The momentum-energy tensor of the electromagnetic field

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The question of the form of the momentum-energy tensor of the electromagnetic field (in a medium) remains debatable to this day. The dilemma of whether the photon momentum in a medium is equal to nhν/c (Minkowski) or hν/nc (Abraham) therefore remains unresolved (n is the refractive index). Simple considerations based on the law governing the motion of the center of gravity of the "field + medium" system lead, however, to a unique choice of Abraham's tensor. The Jones-Richards experiments do not contradict this, although they do not lead to a solution of the problem. In principle, measurements of the Jones-Richards type (of the pressure of light in media) in the pulsed regime would yield the solution of the problem. Considerable space is allotted to an analysis of the question of the "rejection" of Abraham's tensor, a question advanced by Laue and supported by many authors. It is shown that the use of the Laue criterion is based on an error in the very formulation of the question. The arguments advanced in this connection are illustrated by using as an example analogous relations in the case of the motion of a simple static system, namely a charged capacitor. The conservations laws applied to a static electromagnetic field having angular momentum also lead to Abraham's expression for the field momentum density.

CONTENTS

INTRODUCTION

The question of the momentum-energy (m—e) tensor[1)] of an electromagnetic field in an electrically polarizable medium should long ago have been relegated to the archives of classical physics.

During the last decade or two, however, many papers have been published in which the authors returned to this theme. The central point in this literature is occupied by the question of the relative estimate of two tensors, those of Abraham and Minkowski, which were proposed more than half a century ago.

A paradoxical situation has developed in the history of the problem, as a result of which problems concerning certain basic premises of classical electrodynamics have been under discussion to this day. The literature of the question abounds with contradictions and variants, and the result is that concepts which seemingly should not be debatable still remain unclear. We shall return at the end of the article to these contradictions, which often appear to be based on slipshod treatment of the physical meaning of various formal constructions.

A highly adverse influence on the entire subsequent history of the question was exerted by a characteristic episode. In 1950, Laue[2] called attention to certain limiting requirements that must be satisfied by the transformation properties of the components of the momentum-energy tensor of a light wave. These requirements follow (according to his assumption) from the following considerations.

Imagine a light pulse propagating in the form of a wave packet and moving in a medium with velocity c/n (n is the refractive index) relative to an observer who is immobile in this medium. Moving together with this packet (at the same velocity c/n) is a certain object which is situated at all times in the field of the light beam, and consequently remains at all times illuminated by this beam. Obviously, this object will also be seen as illuminated by another observer moving with an arbitrary velocity relative to the first observer. In order for this condition to be satisfied it is necessary that the velocity of the light-wave packet be transformed like the velocity of a material point on going from the reference frame of the first observer (in which he is immobile) to the "second-observer system."

According to Laue's assumption, he had proved that the Abraham tensor should be eliminated from consideration since it contradicts, as it were, the criterion indicated above. Assuming on this basis Minkowski's tensor, Laue drew far-reaching conclusions from this. Inasmuch as Laue's arguments (which from our point of view are inconsistent) continue to be constantly referred to, and inasmuch as analogous considerations leading to rejection of Abraham's tensor were subsequently developed also by Moller[3], this question will be considered by us in greater detail.

This article is not a review of the literature of the problem. A sufficiently complete list of papers devoted to the subject can be found in a recent review[4].

There are a number of papers of formalistic nature, whose authors approach the solution of the question by using generalizations, formulated in postulates, as a basis for subsequent application of a variational method. In most cases this formal approach seems to lead to a solution in the form of the Minkowski tensor (see, e.g., [5]). The authors of another group of papers (de Groot[6] and co-workers) followed the path of microscopic consideration of the problem. This method calls for introduction of an exceedingly cumbersome computation formalism which is difficult to visualize.

It might seem that the just-indicated group of papers should have been prefaced by a fully unambiguously established scheme of concepts within the framework of the Maxwell-Hertz electrodynamics[2)].

The fact that the alternatives in the choice of the form of the m—e tensor of the field (the Abraham tensor or the Minkowski tensor) still remains unresolved leads, however, to an ambiguity in the construction of this scheme of basic premises, inasmuch as the question of the ponderomotive[3)] forces in the electromagnetic field and of the momentum of this field remains debatable.

However, analysis of the question on the basis of simple models leads to the unique conclusion that, within the framework of the phenomenological picture and when applied to limiting cases of idealized dielectric media, the correct solution of the problem was given by Abraham[7] in the form of the tensor proposed by him. This conclusion, under certain conditions and simplifications, can be drawn on the basis of ideas advanced back in 1954 by the Hungarian physicists Marx and Gyorgyi[8], and in part also in an earlier paper by Beck[9] [4)].

If we stipulate satisfaction of the conservation laws in conjunction with the Maxwell-Hertz equation, the result is, as will be shown, a definite and sufficiently strong "selection rule" (for the theories).

A different "selection rule" was used by Laue—one that appeared to follow from his aforementioned criterion—and this has led to erroneous conclusions.

In view of the contradictions in the basic scheme of the concepts, it is advisable at this stage to disregard the complications introduced by taking the laws of thermodynamics into account. We assume for this purpose that the dielectric constant of the medium is a constant that does not depend on the electric field intensity or on the parameters characterizing the state of the medium.

The two models considered in detail below satisfy, for example, a requirement that is necessary in this connection: the density of the medium, expressed in terms of the number of particles (dipoles) per unit volume, remains unchanged (does not depend on the field intensity).

The use of this abstraction is expedient and seemingly also permissible just as, for example, it is permissible and expedient to consider in theoretical mechanics the laws of motion of an absolutely rigid body, disregarding the elasticity and plasticity properties.

In the exposition that follows, only elementary and, inasmuch as possible, illustrative deductions will be used.

For the sake of simplicity, the medium is assumed nonmagnetic and the particular case of the electromagnetic field of a plane wave is considered.

1. PHOTON MOMENTUM AND CONSERVATION LAWS

In connection with the situation outlined above, no clear-cut answer has as yet been given to the question of the value of the photon momentum in a medium, say an ideal one characterized by a constant refractive index (that does not depend on any parameters).

The Minkowski and Abraham tensors lead to essentially different expressions for the density (**g**) of the field momentum.

Anticipating the results of derivations that will be given below, we present these expressions here. In vector form they are given by

$$g^M = (1/4\pi c)\,[\mathbf{DB}] \qquad (1.1)^*$$

according to Minkowski (see, e.g., [1]) and

$$g^A = (1/4\pi c)\,[\mathbf{EH}] \qquad (1.2)$$

according to Abraham (see[7]); here **E**, **H**, **D**, and **B** are the vectors of the intensities and inductions of the electric and magnetic fields.

When speaking of a photon, we imagine a "packet" in the form of a "train" of plane (plane-polarized) waves, carrying an energy equal to $\mathscr{E} = h\nu$. According to Maxwell's equations, the relation H = nE holds in the field of a plane wave. (For simplicity we assume here and throughout $\mu = 1$.)

At an appropriate orientation of the coordinate axes we obtain, according to (1.1) and (1.2),

$$g^M = n^3E^2/4\pi c = nu/c \qquad (1.3)$$

and

$$g^A = nE^2/4\pi c = u/nc \qquad (1.4)$$

(since $D = \epsilon E$ and $\epsilon = n^2$; $u = n^2(E^2 + H^2)/8\pi$); here (and throughout) u is the light wave energy density.

From this we get the following expressions for the "photon" momentum (G^M and G^A) in the medium:

$$G^M = g^M l = nul/c = nh\nu/c \qquad (1.5)$$

according to Minkowski (l is the length of the train, $l = h\nu/u$), and

$$G^A = g^A l = h\nu/nc \qquad (1.6)$$

according to Abraham.

We note first that the question is also unclear from the experimental point of view. Experiment does not yield a direct answer to this question, and this will also be discussed later on.

A unique answer is, however, obtained by turning to the laws of conservation of the momentum and velocity of the center of gravity (as applied to the "field + medium" system). This, in any case, is the situation if we regard as correct the concept of the inertial character of the energy in accordance with the Einstein relation $\mathscr{E} = mc^2$.

From the simple scheme of the "Gedank experiment," which will now be considered, it follows that Minkowski's assumption contradicts the law of constancy of the velocity of the center of gravity of the "radiation + material body" system. This was first indicated in[10].

For the proof, we consider the "packet of light waves + medium [a transparent plate (rectangular parallelepiped) whose faces are parallel to the coordinate plane] system." Let us compare the displacements of the center of gravity of this entire system along the x axis (direction of beam propagation) in two cases:

a) The "packet" is outside the medium moving towards its (left-hand) boundary, passes through the medium, and then, at the instant of time chosen by us, is in a position that is symmetrical to its initial position relative to the medium (Fig. 1).

b) The initial and final positions of the packet relative to the medium are also symmetrical. But the packet has bypassed the medium in moving from the first position to the second. (The fact that the packet now moves as a result along a trajectory parallel to the x axis but not coinciding with it is obviously of no importance for the derivation.) If the medium were immobile during the time of passage of the light packet through it, then the displacement of the entire system ("light + medium") as a whole along the x axis would be the same in the two indicated cases. However, in the former case the time required to displace the light over the length of the light-conducting plate is larger in the first case, since the speed of light c/n is less in the medium than in vacuum outside the medium (we assume that $n > 1$). Since the velocity of the displacement of the center of gravity of the system is the same in both cases, the displacement of the center of gravity along the x axis from its initial to the final position should be larger in the former case. This means that the light-conducting medium will not remain immobile and should be displaced in the light-propagation direction. In other words, this means that when the light enters the medium it transfers to it a momentum in the beam direction, in contradiction to Minkowski's assumption (the momentum of the light $n\mathscr{E}/c$ in the medium is larger than the momentum E/c in vacuum).

Thus, simple qualitative considerations would seemingly be sufficient to exclude Minkowski's hypothesis. On the other hand, if we write down the equations for the conservation of the momentum and the constancy of the velocity of motion of the center of gravity, then we can also obtain a quantitative result.

We denote by X_c the coordinate of the center of gravity of the system, by M the mass of the plate, by μ the mass equivalent to the light energy, and by G the momentum of the wave packet in the medium. From the definition of the center of gravity of the system we obtain, by differentiation, the following equation:

$$dX_c/dt = [M\,(dx_1/dt) + \mu\,(dx_2/dt)]/(M + \mu); \tag{1.7}$$

here x_1 and x_2 are the coordinates of the center of gravity of the plate and of the light packet, respectively[5].

Let $(dX_c/dt)_1$ be the velocity dX_c/dt prior to the entry of the light into the medium:

$$(dX_c/dt)_1 = \mu c/(M + \mu). \tag{1.8}$$

We denote further by $(dX_c/dt)_2$ the value of the velocity for the time interval during which the light passes through the medium. The momentum-conservation equation yields

$$M\,dx_1/dt = (\mathscr{E}/c) - G. \tag{1.9}$$

Here $dx_2/dt = c/n$, and consequently, according to (1.7)

$$(dX_c/dt)_2 = [(\mathscr{E}/c) - G + \mu\,(c/n)]/(M + \mu). \tag{1.10}$$

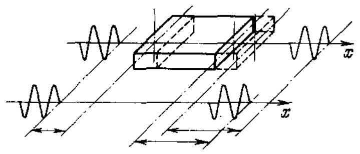

FIG. 1

But the constancy of the velocity of the center of gravity of the system calls for the equality

$$(dX_c/dt)_1 = (dX_c/dt)_2. \tag{1.11}$$

Substituting (1.8) and (1.10) in (1.11), we obtain

$$\mu c = [(\mathscr{E}/c) - G] + \mu\,(c/n). \tag{1.12}$$

Hence

$$G = (\mathscr{E}/c) + \mu\,[(c/n) - c] = \mathscr{E}/nc, \tag{1.13}$$

inasmuch as

$$\mu = \mathscr{E}/c^2.$$

Thus, Abraham's expression (1.6) follows from the conservation laws for the photon momentum.

In these simple calculations we have, of course, tacitly assumed that the energy flux is constant over the entire path of the beam, and consequently that energy losses due to reflection from the boundary of the medium have been excluded. This means that we have tacitly assumed that a "nonreflecting" transition layer with a smooth transition of the refractive index from the value of n in the medium to the value 1 outside the medium is coated on the boundaries of the medium at the points where the beam enters and leaves. Such "nonreflecting" faces of the medium are presently widely used in laser technology. In the theoretical limit (and to some degree also in practice) it is possible to realize conditions under which the reflection losses are reduced to an arbitrarily small value.

In addition, it should be noted that we have identified the group velocity of the light with the phase velocity, something that can also be realized with any desired accuracy if the wavelength band is chosen such that the anomalous-dispersion bands are situated somewhere in a remote region of the spectrum and exert no influence (the refractive index does not depend on the wavelength in this band).

The choice of an expression for the momentum density of the field leads uniquely to a definite conclusion concerning the density of the forces exerted by the wave field on the medium.

2. MAXWELL'S THEOREM, PONDEROMOTIVE FORCES, AND ELECTROMAGNETIC MOMENTUM

We recall the initial data on the ponderomotive forces to which Maxwell's equations lead, and primarily the theorem of the Maxwell stresses, which makes it possible to simplify the derivations presented below and to clarify them to a certain degree.

For the case with static fields, this theorem states that the resultant **K** of forces applied to bodies situated within a certain closed surface S is expressed by the integral over the surface (S) of the Maxwell stresses

$$\int \mathbf{T}\,(n)\,dS = \int \mathbf{f}\,d\Omega = \mathbf{K}, \tag{2.1}$$

where $d\Omega$ is the volume element and **f** is the force density.

Since we shall subsequently have frequent reference to Eq. (2.1), and since the very concept of the stress tensor follows from this equation, we recall some pertinent relations and definitions.

The vector **T**(n) introduced under the integral sign in the left-hand side of the equation is the tension force acting on a surface element dS, the outer normal to which is directed along a specified unit vector **n**. The component T_x(n) of the vector **T**(n) is defined by the relation

$$T_x(n) = t_{xx}\cos(n, x) + t_{xy}\cos(n, y) + t_{xz}\cos(n, z). \qquad (2.2)$$

Analogous relations hold for the components T_y and T_z.

The nine quantities (t_{xx}, t_{xy}, t_{xz}) plus the six others entering in the analogous expressions for T_y and T_z form a three-dimensional symmetrical ("relative") stress tensor.

The four-dimensional m—e tensor (the spatial components of which make up the so-called "absolute" stress tensor[6]) is obtained by generalizing the three-dimensional tensor T_{lm}, which will be discussed later on.

Returning to our interrupted exposition, we refer first to the following calculation result. If in the particular case of the electrostatic problem we deal with a field in the absence of true electric charges in an "ideal" dielectric with a dielectric constant ϵ, then, as shown by calculation, Maxwell's theorem leads to the following expression for the density of the ponderomotive force f acting on the dielectric ([7a], pp. 150—154):

$$\mathbf{f} = -(E^2/8\pi)\,\mathrm{grad}\,\varepsilon. \qquad (2.3)$$

It can easily be shown, taking Maxwell's equations into account, that (2.3) is the equivalent of

$$\mathbf{f} = (\mathbf{P}\cdot\mathrm{grad})\,\mathbf{E} - \mathrm{grad}\,(\mathbf{PE}/2), \qquad (2.4)$$

where

$$(\mathbf{P}\cdot\mathrm{grad}) = P_x(\partial/\partial x) + P_y(\partial/\partial y) + P_z(\partial/\partial z); \qquad (2.5)$$

P is the polarization of the dielectric. Formulas (2.3) and (2.4) will be needed by us later on. Both terms in (2.4) have a simple physical meaning, to which we shall return.

