539.124

RADIATIVE POLARIZATION OF ELECTRONS IN STORAGE RINGS

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Usp. Fiz. Nauk 105, 441–478 (November, 1971)

Polarization of electrons arising as the result of radiation during extended motion in a magnetic field is considered. A calculation is made with the quasiclassical operator method of the probability of a radiative transition in a magnetic field with spin flip, including the case of a nonuniform magnetic field. The kinetic equation is obtained for polarization of electrons in an external field with inclusion of radiation effects, and this is used to analyze the kinetics of radiative polarization. Effects are discussed which lead to depolarization of an electron beam in motion in a nonuniform magnetic field, and means of suppressing these effects and also of intentional depolarization of a beam are pointed out. Means are discussed for measurement of the transverse polarization of high energy electrons, and a description is given of an experiment in which the first indication of the existence of the radiative polarization effect has been obtained.

CONTENTS

1. INTRODUCTION

DURING *extended motion in a magnetic field*, electrons and positrons can be polarized as the result of radiation of photons. The polarization arises because the probability of a radiative transition with spin flip depends on the orientation of the initial spin. This new mechanism for polarization of electrons and positrons of high energy, whose existence in a uniform field was first pointed out by Sokolov and Ternov,[1] is extremely important for the following reasons:

1) This is the only available means of obtaining polarized beams which are immediately of high energy (it begins to be efficient at an energy of several hundred MeV).

2) The polarization process does not change the properties of the beam (intensity, spread in parameters, and so forth), which favorably distinguishes it, let us say, from the method of obtaining polarized beams by means of scattering.

3) Electrons and positrons can be polarized at any specified energy, which removes the very complicated problem of accelerating the polarized particles.

In this way the possibility is opened of setting up experiments with polarized electrons and positrons, which in turn significantly broadens the means for study of electromagnetic interactions at high energies. In experiments in colliding beams the polarization must be taken into account in even the simplest two-particle processes, since the cross section for elastic scattering of an electron by an electron or a positron, and also the cross section for production of pairs of photons, pions, kaons, muons, and so forth depend on the polarization very substantially. Particular interest, however, is presented by experiments with polarized electrons in which the polarization of the final particles will be measured, which evidently will be possible in second-generation experiments.

In order to solve the problem of radiative polarization as a whole it is necessary to discuss the following basic questions:

1) Determination of the probability of a radiative transition with spin flip in a magnetic field, particularly in a nonuniform magnetic field such as exists in storage rings.

2) Establishment of the kinetics of the radiative polarization process, for which it is necessary to find and solve the kinetic equation for the electron spin in an external field with inclusion of radiation effects.

3) Determination of the important depolarizing effects and means of removing them, in order to preserve the radiative polarization which arises. And finally;

4) Measurement of the degree of transverse polarization of high energy electrons moving in a storage ring.

In what follows we will discuss all of these questions, and also the first experiment on measurement of radiative polarization.

2. RADIATIVE TRANSITION WITH SPIN FLIP

Quantum effects in external fields are usually discussed in the so-called Furry representation, in which the radiation process is considered in terms of perturbation theory with use of exact solutions of the wave equations (Dirac, Klein-Gordon) in a given field (i.e., without use of perturbation theory in the external field). However, exact solutions are known for a very limited class of fields (a uniform, constant magnetic field;

crossed electric and magnetic fields; the field of a plane wave; the Coulomb field), and the calculations themselves are extremely complex and tedious. For this reason at high energies the quasiclassical asymptotes of the expressions obtained are used. On the whole this approach turns out to be unjustifiedly complicated.

2.1. Method of Discussion

In recent work by the author and V. M. Katkov, a general method was developed for studying electromagnetic phenomena in external fields.[2-4] This method is based on the fact that quantum effects in motion of high-energy particles in an external electromagnetic field (for definiteness, a magnetic field) have a twofold origin: quantization of the motion itself and quantized recoil of the particle during radiation. In the first case the magnitude of the quantum effects (and, correspondingly, the non-commutation of the dynamical variables of the particle) has an order $\hbar\omega_0/\epsilon$ (where $\omega_0 = v_t/R$, R is the instantaneous radius of curvature, v_t is the velocity component perpendicular to the magnetic field, ϵ is the particle energy, and $\hbar\omega_0$ is the distance between neighboring energy levels of the electron in a magnetic field in the case of large quantum numbers). The quantity

$$\hbar\omega_0/\varepsilon = H/(H_0\gamma^2) \tag{2.1}$$

(where $\gamma = \epsilon/(mc^2)$, H is the magnetic field, $H_0 = m^2c^3/(\hbar e) = 4.41 \times 10^{13}$ G (for an electron) is the critical magnetic field of the quantum effects) is extremely small and decreases with increasing energy. Thus, the motion of an electron in a magnetic field becomes more classical with increasing energy.

The magnitude of the quantum effects from recoil during radiation is of order $\hbar\omega/\epsilon$, where ω is the frequency of the radiated photon. The quantum effects in a magnetic field can be characterized conveniently by the invariant parameter

$$\chi = \frac{\hbar e}{m^3c^4}\sqrt{(F_{\mu\nu}p^\nu)^2} = \frac{\hbar\omega_0\gamma^3}{\varepsilon}\frac{v_t}{c} = \frac{\hbar|\dot{\mathbf{v}}|}{mc^3}\gamma^2 = \frac{H}{H_0}\frac{p_t}{m_c}. \tag{2.2}$$

For $\chi \ll 1$ (just this case occurs in contemporary storage rings) the magnitude of the quantum effects is relatively small, and $\omega \approx \omega_0\gamma^3$. The region $\chi \gtrsim 1$ is essentially a quantum region, and in this case $\hbar\omega \sim \epsilon$. Thus, it is evident that at high energies ($\gamma \gg 1$), for any χ the quantum effects of the first type are negligibly small in comparison with the quantum effects of the radiation. This fact is the basis of the method developed in refs. 2–4, in which quantum effects of the first type are neglected. If there are two types of quantum effects in the theory and we wish from the very beginning not to take one of them into account, the operator formulation of quantum mechanics is particularly convenient for this purpose. Actually, in our case we can neglect the non-commutation of the dynamic variable operators of the particle between themselves (of magnitude $\sim\hbar\omega_0/\epsilon$) and take into account only their commutators with the field of the radiated photon (of magnitude $\sim\hbar\omega/\epsilon$).

The standard form of the matrix element for radiation of a photon in an external field is

$$U_{fi} = \frac{e}{(2\pi)^{3/2}\sqrt{2\hbar\omega}}\int dt\int d^3rF^+_{fs'}(\mathbf{r})\,e^{\frac{i\varepsilon_f t}{\hbar}}(ej)\,e^{i(\omega t-\mathbf{kr})}\,e^{-\frac{i\varepsilon_i t}{\hbar}}F_{is}(\mathbf{r}), \tag{2.3}$$

where $F_{is}(\mathbf{r})$ is the solution of the wave equation in this field with energy ϵ_i and a spin state s, e_μ is the photon polarization vector, and j_μ is the current.

Here and subsequently we use the metric $ab = a_0b_0 - \mathbf{a}\cdot\mathbf{b}$, and a system of units c = 1.

For the states of interest to us with large quantum numbers we can use the approximate representation

$$e^{-i\varepsilon_i t/\hbar}F_{is}(\mathbf{r}) = \Psi_s(\mathcal{P})\,e^{-i\mathcal{H}t/\hbar}|i\rangle, \tag{2.4}$$

where $\Psi_s(\mathcal{P})$ is the operator form of the particle wave function in a spin state s in an external field. This form is obtained from the free wave functions by replacement of the variables by operators; $\mathbf{p} \to \mathcal{P}$, $\epsilon \to \mathcal{H} = \sqrt{\mathcal{P}^2 + m^2}$. The state vector $|i\rangle$ determines the states of the particle in the field (except for the spin s). In Eq. (2.4), interaction terms of the spin-field type are neglected; for example, for particles with spin 1/2 (terms of the form $\boldsymbol{\Sigma}\cdot\mathbf{H}$ and $\boldsymbol{\alpha}\cdot\mathbf{E}$).

We will give an example of Eq. (2.4) for the case of the Dirac equation in an external field

$$(\gamma^\mu\mathcal{P}_\mu - m)\,\psi(x) = 0, \quad \mathcal{P}_\mu = i\hbar\partial_\mu - eA_\mu(x). \tag{2.5}$$

The squared Dirac equation is

$$(\gamma^\mu\mathcal{P}_\mu + m)(\gamma^\mu\mathcal{P}_\mu - m)\,\psi(x) = \left[\mathcal{P}_\mu^2 - m^2 - \frac{\hbar}{2}e\sigma^{\mu\nu}F_{\mu\nu}\right]\psi(x) = 0, \tag{2.6}$$

where $\sigma^{\mu\nu} = i/2[\gamma^\mu, \gamma^\nu]$, and $F_{\mu\nu}$ is the electromagnetic field tensor. If we discard terms of the spin-field type $1/2\sigma^{\mu\nu}F_{\mu\nu} = (-\boldsymbol{\Sigma}\cdot\mathbf{H} + i\boldsymbol{\alpha}\cdot\mathbf{E})$, then Eq. (2.6) goes over to the Klein-Gordon equation in this field. With this accuracy we can represent the solution of Eq. (2.5) in the form

$$\psi(x) = C(\gamma^\mu\mathcal{P}_\mu + m)\,\Phi(x), \tag{2.7}$$

where $\Phi(x)$ is the solution of the Klein-Gordon equation, and C is a normalization constant. Using the standard γ-matrix representation, Eq. (2.7) can be rewritten in the form of (2.4). Thus, in the coordinate representation $|i\rangle$ is the solution of the Klein-Gordon equation in this field, $\Phi_i(x)$.

Substituting (2.4) into (2.3) and converting to Heisenberg operators, we write the matrix element U_{fi} in the form

$$U_{fi} = \frac{e}{(2\pi)^{3/2}\sqrt{2\hbar\omega}}\langle f|\int dt M(t)\,e^{i\omega t}|i\rangle, \tag{2.8}$$

where

$$eM(t) = \Psi^+_{s'}(\mathcal{P})\{(ej),\, e^{-i\mathbf{kr}(t)}\}\Psi_s(\mathcal{P}),$$
$$j_\mu(t) = e^{\frac{i\mathcal{H}t}{\hbar}}j_\mu e^{-\frac{i\mathcal{H}t}{\hbar}}, \quad r_k(t) = e^{\frac{i\mathcal{H}t}{\hbar}}r_k e^{-\frac{i\mathcal{H}t}{\hbar}} \tag{2.9}$$

are the current and particle-coordinate operators, and $\{\ldots\}$ designates the symmetrized product of the operators. It is important that in the large-quantum-number approximation adopted, the order of writing the operators entering into $\Psi_s(\mathcal{P})$ is unimportant. For a particle with spin 1/2

$$M_e(t) = u^+_{s'}(\mathcal{P})\,\alpha e e^{-i\mathbf{kr}(t)}\,u_s(\mathcal{P}), \tag{2.10}$$

where

$$u_s(\mathcal{P}) = \sqrt{\frac{\mathcal{H}+m}{2\mathcal{H}}}\begin{pmatrix}\varphi(\zeta(t))\\ \frac{\sigma\mathcal{P}}{\mathcal{H}+m}\varphi(\zeta(t))\end{pmatrix}; \tag{2.11}$$

here $\varphi(\zeta(t))$ is a two-component spinor describing the spin state of the electron at the moment of time t.

We will be interested in the probability of a transition with radiation of a photon, summed over all final states of the particle (except spin states). Carrying out this summation, we obtain the following expression for the probability of a radiative transition:

$$dw = \frac{e^2}{4\pi\hbar}\frac{d^3k}{(2\pi)^2\omega}\left\langle i\left|\int dt_1\int dt_2 e^{i\omega(t_1-t_2)}M^*(t_2)M(t_1)\right|i\right\rangle, \quad (2.12)$$

where

$$\frac{e^2}{4\pi\hbar}=\alpha=\frac{1}{137}. \quad (2.13)$$

In derivation of Eq. (2.12) we have used the completeness condition*) $\sum_f |f\rangle\langle f| = I$.

