_2 = \frac{\mu}{16\pi}(-\beta + g), \quad (4.5)$$

where a_I is the scattering length for isotopic spin I, and $g = \gamma + \alpha\mu^{-2}$ (see[17]). It follows from (4.3) that

$$\mu a_0 = 0.2, \quad \mu a_2 = -0.06. \quad (4.6)$$

We can appreciate the existing situation regarding the determination of the $\pi\pi$ scattering lengths from Fig. 12. The straight line $2a_0 - 5a_2 = 0.7\ \mu^{-1}$ gives the theoretically expected values that follow from (4.4). The point W corresponds to choosing the parameters (4.3). The shaded areas A, B, C and D give the data (with allowance for errors) of peripheral experiments[45], the region E gives the data obtained from the analysis of the decays $K \to 3\pi$[46], and the band F gives the data from K_{e4} decay[47]. It would evidently be premature to draw a categorical conclusion from Fig. 12 about the existence of a contradiction between (4.3) and the experimental data.

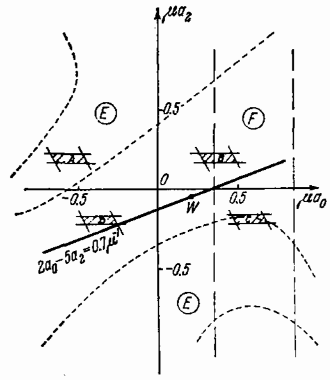

FIG. 12

In analyzing peripheral experiments, the assumption is made from the very beginning that $\alpha \neq 0$, $\beta \approx 0$ and $\gamma \approx 0$, i.e., it is assumed that the function M in (4.2) is practically constant over scales $\sim \mu^2$. Then α and the corresponding scattering lengths are determined. It is only with this procedure that it is possible, with small statistics, to perform a well-defined extrapolation from the physical region of momentum transfers ($t < 0$) to the pion pole ($t = \mu^2$). It is clear that, if one of the sets of parameters (4.3) or (4.4) is actually realized, the scattering lengths obtained in this way (the regions A, B, C and D in Fig. 12) have no relation to the actual values.

Information about the $\pi\pi$ scattering lengths can be obtained from the decays $K^\pm \to \pi^\pm\pi^+\pi^-$ by analyzing the pion spectra and the distributions in the Dalitz plane. However, it is then necessary to analyze the spectra in a narrow band at the peripheries of the Dalitz plane, where until recently the statistics have been poor. The results of[48], where $\sim 10^6$ decays have been detected, have not yet been analyzed in this way. The region E in Fig. 12 is obtained by analyzing a combination of the ratios of $K^\pm \to 3\pi$ and $K^0 \to 3\pi$ decay probabilities[46]:

$$\xi = \frac{1}{4}\frac{\Gamma(K^+ \to \pi^+\pi^+\pi^-)}{\Gamma(K^+ \to \pi^0\pi^0\pi^+)} - \frac{3}{2}\frac{\Gamma(K^0 \to \pi^+\pi^-\pi^0)}{\Gamma(K^0 \to \pi^0\pi^0\pi^0)}. \quad (4.7)$$

The quantity ξ is proportional to $(a_0 - a_2)^2$, where a_0 and a_2 are the $\pi\pi$ scattering lengths. Recent results[49] which improve the accuracy of the probabilities of the neutral decay modes of the kaon make it possible to further narrow down the region E. Nevertheless, the errors remain very large and do not enable us to categorically exclude the values (4.3) and (4.4).

The region F (data from K_{e4} decay) is based on the results of a single work[47]. Owing to the large errors, these results do not strongly contradict the value $\mu a_0 = 0.2$ which follows from (4.3) and do not at all contradict (4.4).

Nevertheless, the entire set of data indicates that the values of the scattering lengths (4.6) and the corresponding set of parameters (4.3) are improbable. However, the values (4.e) and, even more so, the values (4.4) (the line $\mu(2a_0 - 5a_2) = 0.7$ in Fig. 12) cannot as yet be reliably excluded .

Further measurements are required to clarify the situation regarding the $\pi\pi$ scattering lengths, and the processes in Figs. 3, 4, 6 and 7 may serve as a source of important additional information about the structure of the amplitude $T_{\pi\pi}$.

5. THE DECAY $\pi^0 \to 2\gamma$

a) Phenomenological structure. The amplitude for the process $\pi^0 \to 2\gamma$ (see Fig. 2) is of the form

$$T_{\nu\mu} = f(p^2, q_1^2, q_2^2)\, \varepsilon_{\nu\mu\alpha\beta} q_{1\alpha} q_{2\beta}, \tag{5.1}$$

where p is the π^0 momentum, q_1 and q_2 are the photon momenta, ν and μ are the corresponding polarization indices, and $\epsilon_{\nu\mu\alpha\beta}$ is the antisymmetric tensor ($\epsilon_{0123} = 1$)[3)]. The constant

$$f \equiv f(\mu^2, 0, 0) \tag{5.2}$$

determines the lifetime of the π^0 meson:

$$\tau(\pi^0) = \frac{64\pi}{f^2\mu^3}. \tag{5.3}$$

It is natural to adopt the assumption that $f(p^2, q_1^2, q_2^2)$ is slowly varying over scales that are small in comparison with $\sim m_\rho^2$ (as well as the vertex function in Sec. 3a). We can then assume that

$$f \approx f(0), \tag{5.4}$$

where f is defined in (5.2), and $f(0) \equiv f(0, 0, 0)$. This assumption will be of the utmost importance in studying more complex amplitudes with an odd number of pions. The slow variation of the function $f(p^2, q_1^2, q_2^2)$ can in pri principle be verified experimentally by studying the distribution in the effective mass p^2 of the system of two photons near the mass of the π^0 meson[4)], as well as by considering the distribution in the mass q_i^2 of the Dalitz pairs in the reaction $X \to X' + e^+e^- + \gamma$. The dependence on the variables q_1^2 and q_2^2 can be studied in the production of π^0 mesons in colliding e^+e^- beams by the two-photon mechanism (see[4,9]).

b) Numerical value of the constant f. The current value of $\tau(\pi^0)$ is $(0.84 \pm 0.1) \times 10^{-16}$ sec (see[15]). This gives

$$f \approx \frac{0.45\alpha}{\mu}. \tag{5.5}$$

However, there exist data[50] (some of the latest) which imply the value $\tau(\pi^0) \approx 0.56 \times 10^{-16}$ sec. In this case, $f \approx 0.57\alpha/\mu$. We shall use the value (5.5), but, in view of the significant variation in $\tau(\pi^0)$ in the experimental data of recent years, it should be borne in mind that there may be an uncertainty in the choice of the constant f.

It is important to emphasize for what follows that the constant in (5.5) is not small, but is of a "normal" order of magnitude, since it corresponds to a reasonable value for the interaction radius $\sim \mu^{-1}$.

c) Contradiction with the PCAC hypothesis. It was pointed out in[51,52] that the hypothesis (5.4) is incompatible with the relation (2.2). In fact, the matrix element (5.1) can be written in the form (for simplicity, we consider only the case of real photons)

$$\varepsilon_\nu(q_1) T_{\nu\mu} \varepsilon_\mu(q_2) = -(p^2 - \mu^2)\langle q_1, q_2 \,|\, \varphi^3(0) \,|\, 0\rangle, \quad p = q_1 + q_2. \tag{5.6}$$

Using (2.2), we obtain

$$\langle q_1, q_2 \,|\, \varphi^3(0) \,|\, 0\rangle = \frac{ip_\alpha}{\mu^2 F_\pi} \langle q_1, q_2 \,|\, a_\alpha^3(0) \,|\, 0\rangle = \frac{p_\alpha A_{\alpha\nu\mu}}{\mu^2 F_\pi} \varepsilon_\nu(q_1)\varepsilon_\mu(q_2). \tag{5.7}$$

Let us assume that it is possible to make a series expansion in the momenta in (5.7). Taking into account the requirement of Bose symmetry, the pseudotensor $A_{\alpha\nu\mu}$ can then be written in the form

$$A_{\alpha\nu\mu} = k\varepsilon_{\alpha\nu\mu\beta}(q_1 - q_2)_\beta + \varepsilon_{\alpha\nu\beta\sigma} T^{(1)}_{\mu\beta\sigma} + \varepsilon_{\alpha\beta\mu\sigma} T^{(2)}_{\nu\beta\sigma} + \varepsilon_{\beta\nu\mu\sigma} T^{(3)}_{\alpha\beta\sigma} + O(p^4), \tag{5.8}$$

where the tensors $T^{(j)}_{\alpha\beta\gamma}$ must be formed from the momenta and are at least of third order. Gauge invariance requires that the conditions $q_{1\nu} A_{\alpha\nu\mu} = A_{\alpha\nu\mu} q_{2\mu} = 0$ are satisfied, from which it follows that k = 0. Thus, the entire expression (5.6) is of fourth order in the momenta, which contradicts the hypothesis $f(p^2, 0, 0) \approx$ const in (5.1). It is easy to see that (5.6), together with (5.7) and (5.8), implies that $f(p^2, 0, 0) \sim p^2$, i.e., that $f(0) = 0$ and Eq. (5.4) cannot be satisfied.

As a solution of this problem, it is meaningful to consider two possibilities: either 1) we have in fact $f(0) = 0$, in which case Eq. (5.4) is not valid and the function $f(p^2, 0, 0)$ varies rapidly over scales $\sim \mu^2$, or 2) Eq. (5.4) is satisfied, but the hypothesis (2.2) is incorrect in the application to the decay under consideration.

Our further considerations will be based on the second possibility, but in this connection it should be stressed that this choice cannot be well substantiated by theoretical arguments alone at the present time. Meanwhile, the criterion here is an argument of an esthetic nature, connected largely with the result of an investigation of this problem in the σ model.

d) The decay $\pi^0 \to 2\gamma$ in the σ model. The lagrangian of the σ model has the form[53]

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\psi\varphi} + \mathscr{L}_{\varphi\sigma}, \tag{5.9}$$

where $\mathscr{L}_0$ is the free lagrangian, and

$$\mathscr{L}_{\psi\varphi} = g_0 \bar{\psi}_0 (i\gamma_5 \tau_a \varphi_0^a + \sigma_0)\psi_0, \tag{5.10}$$

$$\mathscr{L}_{\varphi\sigma} = -\frac{\varkappa_0^2 - \mu_0^2}{8m_0^2} g_0^2 \left[(\varphi_0^2 + \sigma_0^2)^2 - \frac{4m_0}{g_0}\sigma_0(\varphi_0^2 + \sigma_0^2)\right]. \tag{5.11}$$

Here m_0, μ_0 and κ_0 are the masses of the nucleon, pion and σ particle, respectively. The index zero denotes "bare," nonrenormalized quantities. The vector and axial-vector currents have the form

$$v_\nu^k = \frac{1}{2}\bar{\psi}_0 \gamma_\nu \tau^k \psi_0 - i\frac{\partial\varphi_0^a}{\partial x_\nu} T_k^{ab} \varphi_0^b, \quad T_k^{cd} = -i\varepsilon_{kcd}, \tag{5.12}$$

$$a_\nu^k = \frac{1}{2}\bar{\psi}_0 \gamma_\nu \gamma_5 \tau^k \psi_0 - \frac{m_0}{g_0}\frac{\partial\varphi_0^k}{\partial x_\nu} + \left(\frac{\partial\varphi_0^k}{\partial x_\nu}\sigma_0 - \frac{\partial\sigma_0}{\partial x_\nu}\varphi_0^k\right). \tag{5.13}$$

Their divergences are, respectively,

$$\partial_\nu v_\nu^k = 0, \tag{5.14}$$

$$\partial_\nu a_\nu^k = \mu_0^2 \frac{m_0}{g_0}\varphi_0^k. \tag{5.15}$$

Thus, Eq. (22) is satisifed in the σ model as an exact operator equality. The "bare" $\pi \to e\nu$ decay constant is then $F_{\pi 0} = m_0/g_0$.

By including the interaction of charged particles with the electromagnetic field in (5.9) by means of the usual substitution $\partial_\nu \to \partial_\nu + ieA_\nu$ and naively applying the equations of motion, we obtain

$$\partial_\nu v_\nu^k = -ieT_3^{kn} v_\nu^n A_\nu, \tag{5.14'}$$

$$\partial_\nu a_\nu^k = \mu_0^2 \frac{m_0}{g_0}\varphi_0^k - ieT_3^{kn} a_\nu^n A_\nu. \tag{5.15'}$$

Thus, the condition (5.15) for the neutral components (k = 3) has been preserved, even in the presence of an electromagnetic field. Applying further the arguments of part c of this section to the decay $\pi^0 \to 2\gamma$, we immediately find that f(0) = 0 in each order of perturbation theory in the σ model.

On the other hand, the decay $\pi^0 \to 2\gamma$ is described by the contribution of the triangle diagram in Fig. 13 in the lowest order in g_0. We note that, in the lowest approximation, the $\pi^0 \to 2\gamma$ amplitude in the σ model has the same form as in the conventional theory with a pseudoscalar πN coupling. A direct calculation of the diagram in Fig. 13 gives the non-zero result[54,55]

$$f(0) = -\frac{e^2}{4\pi^2}\frac{g_0}{m_0}, \tag{5.16}$$

which contradicts the condition (5.15), since the latter implies that $f(0) = 0$.

This paradox was formulated in[55]. Its resolution is connected with an insufficiently rapid convergence of the diagram in Fig. 13. This can be seen from the following considerations.

Using the conservation conditions, standard methods can be used to derive the Ward identities in the σ model for the vertices of the vector and axial-vector currents[56]:

$$ip_\nu \Gamma^a_\nu(k', k) = \frac{1}{2}\tau^a (S^{-1}(k) - S^{-1}(k')), \tag{5.17}$$

$$ip_\nu \Gamma^a_{5\nu}(k', k) = \frac{1}{2}\tau^a (S^{-1}(k)\gamma_5 + \gamma_5 S^{-1}(k')) + \frac{m_0 \mu_0^2}{g_0}\frac{\Gamma^a_5(k', k)}{p^2 - \mu_0^2 - \Sigma_0(p^2)}, \tag{5.18}$$

where $p = k' - k$, $\Gamma^a_5(k', k)$ is the pseudoscalar vertex for the emission of a pion, $\Sigma_0(p^2)$ is the mass operator of the pion, and $S(k)$ is the fermion Green's function. To lowest order of perturbation theory with $p \to 0$,

$$\Gamma^a_{5\nu} = \frac{1}{2}\tau^a \gamma_\nu \gamma_5, \quad \Gamma^a_\nu = \frac{1}{2}\tau^a \gamma_\nu, \quad \Gamma^a_5 = i g_0 \tau^a \gamma_5, \quad S^{-1}(k) = -i(\hat{k} - m_0),$$

and the conditions (5.17) and (5.18) are trivially satisfied.

It follows from the relation (5.7) (if it is valid) that the diagram in Fig. 13 should be obtained by adding a factor $ip_\alpha g_0/m_0\mu_0^2$ to the graph of Fig. 14, where the plus sign denotes the vertex of the axial-vector current. This property must follow from the Ward identity for the axial-vector vertex.

The diagram of Fig. 14 is a third-rank tensor. It is convergent and is given by $-(ie^2/8\pi^2)\,\epsilon_{\alpha\nu\mu\sigma}(q_1 - q_2)_\sigma$ to lowest order in the momenta. However, this structure is not transverse in the photon momenta q_1 and q_2. The transversality property is usually a consequence of the Ward identity (5.17). Here this is not the case, since the degree of convergence of the diagram is worsened when the identity (5.17) is applied to the vector vertices and there appears a difference between the two linearly divergent integrals. Thus, despite the fact that the graph in Fig. 14 is convergent, it does not converge rapidly enough and a subtraction (regularization) must be made in it, defining it to within a gauge-invariant structure. The form of the subtraction term is determined unambiguously to lowest order in the momenta, since it must cancel the inadmissible structure $\sim \epsilon_{\alpha\nu\mu\sigma}(q_1 - q_2)_\sigma$. With such a regularization, the PCAC property is lost and it is found that Eq. (5.7) is not satisfied if $\langle q_1, q_2 | a^3_\alpha(0)|0\rangle$ is interpreted as the regularized and gauge-invariant matrix element.

The modification in Eq. (5.7) is connected with the occurrence of the subtraction term $\sim \epsilon_{\alpha\nu\mu\sigma}(q_1 - q_2)$, which must be extracted from the matrix element of the axial-vector current. After multiplying by $p_\alpha = (q_1 + q_2)_\alpha$ in (5.7), the subtraction term takes the form $\sim \epsilon_{\nu\mu\alpha\beta} q_{1\alpha} q_{2\beta}$. As a result, we find that we must write, instead of (5.7), a relation which is a formal consequence of the anomalous PCAC condition (2.10) (if we choose $c = F_{\pi 0} g_0/m_0 = 1$ in (2.10)):

$$\langle q_1, q_2 | \varphi^3_0(0)|0\rangle = \frac{ip_\alpha}{\mu_0^2 F_{\pi 0}}\langle q_1, q_2 | a^3_\alpha(0)|0\rangle_{\text{reg}} - \frac{e^2 g_0}{4\pi^2 m_0 \mu_0^2}\epsilon_{\nu\mu\alpha\beta} e_\nu(q_1) e_\mu(q_2) q_{1\alpha} q_{2\beta}, \tag{5.19}$$

where $\langle q_1, q_2 | a^3_\alpha(0)|0\rangle_{\text{reg}}$ is the gauge-invariant (regularized) matrix element of the axial-vector current, and by definition (see (5.6) and (5.1))

$$\mu_0^2 \langle q_1, q_2 | \varphi^3_0(0)|0\rangle = f(0)\,\epsilon_{\nu\mu\alpha\beta} e_\nu(q_1) e_\mu(q_2) q_{1\alpha} q_{2\beta}.$$

The first term on the right-hand side of Eq. (5.19) is of fourth order in the momenta (see Sec. 5c), so that the quantity $f(0)$ is related entirely to the contribution of the second term in (5.19) and is determined by Eq. (5.16). Thus, the paradox of[55] is resolved in favor of the second of the two possibilities mentioned at the end of Sec. 5c. This solution of the problem was first proposed in[21].

It is clear from the foregoing discussion that the occurrence of the paradox and the violation of the PCAC condition are due entirely to the poor convergence of the triangle diagram in Fig. 14 containing the vertex of the axial-vector current. In[57] a complete classification was made of the diagrams with fermion loops which involve the vertices of the vector and axial-vector currents and in which, as in the graph of Fig. 14, the formal consequences of PCAC and conservation of the vector current do not hold simultaneously. All possible anomalous diagrams are shown in Fig. 15. We see that they contain an odd number of vertices of the axial-vector current. Therefore we might expect anomalous PCAC conditions to occur in amplitudes with an odd number of pions.

