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DEVELOPMENT OF THE CONCEPTS OF PLASMA TURBULENCE

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The review is devoted to exposition of the physical principles on which the current ideas on plasma turbulence are based. A comparison is made of the results obtained in the theory of plasma turbulence and the turbulence of incompressible liquids. The basic physical differences between hydrodynamic and plasma turbulence are pointed out. It is shown how the concepts of turbulent excitations arise in the statistical description of turbulence. The fundamental difference is pointed out between turbulent elementary excitations and elementary excitations describing a state close to thermodynamic equilibrium. Special emphasis is given to explanation of the physical meaning of the concept of effective turbulent collisions. It is shown that inclusion of turbulent collisions does not make possible construction of a theory of weak turbulence on the basis of simple expansions of the interaction in the turbulence energy.

Examples are presented which show that effective turbulent collisions can fundamentally change the theoretical predictions which must be compared with existing experiments. It is shown how the inclusion of effective turbulent collisions permits construction of a theory of correlation functions of turbulent plasma fields. In connection with the discussion of new approaches to the theory of weak turbulence, taking into account effective turbulent collisions, an analysis is carried out of the theories of anomalous electrical conductivity of a plasma in an external electric field.

THE idea of effective turbulent collisions is deeply involved in the mathematical description of the turbulent state of a plasma, and when effective collisions are specifically taken into account, substantial changes occur in the results which must be compared with existing experiments.

1. THE CONCEPT OF ELEMENTARY EXCITATIONS IN THE THEORY OF PLASMA TURBULENCE

a) Introduction. In recent years plasma turbulence has been studied very intensively both experimentally and theoretically (see the reviews of Kadomtsev[1] and Vedenov and others[2]) and has been utilized in many astrophysical problems[3]. This interest in the problems of plasma turbulence is due first of all to the wide range of experimental plasma studies which have demonstrated the important role of turbulent processes in plasma. It frequently turns out that the principal macroscopic characteristics of a plasma such as diffusion, electric conductivity, thermal conductivity, and so forth are determined by just these turbulent processes. This is easy to understand if we take into account that plasma has turned out to an extraordinary degree to be an unstable state in which very small deviations from thermodynamic equilibrium are sometimes sufficient for development of instability. Landau and Heisenberg[4] noted that the origin of the development of turbulence in liquids is instability. The same is true for plasma. It has been shown also that turbulence is possible not only in liquids or plasma but also in solid materials[5]. Therefore the term turbulence itself now has a somewhat different and more general nature (see part a) of Section 4). It is quite interesting that the intensive research on plasma turbulence has shed more light on the nature of turbulence as a free state of matter than has the study of the turbulence of liquids which has been carried on for decades. This is due first of all to the variety of types of collective motions of plasma. A special role is played by those motions which can be roughly characterized by certain natural frequencies. The best known of them are the Langmuir plasma fluctuations. The existence of natural frequencies has played a role of no small importance in the theory of turbulent plasma.

Development of ideas about the nature of plasma turbulence has proceeded along two paths. On the one hand, use has been made of statistical averaging methods similar to those previously used in liquid turbulence[6], the "elasticity" of plasma motions being the basis for assumption of a weak correlation of the fields of turbulent fluctuations[1]. On the other hand, the concept has been used of elementary excitations—turbulent plasmons[7], whose interaction probabilities were found from the correspondence principle.

A synthesis of these approaches has been obtained recently on the basis of the concept of effective turbulent collisions[8]. As it turned out, the refinement of the statistical averaging method of the weak-coupling type and other types does not go beyond the bounds of weak turbulence, but leads only to correct inclusion of turbulent collisions and, in the last analysis, provides equations which are used in the method of elementary excitations.

On the other hand, the physical difference has been clarified between turbulent elementary excitations and excitations describing a plasma state near statistical equilibrium. This difference is due to turbulent collisions. It indicates the existence of unique ambiguities in the energy and momentum of turbulent plasmons. This throws light on the mechanisms which are internally present in the method of turbulent plasmons and which place natural limitations on the accuracy with which answers must be obtained by means of equations describing the interaction of turbulent plasmons. In this path of development it has been possible not only to obtain rigorous bases for use of the method of elementary excitations in turbulent plasma, but also to consider anew the problems of Landau damping[9], stochastic heating, and the irreversibility of processes in the turbulent regime. Finally, but in no way last in importance, is the variation of the effectiveness of different interactions, which is due to turbulent collisions[8,10] and which is directly reflected in the experimentally measured macroscopic parameters of a plasma. Inclusion of turbulent collisions also permits us to obtain the form of the correlation functions[11] measured in most experiments with turbulent plasma. All of these questions, which have been resolved recently, will be the subject of the present article.

b) Comparison of plasma turbulence with the turbulence of incompressible liquids. It is usually considered that the success in development of the theory of plasma turbulence is due mainly to the existence of a small parameter which is not present in the theory of liquid turbulence. In order to make clear what we are talking about, let us recall some well known ideas from the theory of turbulence of incompressible liquids. The instability of a number of liquid flows leads to excitation of vortices. In a developed turbulent state there are present vortices of all possible scales, substantially different from those due to direct excitation of vortices. Subdivision of the scales of vortices occurs. It is due to the nonlinear interaction of vortices of different scales and creates a flow of vortex energy to smaller scales where they disappear as the result of viscosity. According to Kolmogorov[12] this flow is constant in stationary turbulence. This leads to a universal distribution in scale of turbulent vortices, which is well known as the Kolmogorov spectrum[12]. If W is the turbulent vortex energy per cm^3 and k is the vortex wave number (i.e., the quantity inverse to its scale l, $k = 2\pi/l$), then

$$W = \int W_k\, dk$$

and the formula for the Kolmogorov spectrum has the form (Fig. 1)

$$W_k = \mathrm{const}\cdot K^{-5/3} \qquad (1.1)$$

(in Fig. 1 $k \approx k_0$ is the region of excitation of turbulent vortices, $L = 2\pi/k_0$ is the basic scale of turbulence, and $k \gg k_0$ is the region of the Kolmogorov spectrum). Rigorous theoretical derivation of this formula, which has been obtained from dimensional considerations, has not yet been possible, in spite of numerous approaches and many years of work in development of the theory of liquid turbulence (for more detail see ref. 13). In a number of investigations the spectrum (1.1) is derived at the expense of introducing new "principles" such as the requirement of maximal generalized entropy[14] and others[15], which are not contained directly in the initial equations.

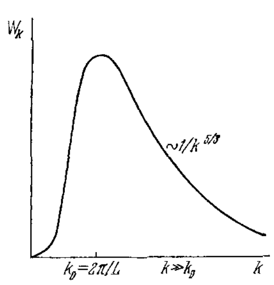

FIG. 1. Turbulence spectrum of an incompressible liquid.

It is considered that these difficulties have a fundamental nature and are due to the fact that in an incompressible liquid the turbulence is strong. This is expressed in the fact that vortices in general do not have any natural frequency and the time of transfer of energy from one vortex to a neighbor is of the order of one revolution of the vortex. In contrast to this, in a plasma there are many collective motions of the fluctuation type which have definite natural frequencies. The time τ for transfer of the energy of these oscillations to neighboring scales (or to neighboring wave numbers) can substantially exceed the natural period of the fluctuations $1/\omega_k$. As a consequence of this "elasticity" of the collective motions, a small parameter can appear,

$$\varepsilon \sim 1/\omega_k\tau \ll 1. \qquad (1.2)$$

It is considered that the existance of this small parameter permits use of regular methods of expansion in the turbulence energy and construction of a theory of weak turbulence. It was just this path which led initially to the so-called quasilinear approximation[16], which does not take into account the interaction of fluctuations with each other, and in addition nonlinear effects have been taken into account[1,7,17]. The method of elementary excitations[7], on the basis of which many specific results have been obtained on the interaction of plasmons with each other and with plasma particles, has been used to determine the spectra of plasma turbulence[18,19]. In a number of wave-number regions these spectra have the nature of a universal power law $1/k^{\nu}$, but ν can be different in different regions, since there is a change in the relative role of different nonlinear processes responsible for shaping the spectrum (Fig. 2; turbulent fluctuations are excited for $k \gg k_{**}$; $k_* = (\omega_{pe}/v_{Te})(m_e/9m_i)^{1/2}$, $k_{**} = (\omega_{pe}/v_{Te})(m_e/9m_i)^{1/5}$; the value of ν depends on the total energy included in the fluctuations and lies in the range $2.8 < \nu < 4$). Spectra of this type for a plasma are the result of solution of the nonlinear equations describing the turbulence (in regard to numerical solutions of these equations and their correspondence with the analytical solutions, see ref. 20). Detailed experimental study of the spectra for development of ion-acoustic turbulence also has shown that the observed spectra are close to power-law[21,22]. We note that in most such experiments the parameter (1.2) is

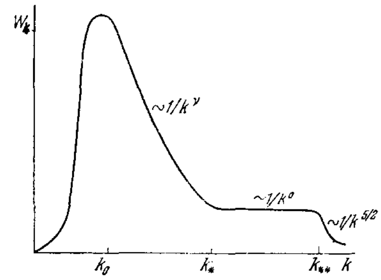

FIG. 2. Schematic representation of the turbulence spectra of Langmuir fluctuations of a plasma.

small. More accurately, a somewhat different quantity turns out to be small—the ratio of the measured turbulent energy W to the thermal energy of the plasma particles nT:

$$\varepsilon' = W/nT \ll 1.$$

Thus, in certain experiments[23] ϵ' was 10^{-1}–10^{-2}. Specifically for ion-acoustic fluctuations it follows from the existing theory[1,10] that ϵ' is of the order of ϵ. Thus, it would appear that there is a small parameter in a plasma for the "elastic" degrees of freedom, that there is a satisfactory means of theoretical treatment of the spectra, and that experiments exist to which this theory should be applicable. However, the further development of the theoretical ideas has led to an understanding that existence of the small parameter (1.2) still does not allow us to expand the interactions in this parameter. The physical meaning of this is associated with the resonance nature of the interactions themselves. In order to make this clear, it is necessary to dwell briefly on the description of the interaction of turbulent pulsations with each other and with the plasma "particles" in the language of elementary excitations.

c) Balance equations for turbulent plasmons. In the general case of anisotropic turbulence the spectrum must be described by the energy density of the turbulence referred to an element $d\mathbf{k}$, i.e.,

$$W = \int W_{\mathbf{k}} d\mathbf{k}. \qquad (1.3)$$

Since the turbulent fluctuations have natural frequencies $\omega_{\mathbf{k}}$, we can introduce the number of quanta $N_{\mathbf{k}}$ ($\hbar = 1$):

$$W_{\mathbf{k}} = \omega_{\mathbf{k}} N_{\mathbf{k}}/(2\pi)^3.$$

For a number of quanta $N_{\mathbf{k}}$ we can write the equation describing their radiation and absorption by particles, scattering by particles, and nonlinear decays of some quanta into others, introducing corresponding probabilities[7]. The equations taking into account both induced and spontaneous processes will be nonlinear in $N_{\mathbf{k}}$ and consequently also in $W_{\mathbf{k}}$. This interaction changes the wave numbers of the fluctuations and creates an energy flux "along k". Thus, the equation describing the radiation of a wave σ by a charged particle moving along a helical line in a magnetic field $\mathbf{H}$ in the quasiclassical limit has the form

$$dN_{\mathbf{k}}^{\sigma}/dt = \gamma_{\mathbf{k}}^{\sigma} N_{\mathbf{k}}^{\sigma} + (2\pi)^3 \omega_{\mathbf{k}}^{-1} Q_{\mathbf{k}}^{\sigma}. \qquad (1.4)$$

The quantity $Q_{\mathbf{k}}^{\sigma}$ is the spontaneous radiation power, referred to the interval $d\mathbf{k}$:

$$Q_{\mathbf{k}}^{\sigma} = \sum_{\nu=-\infty}^{\infty} \int [\omega_{\mathbf{k}}^{\sigma}/(2\pi)^3]\, w^{\sigma}(\mathbf{k}, \mathbf{p}, \nu)\, \Phi_{\mathbf{p}}\, d\mathbf{p}/(2\pi)^3,$$

and

$$\gamma_{\mathbf{k}}^{\sigma} = \sum_{\nu=-\infty}^{\infty} \int w^{\sigma}(\mathbf{k}, \mathbf{p}, \nu)\, \Delta\lambda_i \frac{\partial \Phi_{\mathbf{p}}}{\partial \lambda_i} \frac{d\mathbf{p}}{(2\pi)^3}$$

is the damping or buildup coefficient of the waves; ν is an integer; $i = 1, 2, 3$; $\lambda_1 = p_{\parallel}$, $\lambda_2 = p_{\perp}$, $\lambda_3 = y$, $p_{\parallel} = (\mathbf{pH})/H$, $p_{\perp} = (p^2 - p_{\parallel}^2)^{1/2}$; y is the coordinate of the center of the Larmor circle; $\Delta\lambda_1 = k_{\parallel}(k^2 = k_{\parallel}^2 + k_{\perp}^2)$, $\Delta\lambda_2 = \nu\epsilon\omega_H/p_{\perp}$, $\Delta\lambda_3 = -k_{\perp}/\omega_H\epsilon$, $\omega_H = eH/\epsilon$; $\epsilon = (p^2 + m^2)^{1/2}$, $k_{\parallel} = (\mathbf{kH}/H)$.

