Problems of the theory of linear and nonlinear transformation of waves in inhomogeneous media

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The review is devoted to effects that occur when waves propagate in inhomogeneous media. The types of linear transformation of waves by the inhomogeneities of the medium are classified, and the conditions under which they are realized are indicated. Linear transformation of oscillations in a non-equilibrium medium, leading to wave amplification, is investigated. The linear transformation is consistently analyzed on the basis of a fourth-order differential equation. Attention is called to the singularities of linear wave transformation in a plasma-beam system, particularly the anisotropy of the radiation of transverse waves following passage of a modulated beam through an inhomogeneous plasma. The generation of the second harmonic of an electromagnetic wave in an inhomogeneous plasma and decay processes in inhomogeneous media are considered. Kinetic nonlocal effects (of the echo type) in an inhomogeneous plasma, wherein the focusing action of the inhomogeneity makes possible linear nonlocal effects as well as echo at the summary frequency are described. The indicated nonlinear and kinetic nonlocal effects lead to "transparentization" of the inhomogeneous media. Certain experimental studies dealing with the problems touched upon in the review are discussed.

CONTENTS

I. INTRODUCTION

Electrodynamics of inhomogeneous media has made appreciable progress in the last decade. This progress was due to consistent application of asymptotic methods to the solution of differential equations of second and higher order and to the solution of integro-differential equations. As a result, the electrodynamics of inhomogeneous media has led to qualitatively new consequences. The first pertains to natural modes of the oscillations of such media, and in the second to the eigensolutions. But as to the first consequence, notice should be taken first of the appearance of new oscillation modes that do not occur in homogeneous media, and most importantly, some of the oscillations build up, i.e., instabilities set in. An important division of electrodynamics was created, dealing with the instability of inhomogeneous media and the onset of turbulence. The advances made in this region have already been dealt with in a number of review articles (see, e.g.,[1-4]).

The second qualitatively new consequence of the electrodynamics of inhomogeneous media, to which the present review is devoted, is the coupling of the eigensolutions pertaining to different oscillation modes. The coupling of the eigensolutions leads to phenomena of linear transformation of waves in inhomogeneous media[5-10], and influences in turn the dispersion equation for the natural-oscillation frequencies[11-14]. In addition, significant changes take place here in the nonlinear and the kinetic phenomena (see, for example,[15,16]). Thus, for example, the peculiar interaction between modulated particle beams and waves in a plasma leads to an anomalous increase of the plasma transparency[16]. These effects are important not only from the scientific point of view, but also from the practical one, and are being intensively discussed in the literature of late.

Let us name certain applications of the already mentioned regular linear transformation of waves in an inhomogeneous plasma:

1) Transformation of electromagnetic waves into plasma waves (ionic or electronic) can be used to heat plasma to thermonuclear temperatures. This heating is possible in a stable laminar plasma if the wave field does not exceed a certain critical value[8,17]. This heating can be particularly effective in the microwave band for large-scale thermonuclear reactors[8,18] and can be quite useful when account is taken of surface phenomena in the problem of obtaining high temperatures by focusing laser radiation on a solid target[19].

2) Wave transformation can be used to improve communication, for example "transparentization" of an opaque barrier to waves radiated from a source.

3) The transformation of plasma waves into electromagnetic waves can serve as the basis for the development of plasma sources of electromagnetic radiation. We have in mind essentially the realization, under laboratory conditions, of the effect investigated in[7,20] in connection with the radio emission from the sun and planets, except that we would now be capable of deliberately controlling the plasma parameters by producing optimal conditions for the emission of transverse waves[17,21]. Even the first experiment[22,23] show that such an approach is fruitful.

4) Transformation of unstable modes into stable modes can be used to stabilize instabilities[17,24]. The preliminary experiments are in agreement with the theory[23].

5) Wave transformation in a plasma whose parameters vary with time can be used to transform waves in different frequency bands.

It should further be noted that the theory of oscillations of an inhomogeneous medium (plasma, ferromagnet, mechanical system), developed for the study of wave phenomena, can find interesting applications in the most varied branches of physics. Thus, for example, the theory of the "crossing" of solutions, which is used in the investigation of wave transformation, was initially developed in connection with the study of inelastic atomic collisions[25], where they are used successfully to this day[26,27]. The rapid progress in theory and experiment has led to the recent appearance of review papers on the theory of oscillations in an inhomogeneous plasma and an inhomogeneous liquid[9,13]. Let us consider these papers briefly.

The review[13] deals with the eigenvalue problem for perturbations of inhomogeneous plasma flow or liquid flow in the presence of resonance points, at which the phase velocity of the perturbation coincides with the velocity of the stationary flow. By effectively using the analogy between the indicated resonances and the resonant absorption of wave energy by individual particles (Landau damping), which is well-known in physics, the author of[13] analyzed from a unified point of view a number of classical and new results of the theory of instability of moving continuous media. The paper[9] is devoted to a fundamental investigation of theory and experiment in linear transformation of waves in an inhomogeneous plasma in the region of hybrid resonances, where the electric field of the wave (in the cold plasma) has a singularity. This review contains, in particular, the most complete experimental results on plasma heating with the aid of linear wave transformation, carried out at the Leningrad Physico-technical Institute with the present authors participating.

However, in spite of the availability of the mentioned review articles, as well as of such well-known monographs as[5] and[20], there is at present a noticeable lack of further reviews on the theory of wave propagation in inhomogeneous media. For example, in[9,13] there are no results on nonlinear and kinetic theory of an inhomogeneous medium, while the monographs[5,20], naturally, do not cover the results of the latest years in this direction. In addition, the review literature does not deal, for example, with linear transformation of waves in the case of non-equilibrium media, particularly with allowance for the mutual influence of the non-equilibrium character and the transformation, as a result of which a transformation coefficient larger than 100% can be obtained by using the energy of the non-equilibrium medium. Finally, the wave transformation in the region of hybrid resonances of an inhomogeneous plasma, which is investigated in[9], is only one case of transformation. *At the present time it is possible*, by using a fourth-order differential equation, to analyze different transformation cases from a unified point of view.

The present review is an attempt at partially filling the foregoing gaps in the review literature and in the monographs on the theory of wave propagation in inhomogeneous media, and also to discuss some of the latest experiments on the questions touched upon here. We study the case of regular oscillations. An exception is one example of linear oscillators, which is considered at the end of Sec. 2.

We investigate in the review the case of a weakly-inhomogeneous medium, the behavior of the waves in which can be studied by asymptotic methods (see, e.g.,[28-30] for the case of an abrupt boundary in passive media, and[31] for active media).

II. LINEAR TRANSFORMATION OF WAVES IN INHOMOGENEOUS MEDIA

1. Classification. We consider in the linear approximation different types of "crossing" of the wave solutions for an inhomogeneous media. Since the situation in which the properties of the medium vary little over distance on the order of the wavelength is quite common, we can use the WKB method (the geometrical-optics approximation). If the medium is inhomogeneous along the x axis, we choose the perturbation in the form $\varphi(\mathbf{r}, t) = \varphi(x)\exp(i\omega t - ik_y y - ik_z z)$. We then obtain for $\varphi(x)$ in the general case

$$\sum_{m=0}^{n} \beta^m u_{m+1}(x)\,(d^{2m}\varphi/dx^{2m}) = 0, \tag{1.1}$$

where β is a small parameter characterizing the weak inhomogeneity of the medium. It is assumed for simplicity that the functions $u_m(x)$ have no singularities in the region of interest to us and are well approximated by polynomials ($u_m \sim 1$, with the exception of the vicinity of the points at which they vanish). We have left out from (1.1) the odd derivatives, since in the applications of the theory they frequently contain derivatives of the functions u_m, which in turn yield an extra power of the small parameter β and therefore do not influence the results of the zero-order approximation. It is precisely in this case that it is particularly convenient to classify the different types of "crossing" of the solutions. We seek a solution of (1.1) in the WKB approximation in the form

$$\varphi(x) \sim \Pi(x)\exp\left(\pm i\beta^{-1/2}\int_{x_0}^{x} k(x')\,dx'\right),$$

where $\Pi(x)$ is the pre-exponential factor. The WKB approximation, however, does not hold near points at which either the wave vector $k(x)$ vanishes (turning point) or the wave vectors corresponding to different modes coincide, i.e., $k_s = k_p$ (crossing point of the solutions). Let us consider different cases of crossing of the solutions using the illustrative example of a fourth-order equation. It is convenient here to represent the expression for $k_{1,2}(x)$ in the form

$$k_{1,2} = (1/2)(k_1 + k_2) \pm (1/2)(k_1 - k_2),$$

and also introduce

$$\Delta(x) = u_2^2(x) - 4u_1(x)u_3(x). \tag{1.2}$$

At $\Delta = 0$, the solutions "cross." We can indicate the three most characteristic and qualitatively different cases of transformation[32].

a) Transformation of type I. In this case $u_m(x) \sim 1$ ($m = 1, 2, 3$) and the complex x plane contains complex-conjugate crossing points of the solutions. This type of crossing of the solutions is possible, for example, if $\Delta = x^2 + d^2$.

b) Transformation of type II. On the real x axis there is a point x_0 at which $u_2(x_0) = 0$ or $u_2(x_0) \approx 0$ but $u_1(x_0) \sim 1$. It follows from (1.2) that in this case the point x_0 lies between the two crossing points. Assuming from now on that $|u_3(x)| \ll 1$ (the most characteristic case), we see that in this case the crossing points of the solutions are in the immediate vicinity of x_0. An important example of this type is the transformation of electromagnetic waves into plasma waves, which was investigated theoretically and experimentally in the review[9].

c) Transformations of type III. The function $u_m(x) \sim 1$ ($m = 1, 2, 3$), and Δ can vanish only at one point x_0 (more accurately, from the vanishing of Δ at the point x_0 it does not follow that it is equal to zero at some other point). In the crossing region we have $\Delta = x - x_0$. Attention should be called to the fact that in case (c), unlike case (b), $u_2 \neq 0$ ($u_2 \sim 1$).

Let us examine in greater detail the indicated types of transformation.

2. Transformation of type I. The crossing of the indicated type, in the case when only $k_1 - k_2$ is multiply valued, was first investigated in[25] in a study of inelastic atomic collisions. The method of[25] was used in[33] to solve the problem of mutual transformation of ordinary and extraordinary waves in a magnetoactive plasma. The eigenvalue problem for this type of crossing was considered in[34]. The case when both function $k_1 \pm k_2$ are simultaneously multiply valued was first investigated in[35].

Certain features of this crossing are most conveniently considered using two coupled oscillators as an example. From the formal point of view, such a model differs little from the problem of wave transformation in an inhomogeneous medium. The equations of motion for the considered type are taken in the form

$$\ddot{x} + \omega_1^2(t)\, x = \alpha(t)\, y, \quad \ddot{y} + \omega_2^2(t)\, y = \alpha(t)\, x. \tag{2.1}$$

At constant $\omega_{1,2}$ and α, the equations for the normal coordinates become

$$\ddot{X} + \Omega_1^2 X = 0, \quad \ddot{Y} + \Omega_2^2 Y = 0, \quad (\Omega_1 \neq \Omega_2), \tag{2.2}$$

where Ω_1 and Ω_2 are the frequencies of the normal oscillations, defined by the expressions $\Omega_{1,2}^2 = (1/2)\{\omega_1^2 + \omega_2^2 \pm [4\alpha^2 + (\omega_1^2 - \omega_2^2)^2]^{1/2}\}$. The connection between (x, y) and the normal coordinates (X, Y) is given by

$$X = [(\Omega_1^2 - \omega_2^2)\, x - \alpha y]\, [\alpha^2 + (\Omega_1^2 - \omega_2^2)^2]^{-1/2},$$
$$Y = [\alpha x - (\Omega_2^2 - \omega_1^2)\, y]\, [\alpha^2 + (\Omega_2^2 - \omega_1^2)^2]^{-1/2}. \tag{2.3}$$

The transformation (2.3) corresponds to a normalization in which $X \to x$ and $Y \to y$ if $\alpha \to 0$ and $\omega_1^2 > \omega_2^2$.

In the case of a slow time variation of the coefficients of the system (2.1), the general solution Z for normal coordinates can be represented in the form[36]

$$Z = A_1 X_+ + A_2 Y_+ + A_3 X_- + A_4 Y_-,$$

where

$$X_\pm = \Omega_1^{-1/2} \exp\left(\pm i \int^t \Omega_1(t')\, dt'\right), \quad Y_\pm = \Omega_2^{-1/2} \exp\left(\pm i \int^t \Omega_2(t')\, dt'\right).$$

We denote by **A** and **B** column vectors made up of the constant coefficients of the general solution Z from the left and from the right of the crossing region (resonance region), respectively. Then, by using the analyticity of the exact solution (2.1), it is easy to obtain the matrix M for the column-vector transformation $\mathbf{B} = \hat{M}\mathbf{A}$. Since this conclusion is well known (see, for example,[20]), we present without proof the matrix $\hat{M}$ for the case $\Omega_1 = \Omega_2$[36]:

$$M_{11} = -M_{22}^* = i e^{-i\varphi}(1 - e^{-2\delta})^{1/2}, \quad M_{12} = M_{21} = e^{-\delta},$$

where

$$2\delta = i \oint_{\mathscr{L}} [(\Omega_1 - \Omega_2)/2]\, dt > 0$$

(the contour $\mathscr{L}$ encloses both crossing points), φ is a phase shift which is unknown in the WKB and does not influence the coefficient Q of the transformation between the normal modes X_+, Y_+, and X_-, Y_- ($Q \sim e^{-\delta}$).

