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STRESSES, VISCOSITY, AND OPTICAL ANISOTROPY OF A MOVING SUSPENSION OF RIGID ELLIPSOIDS

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CONTENTS

The general results are presented of the theory of motion of a dilute suspension of rigid ellipsoids, applied to the case when the rotational Brownian motion of the particles becomes significant. In this case the equations of motion are written with the aid of moments of the distribution function, which are the internal variables describing the orientation of the particles in the field and in the stream. The equation of motion contains a finite relaxation time, and consequently the suspension of rigid ellipsoids is an example of a visco-elastic system. Expressions are presented for the viscosity of a suspension under simple shear motion.

1. INTRODUCTION

THE modern theory of the flow of suspensions dates back to 1906, when Einstein published a paper[1] (see also [2], Sec. 22) devoted to the calculation of the viscosity of a suspension of solid spheres in a viscous liquid. Einstein's formulation of the problem and method of its solution served as a pattern followed in the study of flow of the suspension of rigid ellipsoids.[3-6] Just as in the case of a suspension of rigid spheres, it was assumed that the suspension of ellipsoids is an ordinary viscous liquid described by the Navier-Stokes equations, and the presence of particles leads only to a change in the viscosity of the system. In these first investigations, therefore, the problem was raised of calculating the viscosity coefficient as a function of the concentration, of the velocity gradients, and of the dimensions and shape of the particle, but no equations of motion capable of analyzing the behavior of the system under any experimental situation were derived for the suspension; these equations of motion remained unknown until recently.

Since any system (henceforth assumed to be incompressible) satisfies[2] the continuity equation

$$\operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

and the equation of motion

$$\rho \frac{dv_i}{dt} = \frac{\partial \sigma_{ik}}{\partial x_k}, \tag{1.2}$$

the question of determining the equation of motion of the system reduces to finding the stress tensor σ_{ik} as a function of the velocity gradients $\partial v_j/\partial x_s$. In the simplest case the stress tensor is given by

$$\sigma_{ik} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), \tag{1.3}$$

where p is the pressure and η is the viscosity coefficient, which in this case is a material constant of the system. The system (1.1)–(1.3) describes the motion of a viscous liquid. In more complicated cases, the stress tensor cannot be defined in terms of the velocity gradients alone. In particular, suspension of rigid ellipsoids considered in this article, the stress tensor is expressed, in addition to the velocity gradients, also in terms of the moments of the distribution function, for which equations of motion should also be written.

In this article we present the general results of the theory of motion of a dilute suspension of rigid ellipsoids, applicable to the case when the rotational Brownian motion of the particles is significant. The coefficient of rotational diffusion should be large in comparison with the velocity gradients. This requirement imposes a limitation on the particle dimensions. According to the estimate, for ordinary liquids and velocity gradients, the dimensions of the suspended particles should be small compared with 10^{-3}–10^{-4} cm, but of course large in comparison with the molecular dimensions. The motion of the suspension without allowance for the

Brownian motion of the particles is considered in Secs. 3 and 4. The equations of motion of suspension with allowance for the rotational Brownian motion are determined without a field in Secs. 6 and 7, and in an electric field in Secs. 8 and 9. In Sec. 10 we consider the optical anisotropy of a moving suspension.

2. PERTURBATION OF THE FLOW OF A VISCOUS LIQUID BY AN UNDEFORMED ELLIPSOID

Let us consider first, following Jeffrey,[3] the perturbation introduced into the motion of a liquid with viscosity μ by a single particle of ellipsoidal shape, the surface equation of which, in a coordinate system with origin at the center of the particle, is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. It will also be convenient to use for the semiaxes of the ellipsoid the notation $a_1 \equiv a$, $a_2 \equiv b$, and $a_3 \equiv c$.

Let the liquid flow unperturbed by the particle be described by a velocity distribution $\nu_{ik}x_k$, where ν_{ik} is the constant (in the laboratory frame) tensor of the velocity gradients, which, generally speaking, is not symmetrical with respect to the indices. We introduce the symbols γ_{ik} and ω_{ik} for the symmetrized and antisymmetrized parts of the velocity-gradient tensor, so that $\nu_{ik} = \gamma_{ik} + \omega_{ik}$.

The liquid flow perturbed by a rigid ellipsoid is described by a velocity field v_i, which asymptotically approaches the unperturbed velocity field at large distances: $v_i \to \nu_{ik}x_k$. On the boundary surface, the liquid moves together with the surface of the particles.

The perturbed velocity field should be sought as the solution of the system of equations of an incompressible liquid, which have the following form[2] at small Reynolds numbers:

$$\rho\frac{\partial v_i}{\partial t} = -\nabla_i p + \mu\Delta v_i, \quad \operatorname{div}\mathbf{v} = 0. \tag{2.1}$$

The particle in the stream has not only translational but also generally speaking rotational motion. The solution of the problem of the perturbation of the liquid flow by the ellipsoid is conveniently sought in a coordinate system connected with the particle and rotating with angular velocity Ω_{js}.

The conversion from the coordinates of a point in the laboratory system of coordinates x_i to the coordinates of the point in the rotating system of coordinates x'_k and conversely, with a common origin, is described by the equations

$$x_i = a_{ik}x'_k, \; x'_k = a_{jk}x_j, \tag{2.2}$$

where a_{ik} is the cosine of the angle between the i-th axis of the immobile system to the k-th axis of the rotating system, with

$$a_{ik}a_{lk} = \delta_{il}, \; a_{ik}a_{il} = \delta_{kl}. \tag{2.3}$$

Differentiating (2.2) with respect to time and recognizing that the linear velocity of a point that is immobile in the rotating coordinate system is given by the relation

$$u_i = \Omega_{ij}x_j,$$

we obtain the conversion law for the velocity, and then, after differentiating with respect to the coordinates, the law governing the conversion of the velocity-gradient tensor. The symmetrized and antisymmetrized parts satisfy the relations

$$\gamma'_{ik} = a_{jl}a_{ik}\gamma_{jl}, \quad \omega'_{ik} = a_{jl}a_{ik}(\omega_{jl} - \Omega_{jl}). \tag{2.4}$$

In the moving frame, accurate to terms of first order in the velocity gradients, the system of equations of motion (2.1) takes the form

$$\mu\Delta v_i = \nabla_i p, \quad \operatorname{div}\mathbf{v} = 0. \tag{2.5}$$

On the surface of the particle we have $\mathbf{v} = 0$, and at large distances from the particle $v_i \to \nu_{ik}x_k$.