This result is obtained very simply[24] if we use the quantum representation of particle motion in a magnetic field (see refs. 25 and 26); then ν is the difference between the two quantum numbers describing the transition between the Landau levels. The quantity W^{σ} is the diffferential probability of radiation, which in the quasiclassical limit can be found from the correspondence principle[7]. In the limit $N_{\mathbf{k}} \to 0$ in (1.4) there remains only the spontaneous radiation, whose intensity is easily calculated in the classical limit by the Landau method[27] from the action of the field produced by the particle on the particle itself. It has the following specific form:

$$w^{\sigma}(\mathbf{k}, \mathbf{p}, \nu) = (2\pi)^3 \frac{e^2}{\pi} \frac{|\Gamma_{\mathbf{k}} e_{\mathbf{k}}^{\sigma}|^2 \delta(\omega_{\mathbf{k}}^{\sigma} - k_{\parallel} v_{\parallel} - \nu\omega_H)}{\left.\frac{\partial}{\partial\omega}\, {}^{2}\varepsilon_{ij} e_{\mathbf{k},i}^{\sigma *} e_{\mathbf{k},j}^{\sigma}\right|_{\omega=\omega_{\mathbf{k}}^{\sigma}}};$$

here ϵ_{ij} is the linear tensor of the dielectric permittivity and $\mathbf{e}_{\mathbf{k}}^{\sigma}$ is the normal unit vector of the wave σ. The vector $\Gamma_{\mathbf{k}}$ has the following components:

$$\Gamma_1 = \nu v_{\perp} J_{\nu}(\zeta)/\zeta, \quad \Gamma_2 = -i v_{\perp} J'_{\nu}(\zeta), \quad \Gamma_3 = v_{\parallel} J_{\nu}(\zeta); \quad \zeta = k_{\perp} v_{\perp}/\omega_H.$$

It is interesting to note that in this form Eq. (1.4) contains most of the most important plasma instabilities, including the drift instabilities[1] (the derivative with respect to y), and describes also synchrotron radiation, its reabsorption[28], and the synchrotron instability[29]. The similar balance equations for the particle distribution function $\Phi_{\mathbf{p}}$

$$\frac{d\Phi_{\mathbf{p}}}{dt} = \frac{\partial}{\partial p_i} D_{ij} \frac{\partial \Phi_{\mathbf{p}}}{\partial p_j} + \frac{\partial}{\partial p_j} A_j \Phi_{\mathbf{p}} \qquad (1.5)$$

correspond to the general case of quasilinear equations taking into account spontaneous processes:

$$D_{ij} = \sum_{\nu=-\infty}^{\infty} \int \Delta\lambda_i \Delta\lambda_j w^{\sigma}(\mathbf{k}, \mathbf{p}, \nu)\, N_{\mathbf{k}}^{\sigma}\, d\mathbf{k}/(2\pi)^3,$$

$$A_j = \sum_{\nu=-\infty}^{\infty} \int \Delta\lambda_j w^{\sigma}(\mathbf{k}, \mathbf{p}, \nu)\, d\mathbf{k}/(2\pi)^3.$$

These equations also are obtained by elementary means from the conditions of balance of radiation and absorption. They contain as a special case all the results of ref. 30 on generalization of the quasilinear equations to the case of a magnetoactive plasma, the system of quasilinear equations[31] for drift waves, and the quasilinear equations for a relativistic plasma[32] and for the synchrotron instability.

The elementary process which is described by these equations is shown by the diagram of Fig. 3. We note that Eq. (1.4) is linear in $N_{\mathbf{k}}$ and consequently in the turbulence energy $W_{\mathbf{k}}$; in exactly the same way the diffusion coefficient is linear in $W_{\mathbf{k}}$, which results in the name of these equations—quasilinear. In both Eqs. (1.4) and (1.5) the terms quadratic in $W_{\mathbf{k}}$ give the more complex processes of interaction of plasmons with particles, for example, those shown in Fig. 4. The decay process shown in Fig. 5 contributes only to Eq. (1.4). These processes give a nonlinear interaction of

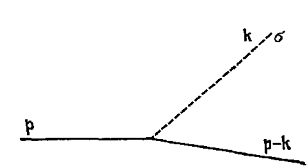

FIG. 3. Process of radiation of a wave by a charged particle.

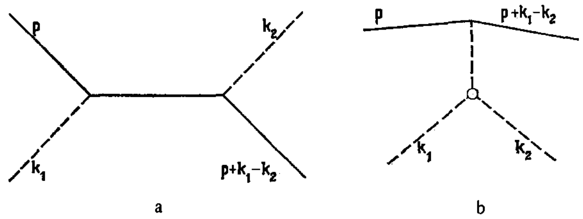

FIG. 4. Processes for scattering of waves by charged particles of a plasma—Thompson scattering (a) and nonlinear scattering (b).

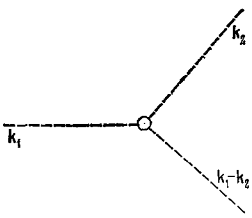

FIG. 5. Diagram of the decay process.

the waves, which leads to a transformation of the turbulent energy along the spectrum. They were obtained in general form in ref. 33. Let's turn our attention to the fact that the scattering process (see Fig. 4) contains two diagrams, which describe the ordinary Thompson scattering and nonlinear scattering. In this connection we will make two important remarks. First, the equations for elementary excitations obtained by this means are similar to the equations for elementary excitations for systems close to statistical equilibrium, although under conditions of developed turbulence the system is far from statistical equilibrium. It is well known that the concept of elementary excitations is fundamental in the current statistical theory of condensed media and can be justified most completely near equilibrium by the Green's-function method[34]. However, as will be evident from what follows, turbulent excitations have a different nature, which is very important for the specific application of the theory.

In the second place, all the interactions of turbulent plasmons have a resonance nature. Thus, radiation occurs only for the condition $\omega_{\mathbf{k}}^{\sigma} - k_{\parallel}v_{\parallel} - \nu\omega_H = 0$, and without a magnetic field for the Cerenkov condition $\omega_{\mathbf{k}}^{0} - \mathbf{kv} = 0$, scattering also requires similar conditions but with the frequency difference of the waves entering instead of the frequency and the difference in wave numbers instead of the wave number. From the quantum point of view these conditions describe the conservation laws in radiation and scattering of waves, and from the classical point of view they describe a resonance between the waves and the particles. Nonlinear interactions in which particles do not participate, for example, decay interactions, also have a resonance nature. Thus, even in the fixed-phase approximation, which is ordinarily used in nonlinear optics[35], the resonance conditions must be satisfied for effective transfer of energy from one mode to another. For random waves such as turbulent plasmons, these conditions are identical with plasmon momentum and energy conservation in decay. For the process illustrated in Fig. 5, these conditions have the form $\omega\mathbf{k}_1 = \omega\mathbf{k}_2 + \omega\mathbf{k}_1 - \mathbf{k}_2$.

We note that the resonance nature of nonlinear interactions raises the question of applicability of the small parameter (1.2) for description of the interactions in a plasma. In fact, near a resonance we would expect that the parameter (1.2) is replaced by

$$\varepsilon'' = 1/\overline{(\omega - \omega_{\mathbf{k}})}\tau,$$

where $\omega_{\mathbf{k}}$ is some resonance frequency and the bar denotes averaging, whose meaning will be clear from what follows: The resonance factor $1/(\omega - \omega_{\mathbf{k}})$ approaches infinity for $\omega \to \omega_{\mathbf{k}}$, but in reality the resonance must be smeared out, if only because of the same nonlinear interactions. Then the average value $\overline{\omega - \omega_{\mathbf{k}}}$ is of the order of $1/\tau$ and consequently ϵ'' is the order of unity. This leads to the idea that the turbulence of liquids and plasma are not so different. In reality, however, the concept of elementary excitations is impossible in liquids, while in a plasma it is possible and correct. The reason for this is just the existence of the parameter (1.2). In addition, the resonance nature of the interactions, strictly speaking, does not permit them to be expanded in a series in the turbulence energy. However, this inability to be expanded effects only the structure of the resonance denominators. In the approximation defined they are approximated by relations which lead to the form used in the balance equations written down above. The possibility of this approximation also determines the possibility of using the concept of elementary excitations.

d) General remarks about the turbulent state of matter. It is necessary to say a few words as to just what is the turbulent state of matter. Turbulence arises only in a nonlinear system having some collective modes of motion. In a plasma this may be plasma fluctuations, in a liquid—vortices, in solid materials—phonons, and so forth. There are important differences in the nature of the collective motions themselves. Thus, in a plasma they usually are accompanied by electromagnetic fields, and in a liquid they are not. Another thing is also important. If a sufficiently large amount of energy is introduced to some of the collective motions in a nonlinear system, the nonlinear interactions, according to the general postulates of statistics, must redistribute it over the other modes—the other degrees of freedom. In this way energy is pumped from those modes in which it is generated, i.e., an energy flow arises. The direction of the flow can also be different. Thus, in liquids energy is transferred to pulsations with larger wave numbers (smaller scale), and in a plasma to smaller wave numbers. This difference is due to the nature of the nonlinear interactions. In liquids energy is transferred from some pulsations to others with conservation of the energy involved in the pulsations. In a plasma energy can be transferred to particles, heating them. The increase in entropy as a result of this heating compensates the loss due to reduction of the phase space occupied by the fluctuations in the process of decrease of their wave numbers.

It is clear that fluctuations in a plasma will die out somewhere. The collective absorption of these fluctuations due to Landau damping[9] or cyclotron damping is particularly effective. If stationary turbulence arises, a balance of production and absorption occurs. It is possible only in the presence of a flow of energy from the production region to the absorption region. Thus, for occurrence of stationary turbulence it is necessary that: 1) a large number of collective degrees of freedom are excited; 2) the energy associated with them is sufficient for occurrence of nonlinear processes which transform the energy from the production region to the absorption region; 3) a separation of the production and absorption regions exists. Nonlinear processes play a composite part in the concept of turbulence. In a plasma they lead not only to mixing of energy in many modes but also to a rapid shakeup of the phase of the plasma fluctuations, making them random and making the plasmons turbulent (strictly speaking, we can discuss the phases only approximately, since in view of the above the fluctuations are not linear). In spite of this, up to the present time the role of nonlinear processes in the theory of turbulence has often been underestimated and the investigations have often been limited only to quasilinear processes. With development of the theory the region of applicability of the quasilinear approximation has steadily become narrower as the nonlinear effects have been evaluated more accurately. Thus, for example, for weak beams of particles the quasilinear approach has turned out to be applicable only if the particle velocities are limited by the inequalities $1 < v/v_{Te} \ll (9m_i v/m_e \Delta v)^{1/4}$. In exactly the same way the role of nonlinear interactions in creation of the anomalous resistance of a plasma to an external electric field[10] has turned out to be significantly greater. The statement is also made that the random nature of the fluctuations can be preserved in the quasilinear stage as a result of the fact that the fluctuations are excited from the thermal level. However, the randomness of the fluctuations under conditions of thermal equilibrium if they are due to binary collisions of particles, corresponds to large time intervals and it remains an open question whether these fluctuations will be random in the small time intervals in which excitation of the fluctuations occurs as the result of instability. In addition, under conditions of a sufficient but not very large level of the fluctuations, inclusion of nonlinearities (which themselves spread the fluctuations in phase) gives a rigorous basis for the quasilinear equations (see below). Under these conditions ordinarily the inclusion of quasilinear effects must be carried out together with nonlinear effects, which in many cases turn out to be more effective.

e) Effective turbulent collisions. The idea of effective turbulent collisions is often introduced by experimenters. It is convenient in connection with the fact that all dissipation processes turn out to be sharply increased in the turbulent regime. In practice one does the following. Consider for example the formula describing the electrical conductivity of a plasma due to binary collision:

$$\sigma = ne^2/m_e\nu_{ei}, \tag{1.6}$$

in which the binary collision frequency ν_{ei} enters. The observed electrical conductivity is many orders of magnitude less than is given by (1.6). Then in (1.6) instead of ν_{ei}, ν_{eff} is substituted, and ν_{eff} is determined from the experimentally observed σ.[36] The same approach is used for description of the anomalous absorption of high-frequency fields which excite turbulence in a plasma[37], for anomalous diffusion[1], and so forth. It is clear that this approach is phenomenological. However, there are deeper physical justifications for introduction of effective turbulent collisions. In the first place, we note that values of ν_{eff} determined phenomenologically turn out to depend on the turbulence energy W [38] and can depend on the angles and other quantities. In this respect the phenomenological approach is taking revenge, since the internal physical mechanisms on which such macroscopic characteristics as the average electrical conductivity (1.6) in the turbulent regime are based are different from binary collisions. The dependence of ν_{eff} on W indicates that they are associated with nonlinear processes. Under the conditions of stationary turbulence we can define the characteristic time of energy transformation over the spectrum, which depends on W. The reciprocal quantity we can call the effective turbulent frequency of collisions ν_{eff}. The existence of these effective collisions enters organically into the concept of turbulence. Since the nonlinear processes responsible for formation of the energy flow can be different, the effective collisions can have a different nature. These collisions must be distinguished from those which are defined phenomenologically. However, the phenomenological values of ν_{eff} are uniquely related to the corresponding nonlinear processes which produce them, in the same way, for example, as ν_{ei} in (1.6) is related to the binary collisions described by the Landau collision integral[39]. However, turbulent collisions also play a more fundamental role, namely that they must be organically taken into account in constructing the theory of turbulence and in justifying the method of elementary excitations (see below). If we proceed from the idea of turbulent collisions, we can easily understand qualitatively a number of simple statements whose proof is given further on.

First, roughly speaking, the resonance denominators should be smeared by turbulent collisions and instead of $1/(\omega - \omega_{\mathbf{k}})$ we should have

$$1/(\omega - \omega_{\mathbf{k}} + i\nu_{eff}).$$

Second, this smearing substantially alters the intensity of the nonlinear interactions themselves. Use of the random-phase approximation simultaneously with the assumption that the resonance denominators have the form $1/(\omega - \omega_{\mathbf{k}} + i\delta)$ is suitable only in the limit of infinitely small amplitude of the waves, $\delta \to 0$, when $\nu_{eff} \to 0$, i.e., under conditions when the nonlinear interactions are negligibly small. However, just this description was used in a number of the early studies on the nonlinear interaction of random waves[17]. This can be correct only when we can use the approximation

$$\mathrm{Im}\,(\omega - \omega_{\mathbf{k}} + i\nu_{eff})^{-1} \approx -\pi\delta(\omega - \omega_{\mathbf{k}}). \tag{1.7}$$

For many interactions this is not valid and it is necessary to recognize that inside the resonance

$$1/(\omega - \omega_{\mathbf{k}} + i\nu_{eff}) \approx -i\nu_{eff}^{-1}.$$

Thus, effective collisions are a real physical process which affects the very interactions which determine them.

Third, the problem of Landau damping[9] arises in a quite different way. In the linear theory this damping is reversible and describes the exchange of energy between the captured particles and the wave. In a turbulent plasma the damping of collective fluctuations turns out to be irreversible. This is due to the fact that just the effective turbulent collisions produce the irreversibility and lead to a quite different, nonlinear treatment of this type of damping.

Fourth, effective turbulent collisions determine the widths of the correlation curves which are measured in experiments on turbulent plasma.