In a plasma, the complete analog of Eqs. (2.1) is, for example, the system of equations describing the behavior of the slow and fast magnetosonic waves in the case of slow time variation of the plasma density. A derivation quite similar to the derivation of the equations for a spatially-inhomogeneous plasma[37] results in the following system of equations:

$$\ddot{u}_1 + k_z^2 v_A^2 u_1 = -\theta k_z^2 v_0^2 u_2, \quad \ddot{u}_2 + k_z^2 c_s^2 u_2 = -\theta k_z^2 v_A^2 u_1; \tag{2.4}$$

here v_A and c_s are respectively the Alfven and the sound velocities, θ is the angle between the wave vector and the magnetic field, $\mathbf{H}_0 = H_0 \mathbf{e}_z$, $v_0 \equiv c_s(0)$, $h_y = H_0 u_1(t)$, $\rho = \rho_0 u_2(t)$, $\rho_0(t)$ is the unperturbed density, and h_y and ρ are the perturbations of the magnetic field and of the density. The crossing occurs in the region $v_A \approx c_s$, and for small angles θ we have $\delta \sim \tau(k_z v_0)\theta^2$, where τ is the characteristic time of the variation of ρ_0. As $\delta \to 0$, the transformation coefficient Q tends to unity. It must be emphasized, however, that we are dealing with transformation of normal oscillations. Therefore a strong transformation of the normal oscillations X and Y in this case does not mean at all a large exchange of energy between the oscillators x and y[38,39]. In fact, assume that as $t \to -\infty$ we have

$$x = A\omega_1^{-1/2} \exp\left(i \int_0^t \omega_1(t')\, dt'\right), \quad y = 0.$$

Then as $t \to +\infty$ we obtain

$$y = B\omega_2^{-1/2} \exp\left(i \int_0^t \omega_2(t')\, dt'\right)$$

where

$$B = (\alpha/2i) \int_{-\infty}^{+\infty} \left[x(t)\, \omega_2^{-1/2} \exp\left(-i \int_0^t \omega_2(t')\, dt'\right)\right] dt = (\alpha A/\omega_0^2)(\pi\omega_0\tau/2) \exp(-i\pi/4)$$

($\omega_1 - \omega_2 = \omega_0 t/\tau$, $\alpha \ll \omega_{1,2}^2$). On the other hand, in this case, as can be easily seen, $2\delta = (\pi/2)[(\alpha/\omega_0^2)(\omega_0\tau)^{1/2}]^2$. It follows therefore that the decrease of δ makes it more difficult to transfer energy from the oscillation x into the oscillation y. In the case of (2.4) this means that if the energy was concentrated in the acoustic oscillations at $t \to \infty$, then it remains concentrated in them at $t \to \infty$ as $\alpha \to 0$ and $\delta \to 0$. Consequently, the adiabatic transition, when $\delta \gtrsim 1$ as a result of the large time τ of the divergence of the branches, is more effective at small α.

It is not difficult to find the transition matrix in the case of the resonance $\Omega_1 = -\Omega_2$, and also on going through a region in which there are simultaneously the resonances $\Omega_1 = \Omega_2$ and $\Omega_1 = -\Omega_2$[35,36].

Referring the reader to[40,41] for details, we present the result of the solution of one case of coupled oscillators, which will be useful later on. Namely, we take the following system of oscillators

$$\ddot{y} = x, \quad -(\ddot{x}/\beta) - [(t^2 + \lambda)/\beta]\, x = y \quad (\lambda, \beta > 0), \tag{2.5}$$

in which x can be treated as an oscillator with negative mass. This leads to the following equation for the oscillator y

$$(d^4y/dt^4) + (t^2 + \lambda)(d^2y/dt^2) + \beta y = 0. \quad (2.6)$$

The frequencies of the normal oscillations of the system (2.5) are

$$\Omega_{1,2}^2(t) = (1/2)[(t^2+\lambda+\sigma)^{1/2} \pm (t^2+\lambda-\sigma)^{1/2}], \quad \sigma = (4\beta-1)^{1/2}.$$

From (2.6) we can easily obtain the invariant

$$|A_1|^2 + |A_2|^2 - |A_3|^2 - |A_4|^2 = \text{invar}; \quad (2.7)$$

here A_1, A_2, A_3, and A_4 are analogous to those introduced above for the solution Z of the system (2.1) and (2.2). We put $\epsilon_{1,2} = (1/2)i\int(\Omega_1 \pm \Omega_2)\,dt = \pi(\lambda \pm \sigma)/2$ and let $\epsilon_1 \to \infty$. We then have

$$A = \begin{Vmatrix} S & D \\ D & S \end{Vmatrix} B, \quad (2.8)$$

where $S_{11} = S_{22} \equiv \gamma = (1 + e^{-2\epsilon_2})^{1/2}\cos\mu$, $S_{21} = -S_{12} \equiv \kappa = (1 + e^{-2\epsilon_2})^{1/2}\sin\mu$, $D_{22} = -D_{11} \equiv \rho = e^{-\epsilon_2}$, $D_{12} = D_{21} = 0$, $\mu = (\epsilon_2/\pi)[1 - \ln(\epsilon_2/\pi)] - \arg\Gamma((1/2) - (i\epsilon_2/\pi))$.

We note that the condition (2.7) can be rewritten in the form $I_1 + I_2 - I_3 - I_4 = \text{const}$, where I_n is an adiabatic invariant in the absence of interaction between the normal oscillations.

In concluding this section, let us consider the following problem. When waves pass through a sufficiently large volume of an inhomogeneous medium, the number of transformation points can be very large. It is natural to assume their distribution over the volume to be random and specified in the form of a certain random function. This raises the question of the evolution of waves in such a medium. As before, it is more convenient, however, to investigate the analogous problem of the passage of a system of oscillators through resonances at random instants of time. Since the adiabatic invariant of each normal oscillation changes appreciably only in a narrow region of the resonance, i.e., in fact jumpwise, we shall speak, in accordance with[36], of collisions of normal oscillations. The trend of the evolution (stability or instability) depends on the form of the invariant of the differential equation describing the system of oscillators in the resonance region. If the solution takes the form $x = \sum_n A_n X_n$, where X_n is a quasinormal oscillation, then the invariant is a quadratic form of the constants A_n:

$$\sum_n s_n |A_n|^2 = \text{invar} \quad (s_n = \pm 1).$$

We assume that all s_n have the same sign, and then $|A_n|$ is bounded from above, i.e., the motion of the oscillators is finite in phase space. In the opposite case, the system can be unstable. This statement is difficult to prove rigorously in general form, but it is confirmed in the particular examples described below.

For one oscillator in a random external field, the resonances are the points at which $\Omega(t) = 0$. The invariant takes the form $|A_+|^2 - |A_-|^2 = \text{invar}$. Thus, one can expect the motion to be unstable. The solution of this problem in[42] confirmed our conclusion. Wave transformation in a medium with random inhomogeneities was considered in[43]. Formally, the situation is equivalent to a system of two coupled oscillators with resonances $\Omega_1 = \Omega_2$. The invariant took the form $|A_1|^2 + |A_2|^2 = \text{invar}$. It was found that, regardless of the initial conditions, the system approaches equilibrium at which $I_1 = I_2$ (in accord with the statements made above). Transformation in the case of random collisions, for an invariant of the system of coupled oscillations in the form (2.7), was considered in[40]. The system of oscillators turned out to be unstable. Let us examine this case briefly, since it is of greatest importance for applications.

Let $X_\pm$ and $Y_\pm$ be the normal oscillations of a system of oscillators with frequencies Ω_1 and Ω_2, respectively. We introduce a column vector Z with components $Z_{1,2} = X$ and $Z_{4,3} = Y$, and consider the auxiliary system of equations

$$dZ/dt = iKZ + \sum_n Q_n Z\delta(t - t_n), \quad (2.9)$$

where K is a diagonal matrix: $K_{11} = -K_{22} = \Omega_1$, $K_{44} = -K_{33} = \Omega_2$, and Q_n are certain fourth-order matrices. The solution of (2.9) experiences discontinuities at the instants of time t_n, with $Z(t_n + 0) = \exp(Q_n)\,Z(t_n - 0)$. If we choose a transition matrix $\exp Q_n$ that coincides with the matrix of the transition between the normal oscillations in the collision, we can consider the equivalent problem of averaging the solutions of the system (2.9) with a random distribution of t_n.

We choose a real solution, $Z_2 = Z_1^*$ and $Z_3 = Z_4^*$, and change over to the real variables $\xi_{1,2} = \text{Re}\,Z_{1,4}$, $\eta_{1,2} = \text{Im}\,Z_{1,4}$, which can be treated as coordinates and momenta of certain oscillators. As is customary in problems of this kind,[44,45], we obtain the following kinetic equation for the distribution function $f(t;\ \xi_1, \xi_2, \eta_1, \eta_2)$ of the coordinates and momenta:

$$\frac{\partial f}{\partial t} + \Omega_1\left(\xi_1\frac{\partial f}{\partial \eta_1} - \eta_1\frac{\partial f}{\partial \xi_1}\right) + \Omega_2\left(\xi_2\frac{\partial f}{\partial \eta_2} - \eta_2\frac{\partial f}{\partial \xi_2}\right) = \text{St}(f),$$

$$\text{St}(f) = -\nu f + \nu\int\int d\sigma\, d\lambda\, w(\sigma, \lambda)\bar{f}, \quad \bar{f} \equiv f(t;\ \bar{\xi}_1, \bar{\xi}_2, \bar{\eta}_1, \bar{\eta}_2): \quad (2.10)$$

here ν is the frequency of the collisions, which are assumed to have a Poisson distribution; σ and λ are the collision parameters with a random distribution $w(\lambda, \sigma)$. As a result of the collision, the phase-space point $(\bar{\xi}_1, \bar{\xi}_2, \bar{\eta}_1, \bar{\eta}_2)$ goes over into the point $(\xi_1, \xi_2, \eta_1, \eta_2)$.

Choosing (2.8) as the matrix for the transition from $(\xi_1, \xi_2, \eta_1, \eta_2)$ to $(\bar{\xi}_1, \bar{\xi}_2, \bar{\eta}_1, \bar{\eta}_2)$ and seeking the solution for the second moments of the distribution function in the form $\sim\exp(\nu\int^t \Lambda(\tau)\,d\tau)$, we obtain the dispersion equation[40]

$$\Lambda^3 + 4(1-\gamma^2)\Lambda^2 + [q^2 + 4q\kappa\gamma + 4(\kappa^2 - \rho^2)]\Lambda - 2\rho^2q^2 = 0, \quad (2.11)$$

where the parameters κ, ρ and γ are given in (2.8), and $q = (\Omega_1 - \Omega_2)/\nu$.

Equation (2.11) always has a positive root Λ_0. For weak collisions ($2\epsilon_2 \gg 1$) its value is $2\exp(-2\epsilon_2) \ll 1$. In the case of strong collisions, when $2\epsilon_2 \ll 1$, we have $\Lambda_0 \approx 2$, i.e., the instability increment is of the order of the collision frequency (we recall that the kinetic equation (2.10) was obtained under conditions that the collision frequency is small in comparison with the frequencies of the normal oscillations). An important conclusion of the proof presented here is that for invariants of the type (2.7) each oscillation in the transformation increases its amplitude as a result of the energy of the medium. This conclusion can be of appreciable importance, in particular, for problems on the stability of plasma accelerators.

3. **Transformation types II and III.** At high tempera-

tures, when the collision temperatures are low, the microwave heating of the plasma is effected by transforming the electromagnetic waves into slow plasma oscillations, which are then damped primarily as a result of collisionless damping. This process, which is of practical importance, occurs in the region of hybrid frequencies of an inhomogeneous plasma and, as indicated in Section 1 above, is a transformation of type II, which can also be called anomalous transformation, since the electric field and the refractive index have singularities in the resonant layer of a cold plasma, and the transformation coefficient can reach unity (see[9] and the references cited therein). The investigation of such a transformation is based on the use of a remarkable property it possesses, namely, the absorption coefficient of an electromagnetic wave in a cold plasma is equal to the transformation coefficient of an electromagnetic wave into a slow plasma mode in a "hot" plasma[10,46-48]. This circumstance makes it possible to simplify the problem greatly, and it is frequently possible to reduce it to the investigation of a second-order differential equation. With the aid of the procedure employed in[9], it became possible to analyze a large number of problems from the theoretical and experimental points of view. It therefore behoves us, from the point of view of type-II transformations, to discuss only the following question: Is an analysis of a transformation of type II on the basis of a fourth-order differential equation indeed necessary, or is it of interest only from the methodological and mathematical points of view, as a supplementary method of proof? The importance of investigating transformations of type I and III on the basis of fourth-order differential equations is obvious in view of the absence of the properties indicated above.

Since the review[9] contains a detailed bibliography of transformations of type II, we shall cite here only the papers (mainly the most recent ones) needed in connection with this question.

Let us list the main reason why, in our opinion, it is important to take into account in the general case the thermal motion and the analysis of the solutions of the fourth-order differential equation[1)]:

1) That wave absorption in a cold plasma, described by an equation of the type $(d^2\varphi/dx^2) + (U(x)\varphi/x^m) = 0$, is equivalent to their transformation into thermal oscillations described by the equation $\alpha(d^4\varphi/dx^4) + x^m(d^2\varphi/dx^2) + U(x)\varphi = 0$, has been proved only for $m = 1$. In the general case of arbitrary m, this equivalence is far from obvious[49,50].

2) If the parameter α in the foregoing fourth-order equation is complex, then additional investigations of the indicated equation are needed even when $m = 1$, in order to verify that the absorption coefficient is equal to the coefficient of transformation of the long-wave and the short-wave oscillations (see[51] concerning the apparent and the real singularities).

3) The truncated equation that yields a correct expression for an absorption coefficient equal to the coefficient of transformation of the long-wave into the short-wave mode coincides with the equation for the waves in a cold plasma only in the limit of sufficiently weak thermal motion[52].

