SOVIET PHYSICS USPEKHI VOLUME 15, NUMBER 4 JANUARY-FEBRUARY, 1973

530.145

Methodological Notes

WHY IT IS IMPOSSIBLE TO INTRODUCE HIDDEN PARAMETERS INTO QUANTUM MECHANICS

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Usp. Fiz. Nauk 107, 463–487 (July, 1972)

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1. THE STATISTICAL CHARACTER OF THE BEHAVIOR OF MICROOBJECTS

No physical theory, including the general theory of relativity, has led to such large gnoseological difficulties as quantum mechanics, since the latter has rejected the determinism of classical mechanics, with which everyone is used, in the description of the behavior of microobjects.[1)] To this we were forced by experiment, which shows that identical microobjects behave differently under completely identical external conditions. Thus, if one shines a convergent light beam on an electron that moves with a definite momentum after the localization of the electron in a definite region of space the momentum of the electron will become uncertain. This means that if one performs a large number of absolutely identical copies of our mental experiment, i.e., if one thinks of a large number of electrons which are strictly in identical states with the same momentum, and the same number of light beams shining on the electrons, then, in spite of the identity of the set-ups, the results of the experiments will not be uniform: some electrons will have one value of the momentum, others will have different values.

However we do not obtain only a spread of the momentum. We note the remarkable fact that as the total number of experiments increases, the ratio of the number of experiments for which a certain value of the momentum is observed to the total number of experiments will tend to a definite value. This means that the observed spread of the momentum values is in fact not random, but is characterized by a definite statistical or probability distribution.

It is natural to relate the appearance of a statistical law of behavior of the electron with an uncontrollable character of the interaction of the beam of light with the electron, or, in a more general formulation, with the uncontrollable character of the interaction of the measuring instrument with the microobject. However, although the action of the instrument on the microobject cannot in principle be reduced to zero (in distinction from what is assumed in classical physics) and the interaction of the instrument with microobjects has an uncontrollable character, one can nevertheless not conclude with logical necessity that a definite statistical distribution of the values of the electron momentum should exist. In fact, the momentum does not play an exclusive role here: an analogous situation, i.e., the existence of a statistical distribution, holds also for the determination of other physical quantities referring to microobjects, i.e., to electrons, atoms and molecules.

Indeed, three different types of behavior of microobjects under measurement are logically conceivable:

1) The measurement of a physical quantity leads with certitude to a definite value.

2) The result of the measurement can only be predicted statistically. But the statistical character is such that if after the first measurement a second one is performed, the succeeding measurement will have a distribution which is in no way related to the first measurement.

3) The result of the measurement can only be predicted statistically, but the succeeding measurement yields a result agreeing with that of the first measurement.

The first possibility corresponds to classical physics in those cases where one can achieve identical experimental conditions.

The second possibility corresponds to classical physics in the case when the experiment is repeated under nonidentical external conditions. For example, if one tosses a large number of coins, on the average, heads will appear in one half of the cases. However, if one

[1)] A clear and detailed discussion of the epistemological problems of quantum mechanics can be found in the works of Fock [1].

selects all those coins which gave heads, a second tossing will not necessarily lead again to heads. The repeated tossing yields the same statistical spread as the first toss.

The third possibility corresponds to classical physics in the case when the different outcomes of the experiment is due to internal causes. Thus, there is a definite probability that a person is color blind. But if one selects all the people that turn out to be color blind in the first test, they will still turn out to be color blind in all succeeding tests.

In quantum physics the first possibility is realized if the wave function which describes the state of the system is an eigenfunction of the operator corresponding to the measured quantity. But in the general case the third possibility is the one realized in quantum physics. Thus, if one passes an unpolarized beam of electrons through an inhomogeneous magnetic field directed along the z axis, then on the average half of the electrons will have a positive spin projection along the z axis. If one now separates the electrons for which the spin projection along the z axis is positive, a repeated measurement of the spin projection along the same direction will always yield a positive result.

It is interesting to note that at the beginnings of the development of quantum mechanics Bohr had assumed that the second possibility is realized (cf.[2]), but this assumption was refuted by Compton's experiment (cf.[3]).

At the present time we can only state that the statistical character in the behavior of microobjects—a statistical character of a special kind—is a remarkable experimental fact, i.e., we can assert that the statistical character lies in the nature of things. But we do not know the deeper reasons lying at its foundation, or more precisely, we cannot say what exactly is the relationship between the statistical character and other deep natural phenomena.

Under these conditions, what kind of problems does quantum mechanics pose? Without attempting to clarify the nature of its statistical character (and remaining in this sense a phenomenological theory) quantum mechanics poses the problem of giving a method of determining the probability distributions for various physical quantities for different states of the microobjects. This problem is closely related with another problem of primary importance: the problem of determination of the spectra of possible values for different physical quantities.

2. THE MATHEMATICAL FOUNDATIONS OF QUANTUM MECHANICS

For the solution of these problems quantum mechanics introduces the concepts of states and observables. It distinguishes pure states form mixtures (or mixed states) and associates to the pure states a function, called the wave function ψ, or the state vector, belonging to a Hilbert space[2], and to mixtures an operator called the density matrix, or statistical operator $\hat{U}$.

To each observable R quantum mechanics associates a hermitian linear operator $\hat{R}$ acting on the Hilbert space. The action of the operator R associates to each vector ψ another vector $\psi' = \hat{R}\psi$.

If $\psi' = r\psi$, where r is a complex number, the vector is called an eigenvector of the operator $\hat{R}$, and the number r is called an eigenvalue. The eigenvalues of a Hermitian operator are real, and the set of its eigenvectors $\psi_1, \psi_2, \psi_3, \ldots$ can be selected to form a basis in the Hilbert space (if there are enough of them—Transl. Note). The transition from one basis to another is implemented by a unitary operator (or transformation).

The eigenvalues of the operators are interpreted in quantum mechanics as the set of possible values of the corresponding observables. This means that the measuring of some quantity by means of a suitable measuring device (called the instrument) will yield one of the eigenvalues of the operator corresponding to the observable under discussion.

As a rule we will obtain different values in different experiments, and what is at stake is the determination of the probability of observing certain eigenvalues of the observable (in the state of the microobject under consideration). It is simplest to find this distribution when the state of the microobject is a pure state. In this case the state vector will evolve in time up to the observation time according to a strictly determined law, governed by the Schrödinger differential equation

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t} + \hat{H}\psi = 0,$$

where $\hat{H}$ is the Hamiltonian operator of the object ($\hbar$ is Planck's constant divided by 2π). In order to find the required probability distribution one needs to know the wave function ψ of the object at the instant just preceeding the measurement of the observable R, and expand this wave function in terms of the eigenfunctions φ_n of the operator $\hat{R}$, corresponding to the observable R:

$$\psi = \sum_n c_n\varphi_n, \qquad \hat{R}\varphi_n = r_n\varphi_n.$$

Then the superposition coefficients c_n of this expansion will play the role of probability amplitudes and the probabilities of finding the values r_n of the observable will be given by the absolute squares $|c_n|^2$ of these coefficients.

The expectation value (average) of R in the state ψ will be

$$\langle R\rangle = \sum_n |c_n|^2 r_n$$

(the vector ψ is assumed to be normalized).

After the first stage of the measurement, consisting in the interaction of the microobject with the measuring instrument, the initial pure state ψ is "destroyed" and replaced by a mixture, described not by a wave function but by a statistical operator

$$\hat{U} = \sum_n |c_n|^2 \hat{P}_{[\varphi_n]}, \qquad (1)$$

[2] A Hilbert space consists of a set of elements called vectors: $\psi, \varphi, \ldots$ for which there are defined the operations of addition, multiplication by complex numbers and scalar (inner) product (φ, ψ). A set of vectors $\psi_1, \psi_2, \ldots$ is called a base of the Hilbert space if any vector ψ of that space can be represented as a superposition $\psi = \sum_i c_i\psi_i$. There are infinitely many linearly independent vectors in Hilbert space. In the sequel, in the discussion of spin states, we shall also consider finite-dimensional spaces, i.e., Euclidean spaces. Rigorously speaking, we are dealing with unitary spaces, or inner-product spaces, where the number of linearly independent vectors can either be infinite (Hilbert space), or finite (Euclidean space).

where $\hat{P}_{[\varphi]}$ is the projection operator on the vector φ in Hilbert space. This operator is defined by

$$\hat{P}_{[\varphi]}\psi = (\varphi, \psi)\varphi,$$

it is Hermitian and idempotent (i.e., equal to its square):

$$\hat{P}^2_{[\varphi]} = \hat{P}_{[\varphi]}.$$

The expectation value of any Hermitian operator in the mixed state (1) is

$$\langle\hat{R}\rangle = \mathrm{Sp}(\hat{U}\hat{R}) = \sum_i w_i \langle\hat{R}\rangle_i, \tag{2}$$

where $w_i = |c_i|^2$ is the probability of finding the system in the state φ_i and $\langle\hat{R}\rangle_i$ is the expectation value of R in the i-th state:

$$\langle\hat{R}\rangle_i = \mathrm{Sp}(\hat{P}_{[\varphi_i]}\hat{R}) = (\varphi_i, \hat{R}\varphi_i).$$

We note that had we considered in place of the mixture a superposition of pure states $\psi = \sum_i c_i\varphi_i$, where $|c_i|^2 = w_i$, then the expectation value of the operator $\hat{R}$ in the state ψ would be

$$\langle\hat{R}\rangle = (\psi, \hat{R}\psi) = \sum_i w_i \langle\hat{R}\rangle_i + \sum_{i\neq j} c_i^* c_j (\varphi_i, \hat{R}\varphi_j). \tag{3}$$

The formula (2) corresponds to classical probability theory, whereas Eq. (3) contains additional interference terms. The presence of the latter describes the phenomenon of interference of probabilities, which is absent in classical mechanics.