In order to calculate the probability (2.12) it is necessary to perform a number of manipulations on the operators which enter into it. According to the statement made above, in Eq. (2.10) for $M_e(t)$ it is necessary to take into account only the commutator of the photon field ($e^{-i\mathbf{k}\cdot\mathbf{r}(t)}$) with the momentum $\mathscr{P}$. The following relations exist:

$$\left.\begin{aligned}\mathscr{P}e^{-i\mathbf{kr}(t)} &= e^{-i\mathbf{kr}(t)}(\mathscr{P}-\hbar\mathbf{k}),\\ \mathscr{H}e^{-i\mathbf{kr}(t)} &= e^{-i\mathbf{kr}(t)}(\mathscr{H}-\hbar\omega),\end{aligned}\right\} \quad (2.14)$$

the first of which is a consequence of the fact that the operator $e^{-i\mathbf{k}\cdot\mathbf{r}(t)}$ is a displacement operator in momentum space, and for derivation of the second relation it is necessary to take into account that

$$[\mathscr{H}, e^{-i\mathbf{kr}(t)}] = -i\hbar\frac{d}{dt}e^{-i\mathbf{kr}(t)}, \quad (2.15)$$

and to carry out an integration by parts in Eq. (2.8).

Using Eq. (2.14), we can bring out the operator $e^{-i\mathbf{k}\cdot\mathbf{r}(t_1)}$ to the left in $M(t_1)$, and the operator $e^{i\mathbf{k}\cdot\mathbf{r}(t_2)}$ to the right in $M^*(t_2)$, and after this consider the combination $e^{i\mathbf{k}\cdot\mathbf{r}(t_2)}e^{-i\mathbf{k}\cdot\mathbf{r}(t_1)}$ which arises. The operators $\mathbf{r}(t_2)$ and $\mathbf{r}(t_1)$ taken at different moments of time do not commute with each other. In solution of the problem with inclusion of all orders in Planck's constant $\hbar$, it is necessary to unfold this combination, it being impossible here to limit ourselves to an expansion in the lowest commutators.

For what follows it is convenient to carry out a substitution of variables in the integral of Eq. (2.12),

$$t=\frac{1}{2}(t_1+t_2), \quad \tau = t_2-t_1. \quad (2.16)$$

Since we will be interested in the transition probability per unit time dw/dt, in the integral (2.12) it is necessary to integrate over the relative time τ and the final photon states. The main contribution to the integral is given by the region $|\dot{\mathbf{v}}|\tau \sim 1/\gamma$ (we will convince ourselves of this below); for this reason we will expand all quantities involved in powers of $|\dot{\mathbf{v}}|\tau$, which corresponds to expansion in powers of $1/\gamma$, and retain only the leading terms of the expansion. In addition, we will neglect quantities

$$\frac{|\dot{\mathbf{H}}|\tau}{|\mathbf{H}|}\ll 1, \quad (2.17)$$

where $|\dot{\mathbf{H}}|$ characterizes the variation of the magnetic field in the trajectory. Physically this criterion means that the field in the trajectory does not change appreciably in the characteristic radiation time. If we introduce a field nonuniformity index

$$n=\left|\frac{\partial \ln H}{\partial \ln r}\right|, \quad (2.18)$$

then criterion (2.17) takes on the form

$$n/\gamma \ll 1. \quad (2.19)$$

Thus, the field must not be too nonuniform; in all practical cases the field satisfies this criterion.

Unfolding of the combination $e^{i\mathbf{k}\cdot\mathbf{r}(t_2)}e^{-i\mathbf{k}\cdot\mathbf{r}(t_1)}$ leads to an identical result for all forms of external fields[2-4]

$$e^{i\mathbf{kr}(t_2)}e^{-i\mathbf{kr}(t_1)} = \exp\left\{i\left[\omega\tau+\frac{\mathscr{H}}{\mathscr{H}-\hbar\omega}(\mathbf{k}\boldsymbol{\rho}-\omega\tau)\right]\right\}, \quad (2.20)$$

where

$$\boldsymbol{\rho}=\mathbf{r}(t_2)-\mathbf{r}(t_1). \quad (2.21)$$

The combination obtained obviously commutes with $\mathscr{H}$ (see Eq. (2.14)). In order to discuss its commutation with the operator $\mathscr{P}$, it is necessary to take into account the fact that, in order to use Eq. (2.14), it is necessary that all operators depend on a single time. Carrying out the appropriate expansions and omitting terms of order $1/\gamma^2$, we find that $e^{i\mathbf{k}\cdot\mathbf{r}(t_2)}e^{-i\mathbf{k}\cdot\mathbf{r}(t_1)}$ commutes with $\mathscr{P}$.

Thus, after these operations have been carried out, all operators in Eq. (2.12) commute with each other with the accuracy adopted. Therefore all of them which stand within the brackets of the initial state (the average in states with large quantum numbers) can be replaced by classical values. In the final result, the square of the matrix element can be written in the following form:

$$\langle i|M^*(t_2)M(t_1)|i\rangle = \exp\left\{i\left[\omega\tau+\frac{\varepsilon}{\varepsilon'}(\mathbf{k}\boldsymbol{\rho}-\omega\tau)\right]\right\}R^*(t_2)R(t_1), \quad (2.22)$$

where

$$R(t)=u^+_{s'}(\mathbf{p}')\boldsymbol{\alpha}\mathbf{e}u_s(\mathbf{p}). \quad (2.23)$$

Here $\epsilon' = \epsilon - \hbar\omega$, $\mathbf{p}' = \mathbf{p} - \hbar\mathbf{k}$. In these expressions ϵ, ϵ', $\mathbf{p}$, and $\mathbf{p}'$ already are not operators, but c-numbers (values of energy and momentum). All the information on the spin and polarization states is contained in the quantity R(t), which has the form of a transition matrix element for free particles (with inclusion of the conservation laws). Consequently, all the features of the radiation in an external field in Eq. (2.22) are contained in the fact that a factor ϵ/ϵ' (inclusion of recoil) appears in the exponential, and in Eq. (2.23) $\mathbf{p} = \mathbf{p}(t)$, where the evolution of momentum in time is taken in this field. The transition to the classical theory is that $\epsilon' \to \epsilon$, $\mathbf{p}' \to \mathbf{p}$ ($\hbar \to 0$), while $R(t) \to \mathbf{e}\cdot\mathbf{j}$, and $\mathbf{j}$ is the classical current.

Using the explicit form (2.11) of the spinors u(p), it is easy to obtain

$$\left.\begin{aligned}R(t) &= \varphi_f^+Q\varphi,\\ Q &= A+i\boldsymbol{\sigma}\mathbf{B},\end{aligned}\right\} \quad (2.24)$$

where

$$A=\frac{\mathbf{ep}}{2}\left[\frac{1}{\varepsilon+m}+\frac{1}{\varepsilon'+m}\right], \quad \mathbf{B}=\frac{1}{2}\left\{\frac{[\mathbf{ep}]}{\varepsilon+m}-\frac{[\mathbf{ep}']}{\varepsilon'+m}\right\}. \quad (2.25)^*$$

Here we have neglected terms of order $1/\gamma$; furthermore, it is everywhere assumed that the final electrons are ultrarelativistic. If we take into account the smallness of the time of radiation in comparison with the

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*In the coordinate representation $\langle f|\ldots|i\rangle = \int d^3r\Phi_f^+(r)\ldots\Phi_i(r)$, $\sum_f \Phi_f(r)\Phi_f^+(r') = \delta(r-r')$.

*$[\mathbf{ep}] \equiv \mathbf{e}\times\mathbf{p}$.

characteristic period of motion T (for example, the period of rotation), $\tau \sim T/\gamma$, and also the fact that the period of spin precession is of the same order as the period of the motion, we can assume that the spin state does not change during the time of radiation, i.e., with an accuracy to terms of order $1/\gamma$ we have

$$\varphi(\boldsymbol{\zeta}(t_1)) = \varphi(\boldsymbol{\zeta}(t_2)) = \varphi(\boldsymbol{\zeta}(t)). \tag{2.26}$$

Then

$$R_2^* R_1 = \mathrm{Sp}\left\{\left(\frac{1+\sigma\zeta_i}{2}\right)(A_2 - i\sigma\mathbf{B}_2)\left(\frac{1+\sigma\zeta_f}{2}\right)(A_1 + i\sigma\mathbf{B}_1)\right\}, \tag{2.27}$$

where $R_i \equiv R(t_i)$, and so forth. Equation (2.27) can be used in discussion of any phenomena, including spin and polarization phenomena, in radiation of photons by an electron in a magnetic field.

2.2. Transition With Spin Flip

Let us discuss now a radiative transition with spin flip, i.e., $\boldsymbol{\zeta}_f = -\boldsymbol{\zeta}_i$. Calculating the trace of Eq. (2.27) in this case, we have

$$R_2^* R_1|_{\text{spin-flip}} = \mathbf{B}_1\mathbf{B}_2 - (\boldsymbol{\zeta}\mathbf{B}_1)(\boldsymbol{\zeta}\mathbf{B}_2) - i(\boldsymbol{\zeta}[\mathbf{B}_1\mathbf{B}_2]). \tag{2.28}$$

Calculation of the function **B** of Eq. (2.25) gives

$$\mathbf{B} = \frac{\hbar}{2(\varepsilon' + m)}[\mathbf{qe}], \tag{2.29}$$

where*

$$\mathbf{q} = \frac{\mathbf{p}\omega}{\varepsilon + m} - \mathbf{k} = \omega\left(\frac{\mathbf{v}}{1 + (1/\gamma)} - \mathbf{n}\right). \tag{2.30}$$

The energies and the field in contemporary accelerators are such that $\chi \ll 1$, i.e., $\hbar\omega \ll \epsilon$. Therefore, all expressions can be represented in the form of a series in Planck's constant $\hbar$ (actually in series in χ (see Eq. (2.2)). In view of the smallness of the expansion parameter, for practical purposes it is sufficient to limit ourselves to terms of lowest order. Obviously, for transitions with spin flip this is order $\hbar^2$. In the expressions (2.28)–(2.29) obtained by us the coefficient of $\hbar^2$ is taken out explicitly. This means that for calculation of terms of order $\hbar^2$ all remaining terms containing $\hbar\omega$ and $\hbar k$ inside the expressions can be omitted, i.e., we must let $\epsilon' \to \epsilon$, $\mathbf{p}' \to \mathbf{p}$ in (2.22) and (2.29). From this it is clear that for solution of the problem of a transition with spin flip to order $\hbar^2$, in general it is not required to carry out an unfolding.

Leaving in Eq. (2.28) the main terms in $\hbar$ and carrying out the summation over photon polarizations, we obtain

$$\sum_\lambda R_2^* R_1|_{sf} = \frac{\hbar^2}{4\varepsilon^2}\left\{\mathbf{q}_1\mathbf{q}_2\left(1 - \frac{(\boldsymbol{\zeta}\mathbf{k})^2}{\omega^2}\right)\right. \tag{2.31}$$
$$\left. + \frac{\boldsymbol{\zeta}\mathbf{k}}{\omega^2}[(\mathbf{q}_1\boldsymbol{\zeta})(\mathbf{q}_2\mathbf{k}) + (\mathbf{q}_2\boldsymbol{\zeta})(\mathbf{q}_1\mathbf{k})] - i\left(\boldsymbol{\zeta} - \frac{(\boldsymbol{\zeta}\mathbf{k})\mathbf{k}}{\omega^2}\right)[\mathbf{q}_1\mathbf{q}_2]\right\}.$$

Here the expansions of the quantities entering into Eqs. (2.22) and (2.31) in powers of the relative time τ have the form

$$\left.\begin{aligned} \mathbf{q}_1 &= (\omega\mathbf{v} - \mathbf{k}) - \frac{\omega\tau}{2}\dot{\mathbf{v}} - \frac{\omega\mathbf{v}}{\gamma} + \ldots, \\ \mathbf{q}_2 &= (\omega\mathbf{v} - \mathbf{k}) + \frac{\omega\tau}{2}\dot{\mathbf{v}} - \frac{\omega\mathbf{v}}{\gamma} + \ldots, \\ \mathbf{r}_2 - \mathbf{r}_1 &= \tau\mathbf{v} + \frac{\tau^3}{24}\ddot{\mathbf{v}} + \ldots \end{aligned}\right\} \tag{2.32}$$

To obtain the total probability of a radiative transition with spin flip per unit time, it is necessary, after substitution of (2.32) into (2.31) and (2.22), to substitute the expression obtained into Eq. (2.12) and perform the integration over the relative time τ and over the final states of the photon. It turns out to be convenient to do this up to the integration over τ by means of the formula

$$\int e^{-iky} f(k_\mu)\frac{d^3k}{\omega} = -f(i\partial_\mu)\frac{4\pi}{(y_0 - i\varepsilon_0)^2 - \mathbf{y}^2}(\varepsilon_0 \to 0), \tag{2.33}$$

where*

$$y_0 = \tau = t_2 - t_1, \quad \mathbf{y} = \mathbf{r}_2 - \mathbf{r}_1, \quad y^2 = y_0^2 - \mathbf{y}^2 = \tau^2\left[(1/\gamma^2) + \frac{\tau^2\dot{\mathbf{v}}^2}{12}\right]. \tag{2.34}$$