4) A joint investigation of the influence of dissipation, thermal motion (within the framework of the fourth-order differential equation) and the nonlinear processes is necessary to ascertain the character of the distribution of the energy of the electromagnetic wave incident on the region with the singularity of the refractive index. In some cases, the situation has been analyzed mainly from the qualitative point of view[26,53].

5) Even in those cases when the investigation of the "cold" equation gives correct information on the transformation of the long-wave into a short-wave one, an investigation of the fourth-order equation can yield additional information concerning this process[14]. This is particularly obvious for the eigenvalue problems considered in[11-13,40,54-55].

We can thus conclude that transformation of a long-wave into a short-wave mode is correctly described by the "cold" equation in the limit of sufficiently weak thermal motion, at low dissipative and nonlinear effects, and in the case of a simple zero of the coefficient of the second derivative for the fourth-order equation. In particular, it follows from the third remark that the "cold" equation is not very useful for quasirelativistic electrons.

Leaving out the detailed analysis of the foregoing remarks, which can be found in the cited papers, we shall illustrate briefly only some of them.

We consider the equation

$$\alpha\beta^2\varphi^{IV} + \beta u_2(x)\varphi^{II} + u_1(x)\varphi = 0 \tag{3.1}$$

with two small parameters α and $\beta \sim (\lambda/L)^2$, where λ and L are respectively the wavelength and the inhomogeneity length. The additional small parameter α is connected with the concrete physical situation and characterizes, for example, the influence of low viscosity or of weak thermal motion, etc. It is convenient to express the wave vector k(x) in the form:

$$k_{1,2} = (1/2)\{[(u_2/\alpha) + 2(u_1/\alpha)^{1/2}]^{1/2} \pm [(u_2/\alpha) - 2(u_1/\alpha)^{1/2}]^{1/2}\}. \tag{3.2}$$

Near the zero of the function $u_2(x)$ we have $u_2 = ux$, $u_1 = u_1(0)$. It is seen here from (3.2) that the crossing of the solutions of (3.1) occurs at the points $x_{1,2} = \pm 2(\alpha u_1(0)/u^2)^{1/2}$, at which the functions $k_1 - k_2$ and $k_1 + k_2$ have branch points. The distance between x_1 and x_2 (the dimension of the singular region) is of the order of $\Delta x \sim L\alpha^{1/2} \ll L$, and $|k\Delta x| \sim (\alpha/\beta^2)^{1/4}$.

Using the Laplace method, we have investigated from a unified point of view the solutions of (3.1) for arbitrary values of the parameter α/β^2[14]. Making the change of variable $x = \beta y$ in the vicinity of the zero of $u_2(x)$, we obtained from (3.1)

$$\varphi^{IV} + \lambda^2(y\varphi^{II} + \gamma\varphi) = 0, \tag{3.3}$$

where $\lambda^2 = \beta^2/\alpha$ and $\gamma \equiv u_1(0)$. The solution of (3.3) is expressed in the form of a contour integral

$$\varphi(y) = \oint_C t^{-2}\exp[(t^3/3\lambda^2) + ty - (\gamma/t)]\,dt. \tag{3.4}$$

The asymptotic properties of the solutions of (3.4) for large values of the parameter λ^2 were investigated in[56]. An analysis of the contour integrals (3.4) shows that at sufficiently large distances from the origin $y = 0$ the asymptotic expressions for $\varphi(y)$ have the same form (are similar) for all values of λ^2.[14] Thus, for example, for a closed contour C circling around the essential singularity $t = 0$, it can be shown that the asymptotic forms of $\varphi(y)$ at distances satisfying the condition $\lambda^2|y|^{3/2} \gg 1$ are similar. In this case the asymptotic form of $\varphi(y)$ is given by[14,56]

$$\varphi(y) \equiv V(y) \approx 2\pi i (y/\gamma)^{1/2} J_1 (2(\gamma y)^{1/2}) \quad (\lambda^2 |y|^{3/2} \gg 1), \tag{3.5}$$

where J_1 is a Bessel function. It is easy to see that the function (3.5) satisfies the abbreviated equation $y\varphi^{II} + \gamma\varphi = 0$ in the entire vicinity of the point $y = 0$. The indicated method illustrates the fifth remark, that it is important to investigate the fourth-order equation. We present one more solution of (3.3), which coincides asymptotically with the solution of the truncated question, but only in a sector of the vicinity of the point $y = 0$. For real y we have[8,14,56,57]

$$U_2 \approx \begin{cases} -\pi |y/\gamma|^{1/2} H_1^{(1)}(2i|y\gamma|^{1/2}), & y<0, \\ i\pi (y|\gamma)^{1/2} H_1^{(1)}(2(y\gamma)^{1/2}) + \dfrac{\pi^{1/2}\exp[i(\pi/4)+i(2\lambda/3)y^{3/2}]}{\lambda^{3/2}y^{5/4}}, & y>0 \end{cases}$$
$$(|y\lambda| \gg 1, \ |y\lambda^{2/3}| \gg 1, \ |y\lambda^2| \gg 1). \tag{3.6}$$

The solution (3.6) provides just the description of the anomalous transition of the transition of the truncated equation into the short-wave equation, and is the most important in the theory of wave transformations[8]. Of course, for the anomalous transformation described by the solution (3.6) it is necessary to satisfy also the inequalities cited above, in addition to the vanishing of the coefficient of the second derivative. In particular, the condition $|y\lambda| \gg 1$ signifies smallness of the singular region in comparison with the dimensions of the system. In addition to the already cited papers, the conditions for the occurrence of anomalous transformation in magnetohydrodynamics were recently repeated in[58]2).

It should be mentioned here that anomalous transformation of waves in the region of the upper hybrid resonance of an inhomogeneous plasma was proved in[40] with the aid of the energy conservation law.

We make two more remarks concerning the properties of the solutions of (3.1). First, it follows from the already noted multiple-valuedness of $k_1 + k_2$ and $k_1 - k_2$ that reflected waves of the same type as the incident ones can appear in the vicinity of the singular region. The solutions (3.5) and (3.6) do not contradict this. It is seen from them, however, that to realize this possibility it is necessary to impart a physical meaning to the solution V(y) (for example, under instability conditions, when growing solutions are meaningful). Second, it is necessary to discuss the classification of the modes, all the more since the treatment of this question is not unambiguous (see, for example, [17,32,58]). In addition to (3.2), we can also use for k(x) the representation

$$k^2 = [u_2 \pm (u_2^2 - 4\alpha u_1)^{1/2}]/2\alpha\beta. \tag{3.7}$$

Expression (3.7) demonstrates the difference between the cases $u_1 > 0$ and $u_1 < 0$. If $u_1 > 0$, the crossing points of the solutions y on the real axis and, in accordance with the customary definition of oscillation modes in a homogeneous plasma (see, for example, [59]), a transition between two different modes occurs at the point $u_2 = 0$. For one of the waves, the components of the group and phase velocities have opposite signs. In the case of $u_1 < 0$, the crossing points of the solutions lies in the complex x plane, and for $k_{1,2}$ far from the point $u_2 = 0$ we have, in accordance with the rule for defining the branches of the oscillations in a homogeneous plasma, the following expressions:

$$k_1^2 \approx \begin{cases} u_2/\alpha\beta, & u_2>0, \\ u_1/\beta u_2, & u_2<0, \end{cases} \qquad k_2^2 \approx \begin{cases} u_1/\beta u_2, & u_2>0, \\ u_2/\alpha\beta, & u_2<0. \end{cases} \tag{3.8}$$

We see from (3.8) that in this case the transition from the rapidly-oscillating solution to the long-wave solution is not a transition, in the sense mentioned above, from one branch of oscillations to another (only $k_2^2 < 0$). On the other hand, if we use the representation (3.2), then the transition from the long-wave solution to the short-wave solution is such a transition for any sine of u_1. In addition, in the asymptotic theory of differential equations with a small parameter preceding the highest-order derivative, one deals with a transition between the singular solutions of the abbreviated equation, obtained from (3.1) as $\alpha \to 0$, on the one hand, and the rapidly-oscillating solutions of (3.1), on the other. Regardless of the method used to define the oscillations modes from the physical point of view it is clear that this transformation leads to an essential change in the properties of the waves, since the expressions for the $k_{1,2}^2$ differ greatly as $x \to \pm\infty$.

It is necessary to call attention next to the fact that the similarity of the asymptotic forms of (3.1) at arbitrary values of λ^2 is due to the choice of the coefficients $u_1(x)$ and $u_2(x)$ in the form $u_2 = ux$ and $u_1 = \text{const}$, and is not a general property of (3.1). In fact, in an investigation of the equation

$$\varphi^{VI} + \lambda_1^2 [x\varphi^{II} + (\beta_1 + \sigma^2 x)\varphi] = 0 \tag{3.9}$$

it was shown in[2] that the transformation properties of (3.9) are equivalent to the absorption properties of the following truncated equation:

$$\xi\varphi^{II} + (\xi + 2\mu)\varphi = 0, \tag{3.10}$$

where $\xi = \sigma x - (\sigma^3/\lambda_1^2)$, $2\mu = (\beta_1/\sigma) + (\sigma^2/\lambda_1^2)$.

As follows from (3.10), the absorption coefficient of the wave is equal to $Q = e^{-2\pi\mu}(1 - e^{-2\pi\mu})$. Since Q depends on μ, i.e., in final analysis on λ_1^2, the transformation coefficient and the asymptotic forms of the solutions of (3.9) depend on λ_1^2. Thus, there is no similarity with respect to λ_1^2 for (3.9). From (3.10) we get the important conclusion that in the general case the abbreviated equation (if it exists at all) does not reduce to the "cold" equation, in accordance with remark (3) (page 68) made above. It must also be noted that it follows from the results of[52] that in the general case the coefficient of transformation of the long-wave mode into a short-wave mode is not equal to the coefficient of transformation of the inverse process.

Examples of equations describing the anomalous transformation in a plasma were given in[9] for the case of hybrid resonances. Let us indicate other examples of anomalous transformation:

1) Transformation of magnetosonic waves in a spatially-inhomogeneous plasma, described by the equation[37]

$$\varphi^{IV} + \left(\frac{\omega^2}{V_A^2} + \frac{\omega^2}{c_s^2} - k_y^2\right)\varphi^{II} + \left[\frac{\omega^2}{V_A^2}\left(\frac{\omega^2}{c_s^2} - k_y^2\right) - \frac{\omega^2 k_y^2}{c_s^2}\right]\varphi = 0. \tag{3.11}$$

In the derivation of (3.11) it is assumed that the plasma density is inhomogeneous along the magnetic field, which is in turn directed along the z axis. If the coefficient of the second derivative vanishes, then, according to the theory developed above, an anomalous transformation between the fast and slow magnetosonic waves is possible[8]. As shown in[58], it is realized in a dense plasma, $V_A^2 \ll c_s^2$, where the solutions of (3.11) "have time" to assume the asymptotic form (3.6).

2) In the region of ferromagnetic resonance of an inhomogeneous ferromagnet, anomalous transformation of

the long-wave and short-wave spin modes is possible[32,39].

3) Anomalous transformation of spin waves into electromagnetic waves and vice versa was investigated in[60].

In concluding this section, let us dwell very briefly on the transformation of the third type. This case was considered in[12], where the rules for bypassing the crossing points of the solutions were obtained and the spectrum of the oscillations was determined for finite problems. Attention was called in[32] to the fact that for propagating waves this case corresponds to 100% transformation. As noted in Section 1 above, the transparency region for both types of waves lie, in the case of a transformation of type III, on both sides of the crossing point, near which the components of the group velocities of the waves take the form

$$(d\omega/dk)_{1,2} \approx \pm 2\,[d\,(k_1 - k_2)/d\omega]^{-1}. \tag{3.12}$$

It follows therefore that the group velocities are antiparallel in the region of the crossing point of the solutions. On the other hand, it is seen from (3.2) that near the crossing point (where $u_2^2 = 4\alpha u_1$), only the internal sign of the wave vector can change, i.e., the transition $k_1 \leftrightarrow k_2$ is realized. Using this fact, as well as formulas (3.12) and the fact that both waves exist only on one side of the crossing point (the attenuate on the other side), we can draw conclusions concerning the character of the transformation process. The incident wave is completely transformed into a wave of another type, which is reflected backwards. This type of transformation can be called reflective. At the present time only one example of this transformation is known. This is the case of potential oscillations of a non-isothermal ($T_i \gg T_e$) plasma in the frequency region $\omega_{Hi} \ll \omega \ll \omega_{He}$ ($k_\perp \gg k_z$, $\omega \gg k_z V_{Te}$, $k_z V_{Ti}$), for which $k^2(x)$ takes the form

$$k_{1,2}^2 = -k_y^2 + \frac{\omega^4}{6\omega_{pi}^2 V_{Ti}^2} \times \left\{1 + \frac{\omega_{pe}^2}{\omega_{He}^2} - \frac{\omega_{pi}^2}{\omega^2} \pm \left[\left(1 + \frac{\omega_{pe}^2}{\omega_{He}^2} - \frac{\omega_{pi}^2}{\omega^2}\right)^2 - 12 k_z^2 v_{Ti}^2 \frac{\omega_{pe}^2 \omega_{pi}^2}{\omega^6}\right]^{1/2}\right\}. \tag{3.13}$$

As follows from (3.13), 100% transformation of the waves is possible in a half-space filled with an inhomogeneous plasma with a pressure that decreases towards the plasma boundary, or with an increasing magnetic field[32].