The solution of (2.5), obtained by Jeffrey,[3] has in the moving frame the form

$$v_i = \nu_{ik}x_k + \frac{\partial}{\partial x_l}(T_j\chi_j) + B_{ik}\varepsilon_{kjs}\frac{\partial\chi_s}{\partial x_j} + x_jA_{jl}\frac{\partial^2\Omega}{\partial x_i\,\partial x_l} - A_{il}\frac{\partial\Omega}{\partial x_l}, \tag{2.6}$$

$$p = p_0 + 2\mu A_{jl}\frac{\partial^2\Omega}{\partial x_j\,\partial x_l}, \tag{2.7}$$

where p_0 is the pressure in the unperturbed liquid, and Ω and χ_i satisfy the Laplace equation and are given by

$$\Omega = \int_\lambda^\infty \left(\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda}\right)\frac{d\lambda}{R},$$

$$\chi_1 = \alpha' yz, \quad \chi_2 = \beta' zx, \quad \chi_3 = \gamma' xy.$$

The quantities T_j, B_{ik}, and A_{ik} are determined by the boundary conditions and can be represented in the form

$$T_1 = -\frac{\gamma_{23}}{\alpha'_0}, \quad T_2 = -\frac{\gamma_{13}}{\beta'_0}, \quad T_3 = -\frac{\gamma_{12}}{\gamma'_0},$$

$$B_{ik} = 2(a_i^2A_{ii} - a_k^2A_{kk}).$$

The diagonal components of the matrix A_{jk} are given by

$$A_{ii} = \frac{3\alpha''_{i0}\nu_{ii} - \sum\alpha''_{k0}\nu_{kk}}{6(\beta''_0\gamma''_0 + \gamma''_0\alpha''_0 + \alpha''_0\beta''_0)}, \tag{2.8}$$

and the off-diagonal components

$$A_{ik} = \frac{\alpha_{i0}(a_k^2 - a_i^2)\gamma_{ik}}{2(\alpha_{i0} - \alpha_{k0})(a_i^2\alpha_{i0} + a_k^2\alpha_{k0})} + \frac{a_k^2\omega_{ik}}{2(a_i^2\alpha_{i0} + a_k^2\alpha_{k0})}. \tag{2.9}$$

We point out that the components of the matrix A_{ik} do not form a tensor.

In expressions (2.8), (2.9), and those following we use the following notation for the integrals:

$$\alpha_0 \equiv \alpha_{10} = \int_0^\infty \frac{d\lambda}{(a^2+\lambda)R}, \quad \alpha'_0 \equiv \alpha'_{10} = \int_0^\infty \frac{d\lambda}{(b^2+\lambda)(c^2+\lambda)R},$$

$$\alpha''_0 \equiv \alpha''_{10} = \int_0^\infty \frac{\lambda\,d\lambda}{(b^2+\lambda)(c^2+\lambda)R}, \tag{2.10}$$

where $R = [(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]^{1/2}$. An analogous notation is introduced for integrals with the other semiaxes of the ellipsoid. The same symbols without zero denote the corresponding integrals with a variable lower limit.

3. MOTION OF ELLIPSOID IN A STREAM

The liquid exerts on a unit surface of the ellipsoid a force $P_i = \sigma_{ik}n_k$, where for a viscous incompressible liquid

$$\sigma_{ik} = -p\delta_{ik} + 2\mu\gamma_{ik}. \tag{3.1}$$

Calculating the velocity derivatives (2.6) at $\lambda = 0$ with the aid of the indicated formulas, we obtain an expression for the force acting on a unit surface area of the particle:

$$P_i = -p_0 g\frac{x_i}{a_i^2} + \frac{8\mu g}{abc}A_{il}\frac{x_l}{a_l^2} - 4\mu g\left(\sum_{j=1}^3 \alpha_{j0}A_{jj}\right)\frac{x_i}{a_i^2}, \tag{3.2}$$

where

$$g=\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{-\frac{1}{2}}.$$

The total force acting on the particle in the liquid is equal to zero, as can be verified by integrating (3.2) over the entire surface of the particle.

Calculating the torque acting on the particle

$$K_{ik}=\oint(P_i x_k-P_k x_i)\,df \tag{3.3}$$

with allowance for the fact that

$$\frac{1}{\Omega}\oint x_i x_k g\,df=a_i^2\delta_{ik}, \tag{3.4}$$

where $\Omega = 4\pi abc/3$ is the volume of the particle and df is the element of the surface area, we obtain

$$K_{ik}=\frac{32}{2}\pi\mu(A_{ik}-A_{ki}). \tag{3.5}$$

The rotational velocity of the moving particle is determined from the requirement that the sum of all the torques acting on the particle be equal to zero. We neglect the intertial forces.

If the particle is acted upon only by hydrodynamic forces of the stream, then the requirement that the moment (3.5) vanish leads to the relation

$$(a_k^2-a_i^2)\gamma'_{ik}+(a_k^2+a_i^2)\omega'_{ik}=0, \tag{3.6}$$

from which we determine with the aid of (2.4) the particle rotational velocity

$$\Omega_{lm}=\frac{a_k^2-a_i^2}{a_k^2+a_i^2}a_{li}a_{ji}a_{mk}a_{sk}\gamma_{js}+\omega_{lm}. \tag{3.7}$$

As a result of relations (2.3), in the general case the description of the position of the particle requires two vectors and the ellipsoid of revolution requires one vector directed along the particle symmetry axis. In the latter case

$$\Omega_{lm}=\lambda(e_m\gamma_{ls}-e_l\gamma_{ms})e_s+\omega_{lm}, \tag{3.8}$$

where we have introduced the notation $a_{i1} = e_i$. The quantity $\lambda = (a^2-b^2)/(a^2+b^2)$ varies from -1 to 1, and its limiting cases are a flat disc and a thin needle. At $\lambda = 0$ the particle degenerates into a sphere.

From (3.8), using the formula $\dot{e}_i = \Omega_{ik}e_k$, we obtain the linear velocity of the end of the vector

$$\dot{e}_l=\lambda(\gamma_{ls}e_s-\gamma_{ms}e_m e_s e_l)+\omega_{lm}e_m. \tag{3.9}$$

The general expression (3.9) was derived by Hand,[7] who generalized Jeffrey's results.[3]

4. STRESS TENSOR

The presence of rigid particles in the liquid changes the local velocity gradients in the liquid. The density of energy dissipation increases because the local velocity gradients increase and decreases because part of the liquid is replaced by solid particles. It will be shown later, however, that the net result is an increase in the energy dissipation and consequently in the effective viscosity of the system.

We assume as before that the velocity and the tensor of the velocity gradients of the liquid are such that at large distances they become equal to the unperturbed velocity and velocity-gradient tensor ν_{ik}. Therefore the observed, smoothed-out, macroscopic velocity is defined as $w_i = \nu_{ik}x_k$. The observed velocity-gradient is $\tilde{\nu}_{ik}$, and is not equal to the average value of the microscopic velocity gradients $(1/V)\int\tilde{\nu}_{ik}\,dV$, as was assumed by several authors without sufficient justification.[1,8,9]

Thus, the entire problem consists of determining, for asymptotically specified values of the velocity gradient ν_{ik}, the averaged stress tensor, which can be calculated in terms of the averaged energy dissipation in the system, a procedure initiated by Einstein,[1] or in terms of the averaged value of the momentum flux tensor $\Pi_{ik} = \rho v_i v_k - \sigma_{ik}$ in the region where the observed velocity gradient is constant. The latter method was proposed by Landau and Lifshitz[2] and turns out to be more convenient, since it does not give rise to divergent expressions.

Changing over to a coordinate system moving with a velocity w_i, we obtain, accurate to first-order terms in the small difference $w_i - v_i$, where v_i is the microscopic value of the velocity,

$$\sigma_{ik}=\frac{1}{V}\int\tilde{\sigma}_{ik}\,dV, \tag{4.1}$$

where $\tilde{\sigma}_{ik}$ is the microscopic value of the stress tensor.

Before we proceed to the calculations, we note that the stress tensor may turn out to be asymmetrical. Indeed, the particle may be acted upon by a certain extraneous torque, which in the approximation under consideration should be balanced by the torque exerted on the particle by the liquid and determined in the general case by expression (3.3). If n is the particle-number density, then

$$\sigma_{ik}-\sigma_{ki}=-nK_{ik}. \tag{4.2}$$

It is convenient to calculate the stress tensor (4.1) by breaking up the region of integration into two regions (liquid and solid). It should be borne in mind here that the integration over the separation surface makes a special contribution to the stress tensor. We can rewrite (4.1) in the form

$$\sigma_{ik}=\frac{1}{V}\int\limits_{V-\Omega}\tilde{\sigma}_{ik}^{S}\,dV+\frac{\varphi}{\Omega}\int\limits_{\Omega}\tilde{\sigma}_{ik}^{L}\,dV-\frac{1}{2}\frac{\varphi}{\Omega}K_{ik}, \tag{4.3}$$

where φ is the volume concentration of the solid phase and Ω the volume of one particle. Indeed, since the tensors $\tilde{\sigma}_{ik}^{L}$ and $\tilde{\sigma}_{ik}^{S}$ are symmetrical, and the torque K_{ik} is antisymmetrical, we obtain (4.2) from (4.3).