After integration over the final photon states, we obtain for the probability of a transition with spin flip per unit time

$$W^\zeta \equiv \frac{dw}{dt} \tag{2.35}$$
$$= \frac{\alpha}{\pi}\frac{\hbar^2}{m^2}\gamma^5|\dot{\mathbf{v}}|^3 \oint \frac{dz}{\left(1 + \frac{z^2}{12}\right)^3}\left[\frac{3}{z^4} - \frac{5}{12z^2} + \left(\frac{1}{z^4} + \frac{5}{12z^2}\right)(\boldsymbol{\zeta}\mathbf{v})^2 - \frac{2i}{z^3|\dot{\mathbf{v}}|}(\boldsymbol{\zeta}[\mathbf{v}\dot{\mathbf{v}}])\right],$$

where we have made the substitution $z = \tau|\dot{\mathbf{v}}|\gamma$, and the integration contour passes below the real axis and is closed in the lower half of the plane. From this it is evident that the main contribution to the integral is made by the region $|\dot{\mathbf{v}}|\tau \sim 1/\gamma$. The contour integrals entering into Eq. (2.35) are easily calculated; as a result we have

$$I_{n,m} \equiv \oint \frac{dz}{z^n\left(1 + \frac{z^2}{12}\right)^m} = \frac{i^n\pi(12)^{\frac{1-n}{2}}}{(m-1)!}\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2} + 1\right)\ldots\left(\frac{n+1}{2} + m - 2\right). \tag{2.36}$$

The following final expression is obtained for the total probability of a radiative transition with spin flip per unit time[5,6]:

$$W^\zeta = \frac{5\sqrt{3}}{16}\alpha\frac{\hbar^2}{m^2}\gamma^5|\dot{\mathbf{v}}|^3\left\{1 - \frac{2}{9}(\boldsymbol{\zeta}\mathbf{v})^2 + \frac{8\sqrt{3}}{15|\dot{\mathbf{v}}|}(\boldsymbol{\zeta}[\mathbf{v}\dot{\mathbf{v}}])\right\}. \tag{2.37}$$

We note that this result applies for an arbitrary magnetic field (if the weak restrictions (2.17) and (2.19) are satisfied).

Let us make an analysis of the expression obtained. For longitudinal polarization $(\boldsymbol{\zeta}[\mathbf{v} \times \dot{\mathbf{v}}]) = 0$ the remaining terms $1 - 2/9(\boldsymbol{\zeta}\cdot\mathbf{v})^2$ do not depend on whether the spin is directed along or opposite to the velocity, so that the radiation does not change the spin states with longitudinal polarization. A different situation arises in the case of transverse polarization $(\boldsymbol{\zeta}\cdot\mathbf{v}) = 0$. In this case the transition probability depends on the spin orientation. For electrons ($e < 0$) the probability of a transi-

*As shown in Appendix B, an effective electromagnetic field $\mathbf{H}_E$ (3.22) acts on a Dirac degenerate spin. In this language the spin is acted on by the term $\sigma\cdot\mathbf{B}$ in Eq. (2.24), and this means that the spin is flipped by the effective electromagnetic field of the radiated wave (compare Eqs. (2.30) and (3.22), taking into account that we are dealing with Fourier components).

*We note that formulas of the type of (2.33) can be used only in the case when the exponent in the integral does not contain terms in $\hbar\omega$. This situation arises directly in calculation of terms of lowest order ($\sim\hbar^0$ in calculation of the total probability, $\sim\hbar^2$ in calculation of the probability of a transition with spin flip). However, these formulas can be useful also for calculation of corrections of the next higher orders, for which it is necessary to expand in a power series the terms in the exponential containing $\hbar\omega$.

tion from a state with spin along the field to a state with spin opposite to the field is higher than the probability of the inverse transition. For positrons ($e > 0$) the opposite situation occurs: the probability of a transition from a state with spin opposite to the field to a state with spin along the field is higher than the probability of the inverse transition. Thus, the resulting polarization (radiative polarization) is transverse* and for electrons is directed opposite to the field and for positrons along it. In order to determine the degree of polarization, it is necessary to solve the corresponding kinetic equation, which will be done in the next section.

Sokolov and Ternov[1] carried out a calculation by traditional means with use of exact solutions of the Dirac equation in a uniform magnetic field. The complexity of the calculation did not permit generalization of the results even to a weakly nonuniform field. The derivation presented above follows the articles of V. M. Katkov and the author.[2,3,5,6]

3. THE KINETICS OF RADIATIVE POLARIZATION

The possibility that transverse polarization of electrons and positrons can arise in an external field follows from Eq. (2.37). In order to clarify how this possibility is realized, it is necessary to obtain and solve the kinetic equation for the polarization density matrix with inclusion of the interaction with the radiation field. This group of questions will be considered below.

3.1. Equation for the Spin Vector With Inclusion of Damping

When we take into account the quasiclassical nature of the motion of a high-energy electron in an external field, the equation for the polarization density matrix can conveniently be represented in the form of an equation for the spin vector (twice the average value of the spin operator in the rest system of the electron) $\boldsymbol{\zeta}$. Thus, we are concerned with obtaining an equation of the Bargman-Michel-Telegdi type (BMT)[7] with inclusion of the interaction with the radiation field. We will introduce the Heisenberg electron spin operator in the rest system $\boldsymbol{\sigma}(t)$ ($\boldsymbol{\sigma}^+(t) = \boldsymbol{\sigma}(t)$), whose average value

$$\boldsymbol{\zeta}_0(t) = \langle t_0 | \boldsymbol{\sigma}(t) | t_0 \rangle \tag{3.1}$$

is the spin vector in the rest system of the particle. Without inclusion of interaction with the radiation field, the variation of this vector with time for particles with a given anomalous magnetic moment is determined by the BMT equation (in the quasiclassical limit, i.e., for fields varying weakly in lengths of order $\hbar/mc$ and narrow wave packets).

After inclusion of the interaction with the radiation field (as in the preceding section, we will use the interaction representation) the evolution of the state vector with time is determined by the matrix $U(t, t_0)$†:

$$|t\rangle = U(t, t_0) | t_0 \rangle. \tag{3.2}$$

The change of the average spin value with time when the interaction with the radiation field is included is

$$\begin{aligned} \langle t | \boldsymbol{\sigma}(t) | t \rangle - \langle t_0 | \boldsymbol{\sigma}(t_0) | t_0 \rangle &= \langle t_0 | U^+(t, t_0) \boldsymbol{\sigma}(t) U(t, t_0) | t_0 \rangle \\ - \langle t_0 | \boldsymbol{\sigma}(t_0) | t_0 \rangle &= \langle t_0 | U^+(t, t_0) [\boldsymbol{\sigma}(t), U(t, t_0)] | t_0 \rangle + \langle t_0 | \boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(t_0) | t_0 \rangle. \end{aligned} \tag{3.3}$$

Here the last term determines the change of the average spin in the absence of the radiation field. We will represent the scattering matrix $U(t, t_0)$ in the form of a perturbation theory expansion in the electromagnetic coupling constant e:

$$U(t, t_0) = I + iT(t, t_0) = I + i[T_1(t, t_0) + T_2(t, t_0) + \ldots]. \tag{3.4}$$

From the condition of unitarity of the scattering matrix we obtain

$$T_1 - T_1^+ = 0, \quad i(T_2^+ - T_2) = T_1^+ T_1 = 2 \operatorname{Im} T_2. \tag{3.5}$$

With the help of these relations and Eq. (3.1) we can rewrite Eq. (3.3) in the form

$$\begin{aligned} \boldsymbol{\zeta}(t) - \boldsymbol{\zeta}(t_0) = \langle t_0 | \Big\{ & T_1^+ \boldsymbol{\sigma}(t) T_1 - \tfrac{1}{2} [\boldsymbol{\sigma}(t) T_1^+ T_1 + T_1^+ T_1 \boldsymbol{\sigma}(t)] + \\ & + i [\boldsymbol{\sigma}(t) \operatorname{Re} T_2] \Big\} | t_0 \rangle + \boldsymbol{\zeta}_0(t) - \boldsymbol{\zeta}_0(t_0), \end{aligned} \tag{3.6}$$

in which

$$\boldsymbol{\zeta}(t_0) = \boldsymbol{\zeta}_0(t_0), \tag{3.7}$$

since the interaction with the radiation field is turned on at the moment of time t_0.

Let us turn to calculation of the individual terms in Eq. (3.6). A photon creation (or annihilation) operator enters into the matrix, and therefore the matrix element is given by

$$\langle t_0 | T_1 | t_0 \rangle = 0, \tag{3.8}$$

since the state vector $|t_0\rangle$ describes the state of the electron in the field (without photons). This fact is taken into account in (3.6). In calculation of terms containing the combination $T_1^+ T_1$, it is necessary to take into account that only the matrix elements T_1 for transition to a one-photon state are different from zero, i.e.,

$$\begin{aligned} \langle t_0 | T_1^+ T_1 | t_0 \rangle &= \sum_n \langle t_0 | T_1^+ | n \rangle \langle n | T_1^+ | t_0 \rangle \\ &= \int d^3k \sum_{s_n, \lambda} \langle t_0 | T_1^+ | t_0, k \rangle \langle k, t_0 | T_1 | t_0 \rangle, \end{aligned} \tag{3.9}$$

where the integration is carried out over the photon momenta, and the summation over the electron spins s_n and photon polarizations λ; $\langle k, t_0 | T_1 | t_0 \rangle$ is the transition matrix element to a one-photon state with a photon ($\mathbf{k}, \lambda$) (radiation of a photon, compare (2.8)). In accordance with the results of the preceding section ((2.22)–(2.25)) this matrix element has the form

$$\langle k, t_0 | T_1 | t_0 \rangle = \frac{e}{(2\pi)^{3/2} \sqrt{2\hbar\omega}} \varphi_n^+ \Big[\int_{t_0}^{t} Q(t) e^{+i \frac{\varepsilon}{\varepsilon'} (\omega t - \mathbf{kr}(t))} dt \Big] \varphi_i. \tag{3.10}$$

On the basis of the arguments which led us to Eq. (2.26) and which mean, in the terms used by us, that the characteristic time for change of the matrix elements of the operators T_1 is the radiation time ($\tau \sim T_c/\gamma$), while the characteristic time of change of $\boldsymbol{\sigma}(t)(\boldsymbol{\zeta}(t))$ is T_c (T_c is, for example, the period of rotation), we can neglect the dependence of $\boldsymbol{\sigma}(t)$ on time with an accuracy to terms of order $1/\gamma$. With inclusion of this fact and Eqs. (3.9) and (3.10), we have

*This fact is evident beforehand: the axial vector of the polarization arising can be directed only along the unit axial vector $[\mathbf{v}\dot{\mathbf{v}}]$ (or what is the same thing, $\mathbf{H}$).

†We note that the state vector $|i\rangle$ which has been introduced is a two-component spinor, and the $U(t, t_0)$ matrix is a 2×2 matrix acting in the space of these spinors. In what follows we will understand the symbol Re U to mean the Hermitian part of the matrix, and i ImU will mean the anti-Hermitian part of the matrix.

$$\Delta\boldsymbol{\zeta}_1 \equiv \langle t_0 | \left\{ T_1^+ \boldsymbol{\sigma} T_1 - \frac{1}{2} [\boldsymbol{\sigma} T_1^+ T_1 + T_1^+ T_1 \boldsymbol{\sigma}] \right\} | t_0 \rangle \tag{3.11}$$
$$= \frac{e^2}{4\pi\hbar} \frac{1}{(2\pi)^2} \int \frac{d^3k}{\omega} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \sum_\lambda \mathbf{S} \exp\left\{ -\frac{i\varepsilon}{\varepsilon'} [\omega(t_2 - t_1) - \mathbf{k}(\mathbf{r}_2 - \mathbf{r}_1)] \right\},$$

where

$$\mathbf{S} = \mathrm{Sp}\left\{ \left[Q_2^+ \boldsymbol{\sigma} Q_1 - \frac{1}{2}(\boldsymbol{\sigma} Q_2^+ Q_1 + Q_2^+ Q_1 \boldsymbol{\sigma}) \right] \frac{(I + \boldsymbol{\sigma}\boldsymbol{\zeta})}{2} \right\} \cdot \tag{3.12}$$

By means of the relation

$$Q_1 \boldsymbol{\sigma} = \boldsymbol{\sigma} Q_1 + 2[\mathbf{B}_1 \boldsymbol{\sigma}], \tag{3.13}$$

it is not difficult to calculate the trace of (3.12):

$$\left.\begin{aligned} \mathbf{S} &= \mathbf{S}_A + \mathbf{S}_B, \\ \mathbf{S}_A &= -[(A_1 \mathbf{B}_2 + A_2 \mathbf{B}_1)\boldsymbol{\zeta}], \\ \mathbf{S}_B &= -2i[\mathbf{B}_2 \mathbf{B}_1] + \mathbf{B}_2(\boldsymbol{\zeta}\mathbf{B}_1) + \mathbf{B}_1(\boldsymbol{\zeta}\mathbf{B}_2) - 2\boldsymbol{\zeta}(\mathbf{B}_1 \mathbf{B}_2). \end{aligned}\right\} \tag{3.14}$$