By introducing the concept of electromagnetic momentum, Abraham generalized Maxwell's theorem to the case of alternating and high-frequency fields.

Assume that we are considering electric charges on conducting bodies situated inside a closed surface S in vacuum. In the case of the electrodynamic problem, transformation of the integral of the Maxwell tensions over the surface S leads to the equation

$$\int \mathbf{T}(n)\,dS - \mathbf{K} = \int (1/4\pi c)\,(\partial\,[\mathbf{EH}]/\partial t)\,d\Omega, \qquad (2.6)$$

where E and H are the electric and magnetic field intensities.

As postulated by Abraham, the right-hand side of (2.6) now contains the derivative d**G**/dt, where $\mathbf{G} = \int \mathbf{g}\,d\Omega$ is the electromagnetic momentum of the field in the entire volume inside the surface S and, accordingly, **g** is the density of this momentum.

If there are no true charges inside the surface S, and consequently **K** = 0, then Eq. (2.6) in the case of a field in vacuum takes the form

$$\int \mathbf{T}(n)\,dS = (1/4\pi c)\int (\partial\,[\mathbf{EH}]/\partial t)\,d\Omega = \int (\partial g/\partial t)\,d\Omega. \qquad (2.7)$$

Therefore **g** (in vacuum) is given by

$$\mathbf{g} = (1/4\pi c)\,[\mathbf{EH}]. \qquad (2.8)$$

However, if the space inside the surface S is filled with a homogeneous medium and there are no true charges in this space, as before, then the appropriate transformation of the surface integral into a volume integral yields

$$\int \mathbf{T}(n)\,dS = (\partial/\partial t)\int (\varepsilon\mu/4\pi c)\,[\mathbf{EH}]\,d\Omega, \qquad (2.9)$$

where ϵ and μ are respectively the dielectric constant of the medium and its magnetic permeability. In this case one cannot introduce a priori and in a unique manner an expression for the electromagnetic momentum density in the medium.

If it is assumed that no forces act, even in an alternating electromagnetic field, on a homogeneous electrically uncharged transparent medium (and in the absence of constant magnetic moments), and if it is assumed at the same time that Abraham's interpretation of the expression in the left-hand side of (2.9) remains in force, then, in accord with the meaning of this equation, the integral in the right-hand side yields the momentum of the field. According to Minkowski it is necessary to assume that the density of the ponderomotive forces is zero under the conditions indicated above. This leads to the conclusion that the radiation-momentum density is

$$\mathbf{g}^{M} = (\varepsilon\mu/4\pi c)\,[\mathbf{EH}] \qquad (2.10)$$

(or, if we assume conversely that the density $\mathbf{g}^M$ is given by (2.10), we arrive at the conclusion that there are no ponderomotive forces).

On the other hand, according to Abraham, expression (2.8) for the momentum density remains in force also in the case of a field in a material medium:

$$\mathbf{g}^{A} = (1/4\pi c)\,[\mathbf{EH}]. \qquad (2.11)$$

This means that if the right-hand side of (2.9) is rewritten in the form

$$(1/4\pi c)\left[\int (\partial\,[\mathbf{EH}]/\partial t)\,d\Omega + \int (\partial/\partial t)\,(\varepsilon\mu - 1)\,[\mathbf{EH}]\,d\Omega\right],$$

and consequently

$$\begin{aligned}\int \mathbf{T}(n)\,dS &= (d\mathbf{G}^{A}/dt) + \mathbf{K}\\ &= (d\mathbf{G}^{A}/dt) + \int (\partial/\partial t)\,[(\varepsilon\mu - 1)/4\pi c]\,[\mathbf{EH}]\,d\Omega\end{aligned} \qquad (2.12)$$

(a natural assumption), then we have the density of the ponderomotive force in the integrand of the second term of the second equation of (2.12).

The forces exerted by a light wave on a transparent dielectric (or magnetoelectric) were introduced by Abraham "hypothetically." These "Abraham forces," however, have a simple physical meaning. The question is considered in detail in[8]. We advance here in this connection only the following considerations:

Let us return to expression (2.3) for the density of the force **f** in an electrostatic field. This expression (and the equivalent (2.4)) is valid only if the total (internal) energy can be identified with the "free" energy. Stipulating that we are dealing with an "ideal" dielectric, we assume that this condition is satisfied. The models which will be considered below satisfy the indicated requirements.

We shall henceforth consider the field of a plane electromagnetic wave. The first term of the expression (2.4) does not play any role in this field. In the second term we have the gradient of the quantity $\mathbf{P}\cdot\mathbf{E}/2$, where $\mathbf{P}$ is the polarization and $\mathbf{E}$ is the electric field intensity.

If, for concreteness, we visualize, the dielectric in the form of an assembly of dipoles with an alternating dipole moment of the electric charges, which are kept together by quasielastic forces, $\mathbf{P}\cdot\mathbf{E}/2$ is the polarization energy (per unit volume) stored in the given field at the given instant of time in the form of the energy of these quasielastic forces.

The mechanical (ponderomotive) forces known from electrostatics and determined by the gradient of an energy density equal to[7] $\mathbf{P}\cdot\mathbf{E}/2$, also appear in the field of an electromagnetic wave, but added to them is also the Lorentz force exerted by the magnetic field on the polarization current.

Considering further, for simplicity, a plane wave with a normal directed along the x axis, we can put

$$\mathrm{grad} = \partial/\partial x.$$

In addition, we bear in mind that from the plane-wave equation, as can readily be verified, it follows that

$$\partial/\partial x = -(n/c)\,\partial/\partial t, \tag{2.13}$$

where n is the refractive index of the medium and c is the speed of light in vacuum. The Lorentz-force density is

$$\mathbf{f}_L = (1/c)\,[(\partial\mathbf{P}/\partial t)\,\mathbf{H}]. \tag{2.14}$$

Under the conditions indicated by us we have

$$f_L = (1/c)\,(\partial P/\partial t)\,H. \tag{2.15}$$

The total density of the ponderomotive force is

$$f = f_L + f_P, \tag{2.16}$$

where

$$f_P = -(\partial/\partial x)\,(\mathbf{PE}/2) = (\partial/\partial t)\,(n/c)\,(\mathbf{PE}/2). \tag{2.17}$$

If we take into account Maxwell's equations and the relation

$$H = nE \tag{2.18}$$

(n is the refractive index), which follows from these equations for the field of a plane wave, then, by simple manipulations we can easily verify that

$$f = [(\varepsilon - 1)/4\pi c]\,\partial(EH)/\partial t. \tag{2.19}$$

Taking (2.18) into account, expression (2.19) (again for a plane-wave field) can be rewritten in the form

$$f = [(n^2 - 1)/cn]\,\partial u(t)/\partial t, \tag{2.20}$$

where u(t) is the energy density. We have assumed that $\mu = 1$ and $n^2 = \epsilon$. Formula (2.20) is valid also in the more general case ($\mu \neq 1$) at $n^2 = \epsilon\mu$. The derivation of (2.20) in this more general form is given in Appendix 1 at the end of the article.

Further, according to (2.13), we have

$$f = -[(n^2 - 1)/n^2]\,\partial u(t, x)/\partial x. \tag{2.21}$$

Relation (2.21) can be interpreted as follows: The density of the ponderomotive force is given numerically by the gradient (with minus sign) of the pressure (of the light on the medium), equal to

$$p_{lt} = [(n^2 - 1)/n^2]\,u(t, x). \tag{2.22}$$

If we assume (as will be done from now on) that the medium is a body with a very large elasticity coefficient, and in the limit an ideally rigid body, and in addition that the wave-packet boundaries are located inside the medium, then the pressure at any instant of time and in any cross section is balanced by the elastic stress of the medium:

$$|T_{med}| = [(n^2 - 1)/n^2]\,u(t, x). \tag{2.23}$$

The derivative $\partial u/\partial t$ when averaged over the time (or the derivative $\partial u/\partial x$ when averaged over space) in the field of a plane sinusoidal wave is equal to zero. Consequently, the density of the ponderomotive force (2.21) is also equal to zero on the average.

On the other hand, the pressure of light and (under the boundary conditions indicated above) the elastic stress of the medium, which is equal and opposite to the light pressure, have the absolute value

$$[(n^2 - 1)/n^2]\,\bar{u}, \tag{2.24}$$

where $\bar{u}$ is the time-averaged energy density.

By way of illustration, and bearing in mind the conclusions that will be needed later on, we apply the Maxwell-Abraham theorem in the three situations shown in Fig. 2. We consider the cylinder CD. It is easy to verify that its lateral surface makes no contribution to the integral $\int\mathbf{T}(n)dS$. Of the two end surfaces (whose area is assumed equal to unity) such a contribution is made only by one, the shaded base of the cylinder CD. In this case, consequently, the left-hand side of (2.12) contains the quantity $\int\mathbf{T}(n)dS = \mathbf{u}$, since[8]

$$T(n) = -T_x = -t_{xx} = u(t, x).$$

The right-hand side of the first of the two equations of (2.12) contains the sum of two terms, $\mathbf{F}_{med} + (dG^A/dt)$, where $\mathbf{F}_{med}$ ($\mathbf{K}$ in (2.12)) is the force acting on the medium in the volume of the cylinder, and dG^A/dt is the increment of the electromagnetic momentum per unit time in the same volume. The quantity dG^A/dt can be expressed as the product of the momentum density by the speed of light c/n:

$$dG^A/dt = gc/n = [u(t, x)/cn]\,c/n = u/n^2 \tag{2.25}$$

(according to Abraham's postulate, since $g^A(1/4\pi c)|\mathbf{E}\times\mathbf{H}| = nE^2/4\pi c = u/nc$). Consequently, (see the first equation of (2.12))

$$u = F_{med} + (u/n^2), \quad F_{med} = [(n^2 - 1)/n^2]\,u(t, x). \tag{2.26}$$

The force F_{med} is equal to the light-wave pressure referred to above.

The same considerations, when applied to the cylinder AB, result in an oppositely directed (with magnitude F_{med}) light-pressure force on the trailing edge of the

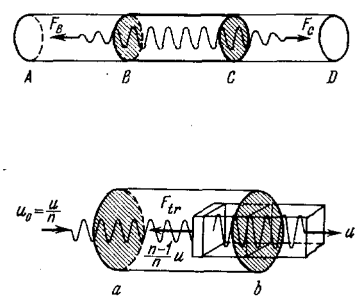

FIG. 2

wave. In (2.26), u(t, x) is the instantaneous value of the field-energy density at the given cross section (x).

Finally, considering the cylinder (ab) of Fig. 2, we obtain the average value of the force F_{tr} acting on the transition layer, in the form of the difference between the average Maxwellian stresses (pressures) on the shaded end surfaces of the cylinder ab:

$$|F_{tr}| = u - u_v = [(n-1)/n]u, \qquad (2.27)$$

where u and u_v are the energy densities in the medium and in vacuum, respectively. (By assumption, $u_v = u/n$: the energy flux uc/n to the medium is equal to the energy flux $n_v c$ in the vacuum.) The force F_{tr} draws the transition layer towards the vacuum in a direction opposite to that of the light beam. The electromagnetic momentum in the volume of the cylinder ab remains, on the average, unchanged. Consequently, the result of the calculation of the force F_{tr} does not depend on any assumption concerning the value of the momentum density, so that this result is reached equally by both hypotheses, Minkowski's and Abraham's. It is easy to calculate F_{tr} also by using relation (2.3) (see Appendix 2).

One more remark concerning the terminology. Above, in connection with Eq. (2.25), we set above the momentum flux equal to gc/n, i.e., equal to the product of the momentum density by the speed of light (or by the momentum transport speed). The flux was calculated as the flow of a liquid with density g. However, the momentum transmitted by the field through a unit cross-sectional area is, generally speaking, larger than the value calculated by the method indicated above. Namely, in this case, if ponderomotive forces of the field are present, then a mechanical momentum equal to the radiation pressure on the medium is also transferred through a unit cross-section area. The total momentum transmitted per unit time through a unit surface (equal to the sum of the two indicated terms) is equal to the Maxwellian pressure (or to the tension with negative sign). In 1908, Planck[11] proposed to interpret the Maxwellian pressure as the density of the total flux of the electromagnetic momentum of the field. Using ensuing terminology, we can state that the total momentum flux is defined in the same manner after Abraham and Minkowski (see (2.9), (2.10), and (2.12)). We shall return to this remark at the end of the article.

3. EXPERIMENTS AIMED AT MEASURING THE PRESSURE OF LIGHT

Before we proceed to consider the forms of the m—e tensor, we shall discuss briefly the experimental verification of the hypotheses of Minkowski and Abraham.

In connection with these questions, the results of Jones and Richards'[12] measurements of the light pressure on a mirror placed in different refractive media have been the subject of a debate. Jones and Richards found that at a given energy flux having the same value for different media, the light pressure on the mirror is proportional to the refractive index of the medium.

If the wave packet referred to above is regarded as a model of the photon, in accord with Minkowski, and a momentum $nh\nu/c$ is assigned to this photon, then the experimental result of Jones and Richards[12] is obtained directly. Reflection imparts to the mirror a momentum equal to double the photon momentum. Since at a given light intensity (in different media) the number of photons per unit surface and per unit time in a unit solid angle is the same for all media, the momentum transferred to the mirror per unit time (in other words, the pressure on the mirror) is proportional to the momentum of an individual photon in the given medium. If Minkowski's expression is assumed, it follows that this momentum is proportional to the refractive index, as is indeed obtained in the experiment[12].

However, regardless of any assumptions concerning the radiation-momentum density in the medium, it follows directly from the theorem on the integral of the Maxwell tensions that the light pressure on the mirror in a certain medium is proportional to the radiation density at the surface of the mirror. And the radiation density is proportional at the same time to the refractive index, if the light intensity is given and remains constant (uc/n = const, hence $u \propto n$).

Whereas Minkowski's assumption concerning the radiation momentum can lead to the correct conclusion that the light pressure in a medium depends on its refractive index, the inverse conclusion, that Minkowski's expression is correct, can be deduced from the experimental data only if it is assumed beforehand, in arbitrary fashion, that the ponderomotive forces due to the propagation of light in a transparent medium are equal to zero.

The fact that no unique conclusion concerning the momentum density of light in a medium can be drawn from observations of stationary fluxes can be seen from the following scheme. The radiation flux from the source S_1, after reflection from mirror R, closes on itself in receiver S_2 (both are in vacuum). We assume that the losses on the boundaries of the medium have been eliminated. The forces exerted by the light pressure on the mirror, on the source, and on the receiver, and also in the boundary layer, are shown in Fig. 3. They do not depend on the assumption concerning the momentum density. The momentum balance conditions are satisfied.

The situation would be different, however, if it were possible to measure, in an experiment with a single light pulse, both the pressure on the mirror upon reflection of a "train" of waves and, simultaneously, the "recoil" acquired by the medium.

For simplicity we consider (not quite rigorously[9)]) the reflection at a certain sufficiently small angle to the normal to the mirror, as shown in Fig. 4. Owing to the pressure applied to the dielectric by the leading and trailing fronts of the waves (reflected and incident), as shown in Fig. 4, the medium acquires a momentum i

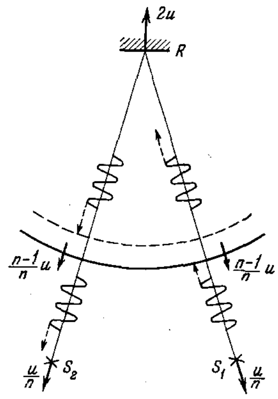

FIG. 3

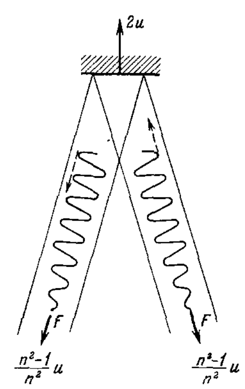

FIG. 4

equal to the product of the indicated pressure by the reflection time τ.