The liquid in which the ellipsoids are suspended is by assumption a Newtonian liquid, the stress tensor of which is given by (3.1). After calculating the velocity gradients and averaging over the volume, we find that the first term in (4.3) is

$$(-p_0+2\mu\gamma_{ik})(1-\varphi). \tag{4.4}$$

The second term in (4.3) can be calculated in terms of the integral over the surface of the particle (see [2], Sec. 2), and is equal to

$$\frac{1}{2}\frac{\varphi}{\Omega}\oint(P_i x_k+P_k x_i)\,df. \tag{4.5}$$

Thus, taking into account the expression for the moment of the forces (3.3), we write

$$\sigma_{ik}=(-p_0+2\mu\gamma_{ik})(1-\varphi)+\frac{\varphi}{\Omega}\oint P_k x_i\,df. \tag{4.6}$$

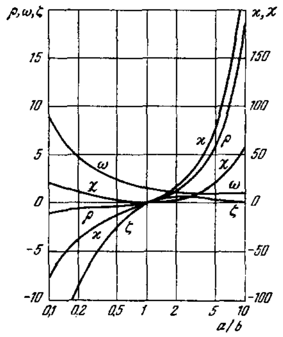

FIG. 1. Coefficients of the equation of motion of a suspension of rigid ellipsoids of revolution as functions of the ratio a/b of the semi-axes of the ellipsoid of revolution.

The force exerted on the surface of the ellipsoid by the liquid is given by (3.2), the use of which, with allowance for (3.4) yields, in accord with formula (4.6)

$$\sigma'_{ik} = -p_0\delta_{ik} + 2\mu(1-\varphi)\gamma'_{ik} + \frac{8\mu\varphi}{abc}A'_{ik} - 4\mu\varphi\left(\sum_{j=1}^{3}\alpha_{j0}A'_{jj}\right)\delta_{ik}. \quad (4.7)$$

The last expression has been written out in a coordinate system bound to the particle, and does not have an explicit covariant form, since $\sum \alpha_{j0} A'_{jj}$ is not a scalar, and the matrix A'_{ik} is not a tensor.

We determine the values of the components of the matrix A'_{ik} with the aid of relation (3.6) and transform (4.7) to the laboratory frame, considering for simplicity ellipsoids of revolution, after which we obtain the stress tensor of a moving suspension of rigid ellipsoids of revolution in the case when the rotational Brownian motion of the particles is disregarded:

$$\sigma_{ik} = -(p_0 + \mu\varphi\rho e_j e_s \gamma_{js})\delta_{ik} + 2\mu(1+\omega\varphi)\gamma_{ik} + \mu\zeta\varphi(e_i e_s \gamma_{sk} + e_k e_s \gamma_{si}) + \mu\chi\varphi e_i e_k e_j e_s \gamma_{js}, \quad (4.8)$$

where

$$\rho = \frac{1}{3ab^4\alpha'_0\beta''_0}\left[2(\alpha''_0 - \beta''_0) + 3ab^2(\alpha_0\alpha''_0 - \beta_0\beta''_0)\right],$$

$$\omega = \frac{1}{ab^4\alpha'_0} - 1, \quad \zeta = \frac{4}{ab^2(a^2+b^2)\beta'_0} - \frac{2}{ab^4\alpha'_0}, \quad (4.9)$$

$$\chi = \frac{2\alpha''_0}{ab^4\alpha'_0\beta''_0} - \frac{8}{ab^2(a^2+b^2)\beta'_0} + \frac{2}{ab^4\alpha'_0}.$$

The values of the constants (4.9) are shown in Fig. 1 as functions of the ratios of the ellipsoid semi-axes.

Expressions (4.8) and (4.9), with slight inaccuracies in the definitions of ρ and ω were indicated by Hand,[7] who obtained these expressions by comparing the results of Jeffrey[3] with the results of Ericksen's phenomenological theory,[10] and not by direct calculation as in the present section.

Thus, the system of equations of motion of a dilute suspension of rigid ellipsoids of revolution, without allowance for the Brownian motion of the particles, consists of the continuity equation (1.1), the equation of motion (1.2) with definition of the stress tensor (4.8), and the equation of motion of the vector of the ellipsoid orientation (3.9). The system of equations is closed and can be used to analyze the motion of a suspension in the case when the average dimensions of the suspended particles exceed 10^{-4}–10^{-3} cm.

5. TRANSLATIONAL AND ROTATIONAL BROWNIAN MOTION

It is known[11, 12] that under the influence of thermal motion small particles execute a disordered motion, called Brownian, which is described effectively as diffusion of the particles. The problem of determining the motion of the particles in space and in time reduces to the problem of finding the probability distribution functions of the positions and orientations of the particles.

The spatial position of a particle of arbitrary shape can be described by a certain vector of the position of its center of gravity, **r**, and the cosines of the angles between unit vectors rigidly connected to the particle and the axes of the laboratory frame. We need two unit vectors to describe the orientation of a particle of arbitrary shape, and one unit vector directed along the symmetry axis to describe a particle having an axis of rotation. Thus, in Brownian motion the probability of the position and orientation of an axially symmetrical particle is determined by the function $W = W(\mathbf{r}, \mathbf{e})$, where e_j is the cosine of the angle between the unit vector and the j-th coordinate axis. In this case $\mathbf{e}^2 = 1$, so that the distribution function is determined only by five independent variables.

If the particle is not acted upon by any forces at all, then the particle has an equal probability of being in any position and at any orientation, and therefore the distribution function is constant. If the particle is acted upon by certain forces and torques, then the result is a nonuniform distribution in space and a predominant orientation of the particles, and consequently we get diffusion flow,[12] which occurs for translational motion under the action of an effective force

$$-T\frac{\partial \ln W}{\partial \mathbf{r}}, \quad (5.1)$$

where T is the temperature in energy units, and for rotational motion under the action of an effective torque, which we shall determine below.

The rotational mobility of the non-interacting particles does not depend on their position in space, and therefore the rotational Brownian motion can be considered separately from the translational one. In this case the end of the unit vector connected with the symmetry axis of the particle can be regarded as a Brownian particle with anisotropic mobility, so that the unit vector can only rotate, since $\mathbf{e}^2 = 1$, and the probability of particle orientation is described by the distribution function W(**e**), which is usually[4, 15, 16] determined from the continuity equation written in a spherical coordinate system. To find the equation for W(**e**) in a Cartesian coordinate system, we assume that the vector **e** can also change its length, i.e., we assume all the components of the vector, which we now denote by **s**, to be independent. Then the diffusion equation for the distribution function W(**s**) can be obtained[12] from the continuity equation

$$\frac{\partial W}{\partial t} + \frac{\partial (u_j W)}{\partial s_j} = 0. \quad (5.2)$$

The average velocity of the end of the vector **u** is determined from the requirement that the sum of all forces acting on the Brownian particle be equal to zero.

If the particle is acted upon by a force **f**, then in a coordinate system connected with the rotating particle the force-equilibrium condition is

$$f_i - T\frac{\partial \ln W}{\partial s_i} - \chi u_k e_k e_i = 0. \quad (5.3)$$

We have added to this condition, in accordance with our assumption, a term proportional to the velocity of the radial motion of the particle $u_k e_k e_i$.