The expression obtained for **S** contains terms of two types: quadratic in $\mathbf{B}_{1,2}$ ($\mathbf{S}_B \sim \hbar^2$ (compare Eq. (2.9)) and linear in $\mathbf{B}_{1,2}$, $A_{1,2}$ ($\mathbf{S}_A \sim \hbar$). These terms lead to different physical consequences, and therefore we will discuss them individually. We will multiply **S** (Eq. (3.14) by $\boldsymbol{\zeta}$:

$$(\mathbf{S}\boldsymbol{\zeta}) = (\mathbf{S}_B \boldsymbol{\zeta}) = 2\{(\mathbf{B}_1 \boldsymbol{\zeta})(\mathbf{B}_2 \boldsymbol{\zeta}) - \zeta^2 (\mathbf{B}_1 \mathbf{B}_2) - i(\boldsymbol{\zeta}[\mathbf{B}_2 \mathbf{B}_1])\} = -2R_2^* R_1 |_{sf}, \tag{3.15}$$

so that the term $(\mathbf{S} \cdot \boldsymbol{\zeta})$ is expressed in terms of the square of the matrix element of the radiative transition with spin flip (2.28). We note that, in contrast to the preceding section where we discussed the problem for one electron ($|\boldsymbol{\zeta}| = 1$), in this section we are carrying out the discussion for an ensemble of electrons (in the language of one of the density matrix representations), so that, generally speaking, $|\boldsymbol{\zeta}| \neq 1$. The further calculation of the integral (3.11) with $\mathbf{S}_B$ is identical to that performed in the preceding section*, since the terms for given structures with ζ are separated uniquely. Thus, the answer follows directly from Eq. (2.37):

$$\frac{\Delta\boldsymbol{\zeta}_{1B}}{\Delta t} = -\frac{1}{T}\left\{ \boldsymbol{\zeta} - \frac{2}{9}(\boldsymbol{\zeta}\mathbf{v})\boldsymbol{\zeta} + \frac{8}{5\sqrt{3}|\dot{\mathbf{v}}|}[\mathbf{v}\dot{\mathbf{v}}] \right\}, \tag{3.16}$$

where

$$\frac{1}{T} = \frac{5\sqrt{3}}{8} \alpha \frac{\hbar^2}{m^2} \gamma^5 |\dot{\mathbf{v}}|^3. \tag{3.17}$$

Let us consider now the term $\Delta\zeta_{1A}/\Delta t$. As can be seen from (3.14), the structure of this term is of the type $[\mathbf{F}_A \cdot \boldsymbol{\zeta}]$†. In contrast to Eq. (3.16), terms of this type describe the rotation of $\boldsymbol{\zeta}$, and not the variation of $|\zeta|$ (obviously $\zeta_{1A}\Delta\zeta_{1A} = 0$, i.e., $\Delta(\zeta_{1A}^2) = 0$).

Using the explicit expressions for (2.25) and (2.29), A and **B** and performing the summation over photon polarization and the expansion (2.32), we obtain, retaining terms to order $1/\gamma^2$ (this is the order of the terms $\sum_\lambda A_1 A_2$ and $\sum_\lambda \mathbf{B}_1 \mathbf{B}_2$),

$$\sum_\lambda (A_1 \mathbf{B}_2 + A_2 \mathbf{B}_1) = \frac{\hbar\omega^2(\varepsilon + \varepsilon')}{2\varepsilon'^2 \gamma}[\mathbf{n}\mathbf{v}]. \tag{3.18}$$

The vector product $\mathbf{n} \cdot \mathbf{v}$ is an odd function of the photon emission angles, and therefore, after substituting $\mathbf{S}_A$ (3.14) with inclusion of (3.18) into the integral (3.11), we see that the integral goes to zero on integration over the final photon states, i.e.,

$$\frac{\Delta\zeta_{1A}}{\Delta t} = 0. \tag{3.19}$$

This fact is not accidental and is due to invariance against time reversal. Actually, for $t_1 \to -t_1$, $t_2 \to -t_2$, we have $\boldsymbol{\zeta} \to -\boldsymbol{\zeta}$, $d\zeta/dt \to d\zeta/dt$, while the integral $\int \sum_\lambda (A_2 \mathbf{B}_1 + A_1 \mathbf{B}_2)$ in Eq. (3.11) does not change sign with this substitution.

Let us turn now to the term with Re T_2. To calculate this term it is necessary to know the Green's function of the electron in a magnetic field (see Appendix A). Using Eqs. (A.1) and (A.5), we obtain for the same assumptions regarding $\boldsymbol{\sigma}(t)$ used in (3.11):

$$\langle t_0 | i[\boldsymbol{\sigma}, \mathrm{Re}\, T_2] | t_0 \rangle = \langle i | \mathrm{Sp}\left[\left(\frac{1 + \boldsymbol{\zeta}\boldsymbol{\sigma}}{2}\right) i[\boldsymbol{\sigma}, \boldsymbol{\sigma}(-\mathrm{Re}\, D)] \right] | i \rangle = \int_{t_0}^{t} dt \frac{2\mu'}{\gamma} [\boldsymbol{\zeta}\mathbf{H}_R]. \tag{3.20}$$

We will recall that $\mathbf{H}_R$ (A.4) is the magnetic field in the rest system of the electron (if in the laboratory system the fields are **E** and **H**), and in the limit $\chi \to 0$, $\mu' = (\alpha/2\pi)(e/2m)$. Thus,

$$\frac{d\boldsymbol{\zeta}_2}{dt} = \frac{2\mu'}{\gamma}[\boldsymbol{\zeta}\mathbf{H}_R] = \left(\frac{g-2}{2}\right)\frac{e}{m\gamma}[\boldsymbol{\zeta}\mathbf{H}_R], \tag{3.21}$$

i.e., we have obtained a rotation term proportional to the anomalous magnetic moment of the electron.

Finally, the difference $\boldsymbol{\zeta}_0(t) - \boldsymbol{\zeta}_0(t_0) = (\Delta\zeta_0/\Delta t)\Delta t$ occurring in Eq. (3.6) describes the change of the electron spin vector in an external field in the absence of interaction with the radiation field. In the quasiclassical limit we can obtain directly from the equation for the spin operator of the Dirac equation (see, for example, ref. 9):

$$\frac{d\boldsymbol{\zeta}_0}{dt} = \frac{e}{\varepsilon}[\boldsymbol{\zeta}\mathbf{H}_E], \quad \mathbf{H}_E = \mathbf{H} + \frac{1}{1 + (1/\gamma)}[\mathbf{E}\mathbf{v}]. \tag{3.22}$$

Thus, the picture of the phenomenon under discussion is the following. In the absence of interaction with the radiation field the spin precesses according to Eq. (3.22). Inclusion of the interaction with the radiation field leads to effects of two types:

1. New forms of rotation terms appear, which are associated with the appearance in the electron, as the result of interaction with the radiation field, of an anomalous magnetic moment (3.21). The sum of (3.21) and (3.22) gives the equation of motion of the spin of an electron with an anomalous magnetic moment in an external field (the BMT equation)*.

2. In addition, terms appear (damping terms (3.16)) which do not reduce to rotation and which change $|\boldsymbol{\zeta}|$.

*It is necessary that the time difference be $t - t_0 \gg \tau \sim 1/|\dot{v}|\gamma$.

†$\mathbf{F}_A$ is the axial vector constructed from the vectors of the problem.

*In this sense the calculation which has been carried out is a direct derivation of the BMT equation. It could have been discussed in the reverse order. Proceeding from the general representation (A.1) and (A.5) for Re T_2 without regard for the coefficients, it is easy to see (3.20) that this is a term of the rotational type. However, then it can be equal only to the term with an anomalous moment in the BMT equation. We note that it follows from (3.19) that the dependence of the coefficients in the BMT equation on the field (χ) enters only through the anomalous magnetic moment of the electron.

Altogether we obtain the following equation for the motion of the spin of an ensemble of electrons in an external field with inclusion of radiation effects (3.16), (3.21), and (3.22):

$$\frac{d\boldsymbol{\zeta}}{dt}=\frac{e}{\varepsilon}[\boldsymbol{\zeta}(\eta\mathbf{H}_R+\mathbf{H}_E)]-\frac{1}{T}\left(\boldsymbol{\zeta}-\frac{2}{9}(\boldsymbol{\zeta}\mathbf{v})\mathbf{v}+\frac{8}{5\sqrt{3}|\dot{\mathbf{v}}|}[\mathbf{v}\dot{\mathbf{v}}]\right), \quad (3.23)$$

where $\mathbf{H}_E$ is defined in (3.22), $\mathbf{H}_R$ in (A.4), $1/T$ in (3.17), $\eta \equiv (g-2)/2 = \alpha/2\pi$. It is necessary to keep in mind that the rotational terms in (3.23) are of order $\hbar^0(\chi^0)$ (we are not taking into account the next corrections in $\hbar(\chi)$ to the rotational terms, since they are small and do not lead to new qualitative effects), while the damping terms are of order $\hbar^2(\chi^2)$, but it is necessary to retain them since they lead to new qualitative effects—a variation of $|\boldsymbol{\zeta}|$.

Nevertheless this difference in the orders of magnitude simplifies to a great extent the solution of the kinetic equation (3.23) and in many cases permits discussion of rotation and damping effects separately.

3.2. Solution of the Kinetic Equation

We will consider the solution of equation (3.23) in a magnetic field ($\mathbf{E} = 0$) for the high-energy case $\gamma \gg 1$, since only in this case does it make sense to include terms associated with damping. It turns out to be convenient to introduce the system of axes (B.18) (Appendix B). Then Eq. (3.23) can be rewritten in the form (compare (B.13))

$$\left.\begin{aligned}\dot{\zeta}_1&=-\Omega\zeta_2-\frac{7}{9T}\zeta_1,\\ \dot{\zeta}_2&=\Omega\zeta_1+\omega\zeta_3-(\zeta_2/T),\\ \dot{\zeta}_3&=-\omega\zeta_2-\frac{1}{T}\left(\zeta_3+\frac{8}{5\sqrt{3}}\right),\end{aligned}\right\} \quad (3.24)$$

where the frequencies Ω and ω are determined by Eqs. (B.17), and $1/T$ by (3.17). The system of equations (3.24) describes the motion of the spin of an ensemble of electrons with inclusion of damping in an arbitrary magnetic field.

As the simplest illustration of the nature of the solutions of the system of equations (3.24), let us consider the motion of an electron in uniform magnetic field for $\mathbf{v} \perp \mathbf{H}$. In this case $\Omega = \eta\gamma\omega_0$ ($\omega_0 = |\dot{\mathbf{v}}| = eH/\epsilon$ is the Larmor frequency), and $\omega = 0$, where Ω and T do not depend on time. Then the solution of the system of equations is as follows:

$$\left.\begin{aligned}\zeta_1&=\zeta_\perp(0)\cos(\Omega t+\varphi_0)e^{-8t/9T},\\ \zeta_2&=\zeta_\perp(0)\sin(\Omega t+\varphi_0)e^{-8t/9T},\\ \zeta_3&=\left(\zeta_3(0)+\frac{8}{5\sqrt{3}}\right)e^{-t/T}-8/5\sqrt{3},\end{aligned}\right\} \quad (3.25)$$

where we have taken into account that $\Omega \gg 1/T$ (terms of order χ^4 have been discarded). Hence it follows that the components $\zeta_1(t)$ and $\zeta_2(t)$ are damped with a characteristic time $t \sim T$ (3.17), while the component $\zeta_3(t)$ survives, so that after a time $t \gg T$ we have

$$\zeta_1=\zeta_2=0, \quad \zeta_3=-\frac{8}{5\sqrt{3}}=-0.924. \quad (3.26)$$

This result does not depend on the initial polarization of the electrons. In particular, if initially the electrons were not polarized, then

$$\zeta_1(t)=\zeta_2(t)=0, \quad \zeta_3(t)=-\frac{8}{5\sqrt{3}}(1-e^{-t/T}). \quad (3.27)$$

Equations (3.25)—(3.27) determine the kinetics of radiative polarization in a uniform field. We note that for electrons ($e < 0$) the vector $\mathbf{v}\times\dot{\mathbf{v}}$ is directed along the field, i.e., the polarization which arises is oriented opposite to the magnetic field $\mathbf{H}$, and for positrons ($e > 0$) the vector $\mathbf{v}\times\dot{\mathbf{v}}$ is directed opposite to the field, and the polarization arising is oriented along the field (see the footnote on page 00). In this way we can actually convince ourselves that the last term in Eq. (3.23) has a nature completely different from the remaining terms: while these terms lead to rotation of the spin without changing its amplitude, the term with $1/T$ in (3.23) changes the modulus $|\boldsymbol{\zeta}|$. The radiative polarization process occurs in such a way that on the fast precession of the spin vector in an external field is superimposed a slow process of damping of the spin components transverse to the field.