If the energy of the wave train is $\mathscr{E}$ (per unit cross section area), then the reflection time is $\tau = (\mathscr{E}/uc)n$, and we obtain for the momentum

$$i = 2\,[(n^2-1)/n]\,\mathscr{E}/c.$$

The momentum I/τ imparted to the mirror per unit time is equal to 2u, and over the entire reflection time its value is $I = 2u\tau = 2\mathscr{E}n/c$ (per unit surface). The momentum balance conditions are satisfied if we assume, after Abraham, that $g = u/cn$. In fact, let G denote the photon momentum as before. The momentum balance equation then yields

$$2G = 2g\,\mathscr{E}/u = I - i = 2\mathscr{E}\,n/c - (2\mathscr{E}/c)\,[(n^2-1)/n] = 2\,\mathscr{E}/nc;$$

here $\mathscr{E}/u$ is the length of the wave "train."

Measurement of I and i would make it possible to determine g. As shown by the foregoing calculations, the recoil forces can be of the same order of magnitude as the pressure on the mirror. Since light pressure can be measured, the Abraham force can also be measured in principle. Consequently, although the references usually made in textbooks and monographs to the effect that these forces are negligibly small cannot be regarded as correct, the possibility of realizing the described experiment in a pulsed regime is nevertheless doubtful. However, the considerations presented above, which appear to be perfectly "lucid" and resort only to conservation laws, would seem to enable us to predict the results of such a Gedank experiment. If this is so, the radiation momentum, which in principle could be obtained experimentally, is equal to $\mathscr{E}/nc$.

At the same time, as we have seen, if we introduce after Minkowski a certain auxiliary quantity $\mathscr{E}n/c$ as the "effective" value of the radiation momentum (excluding by the same token the interaction of the light with the transparent medium), then the results obtained by Jones and Richards[12] become directly explicable.

In the quantum theory of the Cerenkov radiation, Ginzburg[13] based his argument on Minkowski's expression $nh\nu/c$. Here, too, this expression gives the correct value of the "effective" momentum. It is somewhat more difficult to employ simple model concepts to reconcile the indicated "effective" value with the true value of the momentum in the case of the quantum effect. We shall return to this question at the end of the article.

4. THE FIELD MOMENTUM-ENERGY TENSOR

We have frequently referred to the theorem of the transformation of the integral of the fictitious Maxwell stresses.

If relation (2.6), which follows from this theorem, and the definitions of the relative stress tensor are applied to a unit volume (a rectangular parallelepiped dx dy dz), then we obtain

$$[(\partial t_{xx}/\partial x) + (\partial t_{xy}/\partial y) + (\partial t_{xz}/\partial z)]\,dx\,dy\,dz = [f_x + (dg_x/dt)]\,dx\,dy\,dz;$$

on the left side here we have the sum of the x-components of the tensions along the six faces of the parallelepiped surface, and on the right the product of the corresponding densities by the parallelepiped volume.

Equation (2.6) can therefore be rewritten in differential form:

$$(\partial t_{xx}/\partial x) + (\partial t_{xy}/\partial y) + (\partial t_{xz}/\partial z) = f_x + (\partial g_x/\partial t). \tag{4.1}$$

Analogous equations can also be written for the two other components of f and g.

Introducing the notation

$$S_{l4} = icg_l \quad \text{and} \quad S_{lm} = -t_{lm} \tag{4.2}$$

(where l, m = 1, 2, 3 and $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$) and transferring the term $\partial S_{14}/\partial x_4 = \partial g_1/\partial t$ of Eq. (4.1) to its left-hand side, we rewrite (4.1) in the form

$$-\partial S_{1m}/\partial x_m = f_1 \tag{4.3}$$

or, in general,

$$-\partial S_{lm}/\partial x_m = f_l; \tag{4.4}$$

here, as below, summation is implied over repeated indices (in this case, m).

In (4.4), which can also be written in the form

$$-\mathrm{Div}\, S_{lm} = f_l, \tag{4.5}$$

we have m = 1, 2, 3, 4 and l = 1, 2, 3. The symbol Div denotes the four-dimensional divergence. The components indicated above are also supplemented by

$$S_{4m} = (i/c)\,\Phi_m \quad (m = 1, 2, 3) \tag{4.6}$$

and

$$S_{44} = -u, \tag{4.7}$$

where Φ is the energy flux density and u is the energy density.

If the medium is at rest, then the field forces perform no work. In this case the energy conservation law is expressed by the flux-density continuity equation, in the form:

$$\partial S_{4m}/\partial x_m = 0. \tag{4.8}$$

It is postulated that the aggregate of the quantities indicated above (at l and m = 1, 2, 3, 4) forms a four-dimensional tensor—a "world tensor" (Welttensor).

The components of the four-dimensional vector determined by the divergences of the tensor S_{lm} for the static reference frame (the frame in which the medium is at rest), have been indicated above and can be represented by the following scheme:

$$(f_1,\ f_2,\ f_3,\ 0). \tag{4.9}$$

Here the f_m are, as we have seen, components of the ponderomotive-force density.

If the four-vector components are specified in some single admissible reference frame, then they can also be defined in any other inertial system. As is well known (and as can be verified directly by performing the Lorentz transformations in the given particular case), the fourth (temporal) component of the four-vector of the force density is equal to i/c times the power density dissipated by the field forces and transmitted by the field to the medium flux:

$$f_4 = -\partial S_{4m}/\partial x_m = (i/c)\,fw, \tag{4.10}$$

where f is the density of the ponderomotive force and w is the velocity of the medium.

As already noted, two expressions have been proposed for the density g of the electromagnetic momentum:

$$g^{\mathrm{A}} = (1/4\pi c)\,[\mathbf{EH}] \quad \text{and accordingly} \quad S_{l4}^{\mathrm{A}} = (i/4\pi)\,[\mathbf{EH}]_l \tag{4.11}$$

according to Abraham and

$$g^{\mathrm{M}} = (1/4\pi c)\,[\mathbf{DB}] \quad \text{and} \quad S_{l4}^{\mathrm{M}} = (i/4\pi)\,[\mathbf{DB}]_l \tag{4.12}$$

according to Minkowski. For simplicity, and essentially only to abbreviate the notation, we shall assume $\mu = 1$ and B = H.

If the components S^A_{l4} and S^M_{l4} are different, then the components S^A_{4m} and S^M_{4m}, as well as the remaining nine components S_{lm} of the two tensors, are identical (in the case of a medium at rest).

The Minkowski tensor is asymmetrical, $S^M_{l4} \neq S^M_{4l}$, whereas the Abraham tensor is symmetrical. It can be shown in general form[9][10] that this asymmetry of the tensor contradicts the law of motion (velocity conservation) of the center of gravity of the "radiation plus medium" system (if the principle that the energy is inertial is valid). We have already verified this with the simple example considered above. Thus, Minkowski's tensor fails to satisfy one of the "selection rules" referred to at the beginning of the article.

On the other hand, following Laue, the opinion is still being expressed in the literature that the Abraham tensor, which satisfies the criterion just mentioned, seemingly does not agree with another requirement: i.e., it does not satisfy the Laue criterion, which was also discussed earlier (see the Introduction). From this Laue drew the following conclusions:

First, of the two tensors (Abraham's and Minkowski's), one should accept Minkowski's tensor.

Second, as a result, inasmuch as the mixed (space-time) components of the tensor S^M_{lm} are antisymmetrical ($S^M_{l4} \neq S^M_{4l}$), Planck's postulate must be recognized to be in error (and by the same token, also Einstein's relation $\mathscr{E} = mc^2$).

By Planck's postulate, Laue meant the relation

$$g = \Phi/c^2, \tag{4.13}$$

where Φ is the energy flux density. In connection with the discussion that follows, we shall dwell on this in greater detail.

In the case of the Abraham tensor, S^A_{lm} (just as in Minkowski's case), Φ is the flux density of the electromagnetic energy and is determined by the Poynting vector

$$\Phi = (c/4\pi)\,[\mathbf{EH}]. \tag{4.14}$$

Inasmuch as (4.6) yields $S^A_{4l} = (i/c)\Phi_l$, and (4.2) yields

$$S^A_{l4} = icg_l \text{ and } S^A_{4l} = S^A_{l4} = (i/c)\,\Phi_l = icg_l,$$

relation (4.13) follows directly from Abraham's tensor. At the same time, the asymmetry of Minkowski's tensor ($S_{4l} \neq S_{l4}$) is incompatible with relation (4.13)[11].

The formulas become much simpler if, in the general case of the moving medium, one introduces as a parameter in the expressions for the tensor components, the energy density u_0 in the reference frame in which the medium is at rest. (Here and below, u_0 denotes the energy density in the medium.) This reference frame will be designated as the "zeroth" (or "unprimed").

Confining the analysis to the case of a plane and plane-polarized wave and to the "special"[12] Lorentz transformation, we assume that the normal to the plane of the wave is parallel to the x axis.

In this case, as will be shown below, we can confine ourselves in essence, without loss of generality, to two dimensions: x and t (the spatial coordinates y and z are eliminated; the tensor components corresponding to them are equal to zero).

The general scheme of the tensors is therefore given by the table

$$\begin{pmatrix} X_{11} & X_{14} \\ X_{41} & X_{44} \end{pmatrix}. \tag{4.17}$$

The formulas for the transformation of the components X_{lm} (l, m = 1, 4) take the form

$$X'_{lm} = \alpha_{lk}\alpha_{ms}X_{ks} \qquad (k, l, m, s = 1, 4; \quad x_1 = x, \quad x_4 = ict). \tag{4.18}$$

The coefficients α_{lm} can be found from the following table:

$$\begin{array}{c|cc} & \alpha_{l1} & \alpha_{l4} \\ \alpha_{1l} & \gamma & i\beta\gamma \\ \alpha_{4l} & -i\beta\gamma & \gamma \end{array}, \qquad \gamma = (1-\beta^2)^{-1/2}. \tag{4.19}$$

Then, according to (4.2), (4.6), and (4.7), the Abraham tensor in the zeroth system is expressed in the indicated notation by

$$S^{A0} \qquad \begin{pmatrix} u_0 & i\frac{u_0}{n} \\ i\frac{u_0}{n} & -u_0 \end{pmatrix}. \tag{4.20}$$

The scheme of the components of the asymmetrical Minkowski tensor is

$$S^{M0} \qquad \begin{pmatrix} u_0 & inu_0 \\ i\frac{u_0}{n} & -u_0 \end{pmatrix}; \tag{4.21}$$

Here $n = (\epsilon\mu)^{1/2}$ is the refractive index of the medium.

Transformations in accordance with (4.18) and table (4.19) yield the following results:

$$S^{A'} \qquad \frac{u_0}{n}\gamma^2\begin{pmatrix} n(1+\beta^2)-2\beta & i(1-2\beta n+\beta^2) \\ i(1-2\beta n+\beta^2) & -[n(1+\beta^2)-2\beta] \end{pmatrix}, \tag{4.22}$$

$$S^{M'} \qquad \frac{u_0}{n}\gamma^2\begin{pmatrix} (n-\beta)(1-n\beta) & i(n-\beta)^2 \\ i(1-n\beta)^2 & -(n-\beta)(1-n\beta) \end{pmatrix}. \tag{4.23}$$

In addition to the m–e tensor of the electromagnetic field, we shall consider later on also the corresponding mechanical tensors of the medium and the total tensor of the "quasiclosed" system (field + medium).

The spatial components of the mechanical tensor ("absolute stresses") will be represented as sums of two terms

$$p_{lm} + g_l w_m, \tag{4.24}$$

where the p_{lm} are the components of the "relative" tensor of the elastic stresses, while g_l and w_m are the momentum-density and medium-velocity components.

5. TWO MODELS OF AN IDEALIZED DIELECTRIC MEDIUM

We consider two models corresponding to two possible limiting cases, namely: a) in the "zeroth" system (see above), Eq. (4.24) reduces to the one first term—this is the case of an "ideally rigid" body;

b) in the same "zeroth" system, only the second term of the sum (4.24) differs from zero—this is the case of the so-called "dust-like matter."

Let us determine the forms of the tensors in the first of the just-indicated cases. On the leading and trailing fronts of the wave packet, the medium experiences the pressure of light with a force, as we have seen, equal to $[(n^2-1)/n^2]u_0$. (We now have in mind the zeroth reference frame.)

Since the processes are stationary at any point of the field under these conditions, the pressure of the electromagnetic field (under the indicated boundary conditions) is balanced by the elastic stress of the medium.

We consider the problem as one-dimensional (in the spatial coordinate). The total mechanical m—e transfer in the zeroth system can be written under these conditions in the form

$$X^0_{lm} \qquad \begin{pmatrix} -\frac{n^2-1}{n^2}u_0 & 0 \\ 0 & -\mu_0 c^2 \end{pmatrix}, \tag{5.1}$$

where μ_0 is the mass density of the medium.

The component X^0_{44} should, generally speaking, contain as a term the density of the elastic energy. We have assumed this term to be equal to zero, since we are considering the limiting case of an ideally rigid body, i.e., the case of an arbitrarily large elastic force at an arbitrarily small deformation energy (proportional to the square of the deformation itself).

The mechanical tensor of the medium in the absence of the field reduces in the zeroth system to a single component and takes the form

$$\begin{pmatrix} 0 & 0 \\ 0 & -\mu_0 c^2 \end{pmatrix}. \tag{5.2}$$

This tensor should be subtracted from the tensor (5.1), since it is expedient to consider only that part P_{lm} of the tensor (5.1) which is connected with the electromagnetic field and the reaction of the medium to the field-pressure forces.

The field-dependent component of the mechanical tensor of the medium therefore takes the form

$$P^0_{lm} \qquad p_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.3}$$

Transformation to the primed system leads to the following expression for the tensor:

$$P'_{lm} \qquad \gamma^2 p_0 \begin{pmatrix} 1 & -i\beta \\ -i\beta & -\beta^2 \end{pmatrix}; \tag{5.4}$$

here p_0 is the pressure of the medium and is negative under the conditions of this example: $p_0 = -[(n^2-1)/n^2]u_0$.

To clarify the physical meaning of the components P'_{lm}, it is advantageous to take into account the considerations advanced in Appendix 3.

According to (4.22) and (5.4), we obtain

$$T'_1 = S^{A'} + P'_{lm},$$
$$\frac{u_0}{n}\gamma^2 \left\{ \begin{pmatrix} n(1+\beta^2)-2\beta & i(1-2\beta n+\beta^2) \\ i(1-2\beta n+\beta^2) & -[n(1+\beta^2)-2\beta] \end{pmatrix} - \frac{(n^2-1)}{n}\begin{pmatrix} 1 & -i\beta \\ -i\beta & -\beta^2 \end{pmatrix} \right\}$$
$$= \frac{u_0}{n^2}\gamma^2 \begin{pmatrix} (n\beta-1)^2 & i(n-\beta)(1-\beta n) \\ i(n-\beta)(1-\beta n) & -(n-\beta)^2 \end{pmatrix}. \tag{5.5}$$

We shall soon return to the expression obtained by us for the tensor T'_1.

We consider first, however, another idealized model of a medium and the already-mentioned opposite limiting case, in which the elasticity forces are equal to zero and the components of the tensor of the "absolute" stresses coincide with the product $g_l w_m$. In this case we are dealing with a model in which the dielectric medium takes the form of "dustlike matter." The particles of this matter can be assumed to be arbitrarily massive. (In the limit, we shall set the mass density of the matter $\mu_0 = \infty$.) At the same time, the density determined by the number of particles per unit volume will be assumed to be sufficiently small[13] and the interaction between the particles to be negligibly small[14]. The small particle concentration does not mean that the dielectric constant is small, since the dipole moments of the particles of this artificial medium can be assumed to be arbitrarily large. We neglect the dispersion, as before. We can imagine this rarefied medium to be contained in a cylindrical shell. For convenience, we can assume that the entire system is in a weightless state. The entrance opening of the cylindrical vessel containing the refractive medium is covered with a window that ensures lossless entry of the light beam.