We note that the condition (2.10) can be derived in the σ model by using the equations of motion for the fields that appear in the definition of a^k_ν in (5.13). It is necessary here to employ a gauge-invariant method of defining the singular products of field operators at a single point (see, e.g.,[58,59]). The necessity of such a definition and the possibility that anomalous terms can occur here were first pointed out in[60].

The triangle diagram of Fig. 14 also shows up in the occurrence of anomalous terms in the commutation relations of the vector and axial-vector currents. By using current commutators in accordance with the anomalous condition (2.10), it can be seen in another way that there is no suppression of the decay $\pi^0 \to 2\gamma$ in the σ model (see[61]).

For a detailed analysis of the anomalies associated with the poor convergence of the diagrams with fermion loops in perturbation theory, as well as references to a large number of papers on this problem, see[62]. It is interesting that the $\pi^0 \to 2\gamma$ decay amplitude can be calculated in the σ model with allowance for all the approximations in the coupling constant g_0[63]. Graphs that are more complex than the triangle diagram of Fig. 13 have

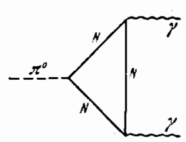

FIG. 13

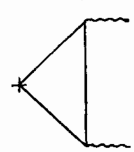
FIG. 14

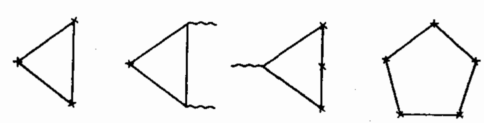
FIG. 15

a better convergence. The corresponding diagrams with vertices of the axial-vector current do not require subtractions and allow the simultaneous application of the Ward identities (5.17) and (5.18). Thus, the normal PCAC condition is found to be satisfied for the contribution of the higher approximations for the amplitude of $\pi^0 \to 2\gamma$. Consequently, the more complex diagrams do not give a separate contribution to f(0), but merely lead to a renormalization of the constants and masses in the skeleton diagram of Fig. 13. The final result has the form

$$f(0) = -\frac{e^2}{4\pi^2}\frac{1}{g_a}\frac{g_{\pi N}}{m}, \tag{5.20}$$

where g_a is the renormalized constant of the axial-vector current (which appears in the β decay of the neutron), $g_{\pi N}$ is the physical πN coupling constant, and m is the nucleon mass. It is remarkable that the quantity (5.20) coincides with the physical $\pi^0 \to 2\gamma$ decay constant in (5.5) with percentage accuracy.

In conclusion, it should be stressed that all the foregoing facts nevertheless do not constitute a proof of the existence of an anomalous PCAC condition in the σ model. This was emphasized in[64]. Other ways of resolving the paradox formulated at the end of part c of this section are also possible.

One of these possibilities was considered in[55]. This ambiguity is connected with the following circumstance.

We have already seen that the matrix elements of perturbation theory without an appropriate regularization and definition may in general not possess the symmetry which is formally built into the lagrangian. (For example, the matrix element in Fig. 14 calculated according to the usual Feynman rules is not transverse, whereas conservation of the vector current is of course contained in (5.9). Another example is the scattering of light by light in ordinary electrodynamics, which is described by a convergent box diagram that is not transverse and requires a subtraction if we wish to preserve gauge invariance in the matrix elements). The variant of the regularization that we considered in connection with the $\pi^0 \to 2\gamma$ problem is, in a sense, a minimal one. In this case, we preserved gauge invariance, but destroyed PCAC. However, the PCAC condition (5.15) can in principle be imposed on the matrix elements by making a further subtraction in the well-convergent diagram of Fig. 13 involving a pseudoscalar vertex. (The subtraction term must be chosen to be equal to $+(e^2g_0/4\pi^2m_0)\,\epsilon_{\nu\mu\alpha\beta}q_{1\alpha}q_{2\beta}$ in order to cancel the contribution of second order in the momenta in the decay $\pi^0 \to 2\gamma$. In this way we obtain f(0) = 0, but then, to be sure, the renormalizability of the σ model is violated). With such a subtraction (at any rate, to lowest order in g_0), Eq. (5.15) is again found to be satisfied. In this sense, the result is dependent on the method that is chosen for calculating the loop diagrams in the framework of the lagrangian (5.9).

Thus, our considerations in the σ model indicate that there are, in general, various possible solutions of the problem of the decay $\pi^0 \to 2\gamma$.

The most natural and simplest possibility is to introduce the anomalous PCAC condition (2.10). In this case, the decay $\pi^0 \to 2\gamma$ is allowed and the condition (5.4) is satisfied. Some of the other possibilities that have been considered lead to nonrenormalizable infinities in the higher approximations of the σ model.

Finally, digressing from the σ model, it should be stressed that the PCAC condition in the form (2.2) is not a strict law. In this respect, the decay $\pi^0 \to 2\gamma$ is sensitive to the corrections. There are no grounds for supposing that the physical PCAC condition (in its application to the matrix elements) cannot be modified, if there is a physical reason for such a modification. As we have already noted, we are assuming that such a modification actually occurs, so that the decay $\pi^0 \to 2\gamma$ is allowed in the sense that $f(0) \neq 0$ and Eq. (5.4) is satisfied. Our considerations in the σ model show concretely how such a modification could occur. However, the explicit form of the anomalous (modified) PCAC condition will not be important in what follows.

e) **Relation to the annihilation $e^+e^- \to$ hadrons.** In connection with the discussion in part d of this section, it is of interest to consider the possible relation between the $\pi^0 \to 2\gamma$ decay amplitude and the asymptotic form of the total cross section for the annihilation $e^+e^- \to$ hadrons, although this will take us far beyond the basic hypotheses of Sec. 2. It is convenient to trace the origin of this relation in the naive quark model.

If we considered the decay $\pi^0 \to 2\gamma$ in the quark model, in which the pion is a composite particle, and assumed, as before, that the PCAC condition is satisfied for the interpolating pion field (which would appear to be necessary), we would obtain, as a result of the contribution of the triangle graph, the expression (5.20) for the constant f(0), with the replacement

$$e^2 \to Le^2, \tag{5.21}$$

where

$$L = \sum_i (Q_{pi}^2 - Q_{ni}^2); \tag{5.22}$$

here Q_p and Q_n are the charges of the proton and neutron quarks. ($Q_p = 2/3$, $Q_n = -1/3$. We note that $Q_p^2 - Q_n^2 = Q_p + Q_n$. The quantity Q_n^2 appears with a minus sign in (5.22), since the coupling of the neutron quark to the π^0 meson in the triangle diagram has the negative sign). The summation Σ_i in (5.22) is carried out over the number of quarks of the neutron and proton types. In the usual quark model (i = 1), we obtain L = 1/3, which gives a $\pi^0 \to 2\gamma$ decay width which is smaller than that required experimentally (or by Eq. (5.20)) by a factor of nine.

There exist models (see, e.g.,[65]) in which the number of quarks is tripled, in accordance with the values of a new quantum number (quark "color" in[65]). In this case, L = 1, which is highly satisfactory from the experimental point of view.

We turn now to the annihilation $e^+e^- \to$ hadrons. The cross section for this process is determined by the imaginary part of the vacuum polarization diagram corresponding to the real transitions $\gamma \to q\bar{q} \to \gamma$. This diagram contains contributions involving the multiple scattering of quarks in the intermediate state and contributions without multiple scattering that involve only the imaginary parts of the exact quark Green's functions.

We shall assume that all the contributions involving multiple scattering fall off at large q^2 (q is the photon momentum). This state of affairs should occur in a theory with finite renormalization constants, as well as in a theory with some (external) cut-off in the transverse momenta of the particles. There is no field-theoretic model in which such a picture could be consistently obtained by summing the graphs of perturbation theory. However, "scaling" or self-similarity, observed in the cross sections for deep inelastic processes, indicates that the actual situation may correspond in some way to models with a finite renormalization.

After discarding the contributions involving the multiple scattering of quarks, the cross section for the process $e^+e^- \to$ hadrons (the imaginary part of the vacuum polarization) will be determined by the integral

$$\int \operatorname{Im} G(k_1) \operatorname{Im} G(k_2)\, \theta(k_{10})\, \theta(k_{20})\, \delta(q-k_1-k_2)\, d^4k_1 d^4k_2 = \pi^2 \int d\varkappa_1^2 \rho(\varkappa_1^2) \int d\varkappa_2^2 \rho(\varkappa_2^2)\, J(\varkappa_1^2, \varkappa_2^2, q^2),$$

where

$$J(\varkappa_1^2, \varkappa_2^2, q^2) = \int \theta(k_{10})\, \theta(k_{20})\, \delta(k_1^2-\varkappa_1^2)\, \delta(k_2^2-\varkappa_2^2)\, \delta(q-k_1-k_2)\, d^4k_1\, d^4k_2.$$

When $q^2 \to \infty$ (but $\kappa^2_{1\mathrm{eff}} \sim \kappa^2_{2\mathrm{eff}} \sim$ const, corresponding to a theory with a finite renormalization), we obtain $J \to$ const (the integral J is, in fact, the phase space of two particles with masses κ_1 and κ_2 and total 4-momentum q). The integrals with respect to κ_i^2 can then be evaluated by exploiting the unitarity condition for the spectral function $\int \rho(\kappa^2)\, d\kappa^2 = 1$, and the total effect of the strong interactions vanishes.

Finally, the total cross section for the annihilation $e^+e^- \to$ hadrons at large q^2 is related to the simplest quark diagram corresponding to the transition $\gamma \to q\bar{q} \to \gamma$ with two free quarks in the intermediate state. The ratio of the cross sections for annihilations into hadrons and into a $\mu^+\mu^-$ pair is then proportional to the sum of the squares of the quark charges:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_i (Q_{pi}^2 + Q_{ni}^2 + Q_{\lambda i}^2). \tag{5.23}$$

In the ordinary three-quark model, R = 2L = 2/3 (L is defined in (5.22)). In the model with nine quarks[65], R = 2L = 2. Thus, certain quark models that give a good description of the decay $\pi^0 \to 2\gamma$ (L = 1) lead to a ratio R = 2, which is in satisfactory agreement with the latest experimental data on the annihilation $e^+e^- \to$ hadrons (see[66]).

The simple, although glaringly nonrigorous, arguments outlined above were recently formulated by Crewther[67], who succeeded in obtaining the relation

$$R = 2L \tag{5.24}$$

independently of the specific quark model[5)]. If we now make use of the experimental value of the constant f(0) (which corresponds to $L_{\mathrm{exp}} \approx 1$), we can obtain the expected value of the asymptotic ratio $\sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$.

In[67], instead of the naive assumptions inherent in the quark model, a hypothesis is adopted about the behavior of the products of current operators at small distances[68] [6)]. However, it is possible that these approaches are equivalent.

The fact that the constant of the low-energy decay $\pi^0 \to 2\gamma$ can determine the asymptotic form (for $q^2 \to \infty$) of the ratio (5.23) is a powerful and exciting statement. Unfortunately, the physical basis for this fact remains unclear.

6. THE PROCESSES $\gamma \to 3\pi$ AND $\gamma\gamma \to 3\pi$ [69-72]

a) Phenomenological structure. The $\gamma \to 3\pi$ amplitude has the form

$$T_\nu^{abc} = -ih(s_{12}, s_{13}, s_{23})\, \varepsilon_{abc} \varepsilon_{\nu\alpha\beta\gamma} p_{1\alpha} p_{2\beta} p_{3\gamma}. \tag{6.1}$$

The form factor $h(s_{12}, s_{13}, s_{23})$ depends on the invariant variables $s_{ij} = (p_i + p_j)^2$ and (off the mass shell) on the virtual masses p_j^2 of the pions.

The matrix element T_ν^{abc} must be a pseudovector and third-rank tensor in the isotopic variables (the photon in the process $\gamma \to 3\pi$ is isoscalar and has no isotopic index). This requirement determines the structure of T_ν^{abc} unambiguously. Bose symmetry determines the obvious symmetry properties of the function h.

We begin with the assumption that h is a slowly varying function and that it can be expanded in a series in the pion momenta. In the zeroth approximation,

$$h(s_{12}, s_{13}, s_{23}) \approx h(0). \tag{6.2}$$

We shall consider below the validity of this assumption and indicate the range of variation of the momenta in which the expansion (6.2) seems to make sense.

Let us consider at the same time the amplitude for the process $\gamma\gamma \to 3\pi$. Its general structure is rather complicated. We shall assume that the non-pole terms of this amplitude are slowly varying functions of the momenta and that they admit a series expansion. In the lowest approximation in the momenta, the matrix element for the process $\gamma\gamma \to 3\pi$ has a simple form

$$T_{\nu\mu}^{abc3} = P_{\nu\mu}^{abc3} + \varepsilon_{\nu\mu\alpha\beta} [B q_{1\alpha} q_{2\beta} (\delta_{3a}\delta_{bc} + \delta_{3b}\delta_{ac} + \delta_{3c}\delta_{ab}) - A (q_1-q_2)_\alpha (p_{1\beta}\delta_{3a}\delta_{bc} + p_{2\beta}\delta_{3b}\delta_{ac} + p_{3\beta}\delta_{3c}\delta_{ac})], \tag{6.3}$$

where A and B are constants, and $P_{\nu\mu}^{abc3}$ is the contribution of the pole terms in Fig. 16. It follows from arguments involving the conservation of G-parity that one of the photons in the process $\gamma\gamma \to 3\pi$ is isoscalar, while the other is isovector. The amplitude $T_{\nu\mu}^{abc3}$ is therefore a fourth-rank tensor in the isotopic variables. In other respects, its structure is determined by the requirements of the conservation of spatial parity and Bose symmetry.

The pole graph in Fig. 16a involves a block of $\pi\pi$ scattering (see (4.1)) and a $\pi^0 \to 2\gamma$ decay vertex (see (5.1)). The contribution of this graph is of order $\sim p^2$ (like the contact term) and is given by

$$-f(0) \frac{\varepsilon_{\nu\mu\alpha\beta} q_{1\alpha} q_{2\beta}}{Q^2-\mu^2} [\delta_{3a}\delta_{bc} M(p_2, p_3, p, -Q) + \delta_{3b}\delta_{ac} M(p_1, p_3, p_2, -Q) + \delta_{3c}\delta_{ab} M(p_1, p_2, p_3, -Q)], \tag{6.4}$$

$$Q = q_1 + q_2 = p_1 + p_2 + p_3.$$

The graph in Fig. 16b involves the $\gamma \to 3\pi$ amplitude. Its contribution is also of order $\sim p^2$ and is given by

$$\Big[-eh(0)\, \varepsilon_{abj}\varepsilon_{3cj} \frac{(2p_3-q_2)_\mu}{(p_3-q_2)^2-\mu^2} \varepsilon_{\nu\alpha\beta\gamma} p_{1\alpha} p_{2\beta} (p_3-q_2)_\gamma + \text{permutation}\ {}^{p_1 \to p_2 \to p_3}_{a \to b \to c}\Big] + \text{permutation}\ {}^{q_1 \to q_2}_{\nu \to \mu}. \tag{6.5}$$

The assumption that the functions $f(p^2, q_1^2, q_2^2)$ (see (5.1)) and $h(s_{12}, s_{13}, s_{23})$ (see (6.1)) are slowly varying is essential here. The difference between, say, f(0) in

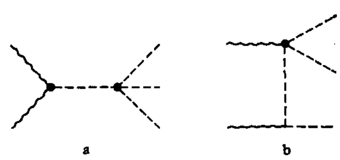

FIG. 16

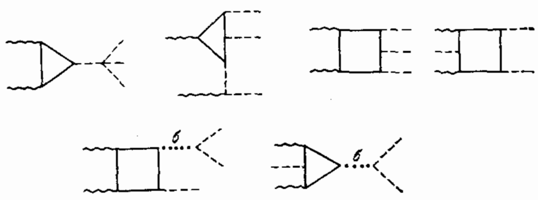

FIG. 17

(6.4) and $f(\mu^2, 0, 0)$ is in fact important in the terms of order $\sim p^4$ (if all $p_i \sim q_i \sim \mu$), which are not taken into account in (6.3).

Gauge invariance ($q_{1\nu}T^{abc3}_{\nu\mu} = 0$) requires the validity of the relation

$$A = eh(0). \tag{6.6}$$

b) **Calculation of the parameters** A, B and h(0). To obtain further restrictions on the parameters A, B and h, we must consider separately the two amplitudes in (6.3): $T^{3113}_{\nu\mu}$ (describing the process $\gamma\gamma \to \pi^0\pi^+\pi^-$) and $T^{3333}_{\nu\mu}$ (describing the process $\gamma\gamma \to 3\pi^0$).

The amplitude for $\gamma\gamma \to \pi^0\pi^+\pi^-$ is proportional to the integral

$$\int dx\, e^{-iq_2x}\langle \pi^0(p_1),\ \pi^+(p_2),\ \pi^-(p_3)\,|\,T(j_\nu(0)\,j_\mu(x))\,|\,0\rangle. \tag{6.7}$$

In the σ model, it is described by the sum of the diagrams in Fig. 17. In this amplitude, we can employ the normal PCAC condition in the form (2.2) for the neutral meson. In fact, by using (2.2) and the reduction formulas for $p_1^2 \neq \mu^2$, we obtain, instead of (6.7),

$$\frac{i(\mu^2-p_1^2)}{\mu^2F_\pi}\int dx\, e^{-iq_2x}\int dx_1 e^{ip_1x_1}\langle \pi^+(p_2),\ \pi^-(p_3)\,|\,T(\partial_\alpha a^3_\alpha(x_1)\,j_\nu(0)\,j_\mu(x))\,|\,0\rangle$$
$$= \frac{\mu^2-p_1^2}{\mu^2F_\pi}\int dx\,dx_1 e^{-iq_2x}e^{ip_1x_1}p_{1\alpha}\langle \pi^+(p_2),\ \pi^-(p_3)\,|\,T(a^3_\alpha(x_1)\,j_\nu(0)\,j_\mu(x))\,|\,0\rangle, \tag{6.8}$$

where we have made use of the fact that a^3_α commutes with the electromagnetic current, which enables us to remove the operation of differentiation from the T-product.

Equation (6.8) implies that the amplitude for $\gamma\gamma \to \pi^0\pi^+\pi^-$ is obtained by adding a factor $p_{1\alpha}$ to the diagrams in which the vertex for emitting only one neutral pion is replaced by the vertex of the axial-vector current. In the σ model, this replacement in the graphs of Fig. 17 does not lead to any "dangerous" diagrams (see Fig. 15), which are poorly convergent and require re-definition. These arguments provide a basis for applying PCAC in the form (2.2) for a single pion when the others are on the mass shell.

