From the History of Physics

JOHANNES KEPLER: FROM THE MYSTERY TO THE HARMONY

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Usp. Foz. Nauk **109**, 175-209 (January 1973)

Though this be madness, yet there is method in't[1]
(*Hamlet*, Act II, scene 2)

'Αναρμονικὸς μὴ κρινέτω [2]

(Inscription on a copy of *Harmony of the Universe* which used to belong to Max Caspar, the biographer of Kepler)

On one of the pages of the book catalog which was published for the Spring 1597 Frankfurt Fair (in the largest center of book trade of the time), there appeared a new strange-sounding name, "Repleus." The unintentional pseudonym, which owed its existence solely to the carelessness of the typesetter, concealed an entirely different but equally unknown name: Johannes Kepler.

The small book, published shortly before the fair (at the end of 1596) had a fancy title: *Precursor of Cosmographic Investigations Containing the Cosmographic Mystery Concerning Marvelous Proportions Between Celestial Orbits and the True Causes for the Number, Magnitudes as well as Periodic Motions of Celestial Spheres Demonstrated with the Aid of the Five Regular Geometric Solids by Johannes Kepler from Wuerttemberg, the Mathematician of the Glorious Province of Styria*.

Aside from the calendars, which the Province Mathematician prepared as part of his duties, the *Precursor*, or, as Kepler himself preferred to call it, *Mysterium Cosmographicum* (The Cosmographic Mystery) was the first of Kepler's publications dealing with astronomy and the only work which went through two editions in Kepler's lifetime. Yielding to the urgings of his friends, Kepler, in the twilight of his years, undertook to publish the second edition "for the benefit of not only book dealers but also scientists." Addressing himself to the new readers, Kepler, who was nearing the end of his life's journey (only the *Rudolphine Tables* remained yet unwritten) proudly wrote in the dedication:

"Nearly twenty-five years have passed since I first published a small book *Cosmographic Mystery*. And although at the time I was still very young and this publication was my first work dealing with astronomy, nonetheless the success which my small book has had in the ensuing years loudly testifies that no one ever yet succeeded to produce a more significant, successful and valuable first work, as far as its subject is concerned. ... It was as if an oracle from heaven had dictated to me the chapters of this small book, so excellent were they, by general admission, and in accordance with the truth.... More than once the chapters of this small book have served me myself—a person who for twenty-five years now has been working on a reform of astronomy (begun by the famous and noble Danish astronomer Tycho Brahe) to illuminate the way. Nearly all astronomical books which I have published in that period have their beginnings in some one of the chapters of my first publication and, therefore, can be considered as a more detailed or more comprehensive presentation of respective chapters. The reason for acting the way I did is not that I, alledgedly, let myself be led by the love for my discoveries...but that the very nature of things, supported by extraordinarily precise observations by Tycho Brahe, led me to conclude that there is no other way to the reform of astronomy, to reliability of calculations, and to the construction of the metaphysical part of the astronomy, which is called heavenly physics, except the way which I either described in detail in that book, or (in cases where deeper insight was lacking) timidly outlined. In order to corroborate the preceding, I refer the reader to *The New Astronomy*, published in 1619, as well as to my *Commentary on the Motions of Other Planets*, which is as yet unpublished, to the five books of *Harmony of the Universe* printed in 1619, and to the fourth book of *Epitomes of Copernican Astronomy* which appeared in 1620. I call as witnesses all those readers who over many years now, since the time *Cosmographic Mystery* was published, have insistently demanded from me by now ragged copies of this book in order to be able to see with their own eyes the way in which it was possible to infer so many important conclusions."

The objective which Kepler set for himself is distinctly formulated in the foreword to the reader which opens the *Cosmographic Mystery*:

"Kind reader! In this small book I have taken upon myself to prove that the All-merciful and the Almighty God, when he created our moving world and when he assigned celestial orbits, took for the basis the five regular solids, which from the time of Pythagoras and Plato and down to our day have won such great fame, and selected the number and proportions of celestial orbits and also selected the ratios between motions in accordance with the nature of the regular solids....

"The essence of three things—why they are arranged just so and not otherwise—has especially interested me, and namely: the number, dimension, and movement of celestial orbits."

Thus, in the foreword of his very first book, Kepler raised a question which is the basic question of modern physics—the question concerning the causes of the phenomena of nature. While this question is quite natural in our days, in the days of Kepler it sounded unusual. It was not raised in either the Ptolemaic or the Copernican astronomy. Following a centuries-long tradition, the astronomers saw the task of their sciences as consisting of describing and predicting as precisely as possible celestial phenomena. In a review of a paper of Kepler's his teacher Maestlin presented to Tuebingen University Rector Hafenreffer, the former noted: "... The subject matter [of this paper] and the method employed are new and nobody thought about them before.... Who would ever even think about ... the number, dimensions, and movement of spheres, then substantiate all this and extract it from the mysterious designs of God, the Creator? However, Kepler dared to undertake such a venture and successfully brought it to the end."

How, then, did Kepler answer his own astonishing questions? How did he succeed in finding the answers? The history of the search is described in detail in the same foreword to the reader.

Initially, Kepler made the assumption that the radii of celestial spheres differ by factors 2, 3, 4, and so on, but verification convinced him of the fallacy of this hypothesis. Having rejected it, he "attempted to try an astonishingly daring way out." Between Jupiter and Mars, and also between Venus and Mercury, Kepler placed "two new planets, invisible because of their small size and ascribed to them certain periods of revolution.... However, one new planet proved to be insufficient to fill the monstrous interspace between Jupiter and Mars." The second attempt similarly did not lead to the desired result and Kepler made a third attempt. This attempt differed from the two preceding ones in greater refinement in the selection of mathematical devices: Kepler made the attempt to represent the distance between the sun and the planet by means of one trigonometric function while representing the "motive force of the planet" [a concept to which the velocity of the planet reduces] by means of another trigonometric function.

Kepler tells of his subsequent attempts as follows: "I spent almost the entire summer at this hard work. Finally, by pure chance I managed to come closer to the truth.... On July 9, 1595, I intended to show my students how the great conjunction [of Jupiter and Saturn] always jumps over eight signs [of the Zodiac] and jumps from one trigon to another. To this end I drew within a circle a multitude of triangles (if only you can call them that) in such a sequence that they would overlap one another. The points at which the sides of the triangles intersected formed a smaller circle, for the radius of a circle inscribed into such a triangle equals one half the radius of the circle that circumscribes it. The relationship between the sizes of the two circumferences which arose before my eyes was completely analogous to the relationship between the sizes of the orbits of Saturn and of Jupiter; but the triangle is the first of the geometric plane figures in the same way that Saturn and Jupiter are the first planets. In exactly the same fashion I attempted to obtain the second distance—between Mars and Jupiter—by using a square, and the next with the aid of a pentagon."

The hypothesis that the distances between planets can be derived with the aid of regular polygons having successively increasing numbers of sides could likewise not stand the test and, morevoer, was unsatisfactory because it did not explain the number of planets known at that time. But "its end simultaneously became the start of the last attempt, which was crowned with success."

Here is how Kepler tells about the success of his undertaking:

"... It occurred to me that if I were to follow the road I chose (if I were to observe the succession of regular polygons) I would never get to the sun and discover the reason why there must be only six planets and not 20 or 100.... I reckoned that arbitrariness would creep in if in order [to explain] the size and proportions of the six orbits assumed by Copernicus it were necessary to find among the remaining infinitely many plane figures five which would differ from the rest by some special distinguishing characteristics. And here I dashed forward with new energy. What bearing do plane figures have for three-dimensional orbits? In this connection, one would rather have to turn to geometric solids. Now, kind reader, you know my discovery and the subject of the entire book! For as soon as a person, even one who poorly knows geometry, utters these words, the five regular solids and the ratios peculiar to them, between the dimensions of the inscribed and circumscribed spheres, come to his mind; that person immediately recalls Euclid's famous addendum to the 18th proposition from Book XIII, where it is proven that there do not exist, and one cannot invent, more than five regular solids. To think only! Although at the time I have as yet not had the slightest idea regarding comparative advantages of regular solids, based on only an intuition and with a knowledge of only the known distances between planets, I have immediately succeeded in realizing my cherished goal—to position the bodies so successfully that later, when I have investigated the causes of things, there was nothing that was in need of being changed."

Further, there follows the formula of the discovery: "The earth[3] is the measure of all orbits. Circumscribe a dodecahedron about it. The sphere circumscribed around the dodecahedron will be that of Mars. Circumscribe a tetrahedron around the sphere of Mars. The sphere which is circumscribed around the tetrahedron is the sphere of Jupiter. Circumscribe a cube around the sphere of Jupiter. The sphere circumscribed around the cube will be that of Saturn. Into the sphere of the earth let us insert an icosahedron. The sphere that can be inscribed in this will be that of Venus. Let us insert into the sphere of Venus an octahedron. The sphere inscribed in this will be that of Mercury."

Here is what, in the opinion of the 23-year-old Kepler, the mystery of cosmography consisted of: The universe turned out to be arranged according to a single geometric principle! But ... was the joy to turn out to be premature? Failures of previous attempts had convinced Kepler of the need for serious verification of each advanced hypothesis, and, restraining the delight that engulfed him at the thought of his marvelous discovery, Kepler undertook to carry out this verification.

"... Now I regretted no longer the lost time, did not feel tired from working, and shunned no calculation, no matter how complicated. I spent days and nights in calculating in order to become convinced as to whether the law formulated by me only verbally will coincide with the orbits of Copernicus or my joy would be cast to the winds. In case everything I conceived is correct, something I did not doubt, I swore to publish at the first opportunity this marvellous example of divine wisdom so that people can become familiar with it. In a few days everything came out right."

The use of a universal geometric principle enabled Kepler to answer two of the three questions he raised: to explain the number of planets known in his day (if regular solids "are to be considered boundaries or partititions..., with their aid one can separate no more than six objects—hence the six planets revolving around the sun") and to deduce the relative distances between them. The answer to the third question (the one about the motion of planets) turned out to be the most difficult one and was obtained only many years later...

Thus, the solution was found and verified. In general outline, the solution was correct and it was now necessary to work out the details and to substantiate it. The latter task was something Kepler's philosophic views forced him toward. In Kepler's opinion, God, "the most perfect of architects had of necessity to produce a creation possessing irreproachable beauty," guiding himself in this, like the builders on this earth do, by number and measurement. "Lines and surfaces do not contain numbers—here the unlimited reigns. The same is true of three-dimensional bodies. Irregular bodies should be eliminated from consideration, for what is at issue here is the basis of the best regulated construction." In a letter to Maestlin dated April 9, 1597, Kepler expressed this thought in the following fashion: "As the eye was created for color, the ear for tone, so was the human intellect created for understanding not just any thing whatsoever but of quantities; the spirit grasps the matter so much more correctly the closer it approaches pure quantities which are at the base of the problem. The further the spirit diverges from quantities, the more room there is for obscurity and error."

Otherwise, everything in this most beautiful of all possible worlds is regulated so perfectly that no room remains in it for chance. Everything, including the solution Kepler found, must have its causes and Kepler undertook to learn these causes with the thoroughness characteristic of him.

First and foremost it was necessary to convince the reader of the correctness of Copernicus' little-known doctrine which by no means enjoyed universal recognition at that time. Kepler himself was "converted" to Copernicanism in his student years.

Kepler tells about his initiation in the following way: "As far back as the days when I had for six years diligently attended in Tuebingen to the instruction of the famous Master Maestlin, I sensed how far from perfect in many respects is the currently prevalent conception of the structure of the universe. Therefore I was so carried away by Copernicus, whom my teacher frequently mentioned in his lectures, that not only did I repeatedly defend Copernican views in student debates, but also thoroughly prepared a paper on the fact that the "first motion" [revolution of the sphere of fixed stars] is called forth by the diurnal revolution of the earth. In this I proceeded from the fact that the motion of the sun should be attributed to the earth not only on mathematical grounds, as Copernicus did, but also on physical grounds, or, if you please, metaphysical grounds. To this end I began to collect little by little, in part from Maestlin's lectures and in part from my own reflections, the advantages which, from a mathematical point of view, Copernicus had over Ptolemy."

The first chapter of Cosmographic Mystery is entitled "Reasons for which the Copernican Doctrine is Correct. Exposition of This Doctrine." No one before Kepler dared to come out so openly in support of a "heretical" doctrine. Even Maestlin, who readily presented Copernicus' views in his lectures, wrote the Epitomes of Astronomy from the positions of the traditional Ptolemaic astronomy. Only in following through the printing of Kepler's Cosmographic Mystery did Maestlin make reference to the author's insufficiently popular style of presentation ("... he [Kepler] judges others using himself as a guide, as if anyone in whose hands this book can come is well familiar with Copernicus' complicated train of reasoning....") and appended to Kepler's book a sort of outline of Copernicus' work Concerning the Revolutions—the so-called "First Narrative" of Rheticus. Similarly, another of Kepler's contemporaries, Galileo, was hesitant to come out publicly in support of Copernicanism. In a letter to Kepler (and Galileo wrote Kepler personally only twice) he openly indicates his reasons for remaining silent:

"... It is many years now that I share the views of Copernicus and by using them as a guide I have discovered the causes of many phenomena of nature which are not explainable on the basis of generally accepted hypotheses. I have written a great deal on this subject about direct and indirect proofs; however, frightened by the fate of Copernicus, our teacher himself, I have as yet not ventured to print them. He has won immortal fame among a few; however, infinitely many (for such is the number of fools) laughed at him and catcalled him. I would dare to publish my line of reasoning if there were more people of thy kind of intellect, but inasmuch as this is not the case, I shall wait."