Multiplying (5.3) vectorially by **s**, we obtain an equation for the moments of the forces

$$\mathbf{L} - T\left[\mathbf{s}\frac{\partial \ln W}{\partial \mathbf{s}}\right] = 0, \tag{5.4}^*$$

where **L** is the sum of the moments of all the forces acting on the Brownian particle, which is balanced by the effective moment of the forces connected with the Brownian motion of the particle,

$$-T\left[\mathbf{s}\frac{\partial \ln W}{\partial \mathbf{s}}\right]. \tag{5.5}$$

The condition (5.4) determines the rotational velocity of the particle.

Being interested here only in the rotational motion of the particles, we shall henceforth put $\chi \to \infty$ in the final results. Then $\mathbf{s} \to \mathbf{e}$ and $W(\mathbf{s}) \to W(\mathbf{e})$.

All the physical quantities are expressed in this case in terms of the moments of the distribution function $W(\mathbf{e})$, so that we shall be faced with the problem of calculating such moments, for example the second-order moments

$$\langle e_i e_k\rangle = \int W(\mathbf{e})\, e_i e_k\, d\mathbf{e}. \tag{5.6}$$

The integration is carried out here under the condition $\mathbf{e}^2 = 1$. The higher-order moments are determined analogously.

It is sometimes more convenient to calculate the moments with the aid of the function $W(\mathbf{s})$ and then go over to the limit, so that, for example:

$$\langle e_i e_k\rangle = \lim_{\chi\to\infty} \int W(\mathbf{s})\, e_i e_k\, d\mathbf{s}. \tag{5.7}$$

We call attention to the fact that the translational ability of a nonspherical particle depends on its orientation relative to the direction of motion. For example,[13] the diffusion coefficients $D^{\parallel}$ and $D^{\perp}$ of an oblate ellipsoid of revolution are different for motion along the symmetry axis and for motion transverse to the symmetry axis. The average value of the diffusion coefficient of ellipsoids of revolution is a tensor and is expressed in terms of the second-order moment of the distribution function:

$$D_{ik} = D^{\perp}\delta_{ik} + (D^{\parallel} - D^{\perp})\langle e_i e_k\rangle. \tag{5.8}$$

The connection between the translational and rotational diffusion for particles of a different type is discussed by Brenner.[14]

We shall not consider translational Brownian motion, and assume that the particle distribution in space is equally probable and remains equally probable during the course of motion.

6. MOTION OF AN ELLIPSOID IN A STREAM WITH ALLOWANCE FOR THE ROTATIONAL BROWNIAN MOTION

When considering the motion of a particle in a stream with allowance for the rotational Brownian motion we start, following [17], from the condition (5.4) and require that the total torque of all the forces acting on the particle be equal to zero. It is necessary here to add to the thermodynamic torque (3.5) the effective torque $-T\mathbf{e} \times \nabla \ln W$.

The condition that the total torque be equal to zero now leads in (3.6) to the relation

$$(a_k^2 - a_i^2)\,\gamma'_{ik} + (a_k^2 + a_i^2)\omega'_{ik} - a_{ji}a_{sk}\frac{3T}{16\pi\mu}(a_i^2\alpha_{i0} + a_k^2\alpha_{k0})\,[\mathbf{e}\nabla \ln W]_{js} = 0, \tag{6.1}$$

whence, changing over to the laboratory frame, we obtain the particle rotational velocity

$$\Omega_{lm} = \frac{a_k^2 - a_i^2}{a_k^2 + a_i^2}a_{li}a_{ji}a_{mk}a_{sk}\gamma_{js} + \omega_{lm} - a_{li}a_{ji}a_{mk}a_{sk}\frac{3T}{16\pi\mu}\frac{a_i^2\alpha_{i0} + a_k^2\alpha_{k0}}{a_k^2 + a_i^2}[\mathbf{e}\nabla \ln W]_{js}.$$

For an ellipsoid of revolution, the last relation, with allowance for (2.3), takes the form

$$\Omega_{lm} = \Omega^0_{lm} + D\,[\mathbf{e}\nabla \ln W]_{lm}, \tag{6.2}$$

where Ω^0_{lm} denotes the velocity of the ellipsoid in the stream without allowance for the rotational Brownian motion, as given by (3.8). The coefficient of rotational diffusion is defined by

$$D = \frac{3T\,(a^2\alpha_0 + b^2\beta_0)}{16\pi\mu\,(a^2 + b^2)}. \tag{6.3}$$

The dimensionless coefficient of rotational diffusion

$$\delta = \frac{4\pi}{3}ab^2\frac{\mu D}{T},$$

is shown in Fig. 2 as a function of the ratio a/b of the semi-axes of the ellipsoid of revolution.

From (6.2) we obtain the linear velocity of the end point of the vector

$$u_j = \dot{e}_j + D\,(e_j e_n \nabla_n \ln W - \nabla_j \ln W), \tag{6.4}$$

where $\dot{e}_j$ is the linear velocity of the motion of the end of the vector, defined by formula (3.9).

From the continuity equation (5.2) we obtain with the aid of (6.4) an equation for the distribution function of the orientation of the symmetry axes of uniaxial ellipsoids moving in a stream

$$\frac{\partial W}{\partial t} + D\left(2e_j\frac{\partial W}{\partial e_j} + e_j e_s\frac{\partial^2 W}{\partial e_j\,\partial e_s} - \frac{\partial^2 W}{\partial e_j^2}\right) + \lambda\,(e_s\gamma_{sl} - e_l e_m e_s\gamma_{ms})\frac{\partial W}{\partial e_l} + \omega_{ls}e_s\frac{\partial W}{\partial e_l} - 3\lambda e_l e_s\gamma_{ls}W = 0. \tag{6.5}$$

From (6.5) we get an equation for the rate of change of the second-order moments of the distribution function

$$\frac{d\langle e_i e_k\rangle}{dt} = -\frac{1}{\tau}\left(\langle e_i e_k\rangle - \frac{1}{3}\delta_{ik}\right) + \lambda\,(\langle e_i e_j\rangle\gamma_{jk} + \langle e_k e_j\rangle\gamma_{ji}) + \omega_{ij}\langle e_j e_k\rangle + \omega_{kj}\langle e_j e_i\rangle - 2\lambda\,\langle e_i e_k e_j e_s\rangle\gamma_{js}. \tag{6.6}$$

The relaxation time is defined here as $\tau = 1/6D$. It

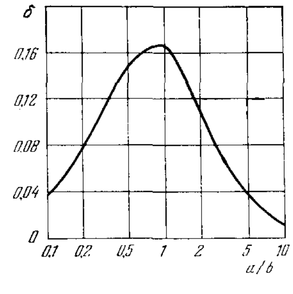

FIG. 2. Dimensionless coefficient of rotational diffusion as a function of the ratio a/b of the semi-axes of the ellipsoid of revolution.

*$[\mathbf{s}(\partial \ln W/\partial \mathbf{s})] \equiv \mathbf{s} \times \partial \ln W/\partial \mathbf{s}$.

is possible to obtain in similar fashion equations for the higher moments of the distribution function. We note that the equations for the rate of change of the second-order moments contains fourth-order moments, and the equation for the rate of change of the fourth-order moments contains sixth-order moments, etc. Thus, the system of equations for the moments is not closed and cannot be solved without some additional assumptions.