We stress that, in going off the mass shell for the three pions simultaneously and consistently applying PCAC for each of them, dangerous diagrams appear (after calculating the current commutators). This means that the structure of the amplitude $T_{\nu\mu}$ and the values of the constants A, B and h(0) are in fact related to the anomalous PCAC condition. However, we make use of arguments which do not require a concrete discussion of the structure of the anomalous terms. These terms are taken into account phenomenologically in the section in which we assume that $f(0) \neq 0$ and that the decay $\pi^0 \to 2\gamma$ is not forbidden (see Sec. 5).

The PCAC hypothesis (2.2), when applied to the neutral pion with momentum p_1, leads to the condition $T^{3113}_{\nu\mu} = 0$ for $p_1 \to 0$ (with $p_2^2 = p_3^2 = \mu^2$). This enables us to express the constant B in (6.3) in terms of the contribution of the diagram in Fig. 16a when $p_1 \to 0$. (By itself, the contribution of the diagram in Fig. 16b vanishes when $p_1 \to 0$). It turns out that

$$B = \frac{f(0)}{F_\pi^2}(1+\gamma F_\pi^2). \tag{6.9}$$

(We are parametrizing the $\pi\pi$ scattering amplitude according to (4.2) and (4.4)).

Thus, we have already expressed the amplitude $T_{\nu\mu}$ in terms of the parameters f(0) and h(0). By using also the PCAC condition (2.2) for one of the neutral mesons in the amplitude $T^{3333}_{\nu\mu}$, we find that $T^{3333}_{\nu\mu} \to 0$ when $p_1 \to 0$ (with $p_2^2 = p_3^2 = \mu^2$). This condition leads to the further relation

$$-2A+3B = \frac{f(0)}{F_\pi^2}(1+3\gamma F_\pi^2). \tag{6.10}$$

It follows from (6.6), (6.9) and (6.10) that[69-72]

$$h(0) = \frac{f(0)}{eF_\pi^2} \approx \frac{f}{eF_\pi^2}. \tag{6.11}$$

The numerical value of h(0) is then found to be

$$h(0) \approx \frac{0.1e}{\mu^3}. \tag{6.12}$$

We note that, if $f(0) = 0$ and the decay $\pi^0 \to 2\gamma$ is forbidden within the framework of PCAC, the amplitude (6.3) is in fact of fourth order in the momenta (while (6.1) is of fifth order) and no predictions can be obtained about the values of the parameters in these amplitudes. In this sense, it is important to test the condition (6.11) experimentally, in order to confirm the hypothesis that anomalous PCAC conditions exist.

Equations (6.9)–(6.11) can be verified by direct calculations in the σ model[70,71,73]. In the lowest approximation, the process $\gamma\gamma \to 3\pi$ is described by the diagrams of Fig. 17, and the process $\gamma \to 3\pi$ by the diagram of Fig. 18. As in the case of $\pi^0 \to 2\gamma$ (see the discussion in Sec. 5d), it can be shown that, in the lowest approximation in the momenta, the diagrams of the following orders in the coupling constant are unimportant, and the amplitudes for $\gamma\gamma \to 3\pi$ and $\gamma \to 3\pi$ are determined in the σ model entirely by the graphs of Figs. 17 and 18, respectively.

Finally, we write the $\gamma\gamma \to 3\pi$ amplitude at $p_i^2 = \mu^2$ for the various charge states, taking into account (6.6), (6.9) and (6.11).

For the process $\gamma\gamma \to 3\pi^0$,

$$T^{000}_{\nu\mu} = -\frac{f(0)}{F_\pi^2}(1+3\gamma F_\pi^2)\frac{\mu^2}{Q^2-\mu^2}\varepsilon_{\nu\mu\alpha\beta}q_{1\alpha}q_{2\beta}. \tag{6.13}$$

For the process $\gamma\gamma \to \pi^0(p_1)\pi^+(p_2)\pi^-(p_3)$,

$$T^{0+-}_{\nu\mu} = \frac{f(0)}{F_\pi^2}\left\{\frac{Qp_1-(1/2)\mu^2(1+\gamma F_\pi^2)}{Q^2-\mu^2}\varepsilon_{\nu\mu\alpha\beta}q_{1\alpha}q_{2\beta}\right.$$
$$\left.+\left[\delta_{\mu\sigma}+\frac{(2p_3-q_2)_\mu p_{2\sigma}}{q_2^2-2q_2p_3}+\frac{(2p_2-q_2)_\mu p_{3\sigma}}{q_2^2-2q_2p_2}\right]\varepsilon_{\nu\sigma\beta\alpha}p_{1\beta}q_{1\alpha}\right\}+\binom{\nu\to\mu}{q_1\to q_2}. \tag{6.14}$$

We see that, owing to cancellations, (6.13) (with $\gamma = 0$) is proportional to $\sim \mu^2$, and it is very small in the case in which $q_i \sim p_i \sim \mu$, $Q^2 \sim 9\mu^2$ and $(1 + 3\gamma F_\pi^2) \sim 1$. Thus, it is not likely that $T_{\nu\mu}(\gamma\gamma \to 3\pi^0)$ can be evaluated correctly within the framework of the approximations that are used. This fact was noted in[74,75].

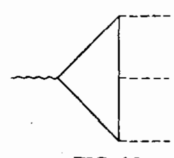

FIG. 18

A correct expression for the $\gamma\gamma \to 3\pi^0$ amplitude was first derived in[74], although the $\gamma\gamma \to \pi^+\pi^-\pi^0$ amplitude was evaluated incorrectly.

The $\gamma\gamma \to 3\pi$ amplitude has been used by a number of authors to estimate the contribution of the 3π intermediate state to the imaginary part of the $K_L \to \mu^+\mu^-$ decay amplitude (see[76,77]).

c) Estimate of the accuracy of the approximation (6.2). In order to compare (6.11) with experiment, we must consider in greater detail the relation between h(0) and the quantity $h(s_{12}, s_{13}, s_{23})$, defined for values of the invariants in the physical region.

Our discussion was based on the assumption (6.2). One can attempt to improve the accuracy of Eq. (6.2) by allowing for a possible dependence of the function h on the invariants, arising from the contribution of the resonance graphs to the amplitude (6.1). In this case, we must add to the resonance graphs the contribution of the nonresonant (background) part of the amplitude in such a way that the total contribution at zero momenta gives the correct limit (6.11). Allowance for the diagrams with ρ and ω exchange (Fig. 19) yields, instead of (6.2), the following extrapolation formula[78]:

$$h(s_{12}, s_{13}, s_{23}) = h(0)\left[1 + \Delta_\rho\left(\frac{s_{12}}{m_\rho^2 - s_{12}} + \frac{s_{13}}{m_\rho^2 - s_{13}} + \frac{s_{23}}{m_\rho^2 - s_{23}}\right) + \Delta_\omega \frac{Q^2}{m_\omega^2 - Q^2}\right],$$
$$\Delta_\rho = \frac{2 f_{\rho\pi\gamma} f_{\rho\pi\pi}}{m_\rho^2 h(0)}, \quad \Delta_\omega = \frac{e\lambda_\omega h_\omega}{m_\omega^2 h(0)}, \tag{6.15}$$

where, as before, $h(0) = f/eF_\pi^2$, $Q^2 = (p_1 + p_2 + p_3)^2$ and $s_{ij} = (p_i + p_j)^2$; m_ρ and m_ω are the masses of the ρ and ω mesons, and $f_{\rho\pi j}$, $f_{\rho\pi\pi}$ and h_ω^2 are coupling constants that can be expressed in terms of the partial widths for the decays of ρ and ω mesons[79]:

$$\left.\begin{aligned} \Gamma(\rho \to \pi^+\pi^-) &\approx \frac{f_{\rho\pi\pi}^2 m_\rho}{48\pi}, \\ \Gamma(\rho \to \pi\gamma) &\approx \frac{f_{\rho\pi\gamma}^2 m_\rho^3}{96\pi}, \\ \Gamma(\omega \to 3\pi) &= \frac{h_\omega^2 m_\omega^7}{2^{12}\cdot 90\pi^3} x, \quad x \approx 0.23, \end{aligned}\right\} \tag{6.16}$$

$-ie\lambda_\omega$ is the vertex for the transition $\gamma \to \omega$, which is related to the $\omega \to e^+e^-$ decay width:

$$\Gamma(\omega \to e^+e^-) = \frac{4\pi}{3}\frac{\alpha^2\lambda_\omega^2}{m_\omega^3}. \tag{6.17}$$

Using the data[15], we obtain

$$\left.\begin{aligned} f_{\rho\pi\pi} &\approx 5.5, \\ f_{\rho\pi\gamma} &\leqslant \frac{3.5\sqrt{\alpha}}{m_\rho}, \\ h_\omega &\approx \frac{4.9}{\mu^3}, \\ \lambda_\omega &\approx 6.7\cdot 10^{-2} m_\omega^2. \end{aligned}\right\} \tag{6.18}$$

This gives

$$\Delta_\rho \leqslant 0.5, \quad \Delta_\omega \approx 3. \tag{6.19}$$

Putting $s_{ij} \sim 4\mu^2$ and $Q^2 \sim 9\mu^2$ in the process $\gamma \to 3\pi$, we obtain a value $\lesssim 25\%$ for the correction term proportional to Δ_ρ in (6.15); the correction from the term proportional to Δ_ω is $\sim 100\%$. The large value of the correction from Δ_ω is due to the anomalously large $\omega \to 3\pi$ decay width: the constant h_ω is almost 10^3 times as large as the expected "natural" value $\sim 1/m_\omega^3$. In this sense, we may hope that the graph of Fig. 19b is distinguished and gives the main Q^2-dependence of the function h.

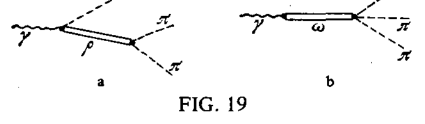

FIG. 19

For the process $\gamma\pi \to \pi\pi$ (with a real photon, i.e., at $Q^2 = 0$), we obtain (for $s_{ij} \lesssim 4\mu^2$) a correction from Δ_ρ smaller than 10%, while the term $\sim \Delta_\omega$ generally gives no contribution in this case.

Thus, we conclude that (6.2) and (6.11) can be used for the process $\gamma\pi \to \pi\pi$ (more precisely, for the description of the reaction $\pi \to 2\pi$ in the Coulomb field of a nucleus); the more accurate formula (6.15) must be used for the process $e^+e^- \to \gamma \to 3\pi$. This claim can be tested experimentally. We note that the corrections in (6.15) depend on the sign of the product of several constants.

d) Estimate of the cross sections. The cross section for the process $\pi \to 2\pi$ in the Coulomb field of a nucleus was calculated in[80-82], and the cross section for the reaction $e^+e^- \to \gamma \to 3\pi$ was calculated in[83-85].

Equation (6.12) can be used to obtain

$$\frac{d\sigma(\pi \to 2\pi)}{dq_\perp^2} \sim \frac{(Z\alpha)^2}{q_\perp^2}\cdot 10^{-29}\ \text{cm}^2 \tag{6.20}$$

where Z is the charge of the nucleus, and $q_\perp^2$ is the square of the momentum transferred to the nucleus.

The differential cross section for the process $\pi \to 2\pi$ on nuclei is related to the cross section $\sigma_{\gamma\pi \to \pi\pi}$ for the process $\gamma\pi^\pm \to \pi^0\pi^\pm$ by the standard formulas of the method of equivalent photons (see, e.g.,[2], p. 464). The differential cross section for the process $\gamma\pi^\pm \to \pi^0\pi^\pm$ has the form

$$\frac{d\sigma_{\gamma\pi\to\pi\pi}}{dt} = \frac{h^2}{128\pi} k^2 \sin^2\theta, \tag{6.21}$$

where k is the momentum of the final mesons, and θ is the scattering angle of the $\pi^\pm$ meson in the c.m.s.;

$$k^2 = \frac{1}{4}(s - 4\mu^2), \quad k^2\cos^2\theta = \frac{s}{4}\frac{(t-u)^2}{(s-\mu^2)^2},$$

where s, t and u are the usual invariants characterizing the two-body process $\gamma\pi \to \pi\pi$. Using (6.11), we obtain $h^2/128\pi \approx (1.5\times 10^{-3}/\mu^6)\alpha/\pi$, which determines the order of the cross sections (6.20) and (6.21).

The total cross section for the reaction $e^+e^- \to 3\pi$ in 250×250 MeV colliding beams turns out to be of the order of 10^{-35} cm^2.

The analytic formula for the differential cross section has the form

$$d\sigma = \frac{\alpha}{\pi}\frac{h^2}{64\pi}\frac{1}{Q^2}[\mathbf{p}_+ \times \mathbf{p}_-]^2 \sin^2\theta\, d\omega_+ d\omega_- d\cos\theta; \tag{6.22}$$

here $Q^2 = 4E^2$ is the square of the total energy in the c.m.s., $\omega_\pm$ and $p_\pm$ are the energies and momenta of the $\pi^\pm$ mesons, and θ is the angle between the vector $p_+ \times p_-$ and the direction of the collision. For h, we must use (6.15) with $\Delta_\rho = 0$.

By integrating Eq. (6.22), the following expression for the cross section for the process $e^+e^- \to \pi^+\pi^-\pi^0$ was obtained in[84,85]:

$$\sigma_{ee\to 3\pi} \approx 3.7\cdot 10^{-10}\mu^{-2}\left(\frac{\sqrt{Q^2} - 3\mu}{\mu}\right)^4.$$

A detailed analysis of the cross section for the process $\gamma\gamma \to 3\pi$, as well as a calculation of the cross section for $\gamma \to 3\pi$ on nuclei of charge Z by the method of equivalent photons, were made in[86]. At a total energy W = 0.5 GeV in the c.m.s. ($W = \sqrt{Q^2}$), we obtain the estimate $\sigma_{\gamma\gamma \to 3\pi}(W) \sim 10^{-33}$ cm^2.

Apart from photoproduction on nuclei, the $\gamma\gamma \to 3\pi$ cross section can be measured in the reaction $ee \to ee + 3\pi$ (see Fig. 8) in high-energy colliding beams. The corresponding calculations can be found in[86,87]. To estimate the cross section for the process in Fig. 8, use can be made of the formula[9]

$$\frac{d\sigma}{dW^2} \approx \left(\frac{\alpha}{\pi}\right)^2 \frac{\sigma_{\gamma\gamma\to 3\pi}(W)}{W^2} \left(\ln\frac{4E^2}{W^2}\ln^2\frac{4E^2\mu^2}{W^2m_e^2} - \frac{1}{3}\ln^3\frac{4E^2}{W^2}\right); \qquad (6.23)$$

2E is the total energy of the e^+ and e^- in the c.m.s., and m_e is the electron mass.

e) **Other methods of calculating the $\gamma \to 3\pi$ amplitude.** The vector dominance model (VDM), in which it is assumed that h(0) is determined by the contribution of the ω-exchange diagram (see Fig. 19b), gives

$$h_{\rm VDM}(0) = \frac{e\lambda_\omega h_\omega}{m_\omega^2}. \qquad (6.24)$$

We shall write h(0) in the form

$$h(0) \equiv \frac{\Lambda e}{\mu^3}. \qquad (6.25)$$

Then it follows from (6.24) that

$$\Lambda_{\rm VDM} \approx 0.33. \qquad (6.26)$$

This exceeds the value of Λ which follows from (6.12) by about a factor of three.

We note that the value $\Lambda = 0.04 \pm 0.15$ was obtained from an analysis of photoproduction data using dispersion relations in[88]. This value is compatible with (6.12).

The $\gamma \to 3\pi$ amplitude has been calculated theoretically by a number of authors in various models. In the table we show the values of Λ obtained in several recent works.

Current algebra and vector dominance are used in[89], modifications of the Veneziano model supplemented by assumptions about the values of a number of parameters of the Regge trajectories are used in[90-93], the hard-pion technique and vector dominance are used in[94], and anomalous Ward identities and again vector dominance are used in[95].

It is interesting that the values of Λ in the table differ sharply. It seems to us that the value $\Lambda \approx 0.1$ which follows from Eq. (6.11) is the best substantiated one from the theoretical point of view. At any rate, the theoretical assumptions on which it is based are most clearly formulated.

Λ	Reference
0.03	89
~ 1	90, 91, 92
0.12	93
0.15	94
0.16–0.24	95

A direct measurement of Λ would be of great interest. Indirect information on Λ can be obtained from an estimate of the contribution of the $\gamma \to 3\pi$ vertex in the process $\pi N \to \pi N\gamma$. In this case, one finds[96,97] $\Lambda \approx 3$. This is in sharp conflict with (6.11), as well as with any of the other values of Λ from the table. It would be of great interest to carry out a more careful re-analysis of the data on the reaction $\pi N \to \pi N\gamma$.

f) **Consequences of (6.11) within the framework of SU(3) symmetry.**

1) Using SU(3) symmetry, we can relate h(0) to the constant $h_{\eta\pi\pi\gamma}$ which appears in the matrix element for the decay $\eta \to \pi\pi\gamma$:

$$T_\nu(\eta \to \pi^+(p_2)\,\pi^-(p_3)\,\gamma(p_1)) = -ih_{\eta\pi\pi\gamma}\varepsilon_{\nu\alpha\beta\sigma}p_{1\alpha}p_{2\beta}p_{3\sigma}. \qquad (6.27)$$

This relation has the form $h_{\eta\pi\pi\gamma} = h/\sqrt{3}$[72,98,99].

There is also an analogous relation between the $\pi \to 2\gamma$ and $\eta \to 2\gamma$ decay constants: $f_{\eta\gamma\gamma} = (1/\sqrt{3})f$[18].

However, it is well known that SU(3) symmetry does not "work" well in describing the decay $\eta \to 2\gamma$. The situation is rectified by allowing for X-η mixing, if an appropriate choice is made for the $X \to 2\gamma$ decay width[18]. (We do not consider the possibility that the experimental data on the $\eta \to 2\gamma$ decay probability may change, as has already happened in recent years).