Numerical values of the polarization time under actual conditions are listed in the table. The data apply to the storage rings VÉPP-2 (R = 150 cm) and VÉPP-3 (R = 750 cm) in Novosibirsk. These times are of the order of the time of operation of the storage rings, so that the polarization effects are quite observable. One additional remark concerns the time dependence of the degree of polarization. It follows from (3.27) that

$$|\zeta_3(\infty)|=0.924, \quad |\zeta_3(T)|=0{,}584, \quad |\zeta_3(T/4)|=0.204, \quad (3.28)$$

so that after comparatively short times (compare with the table) it is already possible to observe polarization effects*.

Now let us consider the case in which the electron is moving in a uniform field along a helical line ($\mathbf{v}\times\mathbf{H} \neq 0$). Then from (B.17) it immediately follows that $\omega = \eta e(H_\parallel/\epsilon) = \eta\omega_0(H_\parallel/H_\perp) = (\Omega/\gamma)(H_\parallel/H_\perp) \ll \Omega$ (if the angle between $\mathbf{v}$ and $\mathbf{H}$ is much larger than $1/\gamma$). In solution of Eq. (3.24) a circumstance new in comparison with (3.25) is the appearance of the undamped component ζ_1, so that the asymptotic polarization vector ($t \gg T$) lies in the (1, 3) plane and forms an angle $\sim \omega/\Omega = (1/\gamma)(H_\parallel/H_\perp)$ with the e_3 axis. The helical motion can be obtained from circular motion by means of a Lorentz transformation along the magnetic field. Since $\boldsymbol{\zeta}^2 = -s^2$ is the square of a 4-vector, the asymptotic degree of polarization for helical motion is the same as for circular motion. This same result follows, naturally, from the system of equations (3.24) if we take into account that in the coefficients for $1/T$ we have retained only the leading terms of the expansion in $1/\gamma$, so that in the solution we should also retain only the leading terms in the expansion in $1/\gamma$.

In discussion of the motion of an electron in a nonuniform field we can, as a rule, neglect the first term in ω (B.17) in comparison with the second term, which contains the gradients of the field. The degree of polarization arising, generally speaking, changes (in comparison with a uniform field) and can be found if $|\omega/\Omega| \gg 1/\gamma$ (in the opposite case the coefficients in Eqs. (3.24) have insufficient accuracy; besides, the corrections are then negligibly small, $\sim 1/\gamma^2$).

In the cases of practical interest the electrons execute small oscillations in a nonuniform field around an

*See Sec. 6.

Time of radiative polarization

Energy, MeV	Magnetic radius of curvature, cm	Time of radiative polarization, minutes	Energy, MeV	Magnetic radius of curvature, cm	Time of radiative polarization, minutes
500	150	177	1000	750	693
600	150	73	2000	750	22
700	150	33	3000	750	0.95

equilibrium (circular) orbit. The ratio $|\omega/\Omega| \ll 1$ and has an order $z_0/R\eta\gamma$ (z_0 is the amplitude of the oscillations, R is the mean radius of the orbit). The system of Eqs. (3.24) can be solved by means of perturbations in the parameter ω/Ω. In the first approximation we have

$$\left.\begin{aligned} \zeta_+(t) &= e^{A(t)}\left[\zeta_+^0(0) + i\int_0^t \omega\zeta_3^0(\tau)e^{-A(\tau)}d\tau\right],\\ \zeta_3(t) &= \zeta_3^0(t) - e^{-\int_0^t \frac{d\tau}{T}}\operatorname{Im}\zeta_+^0(0)\int_0^t \omega(\tau)e^{B(\tau)}d\tau, \end{aligned}\right\} \quad (3.29)$$

where $\zeta_+(t) = \zeta_1(t) + i\zeta_2(t)$, $\zeta_-(t) = \zeta_+^*(t)$; ζ_3^0 and ζ_+^0 are the solutions of the zero approximation (see (3.25)):

$$A(t) = \int_0^t \left(-\frac{8}{9T} + i\Omega\right)d\tau, \quad B(t) = A(t) + \int_0^t \frac{d\tau}{T}. \quad (3.30)$$

The difference from the case of motion in a uniform field lies in the fact that undamped terms of small amplitude ($\sim \omega/\Omega$) appear in ζ_1 and ζ_2, and a damped term which is linear in ω/Ω appears in ζ_3. The undamped correction to ζ_3 appears only in the next approximation, and has the form

$$\Delta\zeta_3 = \frac{8}{5\sqrt{3}}e^{-\int_0^t \frac{d\tau}{T}}\operatorname{Re}\left[\int_0^t \omega(\tau)e^{B(\tau)}\int_0^\tau \omega(\tau_1)e^{-A(\tau_1)}d\tau_1\right]. \quad (3.31)$$

Let us now discuss a specific example. Let the electrons execute small (betatron) oscillations along axis 3 in a field which in the plane $x_3 = 0$ has the form $H_3 = H_0(R/r)^n$, $H_\varphi = H_r = 0$. Then in the oscillator approximation we have

$$x_3 = x_3^0\cos(\omega_3 t + \beta_3), \quad \rho = \rho_0\cos(\omega_r t + \beta_r), \quad \dot\varphi = \omega_0(1-\rho), \quad (3.32)$$

where

$$\rho = \frac{r-R}{r}, \quad \omega_r = \sqrt{1-n}\,\omega_0, \quad \omega_3 = \sqrt{n}\,\omega_0, \quad \omega_0 = \frac{eH_0}{\varepsilon}.$$

For this case

$$\Omega = \Omega_0(1-n\rho), \quad \omega = -\frac{n\dot x_3}{R}, \quad \frac{1}{T} = \frac{1}{T_0}(1-3n\rho). \quad (3.33)$$

Substituting (3.32)–(3.33) into (3.29), we obtain for the spin components

$$\left.\begin{aligned} \zeta_1(t) &= c\,\frac{\Omega_0}{\omega_3}\sin(\omega_3 t + \beta_3)\zeta_3^0(t) + \text{damped terms},\\ \zeta_2(t) &= -c\cos(\omega_3 t + \beta_3)\zeta_3^0(t) + \text{damped terms},\\ \zeta_3(t) &= \zeta_3^0(t) + \text{damped terms}, \end{aligned}\right\} \quad (3.34)$$

where

$$\begin{aligned} \zeta_3^0(t) &= -\frac{8}{5\sqrt{3}} + \left(\zeta_3(0) + \frac{8}{5\sqrt{3}}\right)e^{-t/T_0},\\ c &= \frac{nx_3^0}{R}\,\frac{\omega_3^2}{\omega_3^2-\Omega^2}. \end{aligned} \quad (3.35)$$

The undamped correction to ζ_3 (3.31) has the form

$$\Delta\zeta_3 = \frac{2}{5\sqrt{3}}\left(\frac{nx_3^0}{R}\right)\frac{\omega_3^2}{\omega_3^2-\Omega^2}\cos 2(\omega_3 t + \beta_3). \quad (3.36)$$

The expressions obtained determine the polarization process in a nonuniform field. It is evident, in particular, that the asymptotic degree of polarization changes by an amount $\sim (x_3^0/R)^2$ if we are far from the resonance $\Omega_0 \sim \omega_3$ (see Sec. 6).

The first evaluations of the kinetics of radiative polarization were made by means of the elementary balance equations[1] in the case of a uniform field. The discussion given above follows the articles of V. M. Katkov, V. M. Strakhovenko, and the author.[10,11]

4. DYNAMICS OF POLARIZATION. DEPOLARIZATION EFFECTS

In the preceding section it was shown that during extended motion of electrons (positrons) in a magnetic field they are polarized along the direction $\mathbf{v} \times \dot{\mathbf{v}}$. It is natural that questions arise as to the dynamics of the spin motion, the control of the beam polarization, and also its preservation. In the kinetic equation (3.23) there are terms corresponding to rotation of the spin vector and variation of the modulus of the spin vector. The latter terms lead to appearance of radiative polarization. The characteristic frequencies of the spin motion (see Eq. (B.19) of Appendix B) are $\Omega = 1/t_\Omega \approx \alpha\gamma\omega_0$ (ω_0 is the frequency of rotation of the electron in the field $H_\perp$), while the inverse time ("frequency") of the polarization (3.17) is $1/T \sim \alpha\gamma\omega_0\chi^2$. The ratio of these times is

$$\frac{t_\Omega}{T} \sim \chi^2. \quad (4.1)$$

Since $\chi \ll 1$ with a large margin, in study of the spin motion and the depolarization phenomena associated with it, if the times of these phenomena are much less than T, we can omit the terms with damping and discuss the BMT equation (B.6), (B.13). Below we limit ourselves to discussion of this case.

For motion in a uniform field $\mathbf{H}_0$ (for $\mathbf{E} = 0$, $\mathbf{v} \times \mathbf{H}_0 = 0$) we have from Eq. (B.6)

$$\frac{d(\zeta\mathbf{H}_0)}{dt} = 0, \quad (4.2)$$

i.e., the projection of the spin vector on the direction of the field is conserved*.

In storage rings the role of $\mathbf{H}_0$ is played by the guide field averaged over the orbit. With appearance of field nonuniformities (and the field $\mathbf{H}_\parallel$) this projection is already not conserved. Thus, all effects of change of the direction of polarization (spin rotation) are in one

*Since in this situation the polarization is directed along (opposite to) the vector H, this means conservation of the polarization. In the general case of periodic motion (see Appendix C) the projection of the spin vector is conserved in the periodic solution of Eq. (B.6).

way or another associated with nonuniformity of the magnetic field*. The nonuniform part $|\mathbf{H}_1| \ll |\mathbf{H}_0|$; therefore, generally speaking, the effects of the additional rotation are small. However, if $\mathbf{H}_1$ contains harmonics resonating with the average precession frequency, the spin can rotate by a large angle. If the spin rotation occurs incoherently, this corresponds to depolarization of the beam (dynamic depolarization). An additional depolarization mechanism exists which is associated with the appearance of stochastic elements in the motion (for example, quantum fluctuations of the radiation lead to a stochastic spread in the beam energy). As a result of this, a spread in the average spin value can arise which also leads to depolarization (stochastic depolarization). Below we will discuss in detail the dynamics of motion of the spin ζ, and also the two depolarization mechanisms.

4.1. Dynamics of Spin Motion. Dynamic Depolarization[13,31] †

We will consider the spin motion under conditions of small oscillations of the particles with respect to the equilibrium motion ($|\mathbf{H}_1| \ll |\mathbf{H}_0|$). The problem of dynamic depolarization as a whole reduces to finding those conditions for which the solution of Eq. (B.6) gives an appreciable deviation of the vector ζ from the initial direction (the polarization axis), and also to analysis of the mixing of different deflections of the vector ζ for individual particles and the mixing of the beam depolarization associated with this. The mixing process, as a rule, occurs as a consequence of the spread in the parameters (energy, momentum, coordinates, and so forth) of the particles in the beam. Equations (B.6) are in many ways analogous to the equations for motion of charged particles in an external field and can be solved by the same methods. Here it turns out to be convenient to use the equations for the spin motion in the form (B.13).

a) The nonresonance case. It is shown in Appendix B that under actual conditions $\Omega \gg \omega$. Therefore solutions can be sought by means of perturbations in powers of ω/Ω. We will represent the solution of Eqs. (B.13) in the form

$$\left.\begin{aligned}\zeta_1 &= \sqrt{1-\zeta^2}\cos\Phi,\\ \zeta_2 &= \sqrt{1-\zeta^2}\sin\Phi,\\ \zeta_3 &= \zeta,\end{aligned}\right\} \tag{4.3}$$

where

$$\Phi = \int \Omega\, dt + \varphi. \tag{4.4}$$

The functions ζ and Φ introduced satisfy the equations

$$\dot{\zeta} = -\omega\sqrt{1-\zeta^2}\sin\Phi, \quad \dot{\varphi} = \frac{\zeta\omega}{\sqrt{1-\zeta^2}}\cos\Phi. \tag{4.5}$$