4. Certain features of wave transformation in the interaction of a beam with a plasma. It is well known that a plasma through which a beam of charged particle passes is the simplest example of a nonequilibrium unstable medium (see, for example,[61-63]). In a homogeneous plasma, a nonrelativistic beam generates short-wave longitudinal oscillations which do not satisfy the conditions of propagation in vacuum. In the case of an inhomogeneous plasma, longitudinal and transverse components of the electromagnetic fields are "coupled," and consequently such a system is more convenient from the point of view of drawing energy from the plasma, owing to the possibility of converting quasilongitudinal fields into quasitransverse ones. Although the investigation of transformations of longitudinal oscillations into transverse oscillations in a plasma-beam system is only in the beginning stage (see, for example,[21-23,63,64]), many features of this process have already been revealed, and will be illustrated here with certain characteristic examples.

a) Anisotropy of the radiation of transverse waves. We present here the results of theoretical and experimental investigations of the efficiency of the transformation of longitudinal waves into transverse ones as a function of the sign of the density gradients when a beam moves along an inhomogeneity[23,65]. Thus, in[65] the plasma chamber was a glass tube 50 cm long with inside diameter 2.6 cm, placed in a homogeneous longitudinal magnetic field of intensity up to 2 kOe.

The plasma density could easily increase monotonically in the direction of beam injection, or decrease, depending on the direction from which the working gas was admitted. The oscillations from the plasma were received by external probes located outside the plasma chamber in a longitudinal direction. In addition to using probes, the longitudinal electric component of the oscillations extracted from the plasma by the electron beam were registered with the aid of broad-band helical junctions placed at both ends of the plasma chamber. The results of the experiments show that in the case when the plasma density increases in the beam-injection direction, the generation of both transverse and longitudinal oscillations is observed. A plasma density gradient with direction opposite to the beam propagation changes the picture of the radiation of the oscillations radically. One observes a complete absence of generation of transverse oscillations, as registered by the probes, and a simultaneous presence of radiation of longitudinal components.

Proceeding to the explanation of the experimental results, we must note first, that, with the exception of the region of the resonance of the natural oscillations of a cold plasma at rest and the drift oscillations of the beam (beam mode), the dispersion properties of each of the waves at $\omega_{He} \gg \omega$ are determined, as is well known, from the following respective equations (see, for example,[63]):

$$k_{zc}^2 = k_\perp^2 \omega^2/(\omega_{pe}^2 - \omega^2), \qquad \mathrm{Re}\, k_{zb} = \omega/V \quad (\mathrm{Im}\, k_z \ll \mathrm{Re}\, k_z \ \text{ for } \ n_b \ll n_p); \tag{4.1}$$

here ω_{pe} is the plasma frequency on the axis of the interaction region and varies along the beam propagation direction; k_z and $k_\perp$ are the components of the wave vectors along and across the magnetic field, respectively; n_b and n_p are the beam and plasma densities, and V is the beam velocity. For a small plasma wave excited in a plasma waveguide we have[66] $k_\perp \sim 1/a$ (a is the radius of the waveguide).

When the beam moves in a direction in which the plasma density decreases, the amplified "cold" plasma mode (4.1) propagates in the direction of beam motion (at $\omega_{He} > \omega_{pe}$ the phase and group velocities are parallel), and is transformed in the vicinity of $\omega_{pe} \approx \omega$ in to a rapidly damped "hot" plasma wave with large k_z[5,8], for which the propagation conditions in vacuum $k_z \gg \omega/c$ are not satisfied. It is in this case that no energy of the transverse oscillations is drawn from the beam.

It should be noted that if the plasma density changes noticeably over distances L comparable with the wavelength λ, then wave transformation must be taken into account also in regions where geometrical optics in first order in λ/L is valid[67]. When this remark is taken into account, it becomes clear that if the amplified perturbation moves in the direction of increasing density, when $\mathbf{V} \parallel \nabla n_p$ and the refractive index N decreases, an appreciable fraction of its energy is transformed into

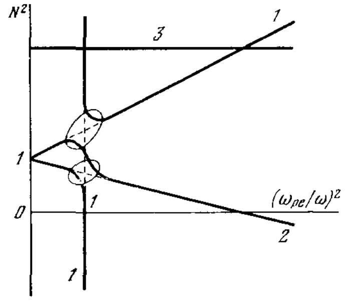

FIG. 1. Dependence of the squares of the refractive indices of the waves on the density in a cold plasma at $\omega_{He} > \omega$. 1) Extraordinary wave; 2) ordinary wave; 3) beam mode without allowance for interaction between the beam and the plasma. The regions of transformation of "cold" plasma modes are encircled.

waves that "flow out" from the plasma chamber to the outside. Such a transition is possible also when $N > 1$[68]. Furthermore, it must be emphasized that if a second "cold" (fast) mode can propagate in the plasma waveguide, the mode with a singularity of the refractive index can be converted into a fast mode that is readily radiated to the outside. In Fig. 1, the transformation regions are encircled. The transformation is effective for angles $\theta \leq (2c/\pi\omega L)^{1/2}$[5] ($\theta$ is the angle between $\mathbf{H}_0$ and $\mathbf{k}$). Since the region of the crossing of the beam and plasma modes is to the left of the transformation region, it is clear that this transition occurs when the beam moves in the direction of increasing plasma density.

We have disregarded above transformation effects due to the transverse density gradient, for in this case $\omega_{He} \gg \omega_{pe} \sim \omega$, meaning that the condition of the indicated transformation, $\omega_{He}^2 + \omega_{pe}^2 \approx \omega^2$, cannot be satisfied.

We note that when a beam moves in the direction of increasing plasma density, the phase velocity of the oscillations increases. The quasilinear regime observed in[69] could in this case lead to a decrease of the effects considered here, for in this case the energy should return to the beam. However, if the inhomogeneity is strong enough, $(V/\omega L) \sim 1$, this circumstance is apparently immaterial.

It is curious to note that the transition radiation of a modulated beam in a plasma with longitudinal density gradient also has a pronounced anisotropy[21]. We confine ourselves in the proof to the case of transition radiation of a uniformly moving charge on a blurred plasma boundary without the magnetic field. Using the standard procedure, we obtain an equation for the magnetic field of the wave with an electron vector lying in the incidence plane (H_x, E_y, E_z)[70]:

$$\frac{d^2 H_{k\omega}}{dz^2} - \frac{d \ln \varepsilon(z, \omega)}{dz}\frac{dH_{k\omega}}{dz} + \frac{\omega^2}{c^2}(\varepsilon - \alpha^2) H_{k\omega} = \frac{ek_\perp}{2i\pi^2 c}\exp\left(-i\frac{\omega z}{V}\right); \quad (4.2)$$

Here $\epsilon = 1 - [\omega_{pe}^2(z)/\omega^2][1 + i(v/\omega)]$ is the dielectric constant of the cold plasma, ν is the collision frequency, $k_\perp$ is the transverse component of the wave vector, and v is the velocity of the charge. If the density varies linearly in the vicinity of the resonance point $\epsilon = 0$ we have $\epsilon \approx -[(z/L) + i(v/\omega)]$. Using the results of[70], we can easily show that the amplitude of the field of the transition radiation of the charge is proportional to the integral I:

$$I = \int_{-\infty}^{+\infty} \exp[-i(\omega z/v)][H_1(z)/\varepsilon(z)]\, dz, \quad (4.3)$$

where $H_1(z)$ is a solution of Eq. (4.2) without the right-hand side, and attenuates as $z \to +\infty$. It is seen from (4.3) that I vanishes when $v < 0$ (when the charge moves against the density gradient), since the pole of the integrand $\epsilon = 0$ lies in the lower half-plane of z. This result can be understood from qualitative considerations if one recognizes the equivalence, noted in Sec. 2, of the thermal motion to dissipation in the region of plasma resonance. The transition radiation process can be then represented as the onset of plasma oscillations as the result of synchronism with the moving charge, followed by their transformation into transverse waves. If the dispersion of the plasma waves is taken into account, it is easy to see that the synchronism point, at which the phase velocity of the plasmon is equal to the charge velocity, is located to the left of the transformation point $\epsilon = 0$. Therefore when the charge moves in the direction in which the density decreases, the plasma waves excited by it "drift" away from the point $\epsilon = 0$, and a transverse electromagnetic wave can arise only as a result of over-the-barrier effects with amplitude of the order of $\exp(-\omega L/v)$. In the opposite case when the charge moves in the direction of increasing density, the plasma oscillations reach the point $\epsilon = 0$ and are transformed into transverse waves that are radiated to the outside. The amplitude of the transition-radiation field is then of the order of $\exp(-L\theta^3/\lambda)$, where θ is the angle between the density gradient in the radiation direction and λ is the wavelength of the transverse oscillations.

b) Remarks concerning the problem of heating and interruption of instabilities in a plasma. We discuss briefly certain possibilities connected with wave transformation:

1) Besides being of independent interest, wave transformation can be important for problems of plasma stability[17]. The correctness of this statement can be demonstrated by the following reasoning[17]. Let one of the "coupled" waves (with wave vector k_1) oscillate at infinity, and let the other (with wave vector k_2) attenuate. We assume that a localized perturbation ("packet"), made up of waves with wave vector k_2 and increasing with time, has been produced in the plasma. If the rate at at which the energy goes off to infinity as a result of the transformation into the wave k_1 now exceeds the rate of energy influx to the perturbation from the instability sources, then no instability develops, and the plasma can serve as a generator of oscillations that go off to infinity. Owing to the inhomogeneity, the unstable mode can be "coupled" with the stable one or with a more stable one. It is clear that in this case the development of the instability can be difficult. An example of such a situation was analyzed in[24], where it was shown that the crossing of the "hot" and "cold" plasma modes in the vicinity of the upper hybrid resonance causes the increase of the critical current, above which two-stream instability develops, to oscillate in the range from (V/v_{Te}) to $(V/v_{Te})^3$ times, where V is the beam velocity and v_{Te} is the thermal velocity of the plasma electrons.

2) At present there is no meeting of minds concerning the most promising method of plasma heating to the thermonuclear temperatures. It is already clear, however, that high frequency and microwave heating methods are quite important. It is necessary to determine the frequency range which is the most acceptable. At the very utmost, the heating must not violate the plasma containment conditions. Since the diffusion coefficient is $D \sim \lambda^2$ (λ is the wavelength), heating at shorter wave-

lengths seems to be more promising. This is confirmed by the results of[23], where they succeeded (by varying the density distribution and applying modulating signals) in verifying the heating and containment of a plasma by exciting different sections of the wave spectrum in a plasma-beam discharge. It turns out that up to ion-cyclotron waves, the excited low-frequency oscillations, while heating the plasma strongly (30% of the beam energy is given up), lead to a rapid drift of the particles and of the energy from the apparatus, with a diffusion coefficient close to the Bohm value. On the other hand, when the heating is produced by waves in the region of the lower hybrid frequency, the diffusion is small, for in addition to the small wavelength, an important role is apparently played by the fact that ion scattering by the field pulsations should cause an increase of the ion-ion viscosity. We note that this frequency range is convenient also because both electrons and ions are adequately heated in it.

In concluding Chap. II, we mention briefly one more linear transformation in which waves with opposite signs of the energy propagate simultaneously in an inhomogeneous medium. As is well known[71], oscillations with negative energy are possible in a non-equilibrium medium. It follows then from general physical considerations[71,72] that interaction of waves with opposite energies should lead to their mutual amplification and to instability. It is clear therefore that if waves with opposite signs of the energy are present in the inhomogeneous medium, linear transformation effects similar to those considered above can lead to a buildup of oscillations. By way of example we mention the instabilities observed in[73,74] and connected with the outflow of waves from an inhomogeneous-plasma region, in which their energy is negative. It follows from the foregoing also that the coefficient of linear transformation of the waves can exceed 100% in a nonequilibrium inhomogeneous medium.

III. CERTAIN NONLINEAR AND KINETIC NONLOCAL EFFECTS PRODUCED WHEN WAVES PROPAGATE IN AN INHOMOGENEOUS PLASMA

We consider kinetic nonlocal effects produced when waves propagate in an inhomogeneous plasma. By "nonlocal" we mean throughout effects of the spatial-echo type in a homogeneous plasma, which were investigated in[75–79].

We recall that the echo effect in a plasma is based on phase coherence of particles of different energies, and satisfaction of this coherence gives rise to arithmetic addition of microscopic currents of individual plasma particles, leading to a burst of macroscopic current, meaning also field. In a homogeneous plasma, however, owing to the thermal scatter of the particle velocities, the initial perturbation attenuates irreversibly with time or with distance from the source. We have in mind a process analogous to the decay of the wave packet as the result of random diffusion of the plasma particles, which was investigated in detail in[80]. For this reason, the echo effect in a homogeneous plasma is nonlinear. In the case of an inhomogeneous plasma, when the wave in each layer of the plasma interacts resonantly with some definite particle group, the phase coherence can be satisfied in the approximation linear in the field amplitude, and consequently it is possible to observe linear nonlocal effects. This was first done in[16,81,82]. We proceed to consider the nonlocal effects.

5. **Linear nonlocal reflection of waves in an inhomogeneous plasma.** a) We first describe briefly the method proposed in[83] for the investigation of nonlocal effects. For one-dimensional longitudinal electronic oscillations of an inhomogeneous plasma, the basic equations are

$$\frac{\partial f_1}{\partial t}+v\frac{\partial f_1}{\partial x}-\frac{\partial\Phi}{\partial x}\frac{\partial f}{\partial v}=\frac{e}{m}vE_1\frac{dF}{d\mathscr{E}},\qquad \frac{\partial E_1}{\partial t}+4\pi j_1=0,\tag{5.1}$$

where $f_1(x, v, t)$ and $F(\mathscr{E})$ are respectively the perturbed and equilibrium distribution functions, $\mathscr{E} = (v^2/2) + \Phi(x)$ is the normalized energy, $\varphi(x) = -(m/e)\Phi(x)$ is the effective static potential that maintains the inhomogeneity of the plasma and is assumed to a monotonically decreasing function of x, with $F(-\infty) = 0$. The current density is given by

$$j_1(x, t)=j_{\text{ext}}(x, t)-en_0\int\limits_{\Phi(x)}^{\infty}d\mathscr{E}\,(dF/d\mathscr{E})\,[f_1^{(+)}(x, \mathscr{E}, t)-f_1^{(-)}(x, \mathscr{E}, t)];\tag{5.2}$$

Here $f_1^{(+)}(x, E, t)$, $f_1^{(-)}(x, \mathscr{E}, t)$ are the values of $f_1(x, v, t)$ for $v > 0$ and $v < 0$, respectively, $j_{\text{ext}}(x, t)$ is the current density of the external source, and n_0 = const.