If allowance is made for X-η mixing within the framework of SU(3) symmetry, (6.11) implies the relation[99]

$$h_{\eta\pi\pi\gamma} = \frac{f_{\eta\gamma\gamma}}{eF_\pi^2}(1-\Delta), \qquad (6.28)$$

where $f_{\eta\gamma\gamma}$ is the $\eta \to 2\gamma$ decay constant, defined in the same way as f in (5.1). The parameter Δ takes into account the mixing:

$$\Delta = -\operatorname{tg}\theta\,\frac{(f_{X\gamma\gamma}/f_{\eta\gamma\gamma}) - (h_{X\pi\pi\gamma}/h_{\eta\pi\pi\gamma})}{1+\operatorname{tg}\theta\,(h_{X\pi\pi\gamma}/h_{\eta\pi\pi\gamma})}, \qquad (6.29)$$

where θ is the mixing angle. By using the value $\theta \approx \pm 10°$ that follows from the mass formulas in the pseudoscalar octet, choosing the ratio $f_{X\gamma\gamma}/f_{\eta\gamma\gamma} = \mp 2.7$ required for a correct description of the decay $\eta \to 2\gamma$ in SU(3) symmetry, and making use of the bounds on the $X \to \pi\pi\gamma$ decay probability, we can obtain an estimate of Δ:

$$0.1 \leqslant \Delta \leqslant 0.5. \qquad (6.30)$$

Taking into account (6.30), Eq. (6.28) is in rather good agreement with the experimental data. If we introduce the ratio $R = h_{\eta\pi\pi\gamma}/f_{\eta\gamma\gamma}$, it also follows[15] from (6.28) that

$$\frac{R_{\rm exp}}{R_{\rm theor}} \approx \frac{.85}{1-\Delta} \approx 1. \qquad (6.31)$$

2) The $\gamma \to 3\pi$ vertex and the form factor of the vector current in K_{e4} decay are also related in the framework of SU(3).

The matrix element of the weak vector current in K_{e4} decay is of the form

$$\frac{G}{\sqrt{2}}\langle\pi^+(p_1),\pi^-(p_2)\,|\,j_\nu^W(0)\,|\,K^+(p)\rangle = \frac{G}{\sqrt{2}}f_4\varepsilon_{\nu\alpha\beta\sigma}p_\alpha p_{1\beta}p_{2\sigma}. \qquad (6.32)$$

The relation in question takes the form

$$f_4 = \frac{\sqrt{2}}{e}h. \qquad (6.33)$$

Using (6.11), it follows from (6.33) that[72,99]

$$f_4 = \frac{\sqrt{2}f}{e^2F_\pi^2} \approx \frac{6}{m_K^3}, \qquad (6.34)$$

where m_K is the kaon mass.

Experimental information on the value of f_4 is derived from the study of the P-odd asymmetry of the positron emission in K_{e4} decay, the result being[47] $f_4 \approx (9 \pm 3.6)/m_K^3$. Thus, (6.34) is in excellent agreement with experiment.

We note that the relation between f_4 and the $\eta \to \pi\pi\gamma$ decay constant was considered in[100]. This relation, together with (6.28), can also be used to estimate f_4. (However, a factor $\sqrt{4\pi}$ that was omitted in[100] must be taken into account; see[99]).

3) It is interesting that the sign of the constant f can also be determined by using the data on the spectra of K_{e4} decay. A discussion of this point follows.

The consistency of the whole approach based on the utilization of the anomalies of the simplest diagrams with fermion loops requires that the sign of f corresponds to the contribution of the triangle diagram. The graph with a nucleon loop leads to the results (5.16) and (5.20), from which it follows that

$$fg_{\pi N} < 0, \tag{6.35}$$

singe $g_a = 1.18 > 0$.

In the general case (of an arbitrary quark fermion loop), the right-hand side of (5.20) contains an additional factor $2\overline{Q}$, where $\overline{Q}$ is the average charge of the fundamental fermion isomultiplet. Thus, according to (5.20), the sign of f depends on the quantum numbers of the fundamental fermions and is different in different field-theoretic models.

There exist a number of experimental (see[101,102]) and theoretical (see[103–105]) arguments that the sign of f corresponds to the diagram in Fig. 13 with a fermion (nucleon) loop. However, the theoretical arguments are based on model-dependent considerations. With the present level of statistics, the experimental photoproduction data[101], on the basis of which a conclusion was drawn about the sign of f in[105], admit a different interpretation (see[106]). The information on the sign of f derived from the data on the Compton effect on the nucleon at low energies[102] is also based on poor statistics and makes use of an analysis of the amplitude for the Compton effect[107] which seems to involve poorly justified assumptions. However, it is possible in principle (with high statistics) to extract unambiguous information about the sign of f from data on the reactions $\gamma p \to p\pi^0$ and $\gamma p \to \gamma p$. As pointed out in[108], this sign can also be determined in the reaction $\pi^- p \to n\gamma\gamma$.

The experimental determination of the sign of f would be a distinctive, although weak, way of testing the basic relations discussed in this section, since a result concerning the sign of f follows directly from (6.34) (see[109]), namely that the condition (6.35) is satisfied.

Parametrizing the contribution of the axial-vector current in K_{e4} decay in the form (cf. (6.32))

$$i\frac{G}{\sqrt{2}}[f_1(p_1+p_2)_\nu + f_2(p_1-p_2)_\nu + f_3(p-p_1-p_2)_\nu], \tag{6.36}$$

it follows from the data[47] that $f_1 \approx -(4.91 \pm 0.2)/m_K$, provided that $f_4 > 0$ in (6.32). Thus, $f_1/f_4 < 0$. The constant f_1 is related to the form factor f_+ of the vector current in K_{e3} decay[110]. This relation has the form $f_1 = f_+/\sqrt{2}F_\pi$, and $f_+ = 1$ in the framework of SU(3) symmetry.

Thus, we obtain

$$\frac{f_4}{F_\pi} < 0. \tag{6.37}$$

By using (6.34) and (6.37), we obtain $f/F_\pi < 0$, from which, taking into account the Goldberger-Treiman relation (2.9), we have the condition (6.35).

7. THE PROCESSES $\gamma \to (2n+1)\pi$ AND $\gamma\gamma \to (2n+1)\pi$ $(n \geqslant 1)$

a) **Phenomenological Lagrangian.** Relations among the amplitudes for the processes $\gamma \to (2n+1)\pi$ and $\gamma\gamma \to (2n+1)\pi$ with an arbitrary number of pions are conveniently derived by employing the technique of phenomenological Lagrangians. The main idea of such an analysis is based on the following assumption. After incorporating the minimal electromagnetic interaction $(\partial_\nu \to \partial_\nu + ieA_\nu)$ in the σ model, the amplitudes under consideration can be calculated to lowest order in the momenta of the participating particles and to lowest order in the strong interaction constant. Moreover, it is assumed that the contributions of higher order in the strong interaction lead only to a re-definition of the coupling constants (renormalization) in the resulting expression. In this case, the ratio of the constants is unchanged, since it is determined by the symmetry. Thus, it is claimed that the relations among the amplitudes that are obtained in the lowest approximation are valid when allowance is made for all orders in the strong interaction.

This procedure can be substantiated by requiring $SU(2) \times SU(2)$ symmetry in considering the interactions in the πN system. The situation is not so clear in the presence of an electromagnetic field, since there is, as yet, no rigorous proof that the result is independent of the detailed structure of the σ model.

Nevertheless, as we have already seen, the relations which arise when using the foregoing procedure for the simplest processes $\pi \to 2\gamma$, $\gamma\gamma \to 3\pi$ and $\gamma \to 3\pi$ are actually reproduced in the σ model even with a phenomenological model. To lowest order, these processes are described in the σ model by the contributions of the diagrams in Figs. 13, 17 and 18, respectively. As we pointed out earlier, the contributions of higher order in the coupling constant are described by diagrams which are better convergent and which permit the application of the PCAC condition to all the pions simultaneously, so that these diagrams do not contribute to lowest order in the momenta. The complete result is related to the anomalous properties of the simplest diagrams with fermion loops. Thus, these examples justify the utilization of the technique of phenomenological lagrangians in the sense indicated above.

Turning to more complex processes, we see that the number of diagrams in the σ model begins to grow rapidly (their number is already rather large for the process $\gamma\gamma \to 3\pi$; see Fig. 17). An important contribution of lowest order in the momenta then occurs, owing to non-trivial canaellations between the various diagrams. Therefore (just as in considering πN interactions) it is convenient to begin with a non-linear modification of the σ model[111]. The corresponding lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_\pi + \mathcal{L}_{NN\pi} + \mathcal{L}_{NN\pi\pi}, \tag{7.1}$$

where

$$\mathcal{L}_N = \bar{\psi}(i\hat{\partial} - m)\psi, \tag{7.2}$$

$$\mathcal{L}_\pi = \frac{1}{2}D_\mu\pi^a D_\mu\pi^a - \frac{1}{2}\mu^2\pi^2, \tag{7.3}$$

$$\mathcal{L}_{NN\pi} = \lambda\frac{G_A}{G_V}\bar{\psi}\chi_\mu\gamma_5\tau^a\psi D_\mu\pi^a, \tag{7.4}$$

$$\mathcal{L}_{NN\pi\pi} = -\lambda^2\bar{\psi}\gamma_\mu\tau^a\psi\varepsilon_{abc}\pi^b D_\mu\pi^c, \tag{7.5}$$

with

$$D_\mu \pi^a = \frac{\partial_\mu \pi^a}{1+\lambda^2\pi^2}, \quad \lambda = \frac{1}{2} F_\pi. \tag{7.6}$$

If we incorporate the electromagnetic field

$$\begin{aligned} \partial_\mu \psi &\to \left(\partial_\mu + \frac{ie}{2}(1+\tau_3) A_\mu\right)\psi, \\ \partial_\mu \pi^a &\to (\partial_\mu \pi^a + e\varepsilon_{3ab}\pi^b A_\mu), \end{aligned} \tag{7.7}$$

there appears an additional interaction:

$$\mathscr{L}_{\rm int} = \mathscr{L}(\psi\psi A) + \mathscr{L}(\pi\pi A) + \mathscr{L}(\psi\psi\pi A) + \mathscr{L}(\psi\psi\pi\pi A), \tag{7.8}$$

where

$$\mathscr{L}(\psi\psi A) = -e\bar{\psi}\cdot\frac{1}{2}(1+\tau_3)\gamma_\mu\psi A_\mu, \tag{7.9}$$

$$\mathscr{L}(\pi\pi A) = -\frac{e\varepsilon_{3ab}\pi^a\partial_\mu\pi^b A_\mu}{(1+\lambda^2\pi^2)^2} + \frac{e^2}{2}A_\mu^2\frac{\pi^2-\pi_3^2}{(1+\lambda^2\pi^2)^2}, \tag{7.10}$$

$$\mathscr{L}(\psi\psi\pi A) = \lambda e\frac{G_A}{G_V}\bar{\psi}\gamma_\mu\gamma_5\tau^a\psi\frac{\varepsilon_{3ab}\pi^b A_\mu}{(1+\lambda^2\pi^2)}, \tag{7.11}$$

$$L(\psi\psi\pi\pi A) = -e\lambda^2\bar{\psi}\gamma_\mu\tau_a\psi\varepsilon_{abc}\varepsilon_{3cd}\frac{\pi^b\pi^d}{1+\lambda^2\pi^2}A_\mu. \tag{7.12}$$

As the Lagrangian (7.1) is clearly nonrenormalizable, it is not completely clear whether the procedure of incorporating the minimal coupling of the electromagnetic field in (7.1) is equivalent to that of including the field in the initial Lagrangian of the linear σ model. (We note that there is even no rigorous proof of the equivalence of (7.1) and the Lagrangian of the initial linear σ model). However, direct calculations of the simplest processes in the lowest approximation indicate that such an equivalence seems to exist if the low-energy approximation is considered.

b) **The $\gamma \to (2n+1)\pi$ vertex.** Let us consider the processes $\gamma \to (2n+1)\pi$ within the framework of the Lagrangian $\mathscr{L} + \mathscr{L}_{\rm int}$ (see (7.1) and (7.8)). We shall construct the corresponding amplitudes to lowest order in the momenta, in this case the third order. The possible diagrams are shown in Figs. 20 and 21. The single-meson hadron vertex here corresponds to the interaction (7.4) and actually involves the set of vertices with emission from a single point of three pions, etc. The two-meson hadron vertex corresponds to the interaction (7.5) and describes the emission of two pions, four pions, etc. It is significant that the photon in the processes $\gamma \to (2n+1)\pi$ is isoscalar, as implied by the conservation of G-parity. It is therefore not necessary to take into account the diagrams in which a photon is emitted from a pion line.

Since each meson vertex of (7.4) and (7.5) involves an external momentum, it is not necessary, in the approximation in question, to take into account diagrams that contain more than three pion vertices.

Moreover, we note that the diagrams in Fig. 20 actually give no contribution. The diagrams in Fig. 20a are absent because the double vertex $\mathscr{L}(\psi\psi\pi A)$ and $\mathscr{L}(\psi\psi\pi\pi A)$ occurs only for an isovector photon. The diagram in Fig. 20b is of at least fifth order in the momenta. In fact, this diagram contains three vector vertices and one axial-vector vertex (a cross on a fermion line denotes the axial-vector vertex $\gamma_\mu\gamma_5$, and a circle denotes the vector vertex γ_μ). Therefore this diagram does not refer to the number of anomalous diagrams (see Fig. 15) which violate the formal consequences of the conservation conditions imposed on the Lagrangian. In our case, the integral over the fermion loop must be transverse, by virtue of the condition of conservation of the isoscalar hadronic current, so that

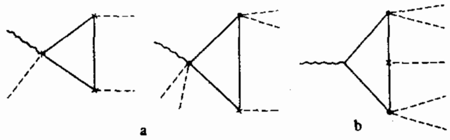

FIG. 20

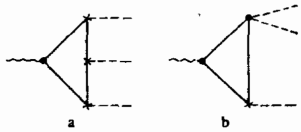

FIG. 21

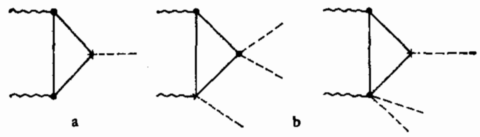

FIG. 22

it involves the external momenta. The presence of three additional external momenta at the three meson vertices renders this diagram unimportant.

Thus, we obtain the effective Lagrangian of the system $\gamma \to (2n+1)\pi$ by considering only the contributions of the diagrams in Fig. 21:

$$\begin{aligned} \mathscr{L}(\gamma \to (2n+1)\pi) = {} & c_1\varepsilon_{\mu\nu\rho\sigma}A_\sigma\varepsilon_{abc}D_\mu\pi^a D_\nu\pi^b D_\rho\pi^c \\ & + c_2\varepsilon_{\mu\nu\rho\sigma}D_\mu\pi^a\varepsilon_{abc}[\partial_\nu A_\rho\pi^b D_\sigma\pi^c + A_\sigma\partial_\nu(\pi^b D_\rho\pi^c)]. \end{aligned} \tag{7.13}$$

Here c_1 is a constant which appears from the integral over the fermion loop in Fig. 21a, and c_2 is the corresponding constant in the diagram in Fig. 21b. This constant will be determined below.

c) **The $\gamma\gamma \to (2n+1)\pi$ vertices.** For the processes $\gamma\gamma \to (2n+1)\pi$, it follows from the conservation of G-parity that one of the photons is isoscalar and the other is isovector. To obtain the physical amplitudes, we must therefore consider the contact terms which occur as a result of the fermion loops, as well as all possible pion pole graphs in which a photon is emitted by a pion.

Simple arguments based on the determination of the number of external momenta in the diagram (cf. the analysis in part b of this section) show that to lowest order in the momenta—in our case, second order—the contact terms are determined entirely by the contributions of the three diagrams in Fig. 22.

We note that the graphs in Fig. 22 involve identical integrals over the fermion loop. On the other hand, the graph in Fig. 22a determines the $\pi^0 \to 2\gamma$ decay amplitude. This enables us to express the corresponding integral in terms of the constant f, defined in accordance with (5.1). The final result for the contribution of the contact terms to $\gamma\gamma \to (2n+1)\pi$ has the form

$$\mathscr{L}(2\gamma \to (2n+1)\pi) = \frac{f}{4}\varepsilon_{\mu\nu\rho\sigma}F_{\nu\rho}A_\sigma\frac{\pi^0\partial_\mu(\lambda^2\pi^2) + (1-\lambda^2\pi^2)\partial_\mu\pi^0}{(1+\lambda^2\pi^2)^2}; \tag{7.14}$$

here $F_{\nu\mu} = -\partial_\nu A_\mu + \partial_\mu A_\nu$.

Next, we note that the graph of Fig. 21b involves the same integral as the diagram in Fig. 22a. Therefore c_2 in (7.13) can be determined, and is found to be

$$c_2 = -\frac{f\lambda^2}{2e}. \tag{7.15}$$

The constant c_1 in (7.13) can be determined from the

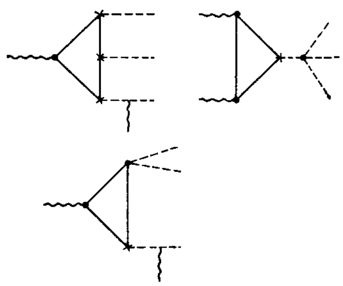

FIG. 23

condition of gauge invariance of the $\gamma\gamma \to \pi^+\pi^-\pi^0$ amplitude, which is described by the contribution of the diagrams in Fig. 23. We obtain in this way

$$c_1 = \frac{1}{3e} f\lambda^2. \tag{7.16}$$

Thus, $\mathscr{L}(\gamma \to (2n+1)\pi)$ in (7.13) and $\mathscr{L}(\gamma\gamma \to (2n+1)\pi)$ in (7.14) are expressed in terms of the single parameter f—the $\pi^0 \to 2\gamma$ decay constant.

As we have pointed out, we must add the contribution of the pole terms, if they exist, in order to calculate the $\gamma\gamma \to (2n+1)\pi$ amplitudes in (7.13) and (7.14). This is equivalent to introducing the total Lagrangian

$$\mathscr{L}_{\text{tot}} = \mathscr{L}(\gamma \to (2n+1)\pi) + \mathscr{L}(\gamma\gamma \to (2n+1)\pi) + \mathscr{L}(\pi\pi A), \tag{7.17}$$

where $\mathscr{L}(\pi\pi A)$, $\mathscr{L}(\gamma \to (2n+1)\pi)$ and $\mathscr{L}(\gamma\gamma \to (2n+1)\pi)$ are defined in (7.10), (7.13) and (7.14) (see also (7.15) and (7.16)).

After algebraic manipulation, $\mathscr{L}_{\text{tot}}$ can be represented as

$$\begin{aligned}\mathscr{L}_{\text{tot}} = &- e\frac{\varepsilon_{3ab}(\pi^a\partial_\mu\pi^b)A_\mu}{(1+\lambda^2\pi^2)^2} + \frac{2f\lambda^2}{3e}\varepsilon_{\mu\nu\rho\sigma}A_\mu\varepsilon_{abc}D_\nu\pi^a D_\rho\pi^b D_\sigma\pi^c \\ &- \frac{f}{8}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\frac{\pi^0}{1+\lambda^2\pi^2} + \frac{f}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\nu\rho}A_\sigma\frac{(\partial_\mu\pi^0)}{(1+\lambda^2\pi^2)^2}.\end{aligned} \tag{7.18}$$

Here (7.18) should be supplemented by the prescription that the amplitudes for the processes are to be determined from $\mathscr{L}_{\text{tot}}$ by the diagrams in the "tree" (pole) approximation.