Fifth and finally, turbulent collisions create an ambiguity between the frequency and the wave number of a turbulent plasmon, i.e., an ambiguity in the relation between the energy and momentum of the elementary excitations of a turbulent plasma. A similar ambiguity occurs also under conditions close to thermal equilibrium, but it is due to linear damping (and is discussed in the same way as done by Landau for plasma waves in linear theory). In the present case ambiguity is associated with the nonlinear process. From this difference the fundamental distinction between a turbulent plasmon and a plasmon describing a state close to a static equilibrium becomes clear. In addition, we can speak of plasmons and of quasiparticles in the turbulent regime when ν_{eff} is less than the natural frequency of the plasmon:

$$\nu_{\text{eff}}/\omega_{\mathbf{k}} \ll 1. \tag{1.8}$$

This condition is identical with (1.2).

f) The balance equation and the concept of elementary excitations of a turbulent plasma. We will now discuss qualitatively the question of what place in the general theory of turbulence is occupied by the balance equations for turbulent plasmons, and what physical criteria and approximations must be used to describe turbulence in the language of turbulent plasmons. Assume for the sake of simplicity that turbulent fluctuations are electrostatic longitudinal plasma oscillations which can be described by means of the potentials φ of the electric fields. Ordinarily the potential correlation function $|\varphi_{\mathbf{k},w}|^2$ is investigated. If we separate the regular part φ^r and the stochastic part φ^{st} in the complete potential:

$$\varphi = \varphi^r + \omega^{st}, \quad \langle \varphi^{st} \rangle = 0,$$

then the correlation function for stationary turbulence is determined by the relation

$$|\varphi_{\mathbf{k},\omega}|^2 = (2\pi)^{-4} \int \langle \varphi^{st}(\mathbf{r}, t)\, \varphi^{st}(\mathbf{r}', t') \rangle\, e^{i\mathbf{k}(\mathbf{r}'-\mathbf{r}) - i\omega(t'-t)}\, d(\mathbf{r}-\mathbf{r}')\, d(t-t').$$

This correlation function is related to the quantity $W_{\mathbf{k},\omega}$, the turbulence energy density for a wave number interval $d\mathbf{k}$ and frequency interval $d\omega$:

$$W = \int W_{\mathbf{k},\omega}\, d\mathbf{k}\, d\omega,$$

by the relation

$$W_{\mathbf{k},\omega} = \alpha_{\mathbf{k}}\, |\varphi_{\mathbf{k},\omega}|^2,$$

where $\alpha_{\mathbf{k}}$ for weak turbulence is a known function of $\mathbf{k}$ (for Langmuir fluctuations, for example, $\alpha_{\mathbf{k}} = k^2/4\pi$). It is evident that $W_{\mathbf{k},\omega}$ contains more detailed information on turbulent fluctuations than $W_{\mathbf{k}}$, which is introduced by Eq. (1.3):

$$W_{\mathbf{k}} = \int W_{\mathbf{k},\omega}\, d\omega. \tag{1.9}$$

For fixed $\mathbf{k}$ the distribution in ω is characterized by some finite width determined by the effective turbulent collisions. Let us write down symbolically the equation describing the state of stationary plasma turbulence, in the form

$$\Phi(\mathbf{k}, \omega, W_{\mathbf{k}_1,\omega_1}) = 0, \tag{1.10}$$

where Φ is some linear functional of $W_{\mathbf{k}_1,\omega_1}$. Assume that the specific form of (1.10) is known (see below). From (1.10) we can obtain the consequence–the balance equation. For this we integrate (1.10) over all frequencies:

$$\int \Phi(\mathbf{k}, \omega, W_{\mathbf{k}_1,\omega_1})\, d\omega = 0. \tag{1.11}$$

Here the question arises whether the balance equation (1.11) can be written in such a form that it contains only the integral characteristic $W_{\mathbf{k}}$ (1.9) describing the turbulence spectrum? In other words, can (1.11) be written approximately in the form

$$\tilde{\Phi}(\mathbf{k}, W_{\mathbf{k}_1}) = 0, \tag{1.12}$$

where $\tilde{\Phi}$ is some new nonlinear functional obtained from Φ? It turns out that this is approximately possible if there is elasticity of the collective motions or, more accurately, if parameter (1.8) is used. It is important that Eq. (1.11) contains the integral over all frequencies, which will not be very sensitive to smearing of the resonances by turbulent collisions. Equation (1.12) is obtained if we use the approximation (1.7) for the resonance denominators and neglect the ambiguity in the relation of the frequency and wave number in the correlation functions, i.e., if we set

$$W_{\mathbf{k}_1,\omega_1} \approx W_{\mathbf{k}_1}\delta(\omega_1 - \omega_{\mathbf{k}_1}).$$

The latter is also possible only in the case when the integral over ω_1 enters into (1.11). Thus, Eq. (1.12) arises as the first approximation to (1.11) in the parameter (1.8). This equation is identical to that obtained by the method of elementary excitations. It is clear also that the conservation laws for interaction of elementary excitations must now be satisfied only with an accuracy to quantities ν_{eff}. We will make one remark in connection with a number of attempts to go beyond the framework of weak turbulence, which are known as the weak-coupling approximation[1]. These attempts actually discussed going beyond the approximation in which the nonlinearities are expanded in a series in W. However, in Eq. (1.10) such an expansion is never possible even for weak turbulence, and in regard to (1.11) we are actually discussing the correct description of turbulent collisions, which in the last analysis provide the justification for the method of elementary excitations. These conclusions were not drawn in the work cited, and we will consider these questions in the following sections. We will also note here that in turbulence of incompressible liquids the parameter (1.2) is absent, and therefore it is not possible to separate the correlation effects and to write equations for $W_{\mathbf{k}}$. Under the conditions of a turbulent plasma we can separate the theoretical problems of finding the turbulence spectrum and the correlation functions. The spectra can be sought from the balance equations (1.12). Knowing the spectra, we can return to the approximate solution of Eq. (1.10) for the correlation functions. In the following discussion we will find specifically an equation of the (1.10) type by the method of statistical averaging.

2. THE METHOD OF STATISTICAL AVERAGING FOR DESCRIPTION OF PLASMA TURBULENCE

a) Averaging over a statistical ensemble. In order to avoid complicating the discussion, we will limit ourselves to the simplest case of longitudinal fields and longitudinal waves.

The statistical description of turbulence is based on the idea of stochastic variation of the physical quantity, for example, of the electric potential of the longitudinal fluctuations. This is possible if the results of measuring such a potential are nonreproducible. Let us make clear what the situation is. Let us assume that for the same initial conditions of a macroscopic experiment, at a time t_0 after the beginning of the experiment a measurement is made of the potential of longitudinal fluctuations and a complicated and irregular dependence of the potential on time is obtained. The existence of such irregularity in no way means that the potential is a random stochastic quantity, since it can also describe a complex but regular process. It is necessary to look at the results of many repetitions of the same experiment under the same macroscopic conditions. Nonreproducibility of the irregularities observed in the fluctuations can serve as an indication of randomness. The cause of this is that small changes in the initial conditions for the same macroscopic conditions of the experiment lead to a substantial change in the behavior of the entire pattern. For linear or almost linear fluctuations, the phase can be such an initial condition. However, in the general case fluctuations are nonlinear and the frequently used expression—random-phase approximation—is not completely accurate. More strictly (as ordinarily in any statistical description) it is necessary to use the representation of a statistical ensemble as an ensemble of systems differing in the initial conditions of development of collective fluctuations[40]. For a stochastic potential the average value over a statistical ensemble is equal to zero,

$$\langle\varphi^{\rm st}\rangle = 0. \tag{2.1}$$

In an ergodic system the average over the ensemble is equal to the time average. The basis of the statistical description of turbulence is the general postulates of statistical physics, extended to intensely excited and intensely interacting collective degrees of freedom (for a plasma—collective fluctuations). In such a description it is automatically understood that effective turbulent collisions associated with nonlinear interactions are present. While in an ordinary molecular motion a uniform distribution over the various degrees of freedom and ergodicity are achieved by particle collisions, in collective motions they are achieved by nonlinear interaction of the modes. Therefore many modes are always present in developed turbulence. We can say a few words about the excitation of turbulence. According to Landau[4], this excitation is due to instability. However, in the presence of instability many modes can be immediately excited, but also possibly a small number of modes. In both cases the interaction of the modes can lead to redistribution of energy over many modes, including those not directly excited by the instability. In incompressible liquids only large-scale vortices are usually excited directly, the remaining vortices arising as the result of nonlinear division of vortices. In a plasma, apparently, the so-called hydrodynamical instabilities also can lead initially to excitation of several or even a single mode and only later do the nonlinearities redistribute the energy. In the examples given, the initial stage of development of instability is not stochastic, and only with the passage of time does this system transfer to a stochastic regime.[41]

b) The general equations for the stochastic potential. In the general case the potential will be the superposition of regular and stochastic parts

$$\varphi = \varphi^{\rm r} + \varphi^{\rm st}, \tag{2.2}$$

where $\varphi^{\rm st}$ satisfies (2.1), i.e., $\varphi^{\rm r} = \langle\varphi\rangle$. For example, $\varphi^{\rm r}$ can be the potential of external fields. If we carry out the separation (2.2), it is necessary to do the same with the distribution function:

$$f = f^{\rm r} + f^{\rm st}, \qquad \langle f\rangle = 0,$$

since the plasma particles always take part in the fluctuations. We will use the collisionless kinetic equation and the Poisson equation for f and φ to find the equations for $f^{\rm st}$ and $\varphi^{\rm st}$:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} = e\nabla\varphi\frac{\partial f}{\partial \mathbf{p}}, \qquad \Delta\varphi = -4\pi e n = -4\pi e\int f\,d\mathbf{p}/(2\pi)^3. \tag{2.3}$$

Here for simplicity we have omitted in Eq. (2.3) the sum over the types of charges. By averaging these equations over the statistical ensemble and subtraction of the averaged equations from the initial equations, we obtain the equations for the regular and stochastic components. We will write down these equations with certain simplifications which are not fundamental for what follows: 1) $\varphi^{\rm r} = 0$, 2) the turbulence is stationary and the average of the four-dimensional Fourier components of any two stochastic quantities a_{k_1} and b_{k_2} is proportional to $\delta(k_1 + k_2) = \delta(\mathbf{k}_1 + \mathbf{k}_2)\delta(\omega_1 + \omega_2)$; $k = \{\mathbf{k}, \omega\}$; $dk = d\mathbf{k}\,d\omega$:

$$= \delta(\mathbf{k}_1 + \mathbf{k}_2)\,\delta(\omega_1 + \omega_2);\ k = \{\mathbf{k},\ \omega\};\ dk = d\mathbf{k}\,d\omega:$$

$$\frac{\partial f^{\rm r}}{\partial t} + \mathbf{v}\frac{\partial f^{\rm r}}{\partial \mathbf{r}} = ie\int \mathbf{k}_1\frac{\partial}{\partial \mathbf{p}}\langle\varphi^{\rm st}_{k_1} f^{\rm st}_{k_2}\rangle\,dk_1\,dk_2, \tag{2.4}$$

$$-i(\omega - \mathbf{k}\mathbf{v})\,f^{\rm st}_k - ie\varphi^{\rm st}_k\left(\mathbf{k}\frac{\partial f^{\rm r}}{\partial \mathbf{p}}\right) = ie\int \mathbf{k}_1\frac{\partial}{\partial \mathbf{p}}\left(\varphi^{\rm st}_{k_1} f^{\rm st}_{k_2} - \langle\varphi^{\rm st}_{k_1} f^{\rm st}_{k_2}\rangle\right)\delta(k - k_1 - k_2)\,dk_1\,dk_2, \tag{2.5}$$

$$k^2\varphi^{\rm st}_k = 4\pi e\int f^{\rm st}_k\,d\mathbf{p}/(2\pi)^3. \tag{2.6}$$

These equations are the exact consequences of the initial equations. Equation (2.5) describes the nonlinear relation of $f^{\rm st}_{\rm k}$ and $\varphi^{\rm st}_{\rm k}$ and contains the resonance factor $(\omega - \mathbf{k}\cdot\mathbf{v})$. If (2.5) is solved and this relation is found, then (2.6) gives the nonlinear equation for the stochastic potential, and the right-hand side of (2.4) describes the action of the stochastic potential on the regular part of the distribution function.

c) Expansion in the stochastic-potential amplitudes. The simplest approach to solution of (2.5) under the conditions of weak turbulence, it would appear, consists of expanding $f_{\mathbf{k}}^{st}$ in $\varphi_{\mathbf{k}}^{st}$. The lack of validity of this operation has already been made clear above and, we may note, follows immediately from the result obtained in the first approximation from (2.5):

$$f_k^{st} \approx -\frac{e\varphi_k^{st}}{\omega - \mathbf{kv}} \left(\mathbf{k}\frac{\partial f^r}{\partial \mathbf{p}}\right). \tag{2.7}$$

Of course, the division by $\omega - \mathbf{k}\cdot\mathbf{v}$ is allowable if there is no resonance. However, near a resonance $\omega \to \mathbf{k}\cdot\mathbf{v}$ the result loses any meaning. It would appear that the problem arising here is only one of avoiding a pole, which has been discussed in detail by Landau[9], and the result should lead to the well known Landau damping. Actually this is not completely true, and the entire problem discussed by Landau requires reexamination here. Landau considered the problem only in the linear approximation for weak perturbations and showed that the exact initial formulation of the problem of development of such a perturbation (which is more convenient to discuss by means of a Laplace transform) leads asymptotically to damping of the perturbations, which can be described if we go around the pole $1/(\omega - \mathbf{k}\cdot\mathbf{v})$ in such a way that it contains an infinitely small positive addition, i.e., write $1/(\omega - \mathbf{k}\cdot\mathbf{v})$ in the form $1/(\omega - \mathbf{k}\cdot\mathbf{v} + i\delta)$, $\delta \to +0$. Why is it not possible to transfer these results to (2.7)? It has already been emphasized repeatedly that nonlinearity is a necessary composite element entering into the very idea of turbulence. Therefore it seems evident that under the conditions $\omega - \mathbf{k}\cdot\mathbf{v} \to 0$, when the first term on the left-hand side of (2.5) becomes small, it is not permissible to discard the nonlinear terms. However, we can understand more clearly the substance of the differences between the present discussion and that carried out by Landau if we take into account that Eq. (2.7) was written down for the stochastic components, for which the formulation of the initial problem is impossible in principle. As was emphasized above, it is just the lack of dependence of the stochastic components on the initial conditions and the nonreproducibility of measurements of the stochastic components which are the starting point in the concept of turbulence.

