The laws of conservation of energy and momentum in emission of electromagnetic waves (photons) in a medium and the energy-momentum tensor in macroscopic electrodynamics

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In analyzing processes of emission and absorption of electromagnetic waves in a medium, people currently make widespread use of the laws of conservation of energy and momentum for "photons in the medium," or energy quanta of excitations that propagate in the medium (in different cases, such "photons in a medium" have been termed also photons, real excitons, excitons, polaritons, plasmons, etc.). For example, if we are dealing with emission of one "photon in a medium" as the emitting system (the latter can also be a single moving charged particle) goes from a state of energy E_1 and momentum $\mathbf{p}_1$ to a state of energy E_2 and momentum $\mathbf{p}_2$, then the conservation laws have the form

$$E_1 = E_2 + \hbar\omega, \tag{1}$$

$$\mathbf{p}_1 = \mathbf{p}_2 + \hbar\mathbf{k}, \tag{2}$$

Here $\hbar\omega$ is the energy, and $\hbar\mathbf{k}$ is the momentum of the "photon in the medium" (ω is the frequency and $\mathbf{k}$ is the wave vector of the radiation). Here, for example, in an isotropic medium

$$k = (\omega/c)\, n(\omega), \tag{3}$$

where $n(\omega)$ is the refractive index at the frequency ω. By using the conservation laws (1) and (2) and the relationship (3), or the latter generalized to the case of an anisotropic medium, we can derive the condition for Cerenkov radiation and the formula for the Doppler effect in the medium, and get some information on the nature of the transition (see[1] and below). If, in addition, the energy $\hbar\omega$ and the momentum $\hbar\mathbf{k}$ are small in comparison with E_1 and p_1 (and hence also small in comparison with E_2 and p_2), then to a good approximation, quantization of the excitation energy plays no role, and naturally, the result does not depend on the quantum constant $\hbar$. Of course, under such conditions quantum language is not obligatory, although it is convenient. In classical terms, the corresponding expressions that don't depend on $\hbar$ can be derived under the assumption that the emitted energy $\mathcal{E}$ is related to the emitted momentum $\mathbf{G}$ by

$$\mathbf{G} = (\mathcal{E} n(\omega)/c)\, \mathbf{k}/k. \tag{4}$$

In the given context, this relationship is equivalent to the dispersion equation (3). However, if the dispersion equation (3) arises directly from the field equations and is generally known, then one cannot in any way make such a statement about the relationship between the energy $\mathcal{E}$ and the momentum $\mathbf{G}$ lost in the emission of electromagnetic waves. One can consistently solve the problem of the relation between $\mathbf{G}$ and $\mathcal{E}$ only by using the expressions for the energy-momentum tensor in macroscopic electrodynamics. Moreover, the problem of the energy-momentum tensor in macroscopic electrodynamics has been discussed for more than sixty years, even up to the very present (see[2–5] and the literature cited there; to supplement this literature, we refer to the articles[6,7]). Indeed, we can now consider the situation that no objections can be made against the energy-momentum tensor of the electromagnetic field in the form proposed by Abraham to be well enough established, and it is precisely the correct tensor in the general case. However, the relationship (4) is not derived directly by using the tensor of Abraham, but the energy-momentum tensor in the form proposed by Minkowski. What the problem here essentially is has been elucidated, in particular, in the articles[1,3,5,6]. However, it seems to us pertinent to remark further on this topic. Moreover, we must explain why in the quantum approach to the problem, one can get a correct result, concretely, the conservation laws in the form of Eqs. (1)–(3), with no use at all of any expressions for the energy-momentum tensor of the electromagnetic field.

1. We shall be interested only in the fundamental side of the problem, rather than in deriving the most general expressions. Hence, we shall treat the quantization of the electromagnetic field in a medium[8–11] as applied to an isotropic, nonmagnetic, dispersionless medium (with dielectric constant $\epsilon = n^2$), in the presence of only one non-relativistic charge (an electron). We note immediately that in essence these restrictions are completely inconsequential. Under these conditions, the Hamiltonian function for the field and the particle has the form

$$\mathcal{H} = (1/2m)\,[\mathbf{p} - (e/c)\mathbf{A}(\mathbf{R})]^2 + (1/8\pi)\int (\varepsilon \mathbf{E}^2 + \mathbf{H}^2)\, d\mathbf{r}, \tag{5}$$

Here $\mathbf{p}$ is the momentum and $\mathbf{R}$ is the radius vector of the particle of mass m and charge e, and $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field intensities (the letter E below also denotes the energy, but this should not lead to confusion).

In treating only one particle in the absence of an external field, we can without losing generality consider the scalar potential to be zero, and hence we get

$$\mathbf{E} = -(1/c)\,\partial\mathbf{A}/\partial t, \quad \mathbf{H} = \operatorname{rot}\mathbf{A}, \quad \operatorname{div}\mathbf{A} = 0. \tag{6}$$

We can consider the field to be periodic for a cube of edge L = 1, and use the expansion

$$\mathbf{A} = \sum_{\lambda,\, i=1,\,2} q_{\lambda i}(t)\,\mathbf{A}_{\lambda i}(\mathbf{r}), \quad \mathbf{A}_{\lambda 1} = (8\pi)^{1/2}(c/n)\,\mathbf{e}_\lambda \cos(\mathbf{k}_\lambda \mathbf{r}),$$
$$\mathbf{A}_{\lambda 2} = (8\pi)^{1/2}(c/n)\,\mathbf{e}_\lambda \sin(\mathbf{k}_\lambda \mathbf{r}), \quad e_\lambda = 1, \quad (\mathbf{e}_\lambda \mathbf{k}_\lambda) = 0, \quad n = \varepsilon^{1/2}. \tag{7}$$