We expand the perturbation of the distribution function in a Fourier integral with respect to time:

$$f_1(x, \mathscr{E}, t)=\int\limits_{-\infty}^{+\infty}f_1(x, \mathscr{E}, \omega)\exp(-i\omega t)\,d\omega.$$

We then obtain from (5.1) by the method of characteristics

$$f_1^{(+)}(x, \mathscr{E}, \omega)=(e/m)(dF/d\mathscr{E})\int\limits_{-\infty}^{x}E_1(x', \omega)\exp[i\omega t(x, x')]\,dx',$$

$$\begin{aligned}f_1^{(-)}(x, \mathscr{E}, \omega)&=(e/m)(dF/d\mathscr{E})\int\limits_{x_{\mathscr{E}}}^{x}E_1(x', \omega)\exp[i\omega t(x', x)]\,dx'\\&+(e/m)(dF/d\mathscr{E})\int\limits_{-\infty}^{x_{\mathscr{E}}}E_1(x', \omega)\exp[i\omega t(x_{\mathscr{E}}, x)+i\omega t(x_{\mathscr{E}}, x')]\,dx',\end{aligned}\tag{5.3}$$

where

$$t(x, \xi)=\int\limits_{\xi}^{x}[v(y, \mathscr{E})]^{-1}\,dy,\qquad v(y, \mathscr{E})=\{2[\mathscr{E}-\Phi(y)]\}^{1/2}$$

and $x_{\mathscr{E}}$ is the point of reflection of particles with energy $\mathscr{E}$ ($\Phi(x_{\mathscr{E}}) = \mathscr{E}$). After substituting expressions (5.3) in formula (5.2), we obtain a singular integral equation for the electric field $E_1(x, \omega)$ of the longitudinal oscillations

$$\begin{aligned}E_1(x, \omega)=i(\omega_p^2/\omega)\int\limits_{\Phi(x)}^{\infty}(dF/d\mathscr{E})\Big\{&\int\limits_{-\infty}^{x}E_1(x', \omega)\exp[i\omega t(x, x')]\,dx'\\&+\int\limits_{x}^{x_{\mathscr{E}}}E_1(x', \omega)\exp[i\omega t(x', x)]\,dx'\end{aligned}\tag{5.4}$$

$$-\int\limits_{-\infty}^{x_{\mathscr{E}}}E_1(x', \omega)\exp[i\omega t(x_{\mathscr{E}}, x)+i\omega t(x_{\mathscr{E}}, x')]\Big\}\,d\mathscr{E}+(4\pi i/\omega)\,j_{\text{ext}}(x, \omega),$$

where $\omega_p^2 = 4\pi e^2 n_0/m$.

We consider the natural oscillations of a weakly-inhomogeneous plasma. Seeking the solution of (5.4) by the WKB method, we represent the electric field in the form

$$E_1(x, \omega)=A_1(x, \omega)\exp\Big[i\int\limits_{x_0}^{x}k(x')\,dx'\Big].$$

We put $j_{\text{ext}} = 0$ and integrate in (5.4) twice by parts with respect to x. Introducing the local dielectric constant $\epsilon(\omega, k, \Phi)$ of the inhomogeneous plasma

$$\varepsilon(\omega, k, \Phi)=1-\frac{\omega_p^2}{\omega}\int\limits_{\Phi(x)}^{\infty}\frac{\partial F}{\partial\mathscr{E}}\Big[\Big(k-\frac{\omega}{v}\Big)^{-1}-\Big(k+\frac{\omega}{v}\Big)^{-1}\Big]\,d\mathscr{E},$$

we obtain for the amplitude $A_1(x, \omega)$ the equation

$$A_1(x, \omega)\, \varepsilon(\omega, k, \Phi) - i\, (\partial\varepsilon/\partial k)^{1/2} (d/dx)\, [A_1 (\partial\varepsilon/\partial k)^{1/2}] = 0. \quad (5.5)$$

Equating the terms of the different orders of smallness in the WKG in sequence to zero we obtain the oscillation dispersion equation $\epsilon(\omega, k, \Phi) = 0$ (from which we obtain the complex wave vector $k_\omega(x) \equiv k(\omega, \Phi(x)) = q_\omega(x) + i\kappa_\omega(x)$), and also the electric field amplitude

$$A_1(x, \omega) = \mathrm{const} \cdot [\partial\varepsilon(\omega, k_\omega(x))/\partial k_\omega(x)]^{-1/2}.$$

We note that for slowly-damped waves, integration by parts in (5.4) leads to small denominators $(k - (\omega/v))$. In this case the contribution of the resonant particles is calculated by the saddle-point method[83] and enters in the form of the usual anti-Hermitian part of the dielectric constant $\epsilon(\omega, k)$.

b) We consider now nonlinear nonlocal wave reflection in a weakly inhomogeneous plasma. The physical mechanism of the nonlocal reflection consists in the following: The plasma particles with energy $\mathscr{E}$ moving in the propagation direction of the incident wave with frequency ω interact resonantly in the vicinity of the point $x_s(\mathscr{E})$ with the wave $q_\omega(x_s) = \omega/v(x_s, \mathscr{E})$ and absorb its energy. The interaction between the wave and the plasma produces modulated beams of particles with different energies. After reflection from the potential that maintains the inhomogeneity of the plasma, the particles again return to the region $x \approx x_s$, where they are now at resonance with the reflected wave, and consequently they can radiate the reflected wave. However, the effective radiation of the reflected wave occurs only in the absence of interference between the radiation of particles with different energies. To this end it is necessary to satisfy a certain condition, called the "phase-coherence condition"[82], which states that the times required for the particles with different energies $\mathscr{E}$ to return to the corresponding resonance point $x_s(\mathscr{E})$ should be equal, i.e.,

$$\left.\frac{\partial\tau(x, x_{\mathscr{E}})}{\partial\mathscr{E}}\right|_{x=x_s} \equiv 2\left.\left(\frac{\partial}{\partial\mathscr{E}} \int_x^{x_{\mathscr{E}}} \frac{dx'}{v(x', \mathscr{E})}\right)\right|_{x=x_s} = 0. \quad (5.6)$$

The phase-coherence condition (5.6) determines the energy $\mathscr{E}_0$ of the particles that generate the reflected wave, and at the point of generation we have $x_c = x_s(\mathscr{E}_0)$. When (5.6) is satisfied, the particles with energies $\mathscr{E}$ near $\mathscr{E}_0$ emit the reflected wave in the vicinity of the point x_c coherently. In the opposite case, the nonlocal reflection is an over-the-barrier effect.

To calculate the coefficient of linear nonlocal reflection, we represent the field $E_1(x, \omega)$ in the form of a sum of incident and reflected waves:

$$E_1(x, \omega) = A_+(x, \omega) \exp\left(i \int_{x_0}^{x} k_\omega(x')\, dx'\right) + A_-(x, \omega) \exp\left(-i \int_{x_c}^{x} k_\omega(x')\, dx'\right). \quad (5.7)$$

where $A_+(x, \omega) = \Pi_\omega(x)/\Pi_\omega(x_0)$ is the amplitude of the incident wave and $\Pi_\omega(x) \equiv [\partial\epsilon(\omega, k_\omega(x))/\partial k_\omega(x)]^{-1/2}$. Substituting (5.7) in (5.3) and recognizing that the main contribution to the integral comes from the saddle point $x_s(\mathscr{E})$, we obtain the oscillations of the distribution function of the reflected particles:

$$\delta f_1^{(-)} = \frac{e\omega\varkappa_s}{m\omega_p^2} \left(\frac{2\beta_s}{\pi}\right)^{1/2} \frac{dx_s/d\mathscr{E}}{\Pi_\omega(x_s)\, \Pi_\omega(x_0)} \times \exp\left[i \int_{x_0}^{x_s} k_\omega(x')dx' + i\omega t(x_{\mathscr{E}}, x) + i\omega t(x_{\mathscr{E}}, x_s) - 3i\frac{\pi}{4}\right];$$

here

$$\beta_s \equiv \left\{\frac{d}{dx}\left[q_\omega(x) - \frac{\omega}{v(x, \mathscr{E})}\right]\right\}\Bigg|_{x=x_s} = -q_\omega^3(x_s)\, \omega^{-2} \left(\frac{dx_s}{d\mathscr{E}}\right)^{-1},$$

$$\varkappa_s \equiv \varkappa_\omega(x_s).$$

We next obtain from (5.4) an equation for the amplitude of the reflected waves (5.5) with a right-hand part

$$\frac{d}{dx}\left[\frac{A_-(x, \omega)}{\Pi_\omega(x)}\right] = \frac{4\pi e n_0}{\omega} \Pi_\omega(x) \int_{\Phi(x)}^{\infty} \delta f_1^{(-)}(x, \mathscr{E}, \omega)\, d\mathscr{E} \exp\left[i \int_{x_c}^{x} k_\omega(x')\, dx'\right]. \quad (5.8)$$

We see that the source of the nonlocal reflection, the right-hand side of (5.8), contains information on the past of the particles, namely on their interaction with the field of the incident wave prior to reflection. The solution of (5.8) reduces to[82]

$$\frac{A_-(x, \omega)}{A_+(x, \omega)} = 2i \exp\left[i \int_{x_0}^{x} k_\omega(x')\, dx' + i \int_{x_c}^{x} k_\omega(x')\, dx'\right] \int_{-\infty}^{+\infty} \varkappa_s \exp[i\Psi(x, x_s)]\, dx_s, \quad (5.9)$$

where

$$\Psi(x, x_s) = \omega t(x_{\mathscr{E}}, x) + \int_x^{x_s} \{k_\omega(x') - [\omega/v(x', \mathscr{E})]\}\, dx'.$$

In the case of weakly inhomogeneous plasma, the integrand in (5.9) contains a rapidly-oscillating function, and its value is therefore determined by the contribution of the saddle point $x_s = x_c$, at which the phase $\Psi(x, x_s)$ has an extremum. It is easy to show that the extremum condition $\partial\Psi(x, x_s)/\partial x_s = 0$ can be recast in the form (5.6), so that this condition ensures phase coherence. As a result of the calculations we obtain for the nonlocal-reflection coefficient $R = |E_-(x, \omega)/E_+(x_0, \omega)|^2$ the expression

$$R = 4\pi(\varkappa_c^2/|\gamma|) \exp\left[-2 \int_{x_0}^{x_c} \varkappa_\omega(x')\, dx' - 2 \int_{x}^{x_c} \varkappa_\omega(x')\, dx'\right]; \quad (5.10)$$

Here $\kappa_c \equiv \kappa_\omega(x_c)$, $\gamma = \beta_c + (\omega/2)(d^2\tau(x, x_{\mathscr{E}})/dx_s^2)|_{x=x_c}$. In order of magnitude, we have $(\gamma \sim \beta \sim q/L_q)$

$$R \sim [\varkappa(L_q/q)^{1/2}]^2 \exp\left(-2 \int_{x_0}^{x_c} \varkappa_\omega(x')\, dx' - 2 \int_{x}^{x_c} \varkappa_\omega(x')\, dx'\right).$$

The region of applicability of formula (5.10) obtained above is determined by the estimate[82]

$$(qL_q)^{1/2} \gg 1, \quad (q/\varkappa)^2 \gg (qL_q)^{1/2} \gg (\omega/qv_{Te})^2.$$

In addition, the dimensions of the region where particles of given energy are at resonance with the wave $(L_q/q)^{1/2}$ should be small in comparison with the wave damping length κ^{-1}.

We investigate now the condition of phase coherence (5.6) as a function of the potential $\Phi(x)$. It is easy to show that the phase coherence is satisfied if $\Phi(x) = 0$ at $x < 0$ and $\Phi(x) = v_T^2 x/L$ at $x > 0$. In the case when $\Phi(x) = v_T^2(x/L)^n$ (with $n > 0$), phase coherence is possible if the following equation has a real solution $(\zeta = \mathscr{E}/\Phi)$

$$[\zeta/(\zeta - 1)]^{1/2} = [(1/2) - (1/n)]\, \zeta^{1/n} \int_1^{\zeta} y^{-1/n} [y(y - 1)]^{-1/2}\, dy$$

It follows therefore that we must have $n > 2$.

In concluding this section, we make two remarks. First, in the case of a quadratic potential $\Phi(x) = \Phi_0(x/L)^2$, all the particles are trapped and have the same period of oscillations. The nonlocal wave reflection is then highly effective and, as shown in[84], leads to the onset of "regenerative" plasma oscillation modes

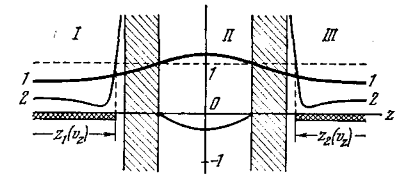

FIG. 2. Coordinate dependence of the magnetic field eH(z)/mc (curve 1) and of the function $\alpha(z) = [\omega - \omega_{He}(z)]/cq_\omega(z)$ (curve 2). The non-transparency regions of the extraordinary wave are singly-hatched.

with frequency equal to double the frequency of the particle oscillations in the potential well $\Phi(x)$. Second, phase coherence of the particles exists not only under the condition (5.6). It is automatically fulfilled if the time of return of the resonant particles is small in comparison with the period of the wave, $\omega\tau(x_s, x_g) \ll 1$, for in this case the distance between the reflection point and the resonance point is so small that there is no phase mixing of the particles generating the reflected wave.