Equations (7.13)–(7.18) were derived in[112]. The corresponding Lagrangian for processes with neutral pions was obtained correctly in[74].

In[113] the total phenomenological Lagrangian was constructed by using the transformation properties with respect to the group SU(2) × SU(2). When the electromagnetic field is incorporated in the phenomenological pion Lagrangian of the non-linear σ model, the symmetry group SU(2) × SU(2) is violated in a completely determined way, and this enables us to calculate the variation of the total Lagrangian under a chiral transformation (a transformation from the group SU(2) × SU(2)). The Lagrangian of[113] coincides with (7.18) with an accuracy up to a canonical transformation of the pion field. The general problem of soft pion production by external axial-vector and vector currents was solved in[72], where, in particular, a Lagrangian was obtained for the processes $\gamma \to (2n+1)\pi$ and $\gamma\gamma \to (2n+1)\pi$ that differs from (7.18) by a unitary transformation on the pion field.

d) **Suppression of the processes $\gamma\gamma \to (2n+1)\pi^0$ with $n \geq 1$.** We have already pointed out (see Sec. 6) that the $\gamma\gamma \to 3\pi^0$ amplitude vanishes at $\mu^2 = 0$. This assertion, however, has a more general character and applies to all the processes $\gamma\gamma \to (2n+1)\pi^0$ with $n \geq 1$[75]. (We note that the processes $\gamma \to (2n+1)\pi^0$ are also forbidden. This prohibition is strict and is connected with the conservation of charge parity).

The Lagrangian (7.1) for neutral pions ($\pi^a = \delta_{3a}\pi^0$) with $\mu^2 = 0$ can be rewritten in the form

$$\bar{\psi}(i\hat{\partial} - m)\psi + \frac{G_A}{G_V}\bar{\psi}\gamma_\mu\gamma_5\tau_3\psi\partial_\mu \operatorname{arctg}\lambda\pi^0 + \frac{1}{2\lambda^2}(\partial_\mu \operatorname{arctg}\lambda\pi^0)^2. \tag{7.19}$$

Redefining the π^0 meson field as

$$\tilde{\pi}^0 = \frac{1}{\lambda}\operatorname{arctg}\lambda\pi^0, \tag{7.20}$$

we arrive at the Lagrangian

$$\bar{\psi}(i\hat{\partial} - m)\psi + \lambda\frac{G_A}{G_V}\bar{\psi}\gamma_\mu\gamma_5\tau_3\psi\partial_\mu\tilde{\pi}^0 + \frac{1}{2}(\partial_\mu\tilde{\pi}^0)^2. \tag{7.21}$$

It is obvious that the Lagrangian (7.21), after including the electromagnetic field, yields, to lowest (second) order in the momenta, the unique non-vanishing $\pi^0 \to 2\gamma$ amplitude described by the diagram in Fig. 22a.

8. The $\gamma\gamma \to \pi\pi$ AMPLITUDE

a) **Phenomenological structure.** The $\gamma\gamma \to \pi\pi$ amplitude contains the contributions of the pion pole diagrams (Fig. 24) and a non-pole (contact) part. At small particle momenta $q_i \sim p_i \sim \mu$, we may expect that the contact part is slowly varying and that it can be expanded in a series. We retain in this part the leading terms (of zero order in the momenta) and the first correction (the terms of second order in the momenta). We must then also include in the pole diagrams the next correction in the momenta at the electromagnetic vertex of the pion (see Sec. 3). When this is done, the amplitude for the process can be written in the form[114]

$$T^{ab}_{\nu\mu}(p_1, p_2; q_1, q_2) = P^{ab}_{\nu\mu} + K^{ab}_{\nu\mu} + R^{ab}_{\nu\mu}, \tag{8.1}$$

where $P^{ab}_{\nu\mu}$ is the contribution of the pole diagrams in Fig. 24:

$$\begin{aligned}P^{ab}_{\nu\mu} = -e^2(\delta_{ab} - \delta_{3a}\delta_{3b})\Big[&\frac{\mathscr{T}_\nu(p_1, q_1 - p_1)\mathscr{T}_\mu(q_2 - p_2, p_2)}{(p_1 - q_1)^2 - \mu^2} \\ &+ \frac{\mathscr{T}_\mu(p_1, q_2 - p_1)\mathscr{T}_\nu(q_1 - p_2, p_2)}{(p_1 - q_2)^2 - \mu^2}\Big]\end{aligned} \tag{8.2}$$

The vertex functions $T_\nu(p, p')$ are defined in (3.8).

In Eq. (8.1), $K_{\nu\mu}$ and $R_{\nu\mu}$ are contact terms, where $K_{\nu\mu}$ is chosen in such a way as to define the contribution of the pole diagrams up to a gauge-invariant structure, while $R^{ab}_{\nu\mu}$ is a manifestly transverse function in the photon momenta ($q_{1\nu}R^{ab}_{\nu\mu} = R^{ab}_{\nu\mu}q_{2\mu} = 0$). To second order in the momenta, the unique transverse structure has the form

$$R^{ab}_{\nu\mu} = 2e^2[\beta(\delta_{ab} - \delta_{3a}\delta_{3b}) + \beta_0\delta_{3a}\delta_{3b}](q_1q_2\delta_{\nu\mu} - q_{1\mu}q_{2\nu}); \tag{8.3}$$

here β and β_0 are arbitrary constants. The isotopic structure of $R^{ab}_{\nu\mu}$ is determined by the conservation of G-parity, from which it follows that both photons in the process $\gamma\gamma \to \pi\pi$ are either isoscalar or isovector.

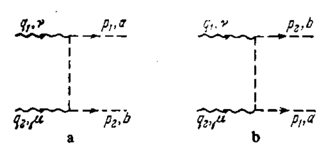

FIG. 24

The tensor $K^{ab}_{\nu\mu}$ in (8.1), with an accuracy up to second order in the momenta, is uniquely determined by the Ward identities:

$$q_{1\nu}T^{ab}_{\nu\mu}(p_1, p_2; q_1, q_2) = -e^2(\delta_{ab}-\delta_{3a}\delta_{3b})\left[\frac{p_1^2-\mu^2}{(p_1-q_1)^2-\mu^2}\mathscr{T}_\mu(q_2-p_2, p_2) + \frac{p_2^2-\mu^2}{(p_2-q_1)^2-\mu^2}\mathscr{T}_\mu(q_2-p_1, p_1)\right]. \quad (8.4)$$

The identity (8.4) is a consequence of the conservation of the electromagnetic current. It follows from (8.4) that $K^{ab}_{\nu\mu}$ must be chosen in the form

$$K^{ab}_{\nu\mu} = 2e^2(\delta_{ab}-\delta_{3a}\delta_{3b})\left\{\delta_{\nu\mu}+\frac{\langle r^2\rangle}{6}[(q_1^2+q_2^2)\delta_{\nu\mu}-q_{1\nu}q_{1\mu}-q_{2\mu}q_{2\nu}]\right\}. \quad (8.5)$$

The sum $P_{\nu\mu} + K_{\nu\mu} + R_{\nu\mu}$ in (8.1) will then satisfy the relation (8.4).

After regrouping the terms, the final result for $T_{\nu\mu}$ in (8.1) with $p_i^2 = \mu^2$ can be represented in the form

$$T^{ab}_{\nu\mu} = e^2(\delta_{ab}-\delta_{3a}\delta_{3b})F(q_1^2)F(q_2^2) \times\left[\frac{(2p_1-q_1)_\nu(2p_2-q_2)_\mu}{(p_1-q_1)^2-\mu^2}+\frac{(2p_1-q_2)_\mu(2p_2-q_1)_\nu}{(p_1-q_2)^2-\mu^2}+2\delta_{\nu\mu}\right]+R^{ab}_{\nu\mu}, \quad (8.6)$$

where $F(q^2)$ is the form factor (3.3), and $R^{ab}_{\nu\mu}$ is defined in (8.3).

b) **Relation to the polarizability of the pion**[114,115]. Let us consider the $\gamma\gamma \to \pi\pi$ amplitude at small q_i^2 and $p_i^2 = \mu^2$ ($q_1^2 \sim q_2^2 \ll \mu^2$). In this region, it takes the simpler form

$$e^2(\delta_{ab}-\delta_{3a}\delta_{3b})\left[-\frac{(2p_1-q_1)_\nu(2p_2-q_2)_\mu}{2p_1q_1}-\frac{(2p_2-q_1)_\nu(2p_1-q_2)_\mu}{2p_1q_2}+2\delta_{\nu\mu}\right]+R^{ab}_{\nu\mu}. \quad (8.7)$$

Equation (8.7) corresponds to the following phenomenological Lagrangian for the interaction of the pion with an external field:

$$\mathscr{L} = -ieA_\mu\left(\frac{\partial\varphi^*}{\partial x_\mu}\varphi-\frac{\partial\varphi}{\partial x_\mu}\varphi^*\right)+2e^2A_\mu^2\varphi^*\varphi-\frac{e^2}{2}\left(\beta\varphi^*\varphi+\frac{1}{2}\beta_0\varphi_0\varphi_0\right). \quad (8.8)$$

Here φ and φ_0 are the quantized fields of the π^- and π^0 mesons. In the nonrelativistic approximation, Eq. (8.8) corresponds to the following effective single-particle hamiltonian for the π^- meson in an electromagnetic field:

$$\mathscr{H}(\pi^-) = \frac{(p-eA)^2}{2\mu}-\frac{1}{2}\alpha_\pi E^2, \quad (8.9)$$

where

$$\frac{1}{2}\alpha_\pi = \frac{e^2\beta}{2\mu}. \quad (8.10)$$

The parameter α_π is known as the polarizability of the pion (the dipole moment of the pion in an external electric field $\mathbf{E}$ is $\alpha_\pi\mathbf{E}$).

Corresponding to this, we obtain for the neutral pion

$$\mathscr{H}(\pi^0) = \frac{p^2}{2\mu}-\frac{1}{2}\alpha_\pi^0 E^2, \quad (8.11)$$

where

$$\frac{1}{2}\alpha_\pi^0 = \frac{e^2\beta_0}{2\mu} \quad (8.12)$$

is the polarizability of the π^0 meson.

We note that the relation between the polarizability of the nucleon and the parameters characterizing the Compton scattering amplitude at low energies was discussed in[116-118]. The polarizability of the nucleon was studied in detail in[118].

c) **The polarizability of the π^0 meson.** Let us consider $T^{ab}_{\nu\mu}(p_1, p_2; q_1, q_2)$ as $p_1 \to 0$. The appropriate limit can

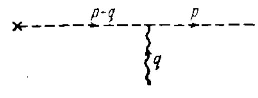

FIG. 25

be evaluated by using the reduction formulas and the PCAC condition (2.2):

$$T^{ab}_{\nu\mu}(0, p_2; q_1, q_2) = -\frac{e^2\varepsilon_{3ac}}{F_\pi}[\tau^{bc}_{\nu\mu}(p_2, q_2)+\tau^{bc}_{\mu\nu}(p_2, q_1)], \quad (8.13)$$

where

$$\tau^{bc}_{\nu\mu}(p, q) = i\int dx\, e^{-iqx}\langle\pi^b(p)|T(a^c_\nu(0)v^3_\mu(x))|0\rangle. \quad (8.14)$$

Equation (8.13) (with $p_2^2 = \mu^2$) follows from the chain of equalities

$$\begin{aligned} T^{ab}_{\nu\mu}(p_1, p_2; q_1, q_2) &= \frac{-p_1^2+\mu^2}{\mu^2F_\pi}\int dx\,dy\, e^{-iq_2x}e^{ip_1y}\langle\pi^b(p_2)|T(\partial_\alpha a^a_\alpha(y)j_\nu(0)j_\mu(x))|0\rangle \\ &= \frac{-p_1^2+\mu^2}{\mu^2F_\pi}\Big\{-ip_{1\alpha}\int dx\,dy\, e^{-iq_2x}e^{ip_1y}\langle\pi^b(p_2)|T(a^a_\alpha(y)j_\nu(0)j_\mu(x))|0\rangle \\ &\quad -\int dx\,dy\, e^{-iq_2x}e^{ip_1y}\langle\pi^b(p_2)|T(\delta(y_0)[a^a_0(y), j_\nu(0)]j_\mu(x) \\ &\quad +\delta(y_0-x_0)[a^a_0(y), j_\mu(x)]j_\nu(0))|0\rangle\Big\}. \end{aligned} \quad (8.15)$$

Putting $p_1 = 0$ in (8.15) and evaluating the current commutators, we arrive at the relation (8.13), if we make use of the fact that $\langle\pi^b|(T(a^c_\nu j_\mu)|0\rangle = e\langle\pi^b|T(a^c_\nu v^3_\mu)|0\rangle$ by virtue of the conservation of G-parity.

A more phenomenological expression for $\tau_{\nu\mu}(p, q)$ at $q^2 = 0$, $p^2 = \mu^2$, with an accuracy up to second order in the momenta, is of the form

$$\tau^{bc}_{\nu\mu}(p, q) = \varepsilon_{3bc}\left[F_\pi\frac{2p_\mu(p-q)_\nu}{(p-q)^2-\mu^2}-F_\pi\delta_{\mu\nu}-h_A(pq\delta_{\mu\nu}-p_\mu q_\nu)\right]. \quad (8.16)$$

(It is intended that at $q^2 = 0$ the matrix element $\tau_{\nu\mu}$ is multiplied by the photon polarization vector $\epsilon_\mu(q)$, so that the terms proportional to q_μ are not written in (8.16)). We have explicitly separated the contribution of the pion pole graph (see Fig. 25, where the cross denotes the vertex of the axial-vector current: $\langle\pi^a(p)|a^b_\mu(0)|0\rangle = -ip_\mu\delta_{ab}F_\pi$). The contact term $-F_\pi\delta_{\mu\nu}$ in (8.16) adds a gauge-invariant structure to the contribution of the pole graph and guarantees the equality $q_\mu\tau^{bc}_{\nu\mu}(p, q) = -\epsilon_{3bc}F_\pi p_\nu$, which is a consequence of the conservation of the vector current and the commutation relation (2.6).

Equations (8.13) and (8.16) imply, in particular, that $T^{ab}_{\nu\mu} \sim \delta_{ab} - \delta_{3a}\delta_{3b}$, and, if (8.3) is taken into account, it follows from this that $\beta_0 = 0$. Equation (8.12) then indicates that

$$\alpha_\pi^0 = 0. \quad (8.17)$$

The condition (8.17) means that the polarizability of the π^0 meson vanishes to second order in the momenta in $T^{ab}_{\nu\mu}$, i.e., $\alpha_\pi^0 \sim \alpha_\pi\mu^2/m_\rho^2 \ll \alpha_\pi$, where $m_\rho \gg \mu$ is the characteristic range of variation of the hadronic amplitudes.

d) **Relation to the decay $\pi \to e\nu\gamma$. Calculation of the polarizability of the π^- meson**[114,115]. The $\pi^- \to e\nu\gamma$ decay amplitude has the form

$$T_\mu = ieG_V\cos\theta[M_{\mu\nu}l_\nu+F_\pi\bar{u}_e(k_e)\gamma_\mu(\hat{k}_e+\hat{q}-m_e)^{-1}\hat{p}(1-\gamma_5)u_\nu(k_\nu)], \quad (8.18)$$

$$M_{\mu\nu} = ih_V\varepsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta-F_\pi\left(\delta_{\mu\nu}+\frac{p_\mu(p-q)_\nu}{pq}\right)-h_A(pq\delta_{\mu\nu}-p_\mu q_\nu), \quad (8.19)$$

where G_V is the weak interaction constant, θ is the Cabibbo angle, $l_\nu = \bar{u}_e\gamma_\nu(1-\gamma_5)u_\nu$ is the leptonic current, and p and q are the momenta of the pion and photon. The matrix element $M_{\mu\nu}$ can be written in the form (cf.[119])

$$\sqrt{2}\, eM_{\mu\nu} = -\int dx\, e^{iqx} \langle 0 | T(j_\mu(x),\ v_\nu^{(+)}(0) - a_\nu^{(+)}(0)) | \pi^-(p)\rangle. \quad (8.20)$$

The contribution of the weak vector current in (8.20) after isotopic flip can be reduced to the $\pi^0 \to 2\gamma$ decay amplitude[120]. The parameter h_V in (8.19) is defined here as

$$h_V = -f/2e^2, \quad (8.21)$$

where f is defined in (5.1).

The contribution of the weak axial-vector current in (8.20) after isotopic flip reduces to $e\tau^{12}_{\nu\mu}(p, q)$ (see Eqs. (8.14) and (8.16)). This fact was the reason for choosing a single constant h_A in (8.16) and (8.19).

Using (8.1) and (8.16), it follows from the relation (8.13) with $q_1^2 = q_2^2 = 0$ that

$$\beta = \frac{1}{F_\pi} h_A. \quad (8.22)$$

By employing (8.10), we obtain an expression for the polarizability of the π^- meson:

$$\alpha_\pi = \frac{e^2}{\mu F_\pi} h_A. \quad (8.23)$$

The parameter h_A can be obtained by measuring the differential probability for the decay $\pi \to e\nu\gamma$[121]. In[121] data on the total probability was used to obtain two solutions for the ratio $\gamma = h_A/h_V$:

$$\gamma = 0.4 \text{ or } -2.1. \quad (8.24)$$

Using the value $\gamma = 0.4$ (this value is closer to the prediction of the vector dominance model[119]: $\gamma_{VDM} = F_\pi/2m_\rho^2 h_V \approx 0.55$) and Eqs. (8.21) and (8.23), we obtain

$$\alpha_\pi \approx \frac{0.2\alpha}{\mu^3} \approx 4\cdot 10^{-42}\,\mathrm{cm}^3 = 4\cdot 10^{-3}\ \mathrm{F}^3 \quad (8.25)$$

The polarizability of the pion is found to be almost 5 times the value of the corresponding quantity for the proton[122]7). (The polarizability of the proton, defined in analogy with α_π by the expression for the effective energy of interaction with an external field $\mathscr{H}_{int} = -1/2\alpha_p E^2$, has the value $\alpha_p = 0.9 \times 10^{-3}$ F^3). However, it should be stressed that the values of γ in (8.24) are very sensitive to variations in the constant h_V[114]. In this sense, a change in the experimental data on the pion lifetime or, say, the absence of the relation (8.21) between h_V and f8) could significantly alter our estimate of the value of α_π in (8.25).

An interesting fundamental possibility of directly measuring the polarizability of the π^- meson was discussed in[123–125]. It was proposed to study the level shifts in π-mesonic atoms due to the contribution of the interaction $-(1/2)\alpha_\pi E^2$, where $E = -Ze/r^2$ is the field intensity of the nucleus at the pion orbit. The value of the polarizability (8.25) leads to an energy shift ~ 2 eV in the 6h-5g transition in[81] Tl. A measurement of such effects would require a relative accuracy $\sim 10^{-5}$ in the determination of the transition energy. At the present time, an accuracy $\sim 6\times 10^{-5}$ (see[126]) has already been reached for this transition9).