There are actually two polarization vectors $\mathbf{e}_\lambda$, but this has not been expressed in explicit form for the sake of simplicity. If we substitute (7) into the expression for the energy of the transverse field, with account taken of the relationships in (6), we have

$$\mathcal{H}_{\mathrm{tr}} = (1/8\pi)\int (\varepsilon E^2 + H^2)\, d\mathbf{r} = (1/2)\sum_{\lambda,\, i} (p_{\lambda i}^2 + \omega_\lambda^2 q_{\lambda i}^2),$$
$$p_{\lambda i} = dq_{\lambda i}/dt \equiv \dot{q}_{\lambda i}, \quad \omega_\lambda^2 = c^2 k_\lambda^2/n^2 \equiv c^2 k_\lambda^2/\varepsilon. \tag{8}$$

The introduced variables $p_{\lambda i}$, $q_{\lambda i}$, as well as $\mathbf{p}$ and $\mathbf{R}$ are canonical, i.e.,

$$\dot{q}_{\lambda i} = \partial\mathcal{H}/\partial p_{\lambda i} = p_{\lambda i}, \quad \dot{p}_{\lambda i} = -\partial\mathcal{H}/\partial q_{\lambda i} = -\omega_\lambda^2 q_{\lambda i}$$
$$+ (e/c)\,[\mathbf{p} - (e/c)\mathbf{A}(\mathbf{R})]\,\mathbf{A}_{\lambda i}(\mathbf{R}),$$

$$\dot{\mathbf{R}} \equiv \mathbf{v} = \partial\mathscr{H}/\partial\mathbf{p} = \left(\frac{1}{m}\right)\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{R})\right), \quad \dot{\mathbf{p}} = -\partial\mathscr{H}/\partial\mathbf{R}. \tag{9}$$

We get from this equation of motion for the particle and the equations for the $q_{\lambda i}$:

$$\ddot{q}_{\lambda i} + \omega_{\lambda i}^2 q_{\lambda i} = \left(\frac{e}{c}\right)(\mathbf{v}\mathbf{A}_{\lambda i}(\mathbf{R})). \tag{10}$$

Of course, these equations can also be derived directly from the equation for the potential (the charge is considered to be a point charge, and δ is the delta function):

$$\Delta\mathbf{A} - (\varepsilon/c^2)\ddot{\mathbf{A}} = -(4\pi/c)\,\mathbf{j} = -(4\pi/c)\,e\mathbf{v}\delta(\mathbf{r} - \mathbf{R}(t)). \tag{11}$$

Using the field equations in the form of (10) (this approach is usually called the Hamiltonian method) is convenient in a number of cases, even within the framework of the classical theory. This is especially true of the case of an anisotropic medium,[12] but now we shall restrict ourselves to the example of Cerenkov radiation in an isotropic medium. Here the following expressions take part on the right-hand side of Eqs. (10):

$$(8\pi)^{1/2}(e/n)(\mathbf{v}\mathbf{e}_\lambda)\cos(\mathbf{k}_\lambda\mathbf{v})\,t, \quad (8\pi)^{1/2}(e/n)(\mathbf{v}\mathbf{e}_\lambda)\sin(\mathbf{k}_\lambda\mathbf{v})\,t;$$

$\mathbf{R} = \mathbf{v}t$ is the trajectory of the uniformly moving charge ($\mathbf{v}$ = const.). The condition for steady-state emission amounts to resonance between the frequency of the acting "force" on the right-hand side of Eqs. (10) and the intrinsic frequency of the field oscillators $\omega_\lambda = ck_\lambda/n$. For Cerenkov radiation, we can directly derive therefrom the condition $\mathbf{k}_\lambda\cdot\mathbf{v} = \omega_\lambda = ck_\lambda/n$, which is identical with the usual condition

$$\cos\theta_0 = c/nv, \tag{12}$$

where θ_0 is the angle between the wave vector $\mathbf{k}$ and the velocity $\mathbf{v}$ of the charge. One can also easily derive the Tamm-Frank formula for the power of Cerenkov radiation by an elementary integration of Eqs. (10) and subsequent calculation of the energy of the field $\mathscr{H}_{tr}$.

While the use of the Hamiltonian method in classical electrodynamics is a particular method that has some advantages and defects as compared with other methods of solving the field equations, we can say that use of the Hamiltonian formalism is of fundamental significance upon quantization. Whatever refined methods of quantization have yet been applied, they are based on reducing the problem of quantizing a field (in particular, an electromagnetic field) to quantizing a mechanical system. In the Hamiltonian method, this side of the problem stands out especially prominently and directly. In fact, after one has introduced the canonical variables $p_{\lambda i}$ and $q_{\lambda i}$, the Hamiltonian function of (5) takes on the same form as for a set of an infinite number of oscillators interacting with a mechanical subsystem characterized by the canonical variables $\mathbf{p}$ and $\mathbf{R}$. Quantization of this mechanical subsystem, i.e., going from classical to quantum mechanics, involves treating the quantities $\mathbf{p}$ and $\mathbf{R}$ as operators that satisfy the commutation conditions $p_jR_k - R_kp_j = -i\hbar\delta_{jk}$ ($j, k = 1, 2, 3$; $\delta_{kk} = 1$, $\delta_{j,k\neq j} = 0$). Quite analogously, quantization of the electromagnetic field in a medium involves using the commutation relationships

$$p_{\lambda j}q_{\lambda k} - q_{\mu k}p_{\lambda j} = -i\hbar\delta_{\lambda\mu}\delta_{jk} \qquad (j, k = 1, 2). \tag{13}$$