There were two reasons why Kepler was convinced of the correctness of Copernicus' doctrine: in the first place it was "the excellent agreement between all celestial phenomena and the views which Copernicus held" and in the second place it was the fact that "only Copernicus substantiates in the most excellent fashion the things others teach us to wonder about and in so doing he eliminates the cause for wonderment, i.e. the ignorance of the essence of a phenomenon." Ptolemaic astronomy was powerless to explain, for example, why the epicycle of Mars is of such a large size when compared with the epicycles of Jupiter and Saturn although the deferents (circles along which the centers of the epicycles move) of these planets are significantly greater than the deferent of Mars; why must Mars, Jupiter, and Saturn at the time of opposition to the Sun be at the point of perigee of their respective orbits, why is it that the sun and the moon never retrograde in their orbits, etc. All of these phenomena, as Kepler has shown, find a simple and natural explanation within the framework of Copernican theory.

Further, Kepler explains why the regular solids must be arranged precisely in the order discovered by him. Kepler's reasoning will probably strike us as being completely devoid of logic and we would be prone to suspect the author of dishonesty, of trying to mislead the reader. But Kepler could not be dishonest; what may strike us as a clumsy stratagem was for Kepler convincing arguments.

As a science develops, changes occur not only in its contents but also in its language, and, what is even more important, the logic of proofs changes. The traditional quod erat demonstrandum has a completely different meaning when used by a 16th Century author and when used by a 20th Century author. One must not forget this when reading Kepler today. But Kepler, while trying out different proofs, in the end still gave preference to the comparison of theory with experience. For the sake of accomplishing such an agreement, Kepler can be seen to have many times restructured his views. According to Kepler all solids can be subdivided into two classes: the primary ones (cube, tetrahedron, and dodecahedron) and the secondary ones (icosahedron and octahedron).

In Book IV of the Epitome of Copernican Astronomy the following is said regarding the primary solids: "These solids express particularly clearly the metaphysical opposition of equality and difference. The cube is the embodiment of equality, the other two solids (the tetrahedron and dodecahedron) are the embodiment of difference. The cube is the first solid that arises upon construction, the tetrahedron is the first solid obtained by excision from the cube, and the dodecahedron is the first solid that results from combining the produced their vertices, then in both cases the eye will reject so ugly a sight."

And nonetheless, the incorrigible optimist considers his problem solved: "... If some peasant now asks you what sort of hooks hold up the heavens so that they do not fall, it will be easy for you to answer him." Many years later in the second edition Kepler notes with bitterness in the same spot: "Woe unto me; how terribly mistaken I was."

The phantasy of the author knows no limits. Nothing escapes his attention: neither the marvelous properties of numbers connected with regular solids, nor the "aspects" of which we will have occasion to speak in greater detail below in connection with the discussion of the Harmony of the Universe. And suddenly, in the middle of the book (in Chapter 13), there rings an unexpected warning: "Everything said up to this point serves only to support with likely arguments the law which we have discovered. Now we wish to go on to the definition of planetary orbits and to geometric studies. If the calculated values will not coincide [with those given by Copernicus] then our entire labor will be unquestionably in vain."

First of all it was necessary to solve a subsidiary geometrical problem: to find for each of the five regular solids the ratio of radii of inscribed and circumscribed spheres. Taking the radius of the circumscribed sphere to equal 1,000 parts, Kepler obtained the following values for the radii of inscribed spheres: 577 for the cube, 333 for the tetrahedron, 795 for the dodecahedron, 795 for the icosahedron, and 577 for the octahedron.

With these data it was not possible to undertake the solution of the main problem—"the astronomical proof of the fact that five regular solids are located between the celestial spheres." Before doing this one only had to specify a certain, by no means unimportant, detail (it is frequently forgotten): the spheres that separated the orbits could not be the infinitesimally thin mathematical spheres.

Celestial spheres must possess a certain thickness; this circumstance caused no doubts among physicists: it was necessary to locate between the inner and outer surfaces of the spheroidal layer (the "thick sphere") the eccentric circular orbit of a planet, which orbit was highly essential for explaining the unevenness in the motion of the planets (as yet there was not even a mention of elliptical orbits). But Kepler disagreed with further conclusions: "The physicists assumed that, allegedly, starting with the innermost sphere (that of the moon) and up to the tenth sphere[4)] there were no voids, nothing that was not filled with spheres. The spheres must without fail touch one another: the inner side of the outer sphere must coincide with the outer side of the inner sphere nearest it." Besides, the Ptolemaic astronomy had nothing to say about the relative sizes of the spheres (in solids...." The secondary solids differ "from the primary ones basically in that more than three edges converge in a single vertex...." The earth, on which dwells the crowning piece of Creation, man, deserves to be placed between the primary and secondary solids.

Feeling, apparently, the shakiness of such arguments, Kepler attempts to find yet others: "... The primary regular solids must by their very nature stand upright while the secondary ones must float. If the latter are to be placed on one of the edges and the former on one of the words of Kepler, in this school of astronomy "there was nothing to go by in investigating proportions of celestial orbits."). However, the "weak arguments of physicists in favor of celestial spheres touching each other" could only be refuted by an astronomer who "with the aid of observations or hypotheses could himself soar to the orbits, into the celestial space" (for "one cannot object to one who writes about New India without having walked that country from one end to the other himself!"). However, from the hypothesis of Copernicus and from the assumption about the annual revolution of the earth, it followed that between neighboring spheres there must be empty gaps.

For example, "if the average distance from the earth to the center of the universe is assumed to equal 60 units, the average distance from Venus to the center of the universe would equal $43\,{}^1\!/_6$ units, so that the difference would equal $16\,{}^5\!/_6$ units. At the perigee earth approaches Venus an additional $2\,{}^1\!/_2$ units. At the apogee Venus approaches earth $2\,{}^1\!/_2$ units. Consequently, overall, the two planets come closer to one another by 5 units. However, the distance between both planets, even at the point of their greatest proximity, remains equal to 12."

Kepler's physical intuition rebelled against a space completely filled with celestial spheres. "Why does Nature need such wastefulness?" he exclaims. "How absurd and useless would such wastefulness be! How little of it have we grown accustomed to meet in Nature!"

With trepidation Kepler submitted his results to the "judgement of astronomy." These results, presented in table form, are one of the first comparisons of a model with an experiment ever published in scientific literature.

The experimental data in this case are the spheres of the planets calculated by Copernicus from the data obtained by observation. The ratios of the outer surface of a sphere to the inner surface of the next sphere are determined geometrically. The thickness of a sphere as defined by Copernicus is determined by the difference between the aphelion and perihelion of a planet.

From our point of view, agreement with the experimental data is not bad in the above model. Yet the point to note here is that Kepler's model had no foundation whatever; subsequent discoveries of the three outer planets (Neptune, Uranus, and Pluto) and of the asteroid belt deprived that model of any sense altogether. Thus it was that Nature taught physicists a lesson in being vigilant in dealing with simple models. Kepler, however, was irritated by the last line in the table. When it came to the orbit of Mercury, he corrected the rules of the game on the run.

Having computed an orbit "by the rules," Kepler found it unsuitable. At this point he introduced another orbit which can be arrived at if one takes not the radius of the inscribed sphere but the radius of a sphere that touches the middle of the edges of the octahedron (and not the centers of its faces). "This value," Kepler remarks, "is not too far off from 723." Of course, such an obvious digression from the general rules required some justification, and Kepler devoted a special chapter to the analysis (an unconvincing one, to our mind) of the exceptional case. Concerning this chapter, Kepler is to write in the second edition that "all of this is incorrect," and that "all the arguments are faulty." But nonetheless the correspondence is sufficiently satisfactory and, as Kepler himself notes, "the respective numbers are close to one another. In the case of Mars and Venus they are equal. In the case of earth and Mercury, they do not differ all that much. And only in the case of Jupiter do the numbers diverge, which, after all, is not astonishing if one takes into account the monstrous distance to it."

"It is not difficult to see," Kepler notes, "how great would have been the divergence between numbers if our experiment would have been counter to the nature of the heavens, i.e., if God in creating the world would have been guided by other numerical ratios. Such a close correspondence of numerical ratios characteristic of regular solids and of distances between the planets cannot, of course, be accidental...."

Similar substantiations were accepted as proofs even after Kepler's day. But while convincing the reader Kepler himself could not remain indifferent to the minor discrepancies which he knew remained. Kepler could not eliminate these discrepancies and stay within the framework of his geometric model of the structure of the universe: the scheme contained no free parameters which one could vary to adjust the results.

First and foremost Kepler noticed that the task he was achieving differed from the one Copernicus sought to accomplish. Copernicus' book Concerning the Revolutions of the Heavenly Bodies was more of a cosmographic than an astronomical study. Minor discrepancies in the relative distances were of no particular significance to Copernicus.

Moreover, although Copernicus did claim to consider the sun as being the center of the universe, he, however, "in order to shorten the calculations and not unduly intimidate the zealous readers by excessively large deviations from Ptolemy, did calculate the farthest and the least distances ... and the points in the orbit of a planet where it is farthest or nearest the sun, known respectively as "aphelion" and "perihelion," not with respect to the center of the sun, but with respect to the center of the earth's orbit, as if the latter were the center of the universe...."[5)]

If Kepler were to believe Copernicus, he would have been forced to consider that the eccentric distance, i.e. the distance from the sun to the center of the earth's orbit, is equal to zero (while the eccentric distances of other planets, i.e. the distances between the centers of their orbits and the center of the earth's orbit, would have to remain, as before, other than zero), and, consequently, to assume that "the sphere of the earth as distinguished from [the sphere of] other planets has no thickness. But then the centers of the edges of the dodecahedron on the one hand and the vertex of the icosahedron on the other would be located in the same sphere and the whole world would look more compressed and flattened out." Such corrections in the model were scarcely acceptable to Kepler, for they would assign earth a special role among the other planets.

There remained only one thing to do: to recalculate Copernicus' data, accepting the center of the sun as the center of the universe. It was Kepler's former teacher Maestlin who readily agreed to carry out this labor-consuming task at Kepler's request. The differences, as was to be expected, turned out to be rather substantial. For example, "for Venus, the difference [in the position of the line of apsides] comprised more than three signs of the Zodiac [i.e., more than 90°], for the aphelion [the point in the orbit nearest the sun] lies in Taurus and Gemini while its apogee [the point in orbit nearest the earth] lies in the Capricorn and Aquarius."

Not only did the distances turn out to be different, but also the annual parallaxes of planets in aphelion when computed in three ways: 1) using the scheme of the position of regular solids, 2) according to Copernican distances from the sun, and 3) again according to Kepler's theory but with the moon included in the sphere of the earth (which led to an increase in the thickness of the earth's sphere). Especially markedly different were the results obtained for Mars (40° 9′, 37° 22′, and 37° 52′) and Venus (49° 36′, 47° 51′, and 45° 33′).

Such obvious discrepancies did not shake Kepler's belief in the correctness of the scheme he proposed for the structure of the universe, but they could make an unfavorable impression on the readers of the Cosmographic Mystery and even undermine their confidence in the new theory. Being aware of it, Kepler undertook an extensive analysis of possible causes for the discrepancies in the results.

First of all, it was necessary to find out just how reliable the computations were. Maestlin, in making his calculations, used the so-called Prutenic Tables compiled in 1551 by Reinhold on the basis of the Copernicus model.[6)] Kepler was quite skeptical in evaluating the reliability of these tables: in computing the position of the planets they, in his words, "not infrequently lead to errors," they contain values of eccentric distances for planets which "do not instill trust" and are generally so rough that at times they do not permit one to attain an accuracy exceeding 1/2 degree.

Copernicus' data was also not sufficiently reliable. Therefore, the contradictions that were discovered did not support Copernicus' theory, but they could not be considered decisive enough to discredit it. It remained for Kepler to try another way and to obtain data about relative distances between planets with the aid of considerations which went beyond the framework of the geometry of regular solids. Kepler hoped to obtain from an investigation "of the proportions of motions to orbits" of celestial bodies the necessary data (and along with it the answer to the third question formulated in the Cosmographic Mystery). This idea (intended to save the model) turned out to be truly a lucky one.