Let us determine the distribution function and its second- and fourth-order moments in the stationary case at small velocity gradients, starting from Eq. (6.5). We seek the distribution function in this case in the form of an expansion in invariant combinations of the vector e_i and the tensors γ_{ik} and ω_{ik}. Accurate to second-order terms in the velocity gradients we have

$$W=\frac{1}{4\pi}\left(1+\frac{\lambda}{2D}e_i e_k \gamma_{ik}-\frac{\lambda}{2D^2}e_i e_k \gamma_{si}\omega_{sk}+\frac{\lambda^2}{8D^2}e_i e_k e_s e_j \gamma_{ik}\gamma_{sj}-\frac{\lambda^2}{60D^2}\gamma_{ik}\gamma_{ik}+\ldots\right). \tag{6.7}$$

In the case of a simple shear flow, the distribution function was determined with high accuracy by Peterlin[4] and recently by Workman and Hollingsworth.[15] In the case of simple tension, the distribution function was determined by Takserman-Kroser and Ziabicki.[16]

The distribution function (6.7) can be used to determine[17] the second- and fourth-order moments of the distribution function, which take the form

$$\langle e_j e_s\rangle=\frac{1}{3}\delta_{js}+\frac{\lambda}{15D}\gamma_{js}-\frac{\lambda}{90D^2}(\gamma_{lj}\omega_{ls}+\gamma_{ls}\omega_{lj})-\frac{\lambda^2}{315D^2}\gamma_{ik}\gamma_{ik}\delta_{js}+\frac{\lambda^2}{105D^2}\gamma_{lj}\gamma_{ls}+\ldots, \tag{6.8}$$

$$\begin{aligned}\langle e_j e_s e_m e_n\rangle=&\frac{1}{15}\Delta_{jsmn}+\frac{\lambda}{105D}(\gamma_{js}\delta_{mn}+\ldots)-\frac{\lambda}{630D^2}[(\gamma_{lj}\omega_{ls}+\gamma_{ls}\omega_{lj})\delta_{mn}+\ldots]\\&-\frac{4\lambda^2}{4725D^2}\gamma_{ik}\gamma_{ik}\Delta_{jsmn}+\frac{\lambda^2}{945D^2}(\gamma_{lj}\gamma_{ls}\delta_{mn}+\ldots)\\&+\frac{\lambda^2}{945D^2}(\gamma_{js}\gamma_{mn}+\ldots)+\ldots,\end{aligned} \tag{6.9}$$

Where $\Delta_{jsmn}=\delta_{js}\delta_{mn}+\delta_{jm}\delta_{sn}+\delta_{jn}\delta_{sm}$. The dots in the parentheses of the last formula denote terms with all the remaining combinations of the indices.

7. STRESS TENSOR AND VISCOSITY OF A SUSPENSION WITH ALLOWANCE FOR THE ROTATIONAL BROWNIAN MOTION OF THE PARTICLES

We now take into account the rotational Brownian motion of the particles in the calculation of the stresses arising in the motion of the suspension, following in the general case a method[18] used to calculate the stress tensor in the stationary case.

Equation (4.7) remains valid also when the rotational Brownian motion is taken into account, and the problem consists of determining the components of the matrix A'_{ik} with the aid of the new condition (6.1) and of changing over to the laboratory frame. After performing these operations and averaging with the aid of the distribution W(e), we obtain an expression (determined by another method by Shmakov and Taran[19]) for the stress tensor of a suspension of ellipsoidal particles, with allowance for the rotational Brownian motion of the particles

$$\begin{aligned}\sigma_{ik}=&-(p_0+\mu\varphi\rho\langle e_j e_s\rangle\gamma_{js})\delta_{ik}+2\mu(1+\varphi\omega)\gamma_{ik}+\mu\varphi\varkappa D\left(\langle e_i e_k\rangle-\frac{1}{3}\delta_{ik}\right)\\&+\mu\varphi\zeta(\langle e_i e_j\rangle\gamma_{jk}+\langle e_k e_j\rangle\gamma_{ji})+\mu\varphi\chi\langle e_i e_k e_j e_s\rangle\gamma_{js},\end{aligned} \tag{7.1}$$

where ρ, ω, ζ, and χ have the meaning indicated by formulas (3.9), and

$$\varkappa=\frac{3\lambda}{\delta}=\frac{12(a^2-b^2)}{ab^2(a^2\alpha_0+b^2\beta_0)}. \tag{7.2}$$

The values of κ, ρ, ζ, ω, and χ as functions of the ratio a/b of the semi-axes of the ellipsoid of revolution are shown in Fig. 1.

Thus, the system of equations of motion of a dilute suspension of rigid ellipsoids of revolution consists of the continuity equation (1.1), the equation of motion (1.2) with a definition of the stress tensor (7.1), and an infinite chain of equations for the even-order moments of the distribution function, the first of which is (6.6). The system of equations is not closed, and the analysis of the motion of the suspension cannot be carried out without some approximations. When the particle shape deviates little from spherical, $|\lambda|\ll 1$, we can use successive approximations.

For stationary cases, the expression (7.1) for the stress tensor can be rewritten with the aid of (6.6), in which the time derivative is equal to zero, in the form

$$\begin{aligned}\sigma_{ik}=&-(p_0+\mu\varphi\rho\langle e_j e_s\rangle\gamma_{js})\delta_{ik}+2\mu(1+\varphi\omega)\gamma_{ik}\\&+\mu\varphi\left(\zeta+\frac{1}{6}\varkappa\lambda\right)(\langle e_i e_j\rangle\gamma_{jk}+\langle e_k e_j\rangle\gamma_{ji})\\&+\frac{1}{6}\mu\varphi\varkappa(\omega_{ij}\langle e_j e_k\rangle+\omega_{kj}\langle e_j e_i\rangle)+\mu\varphi\left(\chi-\frac{1}{3}\lambda\varkappa\right)\langle e_i e_k e_j e_s\rangle\gamma_{js}.\end{aligned} \tag{7.3}$$

The advantage of this expression, which has been indicated in [18], lies in the fact that at a given accuracy of the moments it determines the stress with a greater accuracy than expression (7.1). For example, with the aid of moments calculated accurate to terms of second order in the gradients it is possible to determine in the stationary case the stress-tensor components accurate to third-order terms.

The stresses produced for a given stationary flow of particular form can be determined from the stress tensor (7.3) and the definitions (6.8) and (6.9). For example, for the case which, for understandable reasons, attracted great attention on the part of the researchers,[4-6] namely for steady-state simple shear flow, when only one component of the velocity-gradient tensor differs from zero ($\nu_{12}\neq 0$), the indicated relations determine the stresses, of which only the component σ_{23} and σ_{13} are equal to zero, accurate to terms of third order in the velocity gradients. The shear stress $\sigma_{12}=\eta\nu_{12}$ determines the coefficient of effective shear viscosity

$$\eta=\mu\left\{1+\varphi\left[V-S\left(\frac{4\pi ab^2\mu}{3T}\nu_{12}\right)^2\right]\right\}, \tag{7.4}$$

where

$$V=\frac{1}{30}\varkappa\lambda+\omega+\frac{1}{3}\zeta+\frac{1}{15}\chi,\qquad S=\frac{\varkappa^2}{405}\left(\frac{1}{24}\frac{\varkappa}{\lambda}+\frac{\varkappa\lambda}{280}-\frac{\zeta}{28}-\frac{\chi}{35}\right).$$

It can be verified that S is always positive and therefore the coefficient of shear viscosity of a suspension decreases with increasing velocity gradient, a fact connected with the orientation of the particles by the flow. In the general case η can obviously be represented in the form of an expansion in even powers of the velocity gradient.