When there are no charges, or when we neglect interaction of the field with charges, the Hamiltonian function for the field has the form (8), and as is well-known from quantum mechanics, the eigenvalues of the corresponding Hamiltonian operator $\mathscr{H}_{tr}$ are[1)]

$$\mathscr{H}_{tr}\Psi(q_{\lambda i}) = E_{tr}\Psi(q_{\lambda i}), \quad E_{tr} = \sum_{\lambda, i} n_{\lambda i}\hbar\omega_\lambda \qquad (n_{\lambda i} = 0, 1, 2, 3, \ldots). \tag{14}$$

Thus we have introduced energy quanta that are equal to $\hbar\omega_\lambda$. Emission or absorption of one such quantum amounts to a transition of the field to a state distinguished by an increase or decrease by unity of one of the numbers $n_{\lambda i}$. When one uses the standard quantum-mechanical perturbation theory,[13] the probability of emission and absorption processes is determined by the matrix elements of the energy of interaction. In this case, the latter are given by the following (see (5); the term in the interaction energy proportional to A^2 gives rise to two-quantum transitions, in which we shall not be interested):

$$\mathscr{H}' = -(e/mc)\,\mathbf{p}\mathbf{A}(R)$$
$$= -[(2\pi)^{1/2}e/m]\sum_\lambda[(\mathbf{p}\mathbf{e}_\lambda)/n]\,[(q_{\lambda 1} - iq_{\lambda 2})\,e^{i\mathbf{k}_\lambda\mathbf{R}} + (q_{\lambda 1} + iq_{\lambda 2})\,e^{-i\mathbf{k}_\lambda\mathbf{R}}]. \tag{15}$$

We note that we must assume that $n = n(\omega_\lambda)$ upon taking account of dispersion. In the very simple case under discussion of Eq. (5), the Hamiltonian of the unperturbed system has the form $\mathscr{H}_0 = (p^2/2m) + \mathscr{H}_{tr}$, and its eigenfunctions are (see also (14)):

$$\Psi(\mathbf{R}, q_{\lambda i}) = e^{i\mathbf{p}\mathbf{R}/\hbar}\,\Psi_{n_{\lambda i}}(q_{\lambda i}). \tag{16}$$

The matrix elements $\int\Psi_1^*\mathscr{H}'\Psi_2\,dR\,dq_{\lambda i}$ for the perturbation of (15) and for the wave functions of (16) differ from zero only for one-quantum transitions (e.g., $\int\Psi^*_{n_{\lambda i}=0}q_{\lambda i}\Psi_{n_{\lambda i}=1}dq_{\lambda i} = (\hbar/2\omega)^{1/2}$), even when one of the equations $\mathbf{p}_1 - \mathbf{p}_2 \pm h\mathbf{k}_\lambda = 0$ is satisfied. However, these equations are precisely the law of conservation of momentum, respectively, for emission (minus sign) or absorption (plus sign) of one quantum of energy $\hbar\omega_\lambda$.

Thus, the law of conservation of momentum (2) for emission of a "photon in a medium" can be derived without any additional assumptions, and without elucidating the problem of the form of the momentum operator for the field. The law of conservation of energy (1) also automatically follows upon applying the non-steady perturbation theory, which permits one to calculate the intensity of emission. An evident generalization of the Hamiltonian for a particle in a field to the relativistic case using the Kelin-Gordon equation (spin zero), the Dirac equation (spin 1/2), or equations for particles of larger spins, permits us to calculate the intensity of Cerenkov radiation for a particle of any velocity, and if we take account of the spin, with any spin also (see[1] and the literature cited there).

The conservation laws (1) and (2) with the relationship (3) taken into account and with

$$E_{1,2} = (m^2c^4 + c^2p_{1,2}^2)^{1/2}, \quad \mathbf{p}_1 = m\mathbf{v}_1/[1 - (v_1^2/c^2)]^{1/2}$$

give the condition for emission

$$\cos\theta_\lambda = (c/nv_1)\{1 + (\hbar\omega_\lambda/2mc^2)(n^2 - 1)[1 - (v_1^2/c^2)]^{1/2}\}, \tag{17}$$

where θ_λ is the angle between $\mathbf{k}_\lambda$ and $\mathbf{v}_1$. Evidently, if

$$(\hbar\omega_\lambda/2mc^2)(n^2 - 1)[1 - (v_1^2/c^2)]^{1/2} \ll 1, \tag{18}$$

then Eq. (17) is reduced to the classical Cerenkov condition (12) with $\mathbf{v} \approx \mathbf{v}_1 \approx \mathbf{v}_2$.