It follows from the foregoing calculations (2.27) that after the light beam is admitted, the cylindrical shell of the medium moves uniformly in the direction towards the light source, having a mechanical momentum equal to

$$(n-1)\,\mathscr{E}/c, \tag{5.6}$$

where $\mathscr{E}$ is the total energy of the light field.

The "conservation" equations (1.7), (1.9), and (1.11) can be applied to any autonomous system. These equations make it possible to determine in general form the two unknowns—the momentum (Mv) of the material component of the system and the momentum of the radiation.

For the quantity Mv we obtain from the indicated equations, in general form, the expression

$$Mv = (\mathscr{E}/c)(n-1)/n. \tag{5.7}$$

In this example, Mv is the difference between two quantities, the momentum of the medium (i) and the momentum of the shell of the "container." Consequently, according to (5.6) and (5.7),

$$\begin{gathered}(\mathscr{E}/c)(n-1)\,n = i - (\mathscr{E}/c)(n-1),\\ i = (\mathscr{E}/c)\{[(n-1)/n] + n - 1\} = [(n^2-1)/n]\,\mathscr{E}/c.\end{gathered} \tag{5.8}$$

We shall verify later on (see formula (5.13) below) that i is indeed equal to the momentum of the Abraham forces.

Inasmuch as expression (5.6) does not depend on the assumptions concerning the density of the momentum, and the Minkowski tensor as well as the Abraham tensor lead equally well to expression (5.6), this example shows directly that the Abraham forces are obtained as a consequence of the fundamental conservation laws.

We note that the sum of the material momentum coupled with the light wave (i) and the field momentum (G) is equal to

$$G + i = n\,\mathscr{E}/c. \tag{5.9}$$

The total momentum i + G agrees in this case with the field momentum as given by Minkowski.

If the equation of the beam in the zone of a sinusoidal wave is expressed in the form

$$E = E_0 \sin\{\omega[t-(xn/c)]\} \text{ and } H = nE_0 \sin\{\omega[t-(xn/c)]\} \tag{5.10}$$

and if the simplifications stipulated above are taken into consideration, then we can verify by simple calculations[15] that the light pressure imparts to the medium a motion having the following character. At the instant of time t = $(k\pi/\omega) + (xn/c)$ (where k is an integer at a specified x = const) the particle velocity at the point x is equal to zero. In the time interval $(k\pi/\omega) + (xn/c) < t < (k+1)(\pi/\omega) + (xn/c)$, the velocity of motion changes from zero to a certain maximum value and then again to zero. On the average, consequently, the velocity of the translational motion of the particle differs from zero and is positive.

The pressure of the light causes "drift" of the dustlike matter in the direction towards the beam. The drift velocity under our conditions is negligibly small. As we shall see (see (5.21) below), this velocity, henceforth designated $\beta_0 c$, is equal to

$$\beta_0 c = [(n^2-1)/n\mu_0 c]\, u_0 = [(n^2-1)\, n/4\pi\mu_0 c]\, E_0^2 \sin^2\{\omega\,[t-(xn/c)]\}.$$

This particle "drift" is also the reason why a certain condensation of the medium takes place within the limits of the illuminated region, as will be shown subsequently, and the density of the mass either fluctuates or oscillates, becoming dependent on the time[16].

The foregoing follows from the following simple calculation. We have seen (see (2.19)) that force density in a "nonzeroth" system is given by:

$$f = [(n^2-1)/4\pi c]\, \partial (EH)/\partial t. \qquad (5.11)$$

It follows directly from this that the momentum density is

$$g_\mu = \int f\, dt = [(n^2-1)/4\pi c]\,(EH) \qquad (5.12)$$

(the integration constant is equal to zero, since we can put E = H = 0 at t = 0).

Since Maxwell's equations for a plane wave yield H = nE and an energy density $u_0 = nEH/4\pi$, formula (5.12) can be rewritten in the form

$$g_\mu = [(n^2-1)/nc]\, u_0(t); \qquad (5.13)$$

here $u_0(t)$ is the energy density of the field at a given instant of time, and g_μ is the material momentum density.

If the mass density in the absence of the field is designated μ_0 and the density at each instant of time is designated

$$\mu(t) = \mu_0 + \Delta\mu_0, \qquad (5.14)$$

then Eq. (5.13) for the "drift" velocity $(\beta_0 c)$ becomes

$$\beta_0 c = (n^2-1)\, u_0/nc\,[\mu_0 + \Delta\mu_0(t)]. \qquad (5.15)$$

Simple calculation yields[17]

$$\Delta\mu_0 = [(n^2-1)/c^2]\, u_0(t)\ ^*). \qquad (5.16)$$

Going to the limit $\mu_0 = \infty$, we can in the assumed approximation neglect the quantity $\Delta\mu_0$ in the denominator of (5.15). Consequently

$$\beta_0\mu_0 c^2 = [(n^2-1)/n]\, u_0. \qquad (5.21)$$

The product $\beta_0\mu_0 c^2$ remains finite on going to the indicated limit, and the product $\beta_0^2\mu_0 c^2$ tends to zero. We shall therefore neglect terms of order $\beta_0^2\mu_0 c^2$.

Taking (5.16) into consideration, we now rewrite (5.14) in the form

$$\mu(t) = \mu_0 + [n^2-1)/c^2]\, u_0(t). \qquad (5.22)$$

We write down the components X_{lm}^0 of the mechanical ("kinetic") tensor of "dustlike" matter in the field of an electromagnetic wave. The spatial components in this case are the components of the momentum flux, equal to

$$w^2\mu_0 = \beta_0^2 c^2 \mu_0. \qquad (5.23)$$

The abbreviated scheme of the tensor X_{lm}^0 is

$$X_{lm}^0 \qquad \begin{pmatrix} \beta_0^2\mu_0 c^2 & i\frac{n^2-1}{n}u_0(t) \\ i\frac{n^2-1}{n}u_0 & -\mu_0 c^2-(n^2-1)\,u_0(t)\end{pmatrix}. \qquad (5.24)$$

An expression for the term X_{44}^0 follows from (5.22).

Subtracting also in this case the tensor k_{lm} of the "dustlike" matter in the absence of the field,

$$k_{lm} \qquad \begin{pmatrix} 0 & 0 \\ 0 & -\mu_0 c^2\end{pmatrix}, \qquad (5.25)$$

from the tensor X_{lm}, we obtain the "dustlike" part K_{lm}^0 of the kinetic (mechanical) tensor. K_{lm}^0 is connected with the electromagnetic field by

$$K_{lm}^0 = X_{lm}^0 - k_{lm}^0 \qquad \begin{pmatrix} \beta_0^2\mu_0 c^2 & i\frac{n^2-1}{n}u_0(t) \\ i\frac{n^2-1}{n}u_0(t) & -(n^2-1)\,u_0(t)\end{pmatrix}. \qquad (5.26)$$

The "dustlike" kinetic tensor contained an additional energy density $(n^2-1)\,u_0(t)$. This energy density is connected with the "drift" of the matter and with the condensation of the medium due to this drift.

We note that according to (5.24) and (5.21) the momentary density of the medium is given by

$$[(n^2-1)/nc]\, u_0 = \mu_0\beta_0 c. \qquad (5.27)$$

Thus, this is the momentum density of "dustlike" matter (of density μ_0) moving with a very small drift velocity ($\beta_0 c$).

It is interesting that the same momentum density (5.27) can be expressed in an entirely different fashion, namely, as the product of the density of a supplementary mass $(\Delta\mu_0)$ by the velocity (c/n) of "motion" of this mass, transported together with the light-wave field. Indeed, according to (5.20)

$$\Delta\mu_0 c/n = [(n^2-1)/cn]\, u_0,$$

which coincides with (5.27). If now, given the value $u_0(t)$, we go to the limit $\mu_0 = \infty$, then the tensor K_{lm}^0 can be represented in the form[18]

$$K_{lm}^0 \qquad \frac{n^2-1}{n}u_0(t)\begin{pmatrix} 0 & i \\ i & -n\end{pmatrix}. \qquad (5.28)$$

After transforming to the "primed system" we obtain the following formula:

$$K'_{lm} \qquad \frac{n^2-1}{n}u_0(t)\,\gamma^2\begin{pmatrix} n\beta^2-2\beta & i(1+\beta^2-n\beta) \\ i(1+\beta^2-n\beta) & -(n-2\beta)\end{pmatrix}. \qquad (5.29)$$

Just as in the case of the previously considered example, the total tensor is obtained as the sum of the tensors S_{lm}^A and K_{lm}:

$$T'_2 = S^{A'} + K' \qquad u_0\gamma^2\begin{pmatrix} (n\beta-1)^2 & i(n-\beta)(1-n\beta) \\ i(n-\beta)(1-n\beta) & -(n-\beta)^2\end{pmatrix}. \qquad (5.30)$$

Returning to the preceding example and comparing expressions (5.5) and (5.30) for the tensors T'_1 and T'_2, we verify that in these two opposite cases of limiting conditions, the tensors that were formed as indicated differ only by a factor $1/n^2$. A table of the components of the tensor T_2^0 is obtained from (5.30) by putting $\beta = 0$ in (5.30).

The model of the "dustlike" matter is convenient for comparison of the Abraham and Minkowski tensors. We shall return to their comparison at the end of the article. Now we call attention to the following. Since the components T'_{44} and T'_{41} are given, we can determine the velocity of the energy flux c* by dividing the energy flux density $(c/i)T'_{41}$ by the density of the energy itself, T'_{44}.

Both tables (both in (5.5) and in (5.30)) give one and the same value for the indicated velocity, which can also be defined as the ratio icT'_{11}/T'_{14}:

$$c^* = -(c/i)\, T'_{14}/T'_{44} = icT'_{11}/T'_{14} = c\,(1-n\beta)/(n-\beta). \qquad (5.31)$$

Inasmuch as in the zeroth system $c^* = c/n$, formula (5.31) means that the velocity c* transforms like the velocity of a material point. This suggests the conclusion that in order to obtain the correct value of the transport velocity of the light energy, it is necessary to consider, as was done above, the tensor of the total energy (and the corresponding momentum), including in this tensor also the mechanical forms of energy and momentum co-moving with the light field. Incidentally, this thought was already advanced in passing by one author (see[15], page 91) and soon refuted by another ([4a], page 40).

We shall show in the next chapter that the assumption just stated can be corroborated. Inasmuch as for a long time this question remained (and still remains) debatable, we shall dwell in greater detail on the analysis of the conclusions usually cited in this connection, although the arguments that will be developed are in essence quite trivial.

6. ENERGY FLUX AND PROPAGATION VELOCITY OF THE ENERGY OF A LIGHT WAVE

The question (or paradox) referred to above is resolved by recognizing that when a light wave propagates in a medium the fluxes of the electromagnetic energy and momentum, on the one hand, and the fluxes of the co-moving mechanical (in particular, elastic) energy and momentum, on the other hand, are interrelated.

It is possible that the calculations pertaining to the propagation of light, which will be presented below, will turn out to be more convincing if the gist of the question is first explained by using a simple example, that of a moving "electrostatic" system. We have in mind the following example. A charged capacitor moves uniformly perpendicular to the direction of the force lines (we have in mind the direction of the field in its central part). The rate of energy transport in this case is specified beforehand. It is equal to the rate of motion of the capacitor. This velocity can be comparable with the ratio of the energy flux density and the density of the energy itself. On the forward face of the moving capacitor, as shown in Fig. 5, the electromagnetic forces perform work against the elastic forces of the dielectric plate. On the rear face of the capacitor, this energy is returned to the field as a result of the work of the elastic forces against the electric forces. It is therefore clear that the flux of the electromagnetic energy through an immobile transverse cross section plane (defined by the component W_{41} of the m—e tensor of the field) is not equal to the product of the density of the energy of the field by the velocity of the capacitor, but is larger than this product. Consequently, the quotient of the electromagnetic-energy flux divided by the density of this energy is not equal to the velocity of motion of the capacitor (the energy transport velocity) but is larger.

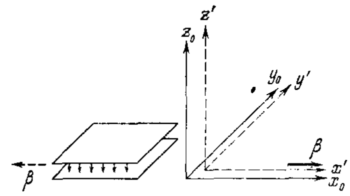

FIG. 5

Let us examine in detail the circulation of the energy and momentum fluxes that exist in the moving capacitor.

Just as in the case of the light-wave field, the directions of the magnetic and electric fields are mutually perpendicular and perpendicular to the direction of motion, which is the direction of the energy transport.

The parallel-plate capacitor is made up of a dielectric plate onto the faces of which (parallel to the plane X_0Y_0) a metal has been evaporated. The charged capacitor moves in the direction of the negative x' axis with constant velocity (equal to $w = -\beta c$) relative to the primed system (x', y', z') in Fig. 5. We shall henceforth (up to formula (6.15)) omit the factor c. The capacitor is at standstill in the zeroth system.

The schemes of the m—e tensor W can be represented (in this case, equally well for the Abraham and Minkowski tensors) in the case of the two reference frames indicated above in the following respective forms:

$$W^0_{lm}$$
$$\frac{\varepsilon E_0^2}{8\pi}\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (6.1)$$

$$W'_{lm}$$
$$\frac{\varepsilon E_0^2}{8\pi}\gamma^2\begin{pmatrix} (1+\beta^2) & 0 & 0 & -2i\beta \\ 0 & \frac{1}{\gamma^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\gamma^2} & 0 \\ -2i\beta & 0 & 0 & -(1+\beta^2) \end{pmatrix}. \qquad (6.2)$$

We consider the capacitor cross section plane y'z', which is at standstill in the primed reference frame. (The plane is at standstill and the capacitor moves relative to it.)

The flux of electromagnetic energy through this plane at a given instant of time t', determined by the component W'_{41} of the m—e tensor W', is equal to

$$-(2\beta\varepsilon E_0^2/8\pi)\,\gamma^2 \qquad (6.3)$$

(according to (6.2); the cross-section area of the capacitor is assumed equal to unity). The energy flux density Φ' in this example is also equal to $(1/4\pi)\mathbf{E}'\times\mathbf{H}'$.

Dividing the energy flux density by the energy density, we obtain the velocity

$$w^* = -2\beta/(1+\beta^2). \qquad (6.4)$$

The velocity w* is not equal to the velocity $-\beta$ of the capacitor motion, $|w^*| > \beta$.

We shall speak arbitrarily of an energy "flux front." We have in mind an arbitrary field boundary, a certain plane y_0z_0 outside the capacitor, moving in the (x', y', z') system together with the capacitor. The question is: what is the velocity dx_0/dt' of motion of the "flux front" as seen by an observer who is at standstill in the (x′, y′, z′) system?

In the energy balance equation it is necessary to take into account the work L of the ponderomotive force F′ acting on the front edge of the dielectric of the capacitor. In the zeroth reference frame, the force F_0, determined by the Maxwell tensions (in this case, by the pressure) is numerically equal to the energy density u_0. As is well known, the longitudinal component of the force remains unchanged on going to the moving system:

$$F' = F_0 = u_0. \tag{6.5}$$

Hence

$$L' = u_0\beta. \tag{6.6}$$

(The cross-section area of the capacitor is assumed equal to unity.)