We note that information on the sign of α_π can be deduced from (8.23). In fact, by expressing α_π directly in terms of the $\pi^0 \to 2\gamma$ decay constant by means of (8.21) and (8.23), we obtain

$$\alpha_\pi = -\gamma \frac{f}{2\mu F_\pi}.$$

Since $f/F_\pi < 0$ (see (6.34) and (6.37)), we have $\alpha_\pi > 0$ for $\gamma = 0.4 > 0$. This corresponds to an increase in the level spacing in π-mesonic atoms.

We emphasize that the sign of the polarizability cannot be determined a priori in a relativistic theory. In nonrelativistic quantum mechanics, the polarizability of a particle can be represented in the form

$$\alpha_\pi = \frac{e^2}{\mu}\frac{\langle r^2\rangle}{3} + \sum_{n,j} \frac{1}{3} \frac{|\langle \pi^-(0)| \int j_0(\mathbf{x}, 0)\, x_j\, d^3x\, | n\rangle|^2}{E_n - \mu},$$

where the sum Σ_n extends over the states with zero momentum p = 0 and $E_n > \mu$, and $\langle r^2\rangle^{1/2}$ is the radius of the pion. It follows from this formula that $\alpha_\pi > 0$. However, the picture is changed if allowance is made for relativistic effects (pair production). The sum Σ_n must be replaced by the expression

$$\sum_n{}' = \sum_n - S_{vac},$$

where S_{vac} is the contribution of the disconnected vacuum diagrams, corresponding to a sum of the form

$$S_{vac} \sim \sum_{n,n'} \frac{|\langle \pi^- | n\rangle|^2 |\langle 0| \int j_0(\mathbf{x}, 0)\, x_j\, d^3x\, | n'\rangle|^2}{E_n + E_{n'} - \mu}.$$

Let $|n\rangle$ and $|n'\rangle$ be states that include a nucleon-antinucleon pair ($|n\rangle = |p, \bar{p}\rangle$, $|n'\rangle = |p', \bar{p}'\rangle$, where p and p′ are the momenta of the particles, and $\bar{p}$ and $\bar{p}'$ are the momenta of the antiparticles). If $p \neq p'$, the contribution corresponding to S_{vac} is also contained in Σ_n and cancels in the difference $\Sigma_n - S_{vac}$. But if $p = p'$, the corresponding contribution is contained in S_{vac} but is absent in Σ_n, owing to the Pauli principle. As a result, the non-cancelled contribution of the states with p = p′ in S_{vac} leads to a negative component in the expression for the polarizability, as compared with the manifestly positive expression corresponding to nonrelativistic mechanics.

A straightforward analysis in relativistic perturbation theory (in the theory with γ_5 coupling) shows that the polarizability of the π^0 meson is negative, while it is equal to zero in the σ model. The polarizability of the π^- meson is positive, owing to the term $\langle r^2\rangle/3$. The contribution corresponding to $\Sigma_n - S_{vac}$ separately is also found to be negative.

e) The decay $\pi \to e\nu e^+e^-$. We see that the decay $\pi \to e\nu\gamma$ is of interest from the point of view of measuring the polarizability of the pion. It is significant that one can determine the value of the constant h_A by studying the electron and photon spectra in this decay and thus choose between the two solutions in (8.24) (see[127]).

In connection with the discussion of Sec. 3, we note that the process $\pi \to e\nu\gamma$ with a virtual photon (the decay $\pi \to e\nu e^+e^-$) is also of interest from the point of view of the possible measurement of the radius of the pion[128]. The $\pi \to e\nu e^+e^-$ decay amplitude has the form

$$T = \frac{e}{q^2}\bar{u}_e\gamma_\mu u_e T_\mu - \text{permutation of the electrons}, \quad (8.26)$$

where T_μ is defined in (8.18), in which one must use Eq. (8.20) for $M_{\mu\nu}$ with $q^2 \neq 0$. By using in (8.20) the condition of conservation of the vector current and partial conservation of the axial-vector current, as well as the commutation relations (2.5), we obtain

$$q_\mu M_{\mu\nu} = -F_\pi p_\nu, \tag{8.27}$$

$$(p-q)_\nu M_{\mu\nu} = F_\pi \left[p_\mu + \frac{\mu^2}{(p-q)^2-\mu^2} \mathcal{T}_\mu (p, q-p) \right], \tag{8.28}$$

where $\mathbf{T}_\mu$ is the vertex function (3.8). With an accuracy up to second order in the momenta, it follows from (8.27) and (8.28) that[119,128]

$$\begin{aligned} M_{\mu\nu} &= ih_V \varepsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta - F_\pi \left[\frac{(p-q)_\nu (2p-q)_\mu}{2pq-q^2} \left(1 + \frac{\langle r^2 \rangle}{6} q^2\right) + \delta_{\mu\nu} \right] \\ &- h_A [q(p-q)\delta_{\mu\nu} - q_\nu (p-q)_\mu] - \frac{\langle r^2 \rangle}{6} F_\pi (2q^2 \delta_{\mu\nu} - (p-q)_\nu q_\mu - 2q_\nu q_\mu). \end{aligned} \tag{8.29}$$

Thus, the $\pi \to e\nu e^+e^-$ decay amplitude is completely determined by a knowledge of the $\pi^0 \to 2\gamma$ decay constant (see (8.21)), the polarizability of the pion (see (8.23)) and the pion radius $\langle r^2 \rangle$.

f) **Relation to the current spectral functions.** We note that there is a relation between the parameter β in (8.3) and the spectral functions of the vector and axial-vector currents. This relation appears when we consider the quantity $T^{ab}_{\nu\mu}(p_1, p_2; q_1, q_2)$ as $p_1 \to 0$ and $p_2 \to 0$. By making use of the PCAC condition (2.2) and the commutation relations (2.3)–(2.5), we can (in analogy with the way in which we calculated the amplitude $T_{\nu\mu}(0, p_2; q_1, q_2)$ in (8.15)) derive the expression[129,130][10)]

$$\begin{aligned} &T^{ab}_{\nu\mu}(0, 0; q, -q) \\ &= \frac{2ie^2}{F_\pi^2} (\delta_{ab} - \delta_{3a}\delta_{3b}) \int e^{iqx} \langle 0 | T(a^3_\nu(0) a^3_\mu(x) - v^3_\nu(0) v^3_\mu(x)) | 0 \rangle dx. \end{aligned} \tag{8.30}$$

By using the spectral representation for the Green's functions of the currents in (8.30)[131] and exploiting the relation

$$\begin{aligned} &T^{ab}_{\nu\mu}(0, 0; q, -q) = \\ &= 2e^2 (\delta_{ab} - \delta_{3a}\delta_{3b}) \left[\delta_{\nu\mu} - \frac{q_\nu q_\mu}{q^2-\mu^2} + \left(\frac{\langle r^2 \rangle}{3} - \beta\right) (q^2 \delta_{\nu\mu} - q_\nu q_\mu) \right], \end{aligned} \tag{8.31}$$

which follows from the representation (8.1) (with $\beta_0 = 0$), we obtain

$$\beta = \frac{\langle r^2 \rangle}{3} + \frac{1}{F_\pi^2} \int \frac{\rho^A(\varkappa^2) - \rho^V(\varkappa^2)}{\varkappa^4} d\varkappa^2, \tag{8.32}$$

where ρ^A and ρ^V are the spectral functions of the currents:

$$\rho^V(p^2) = -\frac{(2\pi)^3}{3} \sum_n \delta(p_n - p) |\langle 0 | v^3_\nu(0) | n \rangle|^2, \tag{8.33}$$

$$\rho^A(p^2) = -\frac{(2\pi)^3}{3} \sum_n \delta(p_n - p) |\langle 0 | a^3_\nu(0) | n \rangle|^2, \tag{8.34}$$

and Σ_n in (8.34) does not include the state with a single pion.

In principle, the functions ρ^V and ρ^A can be measured in the reactions $e^+e^- \to 2\pi$ + (everything else) and $e^+e^- \to \pi$ + (everything else)[132].

From (8.22) and (8.32) there follows an expression for the parameter h_A in terms of the current spectral functions, which was first derived in[119]. The utilization of the VDM in Eq. (8.32) leads to the value

$$\beta = \frac{1}{2m_\rho^2}, \tag{8.35}$$

where m_ρ is the mass of the ρ meson.

g) **Quantitative estimates.** The differential cross section. From (8.22) and (8.24) with $\gamma = 0.4$, there follows a value for the parameter β:

$$\beta \approx \frac{1.2 \cdot 10^{-2}}{\mu^2}. \tag{8.36}$$

We note that $\beta \sim 1/m_\rho^2$, where m_ρ is the mass of the ρ meson. The relative contribution of the term of second order in the momenta in (8.1) is then of order $\sim \mu^2/m_\rho^2$, which confirms the expected range of the effective expansion parameter in (8.1). An estimate of the terms of fourth order in the momenta in the amplitude $T^{ab}_{\nu\mu}$, which are associated with exchange of the ρ meson, indicates that their range is $\sim 10^{-3} \sim (\mu^2/m_\rho^2)^2$ (see[114]). Thus, we may expect that Eq. (8.1), with the value of β given by (8.22), actually determines the amplitude for the process $\gamma\gamma \to \pi\pi$ with an accuracy up to radiative corrections.

It follows from (8.1) and (8.36) that the amplitude $T^{ab}_{\nu\mu}$ at low momenta is determined by the contribution of the pole graphs with an accuracy of the order of a few percent. This fact has been pointed out in a number of papers (see, e.g.,[133,134][11)]).

The total cross section for the process $\gamma\gamma \to \pi^+\pi^-$ with real photons (the $\gamma\gamma \to 2\pi^0$ cross section is equal to zero as a consequence of (8.1), since $\beta_0 = 0$) is of the form

$$\sigma = \sigma_B + \sigma', \tag{8.37}$$

where σ_B is the contribution of the terms (8.2) and (8.5) (see[136], p. 644; this contribution corresponds to the Born approximation), and σ' is the correction due to the term (8.3):

$$\sigma' = \frac{4\pi\alpha^2}{s} (\mu^2\beta) \ln \frac{1+R(s)}{1-R(s)}, \quad R(s) = \sqrt{1 - \frac{4\mu^2}{s}}; \tag{8.38}$$

here $s = (q_1 + q_2)^2$.

When $4\mu^2/s \sim 1$, the cross section σ_B is of order 10^{-29}–10^{-30} cm^2. For the value of β from (8.36), we obtain

$$\frac{\sigma'}{\sigma_B} \sim 2 \cdot 10^{-2}. \tag{8.39}$$

The cross section for the process $\gamma\gamma \to \pi^+\pi^-$ in the region $s \sim t \sim \mu^2$ can be measured in the reaction $e^+e^- \to e^+e^-\pi^+\pi^-$ with colliding beams of high energy, when the diagram in Fig. 26 dominates. The contribution of the diagram in Fig. 26 with the block $\gamma\gamma \to \pi\pi$ was calculated in the pole approximation in[10,133]. In the leading, logarithmic approximation, the cross section for the reaction $e^+e^- \to e^+e^-\pi^+\pi^-$ has a simple form (cf.[9]):

$$\frac{d\sigma}{ds} = \left(\frac{\alpha}{\pi}\right)^2 \frac{\sigma_{\gamma\gamma\to\pi\pi}(s)}{s} \left(\ln \frac{4E^2}{s} \ln^2 \frac{4E^2\mu^2}{sm_e^2} - \frac{1}{3} \ln^3 \frac{4E^2}{s} \right); \tag{8.40}$$

here $2E$ is the total energy in the c.m.s., m_e is the electron mass, and $\sigma_{\gamma\gamma \to \pi\pi}$ is the cross section for the process $\gamma\gamma \to \pi^+\pi^-$ with real photons. Estimating the cross section (8.40) for $s \sim 4\mu^2$ and $2E \approx 7$ GeV, we find $\sigma \sim 10^{-33}$–10^{-34} cm^2. Eq. (8.40) is valid in the asymptotic region, where $\ln(4E^2/s) \gg 1$, and it has an accuracy of $\sim 30\%$ for $2E \sim 7$ GeV.

It has been proposed in a number of papers (see[12,133,137]) to study the reaction $\gamma\gamma \to \pi\pi$ at low and moderate energies in connection with the extraction of information on the $\pi\pi$ scattering phase shifts. We note that, within the framework of the hypotheses of Sec. 2 (see also Sec. 4), pion rescattering gives a contribution

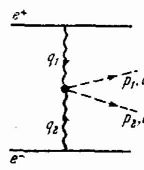

FIG. 26

of second order in the momenta in the imaginary part of the $\gamma\gamma \to 2\pi$ amplitude and, accordingly, a fourth-order contribution to the total cross section (see [114]). Therefore, for $s \sim 4\mu^2$, the correction σ' (see (8.38)) should give a larger contribution than $\pi\pi$ rescattering.

h) **The process $\gamma\gamma \to \pi\pi$ with highly virtual photons.** Equation (8.30) can be used to derive an expression for the amplitude $T^{ab}_{\nu\mu}(p_1, p_2; q_1, q_2)$ in the region $q_1^2 \sim q_2^2 \gg \mu^2$, $p_i \sim \mu$. It follows from (8.30)[70,134] that in this region

$$T^{ab}_{\nu\mu} \approx 2e^2(\delta_{ab} - \delta_{3a}\delta_{3b})\left(\delta_{\nu\mu} - \frac{q_\nu q_\mu}{q^2}\right) R(q^2), \tag{8.41}$$

where

$$R(q^2) = \frac{1}{F_\pi^2} \int \frac{d\varkappa^2}{\varkappa^2 - q^2}\,[\rho^V(\varkappa^2) - \rho^A(\varkappa^2)]. \tag{8.42}$$

We note that the amplitude for the process $\gamma\gamma \to 2\pi^0$ is equal to zero as a consequence of (8.41). At large q_i, it is far from obvious that the pion momenta in $T^{ab}_{\nu\mu}$ can be neglected, At any rate, we must require the validity of the condition $p_i q \ll m_\rho^2$, where m_ρ is the characteristic range of variation of the hadronic amplitudes. Since we have at the same time $q_1^2 \sim q_2^2 \gg \mu^2$, the range of applicability of Eq. (8.41) may be rather narrow.

In principle, it is possible to measure the amplitude (8.41) in the reaction $e^+e^- \to e^+e^-\pi^+\pi^-$ (see Fig. 26). The bremsstrahlung (single-photon) mechanism of $\pi^+\pi^-$ pair production can be separated, since the corresponding graphs do not interfere with the diagram in Fig. 26 and are readily calculated. The contribution of the graph in Fig. 26 to the differential cross section at high lepton energies ($\epsilon^2 \gg q_i^2$) is

$$d\sigma = \frac{16\alpha^4}{\pi}\,\frac{R^2(q^2)}{q^6}\,\frac{dq^2}{q^2}\,(\omega_+^2 - \mu^2)^{1/2}(\omega_-^2 - \mu^2)^{1/2}\,d\omega_+ d\omega_-, \tag{8.43}$$

where $\omega_\pm$ are the meson energies. Putting $dq^2 \sim q^2 \sim m_\rho^2$, $\omega_\pm \sim d\omega_\pm \sim \mu$ and approximating $R(q^2)$ by the contribution of the ρ and A_1 mesons (see, e.g.,[136]; here $\rho^V = g_\rho^2\delta(\kappa^2 - m_\rho^2)$, $\rho^A = g_\rho^2\delta(\kappa^2 - m_\rho^2)$, $m_A^2 = 2m_\rho^2$, $g_\rho^2 = 2F_\pi^2 m_\rho^2$ and $R(q^2) \approx 2m_\rho^4[(q^2 - m_\rho^2)(q^2 - 2m_\rho^2)]^{-1}$), we obtain $\sigma \sim 10^{-38}$ cm^2. Unfortunately, the cross section is very small.

9. THE PROCESSES $\gamma \to (2n)\pi$ AND $\gamma\gamma \to (2n)\pi$ ($n > 1$)

a) **The phenomenological Lagrangian.** To lowest order in the momenta, these processes are described by the contributions of the pion pole diagrams. The calculation of the latter presupposes a knowledge of the amplitudes for $2\pi \to 2\pi$, $2\pi \to 4\pi$, etc., which can be evaluated in the framework of current algebra and involve no arbitrary parameters if the condition (2.11) is satisfied. All of them are contained in the phenomenological Lagrangian (7.1). To lowest order in the momenta, all of the amplitudes for $\gamma \to (2n)\pi$ and $\gamma\gamma \to (2n)\pi$ are described by the phenomenological Lagrangian (7.18). However, the expressions obtained on the basis of (7.18) may be rather remote from the true physical amplitudes, since they are valid only in the limit in which the momentum of each of the mesons tends to zero. (Analogous remarks also hold, of course, for the processes $\gamma \to (2n+1)\pi$ and $\gamma\gamma \to (2n+1)\pi$ with a large number of pions). The comparison with experiment is a very difficult problem here, since it presupposes a prior study of the extrapolation formulas that

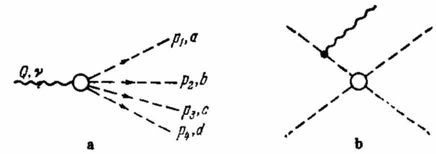

FIG. 27

would make it possible to continue results obtained at low momenta into the physical region.

We shall consider the processes $\gamma \to 4\pi$ and $\gamma\gamma \to 4\pi$ in somewhat greater detail, since in these cases we may still hope that allowance for the lowest and the next order in the expansion in the momenta is sufficient for a continuation of the amplitude into the physical region.

b) **The $\gamma \to 4\pi$ amplitude**[12)]. It follows from the conservation of G-parity that the photon in this process is isovector. Therefore the general structure of the amplitude has the form (Fig. 27)

$$T_\nu^{3abcd} = -ie\delta_{ab}\varepsilon_{3cd}M_\nu(p_1, p_2, p_3, p_4) + \text{permutations} \tag{9.1}$$

We note that it is not necessary to write any isotopic structures of the type $\delta_{3a}\epsilon_{bcd}$, since $\delta_{3b}\epsilon_{acd} = \delta_{ab}\epsilon_{3cd} + \delta_{bd}\epsilon_{3ac}$.