If we neglect recoil (i.e., under the condition (18)), we can write directly

$$\Delta E = E_2 - E_1 = (\partial E/\partial\mathbf{p})\,\Delta\mathbf{p} = (c^2\mathbf{p}/E)\,\Delta\mathbf{p} = \mathbf{v}\Delta\mathbf{p}, \quad \Delta\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1.$$

If we assume here, in agreement with (1)–(3), that $\Delta E = h\omega$ and $\Delta\mathbf{p} = h\mathbf{k} = (\hbar\omega n/c)\mathbf{k}/k$, we get directly the condition (12) for the angle θ_0 between $\mathbf{k}$ and $\mathbf{v}$. Of course, the absence in this condition of the quantum constant $\hbar$ is not fortuitous. The point is that we get the same result if we assume that $\Delta E = \varepsilon$ and $\Delta\mathbf{p} = (\varepsilon n/c)\mathbf{k}/k$, that is, we use the relationship (4) between the emitted energy $\mathcal{E}$ and the momentum **G**, with the values of $\mathcal{E}$ and **G** themselves not quantized (being classical quantities). Use of the conservation laws (1)—(3) as applied to emitters having internal degrees of freedom leads to the formulas for the Doppler effect in a medium.[1,14] Here, one can show, even from the conservation laws alone, that in the normal Doppler effect (for $\theta > \theta_0$) an emitter goes with emission of a photon from the upper to the lower level. However, in the anomalous Doppler effect (for $\theta < \theta_0$) emission occurs with excitation of the emitter, which goes from the lower to the upper level of internal motion (the energy of the photon here is taken from the energy of translational motion). In this regard, the conservation laws give more than the classical interference condition, which also leads to Eq. (12) and to the Doppler formula. However, this is a different topic (see[1] and the literature cited there).

Here we have wanted to demonstrate that the quantum theory of emission in a medium directly leads to the conservation laws (1) and (2) using the relationship (3). Of course, this is true not only for the most elementary (but in essence quite exact) form of the theory developed in[8] and above, but also when one uses covariant formulations.[9-11] The fact that the results do not depend on any assumptions about the form of the expressions for the momentum of the field should not arouse any special wonder here. In fact, we have used above both the field equations (in particular, they specifically lead to the relation (3) between the frequency ω and the wave vector **k** of the radiation) and an expression for the Hamiltonian function (5). Moreover, the electrodynamic conservation laws are implied directly by the field equations (in particular, see below), while the expression for the Hamiltonian function determines the form of the Hamiltonian.

We can see clearly from the following fact alone how much "mathematics is smarter than man" in other regards, and concretely, how inessential the relation is between the wave functions used and the eigenfunctions of the field momentum operator. If we expand the field in terms of standing waves (see (7)) in states that correspond to the eigenvalues of the energy operator (see (14)), the mean value of the momentum of the field is zero, while the eigenfunctions of the energy are not eigenfunctions of the momentum operator. The above is directly evident, since the flux of energy in a standing wave is zero. We can convince ourselves of the same formally as well for a vacuum, since in this case the expression for the momentum of the field is well known, and is equal to $\mathbf{G} = (1/4\pi c) \int \mathbf{E} \times \mathbf{H} d\mathbf{r}$. If we try from the onset to work with photons of energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$, then we must expand the field (the potential) in terms of running waves, as is usually done (see, e.g.[13]). We have not proceeded in this way so as to emphasize the independence of the results from the "form" of quantization, and also because an expansion into standing waves leads more directly to canonical variables[13]). Independently of the problem of choosing the expression for the momentum of the field **G** in a medium, the abovesaid also applies to the case of a field in a medium because the expressions of Minkowski and of Abraham for the momentum of the field are both proportional to $(1/4\pi c) \int \mathbf{E} \times \mathbf{H} d\mathbf{r}$, and they differ only in a coefficient (in Abraham's case, this coefficient is unity, but in Minkowski's case it is $\epsilon = n^2$, see below).

As we see it, these remarks leave no doubt on the correctness of the results of the quantum theory of emission in a medium, and in particular, on the correctness of the conclusions involving the use of relationships (1)—(3). However, all these results and conclusions must, of course, be also derivable by a correct application of the expressions for the energy-momentum tensor of the field in a medium. This will be demonstrated below.

2. Preliminarily, however, we shall present some relationships and expressions involving derivation and use of the energy-momentum tensor in macroscopic electrodynamics (for more details, see[2-6]).

For the very simple case discussed above (a nonmagnetic, dispersionless medium at rest), the field equations have the form

$$\operatorname{rot} \mathbf{H} = (4\pi/c)\, \mathbf{j} + (\varepsilon/c)\, \partial \mathbf{E}/\partial t, \tag{19}$$

$$\operatorname{rot} \mathbf{E} = -(1/c)\, \partial \mathbf{H}/\partial t, \tag{20}$$

$$\operatorname{div} (\varepsilon \mathbf{E}) = 4\pi\rho, \tag{21}$$

$$\operatorname{div} \mathbf{H} = 0. \tag{22}$$

Let us find the scalar product of Eq. (19) with **E**, and that of Eq. (20) with **H**. If we subtract the expressions obtained from one another, and use the identity $\mathbf{E} \cdot \operatorname{curl} \mathbf{H} - \mathbf{H} \cdot \operatorname{curl} \mathbf{E} = -\operatorname{div} (\mathbf{E} \times \mathbf{H})$, we get

$$(1/8\pi)\, [\partial (\varepsilon E^2 + H^2)/\partial t] + \mathbf{jE} = -\operatorname{div} \mathbf{S}, \quad \mathbf{S} = (c/4\pi)\, [\mathbf{EH}]. \tag{23}$$