If interference of probabilities would occur for classical objects, paradoxical situations would arise. An example of such a situation is the so-called "Schrödinger cat." It consists of the following: a box contains a cat, a Geiger counter and a hammer, which, when the counter responds, smashes a vial of prussic acid. Near the counter there is a quantity of radioactive material such that the probability of one radioactive decay in one hour equals ½. If probability interference were possible for classical objects, after one hour the box would contain a superposition with equal weights of a live and dead cat, i.e., we would be dealing with a cat which is half-alive and half dead, which is clearly absurd.

During the second stage of the measurement process the measuring device is capable of separating the mixture into pure states described by the eigenfunctions φ_n of the measured quantity R. If one now measures again the quantity R in one of these states φ_n we no longer find a dispersion of its values, but obtain a definite value: the eigenvalue r_n of the observable, corresponding to the state φ_n under consideration.

3. THE MEASURING PROCESS

Mutually commuting operators have joint eigenfunctions, therefore the corresponding observables (which we will call compatible) can be simultaneously determined with arbitrarily good accuracy. If the operators do not commute, the exact knowledge of one of the quantities (i.e., considering a state with a definite value of that quantity) implies that for the second quantity we do not obtain a definite value, but only a statistical distribution which does not depend at all on the measuring device; it is only necessary that the measuring instrument be capable of measuring the quantity in which we are interested and that it be a classical device. The last condition means that it is assumed that the measuring instrument is subject to the laws of classical physics (more precisely, it suffices to assume that the instrument admits a quasiclassical description).

This does not mean that one must necessarily use macroscopic bodies as measuring devices, and that microobjects are excluded. On the contrary, one may use microobjects as measuring devices if under the conditions under discussion they admit of a quasiclassical description. If they are not quasiclassical, the accuracy which can be attained by means of such instruments cannot, in principle, be large; the accuracy will be lower the more "quantal" the measuring object, i.e., the more the measuring device deviates in its properties from a classical object.

In the determination of the various statistical distributions refering to different noncommuting observables one has to keep in mind that for this it is not sufficient to consider only one ensemble of mutually noninteracting microobjects. Such an ensemble is, of course, necessary (for the definition of probability concepts), but it must be equipped with the appropriate measuring "trimmings" which are different for different observables. Therefore we are dealing not with one, but with several ensembles for different observables, and since these "trimmings" are mutually incompatible for noncommuting observables, the ensembles cannot be unified into a single larger ensemble.

We have said above that the measurement process is divided into two stages: the interaction of the quantum object with the measuring instrument, which has as a result the transformation of the pure state into a mixture, and the act of recording the result, which separates a pure state from the mixture. In order to clarify these stages let us consider, following Heisenberg[4], a beam of excited atoms moving in a strongly inhomogeneous magnetic field H_y (the Stern-Gerlach experiment, cf. Fig. 1). If the magnetic moment of the atom in the n-th state is μ_n, the interaction energy of the atom with the field is $E_n = \mu_n H_y(y)$, and the force acting on the atom is $-\mu_n(dH_y/dy)$. Since different states of the atom have different values of the magnetic moment, the beam will be split by the magnetic field. The deflection angle will be $\alpha_n = (dE_n/dy)T/p_x$, where T is the transit time in the apparatus and p_x is the projection of the momentum of the atoms on the direction of the incident beam.

Making use of the uncertainty relation one can determine the natural spread of the beam directions

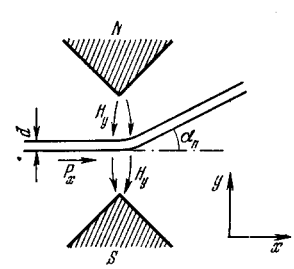

FIG. 1

$$\Delta\alpha \sim \lambda/d = h/p_x d,$$

where d is the width of the beam. In order to be able to detect a deviation of the beam it is necessary that $\alpha_n \gg \Delta\alpha$, or

$$(dE_n/dy)\ Td \gg h.$$

On the other hand, the phase of the wave function ψ of the atom in the n-th excited state is

$$\varphi_n = -(2\pi E_n/h)\ t.$$

Therefore the uncertainty of the phase in the beam $\Delta\varphi_n = (d\varphi_n/dy)d$ will be

$$\Delta\varphi_n \sim (2\pi/h)\ T\ (dE_n/dy)\ d,$$

i.e., $\Delta\varphi_n \gg 2\pi$.

Thus, as a result of the measurement the phase becomes completely undetermined, in other words, the phase relations between the atoms are destroyed. This is the "uncontrollable interaction" between the measuring instrument and the measured object.

If this interaction is not taken into account one arrives at paradoxes. Let us consider, for instance[4a], a beam of excited atoms passing through two inhomogeneous magnetic fields H_1 and H_2 (Fig. 2). Assume that before entering the field H_1 the beam contains only atoms which are in the state n. Let us denote by $A^{(1)}_{n\to m}$ the transition amplitude of the atom from the state n into the state m under the influence of the field H_1. Then the probability of finding the atom in the state m after it has passed through the field H_1 is $|A^{(1)}_{n\to m}|^2$. Let further $A^{(2)}_{m\to l}$ denote the transition amplitude from the state m into the state l under the influence of the field H_2. Then the probability of finding the atom in the state l after passing through the field H_2, with the condition that before entering the field it was in the state m, will be $|A^{(2)}_{m\to l}|^2$. Therefore the probability that the atom is in the state l after passing through both fields H_1 and H_2 is

$$P_{n\to l} = \sum_m |A^{(1)}_{n\to m}|^2 |A^{(2)}_{m\to l}|^2. \qquad (4)$$

On the other hand, if one considers the fields H_1 and H_2 as a single field H_{12}, the amplitude describing the transition from the state n into the state l after passing through the field H_{12} will be determined by

$$A^{(12)}_{n\to l} = \sum_m A^{(1)}_{n\to m} A^{(2)}_{m\to l},$$

and therefore the probability of finding the atom in the state l after it has passed the field H_{12} must be

$$|A^{(12)}_{n\to l}|^2 = \Big|\sum_m A^{(1)}_{n\to m} A^{(2)}_{m\to l}\Big|^2. \qquad (5)$$

But this quantity is not equal to $P_{n\to l}$. The apparent contradiction disappears if we realize that the results (4) and (5) refer in fact to different experiments. The expression (4) is correct if between the fields H_1 and H_2 one carries out a measurement allowing one to determine the stationary state in which the atom is. If such a measurement is performed it is inevitable that the phase of the Schrödinger wave belonging to the m-th state is changed by an undetermined amount, and therefore each term $A^{(1)}_{n\to m} A^{(2)}_{m\to l}$ in the sum (5) has to be multiplied by the phase factor $\exp(\pm i\chi_m)$ with an unknown phase χ_m, over which the whole expression has to be averaged. As a result of this averaging Eq. (5) goes over into the expression (4). If the intermediate experiment determining the state of the atom between the fields H_1 and H_2 is not performed, the result of the final experiment is given by Eq. (5). In this case Eq. (4) is not valid, since it is meaningless to say that the atom was in a definite state m between the fields H_1 and H_2.

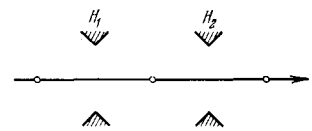

FIG. 2

Thus one has to distinguish strictly three experiments:

First experiment. Between the fields H_1 and H_2 the atoms are not subjected to any perturbation. The probability of finding the atom in the state l after it passes through the field H_2 will be

$$\Big|\sum_m A^{(1)}_{n\to m} A^{(2)}_{m\to l}\Big|^2.$$

Second experiment. Between H_1 and H_2 there is an action on the atoms which allows one to determine their stationary state, but the result of this measurement is not recorded. In this case there appears a "mixture," and the probability of finding the atom in the state l after passing through the field H_2 will be

$$\sum_m |A^{(1)}_{n\to m}|^2 |A^{(2)}_{m\to l}|^2.$$

Third experiment. Between H_1 and H_2 the atoms are subject to an interaction which allows one to determine their stationary states, but the result of the measurement is recorded. The probability of finding the atom in the state l after it passes through the field H_2 will then be

$$|A^{(2)}_{m\to l}|^2.$$

The difference between the second and the third experiments is already familiar in classical theories. The fundamental difference between the first and second cases, or its suitable generalizations, forms the central point of quantum theory.

In the words of Heisenberg, "the measurement process is divided into two strictly distinct acts. The first act of measurement consists in subjecting the system to an external, physically real, interaction which changes the way things are, e.g., one shines a beam of light on the object, or switches on a field. This interaction has the effect that the observed system goes over into a "mixture" of states, in general, of infinitely many. The second act of measurement selects from the infinite number of states in the mixture a certain definite one, which is in fact realized. This second step is a process which does not itself change the way things are, but only changes our knowledge of real relationships[4a].

4. THE HYPOTHESIS OF "HIDDEN" PARAMETERS

But did quantum mechanics indeed overturn Laplace's determinism? There are statistical laws even in classical physics, i.e., in the presence of complete determin-

ism. It suffices to indicate the regularities in the fluctuations of various quantities which refer to macroscopic bodies. Moreover, Boltzmann and Gibbs have shown that the whole of thermodynamics can be considered as part of statistical mechanics, and that starting from the laws of mechanics and taking into account the atomic structure of macroscopic bodies, one can average various quantities referring to these bodies with respect to unobservable "hidden" parameters (the coordinates and momenta of the atoms) and obtain all the thermodynamic properties of the macroscopic bodies.