6. Linear regeneration of extraordinary waves in an inhomogeneous magnetic field. As follows from the results of Sec. 5, linear nonlocal effects result from the focusing of particles by the plasma inhomogeneity. However, investigation of the phase-coherence condition show that in an inhomogeneous plasma without a magnetic field these effects have the character of nonlocal reflection. At the same time, particular interest attaches to the transport of the waves by the particles through the non-transparency regions and through regions of strong collisionless damping "forward". As shown in[16], linear "forward" transport of waves by particles, which we shall call regeneration, is possible in a plasma situated in an inhomogeneous magnetic field.

We examine the propagation of an extraordinary wave along a weakly inhomogeneous magnetic field H(z) in the form of a hump, with $H_{min} < mc\omega/e < H_{max}$. The function H(z) is shown schematically in Fig. 2. We recall that the condition under which plasma particles having a velocity v_z along the magnetic field are in cyclotron resonance with an extraordinary wave of frequency ω and wave vector $k_\omega = q_\omega + i\kappa_\omega$ can be expressed in the form

$$\alpha(z) \equiv [\omega - \omega_{He}(z)]/cq_\omega(z) = v_z/c, \qquad (6.1)$$

from which we obtain the function $z(v_z)$ (which is multiply valued in our case) that determines the position of the cyclotron-resonance point as a function of the particle velocity. A plot of $\alpha(z)$ at $\omega_{pe}^2 \ll \omega^2$ is also shown in Fig. 2 (curve 2).

Let the extraordinary wave be excited by a source located at a point z_0 to the left of region I and let it propagate in the positive z direction. The plasma particles moving in the same direction interact resonantly with the wave at points $z_1(v_z)$ of region I. Absorbing part of the wave energy, they pass through region II into region III, where the conditions of cyclotron resonance with the extraordinary wave traveling in the same direction can again be satisfied for the particles (it is assumed that the incident wave does not penetrate into region III as a result of cyclotron damping and of the nontransparency barriers). Consequently at the cyclotron-resonance points $z_2(v_z)$ the plasma particles radiate (regenerate) an extraordinary wave. Coherent regeneration occurs upon satisfaction of the condition[16,85]

$$\frac{d\Psi(v_z)}{dv_z} \equiv \frac{d}{dv_z}\int\limits_{z_1(v_z)}^{z_2(v_z)}\left[q_\omega(z) - \frac{\omega - \omega_{He}(z)}{v_z}\right]dz = 0, \qquad (6.2)$$

which is the equation for the velocity v_0 of the particles regenerating the extraordinary wave.

The regeneration efficiency is determined by the ratio $d(z, z_0) = S_R(z)/S_i(z_0)$ of the energy fluxes in the regenerated and initial waves. Calculations similar to those performed in the preceding section lead to the following expression for $d(z, z_0)$[85]:

$$d = \left|\frac{8\pi}{\rho}\varkappa_1\varkappa_2\frac{dz_1(v_0)}{dv_0}\frac{dz_2(v_0)}{dv_0}\right|\exp\left[-2\int\limits_{z_0}^{z_1(v_0)}\varkappa_\omega(z')\,dz' - 2\int\limits_{z_2(v_0)}^{z}\varkappa_\omega(z')\,dz'\right], \qquad (6.3)$$

where $\rho = d^2\Psi(v_0)/dv_0^2$, $\kappa_n \equiv \kappa_\omega(z_n(v_0))$ is the cyclotron damping constant of the extraordinary wave.

From (6.1) we have

$$\left\{\frac{d}{dz}\left[q_\omega(z) - \frac{\omega - \omega_{He}(z)}{v_z}\right]\right\}\Big|_{z=z(v)} = -\frac{q_\omega(z(v))}{v\,(dz/dv)} \sim \frac{q}{L_q}.$$

Consequently, the dimension of the region where particles of given velocity are in cyclotron resonance with the wave is of the order of $\Delta z \sim (L_q/q)^{1/2}$. We then obtain from (6.3), in order of magnitude,

$$d\,[z_2(v_0),\ z_1(v_0)] \sim [\varkappa\,(L_q/q)^{1/2}]^2.$$

We see therefore that when the inhomogeneity length L_q increases the regeneration effect increases and we can expect $d[z_2(v_0), z_1(v_0)]$ to become of the order of unity at $[\kappa(L_q/q)^{1/2}]^2 \sim 1$. Strictly speaking, formula (6.3) was derived under the assumption that $\kappa(L_q/q)^{1/2} \ll 1$.

We consider in greater detail the conditions under which this effect exists. First, the phase-coherence condition (6.2) reduces to the form

$$\int\limits_{z_1(v_0)}^{z_2(v_0)}[\omega - \omega_{He}(z)]\,dz = 0,$$

for which it follows that the function $[\omega - \omega_{He}(z)]$ should go through zero.

Second, cyclotron absorption of the extraordinary wave by the plasma particles is possible if the following inequalities are satisfied[85]:

$$(\omega_{pe}/\omega)^2 < 2v_0/3\sqrt{3}\,c, \quad eH_{min}/mc\omega < 1 - (3\omega_{pe}^2/2\omega^2).$$

Third, in an inhomogeneous magnetic field, the longitudinal plasma-particle velocity v_z depends on the transverse velocity. Therefore when the particles pass between the wave absorption and regeneration regions, an additional phase mixing occurs, generally speaking, owing to the thermal scatter of the particles with respect to the transverse velocities. This mixing does not have time to occur if $|v_\perp\partial\Psi/\partial v_\perp| \ll 1$. In the opposite case, the phase-coherence condition (6.2) must be supplemented with the requirement that the phase Ψ be extremal with respect to the transverse velocity, otherwise the regeneration will have the character of an over-the-barrier effect.

7. Nonlinear "transparentization" of an inhomogeneous plasma. It was shown in the preceding section that even in the linear approximation, owing to the presence of "memory" in the system, the extraordinary wave can penetrate anomalously in a plasma situated in an inhomogeneous magnetic field. It is therefore of interest to investigate effects of nonlinear "transparentization"

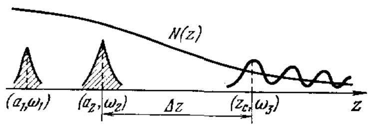

FIG. 3. Echo points z_c and plot of the plasma density N(z) as functions of the locations of the sources.

of an inhomogeneous plasma. We consider the case of an isotropic plasma.

There is a well-known statement that electromagnetic waves do not penetrate into a plasma region where $\omega_{pe}^2 > \omega^2$. We shall show that penetration is actually possible in effects of the nonlinear-echo type.

a) Proper longitudinal echo of transverse waves in an inhomogeneous plasma. Assume that there are two transverse external sources

$$j_{ext}(z,\ t) = \mathbf{e}_y \sum_{s=1}^{2} j_s \delta(z - a_s) \cos \omega_s t$$

in a plasma that is inhomogeneous along the z axis. We assume for simplicity the fundamental part of the plasma to be almost "cold" with density N(z) that decreases monotonically with increasing z, to which a "hot" component with a Maxwellian velocity distribution and homogeneous density n_0 is added. The frequencies $\omega_{1,2}$ of the external sources belong to the non-transparency region, by virtue of which the fields are contained in the linear approximation in skin layers near the points $z = a_{1,2}$. Under these conditions, in second order in the field amplitude, a longitudinal echo current of the "hot" component, with difference frequency $\omega_3 = \omega_2 - \omega_1 > 0$, is produced at a distance $\Delta z = (a_2 - a_1)\omega_1/\omega_3$ from the second source ($a_2 > a_1$). If the echo point z_c is located in the transparency region of a wave with frequency ω_3 ($\omega_3^2 > 4\pi e^2 N(z_c)/m$), then the echo current excites a natural longitudinal oscillation of the "cold" plasma component, and this oscillation propagates towards the plasma boundary. The locations of the sources, the echo points z_c, and a plot of the density N(z) are shown schematically in Fig. 3.

In the case when the echo point z_c coincides with the reflection point of the plasma wave $\omega_3^2 = 4\pi e^2 N(z_c)/m$, the expression for the electric field of the echo takes the form[85]

$$E_z^{(2)} = \frac{H_1 H_2 \omega_p^2}{H_* \omega_2^2} \left(\frac{\varepsilon_0}{\varepsilon(z)}\right)^{1/4} \exp\left[\frac{i\omega_3}{V_T} \int_{z_c}^{z} \varepsilon^{-1/2}(z')\,dz' - i\omega_3 t - 3i\frac{\pi}{4}\right],$$

where H_1 and H_2 are the magnetic fields near the sources at the frequencies ω_1 and ω_2, $H_* = 4\pi^{1/2} mc^2 \epsilon_0/ed$, $d = z_c - a_1$, $\epsilon_0 = (V_T/\omega_3 V)^{-2/3}$, V_T is the thermal velocity of the "cold" component, L is the length of the inhomogeneity of the density N(z) near the point z_c, $\omega_p^2 = 4\pi e^2 n_0/m$, and $\epsilon(z) = 1 - (4\pi e^2/m\omega_3^2)N(z)$ is the dielectric constant of the "cold" component at the frequency ω_3.

This effect can be used to transmit to the plasma boundary, in the form of plasma waves, information concerning oscillations from emitters located in the non-transparency region. It should be noted here that the longitudinal plasma wave (which propagates almost normally to the abrupt plasma boundary) is transformed, with the transformation coefficient on the order of unity, into a transverse wave which is radiated into the vacuum[86,87].

b) Echo at the summary frequency. The number of types of frequency spectra of the echo becomes much larger in an inhomogeneous plasma. Whereas in a homogeneous plasma the echo is nonlinear, and in a homogeneous isotropic plasma it is possible in the second order in the field amplitude only at the difference frequency, linear echo effects are possible in an inhomogeneous plasma (as shown in Secs. 5 and 6). We shall now show that a nonlinear echo at the summary frequency of the external signals is possible in an inhomogeneous plasma without a magnetic field. This effect is analogous fundamentally to the nonlinear local reflection, since in both cases the phase coherence of the particle is satisfied because of the focusing action of the plasma inhomogeneity.

We consider by way of example two longitudinal sources $j_{ext}(x,\ t) = \sum_{s=1}^{2} j_s \delta(x - a_s) \cos \omega_s t$ in a plasma with the inhomogeneity described in Sec. 5. Particles traveling in the positive x direction interact at the point a_1 and a_2 with the external source, as a result of which modulated streams can be produced in the plasma. We assume that $a_1 < a_2 < x_{\mathscr{E}}$, where $x_{\mathscr{E}}$ is the point of reflection of the particles with energy $\mathscr{E}$: After the particles are reflected from the potential $\Phi(x)$ that contains the inhomogeneous plasma, the phase of the oscillations of the second-approximation distribution function takes the form

$$\Psi(x,\ \mathscr{E}) = (\omega_1 + \omega_2)\left[\int_{a_2}^{x_{\mathscr{E}}} \frac{dx'}{v(x',\ \mathscr{E})} + \int_{x}^{x_{\mathscr{E}}} \frac{dx'}{v(x',\ \mathscr{E})}\right] + \omega_1 \int_{a_1}^{a_2} \frac{dx'}{v(x',\ \mathscr{E})},$$

where $v(x,\ \mathscr{E}) = \{2[\mathscr{E} - \Phi(x)]\}^{1/2}$. Under the phase-coherence condition $\partial\Psi(x,\ \mathscr{E})/\partial\mathscr{E} = 0$, a macroscopic current of plasma particles begins to flow and excites a natural plasma oscillation at the summary frequency $\omega_4 = \omega_1 + \omega_2$, if the resonance point $x_s(\mathscr{E})$, at which $\omega_4 = q(\omega_4,\ x_s)v(x_s,\ \mathscr{E})$ is situated near the phase-coherence point of the particles with energy $\mathscr{E}$. Thus, coherent excitation of a longitudinal wave at the summary frequency occurs under the condition

$$\left.\frac{\partial\Psi(x,\ \mathscr{E})}{\partial\mathscr{E}}\right|_{x=x_s(\mathscr{E})} = 0. \qquad (7.1)$$

Equation (7.1) determines the energy $\mathscr{E}_0$ of the particles generating a wave with frequency ω_4, and also the position of the echo point $x_c = x_s(\mathscr{E}_0)$. In this example, greatest interest from the experimental point of view attaches to the case $2\omega_{1,2} < \omega_{pe}(a_{1,2})$, $\omega_{pe}(x_c) < \omega_4$, when the fields near the emitters are confined to the skin layers in the first and second approximations, and transparency occurs in the echo region $x \sim x_c$ for the wave at the summary frequency.

In the case of a single source with $j_{ext} = j\delta(x - a) \times \cos \omega t$, the echo at the summary frequency leads to nonlocal generation of the second harmonic. The phase-coherence condition (7.1) means in this case that the travel times of the particles (with reflection) from the source to the resonance points x_s should be the same. Assuming $\Phi(x) = 0$ at $x < 0$ and $\Phi(x) = v_{Te}^2 x/L$ for $x > 0$, and assuming that the source and the echo points at double frequency are located in the region $x < 0$ where the plasma is homogeneous, we obtain as a result of the calculations the following expression for the electric field of the second harmonic[85]:

$$|E_x^{(2)}| = (eE_*^2/mv_T\omega)(\varkappa_2 v_T/\omega),$$

where $E_* = 2\pi j/v_T$, and $\kappa_2 \equiv \mathrm{Im}\, k\,(2\omega)$ is the decrement of the spatial damping of the longitudinal wave at the frequency 2ω. We note that the method indicated for harmonic generation in an inhomogeneous plasma can be quite effective for diagnostic purposes.