The function M_ν can be written in the form

$$M_\nu = P_\nu(p_1, p_2, p_3, p_4) + B_\nu(p_1, p_2, p_3, p_4), \tag{9.2}$$

where P_ν is the pole term (see Fig. 27):

$$\begin{gathered} P_\nu(p_1, p_2, p_3, p_4) = \\ = \frac{\mathcal{T}_\nu(p_3, -p_3')}{p_3'^2 - \mu^2} M(p_1, p_2, p_3', p_4) - \frac{\mathcal{T}_\nu(p_4, -p_4')}{p_4'^2 - \mu^2} M(p_1, p_2, p_3, p_4'), \\ p_3' = -(p_1 + p_2 + p_4), \quad p_4' = -(p_1 + p_2 + p_3). \end{gathered} \tag{9.3}$$

B_ν is the contact term, in which we retain the terms of lowest order (linear in the momenta) and of the next order (cubic in the momenta). The function $M(p_1, p_2, p_3, p_4)$ in (9.3) is the $\pi\pi$ scattering amplitude, defined in accordance with (4.2). In the approximation of interest to us, we must take into account in $M(p_1, p_2, p_3, p_4)$ the terms up to fourth order in the momenta, inclusive. The corresponding expression has the form

$$\left.\begin{aligned} M(p_1, p_2, p_3, p_4) &= \alpha - \left(\frac{\beta}{2} + \gamma\right)(\sigma_{13} + \sigma_{14} + \sigma_{23} + \sigma_{24} + 2\mu^2) - \\ &\qquad - \gamma(\sigma_{12} + \sigma_{34} + \mu^2) + \sum_{j=1}^{6} c_j R_j, \\ R_1 &= \sigma_{12}^2 + \sigma_{34}^2 - \mu^4, \quad R_2 = \sigma_{13}^2 + \sigma_{14}^2 + \sigma_{23}^2 + \sigma_{24}^2 - 2\mu^4, \\ R_3 &= \sigma_{12}\sigma_{34}, \quad R_4 = \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}, \\ R_5 &= (\sigma_{12} + \sigma_{34})(\sigma_{13} + \sigma_{14} + \sigma_{23} + \sigma_{24}) - 2\mu^4, \\ R_6 &= (\sigma_{14} + \sigma_{23})(\sigma_{13} + \sigma_{24}) - \mu^4; \end{aligned}\right\} \tag{9.4}$$

here $\sigma_{ij} = p_i p_j$ with $i \neq j$ are the independent invariant variables in the amplitude $M(p_1, p_2, p_3, p_4)$. (In (9.4) we do not write the contribution of the imaginary part due to $\pi\pi$ rescattering, which at $\alpha = 0$ is also of fourth order in the momenta).

The self-consistency condition for the amplitude (9.4), namely $M(p_1, p_2, p_3, p_4) \to 0$ as $p_1 \to 0$ if $p_2^2 = p_3^2 = p_4^2 = \mu^2$, leads to the condition

$$\alpha = 0. \tag{9.5}$$

With the additional assumption (2.11) about the structure of the σ commutator, we also find that $M(0, p_2, p_3, p_4) = 0$ if $p_2^2 = p_4^2 = \mu^2$ and $p_3^2 \neq \mu^2$. This leads to the conditions

$$c_1 + 2c_2 - c_6 = 0, \tag{9.6'}$$

$$\gamma = -2\mu^2 (c_6 + c_8 - 2c_2). \qquad (9.6'')$$

We shall next discuss briefly the consequences of the assumptions A, B, C and (2.11) (see Sec. 2) for the amplitude (9.2).

Gauge invariance (the transversality condition $Q_\nu M_\nu = 0$, where $Q = p_1 + p_2 + p_3 + p_4$) leads to the following constraints on the contact term B_ν in (9.2). The contact term contains two arbitrary constants a_1 and a_2, which appear as the coefficients of the two manifestly transverse structures formed from the momenta. Otherwise, the structure of B_ν is completely determined by specifying the constants which appear in the amplitude M in (9.3) and (9.4). The function of the contact term is to add a transverse structure (on the mass shell) to the expression for the pole term (9.3).

The self-consistency condition, when applied to the amplitude (9.2) ($M_\nu(0, p_2, p_3, p_4) = 0$ if $p_2^2 = p_3^2 = p_4^2 = \mu^2$ with $a = b = 3$, $c \neq d \neq 3$), leads to the condition $a_1 = a_2 = 0$. Thus, we have the very important fact that the process $\gamma \to 4\pi$ up to third order in the momenta is completely determined by the structure of the $\pi\pi$ scattering amplitude and involves no new parameters.

We note also that by making use of current algebra (2.3)–(2.5) in conjunction with the condition (2.2) we can obtain (for $a = b = 3$, $c \neq d \neq 3$)

$$T_\nu(0, p_2, p_3, p_4) \to 0 \text{ при } p_3 \to 0,\ p_2^2 = p_4^2 = \mu^2, \qquad (9.7')$$

$$T_\nu(p_1, p_2, 0, p_4) \to -\frac{ie}{F_\pi^2} \mathscr{T}_\nu(p_2, p_4) \text{ при } p_1 \to 0,\ p_2^2 = p_4^2 = \mu^2, \quad (9.7'')$$

where $T_\nu(p', p)$ is the electromagnetic vertex (3.8). It follows from these conditions, using (9.5), (9.6′) and (9.6″), that

$$\beta = F_\pi^{-2} - 8\mu^2 (c_6 - c_2). \qquad (9.8)$$

It is useful to compare the parameters α, β and γ with those of (4.3), obtained in the lowest approximation. Thus, it follows from (9.6) and (9.8) that $\pi\pi$ scattering up to fourth order in the momenta is described by five arbitrary constants (which can be chosen as c_1, c_2, c_3, c_4 and c_5) and that the same constants appear in the amplitude for $\gamma \to 4\pi$. Moreover, as is usually done in the low-energy technique, it is assumed that the relative contribution of the terms $\sim c_i$ is small ($\sim 4\mu^2/m_\rho^2$), so that in the leading approximation the amplitude T_ν is known and is determined by the constant $\beta = F_\pi^{-2}$.

If no restrictions are imposed on the $\pi\pi$ scattering amplitude other than the possibility of expanding it in a series in the momenta, then in the leading approximation the amplitude for the process $\gamma \to 4\pi$ will have the form (9.1)–(9.3), where for the amplitude M in (9.3) we must employ the expression (4.2), while the contact term B_ν in (9.2) is given by

$$B_\nu(p_1, p_2, p_3, p_4) = -2\gamma (p_3 - p_4)_\nu. \qquad (9.9)$$

Thus, the $\gamma \to 4\pi$ vertex is of interest from the point of view of studying the parameters of $\pi\pi$ scattering.

c) **The process $\gamma\gamma \to 4\pi$.** The amplitude for this process, $T_{\nu\mu}^{abcd}(q_1, q_2; p_1, p_2, p_3, p_4)$, can also be written in the form of a sum of the pole term (Fig. 28b), which is determined with an accuracy up to second order in the momenta by the form of the amplitude $T_\nu(p_1, p_2, p_3, p_4)$ in (9.1), and the contact term (Fig. 28a).

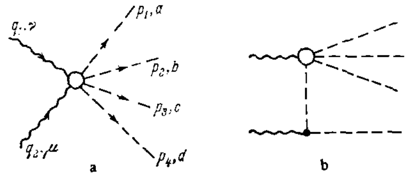

FIG. 28

With an accuracy up to second order in the momenta, the contact term contains a single manifestly transverse structure $\tilde{\beta}(q_1 q_2 \delta_{\nu\mu} - q_{1\mu} q_{2\nu})$ involving an arbitrary parameter $\tilde{\beta}$; in other respects, its form is uniquely determined by the structure of the pole graph, as a consequence of the transversality conditions for the total amplitude: $q_{1\nu} T_{\nu\mu} = 0$ and $p_i^2 = \mu^2$.

By using current algebra, Eqs. (2.3)–(2.5), we can obtain the condition[129]

$$T_{\nu\mu}^{+-+-}(q, -q; 0, 0, 0, 0) = \frac{6ie^2}{F_\pi^4} \int e^{iqx} \langle 0 | T(v_\nu^3(x) v_\mu^3(0) - a_\nu^3(x) a_\mu^3(0)) | 0 \rangle dx, \qquad (9.10)$$

which is sufficient to calculate the parameter $\tilde{\beta}$. This parameter turns out to be related to the radius and polarizability of the pion, if use is made of the spectral representation[131,138] for the Green's function of the currents in (9.10), as well as Eq. (8.32) for the integrals of the spectral functions (cf. the analogous calculations in connection with the process $\gamma\gamma \to \pi\pi$ in Sec. 8).

Thus, the $\gamma\gamma \to 4\pi$ amplitude is also uniquely determined to second order in the momenta by the structure of the $\pi\pi$ scattering amplitude (which appears through the function T_ν, Eq. (9.1), in the pole graph), provided that the radius and polarizability of the pion are known.

In the leading approximation (zero order in the momenta), $T_{\nu\mu}^{abcd}$ is determined entirely by the contribution of the pole graph, which involves the $\pi\pi$ scattering amplitude in the lowest approximation, Eqs. (4.2) and (4.3).

We note that the $\gamma\gamma \to 4\pi^0$ amplitude vanishes in the leading approximation and in the next approximation in the momenta (like the $\gamma\gamma \to 2\pi^0$ amplitude; see Sec. 8).

10. CONCLUSIONS

Thus, the large class of processes $\gamma \to (n)\pi$ ($n \geq 3$) and $\gamma\gamma \to (n)\pi$ ($n \geq 2$) can be described theoretically within the framework of the assumptions formulated in Sec. 2 and can be expressed in terms of a small number of parameters, which are known or measurable in principle in other phenomena. The assumptions of Sec. 2 are used to different extents in calculating the $\gamma(2\gamma) \to (2n)\pi$ and $\gamma(2\gamma) \to (2n+1)\pi$ amplitudes. Thus, the structure of the amplitudes with an odd number of pions is related to the anomalous properties of the triangle diagram (see Fig. 13) and the hypothesis that there is no prohibition of the decay $\pi^0 \to 2\gamma$ in the framework of PCAC. Unfortunately, there is at present no experiment which could serve as a direct test of the resulting relations, although the initial hypotheses have an experimental basis.

The study of the amplitudes for the interaction of soft photons with pions deserves special attention in connection with the ambiguous situation regarding the determination of the parameters of $\pi\pi$ scattering (see

Sec. 4). If the $\pi\pi$ scattering lengths are determined correctly in peripheral experiments, we encounter a serious contradiction, which requires an explanation. In this situation, it is important to have the independent information on $\pi\pi$ scattering that can be obtained in studying photon-meson interactions.

We enumerate below the basic experiments which are required primarily for testing the main results in the area of physics under consideration:

1) More accurate data on the form factor and radius of the pion (see Sec. 3).

2) More accurate data on the $\pi\pi$ scattering amplitude in peripheral experiments, in $K \to 3\pi$ decays and in K_{e4} decay (see Sec. 4).

3) Measurement of the cross section for the reaction $e^+e^- \to 3\pi$ (the determination of the properties of the $\gamma \to 3\pi$ vertex, a test of the basic relation (6.11), a study of the momentum dependence of the $\gamma \to 3\pi$ amplitude, and a test of the extrapolation formula (6.15); see Sec. 6).

4) Measurement of the cross section for the reaction $e^+e^- \to e^+e^- + 3\pi$ (a study of the $\gamma\gamma \to 3\pi$ amplitude, a test of Eqs. (6.13) and (6.14), a determination of the parameter γ, and a study in this way of the structure of the commutator (2.11); see Sec. 6).

5) Measurement of the cross section for the reaction $\pi \to 2\pi$ in the Coulomb field of a nucleus (an independent determination of the $\gamma \to 3\pi$ vertex and a test of Eqs. (6.11), (6.15) and (6.21); see Sec. 6).

6) Measurement of the $\gamma\gamma \to \pi\pi$ amplitude either in the reaction $e^+e^- \to e^+e^-\pi^+\pi^-$ or in the process $\gamma \to \pi^+\pi^-$ in the Coulomb field of a nucleus (a test of the validity of the representation (8.1), which determines the $\gamma\gamma \to \pi\pi$ amplitude with an accuracy of the order of a few percent, and a determination of the polarizability of the pion; see Secs. 8a, 8b and 8g).

7) Measurement of the polarizability of the pion according to the level shifts in π-mesonic atoms (see Sec. 8d).

8) Measurement of the contributions of the vector and axial-vector currents in the decay $\pi \to e\nu\gamma$ (an independent determination of the pion polarizability, and a test of Eq. (8.23); see Sec. 8d).

9) A study of the decay $\pi \to e\nu e^+e^-$ in order to determine the radius and polarizability of the pion (see Sec. 8e).

10) More accurate data on the $\pi^0 \to 2\gamma$ decay probability, and a study of the $\pi^0 \to 2\gamma$ vertex as a function of the virtual masses of the particles (a test of the basic hypothesis that the vertex $\pi^0 \to 2\gamma$ is slowly varying as a function of the momenta, and a more accurate value of the constant f (see (5.5)), which appears as a basic parameter in many relations; see Secs. 5a–5c).

11) Measurement of the $\pi \to 3\pi$ amplitude in the Coulomb field of a nucleus (a study of the properties of the $\gamma \to 4\pi$ amplitude, a test of the validity of the extrapolation formulas (9.1)–(9.3) and (9.9), and a study of the parameters of $\pi\pi$ scattering; see Sec. 9b).

12) Measurement of the cross section for the reaction $e^+e^- \to e^+e^- + 4\pi$ (a study of the properties of the $\gamma\gamma \to 4\pi$ amplitude; see Sec. 9c).

13) A check of the fact that the amplitudes for $\gamma\gamma \to (2n+1)\pi^0$ and $\gamma\gamma \to (2n)\pi^0$ with $n \geq 1$ are suppressed (see Secs. 7d, 8c, 8g and 9).

14) Measurement of the sign of the constant f (see Sec. 6f).

---

1)The right-hand sides of Eqs. (2.3)-(2.6) generally contain schwinger terms proportional to the spatial derivatives of the δ-function, which we do not write. The detailed structure of these terms depends on the model of the strong interaction and the formalism that is chosen. These terms are, as a rule, unimportant in obtaining the physical results. Their role is to add a covariant structure to the expressions for the matrix elements in terms of T-products of currents. Such an addition, however, is unambiguously reproduced in the phenomenological form of the matrix elements. Usually, these terms are simply not considered, since the majority of the physical results are determined by Eqs. (2.3)-(2.6) after integrations with respect to one of the coordinates x or x′.

2)After this review was sent to press, there appeared a paper [139] in which the spectra in K_{e4} decay were analyzed with relatively high statistics. It was found that $\mu a_0 = 0.17 \pm 0.13$, which, within the errors, is an agreement with the prediction of Weinberg [22] (see (4.6)).

On the other hand, the spectra of $K \to 3\pi$ decays obtained in [48] have been analyzed in [140,141]. The corresponding results are: $\mu a_0 = 0.72 \pm 0.07$, $\mu a_2 = 0.09 \pm 0.09$ [140] and $\mu a_0 = 0.6^{+0.1}_{-0.2}$, $\mu a_2 = -0.1 \pm 0.1$ or $\mu a_0 = -0.5^{+0.3}_{-0.1}$, $\mu a_2 = 0 \pm 0.1$ [141]. This is incompatible with (4.6). However, as noted by Volkovitskiĭ and Dakhno [141] in particular, small values of μa_0 can be obtained from the data of [48] at a confidence level $\lesssim 30\%$. Moreover, the experimental results of [48] do not exclude systematic errors, whose probability is particularly large in the region near the boundaries of the Dalitz plane, which is most important for the analysis of $\pi\pi$ scattering.

3)As a rule, we shall not pay attention to the positions (upper or lower) of the vector indices. However, in connection with Eq. (5.1), it should be stressed (and this is important for the subsequent discussion of the sign of the constant f) that a contraction with the tensor $\epsilon_{\nu\mu\alpha\beta}$ should be interpreted as $\epsilon_{\nu\mu\alpha\beta} q_1^\alpha q_2^\beta$. Henceforth contractions with the tensor ϵ must be interpreted in this way, so that only lower indices appear in the tensor $\epsilon_{\nu\mu\alpha\beta}$.

4)It is proposed to consider the process $X \to X' + 2\gamma$. A pronounced variation of the function $f(p^2)$ over scales $p^2 \sim \mu^2$ that are small in comparison with the characteristic scale of the strong interactions would lead to a pronounced variation of the nonresonant background under the peak corresponding to the production and decay of a real pion. A pronounced variation of the function $f(p^2)$ would show up in the behavior of the background if the mechanism $X \to X' + \pi^0 \to X' + 2\gamma$ gives a significant contribution to the background part of the amplitude for $X \to X' + 2\gamma$. We should expect that this mechanism can be isolated, owing to the small pion mass and, accordingly, the large value of the resonance factor $(p^2 - \mu^2)^{-1}$ at $p^2 \sim \mu^2$.

5)In fact, the more general relation $(3/2)\,L = KR'$ was derived in [67], where R′ is the contribution of the isovector part of the current in the process $e^+e^- \to$ hadrons, and K is the constant in the commutation relations of the spatial components of the electromagnetic current, which can be measured independently. Equation (5.24) is obtained under additional assumptions: a) the usual quark structure of the current commutators (corresponding to $K = 1$), and b) the octet character of the electromagnetic current (in which case $R' = (3/4)\,R$).

6)We showed in part d that the decay $\pi^0 \to 2\gamma$ is in fact determined by the high-energy contribution in the triangle diagram with two vector vertices and one axial-vector vertex. The constant f(0) turned out to be equal to the subtraction term, giving a correction to the high-momentum contribution up to a gauge-invariant structure. This makes it possible (see [68]) to relate f(0) to the matrix element of the currents at small distances. On the other hand, as is well known, the process $e^+e^- \to$ hadrons for $q^2 \to \infty$ is also determined by the contribution of small distances.

7)Such a difference seems natural, for the following reason. One may expect that the constant β in (8.3) or (8.8) must be of the same order for the pion and for the nucleon, since it is the coefficient in the covariant contact amplitude, which corresponds to Feynman graphs of the same type for both $\gamma\pi$ and γp scattering. The validity of this statement can hardly be significantly affected by the fact that there are additional structures besides $q_1q_2\delta_{\nu\mu} - q_{1\mu}q_{2\nu}$ to second order in the q_j in the γp amplitude. We note also that the expression for the polarizability in terms of the constant β (see (8.10)) contains an additional factor μ^{-1}—the Compton dimension of the particle. Therefore we might expect the polarizability of the pion to be greater than that of the nucleon by a factor $m_p/\mu \approx 7$.

8)This possibility has recently been considered by D. Bardin and S. Bilen'kiĭ (private communication); see also [127].