The last equations contain ω in the right-hand side and therefore are suitable for solution by successive approximations. In the zero approximation $\omega = 0$, $\zeta = \zeta_0 = \text{const}$, $\varphi = \varphi_0 = \text{const}$. In this case it follows from (4.3) that the projection of the spin vector on axis 3 is conserved, and the projection perpendicular to axis 3 rotates with an average frequency $\bar{\Omega}\left(\bar{\Omega}t = \int_0^t \Omega\, dt\right)$, i.e., a precession of the spin vector ζ occurs around axis 3 with this average frequency. In the first approximation it follows from (4.5) that

$$(\Delta\zeta)_1 = -\sqrt{1-\zeta_0^2}\,\psi_s, \quad (\Delta\varphi)_1 = \frac{\zeta_0}{\sqrt{1-\zeta_0^2}}\psi_s, \tag{4.6}$$

where

$$\psi_s = \int^t \omega\sin\Phi_0\, dt, \quad \psi_c = \int^t \omega\cos\Phi_0\, dt, \quad \Phi_0 = \int \Omega\, dt + \varphi_0, \tag{4.7}$$

so that the complete solution in the first approximation is

$$\left.\begin{aligned}\zeta_1^1 &= \sqrt{1-\zeta_0^2}\cos\Phi_0 + \zeta_0\alpha_1,\\ \zeta_2^1 &= \sqrt{1-\zeta_0^2}\sin\Phi_0 + \zeta_0\alpha_2,\\ \zeta_3^1 &= \zeta_0 - \psi_s\sqrt{1-\zeta_0^2},\end{aligned}\right\} \tag{4.8}$$

where

$$\alpha_1 = \psi_s\cos\Phi_0 - \psi_c\sin\Phi_0, \quad \alpha_2 = \psi_s\sin\Phi_0 + \psi_c\cos\Phi_0. \tag{4.9}$$

From the solution (4.8) it follows that small oscillations are superimposed on the spin precession (Fig. 1). Formally the solution of the first approximation can be obtained from the solutions of the zero approximation by means of the transformation matrix S:

$$\zeta^1 = S\zeta^0, \quad S = \begin{pmatrix} 1 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \\ -\alpha_1 & -\alpha_2 & 1 \end{pmatrix}. \tag{4.10}$$

With an accuracy to second order this matrix is orthogonal ($S^T = S^{-1}$, det S = 1), i.e., corresponds to real rotations and represents a succession of rotations by a small angle α_1 in the (1, 3) plane and by a small angle α_2 in the (2, 3) plane, so that in transformation of the coordinates by means of the matrix S the 3 axis rotates by an angle $\sqrt{\alpha_1^2 + \alpha_2^2}$.

The result obtained is applicable if the correction terms found by perturbation theory are small. For ω = const the order of magnitude of these terms is obviously $\omega/\bar{\Omega} \ll 1$. In the general case it is necessary to expand $\omega(t)$ in Fourier series. If among the harmonics of the expansion there are harmonics which are multiples of $\bar{\Omega}\left(\bar{\Omega}t = \int_0^t \Omega\, dt\right)$, small denominators will appear in the quantities (4.7), i.e., these quantities can become large. The harmonics of $\omega(t)$ are multiples of all characteristic frequencies of the problem, i.e., if

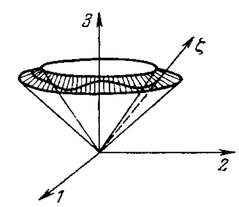

FIG. 1

*In the general case, particle oscillations occur relative to the equilibrium (periodic) trajectory and, as is shown in Appendix C, the corresponding motion of the spin vector is along the periodic solution n of equation (B.6), the role of the vector $\mathbf{H}_0/|\mathbf{H}_0|$ being played by n. For definiteness the further discussion will be carried out in terms of $\mathbf{H}_0$ and $\mathbf{H}_1$.

†This group of questions has also been discussed by a number of authors. [12,34-38]

$$\bar{\Omega} = N\omega_0 + N_1\omega_z + N_2\omega_x + N_3\omega_s \quad (4.11)$$

(where ω_0, ω_z, ω_x, and ω_s are the angular frequencies of z and r betatron and synchrotron oscillations), the solution obtained above is inapplicable. This condition is the condition of resonance.

b) Spin motion near resonance. Let the average frequency $\bar{\Omega}$ be close to some resonance harmonic Ω_n (4.11):

$$\bar{\Omega} = \Omega_n + \delta. \quad (4.12)$$

In solution of the problem of motion of the spin vector $\boldsymbol{\zeta}$, it is frequently convenient to find the system of coordinates in which the spin projection on some direction is conserved, and then to discuss the motion of this system relative to the initial system. In the nonresonance case this transition is accomplished by the matrix S^T. However, this approach is particularly useful in the resonance region. Let us go to a system rotating with a frequency Ω_n relative to our initial system:

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} \cos\Phi_n & -\sin\Phi_n & 0 \\ \sin\Phi_n & \cos\Phi_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_x \\ \zeta_y \\ \zeta_z \end{pmatrix}, \quad (4.13)$$

where $\Phi_n = \psi_n + \varphi_n$, $\psi_n = \Omega_n t$. The phase φ_n is constant and will be chosen below. If ζ_1, ζ_2, and ζ_3 satisfy the system of equations (B.13), then ζ_x, ζ_y, and ζ_z satisfy the equations (in what follows we take into account one harmonic)

$$\left.\begin{aligned} \dot{\zeta}_x &= -\delta\zeta_y + \omega\sin\Phi_n\zeta_z, \\ \dot{\zeta}_y &= \delta\zeta_x + \omega\cos\Phi_n\zeta_z, \\ \dot{\zeta}_z &= -\omega\sin\Phi_n\zeta_x - \omega\cos\Phi_n\zeta_y. \end{aligned}\right\} \quad (4.14)$$

In the right-hand part of this system there are only the low frequencies δ and ω (the variables ζ_x, ζ_y, and ζ_z are slow). In this situation it is appropriate to use Bogolyubov's method of averaging,[14] which consists of averaging the coefficients of the small parameters (in our case the frequencies) over time* (i.e., the zero terms in the Fourier series expansions are retained). Carrying out this averaging, we have

$$\left.\begin{aligned} \overline{\omega\sin\Phi_n} &= \omega_c\sin\varphi_n + \omega_s\cos\varphi_n, \\ \overline{\omega\cos\Phi_n} &= \omega_c\cos\varphi_n - \omega_s\sin\varphi_n, \end{aligned}\right\} \quad (4.15)$$

where

$$\omega_c = \overline{\omega\cos\psi_n}, \quad \omega_s = \overline{\omega\sin\psi_n}. \quad (4.16)$$

Choosing the phase φ_n so that

$$\sin\varphi_n = \frac{\omega_s}{\omega_n}, \quad \cos\varphi_n = -\frac{\omega_c}{\omega_n}, \quad \omega_n = \sqrt{\omega_c^2 + \omega_s^2}, \quad (4.17)$$

we obtain

$$\overline{\omega\sin\Phi_n} = 0, \quad \overline{\omega\cos\Phi_n} = -\omega_n. \quad (4.18)$$

After averaging (4.18), Eq. (4.14) acquires the form

$$\left.\begin{aligned} \dot{\zeta}_x &= -\delta\zeta_y, \\ \dot{\zeta}_y &= \delta\zeta_x - \omega_n\zeta_z, \quad \dot{\boldsymbol{\zeta}} = [\boldsymbol{\zeta}\mathbf{u}], \\ \dot{\zeta}_z &= \omega_n\zeta_y. \end{aligned}\right\} \quad (4.19)$$

These equations formally coincide with the equations of

*The discarded oscillator terms are taken into account in the higher approximations of the averaging method, which we will not discuss here.

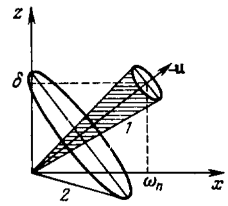

FIG. 2

motion in a magnetic field whose vector lies in the (x, z) plane: $\mathbf{u} = \mathbf{u}(-\omega_n, 0, -\delta)$. To obtain a solution for a constant frequency difference δ, we can introduce a system of coordinates whose z axis coincides with $\mathbf{u}$, in which the solution has the obvious form

$$\left.\begin{aligned} z_1 &= \sqrt{1-z^2}\cos(\omega_p t + \varphi_s), \\ z_2 &= \sqrt{1-z^2}\sin(\omega_p t + \varphi_s), \\ z_3 &= z = \text{const}, \end{aligned}\right\} \quad (4.20)$$

where $\omega_p = |\mathbf{u}| = \sqrt{\omega_n^2 + \delta^2}$, and the phase φ_s is given by the initial conditions. For transformation to the x, y, z coordinate system it is necessary to carry out rotations

$$\begin{pmatrix} \zeta_x \\ \zeta_y \\ \zeta_z \end{pmatrix} = \begin{pmatrix} \delta/\omega_p & 0 & \omega_n/\omega_p \\ 0 & 1 & 0 \\ -\omega_n/\omega_p & 0 & \delta/\omega_p \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad (4.21)$$

and substitute the results obtained into (4.13).

The results obtained have a simple intuitive meaning. The spin vector slowly precesses around the axis $\mathbf{u}$, whose direction is determined by the interaction between ω_n and δ. If $\delta \gg \omega_n$, the vector $\mathbf{u}$ is directed almost along the z axis, so that we reach a nonresonant case: the vector $\mathbf{u}$ precesses around the z axis, and the spin vector slowly precesses around $\mathbf{u}$, all precession angles being small. However, the situation changes for $\delta \lesssim \omega_n$, when the vector $\mathbf{u}$ is directed at a large angle to the z axis. Then the precession angles of the vector $\boldsymbol{\zeta}$ around $\mathbf{u}$ can be large (see Fig. 2, where 2 corresponds to a polarization initially directed along the z axis (3) and the vector $\mathbf{u}$ in turn precesses rapidly around the z axis. Since δ is the magnitude of the frequency difference (the distance from resonance), it is natural to assume as an effective width of the resonance $\delta = \omega_n$. If the spin vector falls inside the resonance, it is strongly deflected from its initial position. In view of the fact that different particles are deflected by different amounts (and can have different phases), depolarization of the beam can occur.*

c) Rapid traversal of a resonance. A real situation is the crossing of a spin resonance by a particle, i.e., when the precession frequency Ω varies, at a certain moment $\Omega(t) = \Omega_n$ ($\delta = 0$) (see (4.11) and (4.12)). This situation is realized in acceleration of polarized particles, and also in oscillations of the energy in a storage ring.

We will first discuss the case of a rapid traversal, in which the frequency difference δ changes rather

*We note, however, that the dynamic mixing (dynamic depolarization), both in motion near a resonance and in the cases discussed below of traversal of a resonance, as a rule does not give $\zeta_3 = 0$. For complete disappearance of the polarization, the stochastic mechanism must play in the problem (see Sections 4.1e and 4.2).

rapidly and the terms containing ω in Eq. (4.14) can be considered as a perturbation and we can apply to them the Bogolyubov method of averaging.[14] Then in the zero approximation

$$\left.\begin{aligned}\zeta_1 &= \zeta_\perp^0 \cos\left(\int \delta\, dt + \varphi_f\right),\\ \zeta_2 &= \zeta_\perp^0 \sin\left(\int \delta\, dt + \varphi_f\right),\\ \zeta_3 &= \zeta_0 = \text{const}, \quad \zeta_\perp^0 = \sqrt{1-\zeta_0^2},\end{aligned}\right\} \tag{4.22}$$

so that the spin projection on the 3 axis does not change, $\zeta_3^{+\infty} = \zeta_3^{-\infty}$. Applying this method of averaging (see Eq. (4.18)) to Eq. (4.14), we obtain in the first approximation of perturbation theory the correction to ζ_3:

$$\delta\zeta_3(\infty) = \zeta_\perp^0 \omega_n \int_{-\infty}^{+\infty} dt \sin\left[\int_0^t \delta(t)\, dt + \varphi_n\right], \tag{4.23}$$

where

$$\varphi_n = \int_{-\infty}^{0} \delta\, dt + \varphi_f. \tag{4.24}$$

The expression (4.23) obtained is universal for rapid traversal through a resonance. In order to obtain specific results, it is necessary to provide a definite model $\delta(t)$. For the simplest case $\delta = \Gamma_0 t$, $\Gamma_0 = \text{const}$, we have

$$\zeta_3^{+\infty} = \zeta_3^{-\infty} + \delta\zeta_3(\infty), \tag{4.25}$$

where

$$\begin{aligned}\delta\zeta_3(\infty) &= \zeta_\perp^0 \omega_n \int_{-\infty}^{+\infty} \sin\left(\frac{\Gamma_0 t^2}{2} + \varphi_n\right) dt\\ &= \zeta_\perp^0 \sqrt{\frac{2\pi\omega_n^2}{\Gamma_0}} \sin(\varphi_n + \pi/4).\end{aligned} \tag{4.26}$$