If, in spite of the above, we nevertheless continue expansion of $f_{\mathbf{k}}^{st}$ in $\varphi_{\mathbf{k}}^{st}$, then from Eq. (2.5) it is easy to find the next terms of the expansion, which contain $\varphi_{\mathbf{k}_1}^{st}\varphi_{\mathbf{k}_2}^{st}$ and $\varphi_{\mathbf{k}_1}^{st}\varphi_{\mathbf{k}_2}^{st}\varphi_{\mathbf{k}_3}^{st}$ and so forth. We will not write them out here.

d) The quasilinear approximation and nonlinearity of the diffusion of particles in turbulent fluctuations. If, in spite of what has been said above, we limit ourselves to approximation (2.7), assuming that $1/(\omega - \mathbf{k}\cdot\mathbf{v})$ is $1/(\omega - \mathbf{kv} + i\delta)$, $\delta \to +0$, then, substituting (2.7) into (2.4), we obtain the well known quasilinear equations

$$\frac{\partial f^r}{\partial t} + \mathbf{v}\frac{\partial f^r}{\partial \mathbf{r}} = \frac{\partial}{\partial p_i} D_{ij} \frac{\partial f^r}{\partial p_j}, \tag{2.8}$$

$$D_{ij} = e^2\pi \int k_i k_j |\varphi_k|^2 \delta(\omega - \mathbf{kv})\, dk,$$

$$\langle \varphi_k^{st}\varphi_{k'}^{st}\rangle = |\varphi_k|^2 \delta(k + k'). \tag{2.9}$$

We will see subsequently that ν_{eff} enters into the resonance denominator and that the operation carried out has some meaning when the diffusion coefficient is approximately independent of ν_{eff}. We note immediately that the equation obtained is identical with that which follows from the balance equations for elementary excitations and describes the processes of Cerenkov radiation and absorption of the waves. If we take into account the next terms of the expansion of $f_{\mathbf{k}}^{st}$ in $\varphi_{\mathbf{k}}^{st}$, we can find the nonlinear terms of particle diffusion in turbulent fluctuations, in particular, terms containing $\langle \varphi_{\mathbf{k}_1}^{st}\varphi_{\mathbf{k}_2}^{st}\varphi_{\mathbf{k}_3}^{st}\rangle$ and $\langle \varphi_{\mathbf{k}_1}^{st}\varphi_{\mathbf{k}_2}^{st}\varphi_{\mathbf{k}_3}^{st}\varphi_{\mathbf{k}_4}^{st}\rangle$. Without going into the details of the calculations (see Ref. 42), we will emphasize here a number of factors which are important for what follows. If a resonance $\omega_{\mathbf{k}} = \mathbf{k}\cdot\mathbf{v}$ is possible, the nonlinear terms in the diffusion contain higher powers of the resonance denominators $1/(\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{v})$ and it turns out that the approximation in which these nonlinear terms do not depend on ν_{eff} usually does not exist, i.e., perturbation theory cannot be used. If a resonance $\omega_{\mathbf{k}} = \mathbf{k}\cdot\mathbf{v}$ is impossible, there is no division by zero. However, in the higher approximations resonance arises in the beats:

$$\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}} = (\mathbf{k}_1 - \mathbf{k})\,\mathbf{v}, \tag{2.10}$$

which can be generally satisfied even if $\omega_{\mathbf{k}} \neq \mathbf{k}\cdot\mathbf{v}$. Nevertheless it remains an open question how to regard the resonance denominators $1/[\omega - \omega_1 - (\mathbf{k} - \mathbf{k}_1)\mathbf{v}]$ and what to insert, $+i\delta$ or $-i\delta$? The answer to this question cannot be obtained within the framework of perturbation theory, since it is necessary to take into account turbulent collisions. If nevertheless a resonance denominator of this form enters in the first power, we can hope that in a certain approximation we can use the approximation (1.7) and obtain a result approximately independent of ν_{eff}. If we follow this course of action, it is possible to obtain the nonlinear terms in the diffusion coefficient of (2.8), which contain $|\varphi_{\mathbf{k}_1}|^2 |\varphi_{\mathbf{k}_2}|^2$. This is obtained if in the term containing $\langle \varphi_{\mathbf{k}_1}^{st}\varphi_{\mathbf{k}_2}^{st}\varphi_{\mathbf{k}_3}^{st}\varphi_{\mathbf{k}_4}^{st}\rangle$, the average of the four potentials is broken down into the possible averages of two potentials and if condition (2.10) is used to prove that in the right-hand side of Eq. (2.4) all derivatives with respect to momenta of power higher than second go to zero. It is interesting that the result arising from this nonlinear term coincides exactly with that obtained in the theory of elementary excitations if we take into account only the one scattering process shown in Fig. 4a. This is, so to speak, scattering by a bare charge. However, there still remains a nonlinear term containing $\langle \varphi_{\mathbf{k}_1}^{st}\varphi_{\mathbf{k}_2}^{st}\varphi_{\mathbf{k}_3}^{st}\rangle$. If we completely neglect correlations, it goes to zero. Thus it is necessary to express this average approximately in terms of the average of four $\varphi_{\mathbf{k}}^{st}$.

We will make use of the nonlinear equation for the stochastic potential, which is obtained from (2.6) by substitution of the expansion of $f_{\mathbf{k}}^{st}$ in $\varphi_{\mathbf{k}}^{st}$:

$$\begin{aligned} k^2\varepsilon_k \varphi_k^{st} = {} & \int S_{k,k_1,k_2} (\varphi_{k_1}^{st}\varphi_{k_2}^{st} - \langle \varphi_{k_1}^{st}\varphi_{k_2}^{st}\rangle)\, \delta(k - k_1 - k_2)\, dk_1\, dk_2 \\ & + \int \Sigma_{k,k_1,k_2,k_3} (\varphi_{k_1}^{st}\varphi_{k_2}^{st}\varphi_{k_3}^{st} - \varphi_{k_1}^{st}\langle \varphi_{k_2}^{st}\varphi_{k_3}^{st}\rangle - \\ & - \langle \varphi_{k_1}^{st}\varphi_{k_2}^{st}\varphi_{k_3}^{st}\rangle)\, \delta(k - k_1 - k_2 - k_3)\, dk_1\, dk_2\, dk_3, \end{aligned} \tag{2.11}$$

where

$$\varepsilon_k = 1 + \frac{4\pi e^2}{k^2} \int \frac{1}{\omega - \mathbf{k}\mathbf{v}} \left(\mathbf{k}\frac{\partial f^{\mathrm{r}}}{\partial \mathbf{p}}\right) \frac{d\mathbf{p}}{(2\pi)^3}, \quad (2.12)$$

$$S_{k,k_1,k_2} = 4\pi e^3 \int \frac{1}{\omega - \mathbf{k}\mathbf{v}} \left(\mathbf{k}_1 \frac{\partial}{\partial \mathbf{p}}\right) \frac{1}{\omega_2 - \mathbf{k}_2\mathbf{v}} \left(\mathbf{k}_2 \frac{\partial f^{\mathrm{r}}}{\partial \mathbf{p}}\right) \frac{d\mathbf{p}}{(2\pi)^3},$$

$$\Sigma_{k,k_1,k_2,k_3} = -4\pi e^4 \int \frac{1}{\omega - \mathbf{k}\mathbf{v}} \left(\mathbf{k}_1 \frac{\partial}{\partial \mathbf{p}}\right) \frac{1}{\omega - \omega_1 - (\mathbf{k} - \mathbf{k}_1)\mathbf{v}} \quad (2.13)$$

$$\times \left(\mathbf{k}_2 \frac{\partial}{\partial \mathbf{p}}\right) \frac{1}{\omega_3 - \mathbf{k}_3\mathbf{v}} \left(\mathbf{k}_3 \frac{\partial}{\partial \mathbf{p}}\right) f^{\mathrm{r}} \frac{d\mathbf{p}}{(2\pi)^3}.$$

Here we consider the fact that ϵ_{k} is not a linear permittivity, since it contains f^{r} which depends, generally speaking, on W, and does not contain the initial distribution function. In the first approximation we can use (2.11) to express $\varphi_{\mathrm{k}}^{\mathrm{st}}$ in terms of the two $\varphi_{\mathrm{k}}^{\mathrm{st}}$:

$$\varphi_k^{\mathrm{st}} \approx (k^2\varepsilon_k)^{-1} \int S_{k,k_1,k_2} (\varphi_{k_1}^{\mathrm{st}}\varphi_{k_2}^{\mathrm{st}} - \langle\varphi_{k_1}^{\mathrm{st}}\varphi_{k_2}^{\mathrm{st}}\rangle)\, \delta(k - k_1 - k_2)\, dk_1\, dk_2. \quad (2.14)$$

If we substitute this into the term with nonlinear diffusion, which is proportional to $\langle \varphi_{\mathrm{k}_1}^{\mathrm{st}} \varphi_{\mathrm{k}_2}^{\mathrm{st}} \varphi_{\mathrm{k}_3}^{\mathrm{st}} \rangle$, the final result of the calculation corresponds in the balance equation for elementary excitations to the interference term of scattering from the two processes shown in Figs. 4a and b. It still remains, however, to "find" the term corresponding to the square of the matrix element of the process shown in Fig. 4b. Before turning to this question, let us consider the fact that in (2.14) we have made one further illegal operation, namely division by ϵ_{k}. The quantity ϵ_{k} goes to zero near a resonance corresponding to the natural mode of fluctuations of the plasma $\omega_{\mathbf{k}}$, i.e., for $\omega \to \omega_{\mathbf{k}}$:

$$\varepsilon_k \approx \left.\frac{\partial \varepsilon_k}{\partial \omega}\right|_{\omega=\omega_{\mathbf{k}}} (\omega - \omega_{\mathbf{k}}).$$

In reality, as can easily be seen, the interference term involves $1/(\epsilon_{\mathrm{k-k_1}})$. The approach to zero of $\epsilon_{\mathrm{k-k_1}} = \epsilon_{\mathbf{k}-\mathbf{k}_1,\omega-\omega_1}$ means that

$$\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} = \omega_{\mathbf{k}_2}, \quad \mathbf{k} - \mathbf{k}_1 = \mathbf{k}_2. \quad (2.15)$$

However, this is precisely the condition for decay. In this process the plasma particles do not exchange energy and momentum with the fluctuations, and D_{ij} under the resonance conditions (2.15) actually turns out to be equal to zero. Proof of this can be obtained if we use near resonance the relation

$$\mathrm{Im}\, \varepsilon_k^{-1} \approx -\pi\delta(\varepsilon_k), \quad (2.16)$$

which, as we will see, has only an approximate meaning. Of course, the existence of effective turbulent collisions of particles and waves creates an imaginary part in ϵ_{k}, but in this case the effective collisions, which are associated with the decay process itself (see below), turn out to be more important. It is important that (2.16) has in principle an approximate nature.

4) Nonlinear scattering and stochastic heating. Let us turn now to the clarification of how purely nonlinear scattering is obtained. This is a very important question which requires explanation, since up to the present time it has not been completely understood even in the latest publication[43]. This in turn concerns the problem closely related to nonlinear scattering of the so-called stochastic heating. The progress achieved in this question is extremely important for interpretation of experiments on stochastic heating (both by turbulence and by external stochastic fields).

We will begin at once with a formal answer to the question of just what term describes nonlinear scattering. The fact is that nonlinear scattering arises formally from the quasilinear diffusion coefficient (2.9). This statement is at first glance at least not obvious. In fact, we have specified that the resonance $\omega_{\mathbf{k}} = \mathbf{k}\cdot\mathbf{v}$ is not satisfied, but Eq. (2.9) contains $\delta(\omega - \mathbf{k}\cdot\mathbf{v})$. Here we come to one of the important properties of any stochastic fields, namely, that they cannot have a definite unique relation between ω and $\mathbf{k}$ corresponding, for example, to the mode $\omega = \omega_{\mathbf{k}}$. However, for a given $\mathbf{k}$ the potential correlator $|\varphi_{\mathrm{k}}|^2$ is described as a function of frequency by some curve with a maximum near $\omega = \omega_{\mathbf{k}}$. The half-width of this curve determines the characteristic time of the correlations $\Delta\omega^{-1}$, which is of the order of $1/\nu_{\mathrm{eff}}$ (determined mainly, as will be shown below, by turbulent collisions due to interaction of the waves with each other). For weak turbulence $\Delta\omega \sim \nu_{\mathrm{eff}} \ll \omega_{\mathbf{k}}$ and the maximum is sharply expressed. However, this in no way means that the correlation curve $|\varphi_{\mathrm{k}}|^2$ does not extend beyond the limits of $\Delta\omega$ to smaller and larger frequencies, to those frequencies which are separated from $\omega_{\mathbf{k}}$ by amounts of the order of $\omega_{\mathbf{k}}$ or more. Thus, we are discussing the remote tails of the correlation functions. The structure of the correlation curves in their central portion cannot be continued to their remote tails. For example, in the center one attempts to describe the correlation curves by a Lorentz formula[44]

$$\nu_{\mathrm{eff}}\, [(\omega - \omega_{\mathbf{k}})^2 + \nu_{\mathrm{eff}}^2]^{-1}, \quad (2.17)$$

which corresponds to a field disordered by random impulses with a frequency ν_{eff}. The same formula is then used for the remote tails, for example $\omega \ll \omega_{\mathbf{k}}$, and $\nu_{\mathrm{eff}}/\omega_{\mathbf{k}}^2$ is obtained. The error in such treatments lies in the fact that it is assumed a priori that ν_{eff} is the same in the center and in the tails of the correlation curve. In reality ν_{eff} turns out to be substantially different. Of course, it would be possible in Eq. (2.17) to consider ν_{eff} as a function of ω and k (which actually must be done anyway), but in this case the meaning of (2.17) as a Lorentz formula is completely lost. The interest in these tails of the correlation curves is immediately associated with the problem of stochastic heating. For example, for stochastic high-frequency fields the condition $\omega_{\mathbf{k}} = \mathbf{k}\cdot\mathbf{v}$ is not satisfied. In the tails of the correlation curves in general there is no unique relation between ω and $\mathbf{k}$, and the condition $\omega = \mathbf{k}\cdot\mathbf{v}$ can be satisfied.

It turns out that the structure of these correlation-curve tails can be obtained in very general form if we utilize only the condition of weak turbulence, i.e., the parameter (1.2). Here we must be convinced that the structure of these tails does not depend on those ν_{eff} which determine the centers of the correlation curves. For this purpose we turn to Eq. (2.11), multiply it by $\varphi_{\mathrm{k}}^{\mathrm{st}}$, and integrate over k'. Using (2.14), we obtain an equation for the correlation function $|\varphi_{\mathrm{k}}|^2$. Furthermore, since we are interested in the region of ω and $\mathbf{k}$ far from resonance, ϵ_{k} is not close to zero but is of the order of or substantially greater than unity.

Therefore terms proportional to $|\varphi_{\mathbf{k}}|^2$ in the right-hand-side are at least W/nT times less than $\epsilon_{\mathbf{k}}$ and can be discarded; this gives

$$\varphi_k{}^2 = (2/k^2) \int |S_{k,k_1,k_2} + S_{k,k_2,k_1}|^2 |\varepsilon_k|^{-2} \times |\varphi_{k_1}|^2 |\varphi_{k_2}|^2 \delta(k - k_1 - k_2)\, dk_1\, dk_2. \qquad (2.18)$$

The formula can be used for explicit calculation of the correlation tails if we neglect the correlations in the right-hand side. This approximation is the first term of an expansion in $\nu_{\mathrm{eff}}/\omega_{\mathbf{k}}$, where the correlation tails in the right-hand side are already making a negligible contribution. By substituting (2.18) into the quasilinear diffusion coefficient, we obtain exactly the nonlinear scattering described in the method of elementary excitations by the diagram of Fig. 4b. Thus, we have actually shown that all terms describing the sum of the matrix elements of nonlinear and Thompson scattering appear. However, important consequences also follow from this.