In quantum mechanics the situation is somewhat different. Here the microsystems, as far as we can tell, do not consist of a large number of smaller objects. But maybe there are nevertheless some "hidden" variables or parameters, which are analogous to the coordinates and momenta of the individual atoms in kinetic theory, parameters which are responsible for the observed statistical behavior of individual microobjects? In order to answer this question at the present time, when there is no other self-consistent and noncontradictory theory but quantum mechanics, one has to investigate whether there is room for "hidden" parameters in present-day quantum mechanics.

For the first time the question of "hidden" parameters was raised by J. von Neumann and also solved by him. The answer given by von Neumann is that quantum mechanics is a logically closed theory, which does not have room for "hidden" parameters (von Neumann's theorem). In other words, in order to introduce "hidden" parameters a fundamental break with quantum mechanics becomes necessary[3)].

In recent years there has been a revival of interest in the epistemological foundations of quantum mechanics, and particularly in von Neumann's theorem. This interest is explained, possibly, by the fact that in spite of all efforts, so far it was impossible to construct a theory of subnuclear matter and of the fundamental interactions to which it is subject. Since this central problem of contemporary physics does not yield to attempts at solving it, it seems natural to try to reanalyze the foundations of the existing theory of the microworld. As a result of these analyses a tremendous amount of literature on the von Neumann theorem has appeared. Below we review this literature and give the simplest and clearest proofs of this fundamental theorem. But before this we describe here the two best-known models of hidden parameters: the hydrodynamic model and the Wiener-Siegel model, and explain the inconsistencies of these models.

The hydrodynamic model[6a,c,7] starts out from the analogy between the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi+V(\mathbf{r},t)\Psi$$

and the equation of motion of an ideal fluid. In order to establish this analogy we set

$$\Psi(\mathbf{r},t)=[\rho(\mathbf{r},t)]^{1/2}e^{iS(\mathbf{r},t)/\hbar}.$$

We then obtain the system of equations

$$\begin{cases}\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0,\\ m\frac{\partial\mathbf{v}}{\partial t}+m(\mathbf{v}\nabla)\mathbf{v}=-\nabla V+\frac{\hbar^2}{2m}\nabla\left(\frac{\Delta\rho^{1/2}}{\rho}\right),\end{cases}$$

where $\mathbf{v}=\nabla S/m$. These equations describe in an obvious way the motion of a fluid of density $\rho(\mathbf{r},t)$ and velocity $\mathbf{v}(\mathbf{r},t)$, where in addition to the external field $V(\mathbf{r},t)$ the fluid is subject to a specific "quantum-mechanical" potential

$$U(\mathbf{r},t)=-(\hbar^2/2m)\,\Delta\,[\rho(\mathbf{r},t)]^{1/2}/\rho(\mathbf{r},t).$$

In the hydrodynamic model the velocity $\mathbf{v}$ (or the momentum $m\mathbf{v}$) is interpreted as a "hidden" parameter, and it is assumed that the momentum of the particle after the measurement of $\mathbf{p}$ differs from the "genuine" value $m\mathbf{v}$. As regards the particle coordinate, it is assumed that its value after the measurement agrees with the genuine value of the coordinate $\mathbf{r}$. One can show[6b] that in this approach we obtain a correct momentum distribution of the particle after the measurement, i.e., a distribution which corresponds to quantum mechanics[4)]. However, this approach gives a preferential role to the coordinate as a dynamical variable, whereas the laws of quantum mechanics should be invariant with respect to the selection of a basis in Hilbert space. The hydrodynamic model[1a,4b,6c,8a] of quantum mechanics does not satisfy this fundamental requirement of quantum mechanics[5)].

In the Wiener-Siegel model[11] the state of a quantum-mechanical system is described by two wave functions: the usual wave function ψ and a "hidden" wave function ξ. The latter is introduced in order that one be able to say exactly which of the eigenvalues of the observable will be obtained when it is measured. More precisely, it is assumed that if one considers a quantity R to which one associates a set of eigenvectors $\{\varphi_i\}$, then ψ is to be expanded in terms of this basis:

$$\psi=\sum_i\psi_i\varphi_i,$$

whereas ξ is to be expanded in terms of the adjoint (conjugate) basis

$$\xi=\sum_i\xi_i\varphi_i^+$$

(φ^+ is the vector which is Hermitian conjugate, or adjoint, to the vector φ). After this one has to find the maximal value of the ratio $|\psi_i|/|\xi_i|$. If this maximum is attained for $i=k$, the measurement of R yields the eigenvalue r_k.

Since in quantum mechanics the probability of obtaining the value r_k equals $|\psi_k|^2$, the distribution of the hidden parameters $\{\xi_i\}$ has to be defined in such a manner that this requirement of quantum mechanics be satisfied in this model also. This is achieved by means of the assumption that the quantities $\{|\xi_i|\}$ have the distribution law[11b,12]

$$f(|\xi_i|)=\exp(-|\xi_i|^2/2)\,|\xi_i|.$$

The problem now reduces to the determination of the probability that the maximal value of the ratio $|\psi_i|/|\xi_i|$ be attained for $i=k$. This probability is obviously equal to

[3)] A detailed discussion can be found in the book [5a].

[4)] The Schrödinger equation can also be obtained classically as the equation of motion of a particle subjected to random forces [6d,8]. But the fundamental difficulty, which, as we shall see, cannot be circumvented, consists in the impossibility of constructing a classical model for quantum-mechanical measurements.

[5)] For other objections to the hydrodynamic model, cf., e.g., [5a,10].

$$p_k = \int \dots \int_{D_k} \prod_i |\xi_i| \exp(-|\xi_i|^2/2)\, d|\xi_i|,$$

where the integration region D_k is determined by the inequalities

$$|\xi_i| \geqslant |\psi_i|\,|\psi_k|^{-1}\,|\xi_k|.$$

Therefore

$$p_k = \int_0^\infty |\xi_k|\, e^{-|\xi_k|^2} d|\xi_k| \prod_{i\neq k} \int_{|\psi_i||\psi_k|^{-1}|\xi_k|}^\infty |\xi_i| \exp(-|\xi_i|^2/2)\, d|\xi_i|,$$

whence, as required,

$$\text{for} \quad p_k = |\psi_k|^2.$$

However, the Wiener-Siegel model, as well as the hydrodynamic model, turns out to be noninvariant with respect to a change of basis in the Hilbert space[13] and therefore is to be rejected[6)]. In order to verify this, we consider in a three-dimensional Hilbert space the projection operator $\hat{P}_3$ onto the 3-axis. If one chooses as a basis the eigenvectors of this operator

$$\varphi_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \varphi_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \varphi_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \hat{P}_3\varphi_i = \lambda_i\varphi_i, \quad \lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 1,$$

then the operator has the diagonal form

$$\hat{P}_3 = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

We note that the projection operators $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$ have physical meaning, since they are simply related to the projections of the operator of angular momentum one. In the representation in which the z-projection of the angular momentum is diagonal, these operators have the form

$$\hat{J}_x = \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}, \quad \hat{J}_y = \frac{1}{\sqrt{2}}\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix}, \quad \hat{J}_z = \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}.$$

If one obtains the x, y, z coordinate system from the 1, 2, 3 coordinate system by means of a rotation by an angle of 45° around the 2 axis (perpendicular to the plane of the figures) which is coincident with the z axis (Fig. 3), it is easy to see that

$$\hat{P}_1 = 1 - \hat{J}_x^2, \quad \hat{P}_2 = 1 - \hat{J}_z^2, \quad \hat{P}_3 = 1 - \hat{J}_y^2.$$

According to the general scheme of Wiener-Siegel, the measurement of $\hat{P}_3$ leads to the value λ_k if among the ratios

$$\frac{|\psi_1|}{|\xi_1|}, \quad \frac{|\psi_2|}{|\xi_2|}, \quad \frac{|\psi_3|}{|\xi_3|} \tag{6}$$

the ratio $|\psi_k|/|\xi_k|$ will be largest. Since $\lambda_1 = \lambda_2$, a rotation of the coordinate axes by an angle θ around the 3-axis (Fig. 4):

$$\psi_1 \to \psi_1' = \psi_1\cos\theta + \psi_2\sin\theta, \quad \psi_2 \to \psi_2' = -\psi_1\sin\theta + \psi_2\cos\theta, \quad \psi_3 \to \psi_3' = \psi_3,$$

$$\xi_1 \to \xi_1' = \xi_1\cos\theta - \xi_2\sin\theta, \quad \xi_2 \to \xi_2' = \xi_1'\sin\theta + \xi_2\cos\theta, \quad \xi_3 \to \xi_3' = \xi_3,$$

will not change the form of the operator $\hat{P}_3$. Therefore if $\hat{P}_3$ is measured the value λ_k is obtained if the ratio $|\psi_k'|/|\xi_k'|$ is maximal among the ratios

$$\frac{|\psi_1'|}{|\xi_1'|}, \quad \frac{|\psi_2'|}{|\xi_2'|}, \quad \frac{|\psi_3'|}{|\xi_3'|}, \dots \tag{7}$$

However, it is easy to see that if in the sequence (6) the maximal ratio was $|\psi_k|/|\xi_k|$ it does not at all follow that in the sequence (7) the maximal ratio will be $|\psi_k'|/|\xi_k'|$. It suffices to assume that, e.g., $\psi_1 = 5$, $\psi_2 = -1$, $\psi_3 = 2$, $\xi_1 = 3$, $\xi_2 = -1$, $\xi_3 = 1$, $\theta = 45°$. Then the sequence (6) takes the form 5/3; 1; 2. Since among these numbers the third is the largest, we shall obtain the third eigenvalue when $\hat{P}_3$ is measured, i.e., $(\hat{P}_3)_\xi = 1$. On the other hand, for $\theta = 45°$, the sequence (7) has the form 1; 3; 2. In this case the second number is the largest, and a measurement of $\hat{P}_3$ yields the second eigenvalue, i.e. $(\hat{P}_3)_{\xi'} = 0$.