It is evident that the main contribution to the integral is from the region $t \sim 1/\sqrt{\Gamma_0}$, while the time of spin rotation (in the "slow" variables (4.14)) is $\sim 1/\omega_n$. The condition of applicability of the approximation used is:

$$1/\sqrt{\Gamma_0} \ll 1/\omega_n. \tag{4.27}$$

This also is the condition for rapid passage through the resonance, and in this situation the spin is not able to rotate during the time of passage. If the passage involves a change in energy, then $\Gamma_0 = \eta\omega_0(d\gamma/dt)$. The strong dependence of the result (4.23) and (4.25) on phase has a clear physical meaning—a perturbation of $\sim\omega$ is superimposed on the main motion (4.22) and shifts ζ_3 as a function of the phase of the main motion (and the encountered phase). We will again find an addition of the first approximation to the transverse components of (4.22). Carrying out this calculation accurately as in (4.22)–(4.26), we obtain

$$|\delta\zeta_+(\infty)| = \zeta_0 \sqrt{\frac{2\pi\omega_n^2}{\Gamma_0}}. \tag{4.28}$$

d) Slow passage through resonance. Now we will discuss passage through resonance when the inequality inverse to (4.27) is satisfied*:

$$\sqrt{\Gamma_0} \ll \omega_n. \tag{4.29}$$

In this case, during the time of crossing, the spin completes many revolutions around the u axis, i.e., we can assume that the spin precesses at each moment of time about the instantaneous axis **u**, but the precession axis itself rotates by an angle of order π. At any given moment in the zero approximation we can use the expressions (4.20) and (4.21) obtained above, which were obtained on the assumption that $\delta = \text{const}$. In rotation of the u axis by an angle π, the spin adiabatically follows the u axis, i.e.,

$$\zeta_z^{+\infty} = -\zeta_z^{-\infty}. \tag{4.30}$$

Thus, for slow passage through resonance, the spin flips. We can find nonadiabatic corrections by means of perturbation theory applied to Eqs. (4.19) with the zero approximation solutions (4.20) and (4.21). If we take into account that the increase in the precession angle $\cos\beta = \delta/\omega_p$ is

$$\frac{d}{dt}\cos\beta = \frac{\Gamma_0\omega_n}{\omega_n^2 + \Gamma_0^2 t^2}\sin\beta, \tag{4.31}$$

then from (4.19)–(4.21) we have in the first approximation of perturbation theory

$$\delta\zeta_z = -\zeta_\perp^0 \int_{-\infty}^{+\infty} \frac{\Gamma_0\omega_n}{\omega_n^2 + \Gamma_0^2 t^2} \cos\left[\int_0^t \sqrt{\omega_n^2 + \Gamma_0^2 t^2}\, dt + \varphi_r\right], \tag{4.32}$$

where

$$\varphi_r = \int_{-\infty}^{0} \omega_p\, dt + \varphi_s. \tag{4.33}$$

The main contribution to the integral in the argument of the cosine in Eq. (4.32) is from the interval of t where this argument is ~ 1, i.e., $t \sim 1/\omega_n \ll 1/\sqrt{\Gamma_0}$. However, then $\Gamma_0^2 t^2 \ll \Gamma_0 \ll \omega_n^2$ and the argument of the cosine is $\omega_n t$. Calculating the integral (4.32) we obtain (Fig. 3)

$$\zeta_z^{+\infty} = -\zeta_z^{-\infty} - \zeta_\perp^0 \pi e^{-\frac{\omega_n^2}{\Gamma_0}} \cos\varphi_r, \tag{4.34}$$

where $\omega_n^2/\Gamma_0 \gg 1$. The exponentially small value of the nonadiabatic corrections is due to the symmetric passage (the velocities of traversal Γ_0 before and after the crossing are the same). In the opposite case we obtain a power-law smallness.

e) An example of intentional depolarization of the beam. In the experimental study of the degree of polarization of a beam, and also in the performance of experiments with polarized particles, it is desirable to be able to depolarize the beam. From the discussion presented above it follows that there are several means of dynamic polarization (with participation of the stochastic mechanism (see the footnote to the preceding page):

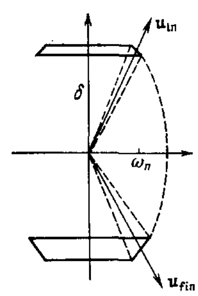

FIG. 3

*The general case of passage through a resonance (including the case $\sqrt{\Gamma_0} \sim \omega_n$) has been discussed by Derbenev et al. [13]

1) setting the average frequency of spin motion $\overline{\Omega}$ near a resonance (4.11);

2) repeated rapid passage through resonance, (4.25) and (4.26), with mixing in phase;

3) slow passage through resonance with mixing in phase (4.34) and in the time of traversal.

It is also possible to shift artificially to a resonance by introduction of an external electromagnetic field or to introduce an external depolarizing electromagnetic field. We will discuss an example of the latter.

Let us introduce an external field $H_{\parallel}$ in a length l (the length of the orbit is 2L):

$$H_{\parallel} = H_{\parallel}^{0} \sin \omega_h t, \tag{4.35}$$

where

$$\omega_h = \omega_h^0 + \delta\omega \sin \Omega_m t. \tag{4.36}$$

The frequency ω_h^0 can be adjusted to the spin frequency (compare (4.11)):

$$\omega_h^0 = \overline{\Omega} - n\omega_0, \tag{4.37}$$

where $\delta\omega \sin\Omega_m t$ is the detuning, which changes sign with the frequency Ω_m. Modulation of the frequency assures multiple traversal of the resonance.

The frequency ω (B.17) is

$$\omega = (g/2)\,\omega_0 \frac{H_{\parallel}^0}{H_z} \sin \omega_h t. \tag{4.38}$$

The frequency ω_n is calculated from Eqs. (4.16) and (4.17):

$$\omega_n = \frac{g}{2}\,\omega_0 \frac{H_{\parallel}^0}{H_z} \frac{\sin\left(n\pi \frac{l}{L}\right)}{2\pi n} \approx \frac{g}{2}\,\omega_0 \frac{H_{\parallel}^0}{H_z} \frac{l}{2L}. \tag{4.39}$$

For rapid traversal it is necessary that, in agreement with (4.27), we satisfy the inequality

$$\dot{\delta}(t = 0) = \delta\omega\Omega_m \gg \omega_n^2. \tag{4.40}$$

For a single traversal, if the initial phases of the spin are not correlated, we have (see (4.26))

$$\overline{\delta\zeta_z^2} = \zeta_\perp^2 \frac{\pi\omega_n^2}{\Gamma_0}. \tag{4.41}$$

Taking into account that the time of a single passage through the resonance is $T_r = \pi/\Omega_m$ and that the deviations add quadratically (which means that the spin averaged over the ensemble $\langle \boldsymbol{\zeta} \rangle$ is scattered), we obtain from (4.41) for the depolarization time

$$\tau_{\text{dep}} \approx T_r \frac{\Gamma_0}{\pi\omega_n^2} = \frac{\dot{\delta}(0)}{\Omega_m \omega_n^2} = \frac{1}{\omega_0} \frac{\delta\omega}{\omega_0} \frac{1}{\left(\frac{H_{\parallel}^0}{H_z} \frac{l}{2L}\right)^2}; \tag{4.42}$$

if we choose $\delta\omega/\omega \sim 10^{-4}$, $H_{\parallel}^0/H_z \sim 10^{-3}$, and $l/24 \sim 10^{-3}$, then the depolarization time is several seconds under typical electron-storage-ring conditions.

4.2. Stochastic Depolarization[15]

In addition to the depolarization mechanisms discussed above, there is a specific mechanism associated with the stochastic nature of the radiation process. This mechanism also acts only in nonuniform fields, but fulfillment of the resonance conditions (4.11) is now not essential. The effect is due to the fact that the jumps in energy, angle, and so forth, associated with the quantum nature of the radiation process (or any other stochastic process, for example, scattering in a gas), when expanded in a Fourier integral, contain, in particular, harmonics which provide resonance (4.11). Therefore, the basis of the effect is a resonance with the unpleasant feature that, generally speaking, it cannot be avoided by selection of parameters (energy ϵ, field $\mathbf{H}$, and so forth).

We will consider the stochastic depolarization process far from the dynamic resonances (4.11). Then the spin motion is described to first order in ω/Ω by Eqs. (4.8). Construct

$$\left.\begin{aligned} \zeta_\perp^2 &= \zeta_1^2 + \zeta_2^2 = \zeta_\perp^{02} + 2\zeta_\perp^0 \zeta_3^0 \psi_s, \\ \zeta_3^2 &= \zeta_3^0 - 2\zeta_\perp^0 \zeta_3^0 \psi_s. \end{aligned}\right\} \tag{4.43}$$

The quantities ζ_3^0 and $\zeta_\perp^0$ are constant for constant energy. The quantity ζ_3 and $\zeta_\perp$ are instantaneous projections of the spin on the 3 axis and on the plane perpendicular to it; ζ_3^0 and $\zeta_\perp^0$ are averages about which oscillations ζ_3 and $\zeta_\perp$ occur (see Sec. 4.1).

In radiation of a photon, the electron energy undergoes a jump $\Delta\epsilon$, while the relative probability of spin flip in a radiative transition is of order χ^2 (see Sec. 2) and is negligibly small in comparison with the probability of a radiative transition without spin flip. Therefore the values of ζ_3 and $\zeta_\perp$ do not change at the moment of radiation. However, at the moment of radiation a discontinuous change occurs in the value ψ_s in Eq. (4.43), which is proportional to the energy jump $\Delta\epsilon$. Since ζ_3 and $\zeta_\perp$ do not change, this means a discontinuous change in the averages ζ_3^0 and $\zeta_\perp^0$. Compensation of the loss by radiation returns ψ_s (more accurately, the amplitudes in (4.43)) to the initial values. However, a set of such jumps leads to a stochastic buildup of ζ_3^0 and $\zeta_\perp^0$, and therefore of ζ_3 and $\zeta_\perp$ (spin diffusion).

The same reasoning can be applied to angular jumps during radiation, and also to any other stochastic mechanism (for example, multiple scattering in a gas) in which the relative probability of a transition with spin flip is small in the individual events.

The reasoning presented above permits us to obtain general formulas characterizing stochastic (quantum) depolarization. Let a parameter ξ (energy, angle, and so forth) undergo a discontinuous change. Then

$$\Delta\psi_s = \frac{d\psi_s}{d\xi}\Delta\xi = \left(\frac{d}{d\xi}\int \omega \sin\Phi_0\, dt\right)\Delta\xi; \tag{4.44}$$

here the values of ζ_3^0 and $\zeta_\perp^0$, i.e., the angle of inclination of the spin vector ($\tan\theta = \zeta_\perp^0/\zeta_3^0$), undergo jumps, and from (4.43) it follows that

$$\Delta\psi_s = \Delta\theta. \tag{4.45}$$

Then the time of depolarization (the time of spin drift by an angle of order unity) is

$$1/\tau_{\text{dep}} \approx \frac{d(\Delta\theta)^2}{dt} = \left(\frac{\overline{d\psi_s}}{d\xi}\right)^2 \frac{(\Delta\xi)^2}{dt} = \left(\frac{\overline{d\psi_s}}{d\xi}\right)^2 \frac{1}{2}\int (\Delta\xi)^2\, dW(\Delta\xi), \tag{4.46}$$

where $dW(\Delta\xi)$ is the probability per unit time of a transition with a given parameter jump $\Delta\xi$.