First, stochastic heating owes its existence not only to the tails of the correlation curves describing nonlinear scattering, but also to Compton scattering and to their interference. In addition we can say that there is no other stochastic heating than that described by induced scattering (under the conditions which were the subject of the discussion carried out above, in which there is no strong linear or nonlinear absorption of turbulent fluctuations).

Second, it must be kept in mind that the scattering is described by the square of the sum of the nonlinear and Compton scattering matrix elements. The terms of this sum often have opposite signs, and for electrons the effect is reduced by a factor $k^2 v_{\mathrm{Te}}^2/\omega_{\mathrm{pe}}^2$.

Third, the stochastic heating of ions can be no less than that of electrons, since the nonlinear scattering, which for them is much greater than the Thompson scattering, depends only on the electron mass. These conclusions have great practical significance for plasma heating by stochastic fields.

f) Polarization "coats" of particles and Fermi excitations of a turbulent plasma. No less important is a general conclusion concerning the concept of elementary excitations. In obtaining the same equations as in Section 1, we have brought out the meaning of what in elementary-excitation theory is called a "particle" of a plasma. And in fact these are also elementary excitations—electronic and ionic, clothed in polarization "coats" of charges of another sign. For just this reason we have appearance of nonlinear scattering, which is absent for individual particles in a vacuum. This is scattering by the polarization coat. These electronic and ionic excitations are described by the regular part of the distribution function, i.e., the function Φ entering into Section 1 is

$$\Phi = f^r.$$

In dividing the microdistribution function f into f^r and f^{st} we actually carried out a renormalization of the distribution function, that is f^r describes elementary Fermi excitations, and f^{st} Bose excitations—plasmons, the separation of f^{st} being harmless since f^r describes already clothed particles. It is clear also that the true particles taking part in the micromotions are described both by f^r and f^{st}, i.e., they part both in the fluctuations and in the scattering of these fluctuations. The order of the entire picture obtained as a result of this division and, in particular, the fact that the processes of interaction of plasmons and quasiparticles are described by positive probabilities containing the square of the moduli of the matrix elements, are a good illustration of the simplicity of the physical concepts established in the method of elementary excitations.

g) The equation for correlation functions of a turbulent potential. It still remains for us to carry out the second part of the program, namely, to obtain an equation for the number of plasmons $N_{\mathbf{k}}$. For this purpose we will carry out a number of additional almost illegal operations whose justification will be given somewhat below. By multiplying Eq. (2.11) by $\varphi_{\mathbf{k}'}^{st}$ we form an equation for $|\varphi_{\mathbf{k}}|^2$. Here we will use Eq. (2.14) for calculation of $\langle \varphi_{\mathbf{k}}^{st} \varphi_{\mathbf{k}_1}^{st} \varphi_{\mathbf{k}_2}^{st} \rangle$. We have already discussed to what extent this is not valid near $\epsilon_{\mathbf{k}} = 0$. In addition we will write down the equation which is obtained by this means:

$$k^2(\varepsilon_k + \varepsilon_k^N)|\varphi_k|^2 = 2\int |S_{k,k_1,k_2} + S_{k,k_2,k_1}|^2 \varepsilon_{-k}^{-1} |\varphi_{k_1}|^2 |\varphi_{k_2}|^2 \delta(k - k_1 - k_2)\, dk_1\, dk_2, \qquad (2.19)$$

where

$$\varepsilon_k^N = \int \Sigma_{k,k_1} |\varphi_{k_1}|^2\, dk_1, \qquad (2.20)$$

$$\Sigma_{k,k_1} = \Sigma_{k,k_1,k,-k_1} + \Sigma_{k,k_1,-k_1,k} + \varepsilon_{k-k_1}^{-1} |\mathbf{k} - \mathbf{k}_1|^{-2} S_{k-k_1,k,-k_1}(S_{k,k_1\,k-k_1} + S_{k,k-k_1,k_1}). \qquad (2.21)$$

Equation (2.19), in contrast to Eq. (2.18) of which it is a generalization, is, generally speaking, meaningless. Actually, near a resonance $\omega = \omega_{\mathbf{k}}$ the denominator of the right-hand side of (2.19) goes to zero. It is clear that this has occurred as a result of the illegal use of (2.14) near resonance. At the same time, the balance equations, which are obtained from this equation on the assumption of (2.16), turn out to be correct. Above we wrote the equation for the correlation functions symbolically in the form of the functional (1.10). Now this functional can be written in explicit form, transferring all terms of (2.19) to the left-hand side of the equation. Then it is necessary to integrate over frequency as indicated in (1.11) and, having taken the imaginary part, to neglect correlations, assuming approximately $|\varphi_{\mathbf{k}}|^2 \approx |\varphi_{\mathbf{k}}|^2 \delta(\omega - \omega_{\mathbf{k}})$. Division by $\epsilon_{\mathbf{k}}$ then takes on a certain meaning if we use (2.16). In reality we must remember that (2.16) is only an approximation of more accurate resonance denominators which take into account turbulent collisions. The result is the balance equations, which are identical with the equations of Section 1 if we take into account the processes of radiation, scattering, and decay. We will leave this verification to the reader, and emphasize here only a number of fundamental factors.

Second, the terms with Σ in (2.21) describe induced Thompson scattering; the second term of (2.21) contains both linear scattering and the interference of nonlinear and Thompson scattering. Thus, in the equations for the plasmons, all terms describing the sum of two scattering processes also appear. Second, decays arise when $\epsilon_{\mathbf{k}-\mathbf{k}_1}$ is close to zero. In that case we cannot speak of nonlinear scattering; in the second term of (2.21) $1/\epsilon_{k-k_1}$ has the greatest imaginary part (i.e., it is necessary to write Eq. (2.16) for it). Then we can neglect the first term of (2.21) and convince ourselves that the second gives induced decays, while the right-hand side of (2.19) gives spontaneous decays.

Thus, with a number of assumptions, and using a number of approximate relations which require justification and proof, we have obtained all the relations of elementary-excitation theory of Section 1. Similar but much more complicated calculations can be carried out for the more general case of arbitrary modes (not only longitudinal waves, as in this section), strong magnetic fields, nonuniform plasma, plasma with admixture of relativistic particles, relativistic plasma, and so forth. The method of elementary excitations, which permits calculation by independent means of the probability of various processes, not only has a heuristic value for numerous generalizations, but also permits many errors, both theoretical and physical, to be avoided (for example, inclusion of only nonlinear scattering in the problem of stochastic heating, as was done in ref. 44, or inclusion of only Thompson scattering in the problem of electron momentum loss in a plasma with Langmuir turbulence, as was done erroneously in ref. 43). Presentation of the final result, when the method of statistical averaging is used, in a form containing the square of the matrix element of the corresponding probability, sometimes requires rather laborious calculations and, where cancellations occur as the result of the difference in sign of the matrix elements, extremely high accuracy in the calculations.

3. EFFECTIVE TURBULENT COLLISIONS FOR INTERACTION OF PARTICLES AND WAVES, AND IRREVERSIBILITY IN THE THEORY OF PLASMA TURBULENCE

a) The turbulent collision operator. As has already been remarked, turbulent collisions must be integrally contained in turbulence theory and are due to nonlinear processes. In view of the fact that various types of nonlinear interactions exist, and also various resonances, for example, between waves and particles, between different waves without participation of particles, and so forth, there are also various turbulent collisions. In particular, the quantity ν_{eff} which we have used above is generally different for different processes.

We will begin here with choice of turbulent collisions of the simplest type, due to interaction of particles and waves in radiation, absorption, and scattering. In other words, we will consider the resonance $\omega = \mathbf{k}\cdot\mathbf{v}$. The question of the broadening of such resonances was taken up by Dupree[45,46] (see also Weinstock[47] for the purpose of describing strong turbulence. In reality, however, under conditions where the parameter (1.2) exists, the turbulence is always weak. In addition Dupree[45,46] obtained incorrect results concerning stochastic heating. A more systematic discussion of the problem has been given by Rudakov and Tystovich[8]. We have already noted that the problem of avoiding the pole $\omega = \mathbf{k}\cdot\mathbf{v}$ is fundamental and cannot be solved here by the path used by Landau (because the initial values of the stochastic quantities cannot be given). We will discuss this question here on the basis of the solutions found in ref. 8. We note that near the resonance $\omega = \mathbf{k}\cdot\mathbf{v}$ it is necessary to take into account nonlinear terms in the equation for the stochastic potential.

Let's turn again to Eq. (2.5). It's right hand side contains nonlinear terms. This equation cannot be solved in general form. Generally speaking, it is necessary to take into account the terms of the entire expansion of $f_{\mathbf{k}}^{\text{st}}$ in $\varphi_{\mathbf{k}}^{\text{st}}$. However, it is necessary to make a definite selection of the most important terms, using the small parameter (1.2). In fact, although turbulent collisions cannot be neglected under resonance conditions, nevertheless $\nu_{\text{eff}} \ll \omega_{\mathbf{k}}, \mathbf{k}\cdot\mathbf{v}$. The entire problem is similar to the theory of the natural width of spectral lines in quantum electrodynamics[26]. Perturbation theory cannot be used to describe this problem, but if we take into account the fact that the width of a line is much less than the distance between energy levels, we can sum to the end the most important terms. In this case the indicated width depends functionally on the level of turbulence or, more accurately, the correlation function $|\varphi_{\mathbf{k}}|^2$. Let us inquire into the nature of the nonlinear terms contained in the right-hand side of (2.5). The integrals over $\mathbf{k}_1$ and $\mathbf{k}_2$ are best represented in the form of sums over the possible modes $\mathbf{k}_1$ and $\mathbf{k}_2$ if we place the entire system in a large cubical box of dimensions L. In this sum we then encounter those terms which are proportional to $f_{\mathbf{k}}^{\text{st}}$, which occurs in the left-hand side of the equation, and those which contain other $f_{\mathbf{k}_1}^{\text{st}}$ with $\mathbf{k}_1 \neq \mathbf{k}$ (here it is necessary to keep in mind that $f_{\mathbf{k}}^{\text{st}}$ is related to $\varphi_{\mathbf{k}}^{\text{st}}$). Terms proportional to $f_{\mathbf{k}}^{\text{st}}$ we will call diagonal, and the remaining terms nondiagonal. It is clear that the nondiagonal terms play the role of an external inducing force and cannot lead to the desired broadening of the resonance, while the diagonal terms lead to broadening. Among all of the diagonal terms it is necessary to retain only the first term of the expansion in $\nu_{\text{eff}}/\omega_{\mathbf{k}}$. In order to accomplish this mathematically, it is necessary to rewrite Eq. (2.5), separating immediately the diagonal terms:

$$-i(\omega-\mathbf{k}\mathbf{v}+i\hat{\nu}_k(\mathbf{p}))f_k^{\text{st}}-ie\varphi_k^{\text{st}}\left(\mathbf{k}\frac{\partial f^{\text{r}}}{\partial\mathbf{p}}\right)$$
$$=\hat{\nu}_k(\mathbf{p})f_k^{\text{st}}+ie\int\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)(\varphi_{k_1}^{\text{st}}f_{k_2}^{\text{st}}-\langle\varphi_k^{\text{st}}f_{k_2}^{\text{st}}\rangle)\,\delta(k-k_1-k_2)\,dk_1\,dk_2; \tag{3.1}$$

here $\hat{\nu}_{\mathbf{k}}(\mathbf{p})$ is an operator acting on the momenta $\mathbf{p}$ of the function $f_{\mathbf{k}}^{\text{st}}$ and describing the effective turbulent collisions. As will be evident, it depends functionally on the correlation function $|\varphi_{\mathbf{k}}|^2$. Actually, in the right-hand side of (3.1) the diagonal terms should be contracted (the first term combined with the diagonal part of the second). If we are interested in the specific form of $\hat{\nu}_{\mathbf{k}}(\mathbf{p})$ with accuracy to terms of first order in $\nu_{\text{eff}}/\omega_{\mathbf{k}}$, then the contraction in the right-hand side should occur only with this accuracy. In Eq. (3,1) we can already use the new perturbation theory, whose initial approximation will be neglecting the nonlinear terms of the right-hand side of (3.1), i.e., we can write[8]

$$f_k^{\text{st}}\approx-\frac{e\varphi_k^{\text{st}}}{\omega-\mathbf{k}\mathbf{v}+i\hat{\nu}_k(\mathbf{p})}\left(\mathbf{k}\frac{\partial f^{\text{r}}}{\partial\mathbf{p}}\right). \tag{3.2}$$

This expression, in contrast to Eq. (2.7), already has no singularity at $\omega = \mathbf{k}\cdot\mathbf{v}$. However, nothing has yet been achieved, since we do not know $\hat{\nu}_{\mathbf{k}}(\mathbf{p})$. Before turning to the search for this quantity, it is convenient to write (3.2) in a simpler form, introducing the inverse operator $\hat{g}_k(p)$:

$$\hat{g}_k(\mathbf{p})(\omega - \mathbf{kv} + i\hat{\nu}_k(\mathbf{p}))f^{\,t} = f_k^{st}. \qquad (3.3)$$

By means of this operator we can rewrite Eq. (3.1) in the form

$$f_k^{st} + e\varphi_k^{st}\hat{g}_k(\mathbf{p})\left(\mathbf{k}\frac{\partial f^r}{\partial \mathbf{p}}\right) = i\hat{g}_k(\mathbf{p})\hat{\nu}_k(\mathbf{p})f_k^{st}$$
$$- e\hat{g}_k(\mathbf{p})\int\left(\mathbf{k}_1\frac{\partial}{\partial \mathbf{p}}\right)(\varphi_{k_1}^{st}f_{k_2}^{st} - \langle\varphi_{k_1}^{st}f_{k_2}^{st}\rangle)\,\delta(k-k_1-k_2)\,dk_1\,dk_2. \qquad (3.4)$$

In order to find $\hat{\nu}_k(p)$ we must require that in the right-hand side of (3.4) the diagonal term is absent in the first approximation. For this reason it is necessary to substitute f_k^{st} from (3.4) into the last term of the right-hand side of (3.4) and separate in it the diagonal term in the first approximation in ν_{eff}/ω_k. We obtain[8]

$$\hat{\nu}_k(\mathbf{p}) = -ie^2\frac{\partial}{\partial p_i}\int dk_1 k_{1i}k_{1j}|\varphi_{k_1}|^2\hat{g}_{k-k_1}(\mathbf{p})\frac{\partial}{\partial p_j}. \qquad (3.5)$$

This specific expression for the turbulent collision operator contains $\hat{g}_{k-k_1}$, which in turn is related to $\hat{\nu}_k(p)$ in view of (3.3), i.e., (3.3) and (3.5) are a complex system of integral-differential equations, but which permit specific investigation and solution in a number of cases which are important in applications.

b) Perturbation theory with inclusion of turbulent collisions. If such a solution is found, the specific form of the zero approximation (3.2) is known and a systematic perturbation theory can be developed. Substituting (3.2) into the right-hand side of (2.4), we obtain a "new" quasilinear equation. It has the form of (2.8) with a changed diffusion coefficient

$$D_{ij} = ie^2\int k_ik_j\hat{g}_k(\mathbf{p})\,|\varphi_k|^2\,dk, \qquad (3.6)$$

which differs from (2.9) only in the smearing of the resonance $\omega = \mathbf{k}\cdot\mathbf{v}$ as the result of turbulent collisions (associated, in essence, with the quasilinear interaction itself). Then we can systematically find the next terms of the expansion in this perturbation theory and obtain the expansion and consequently also the corrections to the diffusion coefficient (3.6). No divergences or difficulties with the denominators arise here.