Thus, the Wiener-Siegel model of hidden parameters is self-contradictory.

5. INTERFERENCE OF PROBABILITIES AND HIDDEN PARAMETERS

We now start our discussion of proofs of von Neumann's theorem on the impossibility of introducing hidden parameters into quantum mechanics without a radical change of its fundamental principles. The idea of the proof consists in establishing a contradiction between the assumption that hidden parameters exist and the postulates of quantum mechanics.

Thus, assume there are hidden parameters which we denote by ξ (ξ denotes one or several quantities). This means that if one measures an observable R and the result of a single measurement is r, it is asserted that the observed value r is a unique function $f_R(\xi)$ of the hidden parameters ξ:

$$r \equiv (R)_\xi = f_R(\xi).$$

We shall assume that such functions $f_R(\xi)$, $f_Q(\xi)$, ... exist simultaneously for all observables R, Q, ..., not depending on whether the operators $\hat{R}$, $\hat{Q}$, ... corresponding to these observables commute or not. Obviously, for noncommuting operators $\hat{R}$ and $\hat{Q}$ it is impossible to obtain $(R)_\xi \equiv f_R(\xi)$ and Q_ξ in the same experiment, since different measuring instruments are needed for this. When we speak of simultaneous existence of the quantities $(R)_\xi$ and $(Q)_\xi$ we have in mind only a potential possibility: the knowledge of the hidden parameters ξ allows only to predict the results of an arbitrary single experiment.

The postulate of simultaneous existence of the quantities $(R)_\xi$, $(Q)_\xi$, ... for all operators $\hat{R}$, $\hat{Q}$, ..., including noncommutative ones, is essential but far from self-

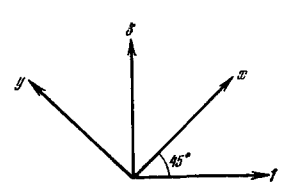

FIG. 3

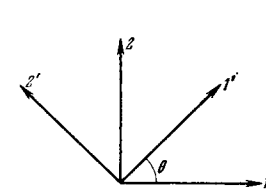

FIG. 4

6)We note that in connection with a proposed modification [14a] of the Wiener-Siegel model (a modification which is also not invariant with respect to a change of base in Hilbert space) which should have led to an observable discrepancy with quantum mechanics in the region of short time intervals, an experiment has been carried out [15], which has confirmed the validity of quantum mechanics [16] up to time intervals of the order of 10^{-15} sec. In [17] it was asserted that interference phenomena dissappear at low photon densities. However, this assertion was based on an experimental error.

evident. Indeed, a logically conceivable possibility is that the quantities $(R)_\xi$, $(Q)_\xi$, ... are defined only in the set of mutually commuting operators[18a]. The following "Gedankenexperiment" of Einstein, Podolsky and Rosen[19] speaks in favor of the postulate of simultaneous existence of all the quantities $(R)_\xi$, $(Q)_\xi$, ... for all operators, including noncommuting ones. Let a system of two electrons which are in singlet state (i.e., has total angular momentum equal to zero) decay into two electrons[7]. Then the projections of their spins on any axis will have opposite signs[8]. Therefore, if there would exist hidden parameters, they would determine the spin projections on any axis, in spite of the fact that projections of the spin operator on different axes do not commute.

Thus, we may assume that it is meaningful to introduce simultaneously the functions $f_R(\xi) \equiv (R)_\xi$, $f_Q(\xi)$ ... for all operators $\hat{R}$, $\hat{Q}$, ..., including noncommuting ones[9].

The quantities $(R)_\xi$ cannot differ from the eigenvalues r_i of the operators $\hat{R}$[10]. Therefore the meaning of the introduction of the functions $f_R(\xi)$ consists in the existence of a one-to-one correspondence between the values of the hidden parameters ξ and the eigenvalues r_i, q_j, ... of the operators $\hat{R}$, $\hat{Q}$, ... A more precise formulation consists in the following: the space of the hidden parameters can be split into regions $(\Delta\xi)_{r_i, q_j}$, ... to which correspond definite eigenvalues r_i, q_j, of the operators R, Q, ... The measure of the region $(\Delta\xi)_{r_i, q_j}$ ... (i.e., a well defined weight attached to the volume of the region $(\Delta\xi)_{r_i, q_j}$ ...) should determine the probability $w(r_i, q_j, ...)$ ($w \geq 0$) of finding the value r_i in a measurement of the quantity $\hat{R}$, of finding the value q_j in a measurement of the quantity Q, etc. It is not assumed that the quantities R, Q, ... are determined in the same experiment, which would, in general, be impossible, since the operators $\hat{R}$, $\hat{Q}$, ... may not be mutually commuting. What is most important here is the existence of a joint probability function $w(r_i, q_j, ...)$ depending on the eigenvalues r_i, q_j, ... of the operators R, Q, ... and determining the outcome of all measurements. In particular, the probability that the measurement of the quantity R yields the value r_i independently of all the other quantities is

$$w(r_i) = \sum_j w(r_i, q_j).$$

We stress that the existence of such a function is a direct consequence of the assumption that hidden parameters exist (i.e., a consequence of the existence of the functional relationships $(\hat{R})_\xi = f_R(\xi)$).

We now show that the existence of a nonnegative function $w(r_i, q_j, ...)$ contradicts the results of quantum mechanics, and in the first place the interference of probabilities[11].

We shall measure the spin projections of the electron on different axes. We denote by $(\hat{s}_i)_\xi$ the result of the measurement of the projection of the electron spin onto the direction i, corresponding to a certain value ξ of the hidden parameter. This quantity should coincide with one of the eigenvalues of the operator s_i, i.e., should equal either $\frac{1}{2}$ or $-\frac{1}{2}$. If hidden parameters exist, there should exist a unique function $w(s_1, s_2, s_3)$ ($s_i = \pm\frac{1}{2}$; i = 1, 2, 3) of the eigenvalues of the spin-projection operators of the electron onto the axes 1, 2, 3, situated arbitrarily with respect to one another, and this function should enable us to predict the results of all measurements of the electron spin.

As is well known from quantum mechanics, if the projection of the electron spin onto the axis i is $\frac{1}{2}$, the probability of finding that the spin projection on another axis j is also $\frac{1}{2}$ equals $\cos^2(\vartheta_{ij}/2)$ where ϑ_{ij} is the angle between the directions i and j.

Let us now assume that an unpolarized electron beam passes through an analyzer which separates the electrons with spin oriented in the direction of the axis 1. Then the probability that an electron passes through the analyzer, i.e., has $s_i = \frac{1}{2}$, will be equal to $\frac{1}{2}$. Let us further pass these electrons through a second analyzer, which selects those electrons which have $s_2 = -\frac{1}{2}$. Then the probability that after passing through the second analyzer the electron has $s_2 = -\frac{1}{2}$ is

$$W(s_1 = 1/2,\ s_2 = -1/2) = 0.5\cos^2[(\pi - \vartheta_{12})/2] = 0.5\sin^2(\vartheta_{12}/2). \quad (8)$$

If hidden parameters exist and their values do not change as a result of the successive measurements, then the probability we have obtained can be derived from the joint probability distribution $w(s_1, s_2, s_3)$ by summing over the possible values of s_3:

$$W(s_1 = 1/2,\ s_2 = -1/2) = w(1/2,\ -1/2,\ 1/2) + w(1/2,\ -1/2,\ -1/2),$$

or

$$0.5\sin^2(\vartheta_{12}/2) = w(1/2,\ -1/2,\ 1/2) + w(1/2,\ -1/2,\ -1/2). \quad (9)$$

Similarly, the following relations should hold

$$0.5\sin^2(\vartheta_{13}/2) = w(1/2,\ 1/2,\ -1/2) + w(1/2,\ -1/2,\ -1/2),$$
$$0.5\sin^2(\vartheta_{32}/2) = w(1/2,\ -1/2,\ 1/2) + w(-1/2,\ -1/2,\ 1/2).$$

and since $w \geq 0$, we have in addition

$$w(1/2,\ -1/2,\ -1/2) \leqslant 0{,}5\sin^2(\vartheta_{13}/2),$$
$$w(1/2,\ -1/2,\ 1/2) \leqslant 0{,}5\sin^2(\vartheta_{32}/2).$$

Substituting these inequalities into (9) we obtain

$$\sin^2(\vartheta_{12}/2) \leqslant \sin^2(\vartheta_{13}/2) + \sin^2(\vartheta_{32}/2). \quad (10)$$

This inequality must be true for any three axes 1, 2, 3, which is impossible. Indeed, let the axes 1, 2 and 3 be situated in the same plane, and assume the axis 3 bisects the angle between axes 1 and 2. Then $\vartheta_{12} = 2\vartheta_{13}$, $\vartheta_{32} = \vartheta_{13}$ and the inequality (10) takes the form

[7] Such a situation could be realized if a neutron knocks out the nucleus from a helium atom which is in the sunglet state.