Kepler's reasoning reduced, apparently, to the following. The difficulty was due to the fact that there was no confidence in the correctness of values obtained for the distances. Direct observations did not yield reliable values. Later on, while studying the motion of Mars, Kepler understood how one should correctly "triangulate the sky", but for the moment he was carried away by another idea. If it is impossible to measure the distances directly, is it perhaps possible to find an indirect method of carrying out such measurements? Indeed, if

Radius of the inner surface *of sphere* = 1000 units	Radius of the outer surface of the neighboring sphere	
	according to Kepler	according to Copernicus (Book V)
Saturn	Jupiter 577	635
Jupiter	Mars 333	333
Mars	Earth 795	757
Earth	Venus 795	794
Venus	Mercury 577 or 707	723

distances between planets are subject to some order, then the periods of their revolution cannot be accidental: they must be governed by some simple law which would establish the interrelation between periods of revolution and distances involved. In such a case computation of the radii would simply reduce to substitution of corresponding periods which were known with a great degree of precision. But there was nobody in Kepler's days who had analyzed the interrelation between radii and periods of revolution. Consequently, it had to be established.

Verification has shown that the periods of the planets are not proportional to the radii of the orbits, for "although the ratios of the periods is similar to the ratio of the distances," the two are nonetheless different. For such a divergence to occur there must be a reason and Kepler advanced two hypotheses: either the motive forces of the planets themselves (Kepler wrote about their "soul") weaken the farther they are away from the sun, or "there exists only a single motive force [again "soul"] which emanates from the center of all orbits, i.e., from the sun, and which acts on all bodies stronger the closer they move toward the sun." Kepler stops to consider the second hypothesis.

In the second edition Kepler makes two comments:

"1. The fact that such souls [of planets] do not exist I have proved in my New Astronomy.

"2. If we replace the word "soul" with the word "force" then we obtain precisely the principle which is at the basis of celestial physics in New Astronomy. At first I firmly believed that the motive force of the planets is the soul.... Now, however, when I have understood that the cause of motion diminishes in proportion to the distance, just as the light of the sun weakens in direct proportion to the distance from the sun, I came to the conclusion that this force must be something material not in the literal sense of the word but ... in the same sense in which we speak about the material nature of light, understanding by it the non-material substance emitted by a material body."

"If the increase in the period of the planet," Kepler says, "were related only to the increases in the size of the orbit, then the motions would be in the same ratios as are the average distances." But such a simple relationship between the period of revolution and the radius of the orbit is distorted by a force that comes from the sun and decreases with the distance from it. "It can be assumed with a great degree of probability that the influence of the sun [on the motion of planets] is subject to the same regularities as is light. We know from optics how the weakening of a light emanating from a point occurs. A small circle [drawn around the Sun] receives the same amount of light as does a larger circle. The light falling on the smaller of the two circles is denser than the light falling on the larger circle. The measure of the weakening of the light as well as of the motive force must be sought in the relationship between the circles. The orbit of Venus is greater than the orbit of Mercury, therefore Mercury moves with greater force, with more haste, more agility and dashing than does Venus. However, the longer the orbit, the greater the time required for the planets to complete a revolution even if they are subject to similar motive forces. Consequently, the increase in the distance from the planet to the Sun influences in two ways the increase in the period of its revolution...."

Planets	Relative distances (Kepler)	Relative distances (Copernicus)
Jupiter: Saturn	0.574	0.572
Mars: Jupiter	0.274	0.290
Earth: Mars	0.694	0.658
Venus: Earth	0.762	0.719
Mercury: Venus	0.563	0.500

Further, Kepler provides an erroneous relation (which he himself corrected in the second edition) between the radii R_1 and R_2 and periods T_1 and T_2 of two planets:

$$T_1 : \frac{T_1+T_2}{2} = R_1 : R_2.$$

This relation, the precursor of Kepler's famous Third Law, allowed him to compute the relative distances between planets. The following results were obtained (here again the Copernican data are regarded as experimental).

"We see how closely we have succeeded in coming to the truth," Kepler stated with joy. But the agreement was still not complete and Kepler could not be content. He anxiously raises the question: "Will we live to see the day when both rows of figures can be brought into complete correspondence with one another?"

Such a day came, although not all that soon. Kepler needed almost a quarter of a century to come to the discovery of the Third Law of motion of the planets. In a comment to the second edition of Cosmographic Mystery, Kepler proudly notes: "We lived to see such a day 22 years later and we, at least I, are happy. I think that Maestlin and others who have read Book V of the Harmony of the Universe share my happiness." And in the same second edition Kepler notes:

"If my numbers turned out to be close to reality, this occurred accidentally.... Perhaps one should not have printed these comments. But now it gives me pleasure to recall how many steep roads I had to travel and what number of walls I have stumbled into in the darkness of my ignorance until I finally found the door which led me to the light.... Thus I have dreamed about the truth."

Work on the Cosmographic Mystery left Kepler full of creative plans. In his mind Kepler conceived a vast new study. In a letter to the Bavarian Chancellor Herwart von Hohenburg dated December 14, 1599, Kepler reports: "I have already developed the method and prepared the first drafts for a book entitled De Harmoniae Mundi dissertatio cosmographica (A Cosmographic Discourse about the Harmony of the Universe). The book will contain five parts or chapters: the first, geometric, concerning the figures which can be constructed by means of a compass and a ruler; the second, the arithmetical one, is about numerical proportions characteristic of regular polyhedra; the third, the musical one, is about the causes of harmony; the fourth, the astrological one, is about the causes of aspects; the fifth, the astronomical one, is about the causes of periodic motions."

However, one serious circumstance interfered with the daring project: Kepler was lacking exact observational data. He couldn't be satisfied by the scanty amount of information which was available to Copernicus. Copernicus himself added to the list of the observations of the ancients only 27 more stars. Only one man in Europe of that time had at his disposal the necessary data. This man's name was Tycho Brahe.

An aristocrat by birth and a scientist by calling, the famous Danish astronomer Tycho Brahe was the first to understand the importance of systematic observations. The precision of observations which he was able to achieve (without a telescope at that!) not only significantly exceeded the precision of observations of his predecessors but also for a long time remained unattainable to those who followed him.

"Everybody must become silent and listen with all the acuity of their intellect to Tycho," Kepler wrote to Herwart von Hohenburg (on April 9/10, 1599), "who devoted to observations 35 years of his life and who saw with his own eyes more than many others. Any one of his instruments is worth more than all of my property and the property of my relations. When compared with him, Ptolemy, Alfonso, and Copernicus would have looked simply as little boys were it not Tycho's custom to credit to them with the greater part of his knowledge and ideas which have served as impetus for his discoveries...."

Tycho was not exclusively an astronomer engaged in observations (and to boot the best astronomical observer of his time), he also was a theoretician, albeit with much less success. Having become convinced of the inadequacies of Ptolemaic astronomy through his own experience that stretched over many years, and at the same time not recognizing the Copernican theory, Brahe advanced his own scheme of the structure of the universe which was of a mixed geo-heliocentric nature. According to Brahe, all planets (except earth) had to revolve around the sun which, in turn, revolved around the earth.

In a letter to Maestlin written soon after Brahe's death (Brahe died December 20, 1601 when he was only 55 years old), Kepler characterized Brahe's scientific work in the following way: "Everything that Tycho accomplished, he accomplished before '97.... Tycho's main achievement was his observations, which consist of as many voluminous tomes as there were years which he devoted to his work. But his Progymnasmata [The Basic Principles] truly spread the fragrance of ambrosia.... Tycho taught about all the planets correctly and conducted thorough investigations but, basically, like Copernicus, mutatis mutandis he did it in the manner of Ptolemy. From this you can see how God distributes his gifts: no one of us has everything. Tycho has performed the same mission as Hipparchus: he laid the foundation of a building and in so doing performed a great deal of work. But no one man is able to carry out everything. Hipparchus needed Ptolemy, who developed the theory of the remaining five planets."

We cannot stop here to consider the rather involved chain of events which preceded the meeting between Kepler and Brahe, Kepler's move, together with his family, to Prague, and the start of cooperation between the two titans. We should only remind the reader that their uneasy alliance was very short-lived. It lasted for only a year and a half. The relationship between Brahe and Kepler was from the start far from cloudless. They each set vastly different goals for themselves. Brahe valued highly the refined analytical mind and the learning of his younger "brother in the observation of the heavens," and Kepler considered it an unusual opportunity to be able to work with the "prince of astronomy," with "our phoenix." Already after Brahe's death, Kepler wrote Maestlin: "If the Lord is in the least bit concerned with astronomy, and my piety commands me to believe this way, then I hope to accomplish in this area certain results, for I see how God has tied with inseparable bonds my fate with Tycho and did not permit us to part even after the gravest of arguments."

Having sent Brahe a copy of Cosmographic Mystery, Kepler hoped to receive in return not only a response but also the data that interested him very much dealing with the eccentricities of orbits and the distances between the orbits. But Brahe intended to use his observations to support the correctness of his own model of the Universe and was in no hurry to share with anyone the treasures which he had accumulated.

Vexed by the unexpected obstacle, Kepler wrote Maestlin (February 26, 1599): "here is my opinion about Tycho. He is rich beyond measure, but like the majority of rich people he does not know what to do with his riches. It is necessary, therefore, to take upon oneself the labor (which I have performed with appropriate delicacy) of depriving him of the accumulated treasures, and to force him to publish the observations without holding back, and all of them at that."

A personal meeting with Brahe at the latter's observatory in Benatky near Prague did not bring the desired result either. In a letter to Herwart von Hohenburg dated July 12, 1600, Kepler reports: "I would have already concluded my researches about the world harmony had not Tycho's astronomy so captivated me that I nearly wen went out of my mind.

"... One of the most important reasons for my visiting Tycho was the desire to receive from him the more precise values for the eccentricities of the planet orbits and with their aid check the Cosmographic Mystery and the already mentioned Harmony of the Universe, because the a priori conclusions must not contradict the obvious but, to the contrary, must be in complete agreement with it. No matter how I tried, I could not find out anything from Tycho. Only at mealtime, in conversations, he mentioned in passing today the apogee of one planet, tomorrow the nodes of another."

To Kepler, who was always generous in sharing not only finished results but also ideas about future work, the thought of precise observations lying under lock and key was unbearable. His visit to Benatky convinced Kepler also of the fact that Brahe alone would not be able to cope with the task of processing a huge mass of observations.

"Tycho possesses the best observations and, consequently, material for the erection of a structure," wrote Kepler. "He has workers and everything else which might be necessary. What he lacks is only an architect who would use all of this according to his, Tycho's, own plans. For no matter how happy a talent Tycho poseses, how skillful he is in architectonics, the diversity of the tasks and the fact that the truth sometimes lies hidden deep within them, hinders his success. Now old age steals upon him, for his spirit and strength weakened or will weaken to the point where it will be difficult for him to accomplish everything alone."

Who, then, must be this architect who shall build from Brahe's observations the magnificent edifice of the new astronomy? On this account Kepler had no doubts: it is no accident that he, Johannes Kepler, came to Benatky precisely at that recent time when Brahe's assistant, Christen Soerensen Longomontanus, attempted (without noticeable success) to construct a theory of the motion of Mars. "If Christen was occupied with another planet," Kepler reports, "I, too, would have had to become interested in it." Kepler saw a meaning in the entire course of events:

"... I again see the will of Providence in the fact that I arrived exactly at the time when he was occupied with Mars. Otherwise we would never have had the opportunity to arrive, on the basis of observations, at the knowledge of the mysteries of astronomy and they would have forever remained hidden from us."

Mars was always an unusual and wily planet, and poor Longomontanus had no end of trouble in attempting to reach a satisfactory agreement with the observations. Having become convinced of the extraordinary abilities of his new assistant, Brahe decided to entrust Mars to Kepler's care (not forgetting to exact from the latter an oath to keep the data entrusted to him secret) and to assign Longomontanus the theory of the moon.

To Kepler, who considered himself by no means a novice in astronomical calculations, the new task at first appeared to be not very difficult and he even made a bet, rashly promising to complete the theory of Mars in eight days.

We don't know how high the bet was, but we know that Kepler lost the bet. Work on the theory of Mars, instead of the eight days initially promised, took up (to be sure, with interruptions) almost six years.

"The fruits of titanic efforts were published only in 1609 in a book which was called "The New Astronomy Based on the Study of Causes or Celestial Physics Presented in Commentaries About the Motion of the Planet Mars on the Basis of the Observations of the Noble Tycho Brahe by the Order, and with the Patronage of, Rudolph II, The Roman Emperor, etc., Which Was Developed in Prague in the Course of Investigations of Many Years Duration by the Mathematician of His Holy Imperial Majesty, Johannes Kepler."

Kepler succeeded in accomplishing what no one of his precursors could, which was "to shackle Mars with the chains of computations."