This theorem by Poynting corresponds to the law of conservation of energy, and it is given in any textbook. The derivation from the field equations of the law of conservation of momentum is somewhat less popular. In order to derive it, we shall find the vector product of Eq. (19) by **H**, and that of Eq. (20) by **E**. By adding the obtained expressions, we get

$$\frac{1}{4\pi} \{[\mathbf{H} \operatorname{rot} \mathbf{H}] + [\mathbf{E} \operatorname{rot} \mathbf{E}]\} = -\frac{1}{c} [\mathbf{jH}] - \frac{1}{4\pi c} \frac{\partial}{\partial t} [\mathbf{EH}] - \frac{\varepsilon - 1}{4\pi c} \left[\frac{\partial \mathbf{E}}{\partial t} \mathbf{H}\right].$$

Now let us add to the right and left-hand sides of this relationship the expression $-\rho\mathbf{E} - [(\epsilon - 1)/4\pi c] \times (\partial\mathbf{H}/\partial t)$. On the left-hand side, we shall transform this additive term with the aid of (20) and (21) to the form $-\mathbf{E} \operatorname{div} (\epsilon\mathbf{E}) + [(\epsilon - 1)/4\pi]\, \mathbf{E} \times \operatorname{curl} \mathbf{E}$. Consequently we get

$$\frac{1}{4\pi} \{[\mathbf{H} \operatorname{rot} \mathbf{H}] + \varepsilon [\mathbf{E} \operatorname{rot} \mathbf{E}] - \mathbf{E} \operatorname{div} \varepsilon \mathbf{E}\} + \frac{1}{4\pi c} \frac{\partial}{\partial t} [\mathbf{EH}] = -\left\{\rho \mathbf{E} + \frac{1}{c} [\mathbf{jH}] + \frac{\varepsilon - 1}{4\pi c} \frac{\partial}{\partial t} [\mathbf{EH}]\right\}. \tag{24}$$

The terms taking part on the right-hand side here are the density of the Lorentz force $\mathbf{f}^L = \rho\mathbf{E} + (1/c)\, \mathbf{j} \times \mathbf{H}$, and the density of the volume force

$$\mathbf{f}^A = [(\varepsilon - 1)/4\pi c]\, \partial\, [\mathbf{EH}]/\partial t, \tag{25}$$

which is sometimes called the Abraham force. The minus sign on the right-hand side of (24) involves the fact that $\mathbf{f}^L + \mathbf{f}^A$ is a force acting on the medium, while Eq. (24) defines the balance of forces and the momentum as referred to the field, with

$$\mathbf{g}^A = (1/4\pi c)\, [\mathbf{EH}] = \mathbf{S}/c^2 \tag{26}$$

as the momentum density of the field (it is precisely this expression, which is the same both in a vacuum and in a medium, that corresponds to the choice of the energy-momentum tensor in the form of Abraham).

If we assume a homogeneous medium for the sake of simplicity (with ϵ = const),[2)] then it is especially easy to transform Eq. (24) to the standard form

$$\partial\sigma_{\alpha\beta}/\partial x_\beta - \partial g^A_\alpha/\partial t = f_\alpha,\ f_\alpha = f^{II}_\alpha + f^A_\alpha\ (\alpha, \beta = 1, 2, 3), \qquad (27)$$

where $\sigma_{\alpha\beta}$ is the Maxwell tensor of the field intensities

$$\sigma_{\alpha\beta} = (1/4\pi)\,[\varepsilon E_\alpha E_\beta + H_\alpha H_\beta - (1/2)\,(\varepsilon E^2 + H^2)\,\delta_{\alpha\beta}]. \qquad (28)$$

Thus, the law of conservation of momentum (27) follows from the field equations without additional assumptions. If we combine this law and the law of conservation of energy (23) into one four-dimensional relationship, the law of conservation of energy-momentum we also arrive at an expression for the energy-momentum tensor T_{ik}:

$$T^A_{ik} = \begin{pmatrix} \sigma_{\alpha\beta} & -icg^A \\ -\frac{i}{c}\mathrm{S} & W \end{pmatrix},\quad W = \frac{\varepsilon E^2 + H^2}{8\pi},\quad \mathrm{S} = \frac{c}{4\pi}\,[\mathrm{EH}] = c^2 g^A \qquad (29)$$

$$(i, k = 1, 2, 3, 4;\ \alpha, \beta = 1, 2, 3;\ x_4 = ict),$$

$$\partial T^A_{ik}/\partial x_k = f_i,\quad f_\alpha = f^{II}_\alpha + f^A_\alpha,\quad f_4 = (i/c)\,(\mathrm{jE}). \qquad (30)$$

The tensor (29) is the Abraham tensor for a medium at rest; for a medium in motion, this tensor looks somewhat more complicated (see[3,5,7]).