In concluding this section, we make a few remarks. The nonlocal effects considered above occur as a result of phase coherence of the particles focused by the plasma inhomogeneity, and lead to transport of the wave over a distance on the order or larger than the inhomogeneity length of the phase velocity of the waves. At sufficiently short distances, however, as noted in[82], transfer of information concerning the wave motion can occur also in the absence of focusing, as a result of incomplete cancellation of the currents of the individual particles. Such effects were investigated for a piecewise-homogeneous plasma in[88,89]. In a weakly inhomogeneous plasma, they lead to transport of information over a distance much shorter than the inhomogeneity length of the phase velocity of the waves[85]. In addition to the linear nonlocal effects considered above, a new type of transformation is possible in an inhomogeneous plasma, namely linear nonlocal transformation of waves belonging to different plasma oscillation modes[85,88]. As shown in[90,91], the echo is sensitive to the action of Coulomb collisions and to microturbulence, which smooth out the distribution-function perturbations that oscillate rapidly with the particle velocity. Therefore the foregoing investigation of nonlocal effects is valid in a sufficiently rarefied plasma with a low turbulence level, when the characteristic lengths of the processes considered here are small in comparison with the transport length of the oscillations of the distribution function. Furthermore, at sufficiently large distance, owing to a phase-mixing mechanism similar to that investigated in[92,93], the echo effects become saturated and the field of the echo wave has an asymmetrical dependence of the source amplitudes[94–96]. Nor do we consider here effects of temporal echo in an inhomogeneous plasma.

Temporal echo for the case of the upper hybrid resonance of a inhomogeneous plasma was investigated in[97].

8. Second harmonic generation by electromagnetic wave incident on an inhomogeneous plasma. As is well known, second-harmonic generation is possible in nonlinear homogeneous media only if the following synchronism conditions are satisfied[98]

$$\omega_2 = 2\omega_1, \quad \mathbf{k}_2 = 2\mathbf{k}_1.$$

In a homogeneous plasma without a magnetic field, the dispersion of the electromagnetic waves is such that the synchronism conditions cannot be satisfied. Nonetheless, experiment shows noticeable second-harmonic generation when electromagnetic waves are reflected from the ionosphere, and also an appreciable admixture of the second harmonic and the spectrum of the radio emission of the solar corona[20]. The solar second-harmonic radio emission band duplicates in general outline the features of the first-harmonic radio emission band. Attempts were therefore made to attribute this effect to Rayleigh scattering of the second harmonic of the plasma wave into electromagnetic radiation, or to Raman scattering of plasma waves by thermal fluctuations (see[6,20]). In[15] another second-harmonic generation mechanism was proposed, based on the inhomogeneity of the plasma and preserved in a cold plasma. Let us consider this case briefly.

1) Let an electromagnetic wave be incident on a cold isotropic plasma which is inhomogeneous along the z axis. The electric field vector is in the plane of incidence $E = (0, E_y, E_z)$, and the magnetic field has a component H_x. Assuming the nonlinear effects to be weak and representing the magnetic field of the second harmonic in the form $H_x^{(2)}(z, y, t) = H_2(z)\exp(2i\omega t - 2ik_\perp y)$, we obtain for $H_2(z)$ the equation[15]

$$\frac{d^2H_2}{dz^2} - \frac{d\varepsilon_2}{\varepsilon_2\, dz}\frac{dH_2}{dz} + \frac{4\omega^2}{c^2}(\varepsilon_2 - \alpha^2)\, H_2 = F(z), \tag{8.1}$$

where $\epsilon_2 = 1 - (\omega_{pe}^2(z)/4\omega^2)$ is the dielectric constant of the plasma at the second harmonic, and F(z) is a nonlinear source in the form

$$F(z) = \frac{ie\varepsilon_2}{mc\omega}\left[\frac{ik_\perp}{2\varepsilon_2^2}E_1^2\frac{d\varepsilon_1}{dz} - \frac{2ik_\perp}{\varepsilon_1\varepsilon_2}E_z^2\frac{d\varepsilon_1}{dz} - \frac{d}{dz}\left(\frac{E_yE_z}{\varepsilon_1\varepsilon_2}\frac{d\varepsilon_1}{dz}\right)\right]; \tag{8.2}$$

here $\epsilon_1 = 1 - (\omega_{pe}^2(z)/\omega^2)\,[1 + i(\nu_{eff}/\omega)]$ is the dielectric constant at the fundamental frequency, and E_z and E_y are the components of the electric field of the first harmonic. We call attention to the fact that F(z) is proportional to the density gradient and vanishes in a homogeneous plasma (in a homogeneous isotropic plasma the nonlinear currents induced by the incident wave are purely longitudinal). We note that the electric field of the first harmonic has a singularity at the plasma-resonance point where the dielectric constant ϵ_1 vanishes[5,6].

The second-harmonic radiation field is proportional to the matrix element

$$V = \int_{-\infty}^{+\infty} [F(z)\, H(z)/\varepsilon_2(z)]\, dz,$$

where H(z) is a solution of (8.1) without the right-hand side. In a weakly-inhomogeneous plasma, V is determined by the contribution of the integrand singularity closest to the real axis, in our case by the contribution of the resonance point (in a magnetic field, the synchronism conditions can be satisfied, so that in addition to the contribution of the singularities it is necessary to take into account the competing contribution of the synchronism point). Near the resonance point we have $\epsilon_1 = -[(z/L) + i(\nu_{eff}/\omega)] \equiv -\zeta/L$, and the magnetic and electric fields of the first harmonic take the form

$$\begin{aligned} &H_x^{(1)} = -H(0)\, k_\perp\zeta H_1^{(1)}(k_\perp\zeta), \quad H_x^{(1)}(0) = H(0), \\ &E_y^{(1)} \approx i\rho\alpha^2 H(0)\,[C + \ln(k_\perp\zeta/2)], \quad E_z^{(1)} = -\alpha L H(0)/\zeta; \end{aligned} \tag{8.3}$$

here C is the Euler constant. For a weakly-inhomogeneous plasma, the quasiclassical parameter $\rho = \omega L/c$ is large, so that noticeable generation of the second harmonic occurs at small angles of incidence of the wave on the plasma, when $\alpha = (ck_\perp/\omega) \ll 1$. In this case we obtain from (8.1)–(8.3), neglecting small terms of the order of α and ρ^{-1}, but assuming that $\rho\alpha^3 \gg 1$,

$$\frac{d^2H_2}{dz^2} + k_2^2H_2 = -\frac{eH^2(0)\, L^2\alpha^3}{mc^2}\frac{d}{dz}\left[\frac{(1+C) + \ln(k_\perp\zeta/2)}{i\zeta^2}\right], \tag{8.4}$$

where $k_2^2 = (4\omega^2/c^2)(\epsilon_2 - \alpha^2)$. To solve (8.4), we formulate the boundary conditions for the radiation

$$H_2(z) \to C_\pm \exp(\mp ik_2z) \quad \text{as} \quad z \to \pm\infty.$$

Then the constant $C_\pm$ are given by the formulas

$$C_\pm = \frac{i}{2k_2}\int_{-\infty}^{+\infty} F(z)\exp(\pm ik_2z)\, dz. \tag{8.5}$$

Since the singular point of the function F(z) lies in the lower half of the z plane, the constant C_+ is equal to zero. This means that the second harmonic is omitted from the resonance region "backward" and is present only in the reflected signal. Calculation of C_- yields

$$C_- = -\pi\sqrt{3}\,\frac{eH^2(0)}{mc\omega}\,\rho^2\alpha^3\left[2+i\frac{\pi}{2}+\ln\left(\frac{\alpha}{2\sqrt{3}}\right)\right]\exp\left(-\rho\sqrt{3}\,\frac{\nu_{\text{eff}}}{\omega}\right). \quad (8.6)$$

Let us compare the energy fluxes at the first and second harmonics. Their ratio is equal to

$$\frac{S_z^{(2)}}{S_z^{(1)}} = 3\left(\frac{e\alpha LH_0}{mc^2}\right)^2\left\{\pi^2+\left(2+\ln\left(\frac{\alpha}{2\sqrt{3}}\right)\right)^2\right\} \times \exp\left(-2\sqrt{3}\,\frac{\rho\nu_{\text{eff}}}{\omega}-\frac{8}{3}\rho\alpha^3\right), \quad (8.7)$$

where H_0 is the magnetic field of the first harmonic in vacuum. We estimate in our approximation the results for the maximum possible generation of the second harmonic. According to[15], the criterion for the smallness of the nonlinearity is

$$eH(0)/mc\omega \ll (\rho/\alpha)\,(\nu_{\text{eff}}/\omega)^{5/2}. \quad (8.8)$$

Substituting (8.8) in (8.7), we obtain

$$\frac{S_z^{(2)}}{S_z^{(1)}} \leqslant \alpha^2\left(\rho\frac{\nu_{\text{eff}}}{\omega}\right)^5\exp\left(-2\rho\sqrt{3}\,\frac{\nu_{\text{eff}}}{\omega}-\frac{4}{3}\rho\alpha^3\right) < \alpha^2\exp\left(-\frac{4}{3}\rho\alpha^3\right).$$

For comparison we indicate the fraction of the energy absorbed in the resonance region. As $\nu_{\text{eff}} \to 0$ and at $\rho\alpha^3 \gg 1$, it is equal to $W_\nu = 2S_z^{(1)}\exp(-4\rho\alpha^3/3)$. Thus, in the case investigated by us, a small fraction of the energy absorbed in the region $\epsilon_1 \approx 0$ is transformed into the second harmonic. One can expect them to become comparable, however, when $4\rho\alpha^3/3 \sim 1$ and $(eH(0)/mc\omega) \sim (\rho/\alpha)(\nu_{\text{eff}}/\omega)^{5/2}$ (for details see[15,32,39]).

2) We consider now second-harmonic generation under conditions when the collisions are sufficiently rare and effects of the finite temperature of the plasma become significant. When the thermal motion in the resonance region is taken into account in the resonance region $\epsilon_1 \approx 0$, linear transformation of the incident electromagnetic wave gives rise to plasma oscillations[5,6,47]. They carry energy away from the resonant region, by the same token limiting the electric field. As indicated in[5], it is possible to introduce in this case an effective dissipation for which $\nu_{\text{eff}} = \omega(r_D/L)^{2/3}$, where r_D is the Debye radius and L is the plasma-density inhomogeneity length in the region $\epsilon_1 \approx 0$. As seen from (8.6), the second-harmonic emission intensity does not depend on ν_{eff} at $\rho(\nu_{\text{eff}}/\omega) \ll 1$. It is unnatural to expect the results on second-harmonic generation to remain in force in a thermal collisionless plasma[15]. We shall demonstrate this by direct calculation of the second-harmonic emission.

First, in place of formula (8.2) it is necessary to take for a nonlinear source F(z) the expression

$$F(z) = -\frac{ie}{mc\omega}\frac{d}{dz}\left[2ik_\perp E_z^2+\frac{d}{dz}(E_yE_z)\right]. \quad (8.9)$$

Further, the longitudinal electric field is given by[47]

$$E_z^{(1)} = i\alpha H(0)\,(\rho/\beta)^{2/3}\int_0^\infty \exp\left[iq\xi+(iq^3/3)\right]dq, \quad (8.10)$$

where

$$\beta = v_{Te}/c,\quad z = (c/\omega)\,(\rho\beta^2)^{1/3}\xi.$$

For the transverse component of the electric field we obtain

$$E_y^{(1)} = i\rho\alpha^2 H(0)\left[A-\int_0^\infty q^{-1}\left(e^{iq\xi}-1\right)e^{iq^3/3}\,dq\right]; \quad (8.11)$$

here $A = i(\pi/3) + (1/3)C + \ln(\alpha/2) + (1/3)\ln(\rho\beta^2/3)$.

Substituting (8.9)—(8.11) in (8.5) and assuming that $(\rho\beta^2)^{1/3} \ll 1$, we obtain by calculation the same expression (8.6) for the constant C_- as before, while the constant C_+ turns out to be equal to zero. Thus, the second-harmonic emission does not depend on the concrete mechanism that limits the electric field in the resonance region. In other words, the equivalence of thermal motion to dissipation, which was demonstrated above, is preserved also in the nonlinear case[39]. This conclusion remains in force for an inhomogeneous plasma in a magnetic field. It is therefore necessary to point out the published incorrect result[99] on the generation of the second harmonic of an electromagnetic wave incident on an inhomogeneous plasma situated in an inhomogeneous magnetic field. According to[99], the direction of the second-harmonic radiation from the hybrid resonance region $\omega_{pe}^2 + \omega_{He}^2 = \omega^2$ varies, depending on whether the longitudinal electric field is limited in the resonance region by collisions or by the loss of energy through thermal oscillations. The error in that paper lies in the incorrect choice of the integral representation for the electric field of the plasma wave emerging from the region of the hybrid resonance.

We call attention to a qualitative difference between the generation of the second harmonic of the electromagnetic wave in a cold collision-governed plasma and the case of generation in a thermal collisionless plasma. In a cold plasma, the second-harmonic generation is connected with singularities of the field of the initial wave.

In a thermal plasma, the wave fields are analytic functions (see formulas (8.10)—(8.11)). Thermal oscillations (plasmons) are then produced in the resonance region as a result of linear transformation of the incident electromagnetic waves. It is the nonlinear interaction of two plasmons, and also of a plasmon with an incident wave, which leads to second-harmonic generation.

In concluding this section, we make the following remarks. First, second-harmonic generation in normal incidence of an extraordinary wave on a cold inhomogeneous plasma situated in a homogeneous magnetic field was investigated in[100]. The results do not differ in principle from those given here. Second, the hysteresis phenomena investigated in[101] that are parasitic for second-harmonic generation and are produced in the resonance region in a sufficiently strong high-frequency field. In our case they are of no importance if the criterion (8.8) is satisfied.