In the dedication, presenting the Emperor Rudolph II a conquered Mars, Kepler tells in allegorical form about the vagaries of a protracted "war" with the latter. With the triumph of a conqueror, Kepler speaks about the bad reputation which his "noble captive" has acquired among the astronomers:

"He (Mars) invariably had the upper hand over human intuition, he ridiculed all the contrivances of astronomers, destroyed everything that they have created, and routed enemy troops. The mystery of his might Mars preserved over all the past centuries and made his voyage without restraining his freedom in any way, which gave Pliny, the most famous of the Latins, the priest of Nature, reason to exclaim: "Mars is a planet which does not submit itself to observations!"

No less colorful is another example of the difficulties connected with the creation of a theory of Mars which Kepler brings up. This is an incident which occurred with Georg Joachim Rheticus, "the most famous of Copernicus' students who was the first to dare and make the renovation of astronomy the purpose of his desires and strove to accomplish what he has conveieved with the aid of observations and ideas of no small importance."

Rheticus, according to the legend, growing desperate in his attempts to understand the motions of Mars, turned to his guardian angel with a request to have the latter help him master the truth. The guardian angel, driven out of his wits with insistent pestering, grabbed his charge by the hair and proceeded to hit Rheticus' head against the ceiling, then threw him on the floor, while mumbling all the time: "Here you have the theory of Mars!"

"Legend is a wicked creature," Kepler adds. "Nothing can inflict greater harm on a good name than a legend for it transmits fiction and fable as strictly as it does declare the truth. However, it is quite possible that the troubled spirit of Rheticus became enraged from the fact that Rheticus' speculations brought no results, and his head began to beat against the wall on its own. Must one be amazed at the fact that Rheticus, who dared to investigate Mars, met the same fate as did the emperor Octavian Augustus who lost five of his legions under the command of Quintilius Varus when these were surrounded by the enemy of Augustus Arminius, an offspring of the German God of War."

Kepler himself did not trust the rumors about the invincibility of Mars, considered them exaggerated, and "always thought about the way to victory."

The first to venture against the formidable enemy was Tycho Brahe, who with his observations laid the groundwork for future victory.

"Absolutely special praise," Kepler wrote, "is deserved by the zeal of Tycho Brahe, the supreme commander-in-chief in this campaign. Supported by the kings of Denmark, Frederick II and Christian, and of recent time by your Imperial Majesty [Rudolph II] as well, he, working nights over the last 20 years unceasingly reconnoitered all the habits of our enemy and his military cunning, he exposed all the enemy's intentions and, before dying, wrote all of the above in his books."

Armed with Brahe's observations, Kepler replaced him in the post of the "Supreme Commander-in-Chief."

The struggle against the perfidious enemy proceeded with variable results. "Oftentimes we lacked machines of war precisely in those places where they were especially necessary," Kepler reports. "Clumsy drivers delivered them by roundabout ways with great delays and at great costs. The enemy, as yet poorly studied, turned out to be in places other than those where he was expected to be. The sun and the moon blinded the gunlayers, thick clouds at times obscured the target. Even more frequently the cannon balls drifted off target because of the damp air."

Sometimes one managed to approach the den of the enemy quite closely, but a successful storm of the den was hindered by the "slanting walls of faulty conclusions." The enemy put up an extremely tenacious and inventive defense. As soon as one bastion was taken, he immediately erected a new one and the old means were insufficient to conquer it and it was necessary to develop new ones. One also felt the accidents and pestilence in one's own camp: "loss of a valiant Commander-in-Chief, divisions, infections, and diseases." Domestic affairs, "pleasant and unpleasant," also took time.

Finally the enemy, no longer capable of enduring the persistent chase, began to think about concluding peace (to be sure, not before he became convinced of the fact that "in all of the Empire he did not have a single reliable refuge left") and asked for mercy. "Mother Nature" became the mediator in negotiations dealing with the terms of capitulation. Having secured for himself the freedom within the confines permitted by the voluntarily assumed fetters, Mars, "under armed escort of arithmetic and geometry, heartened and cheerful," went into captivity, into the camp of the conqueror.

Great were the trophies captured by Kepler in the exhausting "duel" he fought with Mars. Kepler discovered not only the two laws of planetary motion (first the one we call second, and then the first) which have immortalized his name, but he also proved the productivity of an essentially new, physical, approach to the study of celestial phenomena.

The novelty and daring of the conclusions and methods of the New Astronomy could frighten a reader educated in the spirit of the old astronomy (for they signified a break with a two-thousand-year-old tradition!), and Kepler decided to move gradually, forcing the reader to go the road from the old to the new that he had had to travel himself.

"What is, after all, involved here is not just the simplest way to introduce the reader to the essence of the subject matter being presented," Kepler explains the manner of presentation he had selected. "Another thing is important: for what reason, by use of what intricate device or happy accident did I, the author, manage to arrive at what I did. When Christopher Columbus, Magellan, and the Portuguese (the first discovered America, the second the Chinese Ocean, while the Portuguese discovered the sea route around Africa) tell of how they lost their way, we not only do not condemn them, but, to the contrary, are afraid to miss something in their stories, so great is the pleasure we derive from reading them. Therefore I too should not be blamed if out of love for the reader I shall use in my work the same device. Reading about the exploits of the argonauts we do not experience the difficulties which they have suffered, while the difficulties and the thorns on the road of my thought are unfortunately quite perceptible to the reader. Such is the lot of all mathematical compositions. Just as some people derive pleasure from one thing, others from another thing, there will be, I am sure, those who will experience the greatest joy when, having overcome all the places which are difficult to understand in my writings, they will be able to grasp with one look the chain of my discourses."

Kepler starts from afar by relating how the astronomers learned to "distinguish the first motion of the planets from the second, or the motion proper," and how in investigating the second motion they have discovered two inequalities—the first and the second.

"The fact that the motions of the planets are circular," Kepler says, "is attested by their unceasing repetition. The intellect that deduces this truth from experience immediately concludes from here that the planets revolve along ideal circles, for among the plane figures the circle, and among the solids, the celestial sphere, are considered the most perfect ones. However, with more careful investigation, it turns out that the experience teaches us something else, namely the orbits of the planets are different from simple circles. Inasmuch as such a conclusion caused the greatest of amazements, people were finally forced to take upon themselves the task of finding the causes of the aforementioned deviation.

"Thus appeared the kind of astronomy whose objective it was to point out the reasons why the motions of heavenly bodies appear to be disorderly when observed from the earth while in the heavens they are subject to a strict order, to investigate the circles along which the heavenly bodies revolve, and to predict the positions of heavenly bodies and of celestial phenomena for any given point in time."

For the ancient astronomers, who did not know the difference between first and second motion, the orbits of heavenly bodies appeared to be entangled ("like the thread in a ball") and complicated.

"These naive views of astronomy," Kepler notes, not without sarcasm, "based not on the explanation of phenomena by causes but on crude observations not representable in the form of figures and numbers and not applicable to the future (for they are never repeated—the duration of one revolution differs from the duration of another, one spire of an orbit never goes into another spire of the same size), are the views, I say, that attempts are being made to revive by those who are eager to to gain the recognition of the mob and who are not without sucess in obtaining such recognition from the unenlightened. Fortunately, those who are knowledgeable in astronomy consider that such people are either fooling around, or, as Patrizzi[7] used to say about philosophers, are themselves fools, their intellect not withstanding."

Reflecting over the amazingly complicated motion of heavily bodies, the astronomers gradually understood that the motions of stars which are visible from the earth are in reality a superposition of "two simple motions, the first and the second, the general and the particular."

"The first motion is understood to involve the rotation of the entire sky, with all the stars, from the east over the south to the west, and from the west, over the lower portion of the sky, to the east, which occurs every 24 hours. The second motion is the motion of individual planets from west to east, which occurs at greater intervals of time."

Having separated the first motion from the second, the astronomers discovered that in the second motion one can single out two so-called inequalities: the first is the non-uniformity of the angular velocity of the planet (it takes different lengths of time for a planet to traverse similar arcs), the second is the non-uniformity of the direction of the motion of a planet (periods when a planet stands still and when it retrogrades). Methods were developed which allowed both inequalities to be taken into account.

Thus, an eccentric circle (the eccentric) was used in the astronomy of Ptolemy to explain the first inequality, and the epicycle to explain the second inequality.

Following the greatest authority of antiquity and middle ages, Aristotle, the astronomers (not only Ptolemy and his contemporaries but also Copernicus and, of course, Brahe) imbued the circular motion with special properties. In their opinion the exalted nature of everything celestial, and, consequently, of planets as well, demanded that the majestic celestial bodies should revolve in circles without hastening or slowing down their pace. Every violation of the uniformity of movement (every inequality) contradicted their conception of the ideal order governing the heavens and was declared as being apparent or imaginary.

The non-uniformity of the angular velocity of a planet (the first inequality) in Ptolemy's astronomy was attributed to an unfortunate selection of the point of observation: a displacement of the earth relative to the center of the "equant point" A from which the movement of the planet would appear uniform. In order to explain the motion of the Sun it was sufficient to have a "simple eccentric"; in other words, it was assumed that the center O of the circular orbit of the sun coincides with the center of the equant point A. In more complicated cases (for instance, when examining the motion of Mars), the scheme of motion proved to be not all that simple. In order to achieve agreement with observations it was necessary to introduce a hypothesis according to which three points—the center of the universe which coincides with the center T of the earth, the center of the equant point A, and the center O of the circular orbit of the planet, all lie on a straight line and do not coincide one with the other. The non-uniformity of the angular velocity of the planet was explained by the fact that the earth (T) and the center of the equant A lie on different sides of the center O of the circular orbit of the planet. Therefore, from point T the motion of the planet appears to be the slowest in the aphelion and the fastest in the perihelion.

In constructing the theory of Mars within Ptolemaic astronomy, it was necessary to introduce yet another hypothesis, that TO = OA. (The only argument in favor of such a hypothesis was a reference to the analogy between the motion of Mars and the motion of the inner planets Mercury and Venus.)

To explain the second equality (when the motion of planets is arrested and reversed), Ptolemy proposed to use the epicycle, i.e., represent the motion of planets as the result of superposition of two circular motions: uniform revolution of a planet in a small circle, the epicycle, and revolution of the center of the epicycle on the circumference of another circle, the deferent (a complete revolution of the center of the epicycle must occur in the sidereal period of the revolution of a planet). Moreover, the radius vector drawn from the center of the epicycle to the planet must at any one moment in time be parallel to a radius vector drawn from the center of the solar equant to the so-called average sun (this is true for the upper planets). Kepler defines the average sun as follows: "The true position of the sun is that place where we see the sun as the result of the inequality of its motion, the average position is the place where the sun would have been had there been no inequalities," i.e., if the sun were to move uniformly.

In Copernicus' theory the necessity of introducing the epicycle for the explanation of the second inequality was eliminated: the fact that planets stood still and retrograded received a much simpler and more natural explanation. They were considered as being conditioned by the fact that the observations of the movement of planets were conducted from a moving observatory, the earth. However, Copernicus did require epicycles, although for another purpose, to explain the first inequality because the orbits of the planets continued to be considered circular in form. The object of particular pride for Copernicus was the fact that all circular motions (of the planet in the epicycle and of the center of the epicycle along the larger, deferent, circle) were uniform. Copernicus saw in this the correction of Ptolemy's error and a return to the "true principles" of motion.

Copernicus, who approached the study of the motion of planets from the position of a pure mathematician, assumed that his epicycle was completely equivalent to the Ptolemaic equant. Kepler came to a different conclusion. Having demonstrated with the aid of a simple geometric construction the qualitative equivalence of both methods of explaining the first inequality, he, at the same time, did not neglect a slight difference in the position of the planet as computed in accordance with the recipes of Ptolemy's and Copernicus' theories (the difference in the formula for the angle or the true anomaly). And although we will have many more occasions to come across Kepler's amazing power of observation and his unusual "attention to details," this first manifestation of the wonderful gift of a great astronomer cannot but evoke admiration, for Kepler (if one were to use modern-day terminology) observed a discrepancy in the third order of the theory of perturbations along the eccentricity of Mars.

The fact that Kepler had proved approximate equivalence of kinematic schemes used in Ptolemy's and Copernicus' theories for the explanation of the first inequality by no means meant that Kepler considered the physical content of these two theories equivalent. From the point of view of a creator of a new science, celestial physics, Copernicus' theory had undisputable advantages over both the ancient doctrine of Ptolemy and the new theory of Brahe.

Kepler speaks out on this point (in the introduction) in no uncertain way: "... Although the difference between the three points of view with regard to their physical content is established with the aid of hypotheses, the hypotheses are such that in their reliability they are not inferior to hypotheses of doctors about the functions of parts of the human body or some other physical hypotheses.

But even the best among the theories, that of Copernicus, was far from perfection and was in need of corrections.

Copernicus saw nothing strange in the fact that in his theory the center of the universe was not a material body but a certain "empty" point, the center of Earth's circular orbit. Copernicus drew conviction in the correctness of his hypothesis by reference to the high authority of Ptolemy, who had planets moving along epicycles similarly rotating around a "non-physical" point.