The values of the dimensionless initial shear viscosity $V=(\eta_0-\mu)/\mu\varphi$, which does not depend on the velocity gradients, are shown in Fig. 3 as functions of the

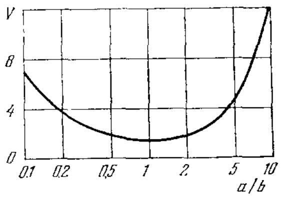

FIG. 3. Initial shear viscosity of the suspension $V = (\eta_0 - \mu)/\mu\varphi$ as a function of the ratio a/b of the semi-axes of the ellipsoid of revolution.

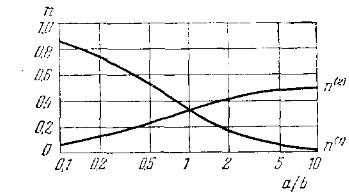

FIG. 4. Depolarization coefficients as functions of the ratio a/b of the semi-axes of an ellipsoid of revolution.

ratio a/b of the semi-axes of the ellipsoid of revolution. The function V + 1 was tabulated in [5,20]. The dimensionless shear viscosity coefficient as a function of the parameters a/b and ν_{12}/D was tabulated in [21].

The properties of the suspension were considered also in other simple particular cases.[16, 18, 22]

We note that if the ellipsoids degenerate into spheres, then the integrals in (2.10) can be easily calculated and formulas (4.9) and (7.2) yield $\omega = 1.5$, $\rho = \zeta = \chi = \kappa = 0$. The stress tensor (7.1), just as (4.8) and (7.3), takes in accordance with the results of the theory of suspension of small spheres[23] the form

$$\sigma_{ik} = -p_0\delta_{ik} + 2\mu(1 + 1.5\varphi)\gamma_{ik} \tag{7.5}$$

and coincides with (1.3). In this case the suspension is an ordinary viscous liquid with a viscosity coefficient

$$\eta = \mu(1 + 1.5\varphi).$$

8. DIELECTRIC ELLIPSOID IN A STREAM AND IN AN ELECTRIC FIELD

In the preceding sections we considered the behavior of a suspension in the case when the ellipsoids are acted upon by the torque due to the hydrodynamic forces, which lead to orientation of the particles by the stream, and the torque due to the effective Brownian force, which causes disorientation of the particles. It is possible to consider in perfectly similar fashion the motion of ellipsoids and the properties of the suspension in the case when the particles are acted upon directly, in addition, by some extraneous torques, for example, the moment produced when an electric field of intensity E is applied. If the particle has a dipole moment $\mathscr{P}$, then the moment of forces acting on the particle is given by the formula[24]

$$\mathbf{G} = [\mathscr{P}\mathbf{E}]. \tag{8.1}$$

Let us consider the case when the particles do not have a constant dipole moment. Let the principal axes of the dielectric tensor coincide with the axes of the ellipsoid. Then the expression for the dipole moment of the ellipsoid in the coordinate frame connected with the axes of the particle is[24]

$$\mathscr{P}'_k = \frac{abc}{3}\frac{\varepsilon^{(k)} - \varepsilon}{\varepsilon + (\varepsilon^{(k)} - \varepsilon)n^{(k)}}E'_k,$$

where $\epsilon^{(k)}$ are the principal values of the dielectric tensor of the particle, ϵ is the dielectric constant of the liquid, and $n^{(k)}$ are the depolarization coefficients, expressions for which can be found in [24], Sec. 4. For ellipsoids of revolution we have $n^{(2)} = n^{(3)} = (1 - n^{(1)})/2$. $n^{(1)}$ and $n^{(2)}$ are shown in Fig. 4 as functions of the ratio a/b of the semi-axes of the ellipsoid.

Confining ourselves henceforth to ellipsoids of revolution, we write an expression for the dipole moment in the laboratory frame

$$\mathscr{P}_k = \frac{ab^2}{3\varphi}(\varepsilon_0 - \varepsilon) + \frac{ab^2}{3}\gamma\left(e_ke_i - \frac{1}{3}\delta_{ki}\right)E_i, \tag{8.2}$$

where

$$\varepsilon_0 = \varepsilon + \varphi\frac{\varepsilon\,[3\varepsilon(\varepsilon^{(2)} - \varepsilon) + \varepsilon(\varepsilon^{(1)} - \varepsilon^{(2)}) + (\varepsilon^{(1)} - \varepsilon)(\varepsilon^{(2)} - \varepsilon)(n^{(2)} + 2n^{(1)})]}{3\,[\varepsilon + (\varepsilon^{(1)} - \varepsilon)n^{(1)}]\,[\varepsilon + (\varepsilon^{(2)} - \varepsilon)n^{(2)}]}, \tag{8.3}$$

$$\gamma = \frac{\varepsilon^2(\varepsilon^{(1)} - \varepsilon^{(2)}) + \varepsilon(\varepsilon^{(1)} - \varepsilon)(\varepsilon^{(2)} - \varepsilon)(n^{(2)} - n^{(1)})}{[\varepsilon + (\varepsilon^{(1)} - \varepsilon)n^{(1)}]\,[\varepsilon + (\varepsilon^{(2)} - \varepsilon)n^{(2)}]}. \tag{8.4}$$

With the aid of (8.2) we now write an expression for the moment of forces acting on the particle in this case

$$G_{is} = \frac{ab^2}{2}\gamma e_kE_k\,[\mathbf{eE}]_{is},$$

which tends to orient the particle along the field.

From the requirement that the net resultant torque acting on the particle be equal to zero, we obtain the rotational velocity of the ellipsoid

$$\Omega_{lm} = \Omega^*_{lm} - a_{li}a_{mk}a_{ji}a_{sh}\frac{ab^2\gamma}{16\pi\mu}\frac{a_i^2\alpha_{i0} + a_k^2\alpha_{k0}}{a_k^2 + a_i^2}e_nE_n\,[\mathbf{eE}]_{js}, \tag{8.5}$$

where Ω^*_{lm} is the velocity of rotation of the ellipsoid of revolution in the stream without an electric field, and is determined by formula (6.2).

For ellipsoids of revolution we obtain from (8.5) the linear velocity of the end of the unit vector

$$u_j = u_j^* - \frac{\gamma\delta}{4\pi\mu}(e_je_ne_mE_nE_m - e_nE_nE_j), \tag{8.6}$$

where u_j^* is the linear velocity of the end of the unit vector in the stream without the electric field, and is determined by formula (6.4).

From the continuity equation (5.2) we obtain with the aid of (8.6) the diffusion equation for the distribution function of the orientations of the symmetry axes of axially symmetrical ellipsoids moving in the stream and in an electric field

$$\frac{\partial W}{\partial t} + D\left(2e_n\frac{\partial W}{\partial e_n} + e_je_n\frac{\partial^2 W}{\partial e_j\,\partial e_n} - \frac{\partial^2 W}{\partial e_i^2}\right) + \lambda(e_s\gamma_{sj} - e_me_se_j\gamma_{ms})\frac{\partial W}{\partial e_j} + \omega_{js}e_s\frac{\partial W}{\partial e_j} - 3\lambda e_je_s\gamma_{js}W - \frac{\gamma\delta}{4\pi\mu}\left[(e_se_je_mE_sE_m - e_sE_sE_j)\frac{\partial W}{\partial e_j} + (3e_je_sE_jE_s - E^2)W\right] = 0. \tag{8.7}$$

From the last equation we obtain a relation for the rate of change of the second-order moments of the distribution function

$$\frac{d\langle e_ie_k\rangle}{dt} = -\frac{1}{\tau}\left(\langle e_ie_k\rangle - \frac{1}{3}\delta_{ik}\right) + \lambda(\langle e_ie_j\rangle\gamma_{jk} + \langle e_ke_j\rangle\gamma_{ji}) + \omega_{ij}\langle e_je_k\rangle + \omega_{kj}\langle e_je_i\rangle - 2\lambda\langle e_ie_ke_se_j\rangle\gamma_{sj} + \frac{\gamma\delta}{4\pi\mu}(\langle e_ie_j\rangle E_jE_k - 2\langle e_ie_ke_se_j\rangle E_sE_j + \langle e_ke_j\rangle E_jE_i). \tag{8.8}$$

Equations for the higher moments of the distribution function can be obtained in similar fashion. Just as for a suspension moving without the field, the equation for the rate of change of the second-order moments contains fourth-order moments, the equation for the rate of change of the fourth-order moments contains sixth-order moments, etc. Thus, the system of equations for the moments is not closed and the moments cannot be obtained without some additional assumptions.