The energy balance equation should therefore be written in the form

$$|dx_0/dt'| = (|\Phi'_{em}| - F'\beta)/u' = (|\Phi'_{em}| - u_0\beta)/u', \tag{6.7}$$

where u′ is the energy density in the primed system, $u' = -W'_{44}$, and consequently, according to (6.2), we have

$$u' = u_0(1+\beta^2)\gamma^2 = u_0(1+\beta^2)/(1-\beta^2). \tag{6.8}$$

At the same time, the component W'_{41} (6.2) yields

$$|\Phi'_{em}| = 2\beta u_0\gamma^2 = 2\beta u_0/(1-\beta^2). \tag{6.9}$$

Substituting (6.8) and (6.9) in (6.7), we obtain

$$|dx_0/dt'| = \{[2\beta u_0/(1-\beta^2)] - u_0\beta\}(1-\beta^2)/[u_0(1+\beta^2)] = \beta, \tag{6.10}$$

as we should.

The electromagnetic-energy flux is partly offset by the counterflow of the elastic energy. According to the scheme (5.4) given above, we obtain for the density $\rho^{M'}$ of the velocity-dependent component of the elastic energy, recognizing that[19] $p' = p_0 = -u_0$,

$$\rho^{M'} = -\beta^2 u_0\gamma^2. \tag{6.11}$$

Hence, according to (6.8) and (6.11), the density of the total energy is

$$\rho' = u' + \rho^{M'} = u_0\gamma^2(1+\beta^2-\beta^2) = u_0\gamma^2. \tag{6.12}$$

The flux density[20] of the elastic energy is, according to (5.4),

$$(1/i)P'_{41} = -p_0\beta\gamma^2 = u_0\gamma^2\beta. \tag{6.13}$$

The direction of this flux is opposite to the direction of motion of the medium.

Finally, in accord with (6.2) and (6.13), the flux density of the total energy is

$$u_0\gamma^2(-2\beta+\beta) = -\beta u_0\gamma^2. \tag{6.14}$$

The velocity of the "front" of the flux of the total energy, obtained as a result of the dividing the total flux (6.14) by the density of the total energy (6.12), is

$$dx_0/dt' = -\beta u_0\gamma^2/u_0\gamma^2 = -\beta, \tag{6.15}$$

as should be the case.

In a more general form, the same considerations were developed essentially by Laue[2a] in connection with a discussion of the classical experiment by Trouton and Noble, who proposed to observe the effect of orientation of a freely suspended parallel-plate capacitor perpendicular to the motion of the earth.

The balance equations determining the relation between the flux density and the energy density in the plane wave are perfectly analogous to those just considered by us, with the moving capacitor as an example. We recall that

$$\partial S'_{4m}/\partial x'_m = i/cL'_{em}, \tag{6.16}$$

where S'_{41} are the "mixed" components of the m–e tensor and L'_{em} is the power dissipated in a unit volume of the field.

Under the conditions of the reference frames considered by us and the special Lorentz transformation, the energy balance equation (6.16) can be rewritten in the form

$$-\partial\Phi'_{41}/\partial x' = (\partial u'/\partial t') + L'_{em}, \tag{6.17}$$

where Φ'_{41} is the energy flux density and u′ is the energy density;

$$L'_{em} = wf' = -\beta c f^0\gamma, \tag{6.18}$$

where f′ and f^0 are the densities of the force in the primed and in the zeroth coordinates, respectively, and $w = -\beta c$ is the velocity of motion of the medium in the primed system.

In the case of a plane wave in a moving medium, the following relation holds:

$$\partial/\partial t' = -c^*\,\partial/\partial x', \tag{6.19}$$

where c* is the phase velocity, which by assumption coincides in our case with the group velocity. As is well known[21]

$$c^* = c(1-\beta n)/(n-\beta). \tag{6.20}$$

Substituting (6.19) in (6.17) we obtain

$$-(\partial/\partial x')(\Phi'_{em} - c^*u') = L'_{em}. \tag{6.21}$$

We take into consideration the following relations. According to (6.18) we have

$$L'_{em} = \gamma\beta c[(n^2-1)/n^2](\partial u_0/\partial x)_t, \tag{6.22}$$

since

$$f_0 = -(\partial/\partial x)[(n^2-1)u_0/n^2]. \tag{6.23}$$

But

$$\left(\frac{\partial u_0}{\partial x'}\right)_{t'} = \left(\frac{\partial u_0}{\partial x}\right)_t\left(\frac{dx}{dx'}\right)_{t'} + \left(\frac{\partial u_0}{\partial t}\right)_x\left(\frac{dt}{dx'}\right)_{t'} = \left(\frac{\partial u_0}{\partial x}\right)_t\frac{\gamma(n-\beta)}{n}. \tag{6.24}$$

Formula (6.24) can easily be obtained on the basis of the formulas of the Lorentz transformation (x, t) → (x′, t′) and by taking (2.13) also into consideration.

The substitution of $(\partial u_0/\partial x)_t$ from (6.24) in (6.22) yields

$$L'_{em} = \beta c[(n^2-1)/n(n-\beta)](\partial u_0/\partial x')_{t'} \tag{6.25}$$

and according to (6.21) we have

$$-(\partial/\partial x')\{\Phi'_{em} - c^*u' + \beta c[(n^2-1)/n(n-\beta)]u_0\} = 0. \tag{6.26}$$

Integrating from x′ to x′ = ∞, we obtain

$$\int_{x'}^{\infty} -(\partial/\partial x')\{\Phi'_{em} - c^*u' + c[\beta(n^2-1)/n(n-\beta)]u_0\} = 0, \tag{6.27}$$

or

$$\Phi'_{em} - c^*u' + [c\beta(n^2-1)/n(n-\beta)]u_0 = 0. \tag{6.28}$$

Equation (6.28) can be verified directly by substituting in (6.28) the expressions for Φ′ and u′, which are given by Table (4.22) of the Abraham tensor $S^{A'}$, namely

$$\Phi'_{em} = u_0c\frac{1-2n\beta+\beta^2}{(1-\beta^2)n}, \qquad u' = u_0\frac{n(1+\beta^2)-2\beta}{(1-\beta^2)n} \tag{6.29}$$

and c* in accordance with (6.20). The quantities Φ'_{em} and u′ pertain to the cross section x′.

As seen from the derivation, the last term in the left-hand side of (6.28) gives the work, with sign reversed (negative if $\beta > 0$[22]) of the electromagnetic forces per unit time in the region of the field from the chosen section x′ to the front of the wave. It is also seen from the derivation that if we have the mean values in mind, then a nonzero contribution to the integral (6.27) is made only by a narrow interval near the wave front, namely the derivative $-\partial u_0/\partial x'$ can be regarded arbitrarily as a δ function (multiplied by u_0), if the front of the wave packet is steep enough. On the other hand, the mean value (over the time and over the space) of the density of the force (of the derivative $\partial u_0/\partial x$) in the zone of a sinusoidal wave is equal to zero.

Thus, $-c\beta[(n^2-1)/n(n-\beta)]u_0$ yields the power (per unit cross section) lost by the field on the front of the wave.

Denoting by c** the velocity of the wave front, we express the energy balance equation in integral form

$$\Phi'_{em}-c^{**}u'-L'_{em}=\Phi'_{em}-c^{**}u'+c\,[\beta(n^2-1)/n(n-\beta)]u_0=0; \quad (6.30)$$

where Φ'_{em} is the flux of the electromagnetic energy through some cross section in the zone of the sinusoidal wave. A comparison of Eqs. (6.30) and (6.28) shows the following.

First, by dividing the electromagnetic-energy flux by the energy density, we obtain a velocity which is not equal to the velocity of the energy flux (to the velocity c** of the wave front).

Second,

$$c^{**}=c^{*}, \quad (6.31)$$

i.e., the velocity of the wave front (the energy transport velocity) under the conditions considered by us is equal to the phase velocity and is consequently transformed like the velocity of a material point.

Analogous equations can be written for the relation between the flux $\Phi^{M'}$ of the energy of the medium, the density $\rho^{M'}$ of the energy of the medium, and the work of the reaction forces of the medium $L^{M'}$ (elastic forces, inertia forces):

$$\Phi^{M'}-c^{*}\rho^{M'}-L^{M'}=0. \quad (6.32)$$

The reaction forces of the medium are equal and opposite to the forces exerted on the medium by the field:

$$L^{M'}=-L'_{em}. \quad (6.33)$$

Therefore

$$\Phi^{M'}-c^{*}\rho^{M'}-\beta c\,[(n^2-1)/n(n-\beta)]\,u_0=0. \quad (6.34)$$

In the cases of examples (5.4) and (5.29) given above, we have in accordance with (5.4):

$$\Phi^{M'}=c\,[(n^2-1)/n^2]\,[\beta/(1-\beta^2)]\,u_0, \quad (6.35)$$

$$\rho^{M'}=-[(n^2-1)/n^2]\,[\beta^2/(1-\beta^2)]\,u_0, \quad (6.36)$$

whereas according to (5.29) we have

$$\Phi^{M'}=c\,[(1+\beta^2-n\beta)(n^2-1)/n(1-\beta^2)]\,u_0, \quad (6.37)$$

$$\rho^{M'}=[(n-2\beta)(n^2-1)/n(1-\beta^2)]\,u_0. \quad (6.38)$$

If we substitute in (6.34) in one case (6.35) and (6.36), and in the other case (6.37) and (6.38), we can verify that (6.34) is satisfied in both cases. Finally, adding (6.30) and (6.34) and assuming

$$\Phi'_{em}+\Phi^{M'}=\Phi', \quad (6.39)$$

$$u'+\rho^{M'}=\rho', \quad (6.40)$$

we verify that the equation

$$\Phi'-c^{*}\rho=0$$

is valid if Φ′ and ρ′ are taken to mean the corresponding densities pertaining to the total energy.

The quotient Φ_{em}/u obtained by dividing the electromagnetic energy flux by its density in a moving medium yields the propagation velocity only if there are no ponderomotive forces of the light wave. This in turn, takes place only in the case of the Minkowski tensor and is not applicable to any other form of the m—e tensor of the field in general.

Therefore, as is indeed noted in the literature, the treatment of the problem in Moller's book[3] is incorrect. In this splendid book the author writes: "This condition (he has in mind $c^* = \Phi_{em}/u$ – D.S.) is satisfied by the Minkowski tensor and not by the Abraham tensor, and this is the strongest argument in favor of Minkowski's theory" ([3], page 207).

In the next chapter, on the basis of the proof given by Moller himself, we shall be able to generalize the derivation just considered.

7. THE MOLLER CRITERION

In the cited book[3], Moller presented a general formal criterion that must be satisfied by any m—e tensor T_{lm} in order that the velocity, defined as

$$u_m^{*}=-(c/i)\,T_{4m}/T_{44}, \quad (7.1)$$

will transform like the velocity of a material point.

The reasoning developed in detail in the preceding chapter, suggests that the vanishing of all four divergences T_{lm} can serve as such a criterion.

Indeed, it can be shown for a plane wave in the general case that the condition

$$\mathrm{Div}\,T_{lm}=0 \quad (7.2)$$

and the aforementioned Moller condition are equivalent. We bear in mind here that the tensor T_{lm} is symmetrical. Moller's condition is written in the following form ([3], page 165):

$$R_{ik}=T_{ik}+(T_{im}U_m^{*}U_k^{*}/c^2)=0; \quad (7.3)$$

here the components U_m^* are defined as

$$U_m^{*}=\frac{u_m^{*}}{[1-(u^{*2}/c^2)]^{1/2}}\ (m=1,2,3)\ \text{and}\ U_4^{*}=\frac{ic}{[1-(u^{*2}/c^2)]^{1/2}},$$

where the u_m^* are defined by formula (7.1) and

$$u^{*2}=u_1^{*2}+u_2^{*2}+u_3^{*2}.$$

It is assumed that the velocity u*, defined in accordance with (7.1), is less than the velocity of light.

According to Moller, (7.3) is the condition necessary and sufficient for the components U* to define a four-vector. At the same time, this is the condition necessary and sufficient for the velocity u* to transform like the velocity of a material point. Moller has shown that if the equation (7.3) is satisfied in some definite admissible reference frame, then it is also satisfied in any other inertial system. This means that if it is satisfied in one definite reference frame, then U* is a four-vector and R_{ik} is a tensor.

Therefore, without affecting the general character of the derivation, we can compare equations (7.2) and (7.3), assuming that the x axis is parallel to the light beam and that, consequently, the tensor T_{il} is represented by a 2×2 pattern. In this reference frame, the sum in expression (7.3) reduces to three terms and the condition (7.3) reduces to two equations (at i = 1 and i = 4). Let us write out these equations:

$$\begin{aligned} T_{11}-\frac{T_{11}T_{41}^2}{T_{44}^2+T_{41}^2}-\frac{T_{14}^2T_{44}}{T_{44}^2+T_{41}^2}&=-T_{44}\,\frac{T_{14}^2-T_{11}T_{44}}{T_{44}^2+T_{41}^2}=0.\\ T_{41}-\frac{T_{11}T_{41}T_{44}}{T_{44}^2+T_{41}^2}-\frac{T_{14}T_{44}^2}{T_{44}^2+T_{41}^2}&=T_{41}\,\frac{T_{14}^2-T_{11}T_{44}}{T_{44}^2+T_{41}^2}=0.\end{aligned} \tag{7.4}$$

At the same time, the two equations obtained from the condition

$$\partial T_{1m}/\partial x_m = \partial T_{4m}/\partial x_m = 0, \tag{7.5}$$

for a symmetrical tensor yield

$$\begin{aligned} \frac{\partial T_{11}}{\partial x'}+\frac{\partial T_{14}}{ic\,\partial t'}&=\frac{\partial}{\partial x'}\left(T_{11}-\frac{c^*}{ic}T_{14}\right)=0,\\ \frac{\partial T_{41}}{\partial x'}+\frac{\partial T_{44}}{ic\,\partial t'}&=\frac{\partial}{\partial x'}\left(T_{41}-\frac{c^*}{ic}T_{44}\right)=0,\end{aligned} \tag{7.6}$$

where c* is the same as in (6.20), and where account is taken of (6.19). In the considered case of the special Lorentz transformation, Eqs. (7.6) show directly that the velocity u* coincides with the phase velocity c* and *is transformed like the velocity of a material point.*

Both systems (7.4) and (7.6) lead in the same fashion to the relations

$$T_{11}T_{44}=T_{41}^2=T_{14}^2, \qquad T_{11}/T_{14}=T_{14}/T_{44}, \tag{7.7}$$

and consequently, in the given reference frame, Moller's conditions (7.3) and condition (7.2) are equivalent. But if (7.3) is satisfied in one defined system, then it is valid also in any other system (x, t). In the same manner, if all four divergences of the tensor are equal to zero in one system, then they are also equal to zero in any other reference system. Consequently, the conditions (7.2) *and (7.3) are equivalent in any coordinate system.*

We have already assumed that the tensor T is symmetrical. By definition, and taking (7.7) into consideration, we have

$$U_1^*=\frac{-(c/i)\,T_{41}/T_{44}}{[1-(1/i)^2\,(T_{41}/T_{44})^2]^{1/2}}=\frac{icT_{41}}{(T_{44}^2+T_{41}^2)^{1/2}}=\frac{icT_{41}}{T_{44}^{1/2}\,(T_{44}+T_{41})^{1/2}}, \tag{7.8}$$

$$U_4^*=icT_{44}/(T_{44}^2+T_{41}^2)^{1/2}=icT_{44}^{1/2}/(T_{44}+T_{11})^{1/2}. \tag{7.9}$$

As shown by (7.8) and (7.9), $T_{44} + T_{11} > 0$. We put

$$T_{44} + T_{11} = \Sigma. \tag{7.10}$$

The quantity Σ in (7.10) is the sum of the diagonal terms (the trace of the tensor T, which is invariant). The fact that the invariant Σ of the tensor T is larger than zero is a consequence of the two assumptions made by us: 1) the tensor satisfies the Moller criterion, 2) the tensor is symmetrical. We note here that the vanishing of all of the divergences is a characteristic feature of the Minkowski tensor. At the same time, Σ vanishes both in the case of the Minkowski tensor and in the case of the Abraham tensor.