It should also be noted that in a magnetoactive plasma the dependence of the phase velocity of the wave on the frequency, plasma density, and other parameters is nonmonotonic, and therefore the synchronism conditions can be satisfied. For example, in the case of normal incidence of an extraordinary wave on a cold magnetoactive plasma, the synchronism between the first and n-th harmonic is satisfied along the curve $\omega_{He}^2\omega_{pe}^2 = (\omega_{pe}^2 - \omega^2)(n^2\omega^2 - \omega_{pe}^2)$ in the plane of the parameters $(\omega_{He}^2, \omega_{pe}^2)$. This circumstance should lead to a "crumbling" of a wave propagating to the synchronism region of an inhomogeneous plasma[21].

9. Nonlinear generation of electromagnetic radiation upon propagation of a plasma wave in an inhomogeneous plasma. As shown in the preceding section, second harmonic generation is produced in a resonant layer with $\epsilon_1 \approx 0$ when an electromagnetic wave is incident on a weakly inhomogeneous plasma. However, the incident wave penetrates into the region with $\epsilon_1 \approx 0$ under the non-transparency barrier, and therefore the generation effect is, generally speaking, small and depends strongly on the angle of incidence of the wave on the plasma. It is therefore of interest to consider the excitation of the second harmonic of an electromagnetic wave by a plasma wave that is directly reflected in the resonant layer. We note that in a homogeneous plasma one plasma wave does not generate an electromagnetic wave[102].

According to[53], where this problem was solved, the second-harmonic magnetic field satisfies equation (8.1) with a nonlinear source F(z) in the form (8.9), in which it is necessary to substitute the perturbations due to the plasma wave. We point out that F(z) vanishes in a homogeneous plasma, for in this case the nonlinear currents induced by the plasma wave are purely longitudinal.

We retain the notation introduced in Sec. 8 and assume that the parameter $\rho\alpha^3$ is large. Under this condition, the linear transformation of the plasma wave into an electromagnetic wave is exponentially small[47]. We consider first the case $\rho\beta^2 \ll 1$, where the characteristic length of variation of the field of the plasma wave in the resonance region $L(\beta/\rho)^{2/3}$ is small in comparison with the second-harmonic wavelength of the electromagnetic wave c/ω. Using the results of the linear theory[47,103], we write for the longitudinal electric field the expression

$$E_z^{(1)} = A\pi^{-1/2}(\rho\beta^2)^{1/6}\int_0^\infty \left[\exp\left(-iq\xi - i\frac{q^3}{3}\right) - \exp\left(iq\xi + i\frac{q^3}{3}\right)\right]dq, \quad (9.1)$$

where A = const, and the normalization is chosen in such a way that in the incident plasma we have

$$E_z^{(1)} = A\left(\frac{\omega}{ck_z}\right)^{1/2}\exp\left[-i\int_0^z k_z(z')\,dz'\right], \quad S_z^{(1)} = \frac{T_e}{2}\left|n_1 v_z^{(1)}\right| = \frac{c\beta^2}{8\pi}|A|^2;$$

Here $k_z = (\omega/v_{Te})(-z/L)^{1/2}$. If $(\rho\beta^2)^{1/3} \ll 1$, the nonlinear source oscillates rapidly at the second-harmonic wavelength, and it can therefore be averaged. After the averaging, we obtain

$$\langle F(z)\rangle = (2e\alpha A^2/mc^2)(\rho\beta^2)^{1/2}(d/dz)(-c/\omega z)^{1/2} \quad \text{if} \quad z < 0,$$
$$\langle F(z)\rangle = 0 \quad \text{if} \quad z > 0.$$

Calculation of the amplitude of the magnetic field of the second harmonic of the electromagnetic wave is analogous to that carried out in Sec. 8, and yields

$$C_- = -(2\pi/3)^{1/2}(e\alpha A^2/mc\omega)(\rho\beta^2)^{1/2}e^{i\pi/4}, \qquad C_+ = -C_-^*. \quad (9.2)$$

Thus, the second harmonic is radiated equally in both directions from the resonant layer. The coefficient of transformation of the plasma wave into the second harmonic is

$$(S_z^{(2)}/S_z^{(1)}) = (8\pi/3)\rho\beta^2(e\alpha A/mv_{Te}\omega)^2. \quad (9.3)$$

In the second limiting case $(\rho\beta^2)^{1/3} \gg 1$, the nonlinear source F(z) is a slowly varying function of z. Therefore the main contribution to the second-harmonic emission comes from the synchronism point $z_0 = -L(k_2 v_{Te}/2\omega)^2$, at which $k_2 = 2k_1 = (2\omega/c\beta)(-z_0/L)^{1/2}$. The dimension of the synchronsim region can be estimated from the condition

$$\left|\int_{z_0}^{z_0+\Delta z}(k_2 - 2k_1)\,dz\right| \sim 1.$$

It follows therefore that $\Delta z \sim (c/\omega)(\rho\beta^2)^{1/2}$. At the same time, F(z) is proportional to the small factor of "transversality" of the nonlinear currents induced by the plasma waves, $|dk_1/k_1^2 dz| \sim (\rho\beta^2)^{-1}$. The final result takes the form

$$C_- = \frac{4}{9}\sqrt{2}\left(\frac{e\alpha A^2}{mc\omega}\right)(\rho\beta^2)^{-1/2}\exp\left(i\frac{\sqrt{3}}{4}\rho\beta^2 - i\frac{\pi}{4}\right), \quad C_+ = -C_-^*. \quad (9.4)$$

The coefficient of transformation of the plasma wave into the second harmonic is

$$S_z^{(2)}/S_z^{(1)} = (2/\rho\beta^2)(4e\alpha A/9mv_{Te}\omega)^2. \quad (9.5)$$

From a comparison of (9.2)—(9.3) with (9.4)—(9.5) we see that in the region $\rho\beta^2 \ll 1$ the generation effect increases with increasing inhomogeneity length L of the plasma density, reaches a maximum at $\rho\beta^2 \sim 1$ when $(S_z^{(2)}/S_z^{(1)}) \sim (8\pi/3)(e\alpha A/mv_{Te}\omega)^2$, and then decreases with further increase of L, in full accord with the case of a homogeneous plasma. We note also in conclusion that generation of electromagnetic radiation with frequency $\omega \approx 2\omega_{pe}$ by plasma oscillations with a broad wave-number spectrum in a homogeneous unbounded plasma was investigated in[104], and in a homogeneous plasma layer in[105].

10. Singularities of nonlinear interactions of waves in an inhomogeneous plasma. The foregoing investigation of the generation of second harmonic of an electromagnetic wave shows that the nonlinear interaction of waves in the presence of plasma inhomogeneity has a number of singularities in comparison with the case of a homogeneous plasma. These will be briefly analyzed here on the basis of the presently available literature[21,53,99,106]. We note that we are not concerning ourselves here with parametric instability of waves in an inhomogeneous plasma.

a) Nonlinear wave shift. A particular case of nonlinear wave shift is the second-harmonic generation considered above[15,53]. In the quasistationary state, the amplitude $\Psi_3(\mathbf{r})$ of the wave excited by the shift is proportional to the matrix element $V = \int \Psi_1\Psi_2\Psi_3^* d\mathbf{r}$, where $\Psi_{1,2}$ describe the fields of the initial waves and are assumed specified. In the quasiclassical region, the functions $\Psi_n(z)$ are given by

$$\Psi_n \sim \exp\left(\pm i\int^z k_n(z')\,dz'\right),$$

and the wave vector $k_n(z)$ satisfies the dispersion equation $\omega = \omega_n(k_n)$. In a weakly inhomogeneous plasma, the question of the estimate of the value of the matrix element V is perfectly analogous to the question of the estimate of the value of the quasiclassical matrix element in quantum mechanics[107]. Considering Ψ_n and k_n in the complex z plane (as an analytic continuation of the real axis), we find that the matrix element V is determined by the competition of contributions made to the integral from the singularities (such as poles and branch points) of the functions $\Psi_n(z)$, and the contributions of the point z_0 at which the decay condition $k_1 + k_2 = k_3$ is satisfied. In the latter case, the dimension Δz of the wave-generation region can be estimated from the condition

$$\left|\int_{z_0}^{z_0+\Delta z}(k_1 + k_2 - k_3)\,dz\right| \sim 1.$$

Thus, decays forbidden in a homogeneous plasma (the decay condition is not satisfied on the real z axis) occur in an inhomogeneous plasma with exponentially small probability if the fields of the interacting waves have no singularities on the real z axis.

Attention should be called to the fact that a change takes place in the very formulation of the problem of limiting the amplitudes of waves growing in an inhomogeneous plasma as a result of nonlinear interactions[53].

In a homogeneous plasma, nonlinear wave interaction leads to a redistribution of the energy over the spectrum of the wave numbers k, but the interaction itself proceeds simultaneously in all of **r**-space. In the case of an inhomogeneous plasma, the decay occurs in a boundary region of space occupied by the plasma. Therefore the decay problem should be formulated in terms of wave emission from the decay region, and at sufficiently small amplitudes the outflow of energy from the region obviously leads to establishment of a quasistationary state. This, in particular, distinguishes decay in an inhomogeneous plasma from decays in a bounded plasma, which were investigated in[108]. By way of illustration, we consider the system[53]

$$\Psi_1''+[(z-z_1)/4\lambda^2]\,\Psi_1=0,\qquad \Psi_2''+z\Psi_2=\Psi_1^2, \tag{10.1}$$

which simulates second-harmonic generation in an inhomogeneous medium. The wave vectors of the first and second harmonic are equal to $k_1 = (z-z_1)^{1/2}(2\lambda)^{-1}$, $k_2 = z^{1/2}$, and the synchronism condition $k_2 = 2k_1$ is satisfied at the point $z_0 = z_1/(1-\lambda^2)$. Since the solutions of (10.1) are analytic functions, second harmonic is generated only in the vicinity of the synchronism point z_0. Taking into account the boundary condition of the radiation $\Psi_2 = Iz^{-1/4}\exp[i(\pi/4)+(2/3)iz^{3/2}]$ as $z\to+\infty$, we obtain $I = \alpha\int_0^\infty v(\beta+\alpha\xi^2)d\xi$, where $v(\xi)$ is an Airy function and $\beta = (z_1^2 z_0)^{1/3}$, $\alpha = [(z_0-z_1)^2/16z_0z_1]^{1/3}$.

If $\beta \gg 1$, then the synchronism point is located in the wave transparency region, far from the reflection points. Then $I\approx(1/2)(\pi\alpha/\beta)^{1/2}\cos(2\beta^{3/2}/3)$. We note that

$$\frac{2}{3}\beta^{3/2}=\int_0^{z_0}k_2\,dz-2\int_{z_1}^{z_0}k_1\,dz.$$

For $\beta < 0$, the synchronism point falls into the nontransparency region, therefore the generation effect becomes exponentially small: $I\approx(1/4)\,|\pi\alpha/\beta|^{1/2}\times\exp(-2|\beta|^{3/2}/3)$.

b) Decay of large-amplitude wave. We now consider the decay of a wave of large amplitude, called pump wave, in a weakly-inhomogeneous medium. Problems of this type were solved in[106,107]. The main features of this decay can be traced with the following simple example. If the reflection points are far from the decay region, the equation for the amplitude $a_{1,2}(z)$ of the growing waves can be represented in the form

$$da_{1,2}/dz=(\varkappa a_0/a_c)\,a_{2,1}^*\exp(ik_0z^2/L), \tag{10.2}$$

where a_0 is the amplitude of the pump wave, which is assumed invariant; κ, k_0, and a_c are positive parameters; the decay condition is satisfied at the point $z = 0$, by virtue of which

$$\int_0^z(k_0(z')-k_1(z')-k_2(z'))\,dz'=\frac{k_0z^2}{L}.$$

For the function $b_1(z) = a_1(z)\exp(-ik_0z^2/2L)$ we obtain from (10.2) the equation

$$(d^2b_1/dz^2)+[(k_0z/L)^2+i(k_0/L)-(\varkappa\,|a_0|/a_c)^2]\,b_1=0. \tag{10.3}$$

As seen from (10.3), the growth of the waves occurs in the region of the barrier $|z| < \kappa L\,|a_0|/k_0 a_c$, the width of which is proportional to the amplitude of the pump wave and to the inhomogeneity length L.

We substitute for Eqs. (10.2) the boundary conditions $|a_1|^2 = 1$ and $|a_2|^2 = 0$ as $z\to-\infty$. The solution of (10.3) is expressed in terms of parabolic-cylinder functions. As a result we have as $z\to+\infty$

$$|a_1|^2=e^{\mu}\,|a_2|^2=e^{\mu}-1,\qquad \mu=(\pi\varkappa^2/k_0)\left|\frac{a_0}{a_c}\right|^2L. \tag{10.4}$$

Thus, in an inhomogeneous medium the decay of the pump wave leads to a finite amplification of the waves a_1 and a_2. From (10.4) we obtain the condition for the termination of the decay instability $L < (k_0\Lambda/\pi\kappa^2)\,|a_c/a_0|^2$, where Λ is a number on the order of five.

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1)We emphasize that if the corresponding parameter in the fourth-order derivative ceases to be small when account is taken of the thermal motion, it becomes necessary to investigate the exact integral equation.

2)Apparently, because of a misunderstanding, the author of [58] has incorrectly interpreted the results of [14,17]. Namely, when referring to [14,17], he does not give all the conditions obtained there, which are needed for anomalous transformation, and only the condition that the coefficient of the second derivative vanish. In addition, the exponential smallness of the transformation coefficient in [37,38], which the author of [58] uses as a supporting argument, is in fact connected not with the conditions for the applicability of the phase-integral method, but with the type of transformation. In the terminology of the present paper, the transformation referred to in [37,38] is of the over-the-barrier type.

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