The Minkowski tensor under the same assumptions as were made in (29) and (30) has the form

$$T^M_{ik} = \begin{pmatrix} \sigma_{\alpha\beta} & -icg^M \\ -\frac{1}{c}\mathrm{S} & W \end{pmatrix},\quad \mathrm{g}^M = \frac{\varepsilon}{4\pi c}\,[\mathrm{EH}] = \varepsilon \mathrm{g}^A, \qquad (31)$$

$$\partial T^M_{ik}/\partial x_k = f^{II}_i,\quad f^{II}_\alpha = \rho E_\alpha + (1/c)\,[\mathrm{jH}]_\alpha,\quad f^{II}_4 = f_4 = \frac{i}{c}\,(\mathrm{jE}). \qquad (32)$$

It is quite evident that the conservation laws (30) and (32) are identical, at least from the formal standpoint: they differ only in the separation of the same sum into terms. Concretely, if we transfer the Abraham force of (25) from the right to the left-hand side of the equation, and combine it with $\partial T^A_{ik}/\partial x_k$, then we get directly the expression $\partial T^M_{ik}/\partial x_k$, and we can treat the Minkowski tensor as the energy-momentum tensor. Such an ambiguity in the choice of expression for the energy-momentum tensor is not so surprising, being very general in nature, and it appears even in the theory of a field in a vacuum (see, e.g.,[16], Sec. 32). Moreover, a field in a medium is an open system; the only closed system is the one consisting of the field and the medium, the latter being characterized by its own energy-momentum tensor T^m_{ik}. The overall tensor $T_{ik} = T^m_{ik} + T^{EM}_{ik}$, where T^{EM}_{ik} is the tensor for the field (e.g., the tensor of (29)), obeys the conservation law $\partial T_{ik}/\partial x_k = 0$. However, neither the tensor T_{ik}, nor a fortiori its components T^m_{ik} and T^{EM}_{ik}, is defined unequivocally in its general form. A completely different matter is the force density, which is an unequivocal and measurable quantity, at least in principle. In this regard, even the outcome of the "dispute" over the Abraham and Minkowski tensors is finally solved by choosing the expression for the force. The Abraham force of (25) is genetically related to the force of the magnetic field acting on the displacement current. We cannot doubt the reality of this force, in spite of the fact that it has not yet been directly measured.[3)] This solves the problem unequivocally "in favor" of the Abraham tensor. It has also been shown in detail in[5] that the various objections found in the literature against choosing the Abraham tensor are ill-grounded. The fact that one can advance no substantial arguments against choosing the tensor T^{EM}_{ik} in the Abraham form is also emphasized in[3]. We shall restrict ourselves here to recalling one of the arguments in favor of choosing the Minkowski, rather than the Abraham tensor. Namely, when one chooses the Minkowski tensor for a quasimonochromatic wave packet in any system of reference,[7] the energy flux $\mathrm{S} = W\mathrm{v}_{gr}$, where v_{gr} is the group velocity. Analogously, it holds specifically for the Minkowski tensor that $\sigma_{\alpha\beta} = -g^M_\alpha \mathrm{v}_{gr,\beta}$ (see[17], p. 114, and the literature cited there). Such relations do not hold when one chooses the Abraham tensor, and for some reason this is deemed to be some sort of defect or difficulty. In fact, as has been shown in especial detail in[5], the entire matter again involves the presence of the volume force $\mathbf{f}^A$ when one uses the Abraham tensor. In a moving medium, this force performs work on the medium, and hence the relationship $\mathrm{S} = W\mathrm{v}_{gr}$ cannot and should not hold. The situation here is fully analogous to that which occurs in a medium at rest,[4)] where the relationship $\mathrm{S} = W\mathrm{v}_{gr}$ breaks down in the presence of absorption, and in general, of any sources of sinks of energy in the medium. As applied to the flux of momentum density $g_\alpha \mathrm{v}_{gr,\beta}$, the above now refers to the case of a transparent medium at rest, since the relationship $\sigma_{\alpha\beta} = -g_\alpha \mathrm{v}_{gr,\beta}$ can hold only in the absence of a volume force. The latter requirement is precisely satisfied by the Minkowski tensor (we consider charges and currents to be absent), for which $\partial T^M_{ik}/\partial x_k = 0$.

All of the aforesaid lets us consider the Abraham tensor to be "correct." However, as it seems to us, one can declare the Minkowski tensor to be "incorrect" only by approaching the problem somewhat formally. In fact, in most situations the results obtained by using the Abraham and Minkowski tensors are quite identical. This makes it possible in suitable cases not only to use the Minkowski tensor, but it even makes this quite expedient, if any simplifications are attained thus. Hence, we shoud hardly declare the Minkowski tensor T^M_{ik} to be "erroneous." Rather, it is a certain auxiliary concept that (like, e.g., pseudotensors) can be used fully. This inflicts no harm at all on the "prestige" of the more fundamental, and if we wish, genuine energy-momentum tensor of an electromagnetic field in a medium T^A_{ik}.

3. Analysis of the problem of the laws of conservation of energy and momentum in emission of electromagnetic waves (photons) in a medium confirms and illustrates the latter remark. In fact, let us see what the momentum of a wave packet in a medium is by using the Abraham and Minkowski tensors, and then we shall apply the conservation laws in the two cases.