It follows from the foregoing derivation that if the sum of the diagonal terms is equal to zero, then we are faced with two alternatives: either the tensor satisfies the Moller criterion and is inevitably asymmetrical (the Minkowski case), or else, if the tensor is symmetrical and the sum of its diagonal terms is equal to zero, then it cannot satisfy the Moller criterion (the Abraham case).

For the given particular case where the axes are arranged in accordance with (7.8) and (7.9), we have

$$U_m^*=icT_{4m}/T_{44}^{1/2}\Sigma^{1/2}. \tag{7.11}$$

Since we know that under the assumptions indicated above the components U_m^* define a four-vector, relation (7.11) should also hold for any orientation of the coordinate axes. From this we can see that the tensor T can be represented as the following product of two four-vectors:

$$T_{lm}=(-\Sigma/c^2)\,U_l^*U_m^*. \tag{7.12}$$

To verify that the relation (7.12) is satisfied at any orientation of the coordinate axes, it suffices to verify, by substituting (7.8) and (7.9) in (7.12), that it is valid in the particular case considered above.

By substituting (7.12) in (7.3), we can check directly that a tensor such as (7.12) satisfies Eq. (7.3). To this end it suffices to note that $U_l^*U_l^* = -c^2$. Relation (7.12) is easy to verify also with the tensors (5.5) and (5.30) as an example.

It is easy to verify that the tensor (7.12) has the following structure:

$$T_{lm} = T_{l4}T_{4m}/T_{44}.$$

Noting that in accordance with (7.1) we have

$$T_{4m}/T_{44}=-(i/c)\,u_m^*, \quad \text{and} \quad T_{l4}=icg_l,$$

where g_l is the momentum density of the component along the axis labeled *l* and u_m^* is the velocity component along the axis with index m, and that thus $T_{lm} = g_l u_m^*$, we see that T_{lm} is the flux density, in the direction of the m axis, of the *l*-component of the field momentum.

The tensor T_{lm} is the field tensor of a current that has no "sources" or "sinks". Only in the case of a tensor with such a structure is the velocity of the energy flux equal to the quotient of the energy flux density divided by the energy density itself.

Equations (7.5) are the flux continuity equations of the components of the momentum and of the energy flux of such a field. These equations should hold true if T_{lm} is the tensor of the total energy of a closed system. It is in general incorrect to require that the m—e tensor satisfy the Moller criterion.

8. THE ANGULAR MOMENTUM OF A STATIC FIELD

If we are dealing with a superposition of electric and magnetic static fields the vectors of which are mutually perpendicular, then such a field can carry angular momentum. Apparently, Poincare was the first to call attention to this[23].

If a dielectric medium is placed in the field, then the values of the total angular momentum of the field as a whole, which can be obtained by calculating them in one case after Abraham, and in another after Minkowski, are different. The law of angular momentum conservation therefore makes it possible to choose also in the present situation between the two expressions (2.10) and (2.11).

We imagine a cylindrical charged capacitor (of sufficient length) situated inside a solenoid that produces a longitudinal (relative to the capacitor axis) homogeneous magnetic field that closes on itself at some large distance away from the capacitor. We assume first that the space inside the capacitor and inside the solenoid is a vacuum. According to (2.10) and also (2.11) the electromagnetic angular momentum relative to the capacitor axis in a layer r + dr − r (per unit length of this axis) is equal to

$$dI_{em} = ([\mathbf{EH}]/4\pi c)\cdot 2\pi r^2\,dr. \quad (8.1)$$

If the capacitor charge is Q, then E = 2Q/r and the total angular momentum per unit length is

$$I_{em} = \int_0^R (HQr/c)\,dr = HQR^2/2c, \quad (8.2)$$

where R is radius of the outer electrode of the capacitor.

When the current that produces the field H is turned off, a force is exerted on the charge Q (by the induced solenoidal field), and its integrated angular momentum is

$$I_{mech} = Q\Delta\Phi/2\pi c; \quad (8.3)$$

$\Delta\Phi$ is the change of the magnetic-induction flux:

$$\Delta\Phi = \Phi = H\pi R^2. \quad (8.4)$$

Substitution of (8.4) in (8.3) yields

$$I_{mech} = I_{em} = HQR^2/2c. \quad (8.5)$$

If the outer electrode of the capacitor can rotate freely and the mechanical system is autonomous, then turning off the current causes the cylindrical electrode of the capacitor to rotate[24)] with a mechanical torque equal to the vanished electromagnetic angular momentum of the field.

We assume now that a cylindrical layer of dielectric (say, solid) is placed inside the capacitor and fills almost the entire volume of the capacitor (the gap between the surface of the dielectric and the outer electrode is negligibly small).

The outer electrode of the capacitor and the dielectric cylinder can rotate about a common axis freely and independently.

When speaking of the mechanical torque of the solenoidal electromotive force, we must now bear in mind the torque produced both by the true charges Q of the capacitor, and by the free charges q_d of the dielectric. Per unit length (along the capacitor axis) we have

$$q_d = -2\pi PR; \quad (8.6)$$

here P is the polarization of the dielectric,

$$P = (D - E)/4\pi, \quad D = \varepsilon E, \quad (8.7)$$

where ϵ is the dielectric constant. According to (8.6) and (8.7)

$$q_d = -(DR/2) + (ER/2) = Q\,(1 - \varepsilon)/\varepsilon, \quad (8.8)$$

since D = 2Q/R. As follows from the foregoing, when the current of the solenoid is opened, the electrodes of the capacitor and the dielectric are acted upon by torques I_{cap} and I_d, in opposite directions:

$$I_{cap} = Q\Delta\Phi/2\pi c, \quad (8.9)$$

$$I_d = q_d\Delta\Phi/2\pi c, \quad (8.10)$$

or, taking (8.8) into account,

$$I_d = Q\,[(1 - \varepsilon)/\varepsilon]\,\Delta\Phi/2\pi c. \quad (8.11)$$

Adding (8.9) and (8.10) and substituting $\Delta\Phi = H\pi R^2$, we obtain

$$I_{cap} + I_{д} = (Q/\varepsilon)\,HR^2/2c. \quad (8.12)$$

Specifying some expression for the momentum density of the electromagnetic field, we can determine the total torque I_{em} of the entire volume of the field.

Conservation of the angular momentum calls for satisfaction of the equation

$$I_{cap} + I_{д} = I_{em}. \quad (8.13)$$

If we assume for the momentum density Abraham's expression

$$g^A = EH/4\pi c, \quad (8.14)$$

where

$$E = 2Q/\varepsilon r, \quad (8.15)$$

then we obtain for I_{em}

$$I_{em} = \int_0^R (2QH/\varepsilon r\cdot 4\pi c)\cdot 2\pi r^2\,dr = QHR^2/2c\varepsilon, \quad (8.16)$$

and, as seen from comparison with (8.12), the balance of the angular momentum is satisfied. At the same time, we verify that Minkowski's hypothesis $g^M = \epsilon EH/4\pi c$ is not satisfied.

We thus arrive again at the same conclusion, that Minkowski's tensor contradicts the conservation laws.

9. COMPARISON OF THE ABRAHAM AND MINKOWSKI TENSORS AND CONCLUDING REMARKS

Likewise on the basis of the conservation laws, but with an erroneous formulation of the problem, Costa de Beauregard[16a] reached the opposite conclusion. Quite recently, however, [16b] (page 164) examining the pressure from the point of view of the laws of motion of the center of gravity, he himself reached the conclusion that Abraham's expression is correct. The considerations on which he bases his note[16b] agree with those developed in that part of the present article (in its beginning) which had already been written before the author became acquainted with Beauregard's note[25)]. But the fact that the Minkowski tensor is incompatible with the law of constancy of the velocity of the center of gravity was noted long ago.

De Beauregard's note was followed by a number of others[16c-d,17]. We shall return to the conclusions of one of them.

After deriving Abraham's expression for the "photon" momentum, the author of[16] advances the hypothesis that this quantity has a dual value, "macroscopic" according to Abraham and "quantum" according to Minkowski.

The literature of this question is characterized by the tendency to accept (in spite of the facts) the correctness of Minkowski's tensor as "canonically" established. The authors of the review and original articles tend to ignore arguments that appear to lead unambiguously to the conclusion that Minkowski's postulate is not acceptable. This tendency prevails, for example, in a review by Brevik[4] recently published in a respectable scientific journal. The author's main thesis and the tenor of his lengthy paper is the statement that "if properly interpreted" the tensors of Abraham and Minkowski are "adequate and equivalent" in most considered simple physical situations ([4b], page 5). The correct interpretation is formulated as follows: "The Abraham force excites dipoles contained in the material and produces a mechanical momentum that is transported together with the field. If this mechanical momentum, together with the Abraham momentum, is regarded as the momentum of the field, then we obtain Minkowski's tensor" ([4b], page 7). And on page 7 of [4a] we read: "The components of the stresses and of the momentum, defined above, lead to a force that can excite a small mechanical momentum of the component particles (dipoles)." "... Comparing with the experiments of Jones and Richards, we find that the suggestion is indeed confirmed."

If we turn to the models considered above, we can verify immediately that such an "interpretation" cannot hold water. Our model of dustlike matter, to be sure, corresponds in part to a situation described by the just-cited quotation (although the corresponding total tensor does not coincide at all with the Minkowski tensor). In a solid dielectric, however, the dipoles are secured and there can be no thought of a resultant momentum (apart from the translational momentum of the medium as a whole).

On page 26 of the review [4b], the author explains the results of a paper by Balazs [10]. Citing two equations—the equation of conservation of the momentum and the equation of the conservation of the velocity of the center of gravity (which, apart from notation, coincides with (1.10)) the author of the review makes the bare statement that "he cannot agree with his (Balazs') conclusion that the Abraham expression is correct," since the first of the just-indicated equations (which coincides with our equation (1.9)) is seemingly incorrect, being "incomplete." Brevik does not explain just how this equation should be completed, leaving the reader to guess at it.

Yet we are dealing with an equation that cannot be written in different manners, depending on the various hypotheses, and the question is essentially not debatable, provided we do not dispense with such basic premises of mechanics as, for example, the fact that the total momentum of a system of particles is equal to the sum of the mass of these particles multiplied by the velocity of their center of gravity.

In his note [16c] (page 1119), Costa de Beauregard approaches this problem differently: how to reconcile Abraham's expression for the momentum ($h\nu/nc$) with Minkowski's "canonical, quantum" expression ($nh\nu/c$). Insofar as can be understood, he bases himself on the fact that when a photon is emitted the source of the light (which is located in the medium) imparts to the medium an additional mechanical momentum equal to

$$[(n^2-1)/n]\, h\nu/c \tag{9.1}$$

(that this is indeed the case will be made clear later on). Adding the two momenta $h\nu/nc$ (of the photon) and $[(n^2-1)/n]h\nu/c$ (of the medium), the author obtains

$$\{[(n^2-1)/n]+(1/n)\}\, h\nu/c = nh\nu/c \tag{9.2}$$

which is Minkowski's expression. It is clear, however, that the two terms of (9.2) cannot be interpreted as components of the photon momentum. The second term corresponds to the momentum density u/nc, which has a perfectly defined meaning of the momentum density of the electromagnetic field (photon). On the other hand, it is quite meaningless to speak of a momentum density corresponding to the first term of (9.2). If, for example, the medium is a solid and firmly secured body, then the momentum (9.1) is transmitted to the earth and the density of this momentum is equal to zero.

At zero momentum density, the momentum flux is not equal to zero. The density of this flux is equal to the pressure p of the light on the dielectric:

$$p = [(n^2-1)/n^2]\, u, \tag{9.3}$$

where u is the energy density of the field.

The momentum flux density (but not the density of the momentum itself) has in this case a definite physical meaning.

During the time that the light is emitted, the medium acquires a momentum equal to

$$[(n^2-1)/n^2]\, u\tau = [(n^2-1)/n^2]\, u\mathscr{E}n/uc = [(n^2-1)/n]\,\mathscr{E}/c, \tag{9.4}$$

i.e., a momentum equal to (9.1) ($\tau = \mathscr{E}n/uc$ is the time of radiation and u is the radiation energy density).

We have dwelt in detail on these relations, since they lead to a clear-cut answer to the question of the similarity and of the difference between the Minkowski and Abraham tensors.

At the end of Chap. 2 we mentioned Planck's suggestion that the Maxwellian pressure be interpreted as the density of the total flux of the field momentum. In this interpretation (after Planck), the two expressions for the momentum flux, Minkowski's and Abraham's (in a static reference frame) come to coincide.

According to Minkowski, there is no pressure force on the medium and the density of the momentum flux for the case of a two-dimensional (two-by-two) tensor scheme is equal to

$$\varphi^M = gc/n = (nu/c)\, c/n = u = S_{11}^M. \tag{9.5}$$

According to Abraham, we have

$$\varphi^A = g\,(c/n) + p = (u/nc)\,(c/n) + [(n^2-1)/n^2]\, u = u = S_{11}^A, \tag{9.6}$$

where p is the pressure of the light on the medium:

$$S_{11}^A = S_{11}^M. \tag{9.7}$$

Consequently, the total momentum flux (in the sense indicated above) is the same according to Abraham as according to Minkowski. However, whereas φ^M is the product of the density (of the momentum) by the velocity, according to Abraham the total flux of the momentum consists of two components, one of which, the "current component," is equal to gc/n, while the other is equal to the pressure of the light $p = [(n^2-1)/n^2]u$.[26)]

Let us consider, from the point of view of two components, the mechanism of transport of the total momentum (of the matter and of the field) by a light wave propagating in dustlike matter. The "current" component of the transport of the mechanical momentum of the medium is negligibly small in the zeroth system; the tensor component is $K_{11}^0 \approx \beta_0^2\mu_0 c^2$ (see (5.26)). The sum of the "current" components (the momentum flux of the field and the medium) is therefore equal in this case to the "current" component of the electromagnetic momentum:

$$(u/cn)\, c/n = u/n^2. \tag{9.8}$$

In the wave field, however, the medium experiences a light pressure equal to $[(n^2-1)/n^2]$. The resultant[27)] F of the light-pressure forces near the front of the wave is a source of mechanical momentum (the momentum connected with the "drift" of the medium).

In a space bounded by a certain cross section plane (say the plane S_2; Fig. 6), the total momentum trans-

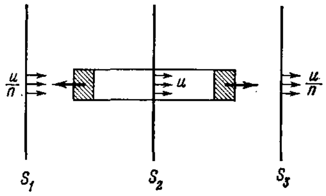

FIG. 6

ferred in a unit time S in the sinusoidal zone of the wave is

$$(u/n^2) + [(n^2 - 1)/n^2]\, u = u = (nu/c)\, c/n, \qquad (9.9)$$

where $[(n^2 - 1)/n^2]u$ is the increment per unit time of the mechanical momentum in the indicated space. The increment of the mechanical momentum in this space is equal to the light pressure acting on the medium on the boundary surface S_2 (see Fig. 6).

On the trailing front of the wave, the pressure directed in the opposite direction produces a resultant force $(-F)$ which serves as a "sink," absorbing the same amount of mechanical momentum. The "drift" motion of the matter is quenched here.

The total momentum is transferred together with the light wave and consequently with velocity c/n, and can be expressed as shown in the right-hand side of (9.9).