Such a choice of a point for the center of the Universe was unacceptable to Kepler, who considered that a "mathematical point, even if it is the center of the universe, can neither move a heavy body nor attract it." In his first book, the Cosmographic Mystery, Kepler expressed the opinion that the sun is the source of the motion of the planets. In Kepler's opinion it is precisely the sun, and not the imaginary "average" sun but the real "true" one, that must be located at the center of the universe. From here followed a very important practical conclusion: the circular orbits should be defined when observing the position of the planets at the point of their opposition with the true and not the average sun (as Ptolemy, Copernicus, and Brahe had proposed). (At the point of opposition with the sun, the earth—planet and sun—planet directions coincide, which makes it possible to eliminate the second inequality which is due to the difference between the geo- and heliocentric longitude.)

This thesis of Kepler's was the cause of an argument with Brahe during one of the first few days Kepler spent at Benatky. Kepler himself describes the discussion with Brahe in the following way: "Having arrived at Brahe's, I immediately noticed that he, along with Ptolemy and Copernicus, defined the first inequality according to the average motion of the Sun. But four years previous to this, supported by physical considerations, I came to the conclusion that one should proceed (as was shown in the Cosmographic Mystery) from the true motion of the sun. When a difference of opinions arose between us, Brahe made reference to the fact that in describing the first inequality he succeeded in achieving just as good a correspondence with observations when using the average motion. I, however, objected to his comment, pointing out that the circumstance just brought up does not prevent one from using, when describing the first inequality, the same observations but with reference to the true motion of the sun, for only the second inequality can decide which one of us is right."

Inasmuch as Brahe and his assistant Longomontanus did, indeed, succeed in achieving satisfactory correspondence with observations (in matters concerning the first equality), Kepler had to prove that the changeover to the true sun leads to only insignificant changes in observed results. The computations which he had carried out had demonstrated that the maximum changes in the heliocentric longitude (because of the small eccentricity of the Earth's orbit) reach 5′ and, consequently, do not lead to contradictions with observations. Kepler carried out all of the subsequent computations in the theory of Mars using the true motion of the sun.

Kepler's conception of the sun as being the source of planetary motion allowed him to introduce an important improvement which, at first sight, may appear to be paradoxical: In explaining the first inequality, Kepler refused to use the epicycle introduced by Copernicus and returned anew to an (almost) Ptolemaic eccentric circle, i.e., he shifted the sun relative to the center of the orbit of the planet. And this Kepler did after he himself demonstrated the approximate equivalence of both methods of representing the first inequality!

In reality this step of Kepler's did not involve an internal contradiction and did not signify a return to Ptolemaic views. Kepler had filled the old Ptolemaic picture of motion with new physical contents: having placed the sun outside the center of a circular orbit, Kepler explained what caused the planets now to accelerate, and now to decelerate their motion. From two mathematically equivalent means of describing the first inequality, Kepler chose the one which made more sense, physically speaking.

Kepler's eccentric circle differed from the Ptolemaic one not just by the fact that, according to Kepler, the true sun was at the center of the Universe and the circular orbit of Mars should be defined according to the planet's opposition with the true sun. Kepler rejected the hypothesis that the center of the circular orbit O of Mars divides in half the segment between the position of the sun S and the center of the equant A (the complete eccentricity). The ratio in which the point O divides the segment AS now had to be computed from observations.

The hypothesis about the circular orbit of Mars similarly had to be confirmed with the aid of Brahe's observations. Kepler had at his disposal a table of ten oppositions compiled by Brahe and entitled "An Exact Description of the Motion of the Planet Mars Along its Eccentric Circle Based on Exact Acronychal Observations Carried Out, as is Clear From the Table Itself, With Particular Meticulousness over 20 Years (from 1580 to 1600) with the Aid of Our Instruments for Various Constellations."

Kepler was unable to use Brahe's data directly for two reasons: First, because the astronomers before Kepler (including Brahe and Copernicus) did not know how to correctly take into account the angle which the plane of the orbit of Mars forms with the plane of the ecliptic, because they considered that this angle fluctuates in time. Second, because Brahe referred all his observations and computations to the average sun, it was necessary to interpolate Brahe's data to obtain dates of actual oppositions and of the position of Mars. The first problem was particularly difficult. In solving it, Kepler again demonstrated the subtlety of his physical intuition. Kepler not only had to solve a difficult problem, but he also had to dispel the myth about the fluctuation of the plane of the orbit of Mars which, before him, was transmitted by astronomers from generation to generation.

Kepler the physicist, as contrasted with Copernicus, flatly rejected from the very start the thought of the plane of Mars' orbit fluctuating without any visible causes. It was necessary only to support this confident belief with computations based on Brahe's observations. Having selected six such observations, Kepler found the longitude of the ascending node (it turned out to be equal to $46\text{-}\frac{1}{3}°$). He then computed the angle of the plane of Mars' orbit relative to the plane of the ecliptic by three different methods.

The calculations which Kepler made enabled him to come to "a completely reliable conclusion: the angle of the plane of the eccentric relative to the plane of the ecliptic does not change" and is equal to $1°50'$.... "Why shouldn't we now generalize this conclusion," Kepler notes, "for there are no reasons why this must be true for only one planet, are there? However, I can prove an analogous conclusion for Venus and Mercury on the basis of observations, too."

Having proved the constancy of the angle between the plane of Mars' orbit and the plane of the ecliptic, Kepler was able to carry out a correct reduction of heliocentric longitudes to the longitudes in the plane of the orbit of Mars. After evaluating for greater precision of calculation the diurnal parallel of Mars and augmenting Brahe's table of oppositions with two more observations of his own (1602 and 1604), Kepler obtained the initial data for verifying his hypothesis about the circular orbit of Mars.

From the days of Ptolemy astronomers knew how to find the position of the line of apsides and the eccentricity of circular orbits. Assuming that the center of the orbit divides the eccentricity in half, three observations were necessary. The solution was obtained by a simple (at least by contemporary standards) geometrical construction. In refusing to accept the hypothesis that the eccentricity is divided in half by the center of the orbit, Kepler made the problem more complicated. To find the value of the proportion, an additional (fourth) observation was needed. The method proposed by Kepler for finding a solution was also completely different from the

Ptolemaic one. Having selected from twelve oppositions four (1587, 1591, 1593, and 1595), Kepler obtained a geometrical solution for the problem of determining heliocentric coordinates.

Presenting his line of reasoning, Kepler notes: "If you, reader, have grown weary from the tiring method, then you have more reason to have sympathy for me, who has done all these calculations at least seventy times, and who has spent on them a great deal of time. I think you will not be amazed when you learn that it is already the fifth year since I began to occupy myself with the theory of Mars, although almost the entire year of 1603 had to be devoted to research in optics."

Kepler the mathematician, who saw in Euclid's Elements the ideal of mathematical structures, was not satisfied with the method because it was not "purely geometric."

Kepler the physicist was proud, and rightly so, of his remarkable accomplishment, and addressing a challenge to mathematicians, he wrote:

There will be artful geometers, men like Vieta, who, if they succeed in proving that my method is not sufficiently skillful, will consider it a special accomplishment of theirs. Vieta has cast, in connection with other problems, an analogous reproach to Ptolemy, Copernicus, and Regiomontanus. Well, these mathematicians can test their strength and find a geometric solution to the problem. I will consider each one of them a great Apollonius. I am satisfied that, proceeding from one premise (four observations and two hypotheses), I can construct four or five conclusions, i.e., find the way out of the labyrinth not with the light of a torch but with the aid of a simple thread (which, however, was in my hands from the very beginning). If it is difficult to understand a method, then it is even more difficult to investigate things without a method."

Having carried out a huge amount of computations, Kepler obtained the data he needed about the orbit of Mars. Thus were found the position of the line of apsides and the ratio in which the center of the orbit divides the full eccentricity (the distance between the center of the equant and the sun). After checking the results obtained on the eight observations which were unused, Kepler addresses the reader:

"You see, industrious reader, that a hypothesis based on methods presented above not only reproduces in the course of computation the four observations used in reaching a conclusion, but also conveys all the remaining observations to within two minutes."

It would seem that the desired objective had been accomplished, but a more rigorous check revealed the illusory nature of the success achieved and the triumph of the victor gave way to the bitterness of failure.

"Who could have thought that such a thing is possible!" Kepler exclaims. "The hypothesis which produced such close correspondence with acronical observations, nonetheless turned out to be erroneous as soon as one switched to the observations of the average or true sun. Ptolemy had warned us against this when he taught that the eccentricity of the equant must be divided in half by the center of the eccentric carrying the planet. I, however, along with Tycho Brahe, did not presume the division of the eccentricity into equal parts. True, Copernicus, too, was not afraid to overlook, from time to time, the division of the eccentricity into equal parts. But Copernicus utilized an extremely small amount of observations, assuming, apparently, that Ptolemy, too, utilized no more observations than those mentioned in his great work. Tycho Brahe succeeded in forging ahead only a little bit further than that."

What further is to be done? "... We have assumed that the orbit along which a planet revolves is an ideal circle and that, besides, there exists on the line of apsides a unique point which is located a specific distance away from the center of the eccentric [which is, precisely, the center of the equant] and relative to which Mars, during equal periods of time, traverses equal angles. One of these two assumptions or both of them are erroneous, more likely both, for the observations I have used are correct." One cannot but admire the merciless logic of Kepler.

Kepler undertook one more attempt to save the circular orbit of Mars with the aid of an old hypothesis according to which the eccentricity of the eccentric is equal to the half of the full eccentricity. However, the computed positions of Mars at eight points failed to coincide with observations, although the deviation was, after all, rather insignificant—about 8′. Having discovered this, Kepler wrote a classic discourse concerning the exactness of theory and experiment.

"Such a small deviation—eight minutes—is the reason why Ptolemy resorted to the division of the full eccentricity in an equal ratio, for the mistake resulting from the division of a full eccentricity in half ... is at most 8′, and this is for Mars, which has the greatest eccentricity. In the case of other planets, the error is even less. Ptolemy says himself that the precision of his observations nowhere exceeds 10′, i.e., 1/6 of a degree. Thus, imprecision, or, as they say another way, permissible deviation in observation, exceeds the error rate of Ptolemaic computations.

"We, however, have been given by Divine Goodness so diligent an observer as Tycho Brahe. His observations exposed the error of Ptolemaic observations so that we could with gratitude recognize this good deed of God and use it, i.e. proceeding from the proven mistaken nature of the hypotheses, we have taken upon ourselves the labor of finally finding the true form of celestial motions. This is the road I wish to travel to give example to others. If one could overlook these 8′, I would have long ago perfected the hypothesis regarding the circular orbit of Mars I formulated in Chapter XVI by introducing into it the division of the full eccentricity in an equal ratio. However, inasmuch as one cannot overlook the aforementioned mistake, these 8′ pointed the way to the reformation of the whole of astronomy. They have served as a building material for the greater part of my work."

So, although the hypothesis that Mars had a circular orbit did permit one to compute the heliocentric longitude of that planet rather precisely, it proved to be inconsistent, despite all the tricks resorted to by Kepler.

"Thus," Kepler concludes, "the edifice, which we have erected using only Tycho's observation, had to be destroyed by us using other observations by the same man. We had to endure this punishment, because in following our predecessors we have accepted hypotheses which seemed plausible but were in reality false."

It was necessary to look for a new approach in establishing the form of the Mars' orbit, and Kepler knew how to find it by selecting a round-about way—through a definition of the form of the Earth's orbit. Kepler hoped to obtain the outline of Earth's orbit by selecting on it a sufficiently dense network of points and calculating the respective distances from these points to the sun. "If in this way one will succeed in finding the key," Kepler wrote, "then the way toward everything else will be open."

And again the physical ideas come to the forefront. Back in Chapter 22 of his Cosmographic Mystery, having expressed possible causes for the non-uniform motion of the planets, Kepler made a special metnion of the fact that his physical considerations, if proven correct, were in equal measure applicable to all planets, including the Earth. In pre-Keplerian theories the earth and the sun have occupied "privileged" positions. It was considered that they uniformly revolved along the so-called simple eccentric (i.e. along the circumference), creating only an illusion of slowing down when approaching the perihelion and of speeding up when approaching the aphelion. The other planets, however, whose equants were considered not to coincide with the center of the eccentric, indeed had to speed up when approaching the sun and slow down when moving away from the sun.

Kepler, in his words, "got the suspicion" that the earth too is similar to the remaining planets.

The method by which Kepler refuted the authorities and proved the existence of the eccentricity of the equant for the earth's orbit too is exceptionally clever. Using our present-day terminology, one could say that, in order to compute the distance between the earth and the sun, Kepler used the method of triangulation. Kepler formulated the task before him in the following manner:

"I intend to find three or more observations of Mars in which the planet is located in one and the same point of the eccentric and trigonometrically calculate the distances from these points of the epicycle or the orbit to the equant. Inasmuch as the circumference is defined by three points, I will establish by three observations the position of the circumference and its line of apsides and will also compute the eccentricity of the equant. If there will be yet a fourth observation, it can be used for verification."