Let us consider a plane wave propagating in a medium:

$$\begin{aligned} \mathbf{E} &= (1/2)\,(\mathbf{E}_0 e^{-i(\mathbf{kr}-\omega t)} + \mathbf{E}^*_0 e^{-i(\mathbf{kr}-\omega t)}),\\ \mathbf{H} &= (1/2)\,(\mathbf{H}_0 e^{i(\mathbf{kr}-\omega t)} + \mathbf{H}^*_0 e^{-i(\mathbf{kr}-\omega t)}). \end{aligned} \qquad (33)$$

If the wave is quasimonochromatic, then $\mathbf{E}_0$ and $\mathbf{H}_0$ are slowly-varying functions of the time t (as compared with the period $2\pi/\omega$). However, for simplicity we shall neglect dispersion, and hence we shall consider the amplitudes $\mathbf{E}_0$ and $\mathbf{H}_0$ to be constant, but shall consider the wave packet to have a cross-section of area unity and length L (for an account of dispersion, see, e.g., the book[17], Sec. 3). If we substitute (33) into the field equations (19) and (20) with real ϵ = const.[5)] and for $\mathbf{j} = 0$, we have

$$\mathbf{E}_0 = -(c/\varepsilon\omega)\,[\mathbf{kH}_0],\quad \mathbf{H}_0 = (c/\omega)\,[\mathbf{kE}_0]. \qquad (34)$$

As the condition for existence of a nontrivial solution, we get from this the dispersion equation (3) with $n^2 = \epsilon$.

Further, we get the following for the time averages (averaged over the high frequency) (see (29) and (31)):

$$\bar{W} = \overline{(\varepsilon E^2 + H^2)}/8\pi = (1/16\pi)(\varepsilon \mathbf{E}_0\mathbf{E}_0^* + \mathbf{H}_0\mathbf{H}_0^*) = (n^2/8\pi)(\mathbf{E}\mathbf{E}_0^*),$$

$$\bar{\mathbf{S}} = (c/4\pi)\overline{[\mathbf{E}\mathbf{H}]} = (c/16\pi)\{[\mathbf{E}_0^*\mathbf{H}_0] + [\mathbf{E}_0\mathbf{H}_0^*]\} = (cn/8\pi)(\mathbf{E}_0\mathbf{E}_0^*)\,\mathbf{k}/k,\quad \overline{\mathbf{g}^{\mathrm{A}}} = \bar{\mathbf{S}}/c^2,\quad \overline{\mathbf{g}^{\mathrm{M}}} = n^2\overline{g^{\mathrm{A}}}, \tag{35}$$

or

$$\mathbf{G}^{\mathrm{A}} = \overline{\mathbf{g}^{\mathrm{A}}}L = (\bar{W}L/cn)\,\mathbf{k}/k = (\mathscr{E}/cn)\,\mathbf{k}/k, \tag{36}$$

$$\mathbf{G}^{\mathrm{M}} = \overline{\mathbf{g}^{\mathrm{M}}}L = (\bar{W}Ln/c)\,\mathbf{k}/k = (\mathscr{E}n/c)\,\mathbf{k}/k, \tag{37}$$

where $G^{A, M}$ and $\mathscr{E} = \mathscr{E}^A = \mathscr{E}^M$ are, respectively, the momenta and the energy of the wave packet.

The relationship (37) coincides with (4), and it leads to the correct expressions in using the laws of conservation of energy and momentum. It is hence now clear, and of course, this has been confirmed by calculations (they have been performed, e.g., in[9]) that one must use the energy-momentum tensor in the Minkowski form in obtaining quanta (photons in a medium) of energy $\hbar\omega$ and momentum $(\hbar\omega n/c)\,\mathbf{k}/k$ with the standard quantization. However, if we use the Abraham tensor, then, as we know, one gets a false result, both in the classical case (see (36), and when one quantizes and accounts for only the momentum G^A. Actually, however, as we should expect, using the Abraham tensor gives the right result, but we must also take into account the action of the Abraham force on the medium that occurs in the process of emission of radiation (the same is true of the absorption process). It is actually necessary to do this, since the force $\mathbf{f}^A$ (see (25)) differs from zero when a wave train is emitted (or, e.g., it enters a medium). Here we are interested not in the force itself, but in the impulse of of the force

$$\mathbf{F}^{\mathrm{A}} = \frac{n^2-1}{4\pi c}\int \frac{\partial}{\partial t}[\mathbf{E}\mathbf{H}]\,dt\,d\mathbf{r} = \frac{n^2-1}{16\pi c}\{[\mathbf{E}_0\mathbf{H}_0^*] + [\mathbf{E}_0^*\mathbf{H}_0]\}L = \frac{(n^2-1)n}{8\pi c}(\mathbf{E}_0\mathbf{E}_0^*)L\frac{\mathbf{k}}{k} = \frac{(n^2-1)\mathscr{E}}{cn}\frac{\mathbf{k}}{k}. \tag{38}$$

Here we have dropped the oscillating terms, and thus are dealing with the time-average quantity.[6] We note that we can treat a more or less arbitrary wave packet from the very onset and calculate, and then compare the integral quantities $\mathscr{E} = \int W\,dt\,d\mathbf{r}$, $G^{A, M} = \int \mathbf{g}^{A, M}\,dt\,d\mathbf{r}$, and $\mathbf{F}^A$. The relations between these quantities remain the same as in (36)–(38) for a train with sharp boundaries.