In exactly the same way we can also obtain an equation for the correlation function, which replaces (2.19):

$$k^2(\tilde{\varepsilon}_k + \tilde{\varepsilon}_k^N)|\varphi_k|^2$$
$$= 2\int\tilde{\varepsilon}_{-k}^{-1}|\varphi_{k_1}|^2|\varphi_{k_2}|^2\,dk_1\,dk_2\,|\tilde{S}_{k,k_1,k_2} + \tilde{S}_{k,k_2,k_1}|^2\,\delta(k-k_1-k_2), \qquad (3.7)$$

where $\tilde{\epsilon}_k$ differs from ϵ_k of (2.12) in that instead of the denominator $1/(\omega - \mathbf{k}\cdot\mathbf{v})$ it involves the expression $\hat{g}_k(p)$. In the same way $\tilde{S}_{k,k_1,k_2}$ differs from S_{k,k_1,k_2} given by (2.13), and $\tilde{\Sigma}_{k,k_1,k_2,k_3}$ from Σ_{k,k_1,k_2,k_3}. Finally, $\tilde{\epsilon}_k^N$ differs from (2.20) in that it involves $\tilde{\Sigma}_{k,k_1}$, which is obtained from (2.21) by replacement of the expressions Σ and S by $\tilde{\Sigma}$ and $\tilde{S}$ and by discarding the second term in (2.21). It is in fact already included in the zero approximation. Thus, the resonances $\omega = \mathbf{k}\cdot\mathbf{v}$ turn out to be smeared in the entire picture. Before turning to analysis of the results of this approach, it is necessary to emphasize a number of factors.

First, this perturbation theory is systematic in contrast to the usual variety which simply leads to divergences or improper integrals whose values begin to depend on the order of integration.

Second, as will be seen, it satisfies the necessary requirement that the integral quantities can be expanded in W/nT, i.e., the next orders of perturbation theory are small in comparison with the preceding ones for $W/nT \ll 1$ (actually the small parameter is $(W/nT)^{1/3}$ and the result cannot be expanded in W/nT).

Third, we have not removed all divergences here, as can be seen directly from the right-hand side of (3.7). In actuality, for example, for $\omega \gg kv_T$, ν_{eff}, $\tilde{\epsilon}_{-k}$ has the usual form $1 - (\omega_{pe}^2/\omega^2)$ and goes to zero for $\omega = \omega_{pe}$. In fact this is already a resonance of the waves, and its nonlinear saturation requires special consideration (see Section 4).

Finally, the approach described does not go beyond the bounds of weak turbulence, but is only a more rigorous proof of it and, in particular, a basis for the quasilinear approximation. We must consider that the smearing of the resonance described by (3.6) is different from that used in the quasilinear equations in their initial form obtained at one time in early investigations by averaging over space and time. Then, instead of denominators of the type $1/(\omega - \mathbf{k}\cdot\mathbf{v})$, there were expressions of the type

$$\gamma_k[(\omega - \mathbf{kv})^2 + \gamma_k^2]^{-1}, \qquad (3.8)$$

where γ_k is the linear increment. Actually, the linear increment γ_k depends on the form of the regular function f^r (and, in particular, in the simplest case it is proportional to $\partial f^r/\partial p$), while neither $\hat{g}_k(p)$ or $\hat{\nu}_k(p)$ in general contain f^r. In addition, for absorption ($\gamma_k < 0$) the quantity (3.8) will be negative, while the quasilinear diffusion coefficient is in reality always positive. In connection with (3.8) we can also encounter incorrect statements that the quasilinear equations are applicable only for unstable modes. It will be shown below that other effective collisions associated with the interaction of modes lead to an expression of the type (3.8) in the quasilinear equation; however, these do not involve γ_k, but rather the effective frequency of the corresponding nonlinear interactions. Thus, (3.8) can be used in the quasilinear integral only if we approximate (3.8) by $-\pi$ times a δ function.

c) Turbulent collisions and irreversibility. Let us turn now to one of the important questions: how the problem of the pole $\omega = \mathbf{k}\cdot\mathbf{v}$ is actually solved in the theory of plasma turbulence. For this purpose we need to find the solution of the equation for $\hat{g}$. It is sufficient to look near resonance, assuming

$$\varepsilon = \eta/\max(\omega_k, kv_T) \ll 1, \qquad (3.9)$$

where $\eta = \omega - \mathbf{k}\cdot\mathbf{v}$. If we introduce instead of ω the variable η, i.e., instead of $\hat{g}_k(p) = \hat{g}_{k,\omega}(p)$ the operator $\hat{g}_{k,\eta}(p)$, it is easy to see that in the first approximation in the small parameter (3.9) it turns out to be diagonal in p, i.e., not an operator but a function. In the first approximation in the parameter (3.9), Eq. (3.3) takes the form

$$\left(\eta + D\frac{\partial^2}{\partial\eta^2}\right)g_{\mathbf{k},\eta}(\mathbf{p}) = 1, \qquad (3.10)$$
$$D \approx ie^2\int(\mathbf{kk}_1)^2|\varphi_{k_1}|^2 g_{\mathbf{k}-\mathbf{k}_1,-\eta_1}(\mathbf{p})\,dk_1,$$

where D is real, and the imaginary corrections to it are of order (3.9) (for more detail see ref. 8). We emphasize that the sign of D actually also determines the sign of the effective turbulent collisions and thereby the by-pass rule. So far this sign is still arbitrary. We can find the formal solution of (3.10) for a given D and substitute it into (3.11). This gives a functional equation for D, which can be solved approximately. It is convenient to carry out a Fourier transformation

$$g_{\mathbf{k},\eta} = (2\pi)^{-1} \int g_{\mathbf{k},\tau} e^{i\eta\tau}\, d\tau. \tag{3.11}$$

Then Eq. (3.10) is written in the form

$$(\partial g_{\mathbf{k},\tau}/\partial\tau) + \tau^2 D g_{\mathbf{k},\tau} = -2\pi i\delta(\tau). \tag{3.12}$$

We can solve the homogeneous equation (3.12) with the boundary condition

$$g_{\mathbf{k},\tau}\,|_{\tau\to+0} - g_{\mathbf{k},\tau}\,|_{\tau\to-0} = -2\pi i. \tag{3.13}$$

However, in solution of these equations it is necessary also to apply the physical condition of finiteness of g. The solution of the homogeneous equation (3.12) is

$$g_{\mathbf{k}} = g_0 e^{-D\tau^3/3}. \tag{3.14}$$

A discontinuity in g is possible only for $\tau = 0$. Thus, if $D > 0$, then for $\tau \to -\infty$ we conclude from (3.14) that $g_0|_{\tau<0} = 0$, and from (3.13) we find $g_0|_{\tau>0} = -2\pi i$, i.e.,

$$g_{\mathbf{k},\eta} = -i \int_0^\infty e^{i\eta\tau - [(D\tau^3)/3]}\, d\tau. \tag{3.15}$$

On the other hand, if $D < 0$, then $g_0|_{\tau>0} = 0$ and $g_0|_{\tau<0} = 2\pi i$, i.e.,

$$g_{\mathbf{k},\eta} = i \int_{-\infty}^0 e^{i\eta\tau - [(D\tau^3)/3]}\, d\tau = i \int_0^\infty e^{-i\eta\tau + [(D\tau^3)/3]}\, d\tau. \tag{3.16}$$

These two solutions are inconsistent. In other words, the nonlinear equations permit either the first or the second solution, but not a sum or combination of them. This is a consequence of the fact that the superposition principle is not satisfied for the nonlinear equations. In order to show the inconsistency of the two solutions, we will substitute one of them, (3.15), into (3.11) and obtain an equation for D:

$$D(\mathbf{k},\mathbf{p}) = (e^2/m^2) \int (\mathbf{k}\mathbf{k}_1)^2 |\varphi_{k_1}|^2\, dk_1 \times \int_0^\infty \exp[-i(\omega_1 - \mathbf{k}_1\mathbf{v})\tau - (\tau^3/3) D(\mathbf{k}-\mathbf{k}_1,\mathbf{p})]\, d\tau. \tag{3.17}$$

In the right-hand side of (3.17) there is an integral over all k_1; the function under the integral sign has a sharp maximum near $\omega_1 = \mathbf{k}_1 \cdot \mathbf{v}$, while $|\varphi_{\mathbf{k}_1}|^2$ is a smooth function of $\mathbf{k}_1$ and ω_1. Therefore in the first approximation in the parameter (3.9) we can replace the function indicated which has a sharp maximum with a δ function in the expression under the integral of (3.17). We obtain

$$D(\mathbf{k},\mathbf{p}) \approx (e^2/m^2)\pi \int (\mathbf{k}\mathbf{k}_1)^2 |\varphi_{k_1}|^2 \delta(\omega_1 - \mathbf{k}_1\mathbf{v})\, dk_1, \tag{3.18}$$

which is also an approximate solution of the functional equation (3.17). It is important that $D > 0$ in agreement with the initial assumption which led to the solution (3.15). If we then use the second solution (3.16), we obtain an equation for D whose solution differs in sign from (3.18) in agreement with (3.16). The two solutions are inconsistent since D cannot be equal to the same quantity with the opposite sign, provided that, of course, $D \neq 0$, i.e., that there is turbulence.

Here there appears a unique combination of irreversibility and equal justification of the two directions of time. It is easy to see that the two solutions for g are asymmetric in τ, i.e., irreversible (τ, as can easily be shown, has the meaning of time), and by the substitution $t \to -t$ are transformed into each other. In fact, for example, the diffusion coefficient (3.6) differs only in the sign, and replacement of t by $-t$ in the quasilinear equation leads the two solutions to the same form. It is necessary to choose one of them, since they are inconsistent, and then to take into account that the positive time direction is not yet determined. This direction is defined as that for which entropy increases. Therefore, if the first solution is chosen, then the quantity t mentioned above is the time, and if the second is chosen, then $-t$ is the time. Thus, the two solutions lead to an identical result (both, incidentally, lead also to a systematic change of entropy).

If we take into account that in the quasilinear diffusion coefficient $\mathrm{Im}\, g_{\mathbf{k}}(\mathbf{p})$ is a function with a sharp maximum, then in the first approximation in the parameter (3.9) we obtain the ordinary quasilinear diffusion coefficient

$$D_{ij} = \pi e^2 \int k_i k_j |\varphi_k|^2 \delta(\omega - \mathbf{k}\mathbf{v})\, dk.$$

However, this approximation is possible only because g occurs under the sign of an integral whose value is insensitive to smearing of the resonance. Exactly as in the first approximation in (3.9), the imaginary part of $\tilde{\epsilon}_{\mathbf{k}}$ describes the ordinary damping of the fluctuations, since g can be replaced approximately by a δ function. In this way the Landau damping problem is solved in the theory of turbulence. It is important, however, that the nature of the approximation here (and particularly $\nu_{\mathrm{eff}}/\omega_{\mathbf{k}}$) is quite different and the damping is irreversible. In order to represent the order of $\nu_{\mathrm{eff}}/\omega_{\mathbf{k}}$, it is necessary to give an estimate for ν_{eff}.

d) Evaluation of the turbulent-collision frequencies. To estimate ν_{eff} we will take into account that, according to (3.51), g is determined by the integral of two functions, of which the first, $e^{i\eta\tau}$, begins to oscillate rapidly and, consequently, the integrand approaches zero for $\tau > \tau_1 = 1/\eta = 1/(\omega - \mathbf{kv})$, and the second function $e^{-D(\tau^3/3)}$ cuts off the integrand for $\tau > \tau_2 = (3/D)^{1/3}$. If $\tau_1 \ll \tau_2$, then $g \approx 1/(\omega - \mathbf{kv})$; if on the other hand $\tau_2 \ll \tau_1$, then

$$g \approx -i \int_0^\infty e^{-(\tau/\tau_2)^3}\, d\tau \approx (\tau_2/3i)\,\Gamma(1/3) = 1/i\nu_{\mathrm{eff}}.$$

The last equation is written as a definition of ν_{eff} for $|\omega - \mathbf{kv}| \ll \nu_{\mathrm{eff}}$, which can be obtained if g is written in the form $1/(\omega - \mathbf{k}\cdot\mathbf{v} + i\nu_{\mathrm{eff}})$. Thus, in order of magnitude

$$\nu_{\mathrm{eff}} \approx D^{1/3}.$$

Assuming that in the spectrum of $|\varphi_{\mathbf{k}_1}|^2$ there is some characteristic k_0 at which the integral (3.18) is concentrated, and that $\delta(\omega_1 - \mathbf{k}_1 \cdot \mathbf{v})$ has order $1/k_0 v_T$, we obtain an estimate for $D^{1/3}$ for $k \sim k_0$*).

*Purely for the sake of simplicity we use here $m = m_e$; similar effects arise also for ions.

$$\nu_{\text{eff}} \approx \omega_{pe} (W/nT)^{1/3} (k_0 v_T/\omega_{pe})^{1/3} \quad (3.19)$$

We emphasize that this value of ν_{eff} does not have an explicit relation to the linear $\gamma_{\mathbf{k}}$.

e) Effect of turbulent collisions on nonlinear processes. Although, as we have noted, the first approximations with accuracy $\nu_{\text{eff}}/\omega_{\mathbf{k}}$ agree with the usual approximations, the existence of ν_{eff} substantially affects the next nonlinear corrections to them. The case of ion-acoustic turbulence is an important example in this respect. In the zero approximation in the presence of electron drift relative to the ions, the ordinary buildup increment of ion-acoustic fluctuations $\gamma_{\mathbf{k}}$ follows from $\tilde{\epsilon}_{\mathbf{k}}$. In the next approximation all nonlinear terms of (3.7) contribute. This gives a certain $\delta\gamma_{\mathbf{k}}$ which depends on W. It turns out that those nonlinearities are most important which are contained in $\tilde{\epsilon}_{\mathbf{k}}$, i.e., those which are corrections for the fact that ν_{eff} is finite.