The moments of the distribution function can be calculated directly if (8.7) is used to determine the distribution function, which can be obtained in the stationary case, at low velocity gradients and in a weak electric field, in the form of an expansion in scalar combinations of the vectors e_i and E_k and of the tensors γ_{ik} and ω_{ik}. Accurate to terms of first order in the velocity gradients and of second order in the field intensity, we have

$$W=\frac{1}{4\pi}\left\{1+\frac{ab^2\gamma}{6T}(3e_je_sE_jE_s-E^2)+\frac{\lambda}{2D}e_je_s\gamma_{js}\right.$$
$$\left.+\frac{ab^2\gamma}{60DT}[5(1-\lambda)e_je_sE^2\gamma_{js}+15\lambda e_ne_le_je_sE_nE_l\gamma_{js}-2\lambda E_jE_s\gamma_{js}]\right\}. \quad (8.9)$$

We note that the distribution function in the particular case of simple shear motion ($\nu_{21}\neq 0$) and an electric field directed along the 1 axis was obtained by Ikeda.[25]

Just as in Sec. 6, we calculate the moments of the distribution function and obtain, with the indicated accuracy

$$\langle e_ie_k\rangle=\frac{1}{3}\delta_{ik}+\frac{ab^2\gamma}{15T}\left(E_iE_k-\frac{1}{3}E^2\delta_{ik}\right)+\frac{\lambda}{15D}\gamma_{ik}$$
$$+\frac{ab^2\gamma}{630DT}\{(7-4\lambda)E^2\gamma_{ik}+2\lambda[3(E_iE_j\gamma_{jk}+E_kE_j\gamma_{ji})-2E_jE_s\gamma_{js}\delta_{ik}]\},$$
$$\langle e_ie_ke_ne_l\rangle=\frac{1}{15}\Delta_{iknl}+\frac{ab^2\gamma}{315DT}[3(E_iE_k\delta_{nl}+\ldots)-2E^2\Delta_{iknl}]$$
$$+\frac{\lambda}{105D}(\gamma_{ik}\delta_{nl}+\ldots)+\frac{ab^2\gamma}{9450DT}\{5(3-2\lambda)E^2(\gamma_{ik}\delta_{nl}+\ldots)$$
$$+\lambda[10(E_lE_n\gamma_{ik}+\ldots)+10(E_sE_k\gamma_{si}\delta_{ln}+\ldots)-16E_jE_s\gamma_{js}\Delta_{iknl}]\}, \quad (8.10)$$

where $\Delta_{iknl}=\delta_{ik}\delta_{nl}+\delta_{in}\delta_{kl}+\delta_{il}\delta_{nk}$, and the dots in the last formula denote the omitted terms with all the remaining combinations of the indices.

9. STRESS TENSOR AND VISCOSITY OF A SUSPENSION IN AN ELECTRIC FIELD

The expression (4.7) which we derived above is applicable to a suspension moving in an electric field. After determining the components of the matrix A'_{ik} for the case in question, changing over to the laboratory frame, and averaging, we obtain expressions for the stress tensor of a suspension moving in an electric field:

$$\sigma_{ik}=-(p_0+\mu\varphi\rho\langle e_je_s\rangle\gamma_{js})\delta_{ik}+2\mu(1+\varphi\omega)\gamma_{ik}$$
$$+\mu\varphi\varkappa D\left(\langle e_ie_k\rangle-\frac{1}{3}\delta_{ik}\right)+\mu\varphi\zeta(\langle e_ie_j\rangle\gamma_{jk}+\langle e_ke_j\rangle\gamma_{ji})+\mu\varphi\chi\langle e_ie_ke_je_s\rangle\gamma_{js}$$
$$+\frac{\varphi\gamma}{8\pi}[(1-\lambda)\langle e_ie_j\rangle E_jE_k+2\lambda\langle e_ie_ke_je_s\rangle E_jE_s-(1+\lambda)\langle e_ke_j\rangle E_jE_i]. \quad (9.1)$$

The constants ρ, ω, ζ, χ, and κ are determined as before by the expressions (4.9) and (7.2), and are shown in Fig. 1 as functions of the ratio of the ellipsoid semi-axes. We note that the stress tensor of a suspension swing moving in an electric field is not symmetrical.

Thus, the system of equations of motion of a dilute suspension of rigid dielectric ellipsoids in an electric field, with allowance for the Brownian motion of the particles, consists of the continuity equation (1.1), the equation of motion (1.2) with definition of the stress tensor (9.1), and an infinite chain of equations for the even-order moments of the distribution function, the first of which is (8.8). Just as for a suspension without a field, the system of equations is not closed and an analysis of the motion of the suspension cannot be carried out without some approximations. When the particles differ little from spheres and $|\lambda|\ll 1$, we can use a successive-approximation method.

Let us consider further the case of steady-state shear flow ($\nu_{12}\neq 0$), when the electric field intensity vector lies in the (1, 2) plane and has components $E_1 = E\cos\psi$, $E_2 = E\sin\psi$, and $E_3 = 0$. Then the expressions (9.1), (8.10), and (8.11) determine the shear stress, accurate to terms of first order in the velocity gradient and of second order in the field intensity, from which we get the coefficient of the effective shear viscosity

$$\eta=\eta_0+\frac{\varphi\gamma E^2}{1680\pi D}\left[23\lambda^2+21\lambda+\frac{6\lambda}{\varkappa}(2\chi+7\zeta)-7\lambda\cos 2\psi\right], \quad (9.2)$$

where η_0 is the initial value of the shear viscosity, which does not depend on the velocity gradients and on the field.

The increase of the effective viscosity of the suspension in the field is due to the hindered rotation of the particles. The dependence of the viscosity on the field direction is obviously connected with the orienting influence of the field on the suspended particles. In a field applied along the flow direction, the viscosity of a suspension of prolate ellipsoids decreases, and that of oblate ellipsoids increases.

The analysis of the motion of a suspension of dielectric ellipsoids in an electric field can be easily generalized to the case when the particle has a constant electric moment **d**. It is necessary to take into account here that the particle is acted upon by an additional torque $\mathbf{d}\times\mathbf{E}$ (**E** is the average value of the field intensity in the system), which must be taken into account when the balance of the forces acting on the particle is set up. All the calculations are performed in the same manner as shown in the preceding sections.

In this more general case, the stress tensor, which was recently determined by Begoulev and Shmakov,[26] is expressed in terms of the first, second, third, and fourth moments of the distribution function. The system of equations of motion of the suspension should include in this case (besides the continuity equation (1.1), the equation of motion (1.2), and the definition of the stress tensor) also an infinite chain of equations for the even moments of the distribution function, and, unlike the cases considered above, also of the odd ones.