Having selected from Brahe's observations positions of Mars which differed each from the other by a whole sidereal period (the sidereal period of Mars, equal to 687 days, was well known), Kepler obtained the basis of triangulation, the segment SM, where S is the true sun and M a point to which Mars returns after complete revolutions. Angles at which the segment SM is seen from the earth were known from observations. It was possible to calculate angles at the vertex S (from the sun) with the aid of a theory of the sun developed by Brahe. From the Mars—earth—sun triangles Kepler found the distance between the sun and the earth, using the distance between the sun and Mars as the unit of measurement; after which he had no difficulty calculating the radius of the circumference (the orbit of the earth, the direction of the line of apsides) and, finally, to achieve the goal of the entire undertaking, which was to determine the distances from the center of the equant and of the sun to the center of the earth's orbit. Both distances turned out to be the same and equal to .018 of the radius of the earth's orbit. Thus, Kepler's idea about the "equality" of the earth with other planets (more precisely, the idea that the earth is not different from other planets) had been proven true.

The result, however, was too important without a rigorous verification of its correctness or on the basis of a single proof. Kepler verified it by different methods.

Only after this did Kepler consider the result obtained as being reliably well-founded and, having used it for the purposes of compiling an extensive table of distances from the earth to the sun (with 1° interval), turned to his favorite topic, the discussion of the physical causes of the motion of planets.

It is precisely at the point in time where one of his guesses (the one about the "equality" of the earth with other planets), which, in the final analysis, was also based on physical considerations, received such brilliant substantiation that Kepler returned to an idea long nurtured in his head and dealing with the connection between the distance from the planet to the sun and the speed of the planet, and he developed his ideas about the forces that cause the planets to revolve around the sun.

In the beginning Kepler discovered that, in the vicinity of apsides, the time it takes a planet to traverse an arc is proportional to the length of a radius-vector drawn from the sun, and he then applied this regularity to other sections of the orbit. Of course, a theoretician so attentive to observations as Kepler did not miss the fact that, outside of aphelion and perihelion, the ratio he had observed is realized only approximately. Proceeding from a partial regularity established on the basis of observation, Kepler came to the conclusion that "the force moving the planets is concentrated in the sun." Kepler's line of reasoning reduces to the following:

"It was proved in the preceding chapter that the time necessary for a planet to traverse equal parts of the eccentric (or equal segments of distance in the celestial space) is proportional to the distance from these segments to a point from which the eccentricity is calculated, or, simply put, a planet revolving round a point assumed to be the center of the universe is less subjected to the action of the force the farther it moves from that point. From here it follows necessarily that the cause of such a weakening lies hidden either in the body of the planet itself and the motive force dwelling within it, or in the point taken to be the center of the universe.

"A commonly accepted axiom of all of natural philosophy says that if some phenomena occur simultaneously, in the same fashion, and always on the same scale, then one of them is either the cause of the other or they are brought about by the action of a cause common to both. In our case, as a planet approaches the center of the universe or moves away from it, there is always a coincidental proportional acceleration or deceleration of the planet's motion. From this I conclude that either the deceleration of the motion is the cause of the planets moving away from the center of the universe, or, conversely, the movement away from the center of the universe causes deceleration, or both phenomena have a common cause."

Having conscientiously examined all three possibilities, Kepler came to the conclusion that in reality only the second one comes about. "Thus," he sums up his reasoning, "there remains only the assumption according to which the cause of deceleration and acceleration of the motion lay hidden ... in the point accepted to be the center of the universe....

"Consequently, inasmuch as with increasing distance from the center of the universe a planet decelerates with increasing distance and accelerates with decreasing distance, the source of the motive force must necessarily lie at the point which we took to be the center of the universe. If such a hypothesis is accepted, then the cause of the observed phenomenon becomes clear: it follows from the hypothesis that the motion of the planet is subject to the law of the lever. Indeed, a planet moving under the influence of a force emanating from the center of power will be heavier (and therefore move more slowly) the less the distance from it to the center. The same occurs when I say that the weight is greater the farther it is from the fulcrum, not by itself, but because of the lever action which is proportional to the length of the arm. In both cases (here the lever and the weight, there the motion of the planet), the force decreases in inverse proportion to the distance.

"... And since the sun is situated in the center of the planetary system, it follows from what has been proven that the seat of motive force is in the sun, for such a seat of power must be located precisely in the center of the universe."

Kepler's conception of the sun as being the source of motion was subsequently made more precise and was supplemented. Kepler saw a very close analogy between properties of the force emanating from the sun and the properties of light, and with the aid of this analogy he once more substantiated the law he had derived from general considerations, and according to which force decreases with distance. The carriers of force, in Kepler's opinion, are special non-material particles released by the sun. For planets to move along the circle, these particles which impart motion to them must themselves move along circles. Consequently, Kepler concluded, the sun, which is the source of motion, must also revolve around its axis. Kepler explains the nature of the sun by means of an analogy comparing it with a magnet.

Kepler presents, in the form of the following six "absolutely correct" axioms, the sum of his reflections on the nature of forces acting on planets and governing their motion.

"1. The body of a planet, where it can be considered isolated from other bodies, tends, by its nature, to remain at rest."

"2. Force emanating from the sun moves the body of the planet from one place on the Zodiac to another."

"3. If the distance from the planet to the sun were to remain constant, the planet would revolve along a circular orbit."

"4. If some one planet were to make two complete consecutive revolutions, each at different distances from the sun, then the periods of revolution would be in the same ratio as the squares of distances or the lengths of circumferences.

"5. The force hidden in the planet itself is insufficient for the planet to move from place to place, for it has neither legs, nor wings, nor fins which it could use to lean against the heavenly ether.

"6. Nonetheless, a change in the distance from the planet to the Sun is conditioned by the force possessed by the planet itself."

Finally the moment came to "translate these speculative conclusions into the language of numbers." It is here that Kepler made one of the discoveries that immortalized his name: he formulated the so-called second law of motion of the planets which now carries his name (chronologically speaking, the second law was discovered before the first).

Again (how many times!) the road to discovery lay through tiring calculations.

"Inasmuch as the periods of time which a planet requires to traverse equal sections of the eccentric stand in the same proportion to one another as do the distances from these sections to the center of the Universe, and these distances change from one point on the semicircle to the next, it cost me no small labor to find in what manner one can obtain the sum of individual distances. Indeed, without knowing the sum of all the distances (and there are infinitely many), it is not possible to indicate the time interval corresponding to a given distance, for any part of the sum stands in the same proportion to its respective period of time as does the entire sum to the entire period of revolution.

"Therefore, I began by dividing the entire eccentric into 360 equal parts (considering them to be, as it were, the smallest) and assumed that within the limits of each such part the distance to the center of the universe does not change."

The table of distances was prepared beforehand. It only remained now to sum them.

"Then," Kepler continued, "I took the period of revolution equal to 365 days and 6 hours, rounded it off, and decided that it must correspond to 360°, or the complete circumference, which signifies to astronomers the average anomaly. I have assumed the ratio of the sum of the distances to the sum of the times to be equal to the ratio of any individually taken distance to the respective time period. Finally, I summed up all the results obtained and carried out all the computations for every degree."

It was extremely difficult to carry out such computations and Kepler, by his own admission, "began to look for a method that would make computations easier."

"Inasmuch as I knew that the eccentric consists of an infinitely large number of points, corresponding to which there are infinitely many distances," Kepler says, "it came to me that all those distances are contained in the area of the eccentric. I remembered that once Archimedes, in striving to find the ratio of the length of the circumference to the diameter, divided the circle in exactly the same manner into infinitely many triangles. Precisely in this lies the innermost meaning of indirect proof proposed by Archimedes. Instead of dividing the circumference, as before, into 360 parts, I divided into as many parts the area of the eccentric circle, drawing rays out of the point from which the eccentricity is calculated...." After this was accomplished, it only remained for Kepler to assume that the area, the sum of the radius vectors, was proportional to time.

Kepler concludes his research into the second inequality with the following words:

"Thus, I have, with the aid of quite reliable observations and proofs, precisely described the cause and measure of the second inequality which, right before our eyes, causes the planets to stop, retrograde, or continue their motion. I have shown that the second inequality affects the first inequality as well, and that the theory of the sun or the earth (according to Copernicus) or that of the epicycle (according to Ptolemy) is in all respects similar to the theory of other planets. I have also found the physical cause of the first inequality and have used this in computing the orbit of the sun."

Having made the orbit of the earth more precise, Kepler could again return to his main task, the establishment of the form of the orbit of Mars.

After selecting several groups of observations (three observations to a group), Kepler computed for each of the observations, on the assumption that the orbit is circular, the parameters of the orbit of Mars—the position of the line of apsides, and the magnitude of the eccentricity. The scatter of the obtained values exceeded by a great deal both the observation errors and the computation errors. "Just how erroneous all these results are," Kepler concludes, "can be seen from the fact that some different result was obtained every time."

Turning to observations, Kepler compares them with distances from Mars to the sun, calculated in accordance with the circular-orbit hypothesis, and arrives at the following results:

Date of observation	Angular distance from the aphelion	Distance calculating according to circular-orbit hypothesis	Distance from observations	Difference
31 October 31, 1590	9°37′	1.66605	1.66255	0.00350
31 December 31, 1590	36°43′	1.63883	1.63100	0.00783
25 October 25, 1595	104°25′	1.48539	1.47750	0.00789

From the preceding Kepler concludes: "The orbit of the planet Mars is not circular in form. The orbit gradually deviates inward on two sides and then returns toward a circular form in the perihelion. It is customary to call an orbit of such form an oval." Then the oval was replaced by an oviform orbit with the blunt end turned toward the aphelion and the pointed end turned toward perihelion (the oviform oval is tangent to the circle at the aphelion and perihelion).

Kepler not only attempted to prove the deviation of the orbit from circular empirically, making references to the observations of Brahe, but also to substantiate it proceeding from the "physical" ideas he had developed about two forces acting on a planet, one of which (the one emanating from the sun) causes the planet to revolve uniformly around the sun and the other (hidden within the planet itself) which forces the planet to move uniformly along a small epicycle thereby causing periodic changes in the distance between the planet and the sun. However, all of this reasoning proved to be faulty and the first to recognize this was Kepler himself:

"As soon as it became clear from the reliable observations of Brahe that the orbit of a planet is not an exact circle but is rather flat on two sides, it seemed to me that I knew the natural cause of such a phenomenon. For I have investigated such questions in great detail in Chapter 30. I would like to ask the reader that, before going on, he carefully reread that chapter. Indeed, in it I have attributed the eccentricity to a certain force hidden within the planet itself, from which it followed that I ascribed the deviation of the planet from an eccentric circle also to the same force. However, what happened to me could not be described better than by the old proverb, "haste makes waste."

What in the beginning appeared to be sufficiently convincing proof turned out to be nothing more than plausible reasoning. Observations had brought to naught all of the artful constructions. Kepler had to begin from the point which he succeeded in attaining without resort to "physics," from the determination of the shape of the orbit point by point. Although each step was possible only at the cost of tremendous computational difficulties, Kepler did not lose his spirit. More than that, he had the strength to be ironic about his (temporary, to be sure) reverse:

"At the time I was drunk with the triumph over the motions of Mars as if the latter were already conquered, thrown in the jail of tables, and fettered by the equations of the eccentric, from different places there began to arrive reports that the victory was only ephemeral and the war flared up with new force. Within the walls of my house, the enemy, whom I had considered my prisoner, tore the fetters of equations and broke out of the jail of tables, for not a single one of the geometric methods... could compare in terms of precision with the temporary hypothesis[8)] (which leads to correct equations although it is based on incorrect assumptions). Outside, the spies positioned along the entire orbit (I have in mind here the true distances) had overpowered the troops of physical causes I had called forth in Chapter 45, and they had overthrown their yoke and had again found freedom. A little bit more and the enemy, who has escaped, would have joined the rebels, which would have led me to despair. Without wasting a minute, I secretely sent out support troops—regiments of new physical causes, and have with all haste reconnoitered in what direction the fugitive departed and began to pursue him, hot on his heels."

So, the "physical reasons" why the orbit of Mars had to be oval, have, as Kepler put it, "gone up in smoke." But did this mean that one couldn't extract any information about the true shape of the orbit from the hypothesis that the orbit is oval? No. Comparing distances computed with the aid of the temporary hypothesis with the distances computed in accordance with the oval-orbit hypothesis, Kepler became convinced that, when far from aphelion and perihelion, the first set of distances invariably turns out to be longer, and the second set shorter, than the distances obtained from observations. From this Kepler draws a conclusion: "The truth lies somewhere in-between."