We consider in addition, and on the basis of the same representations, the balance of the angular momentum in a different situation—in the case of passage of a stationary light flux, without losses, through a refractive solid medium.

We consider the spaces S_2S_3 (bounded by the planes S_2 and S_3) and S_1S_2 (the boundaries of which are the planes S_1 and S_2) (see Fig. 6). The radiation density in the medium is equal to u. The direction from S_2 to S_3 is assumed positive. The flux through the plane S_2 consists of the pressure on the medium, equal to $[(n^2 - 1)/n^2]u$, and the flux ("current") of the radiation momentum (u/n^2).

The total flux entering into the space S_2S_3 through the plane S_2 is

$$u\,\{[(n^2 - 1)/n^2] + (1/n^2)\} = u. \qquad (9.10)$$

In the "sink" F_2 ($F_2 = [(n - 1)/n]u$), the amount of momentum lost per unit time is $[(n - 1)/n]u$, and u/n goes off through the external boundary of the space, namely the plane S_3.

The increment of the momentum in the space between the planes S_2 and S_3 is consequently

$$u\,\{1 - [(n - 1)/n] - (1/n)\} = 0. \qquad (9.11)$$

The momentum balance is thus complete.

We write down the corresponding equation for S_1S_2, assuming now the direction from S_2 to S_1 to be positive. Then the increment (per unit time) of the momentum in the space S_1S_2, in accordance with the same considerations, is expressed by a similar sum of value zero:

$$u\,\{[(n^2 - 1)/n^2] + (1/n^2) - [(n - 1)/n] - (1/n)\} = 0.$$

When writing down the balance equation, we have taken into account only the electromagnetic component of the momentum flux. In a solid medium, however, there is a counterflow of elastic momentum; in a medium in the form of "dustlike" matter, there is no such counterflow.

In the boundary conditions considered above, it was assumed that the radiation is bounded by a region of space situated entirely inside the solid medium. Under these conditions, the "counterflow" of the elastic momentum compensates for the component of the electromagnetic momentum transferred as a result of the light pressure. In other words, the light pressure on the medium is balanced by the tension of the medium.

In the situation just considered, the component of the electromagnetic momentum flux transferred by the light pressure to the medium $[(n^2 - 1)/n^2]u$ exceeds the opposing flux of the elastic momentum $[(n - 1)/n]u$. The excess

$$\{[(n^2 - 1)/n^2] - [(n - 1)/n]\}\, u = [(n - 1)/n^2]\, u \qquad (9.12)$$

together with the "current" component of the electromagnetic momentum u/n^2 yields as a sum the momentum flux density u/n transferred through the plane S_3 to the outside.

It is clear that if we assume Minkowski's expression for the momentum-energy densities of the field, then the momentum balance conditions in the stationary flux will also be satisfied, since the total momentum flux is determined in the same manner by both tensors.

If it were possible to formulate the theory of the Cerenkov effect in terms of the momentum flux, then it would likewise be immaterial for the theory, in this variation, which of the two tensors, Minkowski's or Abraham's, is taken as the basis.

We turn in this connection again to the tendency in Brevik's review[4] to play down the differences between the two compared tensors. In the cited review, as in other papers, it is usually remarked that the resolution of the total tensor of the (field + medium) system into two components is conditional and arbitrary. Of course, it is possible to break up a tensor arbitrarily into two components—subsystem tensors. Within the limits of the scheme considered here (of idealized media), however, the resolution into two components (the m—e tensor of the field and the tensor of the medium) is unique if one adheres to the requirement (or definition) according to which the field tensor S_{lm} should satisfy the condition

$$-\partial S_{lm}/\partial x_m = f_l. \qquad (9.13)$$

(The existence of such a tensor is postulated as a premise). This, in any case, is the situation in the case of the field of the plane wave. This is particularly clearly seen if the tensor scheme is reduced, by choosing the reference frame, to a two-by-two matrix, which, of course, involves no loss of generality.

Indeed, we have verified that within the limits of the indicated approximate scheme the expression for the momentum density (and consequently also the density of the ponderomotive forces) follows from the conservation laws. But if this expression is specified, then the component S_{14} of the tensor S_{lm} is determined by the same token. In addition, there can hardly be any disagreement in the choice of the expression for the term S_{44}, which is equal to the electromagnetic energy density taken with a minus sign. (We are dealing here with the values of the components of the tensor in the reference frame in which the medium is at standstill. The expressions for S_{44}^A and S_{44}^M, those of Abraham and Minkowski, coincide.) Further, S_{41} is obtained from the condition

$$\partial S_{4m}/\partial x_m = 0. \qquad (9.14)$$

Relation (9.14) is a consequence of the fact that the medium is immobile: namely, the work of the field forces, determined by the left side of (9.14), is equal to zero. Equation (9.14) is in this case an expression for the continuity of the energy flux. It follows therefore that S_{41} is the Poynting vector (with the factor i/c). Since $S_{l4} = icg_l = (i/4\pi)[\mathbf{E}\times\mathbf{H}_l]$, we get $S_{41} = S_{14}$, so that the tensor S_{lm} is symmetrical.

Finally, S_{11} is determined (for a plane wave, directly) from the condition (9.13), which yields for the given scheme

$$-\partial S_{1m}/\partial x_m = f\ (m = 1;\ 4).$$

(We recall that the density of the force "f" is given if the expression is given for the momentum density.)

S_{11} can be determined also from another condition: the sum of the diagonal terms (the trace) is equal to zero.

The assumption $\Sigma = 0$ is natural by analogy with the m—e tensor of the field produced by the charges in vacuum, and by analogy with the material particle-flux tensor. In the latter case

$$\Sigma = -\mu_0 c^2,$$

where μ_0 is the density of the rest mass of the particles (equal to zero for a photon field)[28].

Inasmuch as the chain of reasoning followed by us now, which leads to the construction of the tensor S, yields as a result a symmetrical (Abraham) tensor, and since the m—e tensor of the total system (field + medium) should be symmetrical, it follows that the mechanical tensor U_{lm} of the material component of the system (medium) is symmetrical, which is natural.

On the other hand, the asymmetry of the Minkowski tensor implies asymmetry of the mechanical tensor U_{lm} of a material medium.

Summarizing the exposition in this article, we can formulate a few premises, primarily of negative character.

a) The rejection by Laue, Moller, et al. of the Abraham tensor as not satisfying the Laue criterion cannot be regarded as convincing. The proofs of these authors fail if account is taken of the condition for the balance of the two components of the energy flux, electromagnetic and mechanical.

b) Minkowski's tensor is unacceptable as contradicting the principal conservation laws, as well as the fundamental concepts of the interaction of the electromagnetic field with matter (with the exception of the Lorentz forces that act on the polarization currents).

c) Generalizing the conclusions based on the simple model considered above, we can apparently state that the Abraham tensor is the adequate form of the field tensor and satisfies the requirements listed above. As proposed by the Hungarian theoreticians[8], it is expedient to separate from the Abraham tensor a "current" component (S^{cr})[29] satisfying the condition

$$\operatorname{Div} S^{cr} = 0. \tag{9.15}$$

The other component S^{ts} satisfies the requirement

$$-\partial S^{ts}_{lm}/\partial x_m = f_l. \tag{9.16}$$

The Abraham tensor, represented in the form of a sum of these two components (for the particular case of the coordinate axes chosen by us), takes the following form:

$$\underset{S'^{cr}}{\frac{u_0}{n^2}\gamma^2\begin{pmatrix}(n\beta-1)^2 & i(n-\beta)(1-\beta n)\\ i(n-\beta)(1-\beta n) & -(n-\beta)^2\end{pmatrix}} + \underset{S'^{ts}}{u_0\frac{n^2-1}{n^2}\gamma^2\begin{pmatrix}1 & -i\beta\\ -i\beta & \beta^2\end{pmatrix}} = S'^{A} \tag{9.17}$$

Only the first of these two components, S'^{cr}, should be taken into consideration when the light-propagation velocity is defined as the quotient of the energy flux density $[c(u/n^2)\gamma^2(n-\beta)(1-n)]$ by the electromagnetic energy density $[(u/n^2)\gamma^2(n-\beta)^2]$.

The tensor T of the total energy of the system "field + medium", in the considered limiting cases, is represented by the sum $T = S^A + U_{mech}$, if the medium is an ideal dielectric.

In the case considered by Marx and Gyorgyi[8] of an ideal solid (the first of our two models) we have

$$U_{mech} = -S^{ts}, \tag{9.18}$$

$$T^A = S^{cr}. \tag{9.19}$$

In the given particular case, consequently, the "current" component of the Abraham tensor coincides with the total tensor T^A of the entire "field + medium" system as a whole.

In other media, the sum $S^{ts} + U_{mech} \neq 0$ and $T \neq S^{cr}$. The tensor T, generally speaking, depends on the properties of the medium. In our first model, the field forces are balanced by the elastic forces, and in the case of the second model they are balanced by the inertia forces of the medium[30].

d) If we adhere to the phenomenological approach and to the approximation of ideal media, then the problem can be reduced to two questions: 1) the existence of the field m—e tensor, and 2) the ponderomotive forces in the field of the wave propagating in an immobile medium.

If we postulate the existence of the aforementioned tensor and assume expression (2.12), which defines the ponderomotive forces, or expression (2.11) for the momentum density, then there is no room for discussion, since the tensor for a medium at rest is by the same token defined, and its relativistic generalization follows from the general rules for the transformation of the components of four-dimensional tensors.

Incidentally, from the historical point of view it is of interest to note that Abraham[31] constructed his tensor in the general case (for a moving medium) without using relativity theory in explicit form. He followed the way of extrapolation from the relations for a medium at standstill, on the basis of rather arbitrary assumptions and extremely scanty experimental data. The derivation of this tensor is given in his book[7b] (page 360).

If we consider the question of the approximation of ideal media (perhaps in a form more general than ours), then it would be possible to define uniquely a number of general premises—a certain "alphabet," which could and has to be used (without any disagreements) in the treatment of more complicated questions connected with this problem. Unfortunately, historically the situation has developed paradoxically in such a way that so far no order has been introduced into this very "alphabet." The necessary basic premises that could be used have not been established and continue to remain a subject of discussion.

The present article is to a considerable degree a result of discussions between the author and V. L. Ginzburg, who also supplied a bibliography of the question. In addition, Ginzburg undertook to read the manuscript. The same was graciously done, at the author's request, by L. V. Keldysh. The author takes the opportunity to thank them for their help and many useful remarks.

APPENDIX I

The expression for the Abraham forces in general form ($\mu \neq 1$) is obtained if, in addition to the forces considered in the main text, one takes into account also the following terms. First, $-(1/2)\partial(MH)/\partial x$ (where M is the magnetic polarization and H is the magnetic field intensity), and second, the forces f_E of Einstein and Laub

$$f_E = (1/c)\, E\, \partial M/\partial t = [(\mu-1)/4\pi c]\, E\, \partial H/\partial t. \qquad (A1.1)$$

The forces (A1.1) were introduced in [18] from symmetry (or "duality") considerations as the analog of (2.14).

Taking into consideration (2.13) and the relation

$$H = (n/\mu)\, E, \qquad (A1.2)$$

which follows for a plane wave from Maxwell's equations, we rewrite the expression $-(1/2)\partial(PE)/\partial x$ in the form

$$-\frac{1}{2}\frac{n}{c}\frac{\partial}{\partial t}(PE) = -\frac{1}{2}\frac{n}{c}\frac{\varepsilon-1}{4\pi}\frac{\partial E^2}{\partial t} = \frac{n}{c}\frac{\varepsilon-1}{4\pi}E\frac{\partial E}{\partial t} \qquad (A1.3)$$

and then, according to (A1.2) and (A1.3),

$$-\frac{1}{2}\frac{\partial}{\partial x}(PE) = \frac{n}{c}\frac{\varepsilon-1}{4\pi}\frac{\mu}{n}E\frac{\partial H}{\partial t} = \frac{\varepsilon\mu-\mu}{4\pi c}E\frac{\partial H}{\partial t}. \qquad (A1.4)$$

By analogy, using the same procedure, we obtain

$$-(1/2)\,\partial(MH)/\partial x = [(\varepsilon\mu-\varepsilon)/4\pi c]\, H\, \partial E/\partial t. \qquad (A1.5)$$

Finally, assuming the Lorentz forces

$$f_L = (1/c)\,(\partial P/\partial t)\, H = [(\varepsilon-1)/4\pi c]\,(\partial E/\partial t)\, H \qquad (A1.6)$$

and taking (A1.1) into account, we have

$$f_L + f_E = [(\varepsilon-1)/4\pi c]\,(\partial E/\partial t)\, H + [(\mu-1)/4\pi c]\, E\, \partial H/\partial t. \qquad (A1.7)$$

Adding (A1.4), (A1.5), and (A1.7), we obtain

$$f_A = \frac{\varepsilon\mu-\mu+\mu-1}{4\pi c}E\,\partial H/\partial t + \frac{\varepsilon\mu-\varepsilon+\varepsilon-1}{4\pi c}H\frac{\partial E}{\partial t} = \frac{\varepsilon\mu-1}{4\pi c}\frac{\partial(EH)}{\partial t}.$$

The arguments that show that the Lorentz force should be set equal to $(1/c)(\partial P/\partial t)H$ in the given situation (and not $(1/c)(\partial P/\partial t)B$) are developed in [8,18].

APPENDIX 2

Relation (2.3) yields[32)]

$$\mathbf{f} = -(1/8\pi)\, E^2\, \mathrm{grad}\, \varepsilon \qquad (A2.1)$$

under the assumption $\mu = 1$. In the general case ($\mu \neq 1$) the expression for the force density includes also an analogous magnetostatic term ($-H^2\,\mathrm{grad}\,\mu$), and in place of (A2.1) it is necessary in this case to put (omitting henceforth the factor $1/8\pi$)

$$\mathbf{f} = -E^2\,\mathrm{grad}\,\varepsilon - H^2\,\mathrm{grad}\,\mu. \qquad (A2.2)$$

According to Maxwell's equations for a plane wave, we have here the relation

$$H = (n/\mu)\, E. \qquad (A2.3)$$

The continuity condition for the energy flux yields

$$E\cdot H = (n/\mu)\, E^2 = E_0 H_0 = E_0^2, \qquad (A2.4)$$

where E_0 and H_0 are the values of the field variables outside the medium (in vacuum), and E and H are the variables in the medium.

Further, according to (A2.4)

$$\left.\begin{aligned} &E = (\mu/n)^{1/2}E_0, \text{where } n = (\varepsilon\mu)^{1/2},\\ &E^2 = (\mu/\varepsilon)^{1/2}\,E_0^2,\\ &H^2 = (\varepsilon/\mu)^{1/2}\,H_0^2. \end{aligned}\right\} \qquad (A2.5)$$

The force F_{tr} acting on the unit surface of the boundary transition layer is calculated, according to (A2.2), from the formula

$$F_{tr} = -\int\left[\frac{\partial\varepsilon}{\partial x}E^2 + \frac{\partial\mu}{\partial x}H^2\right]dx = -\int\left[\frac{\partial\varepsilon}{\partial x}\left(\frac{\mu}{\varepsilon}\right)^{1/2} + \frac{\partial\mu}{\partial x}\left(\frac{\varepsilon}{\mu}\right)^{1/2}\right]E_0^2\,dx =$$
$$= -\int(\varepsilon\mu)^{1/2}\partial\,\lg\varepsilon\mu/\partial x E_0^2\,dx = -\left|2(\varepsilon\mu)^{1/2}\right|_1^{\varepsilon\mu^{1/2}=n}E_0^2 = -2(n-1)\,E_0^2.$$

Reconstructing the omitted factor $1/8\pi$, we obtain

$$F_{tr} = -(n-1)\,E_0^2/4\pi = -[(n-1)/n]\,u,$$

Since $E_0^2/4\pi = u_0 = u/n$, where u_0 and u are the density of the electromagnetic energy outside the medium and in the medium, respectively.