An expression for the coefficient of effective viscosity of a suspension of ellipsoids was obtained, for the particular case of shear motion, by Saito and Kato[27] and by Chaffey and Mason.[28]

We note that when the ellipsoids degenerate into spheres, the stress tensor (9.1) takes on the form (7.5). Thus, if the spheres have no dipoles, the field does not influence the motion of the suspension, which in this case is an ordinary viscous liquid with a viscosity coefficient $\eta=\mu(1+1.5\varphi)$. However, if the spheres have constant dipole moments, then the stress tensor cannot be reduced to the simple form (1.3), but is determined in terms of the third-order moments. The system of equations for the moments, and consequently the system of equations of motion, is likewise not closed in this case and its analysis calls for approximate methods.

If simple shear motion is considered ($\nu_{12} \neq 0$) in the case of an arbitarily directed field with intensity components $E_1 = E \cos\psi \sin\gamma$, $E_2 = E \sin\psi \sin\gamma$, and $E_3 = E \cos\gamma$, we can obtain an expression for the effective coefficient of shear viscosity of a suspension of spheres in the form of an expansion in even powers of the field intensity. Accurate to fourth-order terms, we have

$$\eta = \mu \left\{1 + 1{,}5\varphi + \varphi\left[\frac{1}{4}\left(\frac{Ed}{T}\right)^2 - \left(\frac{1}{60} - \frac{1}{48}\cos 2\psi\right)\left(\frac{Ed}{T}\right)^4\right]\sin^2\gamma\right\}, \tag{9.3}$$

where E is the mean value of the field intensity in the medium.

The motion of a suspension of spherical particles in an external field without allowance for the rotational Brownian motion of the particles was considered in [29, 30].

All the results of the theory of motion of a suspension in an electric field are valid, with suitable change in notation, for a suspension of magnetic particles moving in a magnetic field, and also for a suspension[30] of buoyant particles moving in a gravitational field and having a mass center that does not coincide with the geometrical center.

10. DIELECTRIC TENSOR

A flowing suspension of non-spherical particles becomes optically anisotropic as a result of the orientation of the particles by the flow. Let us examine this phenomenon, following [17], and starting from the dielectric tensor ϵ_{ik}, defined by the relation[24]

$$D_i = \varepsilon_{ik} E_k,$$

where D_i and E_k are the induction and electric field intensity, averaged over a volume greatly exceeding the volume of the particle.

Since the field $E_k^{(i)}$ inside an ellipsoid placed in a homogeneous field is homogeneous, we can write for a single particle the average values

$$E_k = \Omega E_k^{(i)} + \overline{E_k^{(e)}}, \qquad D_k' = \varepsilon^{(k)}\Omega E_k^{(i)\prime} + \varepsilon\overline{E_k^{(e)\prime}}, \tag{10.1}$$

where $\epsilon^{(k)}$ are the principal values of the dielectric tensor of the particle; by assumption, this tensor is diagonal in a coordinate system connected with the axes of the ellipsoid; ϵ is the dielectric constant of the liquid. In view of the assumed anisotropy, the last formula of (10.1) is valid only in the coordinate system connected with the particle.

Eliminating the averaged field outside the particle from the formulas in (10.1), we obtain

$$D_k' = \varepsilon E_k' + \Omega(\varepsilon^{(k)} - \varepsilon) E_k^{(i)\prime}. \tag{10.2}$$

The expression for the field intensity inside an ellipsoid located in an external field, which we assume in the case when the particle is large compared with the molecular dimensions to be the average macroscopic field E_k (in accordance with [24], Sec. 8), is

$$E_k^{(i)\prime} = \frac{\varepsilon E_k'}{\varepsilon + (\varepsilon^{(k)} - \varepsilon) n^{(k)}}, \tag{10.3}$$

where $n^{(k)}$ are the depolarization coefficients.

From (10.2) and (10.3) we obtain

$$D_k' = \varepsilon E_k' + \Omega \frac{\varepsilon(\varepsilon^{(k)} - \varepsilon)}{\varepsilon + (\varepsilon^{(k)} - \varepsilon) n^{(k)}} E_k'.$$

We now change to the laboratory frame and confine ourselves to ellipsoids of revolution with allowance for (2.3), and obtain after averaging over the particle orientations and taking the total number of particles per unit volume into account an expression for the induction of the electric field, which determines the dielectric tensor

$$\varepsilon_{ik} = \varepsilon_0\delta_{ik} + \gamma\varphi\left(\langle e_i e_k\rangle - \frac{1}{3}\delta_{ik}\right), \tag{10.4}$$

where ϵ_0 and γ are determined by (8.3) and (8.4). The moments of the distribution function are determined in the general case by (6.6) for a suspension without external fields and by (8.8) for a suspension moving in an electric field. For stationary flow it is possible to use in the respective cases the expressions (6.8) and (8.10).

From (10.4) and (6.8) we can obtain a relation between the dielectric tensor and the velocity-gradient tensor. Accurate to first-order terms we have

$$\varepsilon_{ik} = \varepsilon_0\delta_{ik} + \frac{\gamma\varphi\lambda}{15D}\gamma_{ik}. \tag{10.5}$$

Analogously, from (10.4) and (8.10) we obtain, likewise accurate to the first order, the relation

$$\varepsilon_{ik} = \varepsilon_0\delta_{ik} + \frac{ab^2\gamma\varphi}{15T}\left(E_i E_k - \frac{1}{3}E^2\delta_{ik}\right) + \frac{\gamma\varphi\lambda}{15D}\gamma_{ik}. \tag{10.6}$$

The appearance of optical anisotropy is connected with the orienting action of the velocity gradient, and of the field on the particle suspended in the liquid.

The optical anisotropy of the suspension in the particular case of steady-state shear motion was considered by Peterlin and Stuart[31] and with greater accuracy by Scheraga, Edsall, and Gadd,[32] and in the case of oscillatory shear motion by Cerf and Thurston.[33]

The optical anisotropy of a suspension in simple shear motion ($\nu_{21} \neq 0$) in a weak electric field, when the electric field intensity vector is directed along the 1 axis, was considered by Ikeda,[25] who also took the presence of a constant dipole moment of the particle into account.

11. CONCLUDING REMARKS

Thus, in the case when the rotational Brownian motion of particles must be taken into account, i.e., when the dimensions of the suspended particles are smaller than 10^{-3}–10^{-4} cm, the equations of motion of a dilute suspension of rigid ellipsoids of revolution can be expressed with the aid of the moments of the distribution function, which in this case are the internal variables describing the orientation of the particles in the field and in the stream. The equation of motion contains a finite relaxation time $\tau = 1/6D$, and this distinguishes the system under consideration from a viscous liquid or an elastic body, whose respective relaxation times are zero or infinity and do not enter in the equation of motion. This means that a suspension of rigid ellipsoids exhibits visco-elastic properties and is an example of a system whose behavior is in a certain sense intermediate between that of a viscous liquid and a solid.

The relaxation process connected with the disorientation of the particles is not the only one for suspensions. Other relaxation processes, which were recently considered by Afanas'ev and Nikolaevskiĭ[34] for a suspension of small spheres, and which must be taken into account in the study of the suspension of other particles, are those connected with the equalization of the translational and rotational motions of the system.

The theory of flow of suspensions of rigid ellipsoids was initially developed[4,5] to explain the behavior of dilute solutions of macromolecules, and at the present time the results of the theory are widely used to obtain information on the dimensions and shapes of rigid macromolecules.[35] The present theory does not apply to dilute solutions of flexible macromolecules which form statistical coiled balls. A theory of motion of suspensions of deformable particles is presently being intensively developed for the description of the behavior of systems of this type.[8, 36-38]

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