9)After the present review was sent to press, there appeared a paper [142] reporting a measurement of the polarizability of the kaon in connection with a study of transitions in K-mesonic atoms. The result is $\alpha_K = -(4 \pm 11) \times 10^{-3} F^3$. In the framework of SU(3) symmetry, it can be shown that the constant β (see (8.3)) is the same for $\gamma\pi$ and γK scattering and is determined, in particular, by Eq. (8.32) (see below). Therefore the polarizability of the kaon must be smaller than that of the pion by a factor $m_K/\mu \approx 3.5$ (see Eq. (8.10)). From (8.25) we then obtain the estimate $\alpha_K^{\rm theor} \approx 1.2 \times 10^{-43} F^3$.

10)In [129,130] Eq. (8.30) is in fact used as a representation of the amplitude $T_{\nu\mu}$ in the region $p_i \sim q_i \sim \mu$; this is, of course, quite incorrect, since the general phenomenological structure (8.1) contains rapidly varying terms.

11)We note that the $\gamma\gamma \to \pi\pi$ amplitude has been studied in a number of papers (see, e.g., [133,135]) within the framework of dispersion relations. Actually, such investigations are of interest in the region of moderate energies (~1 GeV), since the pole diagram dominates at low energies, as we have pointed out. We do not discuss the results of the dispersion approach here, since this entails a number of approximations that do not fall within the scope of the basic hypotheses of Sec. 2, to which we would like to confine ourselves.

12)V.V. Solov'ev and M. V. Terent'ev (to be published).

1 L. D. Landau and E. M. Lifshitz, Sov. Phys. 6, 244 (1934).
2 A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya élektrodinamika (Quantum Electrodynamics), 3rd edition, Moscow, Nauka, 1969 (Interscience, 1965).
3 F. Calogero and C. Zemach, Phys. Rev. 120, 1860 (1960).
4 F. Low, ibid., p. 582.
5 P. C. DeCelles and J. F. Goehl, ibid. 184, 1617 (1969).
6 V. E. Balakin, V. M. Budnev and I. F. Ginzburg, ZhETF Pis. Red. 11, 559 (1970) [JETP Lett. 11, 388 (1970)]; V. M. Budnev and I. F. Ginzburg, Yad. Fiz. 13, 353 (1971) [Sov. J. Nucl. Phys. 13, 198 (1971)].
7 S. Brodsky, T. Kinoshita and H. Terazawa, Phys. Rev. D4, 1532 (1971).
8 N. Arteaga-Romero, A. Jaccarini and P. Kessler, ibid. D3, 1569.
9 M. V. Terent'ev, Yad. Fiz. 14, 178 (1971) [Sov. J. Nucl. Phys. 14, 99 (1972)].
10 V. Baĭer and V. Fadin, ZhETF Pis. Red. 13, 293 (1971) [JETP Lett. 13, 208 (1971)].
11 É. Choban and V. Shekhter, Yad. Fiz. 14, 190 (1971) [Sov. J. Nucl. Phys. 14, 105 (1972)].
12 C. E. Carlson and W. K. Tung, Phys. Rev. D4, 2873 (1971).
13 V. M. Budnev, I. F. Ginzburg, G. V. Meledin and V. G. Serbo, Preprint TF-67, Novosibirsk, 1972.
14 H. Primakoff, Phys. Rev. 81, 899 (1951).
15 A. Rittenberg, A. Barbaro-Galtieri et al., Rev. Mod. Phys. 43, No. 2 (1971).
16 S. L. Adler and R. E. Dashen, Current Algebras, N. Y., W. A. Benjamin, 1968 (Russ. Transl., Moscow, Mir, 1970).
17 A. I. Vaĭnshteĭn and V. I. Zakharov, Usp. Fiz. Nauk 100, 225 (1970) [Sov. Phys.-Usp. 13, 73 (1970)].
18 J. Bernstein, Elementary Particles and Their Currents, N. Y., W. H. Freeman and Co., 1968 (Russ. Transl., Moscow, Mir, 1970).
19 R. Dashen and M. Weinstein, Phys. Rev. 183, 1261 (1969).
20 V. Grishin, V. Lyuboshitz, V. Ogievetskiĭ and M. Podgoretskiĭ, Yad. Fiz. 4, 126 (1966) [Sov. J. Nucl. Phys. 4, 90 (1967)]; I. C. Aznauryan and A. N. Zaslavsky, Phys. Lett. B39, 226 (1972); L. B. Okun', in Problemy sovremennoĭ fiziki (Problems of Modern Physics), Moscow, VINITI, 1965; T. Das, V. Mathur and S. Okubo, Phys. Rev. Lett. 19, 470 (1967).
21 S. Adler, Phys. Rev. 177, 2426 (1969).
22 S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).
23 M. Gell-Mann, R. Oakes and B. Renner, Phys. Rev. 175, 2195 (1968).
24 V. L. Auslender et al., Yad. Fiz. 9, 114 (1969) [Sov. J. Nucl. Phys. 9, 69 (1969)].
25 J. E. Augustin et al., Phys. Rev. Lett. 20, 126 (1968).
26 V. Balakin et al., Intern. Symp. on Electron and Photon Interactions, Cornell, 1971.
27 C. Bernardini et al., ibid.
28 F. Crowford, Phys. Rev. 117, 1119 (1960).
29 V. G. Grishin, É. P. Kistenev and Mu Chiun, Yad. Fiz. 2, 886 (1965) [Sov. J. Nucl. Phys. 2, 632 (1966)].
30 M. Block et al., Phys. Rev. 169, 1074 (1968).
31 K. Crowe et al., ibid. 180, 1349 (1969).
32 M. Sternheim and R. Hofstadter, Nuovo Cimento 38, 1854 (1965).
33 C. Akerlof et al., Phys. Rev. 163, 1482 (1967).
34 C. Mistretta et al., Phys. Rev. Lett. 20, 1523 (1968); Phys. Rev. 184, 1487 (1969).
35 C. N. Brown et al., Phys. Rev. Lett. 26, 991 (1971).
36 S. Fubini, Y. Nambu and V. Watagin, Phys. Rev. 111, 329 (1958).
37 W. R. Frazer, ibid. 115, 1763 (1959).
38 N. Zagury, ibid. 145, 1112 (1966).
39 S. Adler, Ann. Phys. (N. Y.) 50, 189 (1968).
40 F. A. Behrends, Phys. Rev. D1, 2590 (1970).
41 S. F. Berezhnev, A. V. Demyanov, A. V. Kulikov et al. (JINR, Dubna, USSR), Paper presented to the 16th Intern. Conference on High Energy Physics, Batavia, 1972; S. F. Berezhnev et al., Yad. Fiz. 16, 185 (1972) [Sov. J. Nucl. Phys. 16, 99 (1973)].
42 L. B. Okun' and I. B. Khriplovich, Yad. Fiz. 6, 821 (1967) [Sov. J. Nucl. Phys. 6, 598 (1968)].
43 S. M. Bilen'kiĭ and V. B. Semikoz, Yad. Fiz. 7, 107 (1968) [Sov. J. Nucl. Phys. 7, 79 (1968)].
44 B. V. Geshkenbeĭn and M. V. Terent'ev, Yad. Fiz. 14, 1227 (1971) [Sov. J. Nucl. Phys. 14, 685 (1972)].
45 G. A. Leksin, Usp. Fiz. Nauk 102, 387 (1970) [Sov. Phys.-Usp. 13, 704 (1971)].
46 V. V. Anisovich and P. É. Volkovitskiĭ, Yad. Fiz. 14, 1055 (1971) [Sov. J. Nucl. Phys. 14, 591 (1972)].
47 A. Zylbersztejn, P. Basile et al., Phys. Lett. B38, 457 (1972).
48 W. T. Ford et al., Phys. Rev. Lett. 25, 1370 (1970).
49 V. V. Barmin, V. G. Brykalov et al., Yad. Fiz. 15, 1152 (1972) [Sov. J. Nucl. Phys. 15, 638 (1972)].
50 G. Belletini, C. Bemporad and P. Braccini, Nuovo Cimento A66, 243 (1970).
51 A. Dolgov, A. Vainstein and V. Zakharov, Phys. Lett. B24, 425 (1967).
52 D. Sutherland, Nucl. Phys. B2, 433 (1967).
53 M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1966).
54 J. Steinberger, Phys. Rev. 76, 1180 (1949).
55 J. Bell and R. Jackiw, Nuovo Cimento A60, 47 (1969).
56 M. Gell-Mann, J. Bernstein and L. Michel, ibid. 16, 560 (1960).
57 W. Bardeen, Phys. Rev. 184, 1848 (1969).

[58]B. Zumino, Proc. of the Topical Conference on Weak Interactions, CERN, 1969.
[59]R. Jackiw and K. Johnson, Phys. Rev. 182, 1455 (1969).
[60]J. Schwinger, ibid. 82, 664 (1951).
[61]S. Adler and D. Boulware, ibid. 184, 1740 (1969).
[62]R. Jackiw, Lectures at the Brookhaven Summer School on Elementary Particle Physics, Preprint, MIT, 1971; S. L. Adler, in Lectures on Elementary Particles and Quantum Field Theory, Cambridge, Mass., MIT Press, 1970.
[63]S. Adler and W. Bardeen, Phys. Rev. 182, 1517 (1969).
[64]A. Dolgov and V. Zakharov, Nucl. Phys. B27, 525 (1971).
[65]W. A. Bardeen, H. Fritzsch and M. Gell-Mann, Preprint TH-1538-CERN, July 1972.
[66]F. Ceradini et al., Paper presented to the 16th Intern. Conference on High Energy Physics, Batavia, 1972; R. Averill et al., Paper presented to the 16th Intern. Conference on High Energy Physics, Batavia, 1972.
[67]R. Crewther, Phys. Rev. Lett. 28, 1421 (1972).
[68]K. Wilson, Phys. Rev. 179, 1499 (1969).
[69]M. V. Terent'ev, ZhETF Pis. Red. 14, 140 (1971) [JETP Lett. 14, 94 (1971)].
[70]M. V. Terent'ev, Yad. Fiz. 15, 1199 (1972) [Sov. J. Nucl. Phys. 15, 665 (1972)].
[71]S. Adler, B. Lee, S. Treiman and A. Zee, Phys. Rev. D4, 3497 (1971).
[72]I. Wess and B. Zumino, Phys. Lett. B37, 95 (1971).
[73]T. F. Wong, Phys. Rev. Lett. 27, 1617 (1971).
[74]R. Aviv, N. Dass and R. Saqyer, ibid. 26, 591; R. Aviv and R. F. Sawyer, Phys. Rev. D4, 451 (1971).
[75]E. Abers and S. Fels, Phys. Rev. Lett. 26, 1512 (1971).
[76]E. B. Bogomol'nyĭ, A. D. Dolgov et al., Yad. Fiz. 16, 129 (1972) [Sov. J. Nucl. Phys. 16, 70 (1973)].
[77]S. Adler, G. Ferrar and S. Treiman, Phys. Rev. D5, 770 (1972).
[78]M. V. Terent'ev, Phys. Lett. B38, 419 (1972).
[79]V. M. Shekhter, in Teoreticheskaya fizika i fizika élementarnykh chastits (Theoretical Physics and the Physics of Elementary Particles), Series "Itogi nauki" (Science Reviews), Moscow, VINITI, 1965.
[80]G. S. Iroshnikov and Yu. P. Nikitin, Yad. Fiz. 7, 616 (1968) [Sov. J. Nucl. Phys. 7, 381 (1968)].
[81]G. S. Iroshnikov, Kandidat Dissertation (MIFI, 1970).
[82]V. V. Anisovich and V. M. Shekhter, Yad. Fiz. 13, 369 (1971) [Sov. J. Nucl. Phys. 13, 207 (1971)].
[83]N. Cabibbo and R. Gatto, Phys. Rev. Lett. 4, 313 (1960).
[84]R. Aviv and A. Zee, Phys. Rev. D5, 2372 (1972).
[85]A. Zee, ibid. D6, 900.
[86]J. Smith and N. Stanko, ibid., p. 1332.
[87]R. Köberle, Phys. Lett. B38, 169 (1972).
[88]A. Donnachie and G. Shaw, Ann. Phys. (N. Y.) 37, 333 (1966).
[89]A. Chatterjee, Phys. Rev. 170, 1578 (1968).
[90]G. S. Iroshnikov, Yu. P. Nikitin and A. S. Chernov, ZhETF Pis. Red. 10, 150 (1969) [JETP Lett. 10, 95 (1969)].
[91]G. Murtaza and A. M. Harun-Ar-Rashid, Phys. Rev. D2, 236 (1970).
[92]D. H. Schiller and I. O. Stamatescu, ibid. D4, 1569 (1971).
[93]A. Sundaram, Progr. Theor. Phys. 43, 843 (1970).
[94]M. G. Miller, Phys. Rev. D4, 769 (1971).
[95]A. Ali and F. Hussain, ibid. D3, 1206, 1214.
[96]T. D. Blokhintseva, A. V. Kravtsov and S. G. Sherman, Yad. Fiz. 8, 928 (1968) [Sov. J. Nucl. Phys. 8, 539 (1969)].
[97]V. A. Meshcheryakov, L. L. Nemenov, L. D. Solov'ev et al., Yad. Fiz. 2, 124 (1965) [Sov. J. Nucl. Phys. 2, 87 (1966)].
[98]S. Okubo and B. Sakita, Phys. Rev. Lett. 11, 50 (1963).
[99]V. V. Solov'ev and M. V. Terent'ev, Yad. Fiz. 16, 153 (1972) [Sov. J. Nucl. Phys. 16, 82 (1973)].
[100]V. I. Zakharov, Yad. Fiz. 1, 1053 (1965) [Sov. J. Nucl. Phys. 1, 751 (1965)].
[101]M. Braunschwieg, Phys. Lett. B26, 405 (1968).
[102]P. S. Baranov et al., Yad. Fiz. 5, 1221 (1967) [Sov. J. Nucl. Phys. 5, 873 (1967)].
[103]S. Okubo, Phys. Rev. 179, 1629 (1969).
[104]L. I. Lapidus and Chou Kuang-Chao, Zh. Eksp. Teor. Fiz. 41, 294 (1961) [Sov. Phys.-JETP 14, 210 (1962)].
[105]F. Gilman, Phys. Rev. 184, 1964 (1965).
[106]N. P. Zotov, Yu. A. Rakov and V. A. Tsarev, Kr. soobshch. fiz. (FIAN SSSR), No. 8, 37 (1970).
[107]L. V. Fil'kov, Yad. Fiz. 3, 336 (1966) [Sov. J. Nucl. Phys. 3, 242 (1966)].
[108]L. I. Lapidus and M. M. Musakhanov, Yad. Fiz. 15, 1002 (1972) [Sov. J. Nucl. Phys. 15, 558 (1972)].
[109]M. V. Terent'ev, ibid. 16, 1044 [16, 576 (1973)].
[110]C. Callan and S. Treiman, Phys. Rev. Lett. 16, 153 (1966).
[111]S. Weinberg, ibid. 18, 188 (1967); Phys. Rev. 166, 1568 (1968).
[112]Yu. P. Malakyan, Yad. Fiz. 16, 1035 (1972) [Sov. J. Nucl. Phys. 16, 571 (1973)].
[113]N. D. Hari Dass, Phys. Rev. D5, 1542 (1972).
[114]M. V. Terent'ev, Yad. Fiz. 16, 162 (1972) [Sov. J. Nucl. Phys. 16, 87 (1973)].
[115]M. V. Terent'ev, ZhETF Pis. Red. 15, 290 (1972) [JETP Lett. 15, 204 (1972)].
[116]A. Klein, Phys. Rev. 99, 998 (1955).
[117]A. M. Baldin, Nucl. Phys. 18, 310 (1960).
[118]V. A. Petrun'kin, Zh. Eksp. Teor. Fiz. 40, 1148 (1961) [Sov. Phys.-JETP 13, 808 (1961)]; Tr. FIAN 41, 165 (1968); L. I. Lapidus, Zh. Eksp. Teor. Fiz. 43, 1358 (1962) [Sov. Phys.-JETP 16, 964 (1963)]; V. M. Shekhter, Yad. Fiz. 7, 1272 (1968) [Sov. J. Nucl. Phys. 7, 756 (1968)].
[119]T. Das, V. Mathur and S. Okubo, Phys. Rev. Lett. 19, 859 (1967).
[120]V. G. Vaks and B. L. Ioffe, Nuovo Cimento 10, 342 (1958).
[121]P. Depomier et al., Phys. Lett. 7, 285 (1963).
[122]V. I. Gol'danskiĭ, O. A. Karpukhin, A. V. Kutsenko and V. V. Pavlovskaya, Zh. Eksp. Teor. Fiz. 38, 1695 (1960) [Sov. Phys.-JETP 11, 1223 (1960)].
[123]F. Iachello and A. Lande, Phys. Lett. B35, 205 (1971).
[124]T. Fricson, CERN Preprint No. 1410, Geneva, 1971.
[125]T. Fricson and J. Hufner, Phys. Lett. B40, 459 (1972).
[126]G. Backenstoss et al., ibid. B36, 403 (1971).
[127]D. Bardin and S. Bilen'kiĭ, Yad. Fiz. 16, 557 (1972) [Sov. J. Nucl. Phys. 16, 311,(1973)].
[128]D. Bardin, S. Bilen'kiĭ, G. Mitsel'makher and N. Shumeĭko, Yad. Fiz. 14, 427 (1971) [Sov. J. Nucl. Phys. 14, 239 (1972)].
[129]H. Terazawa, Phys. Rev. Lett. 26, 1207 (1971).
[130]Tsu Yao, Phys. Lett. B35, 225 (1971).
[131]S. Weinberg, Phys. Rev. Lett. 18, 507 (1967).
[132]A. Pais and S. Treiman, ibid. 25, 975 (1970).
[133]D. H. Lyth, Nucl. Phys. B30, 195 (1971).
[134]M. V. Terent'ev, ZhETF Pis. Red. 13, 446 (1971) [JETP Lett. 13, 318 (1971)].
[135]P. S. Isaev and V. I. Khleskov, Yad. Fiz. 16, 1012 (1972) [Sov. J. Nucl. Phys. 16, 559 (1973)].
[136]A. I. Akhiezer and V. B. Berestetskii, Kvantovaya

élektrodinamika (Quantum Electrodynamics), 2nd edition, Moscow, Fizmatgiz, 1959 (Interscience, 1965).
[137]R. L. Goble and J. L. Rosner, Phys. Rev. D5, 2345 (1972).
[138]T. Das et al., Phys. Rev. Lett. 18, 759 (1967).
[139]E. W. Beier et al., ibid. 30, 399 (1973).
[140]S. A. Bunyatov et al., Yad. Fiz. 17, 1307 (1973) [Sov. J. Nucl. Phys. 17, 681 (1973)].
[141]P. É. Volkovitskiĭ and L. G. Dakhno, Yad. Fiz. 19, 182 (1974) [Sov. J. Nuc. Phys. 19, 95 (1974)].
[142]G. Backenstoss et al., Phys. Lett. **B43**, 431 (1973).
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