[8] The paradoxical character of such a correlation, which should occur for arbitrary distances between the particles, has prompted the desire to verify the existence of such a correlation experimentally. The experiment was carried out for photons by Wu and Schachnow (cf. [20]) and has confirmed the presence of correlations (cf. [21a] in this connection).

[9] Some authors have constructed hidden parameter models which forgo this postulate (cf. [21b]). However, this leads to a contradiction with the conclusions of quantum mechanics, and consequently such models cannot be used to explain quantum mechanics in a classical framework. Strictly speaking, one should not even call them models of hidden parameters, for this reason.

[10] We give a rigorous proof of this assertion in Sec. 6.

[11] This assertion was proved independently by Blokhintsev [5b] and by Wigner [22b]. We follow [22b].

$$\sin^2\vartheta_{13} \leqslant 2\sin^2(\vartheta_{13}/2)$$

or $2\cos^2(\vartheta_{13}/2) \leq 1$, which is impossible if $\vartheta_{13} < \pi/2$.

Thus, the existence of hidden parameters contradicts Eq. (8), which was derived from the superposition principle. In other words, the existence of hidden parameters is in contradiction with the interference of probabilities.

The above proof of the nonexistence of hidden parameters has implicitly assumed that the hidden parameters do not undergo any change as a result of repeated measurements. This rather rigid requirement can be removed, as we shall show later.

In conclusion of this section we recall yet another "classical" model of quantum mechanics, which is however equivalent to the usual quantum mechanics. In this model[22a] (cf. also[5c,9b,23]) a joint probability distribution is introduced for the coordinate x and the momentum p

$$f(x, p) = (2\pi)^{-1}\int_{-\infty}^{\infty}\psi^*(x+0.5h\tau)\,e^{ip\tau}\psi(x-0.5h\tau)\,d\tau, \qquad (11)$$

where $\psi(x)$ is the wave function in the coordinate representation. The function f(x, p) yields the probability $dw_x(x)$ of finding the particle in the interval (x, x + dx):

$$dw_x(x) = dx\int f(x, p)\,dp,$$

as well as the probability $dw_p(p)$ that the momentum of the particle be in the interval (p, p + dp):

$$dw_p(p) = dp\int f(x, p)\,dx.$$

However, the expression (11) does not signify that the particle can simultaneously have a definite value of the coordinate and of the momentum, since the function f(x, p) can take on negative values[12].

We also note that Feynman has proposed a "third formulation" of quantum mechanics[25] (the first and second formulations are respectively the Schrödinger equation and Heisenberg's matrix algebra), in which each particle has a stochastic path and the transition probability amplitude is a function-space integral over all possible paths. This formulation, which is equivalent to the Schrödinger and Heisenberg formulations, is not a model for hidden parameters, since in it the amplitudes are added, and not the probabilities.

6. VON NEUMANN'S PROOF OF VON NEUMANN'S THEOREM

The results of the preceding section are based on the very rigid requirement that the hidden parameters do not undergo changes in successive measurements of different quantities. In the proof of von Neumann's theorem which is due to von Neumann himself[26] this requirement is removed. We repeat this proof here.

We start with the fundamental formula $(R)_\xi = f_R(\xi)$ which relates the values of the hidden parameters ξ to the observed value $(R)_\xi$ of the quantity R. Let the distribution of the hidden parameters be given by the probability distribution $w(\xi)$. The function $w(\xi)$ must be determined by the state of the system, in other words, to each pure state ψ, or to each mixture described by a density matrix $\hat{U}$, corresponds its own function $w(\xi)$. (Therefore the function $w(\xi)$ should be equipped with the index ψ or U.)

It is clear that the expectation value $\langle R\rangle$ of the quantity R obtained as a result of a series of independent measurements effected on a system which is in the state ψ will be determined by

$$\langle R\rangle = \int (R)_\xi w_\psi(\xi)\,d\xi.$$

This expression must coincide with the expectation value computed according to the rules of quantum mechanics. In other words, if the system is in the state ψ the following relation must hold

$$\langle R\rangle = (\psi, \hat{R}\psi).$$

If the system is in a mixed state described by the density matrix $\hat{U}$, the following relation will hold

$$\langle R\rangle = \mathrm{Sp}\,(\hat{U}\hat{R}).$$

The quantum-mechanical expectation values satisfy the relation

$$\langle\hat{R} + \hat{Q}\rangle = \langle\hat{R}\rangle + \langle\hat{Q}\rangle.$$

Therefore a model with hidden parameters must satisfy the relation

$$\int (R+Q)_\xi w_U(\xi)\,d\xi = \int (R)_\xi w_U(\xi)\,d\xi + \int (Q)_\xi w_U(\xi)\,d\xi. \qquad (12)$$

Since to different density matrices U correspond different probability distributions w_U, it is natural to require that the following relation hold

$$(R + Q)_\xi = (R)_\xi + (Q)_\xi \qquad (13)$$

for any observables R and Q corresponding to either commuting or noncommuting operators $\hat{R}$ and $\hat{Q}$. To this relation one must add

$$(R^2)_\xi = (R)^2_\xi, \qquad (14)$$

which expresses the dispersion-free character of that quantity, and also the obvious expression

$$(aR)_\xi = a\,(R)_\xi, \qquad (15)$$

where a is an arbitrary complex number.

We now show that these relations cannot hold simultaneously. It is completely irrelevant what exactly the parenthesis $(R)_\xi \equiv (\hat{R})_\xi$ mean, where R is a Hermitian operator. In other words, the system of relations (13)–(15) is self-contradictory in formally algebraic sense. In order to prove this assertion we select a basis $\psi_1, \psi_2, \ldots$ in the Hilbert space, and compute the matrix elements of the operator $\hat{R}$ in this basis:

$$\hat{R}_{mn} = (\psi_m, \hat{R}\psi_n), \qquad \hat{R}^*_{nm} = \hat{R}_{mn}.$$

The quantities $(\hat{R})_\xi$ are functions of the collection $\{R_{nm}\}$ of these matrix elements

$$(\hat{R})_\xi = \varphi(\{R_{mn}\}).$$

It follows from (13) that

$$\varphi(\{R_{mn} + Q_{mn}\}) = \varphi(\{R_{mn}\}) + \varphi(\{Q_{mn}\}).$$

Differentiating this relation with respect to R_{ij} we obtain

[12] We note that such a "quasi-probability" is widely used in quantum kinetic theory [24].

$$\frac{\partial\varphi(\{R_{mn}+Q_{mn}\})}{\partial R_{ij}}=\frac{\partial\varphi(\{R_{mn}\})}{\partial R_{ij}},$$

which implies that the quantities

$$U_{ji}=\frac{\partial\varphi(\{R_{mn}\})}{\partial R_{ij}},\quad U_{ij}=U_{ji}^{*}. \tag{16}$$

do not depend on R_{mn}, i.e., are constants. Since according to postulate (15)

$$\varphi(\{0\})=0,$$

it follows from (16) that

$$\varphi(\{R_{mn}\})=\sum_{ij}U_{ji}R_{ij}. \tag{17}$$

It is obvious that the quantities U_{ji} can be represented in the form

$$U_{ji}=(\psi_j,\ \hat{U}\psi_i),$$

where $\hat{U}$ is a Hermitian operator. With the help of this operator one can write $(R)_\xi$, according to (17), in the form

$$(\hat{R})_\xi=\mathrm{Sp}\ (\hat{U}\hat{R}).$$

In particular, Tr $\hat{U}$ = 1.

We now show that the existence of the density matrix $\hat{U}$ is in contradiction with the dispersion-free character of all Hermitian operators. For this it suffices to select as R a (one-dimensional) projection operator $\hat{P}$. In that case

$$(\hat{P}^2_{[\varphi]})_\xi=\mathrm{Sp}\ (\hat{U}\hat{P}^2_{[\varphi]})=\mathrm{Sp}\ (\hat{U}\hat{P}_{[\varphi]})=(\varphi,\ \hat{U}\varphi).$$

On the other hand, according to (14)

$$(\hat{P}^2_{[\varphi]})_\xi=(\hat{P}_{[\varphi]})^2_\xi,$$

whence

$$(\hat{P}^2_{[\varphi]})_\xi=(\varphi,\ \hat{U}\varphi)^2,$$

i.e.,

$$(\varphi,\ \hat{U}\varphi)=(\varphi,\ \hat{U}\varphi)^2$$

and consequently, the quantity $(\varphi,\ \hat{U}\varphi)$ is either equal to zero or to one.

It is easy to see that the value of $(\varphi,\ \hat{U}\varphi)$ must be the same for all vectors φ of the Hilbert space, since any two vectors ψ_1 and ψ_2 can be continuously transformed into one another, and the transition from zero to one is a discontinuous jump. Zero is obviously excluded. If, on the other hand, $(\varphi,\ \hat{U}\varphi)$ = 1 for all φ, then $1=\sum_{n=1}^{N}(\psi_n,\ \hat{U}\psi_n)=N$, where N is the dimension of the unitary space, which is impossible.

Thus, we have uncovered a contradiction in the system of postulates (13)–(15) on the example of one-dimensional projection operators, have proved the contradictory character of these postulates, and consequently, the impossibility of introducing hidden parameters.