Evidently, in the light of (36)–(38),

$$\mathbf{G}^{\mathrm{A}} + \mathbf{F}^{\mathrm{A}} = \mathbf{G}^{\mathrm{M}} = (\mathscr{E}n/c)\,\mathbf{k}/k. \tag{39}$$

Usually only two factors are essential in the system of applying the law of conservation of energy and momentum: first, the energy and momentum lost upon emission (or gained upon absorption) by the emitting particle or "system"; second, we must know, of course, what field energy is emitted in a given direction. However, the problem of how the momentum of the radiation is distributed or redistributed is not important from this standpoint. In the case being treated, the particle loses the momentum $-G^M$, while the field in the medium gains the momentum G^A, and the medium receives the impulse of force $\mathbf{F}^A = G^M - G^A$. For a medium of particular material acted on by the force $\mathbf{f}^A$, the particles of the medium are accelerated, and $\mathbf{F}^A = G^m$ is the momentum of the medium. However, in the general case, the state of the medium is determined by the corresponding equations of motion, e.g., equations of the theory of elasticity or hydrodynamic equations, in which the density of the volume force is $\mathbf{f}^A$, while in principle it can also contain certain other terms. Of course, we must not think here that $\mathbf{F}^A = G^m = \mathbf{g}^m L$, where $\mathbf{g}^m$ is the momentum density of the medium (precisely this fact is correctly emphasized in[5]).[7] Likewise, we must in general also not state that the density of the Minkowski momentum $\mathbf{g}^M = \mathbf{g}^A + \mathbf{g}^m$. However, as we have seen with regard to the integral quantities (the momenta and the impulse of force $\mathbf{F}^A$), the result (39) does not depend at all on the properties of the medium, and it remains valid even under the assumption (which is generally false) that $\mathbf{g}^M = \mathbf{g}^A + \mathbf{g}^m$. Thus it is actually justified to use the Minkowski tensor in this case, since it not only furnishes a correct result, but also it leads to the goal more directly without treating the action of the volume force. Indeed, it is very simple to account for the action of this force within the framework of the classical approach (see above), but it would apparently seem to be quantum-mechanically a rather cumbersome matter. To our knowledge, such a quantum treatment has not yet been carried out in any way. In non-steady-state problems for whose solution the Abraham tensor is advantageous or even necessary, a corresponding quantum analysis would be justified (although, of course, not necessary as long as the problem is classical, as is probably true with any actual posing of the problem of measuring the Abraham force; in this regard, see[3,5]). As for the above-discussed application of the laws of conservation of energy and momentum in emission of "photons in a medium," it seems to us that the problem can be considered to be quite clear in the light of the presented remarks. Moreover, these remarks are not new, but as it seemed to us, it was still expedient to present and compare them here, in order to supplement somewhat in this regard the analysis of the problem of the choice of the energy-momentum tensor in macroscopic electrodynamics as contained in[5] and also in[2,3]

*$[\mathbf{E}\mathbf{H}] \equiv \mathbf{E}\times\mathbf{H}$.

1)In order to eliminate the zero-point energy, the operator $\mathcal{H}_{tr}$ is written in the form $\mathcal{H}_{tr} = (1/2)\sum_{\lambda,j}(p_{\lambda j} - i\omega_\lambda q_{\lambda j})(p_{\lambda j} + i\omega_\lambda q_{\lambda j})$, which in the classical region is equivalent to Eq. (8).

2)In an inhomogenoues medium, a force also arises, having a density $\mathbf{f}^\epsilon = -(\mathbf{E}^2/8\pi)\nabla\epsilon$, while another force arises if we account for compressibility of the medium (see [15], Secs. 15 and 56; we must emphasize in general that a conservation law like (27) is still insufficient for deriving an unequivocal expression for the force density). In going from (24) to (27), it is convenient to use the identity $\mathbf{a} \times \operatorname{curl} \mathbf{a} = (1/2)\nabla \mathbf{a}^2 - (\mathbf{a}, \nabla)\mathbf{a}$.

3)Such possibilities are discussed in the articles [3] (p. 31) and [5]. Undoubtedly, measuring the Abraham force is quite justified, if only for the sake of "peace of mind," and, moreover, it is hard to doubt that such measurements can be made at the current level of technique.

4)In this case, the volume force acting on the medium does no work, since this work is proportional to the product of the force and the velocity of the medium.

5)To be exact, we must add that the medium is not only considered to be non-absorbing (real ϵ), but also transparent (the condition $\epsilon > 0$).

6)The force acts only while the wave (train) enters the medium or is being emitted. However, while a train of a given length is being propagated in a homogeneous medium, the impulse of the force $\mathbf{F}^A$ is zero.

7)An analogous situation occurs for phonons. The distribution of sound in a solid is not accompanied by displacement of mass, and in this regard the momentum of sound waves is zero (this does not take account of the relativistic effect: the fact that a packet of sound waves of energy ξ has a mass ξ/c^2, and hence has a momentum $(\xi/c^2)s$, where s is the velocity of sound). Hence, we find upon quantizing that sound quanta, or phonons, have an energy $\hbar\omega$ and zero momentum (neglecting the momentum $(\hbar\omega/c^2)\,s$). However, the statement that the momentum of a phonon (say, when emitted by an electron) is $\hbar\mathbf{k} = (\hbar\omega/s)\,\mathbf{k}/k$ actually means that, in the emission of a phonon, the lattice as a whole acquires the momentum $\hbar\mathbf{k}$ (we neglect transfer processes here). From the standpoint of applying the conservation laws during emission, absorption, and scattering of sound, however, nothing is changed if we assume, as is usually done, that the phonons themselves have an energy $\hbar\omega$ and momentum $\hbar k = (\hbar\omega/s)\, \mathbf{k}/k$ (the author is indebted to L. V. Keldysh for this remark). Moreover, the Abraham momentum of the field $G^A = \xi/cn = (\xi/c^2)\, c/n$ (see (36)). That is, it has the same meaning as the "true" momentum of the phonon $(\xi/c^2)\, s$, since the velocity of an electromagnetic pulse is c/n (neglecting dispersion).

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