In Fig. 6 we have shown $\delta\gamma_{\mathbf{k}}/\gamma_{\mathbf{k}}$ as a function of W/nT, as obtained in ref. 8 for ion-acoustic fluctuations. It follows from the figure that $\delta\gamma_{\mathbf{k}}/\gamma_{\mathbf{k}} \ll 1$ if only $W/nT \ll 1$. Recently the question of the role of nonlinear interactions of ion-acoustic fluctuations with electrons has attracted much attention[49]. The statement was made that this interaction can determine the anomalous resistance of a plasma. Crude estimates without inclusion of the broadening of resonances give the dashed straight line in Fig. 6. This shows that $\delta\gamma_{\mathbf{k}}/\gamma_{\mathbf{k}} \sim 1$ for $W/nT \sim m_e/m_i$. In reality at substantially smaller W/nT the broadening of resonances sharply reduces the effect, which nowhere can reach $\delta\gamma_{\mathbf{k}}/\gamma_{\mathbf{k}} \sim 1$, if $W/nT \ll 1$, and therefore the nonlinear interaction of ion-acoustic fluctuations with electrons cannot be responsible for the slowing down of electrons and the appearance of the anomalous resistance of a plasma.

f) Turbulent collisions and stochastic heating. A somewhat different situation occurs under conditions when the resonance $\omega_{\mathbf{k}} = \mathbf{k}\cdot\mathbf{v}$ is impossible, as occurs in Langmuir turbulence. Then it is possible to expand $\hat{g}$ in $\hat{\nu}$. However, $\hat{\nu}$ involves $\hat{g}_{\mathbf{k}-\mathbf{k}_1}$, which already can be resonant. It is important, however, that now we know how to treat such $\hat{g}$. The results of the expansions are identical with those which were obtained in the preceding section if the imaginary part of $g_{\mathbf{k}-\mathbf{k}_1}$ is approximated by $-\pi\delta(\omega - \omega_1 - (\mathbf{k}-\mathbf{k}_1)\mathbf{v})$. This point

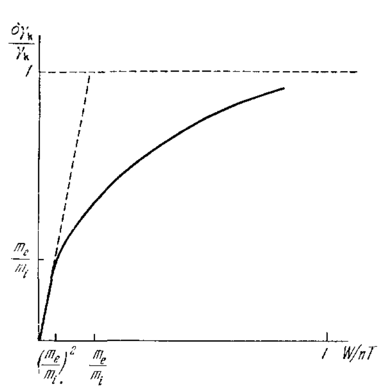

FIG. 6. Nonlinear interaction of ion-accoustic fluctuations with electrons, as a function of the energy W incorporated in turbulent ion-acoustic fluctuations.

clarifies the physical meaning and justifies the concept of stochastic heating. It should be noted that the approach used by Dupree[45] to describe the smearing of the resonance $\omega = kv$ gives for the one-dimensional quasilinear equation a result identical to (3.6) and (3.13); however, $\hat{\nu}$ does not contain $g_{\mathbf{k}-\mathbf{k}_1}$ and for this reason all processes of induced scattering and stochastic heating are lost.

We note, finally, that in $\tilde{\epsilon}_{\mathbf{k}}$ an imaginary part appears and, consequently, the approximate use of (2.16) is justified in the integral balance equations. The balance equations obtained from (2.19) and (3.7) agree in these approximations with each other and with the equations which are obtained by the method of elementary excitations.

4. TURBULENT BROADENING OF RESONANCES OF NONLINEAR DECAY INTERACTIONS, AND CORRELATION FUNCTIONS OF A TURBULENT PLASMA

a) Effective turbulent collisions due to interaction of waves. Turbulent collisions due to interaction of waves lead to an ambiguity in the relation between the frequency and the wave number of a turbulent plasmon.

It has already been noted that not all singularities have been removed in Eq. (3.7), and particularly, $\tilde{\epsilon}_{-\mathbf{k}}$ can go to zero. In reality the problem reduces to inclusion of turbulent collisions, however, not in the Green's function of the particles $1/(\omega - \mathbf{k}\cdot\mathbf{v})$ but in the Green's function of the plasmon $1/\tilde{\epsilon}_{\mathbf{k}}$. The divergence of $1/\tilde{\epsilon}_{\mathbf{k}}$ arises from the illegality of dividing by $\tilde{\epsilon}_{\mathbf{k}}$ in an expression of the type (2.14) if $\tilde{\epsilon}_{\mathbf{k}}$ is close to zero. Smearing of the resonance $\omega = \mathbf{k}\cdot\mathbf{v}$ changes only S into $\tilde{S}$ and ϵ into $\tilde{\epsilon}$, but the operation of division by $\tilde{\epsilon}_{\mathbf{k}}$ remains illegal. The situation in this case is extremely similar to that which was described for the interaction of particles and waves. The point is simply that for $\tilde{\epsilon}_{\mathbf{k}} = 0$ the discarding of the next nonlinear terms becomes inadmissible. A perfected perturbation theory which removes this difficulty was developed first in ref. 11 without inclusion of broadening of particle-wave resonances (i.e., the resonance $\omega = \mathbf{k}\cdot\mathbf{v}$) and, in addition, on the assumption that in terms containing integrals of $1/\epsilon_{\mathbf{k}}$ the turbulent smearing of the resonance is unimportant, since we can use the approximation (2.16). This approximation is sufficient, for example, for discussion of the correlation functions of Langmuir turbulence, but, as analysis shows, does not have the necessary accuracy for ion-acoustic turbulence. A more accurate theory has been developed in ref. 10, but without inclusion of the broadening of the resonance $\omega = \mathbf{k}\cdot\mathbf{v}$. We will set forth here the general case, following most closely the reasoning which has already been used in the preceding section.

Let us separate in an equation of the type (2.11) the term diagonal in $\varphi_{\mathbf{k}}^{\text{st}}$ in the left-hand side, designating

$$\tilde{\varepsilon}_k^N = \int \tilde{\Sigma}'_{k,k_2} |\varphi_{k_2}|^2 \, dk_2, \quad (4.1)$$

where $\tilde{\Sigma}'_{\mathbf{k},\mathbf{k}_2}$ is the unknown quantity which must be found. As will be seen, it is a generalization of the quantity $\Sigma_{\mathbf{k},\mathbf{k}_2}$ introduced above. Then, with inclusion of broadening of the resonance $\omega = \mathbf{k}\cdot\mathbf{v}$ instead of (2.11) we have

$$k^2(\tilde{\varepsilon}_k+\tilde{\varepsilon}_k^N)\varphi_k^{st} = \int \tilde{S}_{k,k_1,k_2}(\varphi_{k_1}^{st}\varphi_{k_2}^{st}-\langle\varphi_{k_1}^{st}\varphi_{k_2}^{st}\rangle)\,\delta(k-k_1-k_2)\,dk_1\,dk_2 + \int\{\tilde{\Sigma}_{k,k_1,k_2,k_3}(\varphi_{k_1}^{st}\varphi_{k_2}^{st}\varphi_{k_3}^{st}-\langle\varphi_{k_1}^{st}\varphi_{k_2}^{st}\rangle\varphi_{k_3}^{st}-\varphi_{k_1}^{st}\langle\varphi_{k_2}^{st}\varphi_{k_3}^{st}\rangle - \langle\varphi_{k_1}^{st}\varphi_{k_2}^{st}\varphi_{k_3}^{st}\rangle)+\tilde{\Sigma}'_{k,k_2,}\varphi_{k_1}^{st}\langle\varphi_{k_2}^{st}\varphi_{k_3}^{st}\rangle\}\,\delta(k-k_1-k_2-k_3)\,dk_1\,dk_2\,dk_3. \quad (4.2)$$

Here we have used $\varphi_{k_2}^{st}\varphi_{k_3}^{st}\rangle = |\varphi_{k_2}|^2\delta(k_2+k_3)$. Then we will require that even in the first approximation in W/nT, the diagonal term disappear in the right-hand side. For this purpose we will multiply Eq. (4.2) by $\varphi_{k'}^{st}$ and obtain an equation for the correlation function $|\varphi_k|^2$. For calculation of $\langle\varphi_{k'}^{st}\varphi_{k_1}^{st}\varphi_{k_2}^{st}\rangle$ we will use

$$\varphi_k^{st}=(\tilde{\varepsilon}_k+\tilde{\varepsilon}_k^N)^{-1}\int\tilde{S}_{k,k_1,k_2}(\varphi_{k_1}^{st}\varphi_{k_2}^{st}-\langle\varphi_{k_1}^{st}\varphi_{k_2}^{st}\rangle)\,\delta(k-k_1-k_2)\,dk_1\,dk_2$$

and require that terms diagonal in $|\varphi_k|^2$ in the right-hand side of the equation obtained be absent in the first approximation in $|\varphi_k|^2$. Hence we obtain a specific expression for $\tilde{\Sigma}'_{k,k_1}$:

$$\tilde{\Sigma}'_{k,k_1}=\tilde{\Sigma}_{k,k_1,k,-k_1}+\frac{\tilde{S}_{k,k_1,k,-k_1}(\tilde{S}_{k-k_1,k,-k_1}+\tilde{S}_{k-k_1,-k_1,k})}{|k-k_1|^2(\tilde{\varepsilon}_{k-k_1}+\tilde{\varepsilon}^N_{k-k_1})}. \quad (4.3)$$

Thus, $\tilde{\Sigma}'_{k,k_1}$ differs from $\tilde{\Sigma}_{k,k_1}$ only in the fact that in the expression in square brackets instead of $\tilde{\epsilon}_{k-k_1}$ we have $\tilde{\epsilon}_{k-k_1}+\tilde{\epsilon}^N_{k-k_1}$. Thus, (4.1) and (4.3) together form a system of integral equations for $\tilde{\epsilon}_k^N$. In a number of cases in view of the fact that (4.1) contains an integral over all values of k_1, the quantity $1/(\tilde{\epsilon}_{k-k_1}+\tilde{\epsilon}^N_{k-k_1})$ can be approximated by the expression

$$-i\pi\delta(\tilde{\varepsilon}_{k-k_1}+\tilde{\varepsilon}^N_{k-k_1})\approx -i\pi\delta(\tilde{\varepsilon}_{k-k_1}), \quad (4.4)$$

i.e., inclusion of $\tilde{\epsilon}_k^N$ is of little importance here. However, neglect of $\tilde{\epsilon}^N$ under the δ-function sign is legal only when the decay processes are well allowed. We have in mind the following. In principle the situation is possible when a decay process is almost allowed, i.e., it is sufficient, for example, to add in the conservation of energy a small amount of energy (to add $\Delta\omega \ll \omega_k$ to the frequency of one of the waves) in order that the decay process change from forbidden to allowed. Such a forbiddenness of a decay process we will call weak. It is clear from this what we understand by the term well allowed decay process (a case opposite to weakly allowed). In a turbulent plasma all of the conservation laws cannot be rigorous; in particular, decay processes must be satisfied only with an accuracy to the effective turbulent collisions. The quantity $\Delta\omega \sim \nu_{eff}$ has a specific meaning and, as will be seen, is equal to the average correlation width. Thus, we can specifically determine what must be understood in a turbulent plasma by the terms weakly forbidden decay and well allowed decay. Specifically, a well allowed decay corresponds to Eq. (4.4), and for a weakly forbidden decay $\tilde{\epsilon}_{k-k_1}$ does not go to zero, while $\tilde{\epsilon}_{k-k_1}+\tilde{\epsilon}^N_{k-k_1}$ goes to zero, i.e., Eq. (4.4) is untrue. It is already evident from this that inclusion of $\tilde{\epsilon}^N$ in (4.3) is very important just for weakly forbidden decays. Ion-acoustic fluctuations[10], as it turns out, are of this type. If we use (4.3), then we can further trace the course of reasoning of ref. 11, construct an integral equation for $\varphi_{k_1}^{st}\varphi_{k_2}^{st}\varphi_{k_3}^{st}\rangle$, and solve it in the approximation (1.2).

The equation for the correlation function takes the form

$$|\varphi_k|^2=2k^{-2}\int\frac{|\tilde{S}_{k,k_1,k_2}+\tilde{S}_{k,k_2,k_1}|^2}{|\tilde{\varepsilon}_k+\tilde{\varepsilon}_k^N|^2}|\varphi_{k_1}|^2|\varphi_{k_2}|^2\,\delta(k-k_1-k_2)\,dk_1\,dk_2. \quad (4.5)$$

The difference of this equation from (3.7) is that in the right-hand side instead of $\tilde{\epsilon}_{-k}$ we have $\epsilon_{-k}+\tilde{\epsilon}^N_{-k}$, while $\tilde{\epsilon}_k^N$ is defined by the system of integral equations (4.3) and (4.1). Thus, the result obtained does not contain divergences.

b) Structure of correlation functions of a turbulent potential. The equation for the correlation functions satisfies the necessary requirement of positiveness of $|\varphi_k|^2$ —a result which it has not been possible to obtain without contradiction in the models of turbulence in incompressible liquids developed up to the present time. If we take into account that the first factor under the integral of (4.5) close to resonance is a slowly varying function, we find that the structure of the correlation function near resonance has the nature of a Lorentz curve

$$|\varphi_k|^2=\mathrm{const}\cdot[(\omega-\omega_k^N)^2+\gamma_N^2]^{-1}. \quad (4.6)$$

Far from resonance it has the previous form (2.18).