The meaning of the established contradiction is very simple: As we shall show immediately, the quantities $(\hat{R})_\xi$ cannot be anything but the eigenvalues of the operator $\hat{R}$[27]. But the eigenvalues of a sum of noncommuting operators are not equal to the sum of eigenvalues of the respective operators, as would be the case if postulate (13) would be valid.

In order to prove that $(\hat{R})_\xi$ is one of the eigenvalues of $\hat{R}$ we first show that for commuting operators $\hat{R}$ and $\hat{Q}$ the following is true

$$(\hat{R}\hat{Q})_\xi=(\hat{R})_\xi(\hat{Q})_\xi. \tag{18}$$

Indeed

$$((\hat{R}+\hat{Q})^2)_\xi=(\hat{R}^2+2\hat{R}\hat{Q}+\hat{Q}^2)_\xi=(\hat{R}^2)_\xi+2(\hat{R}\hat{Q})_\xi+(\hat{Q}^2)_\xi$$
$$=(\hat{R})^2_\xi+2(\hat{R}\hat{Q})_\xi+(\hat{Q})^2_\xi.$$

On the other hand

$$((\hat{R}+\hat{Q})^2)_\xi=(\hat{R}+\hat{Q})^2_\xi=[(\hat{R})_\xi+(\hat{Q})_\xi]^2=(\hat{R})^2_\xi+2(\hat{R})_\xi(\hat{Q})_\xi+(\hat{Q})^2_\xi.$$

Comparing the two expressions we arrive at (18).

We further show that $(\hat{R})_\xi$ equals one of the eigenvalues r_i of the operator $\hat{R}$. We first assume the contrary: $(\hat{R})_\xi\neq r_i$. Then the operator $\hat{R}-(\hat{R})_\xi$ will have an inverse $\hat{Q}$:

$$[\hat{R}-(\hat{R})_\xi]\,\hat{Q}=1.$$

Therefore

$$([\hat{R}-(\hat{R})_\xi]\,\hat{Q})_\xi=1.$$

Since any operator commutes with its own inverse, we have, making use of (18)

$$(\hat{R}-(\hat{R})_\xi)_\xi(\hat{Q})_\xi=1. \tag{19}$$

On the other hand, it follows from the postulates (13)–(15) that

$$(\hat{R}-(\hat{R})_\xi)_\xi=0,$$

and therefore (19) takes on the form 0 = 1, which is absurd. (Note that in our proof the postulate (13) has been used only for mutually commuting operators R and Q.)

7. A PROOF OF VON NEUMANN'S THEOREM WHICH DOES NOT USE THE POSTULATE OF ADDITIVITY OF INCOMPATIBLE OBSERVABLES

The preceding proof of von Neumann's theorem is based on the postulates (13)–(15) for the model of hidden parameters. Since the first of these postulates (the postulate of additivity of incompatible observables) is a very strong requirement[14a,28], there arises the question whether one can prove von Neumann's theorem without using this postulate, the more so, that the relation (13) does not follow directly from (12), particularly if one takes into account the fact that, in principle, it is possible that the probability density $w(\xi)$ depends not only on the state $\hat{U}$, but also on the form of the operator $\hat{R}$.

On the other hand if $\hat{R}$ and $\hat{Q}$ commute, the validity of the relation (13) seems quite natural. Following Kochen and Specker[27], we now show that von Neumann's theorem remains valid even in the case when the postulate (13) is valid only for mutually commuting operators $\hat{R}$ and $\hat{Q}$[13]. The idea of the proof is to consider the totality of projection operators $\hat{P}_{[\varphi_i]}$ (i = 1, 2, ..., s) and by means of the postulates (13)–(15) (the postulate (13) is assumed valid only for mutually commuting operators) one establishes the following properties of these operators:

[13] Misra [29] proved von Neumann's theorem without using the postulate (13) by assuming that if $\hat{R}-\hat{S}=\hat{Q}^2$, then $(R)_\xi\geqslant(S)_\xi$.

$$\left.\begin{aligned}&(\hat{P}_{[\varphi_i]\xi})(\hat{P}_{[\varphi_j]\xi})=0 \quad \text{for } (\varphi_i, \varphi_j)=0,\\ &\sum_{n=1}^{N}(\hat{P}_{[\psi_n]})_\xi=1,\end{aligned}\right\} \tag{20}$$

where $\{\psi_n\}$ is a complete orthonormal system of vectors, N is the dimension of the Euclidean space (for a Hilbert space N = ∞). Further, we show on a concrete example that these properties contradict each other. We shall not give here the full proof, but only illustrate the flow of ideas on a simple example.

Let us assume that in addition to the relations (20) the projection operators satisfy the additional condition: for any noncommuting projection operators $\hat{P}_{[\varphi_i]}$ and $\hat{P}_{[\varphi_j]}$ there exist as a value of the hidden parameter ξ, such that

$$(\hat{P}_{[\varphi_i]})_\xi = (\hat{P}_{(\varphi_j)})_\xi = 1. \tag{21}$$

(This condition is called the separation postulate.) We show that if the postulates (20) and (21) are satisfied one can choose eight vectors in a three-dimensional space for which these relations are not satisfied.

We shall represent each vector φ_i by a point in the plane and the orthogonality relation between the vectors φ_i and φ_j by a line which joins the appropriate points φ_i and φ_j. Thus, in the graph in Fig. 5 the vectors φ_1 and φ_2 are orthogonal and the vectors φ_2 and φ_3 are not orthogonal. According to (20), if the two points φ_i and φ_j are joined by a line, then at least one of the two P_α (α = i, j) is zero ($P_\alpha \equiv (\hat{P}_{[\varphi_\alpha]})_\xi$); if the three points φ_i, φ_j, φ_k are pairwise joined by lines (i.e., form a triangle), then one of the quantities P_α (α = 1, 2, 3) is one and the other two are zero.

It is easy to see that for the set of vectors represented in Fig. 5 these conditions are not satisfied if one assumes $P_1 = P_8 = 1$. Indeed, if $P_1 = 1$, then $P_2 = P_3 = 0$. Similarly, we have that $P_6 = P_7 = 0$. Further, if $P_2 = P_6 = 0$, it follows that $P_4 = 1$. Similarly we find that $P_5 = 1$; therefore $P_4 = P_5 = 1$, which is impossible, since the vectors φ_4 and φ_5 are orthogonal.

Thus, for no choice of the hidden parameter ξ is it possible to realize the equality $P_1 = P_8 = 1$, i.e., the separation postulate is not satisfied.

In this proof we have tacitly assumed that the configuration represented in Fig. 5 is realizable. That this is indeed so can be shown on the following example

$$\varphi_1=(\mathbf{i}-\mathbf{j}+\mathbf{k})/\sqrt{3},\ \varphi_2=(\mathbf{j}+\mathbf{k})/\sqrt{2},\ \varphi_3=(\mathbf{i}+\mathbf{j})/\sqrt{2},\ \varphi_4=\mathbf{i},$$

$$\varphi_5=\mathbf{k},\ \varphi_6=(\mathbf{j}-\mathbf{k})/\sqrt{2},\ \varphi_7=(\mathbf{i}-\mathbf{j})/\sqrt{2},\ \varphi_8=(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3},$$

where **i**, **j**, **k** are three orthogonal unit vectors.

We have thus proved the self-contradictory nature of the postulate (21) and the relation (20). Kochen and Specker[27] have shown that if one omits the postulate (21) the relations (20) are still not satisfied, but in order to show this, one must consider a system of 117 vectors in three-dimensional Euclidean space, rather than only 8 vectors.

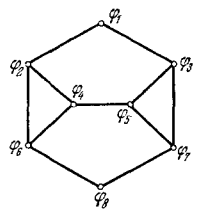

FIG. 5

We note that the simultaneous measurement of quantities corresponding to the projections $\hat{P}_{[\varphi_i]}$, $\hat{P}_{[\varphi_j]}$, $\hat{P}_{[\varphi_k]}$ for mutually orthogonal vectors φ_i, φ_j, φ_k can be realized by investigating the shift of energy levels of an atom with unit angular momentum (J = 1) in a crystal with octahedral symmetry.[30]

8. CLASSICAL AND QUANTUM LOGIC

The two proofs of impossibility of introducing hidden parameters into quantum mechanics were based on contradictions between the postulates of quantum mechanics and the model of hidden parameters. We now show that this impossibility, which is deep-seated, is related to the differences between classical and quantum logic—two theories which cannot be reduced to one another. By a logic we mean here the interrelations between propositions, i.e. a so-called propositional calculus. In classical logic to each proposition A one can associate a set Ω_A of points in a phase space formed by generalized coordinates and momenta of a dynamical system. This set Ω_A is called the support of the proposition A.

For example, if the system is characterized by the Hamiltonian $H = (p^2 + q^2)/2$ then to the proposition A: "the energy of the particle is equal to one" corresponds the set of points Ω_A in phase space which are situated on the circle $p^2 + q^2 = 2$, and to the proposition B: "the particle moves along the positive direction of the q-axis" corresponds the set Ω_B in phase space given by the half-plane $p > 0$.

The different propositions A, B, C, ... are subject to the operations of addition and multiplication[14].

Given two propositions A and B and the proposition C consists in asserting that at least one of the propositions A or B is true, then C is called the sum of A and B, and this relation between propositions is written as:*

$$C = A + B.$$

It is clear that the sum is commutative

$$A + B = B + A \tag{22}$$

and associative

$$(A + B) + C = A + (B + C). \tag{23}$$

Given two propositions A and B and the proposition C consists in asserting that both propositions A and B

[14] We introduce here only those concepts necessary for the proof of the impossibility of introducing hidden parameters into quantum mechanics. A more detailed exposition of classical logic can be found, e.g., in the books [31].