APPENDIX 3

In accordance with the meaning of the tensor P_{lm} we have $P_{44} = -h$, where h is the energy density. Consequently, according to (5.4),

$$h' = \gamma^2 p^0 \beta^2 \qquad (A3.1)$$

is the relativistic term in the expression for the energy density, and depends on the velocity w; under the assumptions made by us, $h^0 = 0$ if $w = 0$.

Let us consider the component P'_{11} of the tensor P'_{lm}. According to (4.24), this component is equal to

$$P'_{11} = p'_{11} + g'_\mu w; \qquad (A3.2)$$

here p'_{11} is the component of the relative-stress tensor

$$w = -\beta c. \qquad (A3.3)$$

According to the meaning of the "absolute stress tensor" (P_{lm}), it follows from (5.4) that the momentum density g'_μ is equal to

$$g'_\mu = (1/ic)\, P'_{14} = -(\beta/c)\, p^0_{11}\gamma^2. \qquad (A3.4)$$

From this we obtain according to (A3.2) and (A3.3)

$$P'_{11} = p'_{11} + [\beta^2 p^0_{11}/(1-\beta^2)]. \qquad (A3.5)$$

On the other hand, according to (5.4) we have

$$P'_{11} = \gamma^2 p^0_{11}. \qquad (A3.6)$$

Equating (A3.5) and (A3.6), we obtain

$$p'_{11} = p^0_{11}\left(\frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2}\right) = p^0_{11}. \qquad (A3.7)$$

as shown by (A3.7), the "longitudinal" (directed along the x axis) component of the relative stresses—in this case the tension of the medium—remains unchanged under the special Lorentz transformation.

Further, according to (4.15), the energy flux is

$$\Phi' = h'w + wp'. \qquad (A3.8)$$

Taking (A3.1) into account for the quantity h', and also the fact that $p' = p_0$, we obtain

$$h'w = [p^0\beta^2/(1-\beta^2)]\,(-\beta c) = -p^0\beta^3 c/(1-\beta^2), \qquad (A3.9)$$

$$wp' = -\beta c p^0. \qquad (A3.10)$$

Substitution of (A3.9) and (A3.10) in (A3.8) yields

$$\Phi' = -[p_0\beta^3/(1-\beta^2)] - \beta c p^0 = -\beta c p^0\gamma^2, \qquad (A3.11)$$

$$(i/c)\,\Phi' = P'_{41} = -ip^0\beta\gamma^2, \qquad (A3.12)$$

in accordance with (5.4).

According to Planck's postulate we have

$$g'_{\mu} = \Phi'/c^2 = -\beta p^0 (1/c)\gamma^2$$

and

$$icg'_{\mu} = P'_{14} = -i\beta p^0\gamma^2 = P'_{41},$$

likewise in accordance with (5.4).

APPENDIX 4

The expressions for the components of the Abraham tensor as functions of the field variables, in the case of a medium moving in the direction of the x axis with velocity w = βc, can be represented in the form of the following symmetrical scheme (see[7b], formulas (199a), (201), and (201a), as well as[1], formula (35), page 666):

$$\begin{aligned}
S_{11} &= -\frac{1}{8\pi}\left\{2E'_xD'_x + 2H'_xB'_x - (\mathbf{E}'\mathbf{D}' + \mathbf{H}'B') + \frac{2\beta}{1-\beta^2}([\mathbf{D}'\mathbf{B}']_x - [\mathbf{E}'\mathbf{H}']_x)\right\},\\
S_{12} &= S_{21} = -(1/4\pi)(E'_xD'_y + H'_xB'_y),\\
S_{13} &= S_{31} = -(1/4\pi)(E'_xD'_z + H'_xB'_z),\\
S_{14} &= S_{41} = (1/4\pi)[1/(1-\beta^2)]([\mathbf{E}'\mathbf{H}']_x - \beta^2[\mathbf{D}'\mathbf{B}']_x),\\
S_{22} &= -(1/8\pi)\{2E'_yD'_y + 2H'_yB'_y - (\mathbf{E}'\mathbf{D}' + \mathbf{H}'\mathbf{B}')\},\\
S_{23} &= S_{32} = -(1/4\pi)(E'_yD'_z + H'_yB'_z) = -(1/4\pi)(E'_zD'_y + H'_zB'_y),\\
S_{24} &= S_{42} = (i/4\pi)[\mathbf{E}'\mathbf{H}']_y,\\
S_{33} &= -(1/8\pi)\{2E'_zD'_z + 2H'_zB'_z - (\mathbf{E}'\mathbf{D}' + \mathbf{H}'\mathbf{B}')\},\\
S_{34} &= S_{43} = (i/4\pi)[\mathbf{E}'\mathbf{H}']_z,\\
S_{44} &= \frac{1}{8\pi}\left\{-(\mathbf{E}'\mathbf{D}' + \mathbf{H}'\mathbf{B}') + \frac{2\beta}{1-\beta^2}([\mathbf{D}'\mathbf{B}']_x - [\mathbf{E}'\mathbf{H}']_x)\right\}.
\end{aligned} \tag{A4.1}$$

The Lorentz transformation formulas yield E' = γ(E − βB) (β is the velocity of the origin of the coordinate system (x', y', z') relative to x, y, z). Next (in the case of a plane plane-polarized wave in a nonmagnetic medium),

$$H' = nE', \quad B' = \frac{n-\beta}{1-\beta n}\frac{H'}{n} = \frac{n-\beta}{1-\beta n}E', \quad D' = n\frac{n-\beta}{1-\beta n}E',$$

where n is the refractive index. If this is taken into consideration, then it is easy to verify that (4.22) corresponds to the expression given above for the components S'_{lm}, where, however, β must be replaced by $-\beta$.

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*[DB] = D × B.

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1) We regard it as possible to adhere to this term (in place of the customary "energy-momentum tensor") since it was used by Pauli in his classical paper[1].

2) To these two names, which mark an epoch in the history of classical physics, we should also add in connection wi h this problem the name of Abraham, who introduced the fundamental concept of electromagnetic momentum into the science.

3) We shall use this term, which was coined in the classical papers but seems to be obsolescent. In the latest literature one usually reads of "volume forces".

4) Whose conclusions, however, are incorrect.

5) At first glance it might seem that the reference to the law of motion of the center of gravity and to the derivation given here is superfluous, since the relation G = μ(c/n) = $\mathscr{E}$ /cn (see (1.5) below) can be regarded as valid a priori. In this case it would also be necessary to exclude a priori the Minkowski tensor, which contradicts this relation, and the problem to which the present article is devoted would have to be regarded as inessential and the literature devoted to this question as the result of misunderstanding. However, in the simple example considered in Chap. 6 below, (a charged capacitor moving uniformly with velocity β), we encounter the following situation: the momentum G of the electromagnetic field is not equal to $\mu\beta$c (here the electromagnetic mass μ is equal to $\mathscr{E}/c^2$, where $\mathscr{E}$ is the field energy). We see from expression (6.2) for the m-e tensor of the field (W) that in this case we have (according to Minkowski as well as according to Abraham)

$$G = 2\beta\mathscr{E}/c\,(1+\beta^2) \neq \mu\beta c.$$

6) This tensor is designated in the same manner as the momentum flux density tensor.

7) Reference is made in this connection, in particular, to Quinke's experiments on the drawing of a liquid dielectric into the field of a capacitor.

8) Here T_{xx} is the (Maxwellian) tension.

9) A more accurate and yet simple calculation could be cited.

10) Assuming that the mechanical m-e tensor of the medium is symmetrical.

11) Both form the historical point of view and in connection with later discussions, let us explain that Planck [11] was the first to point out that relation (4.13) should hold for the flux of energy of any form. In particular, when light propagates in a moving medium an energy wp_{lt} (where w is the velocity of the medium) is transmitted through a unit cross section area in a unit time, in the form of the work of the light pressure (p_{lt}). Therefore the "convective" term (hw, where h is the energy density) in the expression (Φ) for the energy flux should be supplemented with the product wp:

$$\Phi = hw + wp. \tag{4.15}$$

In accordance with Planck's postulate, the expression for the momentum density (g) can consequently be written in the form

$$g = (hw/c^2) + (wp/c^2). \tag{4.16}$$

The components of the vector wp are defined as follows: $(wp)_m = w_l p_{lm}$.

12) We are referring to a transformation (without rotation of the axis) to a system of coordinates whose origin moves parallel to the x axis.

13) In the "zeroth" system, this density and with it the refractive index can be regarded as independent of the field intensity, since the particle mass is arbitrarily large.

14) We have in mind here a mechanical interaction— the constraints imposed by the elastic forces, or exchange of mechanical momentum (upon collision between the particles).

15) To this end we can assume that the front of the wave packet is described by the equations: $E = E_0 \sin\{\omega[t-(xn/c)]\} - E_0(\omega/\omega_0)\sin\{\omega_0[t-(xn/c)]\}\, e^{-\alpha[t-(xn/c)]}$, with H = ne at x < ct/n and E = H = 0 at x > ct/n, where ω is the frequency of the light, ω_0 is the natural frequency of the molecular dipoles, $a \gg 1$, and $\omega_0/\omega \gg 1$. The variables E and H satisfy Maxwell's equations. E and H and their first derivatives are continuous at x = ct/n.

16) We cite in this connection a paper by Tang and Meixner[14] who considered in a certain approximation the question of the m-e tensor of light in a real liquid with the viscosity of this medium taken into account. According to the results obtained by them, the oscillatory motion of the medium must be taken into account when propagation of light in a real liquid is considered. A liquid that is perfectly transparent optically turns out to absorb light to a certain (albeit extremely weak) degree, owing to internal friction. According to the calculations in [14], under the assumptions made by them (for a definite example), a light beam is attenuated because of this effect to half its intensity over a length on the order of several thousand kilometers.

17) We write down the mass-density continuity equation:

$$(\partial/\partial t)(\mu_0 + \Delta\mu_0) + (\partial/\partial x)\, c\,[(\mu_0 + \Delta\mu_0(t))\,\beta_0(t)] = 0. \tag{5.17}$$

We recognize that

$$\partial/\partial x = -(n/c)\,\partial/\partial t. \tag{5.18}$$

According to (5.15) and (5.17) we obtain

$$(\partial/\partial t)[\mu_0 + \Delta\mu_0(t)] = \partial\Delta\mu_0/\partial t = (\partial/\partial t)\{[(n^2-1)/c^2]\,u_0(t)\}. \tag{5.19}$$

Hence, integrating, we obtain

$$\Delta\mu_0(t) = [(n^2-1)/c^2]\,u_0(t) \tag{5.20}$$

(since $u_0 = 0$ and $\Delta\mu_0 = 0$ at t = 0).

18) In (5.24) we have already neglected a term of order $w^2\mu_0$ in the expression for the component X^0_{44}.

19) See Appendix 3. The values of p_0 are not the same here and in (5.4). The fact that, unlike (5.4), the transverse components of the pressure ($p^0_{yy} = p'_{yy}$ and $p^0_{zz} = p'_{zz}$) are not equal to zero does not play any role.

20) The flux density is not equal to the product of the energy density by the velocity of the medium.

21) We recall that (6.20) is a consequence of the invariance of the phase (aee [3], page 57).

22) The medium moves in a direction opposite to the field pressure.

23) Cited by Laue[2b].

24) I_{mech} *is the torque of emf. If we assume that the outer electrode of the* capacitor is a very good conductor (almost a superconductor), then at the initial instant of time I_{mech} is the torque of the carriers, and the apparent motion of the electrode accelerates to the value (8.5) gradually, with attenuating current. At first, it was regarded as obvious that when the current is turned off in some circuit (which does not carry in itself electric charges) the vanishing self-field does not cause this circuit to rotate (and also that the short circuiting of the circuit will not cause a torque to be applied to the circuit).

25) These arguments were reported to the Science Council of the Physics

Institute of the USSR Academy of Sciences, and are included in the stenographic report of the session of this council of November 24, 1969. The communication referred to was made in connection with the discussion of one study by the Oscillations Laboratory of the Physics Institute of the USSR Academy of Sciences.

[26]The momentum (9.1) transferred to the medium during the time τ when the light is emitted from its source is equal to the product $p\tau = [(n^2 - 1)/n]\mathscr{E}/c$.

[27] $F = \int_x^\infty (-\partial p/\partial x)\, dx = [(n^2 - 1)/n^2]\, u$, where x is arbitrarily close to x_0 (x_0 is in the coordinate of the "wave front"), and $\int_{x_1}^{x_2} (-\partial p/\partial x)\, dx = 0$ at $x_1 < x_0$ and $x_2 > x_0$.

[28]However, if we ascribe to the photon an energy equal to the total energy ($-T_{44}$) of formulas (5.5) and (5.30), then the rest mass of such a "photon" is finite (see [8]).

[29]This component (separated from Minkowski's tensor) was considered long ago by Beck [9], who called it the "current component" (Stromungsanteil).

[30]In Brevik's review [4] the tensor coinciding with S^{cr} is designated S_{rad} and is regarded as one of the variants of the field tensor; this, as follows from (9.17), is incorrect.

[31]See Appendix 4.

[32]This relation, derived for a static field, is applicable here, since the "Abraham forces" vanish on the average.

[1] W. Pauli, Relativatätstheorie, Enzyklopadie der mathematischen Wissenchaften Bd. V_2, Hf. 4, Art 19, Lpz., 1921 (translation: Relativity Theory, OGIZ, Gostekhizdat, 1947).

[2] M. Laue, a) Ann. d. Phys. **35**, 524 (1911); b) in: "Albert Einstein: Philosopher-Scientist" (The Library of Living Philosophers, v. 7), Evanston, Ill.), 1949, p. 519; c) Zs. Phys. **128**, 387 (1950).

[3] C. Möller, The Theory of Relativity, Oxford, 1952, 1 1957.

[4] J. Brevik, Mat. Fys. Medd. Dan. Vid. Selsk. **37** a) No. 11 (pt. 1), b) No. 13, (pt. 2) (1970).

[5] U. Schröder, Zs. Naturwiss. **242**, 1356 (1969).

[6] S. de Groot et al., Physica **37**, 284, 297 (1967); **39**, 28, 41, 61, 77 (1968).

[7] a) M. Abraham and A. Föppl, Theorie der Elektrizität, Bd. 1. Lpz., 1912; b) M. Abraham, Theorie der Elektrizität, Bd. 2, Lpz., 1914.

[8] G. Marx and G. Györgyi, Acta Phys. Hung. **3**, 213 (1954); Ann. d. Phys. **16**, 241 (1955).

[9] F. Beck, Zs. Phys. **34**, 136 (1953).

[10] N. Balazs, Phys. Rev. **91**, 408 (1953).

[11] M. Planck, Phys. Zs. **9**, 828 (1908).

[12] R. Jones and J. Richards, Proc. Roy. Soc. (L.) **80**, 190 (1962).

[13] V. L. Ginzburg, Zh. Eksp. Teor. Fiz. **10**, 583 (1940).

[14] C. Tang and J. Meixner, Phys. Fluids **4**, 148 (1961).

[15] S. de Groot and L. Suttorp, Physica **39**, 84 (1968).

[16] O. Costa de Beauregard, a) Nuovo Cimento **48**, 2 (1967); C. R. Ac. Sci. **B274**, b) 164, c) 119, d) 1299 (1972).

[17] C. Imbert and G. Roosen, ibid., p. 1351.

[18] A. Einstein and J. Laub, Ann. d. Phys. **26**, 541 (1908).

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