Kepler's attention was attracted by the lunes between the circle and the inscribed oval. Taking the radius of the circle to be 1, he found that the width of the lunes was 0.00429. Kepler could not get this number off his mind. Again and again he returned to the thought that, in essence, "nothing much is said in Chapter 45, and that precisely therefore the triumph over Mars proved to be a chimera." The solution came, as always, unexpectedly. Kepler calculated the angle (let us call it α) at which the observer, located on Mars, would see a segment from the center of the orbit to the position of the sun. In order to understand Kepler's further search, we shall look at the ellipse. The segment under discussion is the distance from the center of the ellipse to its focus. The width of the lune is the difference between the semimajor axis a and the semiminor axis b (expressed in units of the semimajor axis) so that (a − b)/a = 0.00429. All that we have to know in addition is that the distance from the focus to the end of the minor semiaxis is also a. Kepler noticed another fact which seemed strange to him, namely, when the angle α is at its maximum, its cosecant equals 1.00429, differing exactly by unity from the previously calculated width of the lune. We, however, know that this cosecant equals the ratio of the distance to the planet when it is located at the end of the semiminor axis (when α is maximal) to the length of the semiminor axis:

$$\operatorname{cosec}\alpha = \frac{a}{b} = \frac{a}{a-(a-b)} \approx 1 + \frac{a-b}{a}.$$

This is precisely this that Kepler discovers. The coincidence amazes him. One can only wonder what intuition of a naturalist one must possess to discern a profound law behind these numerical curiosities. For Kepler no coincidences are accidental. In analyzing the correspondences between the angle and the distance for other positions, he comes to equation of the ellipse in the form called to this day the ellipse equation (a = 1):

$$r = 1 + e\cos\beta.$$

In this equation β is the eccentric anomaly, i.e. the angle between the radius vector drawn from the center of the ellipse and the major axis, and e is the eccentricity. As the result of the preceding, Kepler found sufficient reasons for the final rejection of the circular motion of the planets and for the establishment of a new law.

But can one consider the results substantiated if they have not been supported by physical considerations? Of course not. And Kepler again indulges in complicated reasoning to prove, with its aid, that the result is inevitable. Only a short distance remained in the great voyage to the shores of New Astronomy, but even the last leg of this voyage didn't come easy.

"The closer one comes to Nature," Kepler reflects, "the more she frolics and plays pranks with those who want to catch her, to flee him the moment she is almost caught." But the long wished-for shore was near. One more effort, and Kepler found the truth: "For the orbit of a planet there remains no other form except that of an ideal ellipse, for causes deduced from physical principles agree with ... the results of observation and the temporary hypothesis." Thus was discovered Kepler's first law.

The New Astronomy, which Kepler called Celestial Physics, ends with the verification of a new theory, the computation of the coordinates of Mars, which have not yielded to the efforts of Kepler's predecessors. The search for harmony brings Kepler rich finds and confidence in the fact that it is precisely he who is predestined to understand in the end the mechanics of the planets which were concealed from all others. Working frenziedly over the theory of Mars, filling hundreds of pages with computations (sometimes repeating them up to 70 times!), he never forgot about his intention to write a grandiose book, De Harmoniae Mundi, whose plan he presented in a letter to Herwart von Hohenburg as easly as 1599.

Work on Celestial Physics demanded not only patience but also new ideas; it cost Kepler tremendous effort and he simply had no free time to take up the new subject which concerned him very much; however, he never stopped thinking about it. One testimonial as to how unceasingly Kepler was pursued by the thought of searching for the harmony of the universe is contained in the admission which the conqueror of Mars let slip in a letter to the Englishman Heydon (in 1605): "Would but that God free me from astronomy so that I can concentrate all my thoughts on the work Harmony of the Universe!"

The Harmony of the Universe was completed only on May 27, 1618 (though the last, the fifth book of the great work, was subjected to significant revisions during the time of printing and its final version was completed only on February 19, 1619.)

If in Cosmographic Mystery we sense the romantic flight and youthful passion of the author, and if in the New Astronomy we are amazed by the refined intuition of the author allowing him to find the correct way through the labyrinth of observations, then in the Harmony of the Universe we see emerging Kepler the philosopher, concerned with finding the key to the structure of the cosmos, the super principle allowing him to grasp at one look the entire wealth of phenomena and to substantiate the common character of all the members of the solar system.

But it is precisely in this book that the weakness of Kepler's method came to light. Now we clearly see that numerical laws alone, that is to say the arithmetic of harmony, are insufficient for constructing the dynamics of the motion of planets. For this it was necessary to have Newton's equations. Kepler, of course, did not know this. He saw that one must find harmony not only in the geometric properties of the system of planets but also in the laws of their motion. His fantasy is as ever unrestrainable. Overcoming mathematical difficulties[9], he strove to solve the great problem for the solution of which he was predestined by fate.

The book is full of ideas and fantasies; however, it is very difficult for us now to follow the author's line of thought: today we know that his way was doomed to failure from the very start and, in advance, we treat with suspicion those discoveries which appeared valuable to him. But among these discoveries there was also Kepler's third law, whose value is undisputed. We shall not cite in detail everything that is written in the book. Nearly everything there is of no relevance to the third law. We shall "stroll" through the book, stopping to consider only some of its noteworthy aspects.

The lofty objective of the book required a special manner of presentation. Inasmuch as the contents dealt with the uncovering of the innermost plans originated in the course of Creation, all conclusions had to be logically flawless. Therefore Kepler decided to follow the manner of presentation in Euclid's Elements, which mathematicians had considered for many centuries to be the ideal of mathematical rigor.

For Kepler, who saw in geometry "the prototype of the beauty of the world," it was natural to look for causes of harmony not in numerical ratios as was done by the Pythagoreans, but to view the properties of numbers only as a reflection of geometric figures that stood behind them. The leading idea of the entire book is the universal nature of the harmony of the universe, and the role of mathematics in learning this harmony are clearly formulated in the epigraph from Kepler's beloved author of antiquity, Proclus Diadochus, which introduces the first book:

Mathematics makes its greatest contribution in the study of nature by allowing us to discover a harmonious system of ideas in accordance with which the universe is built ... and to represent simple elements, which the heavens are founded on, which assume in different parts appropriate forms in all their harmonius and balanced unity."

In his research on harmonic proportions Kepler used, in many regards, Book X of Euclid's Elements, augmenting Euclidean theory of irrational numbers by their classification according to various degress of "constructibility." (It is curious to note that the very term "irrational" was considered by Kepler to be incorrect and he replaced it with the term "inexpressible.")[10]

Kepler tells in detail in the introduction to Book I of the Harmony of the Universe about his attitudes toward Euclid:

"When I saw that true and authentic differences between geometric figures, from which I intended to deduce the causes of harmonic proportions, are usually completely unknown; when I saw that Euclid, who subjected them to investigation, had been reduced to confusion and suppressed by the malicious criticism of Ramus,[11] that he had been muffled by the cries of arrogant ignoramuses, and that either nobody listens to him or he speaks about the mysteries of philosophy to the deaf; when I saw that Proclus, who has made Euclid accessible to understanding, who has extracted what was hidden in broad daylight, and who has managed to make the most difficult places easily understandable, now serves as the object of scoffing and his commentaries reach no further than the tenth book, it became clear to me what had to be done. I saw my task, first of all, as copying from the Tenth Book of Euclid everything that was particularly important for the plan I have conceived; then, with the aid of certain classification, to put Euclid's ideas in a clear order and indicate reasons for which Euclid overlooked a given part of this sequence, and, finally, to investigate the figures themselves. Inasmuch as what was involved was a quite clear exposition of Euclid, I restricted myself to only giving formulations of respective theorems. Much of what Euclid proved in other ways I had to present anew inasmuch as I pursued a quite definite goal, to compare the constructible and the non-constructible figures. I connected that which was disjointed and changed the order. For the convenience of reference I introduced continuous numbering of definitions, propositions, and theorems as I did in Dioptrics.[12] I did not strive to be particularly precise in the lemmas and was not very much concerned about expressions, for to a greater degree I was concerned with the subject itself acting not as a mathematician in the domain of philosophy but as a philosopher in this area of mathematics."

Next follows an analysis of the properties of geometric figures and their most important property is declared to be the rationality of the ratio of the lengths of their elements and the possibility of constructing them with the aid of a compass and a ruler. This property is made the basis for classifying dividing polygons as those that are constructible and those that are not. Kepler says that "the discussion here concerns very important things, for this is precisely the reason why God did not use a heptagon and other figures of this sort for adorning the world as distinguished from the 'constructible figures which were introduced' earlier."

However, there were infinitely many "constructible" figures and to select from among them a finite number of figures with which one could "substantiate" harmonic proportions was impossible.

Having become convinced of that, Kepler attempted to distinguish figures according to a new property which he called congruence. The Second Book of Harmony of the Universe is, indeed, entitled "Congruence of Harmonic Figures." Kepler defined congruence as the filling of a plane by geometrical figures or the construction of polyhedra from plane figures.

In investigating plane congruences Kepler was one of the first to solve the problem of parquets, i.e., the complete covering (with no uncovered gaps and overlapping) of an area by figures, both by those that are the same and by those of different forms and sizes.

However, it was in investigating three-dimensional configurations that Kepler made his main mathematical discovery. If one is to allow the faces of a regular polyhedron to intersect one another, then one can add to Plato's five regular polyhedra six more having the form of a star. Two of them were discovered by Kepler and are called Keplerian to this day. His contemporaries did not appreciate this discovery of Kepler's. It was forgotten and re-discovered only in the beginning of the last century by the French mathematician Poinsot.

The number of congruent figures turned out to be finite and they all belonged to the class of constructible figures. This appeared to be a good omen.

Now Kepler had to extract from the numerical ratios characteristic of these figures those which could be taken to form the basis of harmony. Initially Kepler took up music, later going on to the planets. But it was already clear to him at that point that the study of the periods of revolution of the planets is the area in which he must apply the theory he was creating.

Book III of the Harmony of the Universe, which is called "The Origins of Harmonic Ratios and Also the Nature and Differences of Musical Intervals", is devoted to the search for harmonic ratios. In the preface to this book Kepler relates the legend which ascribes the discovery of harmonic ratios to Pythagoras: "They say that Pythagoras, passing by a smithy and hearing the harmonic sounds emitted by the hammers, was first to discover that the difference in tone was related to the size of the hammer: the big hammers emit a low tone, the small ones the high tones. Inasmuch as what is called a ratio is connected with numerical quantities, he measured the hammers and without difficulty found the proportions corresponding to harmonic or dissonant, melodic or non-melodic, sound intervals. From hammers Pythagoras went to the lengths of strings for which the ear indicates more precisely which part produces consonance and which part produces dissonance."

Attempts to construct the entire musical scale from some basic intervals were undertaken before Kepler's time as well. Thus, for instance, Plato in "Timaeus" arbitrarily took three intervals as basic: the octave, the fifth and the fourth. Kepler, with his stand on primacy of geometric figures, had before him a much more complicated problem to solve: not only to point out a finite number of intervals with which one could build all the remaining consonances but also to derive basic intervals from the properties of geometric figures.

Having completed computations comparable in volume only to those connected with the empirical determination of the orbit of Mars or of the earth point by point, and having looked over many variants, Kepler (in mid-August 1599) obtained seven harmonic intervals (consonances): the octave (which has 1 : 2 frequency ratio), the major sixth (3 : 5), the minor sixth (5 : 8), the fifth (2 : 3), the fourth (3 : 4), the major third (4 : 5), and the minor third (5 : 6), and derived from them the entire scale, the major and the minor tone, and so on.

"I have first found these seven divisions of the string," Kepler explains, "guided by the ear in number equal to the number of harmonies within the limits of a single octave and only then, not without difficulty, have I deduced, from the deepest foundations of geometry, the causes for individual divisions and of their entirety."

Musical harmony provided Kepler with convenient terminology. However, important as the musical harmonic intervals may be, they represent nothing more than a concrete realization of abstract relationships, which are harmonic in the true sense of the word. The carrier of such "pure" harmonies is the ideal circle and its subdivisions. In Chapter 1, entitled "Concerning the Essence of Sensory Harmonic Ratios as well as Harmonic Ratios Accessible Only to the Spirit," Kepler presented in detail his understanding of harmony:

"It is necessary to distinguish sensory harmonies from pure harmonies, which are analogous to them but which are deprived of everything sensory. The first are numerous and have various carriers. The pure harmonies, however, which lack sensory carriers, are always the same. For example, the type of harmony that arises from a double proportion is always the same. If it is expressed in sounds, it is called an octave; if it is found in angles between two rays, then it is called the opposition. Moreover, in the musical system an octave can be high or low, the harmony can be that of human voices or of sounds emitted by musical instruments. Manifestations of this harmony in meteorology are just as varied; an octave can be the opposition of Saturn and Jupiter or some other pair of planets, it can be observed between the signs of the Zodiac near the points of equinox or between the signs of the Zodiac near solstices."

Yet another realization of harmonies are the so-called configurations which Kepler defines in the following manner:

"Definition I. The word 'configuration' signifies the angle at which rays of some two planets arrive at the earth (considered as a point), or, which is the same, the arc of the great circle passing through the Zodiac, which (arc) serves as the measure of the aforementioned angle ....

"Definition II. A configuration is called efficacious in case the angle between the rays emanating from the two planets possesses special properties to excite animate beings by virtue of their sublunar nature and limited abilities, in such a way that during the time of such a configuration these beings devlop heightened activity."