*Translator's note: In the English-language literature on the subject the sum and product are denoted respectively by V and Λ and called "join" and "meet", respectively; the corresponding set-theoretic union and intersection are denoted, of course, by ∪ and ∩. Cf, e.g., the books by Varadargian and Jauch cited in [33,34].

are simultaneously true, then C is called the product of A and B and this relation is written in the form

$$C = AB.$$

As for addition, multiplication is commutative and associative

$$AB = BA, \quad (AB)\,C = A\,(BC). \tag{24}$$

The operations of addition and multiplication of classical logic can be represented by taking the intersections and unions of the sets which are the supports of the appropriate propositions

$$\Omega_{AB} = \Omega_A \Omega_B, \qquad \Omega_{A+B} = \Omega_A + \Omega_B.$$

These operations are easily represented graphically. In Fig. 6 the circles A and B are the supports of the propositions A and B (in this case the phase space is the plane of the paper). The proposition AB is represented by the cross-hatched intersection and the proposition A + B is the region which is shaded at least once.

In classical logic the distributive law holds:

$$(A + B)\,C = AC + BC. \tag{25}$$

This law is illustrated by Figs. 7 and 8. In Fig. 7 the set representing A + B is shaded horizontally and the set corresponding to the proposition C is shaded vertically. Therefore the proposition (A + B)C is represented by the doubly shaded (crosshatched) set. In Fig. 8 the proposition AC is represented by horizontal shading and the proposition BC is represented by the set with vertical shading. Therefore the proposition AC + BC is represented by the region shaded at least once. We see that the cross-hatched region in Fig. 7 is identical to the region shaded at least once in Fig. 8, as required by distributivity of classical logic.

Among some pairs of propositions one can establish an order relation

$$A \leqslant B,$$

meaning that if proposition A is true B is also true, or that B is a consequence of A.

The relation $A \leq B$ means that the support Ω_A of A is a subset of the support Ω_B of B:

$$\Omega_A \subseteq \Omega_B.$$

Thus, the proposition B "$p > 0$" is a consequence of the proposition C "$p > 1$":

$$C \leqslant B$$

On the contrary, between the propositions A: "$p^2 + q^2 = 2$ and B: "$p > 0$" one cannot establish an order relation, since neither of the two is a consequence of the other.

The order relation is connected with the operations of addition and multiplication of propositions by the ordering laws:

$$A + B \geqslant A, \quad AB \leqslant A \tag{26}$$

and the absorption rule

$$\text{if} \quad A \leqslant B, \text{then } A + B = B \text{ and } AB = A. \tag{27}$$

In quantum logic, for which the basic principles were formulated by Birkhoff and von Neumann[32] (cf. also[33]), to each proposition corresponds a closed subspace L_A of a Hilbert space. For example, to the proposition A "the energy of an atom is E_n", in the absence of degeneracy, corresponds some unnormalized vector ψ of Hilbert space, i.e., the support of the proposition is the one-dimensional subspace L_A: $\psi = C\psi_n$ $((\psi_n, \psi_n) = 1)$ of Hilbert space. For twofold degeneracy the support is a two-dimensional subspace (plane) L_A in Hilbert space:

$$\psi = C_1\psi_1 + C_2\psi_2, \quad (\psi_1, \psi_1) = 1, \quad (\psi_2, \psi_2) = 1,$$

where C_1 and C_2 are arbitrary numbers.

As in classical logic one can construct a propositional calculus in quantum logic, based on the operations of addition, multiplication and implication. The operation of multiplication induces the operation of intersection on the supports (denoted again as multiplication) $L_{AB} = L_A L_B$; the order relation is again represented by inclusion:

$$\text{if} \quad A \leqslant B, \text{then } L_A \subseteq L_B,$$

in the same manner as in classical logic. The operation of addition is no longer represented by the set-theoretic union of the subspaces, but the (orthogonal) direct sum $L_A + L_B$ of the linear subspaces[15]:

$$L_{A+B} = L_A \oplus L_B \neq L_A + L_B.$$

Let, e.g., the proposition A consist in the statement that the magnetic moment μ of an atom is directed along the x axis and the proposition B that this vector points in the y-direction. Then the proposition AB asserts that the vector μ points both in the x and the y directions, which is impossible. Such a proposition which is obviously false will be called the absurd proposition and denoted by Θ. Thus, in the case discussed above, $AB = \Theta$.

The proposition A + B asserts that the vector μ has the form $\mu = C_1\mu_1 + C_2\mu_2$ where μ_1 and μ_2 are vectors respectively along the x and y axes. In other words, the proposition A + B says that the vector μ is in the plane spanned by the x and y axes (in classical logic A + B

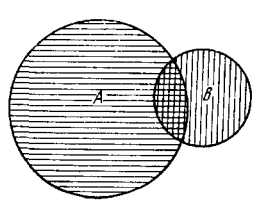

FIG. 6

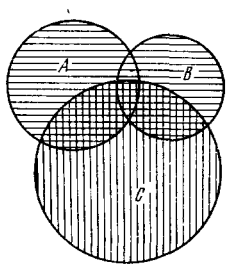

FIG. 7

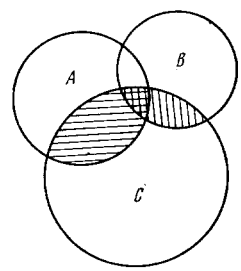

FIG. 8

[15] The direct sum of the spaces L_A and L_B is defined as the set of all possible sums of vectors x + y, with $x \in L_A$, $y \in L_B$. Thus, if L_A and L_B denote respectively the x and y axes, the set-theoretical union $L_A + L_B$ consists of all vectors directed either along the x axis or along the y axis. Their direct sum $L_A \oplus L_B$ consists of all vectors lying in the x, y plane.

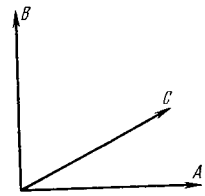

FIG. 9

would mean that μ is either along x or along y). In the case of a two-dimensional space the proposition A + B is always true. Such an obviously true proposition is called trivial and denoted by I.

It is obvious that both in classical and in quantum logic the following relations are true

$$\Theta A = \Theta, \quad \Theta + A = A, \quad IA = A, \quad I + A = I, \quad \Theta \leqslant A \leqslant I,$$

where A is an arbitrary proposition.

The commutative and associative laws (22), (23), (24), the order relation (26) and the absorption law (27) are also valid in quantum logic. However, the distributive law (25) is in general not valid in quantum logic. To see this we consider three vectors A, B, C in a two-dimensional Euclidean space (Fig. 9) and the three propositions asserting that the vector μ is directed along the respective vector. We have A + B = I, (A + B (A + B)C = C. On the other hand, AC = Θ, BC = Θ. Therefore AC + BC = Θ. We see that distributivity is not valid in this case:

$$(A + B)\, C \neq AC + BC.$$

We now give a simple proof of the impossibility of introducing hidden parameters into quantum mechanics, on the basis of quantum logic (this proof is due to Turner[13] [16]).

If hidden parameters would exist, forming a phase space Ω, there would be a mapping $\Omega_A \rightarrow L_A$ of the supports of each proposition A. In other words, one could consider that the state vectors ψ complemented by some hidden parameters ξ form a phase space Ω.

On the other hand, the mapping $\Omega_A \rightarrow L_A$ must conserve the relations of inclusion (the isotony postulate): $\Omega_A \subseteq \Omega_B$ is equivalent to $L_A \subseteq L_B$. Therefore, in order to prove the impossibility of introducing hidden parameters it suffices to prove that the isotony postulate is violated.

For this purpose we consider four directions 1, 2, 1′, 2′ all in the same plane (cf. Fig. 2) of a unitary space. Let the four state vectors $\psi_1, \psi_2, \psi_{1'}, \psi_{2'}$ correspond to these directions. We further introduce the superpositions of states

$$\psi_{12} = C_1\psi_1 + C_2\psi_2, \quad \psi_{1'2'} = C_1\psi_{1'} + C_2\psi_{2'}.$$

Since the four directions 1, 2, 1′, 2′ are coplanar, we have $\psi_{12} = \psi_{1'2'}$. In the language of quantum logic this means the equality of the direct sums

$$L_1 \oplus L_2 = L_{1'} \oplus L_{2'}, \tag{28}$$

where L_1, L_2, $L_{1'}$, $L_{2'}$ denote the subspaces spanned respectively by the four vectors 1, 2, 1′, 2′.

It follows from Eq. (28) that

$$L_1 \subseteq L_1 \oplus L_2.$$

Assume now that there are hidden parameters ξ. Then the state vector ψ_1 and the parameter ξ give rise to a phase subspace Ω_1, $\{\psi_1, \xi\} \in \Omega_1$, and similarly $\{\psi_2, \xi\} \in \Omega_2$, $\{\psi_{1'}, \xi\} \in \Omega_{1'}$, $\{\psi_{2'}, \xi\} \in \Omega_{2'}$. We now take the superposition of states $\psi_{12} = C_1\psi_1 + C_2\psi_2$. Then the following relation should be true: $\{\psi_{12}, \xi\} \in \Omega_1 + \Omega_2$. This relation contains not the direct sum, but the set-theoretic union, since the raison-d'être of hidden parameters is that they be subject to classical logic.