In Eq. (4.6) ω_k^N contains the nonlinear frequency shifts, and γ_N describes the sum of the linear and nonlinear increments. Equation (4.6) obviously indicates that there is no unique correspondence between ω and k in a turbulent plasma, and the measure of this ambiguity is principally γ_N. Since $\tilde{\epsilon}$ contains ν_{eff} (which, for example, is given by Eq. (3.19), it can be shown that γ_N is of the order of this ν_{eff}. However, this is not so for the following reasons. First, $\tilde{\epsilon}$ involves both electronic and ionic components (as agreed, the sum over the charges has been omitted for simplicity). Second, the imaginary part of $\tilde{\epsilon}$, as has been shown, gives a value of the order of the linear increment and to a good approximation does not depend on W. This occurs, for example, for ion-acoustic fluctuations. For Langmuir fluctuations, in view of their nonresonance, it is necessary to carry out an expansion in $\hat{\nu}$, and its value far from resonance, combined with other processes, gives the induced scattering and has nothing in common with (3.19). Induced scattering is also given by ions for ion-acoustic turbulence. Furthermore, the condition of stationary turbulence is that the sum of the linear and nonlinear γ's (i.e., γ_N's) goes to zero. For example, for ion-acoustic fluctuations this would mean compensation of the linear buildup in electrons by induced scattering by ions. The spectrum for ion-acoustic turbulence was obtained for the first time in this way by Kadomtsev and Petviashvili[1,50]. The spectrum of Langmuir turbulence in a definite region is also found from the requirement that γ_N, which is associated with induced scattering, go to zero[19]. Thus, neither the effective frequency associated with the Cerenkov resonance (3.19) nor ν_{eff} of the order of the induced scattering increment characterize the correlation width (4.6). The fact is that all of these processes represent interactions of waves and particles, while (4.6) is determined by the interaction between the waves themselves. Equation (4.5) actually describes a decay process, which is evident also from the δ function. Since each of the $|\varphi_{k_1}|^2$ and $|\varphi_{k_2}|^2$ has a sharp maximum near ω_{k_1} and ω_{k_2}, respectively, the resonance denominator $1/(\tilde{\epsilon}_k + \tilde{\epsilon}_k^N)$ actually placed the third frequency ω inside the decay resonance, i.e., $\omega = \omega_k$. Thus, Eq. (4.5) describes the decay resonance. More accurately, the right hand side of (4.5) contains the so-called spontaneous decays, while the induced decays are contained in ϵ^N. The balance equation with inclusion of decays cannot have the form $\tilde{\epsilon} + \tilde{\epsilon}_k = 0$ from which it also follows that $\gamma_N = 0$), since induced decays must be compensated by spontaneous decays. Thus, γ_N must have the order of the characteristic time of the decay interaction. In other words, each of the interactions creates its own ν_{eff}, and the correlation widths of the waves are determined by the interaction of the waves with each other.

c) Effect of turbulent correlations on nonlinear interactions. How important all of this reasoning is, not only for understanding the physical processes occurring in a turbulent plasma but also for practical utilization of the theory for interpretation of experiments, can be seen in the example of ion-acoustic turbulence. At the present time correlation measurements are carried out in most experiments on plasma turbulence. For ion-acoustic turbulence excited by an external quasistatic field E, they have been measured in detail by Hamberger and Jancarik[38], who showed that the turbulent electrical conductivity depends substantially on the correlation time, which is only 4–10 times the period of the fluctuations. The existence of such a large correlation width indicates, consequently, the existence of some kind of decay interaction. However, it would appear to be forbidden for acoustic waves. On the other hand, this forbiddenness is only weak and the existence of strong correlations removes this forbiddenness, while the decay arising determines the correlations. These arguments were basic in the analysis of ion-acoustic turbulence spectra undertaken by Tsytovich[10].

If we use the experimental value of the correlation width $\Delta\omega$, decay processes already determine the spectrum for all frequencies less than $\sim\omega_{pi}/2$. The spectrum was measured by Paul et al.[22] from ω_{pi} to $10^{-1}\,\omega_{pi}$, and by Jancarik and Hamberger[21] to $10^{-2}\,\omega_{pi}$ and corresponds to $W_\omega \sim \omega^{-1} \ln \Lambda(\omega)$. Thus, almost the entire spectrum falls in the region for which inclusion of correlations in the nonlinear interactions substantially affects the nature of the interaction itself. The spectrum predicted in refs. 10 and 50 agrees with the observed spectrum. The effectiveness of the interaction[10] turns out to be $8T_e/T_i$ times larger than the induced scattering[50] by ions. It turns out that this difference in times, in attempts to interpret measurements[22] of the spectrum of ion-acoustic fluctuations in the front of shock waves, is very important and, in particular, if the spectrum were formed by induced scattering, it would not be possible for it to be established under the conditions of ref. 22, whereas in the case of formation of the spectrum with inclusion of the decay interaction the time of passage of the shock wave front in the plasma is sufficient for establishment of the spectrum. There are also other indications favoring the decay mechanism. Thus, in induced scattering a nonlinear stabilization of the instability leads to outflow of energy from the region of the unstable Cerenkov cone in all other directions with approximately the same probability. This means that if stationary turbulence is actually achieved and the spectrum is formed by nonlinear interaction, and the results of refs. 21 and 22 seem to indicate this, then a significant fraction of the turbulence energy must be present outside the Cerenkov cone (roughly only v_s/u times less than inside the cone). For the decay interaction, on the other hand, the angles in the interaction process are approximately constant and the energy only gradually diffuses to the boundary of the Cerenkov cone with a diffusion step $\Delta\theta \sim (\Delta\omega/\omega)^{1/2}$. Measurements[22] have shown that the turbulent energy is included inside an angular cone whose size is quite close to the Cerenkov cone. Finally, the value of the turbulent electrical conductivity, according to Tsytovich[10], is

$$\sigma = (ne^2/m_e)\,\tau^*, \tag{4.7}$$

$$\tau^* = \frac{v_s \cdot 192\left[\int_0^1 s^2\rho(s)(1-s^2)^{-1/2}\,ds\right]^2}{u\omega_* \int_0^1 \psi(\lambda)\,d\lambda} \approx (v_s/u)\cdot 10^2\,(\omega_{pi})^{-1}, \tag{4.8}$$

$$\sigma = \sigma_0/E^{1/2},\ \sigma_0 \approx 10\omega_{pe}\,(env_s/4\pi\omega_{pi})^{1/2}, \tag{4.9}$$

where u is the directed electron drift velocity, v_s is the velocity of sound, and $\rho(s)$ and $\psi(\lambda)$ are functions characterizing the angular and frequency distributions of ion-acoustic fluctuations, the integrals of which are of the order of or somewhat less than unity, $\omega_* = \omega_{pi}\,(\Delta\omega/\omega)^{1/2}$. The value (4.7) corresponds to that measured both directly and from the width of the shock waves. In obtaining (4.8) it was assumed that the spectrum of turbulence is determined by nonlinear processes. However, for sufficiently small E, quasilinear processes, which have been discussed in ref. 51, can be more important. Here, however,

$$\sigma = env_s/E. \tag{4.10}$$

The numerical coefficient 10^2 in (4.9) substantially reduces the field values for which the turbulent electrical conductivity will be determined by quasilinear effects:

$$E/(4\pi nT)^{1/2} \leqslant 0{,}01 m_e/m_i \equiv E_c/(4\pi nT)^{1/2}. \tag{4.11}$$

In most experiments $E > E_c$ and $u \sim (4\text{–}10)v_s$ (according to (4.9) $u = v_s(E/E_c)^{1/2} > v_s$), but if the factor 10^2 were absent, the inverse inequality would be valid for many experiments. At the same time, it must be clearly pointed out that these conclusions are obtained under conditions in which the correlation effects rather strongly affect the turbulent electrical conductivity. We can attempt to follow how the value of the critical field E_c, which separates the quasilinear electrical conductivity (4.10) and the nonlinear conductivity, will change as the role of correlation effects decreases. In the first place, we must remember that the nonlinear interactions which determine the nonlinear electrical conductivity, as $\Delta\omega$ decreases, will be described over a substantially larger frequency region by induced scattering by ions, which was first discussed by Kadomtsev and Petviashvili[50]. The turbulent electrical conductivity value has been calculated for this case by Sagdeev[52]. Actually, correlation effects strengthen the nonlinear interactions only for $\omega < \omega_{pi}\,(\Delta\omega/\omega)^{1/2}$, and with reduction of $\Delta\omega/\omega$ this region shifts to lower and lower frequencies. In the region $\omega > \omega_{pi}\,(\Delta\omega/\omega)^{1/2}$ the correlation broadening is unimportant, the effectiveness of the nonlinear interaction falls off, and consequently W_k increases.

In the second place, the effective frequency $1/\tau^*$ which determines the turbulent electrical conductivity is proportional to the integral

$$1/\tau^* \sim \int W_k k\, dk. \tag{4.12}$$

Thus, for small $\Delta\omega/\omega$ the main contribution to the integral (4.12) will already come from the region where induced scattering is effective. Then in comparison with (4.10) it is necessary to use Sagdeev's formula[52]

$$\tau^* = (v_s/u)\,(10^2/\omega_{pi})\,T_i/T_e. \tag{4.13}$$

The numerical factor 10^2 in this formula, given by Sagdeev[52], apparently can vary considerably, depending on the details of the angular distribution of the fluctuations. If precisely this value of the numerical coefficient is used in (4.13), the critical field E_c is found to be $T_e\omega/T_i\Delta\omega$ times larger than (4.11). An accurate value of the critical ratio $\Delta\omega/\omega$ at which the main contribution to the integral (4.12) begins to come from induced scattering can be obtained only as the result of detailed numerical analysis of the nonlinear processes near $\omega \sim \omega_{pi}$. The spectrum found by Kadomtsev and Petviashvili, as was emphasized by Sagdeev[59], is valid only for $\omega \ll \omega_{pi}$. In the region $\omega \sim \omega_{pi}$ the transfer is integral and changes the frequencies of the plasmons at once by an appreciable amount, so that it is necessary to extrapolate the spectrum[50] to $\omega \sim \omega_{pi}$. A situation is also possible in which energy can be transferred by one step of a nonlinear transformation to the region where correlation effects are already important (it must also be kept in mind that the maximum of the increment occurs at a frequency $\omega_{pi}/\sqrt{2}$ which is still appreciably less than ω_{pi}). Then in the region $\omega \sim \omega_{pi}$ a high energy density W_k does not arise and the contribution of the region where induced scattering is effective to the value of (4.12) will be small. We can expect that if this integral transfer is absent, then in the transition to the frequency region $\omega > \omega_{pi}(\Delta\omega/\omega)^{1/2}$ a sharp maximum should occur in the spectrum, corresponding to an increase of W_k, if only by T_e/T_i times, In just this case we can use (4.13). In the experiments of refs. 21 and 22 no such maximum was observed, which apparently indicates a small role of induced scattering (which incidentally was found in refs. 21 and 22 to be integral in the region from $\omega_{pi}/\sqrt{2}$ to $\omega_{pi}/2$). It is evident also that the frequency region in which the scattering by ions is integral depends strongly on the ratio T_e/T_i, and therefore the critical value $\Delta\omega/\omega$ for which it is necessary to transfer from (4.8) to (4.13) also depends strongly on T_e/T_i. In addition it was shown by Tsytovich[10] that in development of ion-acoustic turbulence the value of $\Delta\omega/\omega$ increases with time, reaching rather high values. It is evident that more detailed information on the critical values $\Delta\omega/\omega$ and their dependence on T_e/T_i can be obtained in the future only with detailed numerical solution of the nonlinear equations taking into account correlation effects. Thus, the correct inclusion of turbulent broadening of resonances actually plays a major role in the interpretation of existing experiments. In most experiments on plasma turbulence, furthermore, correlation effects are measured directly, and their detailed comparison with a theory which predicts both the shape of the correlation curve (4.6) and the value of the correlation broadening is one of the important problems.

d) Correlation broadenings and the linear electromagnetic properties of a turbulent plasma. The linear electromagnetic properties of a plasma are usually understood to mean its response to an external electromagnetic field[53]. We can expect that the response of a turbulent plasma will be fundamentally different from the response of a quiescent plasma for frequencies $\omega \ll \nu_{eff}$. The low-frequency region presents special interest, since the instabilities most dangerous for plasma containment occur in this region. Their radical change or disappearance under conditions of developed high-frequency turbulence has important signficance. New modes of a turbulent plasma in the low-frequency region were discussed for the first time in refs. 54 and 55 on the basis of a model description of turbulence by means of effective Miller forces. Stabilization of drift instabilities in a turbulent plasma has been discussed by Krivoroutsky and others[60]. Tsytovich[56] has discussed a general method which permits calculation of the dielectric permittivity tensor of a turbulent plasma. It is based on consideration of perturbation of f^r, f^{st}, and φ^{st} on the part of a weak external field φ^r (see Section 2) and a systematic expansion of all quantities in φ^r.[57] This method permits it to be established that a number of new instabilities which arise in a turbulent plasma are formally due to the fact that in the low-frequency region $1/\epsilon_{k_1-k}$ approaches infinity, since $\epsilon_{k_1-k} \to 0$ for $k \to 0$. As we have shown[58], the correct inclusion of turbulent collisions removes this divergence, since the quantity involved is $(\epsilon_{k_1-k} + \epsilon^N_{k_1-k})^{-1}$. However, at the same time the instability is also changed substantially.

In the region of longitudinal plasmon phase velocities $v_p^2 \gg v_{Te}^2(9m_i/m_e)$, according to ref. 58, the increments of the instabilities preserve the form found by Vedenov and Rudakov[54], but here the turbulent energy W must all be concentrated in a very narrow interval of phase velocity and a narrow interval of absolute values, which is practically impossible for those broad spectra which are established as the result of nonlinear interactions (see Fig. 2). It is true that there are also exceptions in this case, an example of which is the turbulence dissipated in radiation at a frequency $2\omega_{pe}$[42] under conditions in which the wave-number region in which the plasmons are generated is close to the wave-number region where they are absorbed, and which are converted to electromagnetic radiation of frequency $2\omega_{pe}$. In the region $v_p^2 \gg v_{Te}^2(9m_e/m_i)$, where the principal turbulence energy is usually accumulated (see Fig. 2), according to ref. 58 the increments of the instabilities are substantially changed by turbulent collisions. These questions, of course, require further and more detailed development, but the examples given show that systematic inclusion of turbulent collisions in derivation of the linear electromagnetic properties of a turbulent plasma can substantially affect the ideas as to the mechanisms of collisionless dissipation of the energy of Langmuir turbulence, which is accumulated in the peak of the spectrum (see Fig. 2).

5. CONCLUSION

The analysis presented here shows that there now exists a systematic picture of the physical processes in a turbulent plasma, although there are of course a number of specific questions which require further calculations, for example, correlation widths for various fluctuations in an external field, the structure of turbulent spectra, the electromagnetic properties of a turbulent plasma, and so forth. The value of the method of elementary excitations is becoming steadily more evident; on the one hand, this method is well justified by direct calculations taking into account turbulent collisions, and on the other hand, since it is very convenient and simple, it can serve to verify various refinements of the theory. Thus, a number of these refinements give corrections which are beyond the accuracy of the method.of elementary excitations. Finally, we have repeatedly emphasized in this article that the concept of effective turbulent collisions, which is important for construction of the general theory, has numerous applications and is important for comparison of theoretical results with experimental observations.

The idea of effective turbulent collisions which we have presented can perhaps help experimenters in a convenient qualitative treatment of the processes in a turbulent plasma, and can help theoreticians in making a clear distinction between turbulent elementary excitations and those which describe states close to statistical equilibrium.

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