The angles between the rays (the "aspects") arriving at the earth from luminaries, in Kepler's opinion, influence the weather. In order to sense the aspects, the earth has to be an animate being, etc.

We shall not stop to investigate the astrological views of Kepler and will limit ourselves to a list of the "efficacious aspects." Kepler has eight of them: the conjunction (1 : 1), sextile (1 : 6), quadrature (1 : 4), trigon (1 : 3), opposition (1 : 2), quintile (1 : 5), trioctant (3 : 8), and biquintile (2 : 5). The chief realization of harmonies is in celestial motions; to it Kepler devotes the fifth book of the *Harmony of the Universe* under the title "The Most Perfect Harmony in Celestial Motions and the Origins of Eccentricities, Radii of Orbits and Periods of Revolution Connected with It."

There is the unfeigned joy of a man who has successfully completed the work of his entire life, who has attained the desired objective toward which he has strived for many years, resounding in a colorful, temperamental, purely "Keplerian" foreword.

"The things about which I only guessed 25 years ago, before the discovery of the five regular solids between celestial orbits; the things of which I was convinced before reading the manuscript of Ptolemy about harmony; the things which I promised my friends, having selected the title of this book before the subject itself was clear to me; the thing which 16 years ago I proclaimed as being the goal of my research in one of my books;[13] the things which drove me to dedicate the best part of my life to astronomical research, to find Tycho Brahe, and to select Prague as my place of residence, ... I have, finally, brought out for judgement.

"... Now, 18 months after the first gleam of daylight, 3 months after a bright day, only a few days after the bright sun of a most highly remarkable spectacle rose, nothing can stop me. I give myself up to sacred ecstasy. Unafraid of the mockery of the mortals, I confess openly. Yes, I have plundered the golden vessels of the Egyptians in order to construct a sacrificial altar for my God far from the borders of Egypt. If you forgive me, I will be happy. If you condemn me, I will bear it. The die has been cast. I have written a book either for my contemporaries or for my posterity; it is all the same to me for whom I have written it. Let the book wait for its reader for hundreds of years: after all, God himself has waited for 6,000 years before a witness has appeared."

Two major ideas form the basis for the Keplerian scheme of the structure of the universe, two principles: a geometric one (the number of planets and distances between them are determined by five Platonic regular solids) and a harmonic one, which governs the eccentricities and periods of revolution. The geometric principle has been presented in detail in *Cosmographic Mystery*, and the first chapter of the astronomical (fifth) book of the *Harmony of the Universe* basically follows the work of Kepler's youth. True, when speaking about the family of regular solids, Kepler makes mention of his new discovery, the star polyhedron, which was previously unknown ("... This family contains also a three-dimensional star which can be obtained if the facets of a dodecahedron are extended to intersect in one point"), but there are no basic changes.

Chapter 2, "Concerning the Connection Between the Harmonic Ratios and Five Regular Solids," was intended by Kepler's design to show that the two principles are not exclusive but rather complement each other. In this chapter mention is made in particular of the following:

"... This connection is quite varied; however, it is basically of four types. It can be seen in the superficial forms of regular solids, or in ratios which arise when one constructs their facets, which are also harmonic, or in ratios of solids already constructed viewed separately or together, or, finally, in ratios which exactly or approximately correspond to the ratios of spheres inscribing or superscribing respective regular solids."

However, although the harmonic ratios have been found, their carrier in celestial motions remained, as before, unknown. Before embarking on a search for the mysterious carrier of celestial harmonies, Kepler considered it necessary to remind the reader of the "basic information necessary in considering celestial harmonies"; to present in 13 theses the state of the astronomical science of his day, and to prepare the ground for further research. This amazing essay opens with a presentation of ideas of Copernicus and Brahe. "First of all, the reader should know that the old astronomical hypotheses of Ptolemy in the form they are presented in Peuerbach's Theoriae [Theories] and in the textbooks of other authors are completely excluded by us from consideration and will not be taken into account, for they incorrectly represent both the positions of celestial bodies and their motions.

"The place of old hypotheses can only be taken by the teachings of Copernicus concerning the structure of the universe, of whose correctness, I would like, if this is at all possible, to convince all the people. Regrettably, the majority of people, when discussion centers on the study of the universe, consider this to be something new. It rings very uncommon to their ears to hear that the earth should be one of the planets and, like the other luminaries, it should revolve around the sun. Therefore everyone for whom the novelty of Copernicus' teachings is a stumbling block must know that the discussion of harmonies that follows is true also for the hypotheses of Tycho Brahe. Indeed this master does not differ with Copernicus as far as the position of heavenly bodies and the explanation of their motion are concerned. It is only that he relates the Copernican annual motion of the earth to the entire system of planetary orbits, including the sun, which by the unanimous opinion of both masters is the center of the entire system."

Following introductory comments (concerning the fact that "all planets, except the moon, revolve around the sun" and that the distances from respective planets to the sun do not remain unchanged, etc.), in the fifth thesis Kepler sums up his research on the theory of Mars:[14)]

"In the fifth place, in order to obtain motions between which harmonies arise, I would particularly like to draw the reader's attention to the things I succeeded in establishing in New Astronomy on the basis of the highly reliable observations of Tycho Brahe. A planet traversing equal arcs of the eccentric corresponding to a period of time, for example a twenty-four hour period, will not do so with the same speed. Unequal intervals of time corresponding to equal arcs of the eccentric are proportional to the distances to the sun, the source of motion. Conversely, if one takes equal time intervals ... then corresponding true arcs of the eccentric orbit are inversely proportional to the distance from the arc to the sun. Furthermore, I have shown that the orbit of the planet has the form of an ellipse and that the sun, the source of motion, is situated in the focus of this ellipse. From this it follows that the planet attains its average distance from the sun between the greatest distance in the aphelion and the least distance in the perihelion, when it covers one fourth of its orbit from the aphelion...."

The novelty begins when Kepler comes to the eighth thesis. Here he formulates his famous third law of the motion of planets. This time Kepler's creative laboratory remains concealed from the reader. We do not see the painful search for the truth, we are not the witnesses of numerous verifications. Kepler strives toward his main goal of fathoming the harmony of the Universe and the third law, for all its importance, remains in the eyes of Kepler only a means toward attaining an end. A little history and the formulation is all that Kepler reports about his discovery.

"... It [the true ratio between the periods of revolution and the sizes of orbits] came into my head on March 8 of this year [1618] when I needed to make certain dates more specific; however, my hand was not successful and I rejected my guess as erroneous. Finally, on May 15, the same thought again occurred to me and, with the second attempt, it dissipated the gloom of my spirit. Between my work of seventeen years on the observations of Tycho Brahe and my present reflections there developed such a complete accord that I began to think that I was dreaming about all of this and that I was taking what I desired for what was real. However, it has been positively and exactly established that the ratio between periods of revolution of any two planets is exactly one and one half times the ratio of their average distances...."

The new weapon was immediately used: in theses 11, 12 and 13, Kepler finds, with the aid of the third law, the relationship between the distances from the sun to the planets in aphelion and perihelion and their greatest and least velocities, and he also determines average velocity from the extremal ones. But it was not this law that was the most important one for the author.

Kepler formulated the main question as follows: "Where in the motion of planets did the Creator imprint the harmonic ratios, and how does this occur?"

After a long search he turned to the ratio of angular velocities of a planet in aphelion and perihelion: "The sun of harmony started to shine in all its brightness"—the ratio of the extremal velocities for the outer planets indeed turned out to be very close to being harmonic (Saturn—4 : 5; Jupiter—5 : 6; Mars—2 : 3).

Kepler considered that harmony arises not only out of the relationship of angular velocities in aphelion and perihelion of a single planet but also out of the relationship between the extremal velocities of two different planets, and he distinguished the two types of harmonies.

"There is a great deal of difference between the harmonies we have introduced for a single planet and the harmonies of two planets. The first cannot arise at some specific moment in time, for the latter this is quite possible. Indeed, if some planet is located in the aphelion, then it cannot simultaneously be in perihelion, which is opposite it. If, however, one is talking about two planets, then one of them can be in the aphelion while the other one is at the same moment in time in the perihelion. In this connection one can make the following analogy. Harmonies formed by single planets relate to harmonies formed by pairs of planets in the same way as the simple, or monophonic singing, called choral, which is the only one known to the ancients, relates to the polyphonic, the so-called figured singing, discovered in the last century."

"Thus, the celestial motion," Kepler concludes, "is nothing but the polyphonic music (perceived not by ear but by the intellect) which never stops for an instant."

The school of New Astronomy was not in vain. While bringing in the most varied proofs in support of the "harmonic theory," Kepler nonetheless did not miss the opportunity to turn to observations, to arrive with their aid at a final judgement. But here, too, the irrepressible phantasy does not abandon him.

Kepler sought to explain a minor discrepancy between, on the one hand, the angular velocities of planets in aphelion and perihelion (relative to an observer located on the sun) contained in tables and, on the other hand, theoretical harmonic relations, deduced from geometrical considerations, as being due to the fact that the heavenly sextet must sound equally in harmony in both the minor and the major keys, and to this end the planets must have the opportunity to tune their instruments.

Proceeding from the harmonic relations between the extremal values of angular velocities, Kepler, using the second and third laws, again calculates the elements of orbits.

As a result Kepler comes to the conclusion that "all the numbers are very close to the distances obtained by me from Tycho's observations."

An explanation was also found for the previously noticed discrepancy between distances computed in Cosmographic Mystery on the one hand and Brahe's observations and distances given by the harmonic theory on the other. The geometry of regular solids defines only the sequence in the position of the planets, then relinquishes its role of the structural principle to harmonic relations: "The geometric cosmos of the most perfect disposition [of regular solids] cannot coexist alongside the most harmonic cosmos."

We can now see that by far not everything was victory. The circle has, indeed, closed and Kepler returned to the point where he began his journey. The irony of fate lay in the fact that Kepler failed to appreciate his great discoveries. He wrote in the Harmony of the Universe about his third law without the usual details. He succeeded in forgetting about the second law altogether. In his search for harmony, Kepler tried to take from it more than it had to offer. But it was beyond his power to see harmony in his two superficially dissimilar laws. To accomplish this, one had to have the equations of mechanics.

After this Kepler did not return to celestial physics. No, he did not stop working. In his closing years he completed the publication of Tabulae Rudolphinae (Rudolphine Tables), a reference book which contains tables for computing the positions of planets and which also contains a catalog of 1005 stars (777 of which were taken from Brahe). His death (in 1630) interrupted work on Somnium (Dream), the first science-fiction novel about a flight to the moon. But not a word more was written about harmony. There were no more over-particular verifications, there were no new hypotheses. Kepler was tired.

"My brain grows tired when I try to understand what I wrote, and I have difficulties in re-establishing the connection between the drawings and the text which once I had found myself...."

This is how the drama came to an end. With Kepler's death his discoveries were forgotten. Even the wise Descartes knew nothing about them. Galileo did not find it necessary to read his book. Only with Newton do Kepler's laws find new life. But Newton was not interested in harmony. He had equations. New times have arrived.

[1]Hamlet was published in the same year Kepler met Brahe.

[2]He, who has no music within him, shall remain silent

[3]In a note to the second edition, Kepler explains that what was meant here was the orbit of the earth.

[4]The eighth sphere contained the fixed stars, while the ninth and tenth spheres were introduced to define the motion of the point of vernal equinox (in Ptolemy's system).

[5]This circumstance frequently escapes many authors. For Copernicus the motion of the planets did not require any kind of a cause and occurred (as we would say nowadays) under its own momentum. Therefore, for him the sun was not the center of energy and the position of the sun did not at all have to coincide with the center of the earth's orbit. The fact that the center of a planetary system was not associated with some material object did not appear strange to Copernicus. Only Kepler in reflecting on the source of motive force understood the role played by the sun. It is precisely this idea that helped him to gradually disentangle the mechanics of planets.

[6]These tables replaced the Alfonsine Tables which were in use since the 13th Century.

[7]Patrizzi is a medieval philosopher.

[8]Kepler called the hypothesis about the circular orbit of Mars, temporary.

[9]We did not mention that Kepler was in fact the first to effectively employ logarithms in his computations. The tables of logarithms which he compiled were still in use in the 18th century.

[10]"Translators into Latin render this term by the word 'irrational,' i.e. they use an expression which holds within it the danger of ambiquity and nonsense. We wish to put an end to the use of the term 'irrational,' for there are many segments of whose existence we are convinced by the most impressive of grounds, although these segments are inexpressible."

[11]Peter Ramus (Pierre de La Ramée) (1515-1570), a mathematician, perished in the Massacre of St. Bartholomew.

[12]The book Dioptrics was written by Kepler during his work on New Astronomy.

[13]Reference here is made to the Cosmographic Mystery.

[14]It is strange that he does not recall his second law here, returning instead to reasoning once rejected.

Translated by A. Andreyewsky