Since the propositions $\Omega_1 + \Omega_2$ and $\Omega_{1'} + \Omega_{2'}$ are not equivalent: $\Omega_1 + \Omega_2 \neq \Omega_{1'} + \Omega_{2'}$ we have

$$\Omega_{1'} \subseteq \Omega_1 + \Omega_2,$$

and consequently the isotony requirement is not satisfied.

Thus, we have shown that the structure of quantum logic cannot be isomorphic to the structure of classical logic.

The tremendous successes of quantum mechanics and the fact that it explains a wide variety of physical phenomena have somehow generated a pragmatic approach to quantum mechanics, where the main emphasis is on the recipes and prescriptions of quantum mechanics, rather than on its foundations and principles.

The importance of von Neumann's theorem is that it convinces us of the logical closedness and self-consistency of quantum mechanics and that no attempts to "correct" it by means of an eclectic mixture of separate elements of the quantum-mechanical formalism and the hypothesis of hidden parameters are possible.

[16] The first proof of the impossibility of introducing hidden parameters into quantum mechanics on the basis of quantum logic was given by Jauch and Piron [34]. However, their proof made use of the so-called "axiom 4": if propositions A and B are true, so is the proposition AB, axiom which is far from obvious in the case of quantum logic [14b]. Proofs based on quantum logic which do not rely on axiom 4 were given by Gudder [18b] and by Zierler and Schlessinger [35]. However, these proofs are too complicated and we do not reproduce them here.

[1] V. A. Fock, a) Usp. Fiz. Nauk 62, 461 (1957); b) Voprosy Filosofii No. 1, 168 (1953); Kvantovaya fizika i stroenie materii (Quantum Physics and the Structure of Matter), Leningrad State University Press, 1965 Secs. 5 and 13; in: Lenin i sovremennoe estestvoznanie (Lenin and Contemporary Natural Science), Moscow, Mysl', 1969, p. 186.

[2] N. Bohr et al., Z. Physik 24, 69 (1924).

[3] É. V. Shpol'skiĭ, Usp. Fiz. Nauk 16, 458 (1936).

[4] a) W. Heisenberg, Physikalische Prinzipien der Quantenmechanik (Russian Transl. Moscow-Leningrad 1932, pp. 48—51, Engl. Transl. The Physical Principles of Quantum Mechanics, Dover, 1950. b) Niels Bohr and the Development of Physics, edit. by W. Pauli, London, Pergamon, 1955, (p. 23 of Russian translation).

[5] D. I. Blokhintsev, a) Printsipial'nye voprosy kvantovoi mekhaniki (Problems Related to the Principles of Quantum Mechanics), Moscow, Nauka, 1966, Secs. 7 and 16; in: "Filosofskie voprosy sovremennoĭ fiziki" (Philosophical Problems of Contemporary Physics), AN SSSR, 1952, p. 381; c) J. Phys. USSR 2, 71 (1940).

[6] D. Bohm, Phys. Rev. 85, a) 166, b) 180 (1952); 89, c) 319, d) 458 (1953); e) Prichinnost' i sluchaĭnost' v sovremennoi fizike (Cause and Chance in Modern

Physics) (Russian Transl. Moscow, IL 1959, p. 164–175).

[7] D. Bohm and J. P. Vigier, Phys. Rev. 96, 208 (1954); D. Bohm et al., Suppl. Nuovo Cimento 1, 48 (1955); D. Bohm and R. Schiller, ibid. p. 67.

[8] B. T. Geĭlikman, Zh. Eksp. Teor. Fiz. 17, 830 (1947); I. Fényes, Z. Phys. 132, 81 (1952); W. Weizel, ibid. 134, 264 (1953); E. I. Adirovich and M. I. Podgoretskiĭ, Zh. Eksp. Teor. Fiz. 26, 150 (1954); D. Kershaw, Phys. Rev. B136, 1850 (1964); J. C. Aron, Prog. Theor. Phys. 33, 726 (1965); E. Nelson, Phys. Rev. 150, 1079 (1966).

[9] a) S. T. Epstein, Phys. Rev. 89, 319 (1953); T. Takabayasi, Prog. Theor. Phys. 8, 143 (1952); b) 11, 341 (1954).

[10] J. P. Vigier, in: La Physique Quantique Resterat-t-elle Indéterministe?, Paris 1953, p. 89; A. S. Davydov et al. in: Filosofskie problemy sovremennoi fiziki (Philosophic Problems of Contemporary Physics), Kiev, AN UkrSSR, 1956, p. 70; H. Freistadt, Suppl. Nuovo Cimento, Ser. 10, 5, 1 (1957).

[11] N. Wiener and A. Siegel, a) Phys. Rev. 91, 1551 (1953); b) Suppl. Nuovo Cimento 2, 982 (1955).

[12] N. Wiener, Nonlinear Problems in the Theory of Stochastic Processes (Russian Transl. IL 1961), Lecture 9.

[13] J. E. Turner, J. Math. Phys. 9, 1411 (1968).

[14] D. Bohm and J. Bub, Rev. Mod. Phys. 38 a) 453, b) 470 (1966).

[15] C. Papaliolios, Phys. Rev. Lett. 18, 622 (1967).

[16] J. H. Tutsch, Rev. Mod. Phys. 40, 232 (1968).

[17] Yu. P. Dontsov and A. I. Baz', Zh. Eksp. Teor. Fiz. 52, 3 (1967) [Sov. Phys.-JETP 25, 1 (1967)].

[18] S. P. Gudder, a) J. Math. Phys. 11, 431 (1970); b) Rev. Mod. Phys. 40, 229 (1968).

[19] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47, 777 (1935) (Russian Transl.: Usp. Fiz. Nauk 16, 440 (1936)); N. Bohr, Phys. Rev. 48, 696 (1935) (Russ. Transl.: ibid. p. 446); in: Albert Einstein: Philosopher-Scientist (The Library of Living Philosophers, V. 7), Evanston, Ill. p. 199 (Russ. Transl. Usp. Fiz. Nauk 66, 571 (1958).

[20] D. Bohm and Y. Aharonov, Phys. Rev. 108, 1070 (1957).

[21] a) A. Peres and P. Singer, Nuovo Cimento 15, 907 (1960); T. B. Day, Phys. Rev. 121, 1204 (1961); b) A. Kh. Levin, Voprosy Filisofii No. 2, 75 (1972).

[22] E. P. Wigner, a) Phys. Rev. 40, 749 (1932); Amer. J. Phys. 38, 1005 (1970).

[23] D. Blokhintsev and P. Nemirovskiĭ, Zh. Eksp. Teor. Fiz. 10, 1263 (1940); G. P. Dishkent, Zh. Eksp. Teor. Fiz. 28, 117 (1955) [Sov. Phys.-JETP 1, 117 (1955)]; R. L. Stratonovich, Zh. Eksp. Teor. Fiz. 31, 1012 (1956); 32, 1438 (1957) [Sov. Phys.-JETP 4, 891 (1956); 5, 1170 (1957)]; G. A. Baker, Phys. Rev. 109, 2198 (1958); G. V. Ryazanov, Zh. Eksp. Teor. Fiz. 35, 121 (1958) [Sov. Phys.-JETP 8, 85 (1959)]; H. Margenau and R. N. Hill, Prog. Theor. Phys. 26, 722 (1961).

[24] J. H. Irwing and R. W. Zwanzig, J. Chem. Phys. 19, 1173 (1951); O. von Roos, Phys. Rev. 119, 1174 (1960); D. C. Kelly, Phys. Rev. A134, 641 (1964); R. Kubo, J. Phys. Soc. Japan 19, 2127 (1964).

[25] R. P. Feynman, Rev. Mod. Phys. 20, 367 (1948); R. P. Feynman and A. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, 1966, Ch. 2 (Russian Transl. 1968).

[26] J. von Neumann, Mathematische Grundlagen der Quantenmechanik, Springer, Berlin 1932, Engl. Translation Princeton 1955, Russ. Translation, 1968.

[27] S. Kochen and E. P. Specker, J. Math. Mech. 17, 59 (1967).

[28] J. S. Bell, Rev. Mod. Phys. 38, 447 (1966); J. Bub. Internat. J. Theor. Phys. 2, 101 (1969).

[29] B. Misra, Nuovo Cimento A47, 841 (1967).

[30] K. W. H. Stevens, Proc. Roy. Soc. A214, 237 (1952).

[31] D. Hilbert and A. Ackermann, Grundzüge der mathematischen Logik, Springer, Berlin, 1932 (Russ. Transl. 1947); P. S. Novikov, Elementy matematicheskoĭ logiki (Elements of Mathematical Logic) Moscow, Fizmatgiz, 1959, Ch. 1.

[32] G. Birkhoff and J. von Neumann, Ann. Math. 37, 823 (1936).

[33] G. Brikhoff, Lattice Theory, AMS Colloq. Publ., Providence, R. I., 1960 (Russian Transl. of older edition, 1952, p. 269–271); V. S. Varadarajan, Commun. Pure Appl. Math. 15, 189 (1962) [Cf. also the book by the same author: Geometry of Quantum Theory, 2 vols. Van-Nostrand, 1968, 1969;—Transl. note]; C. Piron, Helv. Phys. Acta. 37, 439 (1964).

[34] J. M. Jauch and C. Piron, Helv. Phys. Acta 37, 439 (1964) [cf. also the book: J. M. Jaunch, Foundations of Quantum Mechanics, Addison-Wesley, Reading, Mass., 1964—Tr. note].

[35] N. Zierler and M. Schlessinger, Duke Math. J. 32, 251 (1965).

Translated by Meinhard E. Mayer