As an example* we will consider stochastic depolarization as the result of energy jumps $\Delta\epsilon$ in magnetic bremsstrahlung during forced oscillations along the 3

*A number of examples of stochastic depolarization (spin diffusion) for the specific conditions of storage rings have recently been discussed by Derbenev and Kondratenko. [39]

axis for motion of the electron in an axially symmetric magnetic field*. In this case ψ_S (4.7) has the form

$$\psi_{sk} = -\frac{x_{3k}^0}{2R} k^2 \eta\gamma \frac{\sin[\omega_0(k-\eta\gamma)t]}{k-\eta\gamma} + \text{nonresonant terms}, \quad (4.47)$$

where we have taken the perturbation term with the k-th harmonic, and x_{3k}^0 is the oscillation amplitude; in Eq. (4.47) we have explicitly written out only the resonance term providing the main contribution. Substituting (4.47) into (4.44) and (4.46), we obtain

$$1/\tau_{\text{dep}} = \frac{1}{8}\sum_k \left(\frac{x_{3k}^0}{R}\right)^2 \left[\frac{k\eta\gamma}{k-\eta\gamma}\right]^4 \frac{1}{\varepsilon^2}\frac{d\overline{(\Delta\varepsilon)^2}}{dt} = \frac{55}{192\sqrt{3}}\frac{\hbar r_0}{mR^3}\gamma^5 \sum_k \left[\frac{k\eta\gamma}{k-\eta\gamma}\right]^4 \left(\frac{x_{3k}^0}{R}\right)^2, \quad (4.48)$$

where $r_0 = e^2/4\pi\hbar m = 2.8 \times 10^{-13}$ cm is the classical electron radius. The mean-square energy fluctuation is calculated as follows:

$$\frac{1}{\varepsilon^2}\frac{d(\Delta\varepsilon)^2}{dt} = \frac{1}{2\varepsilon^2}\int (\hbar\omega)^2\, dW(\vartheta, \omega) = \frac{55}{48\sqrt{3}}\frac{r_0\hbar}{mR^3}\gamma^5, \quad (4.49)$$

where $dW(\vartheta, \omega)$ is the probability of radiation of a photon per unit time.[1,2]

It is evident from (4.48) that the stochastic depolarization effect depends in the strongest way on energy, and depends strongly on the number of the nearest resonance harmonic, on the distance to the resonance $(k - \eta\gamma)$, and on the amplitude x_{3k}^0. We will estimate the effect for definite parameter values: energy ϵ = 6 BeV, $R = 3 \times 10^3$ cm, $k - \eta\gamma = 1/2$, k = 14, 15 and $x_{3k}^0 = 0.1$ cm. Then τ_{dep} = 25 sec (under these same conditions the polarization time is T = 190 sec).

Thus, in the situation considered it is necessary to take special measures to preserve the polarization (to suppress resonant forced oscillations).

Let us make another estimate of the depolarization in motion near a spin resonance, where the role of the stochastic mechanism is played by radiative damping of the oscillations. Substituting in Eq. (4.7) for ψ_S the harmonic ω closest to the spin resonance (frequency difference δ (4.12), frequency ω_n (4.17)), we obtain from (4.46)

$$1/\tau_{\text{dep}} \approx \overline{(\Delta\psi_s)^2}/\Delta t \simeq (\omega_n/\delta)^2 \tau_0^{-1}; \quad (4.50)$$

τ_0 is the radiative damping time. This formula can be used to evaluate the width of the resonance.

In order to make simple estimates of the influence of depolarizing effects of the stochastic type on the degree of radiative polarization, we will introduce a depolarizing term (of the diffusion type) into the equation for ζ_3 (3.24) for $\omega = 0$:

$$\dot{\zeta}_3^r = -\frac{1}{T}(\zeta_3^r + 8/5\sqrt{3}) - \zeta_3^r/\tau_{\text{dep}}. \quad (4.51)$$

The solution of this equation has the form

$$\zeta_3^r(t) = [\zeta_3^r(0) - \zeta_3^r(\infty)] \exp\left\{-\left(\frac{1}{T} + \frac{1}{\tau_{\text{dep}}}\right)t\right\} + \zeta_3^r(\infty), \quad (4.52)$$

where

$$\zeta_3^r(\infty) = -\frac{8}{5\sqrt{3}}\frac{1}{1+(T/\tau_{\text{dep}})}.$$

It is obvious that in the presence of depolarizing effects the degree of radiative polarization decreases.

4.3. Other Depolarizing Effects

In addition to the effects discussed above, it is necessary to take into account depolarization in scattering by residual gas atoms, bremsstrahlung on residual gas atoms, scattering of particles inside the beam, and so forth. However, elementary estimates show that all these effects are negligibly small. This is due to the fact that the spin flip probability, for example, in scattering or bremsstrahlung at a small angle is suppressed by a factor γ^2.

As an illustration we will give the time of depolarization due to internal scattering of electrons inside the beam[19]:

$$1/\tau_{\text{dep}} = \frac{c\pi r_0^2}{V\gamma^2} N_0 \ln\frac{1}{\theta_0}; \quad (4.53)$$

V is the volume of the beam, θ_0 is the minimum scattering angle, and N_0 is the number of particles in the beam.

For the VÉPP-2 installation, $\tau_{\text{dep}} \sim 10^8$ sec (see the table).

5. MEASUREMENT OF ELECTRON POLARIZATION

The polarization arising as the result of extended motion in a magnetic field must be measured and controlled experimentally. Below we will discuss methods of measuring the transverse polarization of high-energy electrons and positrons moving in a storage ring, which, as will be shown below, have a number of specific features.

5.1. Measurement of Polarization in Experiments on Interaction of High-energy Particles

The cross sections for two-particle reactions are extremely sensitive to electron and positron polarizations. We will give below the cross sections for these reactions for transversely (and antiparallel) polarized electrons and positrons in the center-of-mass system.

The cross section for production of a pair of pseudoscalar particles ($\pi^+\pi^-$, K^+K^-, $K_L^0K_S^0$) in annihilation of transversely (and antiparallel) polarized electrons and positrons has the form[16]*

$$d\sigma_{2p} = d\sigma_{2p}^0[1 + |\zeta_1||\zeta_2|(2\sin^2\varphi - 1)], \quad (5.1)$$

where $|\zeta_1|$ and $|\zeta_2|$ are the degrees of polarization of the positrons and electrons, φ is the angle between the plane of production (the plane passing through the momenta of the initial particle p and the final particle q) and the plane perpendicular to the spin direction (the plane of the orbit), $d\sigma_{2p}^0$ is the cross section for unpolarized particles:

$$d\sigma_{2p}^0 = \frac{r_0^2}{32\gamma^2}\frac{q^s}{\varepsilon^3}\sin^2\vartheta\,|F(4\varepsilon^2)|^2\, d\Omega, \quad (5.2)$$

where $q = \sqrt{\epsilon^2 - m_p^2}$, F(s) is the form factor, $r_0 = \alpha/m$ is

*The spin motion is extremely sensitive also to jumps in the vertical angle, especially with coupling of the z and r oscillations, when the jumps are proportional to the quantum fluctuations of energy.

*In all cross sections we will discard terms of order $1/\gamma^2$.

the classical electron radius ($r_0 = 2.82 \times 10^{-13}$ cm), ϑ is the angle between p and q. If the initial particles are completely polarized, $|\zeta_1| = |\zeta_2| = 1$, then $d\sigma_{2p}$ ($\varphi = 0$) = 0 (the production plane coincides with the orbit plane) and $d\sigma_{2p}$ ($\varphi = \pi/2$) = $2d\sigma^0_{2p}$ (the production plane is perpendicular to the orbit plane, so that the spin vector lies in the production plane). The exclusion of $\varphi = 0$ is the consequence of conservation of parity in annihilation of transversely polarized particles. From considerations related to parity conservation, conservation of helicity in electromagnetic interactions at high energies (with accuracy to order $1/\gamma$), and the one-photon nature of the channel in annihilation of transversely polarized electrons and positrons, a number of other exclusions follow.[17,18] In particular, three pseudoscalar mesons (3π, $K\bar{K}\pi^0$) cannot be produced in the plane in which the spin vector lies. In a real situation the polarization is partial: for a time $t \gg T$ (see Eq. (3.17)) $d\sigma_{2p}$ ($\varphi = 0$) = 0.14 $d\sigma^0_{2p}$, $d\sigma_{2p}$ ($\varphi = \pi/2$) = 1.86 $d\sigma^0_{2p}$ (compare Eq. (3.26)).

The cross section for production of a pseudoscalar meson and photon ($\pi^0(\eta) + \gamma$) has the form[18]

$$d\sigma_{p\gamma} = d\sigma^0_{p\gamma}\left[1+|\zeta_1||\zeta_2|(1-2\sin^2\varphi)\frac{\sin^2\vartheta}{1+\cos^2\vartheta}\right]; \qquad (5.3)$$

here $d\sigma^0_{p\gamma}$ is the cross section for unpolarized particles:

$$d\sigma^0_{p\gamma} = \frac{\alpha}{2\tau m_0^3}\left(\frac{q}{\varepsilon}\right)^3(1+\cos^2\vartheta)\left|\frac{G(4\varepsilon^2)}{G(0)}\right|^2 d\Omega, \qquad (5.4)$$

where τ is the lifetime for decay of the meson into two photons, m_0 is the meson mass, $q/\epsilon = 1 - m_0^2/4\epsilon^2$, and ϑ is the emission angle of the final particles. For completely polarized particles $d\sigma_{p\gamma}$ ($\vartheta = \pi/2$, $\varphi = \pi/2$) = 0 (final particle momentum directed along the spin direction), $d\sigma_{p\gamma}$ ($\vartheta = \pi/2$, $\varphi = 0$) = $2d\sigma^0_{p\gamma}$ (final particle momentum perpendicular to spin direction).

The cross section for production of a pair of fermions with spin 1/2 in annihilation of transversely (antiparallel) polarized electrons and positrons[16] has the form

$$d\sigma_{2f} = \frac{r_0^2}{16\gamma^2}\frac{q}{\varepsilon}\Big\{2|F_1+\mu F_2|^2 \qquad (5.5)$$
$$-\left(\frac{q}{\varepsilon}\right)^2\sin^2\vartheta\left[|F_1|^2-\frac{\varepsilon^2\mu^2}{M^2}|F_2|^2\right][1+|\zeta_1||\zeta_2|(2\sin^2\varphi-1)]\Big\}\,d\Omega,$$

where F_1 and F_2 are the electromagnetic form factors, and μ is the anomalous magnetic moment. For production of a pair of muons ($F_1 = 1$, $F_2 = 0$) we have

$$d\sigma_{2\mu} = \frac{r_0^2}{16\gamma^2}\frac{q}{\varepsilon}\left\{2-\frac{q^2}{\varepsilon^2}\sin^2\vartheta\,[1+|\zeta_1||\zeta_2|(2\sin^2\varphi-1)]\right\}d\Omega. \qquad (5.6)$$

For relativistic muons, $q/\epsilon \approx 1$, and we have for completely polarized particles $d\sigma_{2\mu}$ ($\vartheta = \pi/2$, $\varphi = \pi/2$) = 0 (muon momentum directed along the spin) and $d\sigma_{2\mu}$ ($\vartheta = \pi/2$, $\varphi = 0$) = $2d\sigma^0_{2\mu}$ (muon momentum perpendicular to the spin). In production of a pair of baryons the polarization effects are distorted by the form factors.

The cross section for two-quantum annihilation has the form

$$d\sigma_{2\gamma} = \frac{r_0^2}{4\gamma^2(1-v^2\cos^2\vartheta)}\{1+\cos^2\vartheta+|\zeta_1||\zeta_2|\sin^2\vartheta(1-2\sin^2\varphi)\}, \qquad (5.7)$$

where v is the initial-particle velocity. For completely polarized particles we have $d\sigma_{2\gamma}$ ($\vartheta = \pi/2$, $\varphi = \pi/2$) = 0, $d\sigma_{2\gamma}$ ($\vartheta = \pi/2$, $\varphi = 0$) = $2d\sigma^0_{2\gamma}$.

It must be kept in mind that for the case of production of a pair of pseudoscalar mesons, the exclusions exist for planes (do not depend on the production angle of the final particles), while for all remaining reactions they exist only if the final particle momentum is perpendicular to the momentum of the initial particles.[17]

We will give for reference the cross section for elastic scattering of transversely (antiparallel) polarized electrons and positrons:

$$d\sigma_{e^+e^-} = \frac{r_0^2}{16\gamma^2}\left(\frac{3+\cos^2\vartheta}{1-\cos\vartheta}\right)^2\left[1+\frac{|\zeta_1||\zeta_2|\sin^4\vartheta}{(3+\cos^2\vartheta)^2}(1-2\sin^2\varphi)\right]. \qquad (5.8)$$

For this process the azimuthal asymmetry is also maximal for $\vartheta = \pi/2$, but the degree of asymmetry is appreciably smaller than for the processes discussed above, since the term depending on spin orientation enters with an additional factor 1/9.

5.2. Internal Scattering Effects and Polarization Measurement[19]

We have discussed means of polarization measurement based on measurement of the interaction cross section at high energies. However, we must keep in mind that in this case:

a) The electron spin is in the additional field of the colliding beam and the depolarizing effects associated with this must be investigated (by the methods set forth in the preceding chapter).

b) In one form or another the question may arise of the origin of observed effects which are assigned to polarization. Therefore it is desirable to have independent means of measuring the polarization of each of the beams. We will discuss such means below.

We will begin with a method utilizing the interaction of particles inside the beam. It is well known that an important cause of the loss of particles in storage rings with high intensity is elastic scattering of electrons inside the bunches.[20] If this scattering occurs into a sufficiently large angle and is such that particles with a large transverse momentum and small longitudinal momentum (in the rest system of the beam) acquire a large longitudinal momentum, then in conversion to the laboratory system the longitudinal momentum is subject to the relativistic transformation and can turn out to be larger than the permissible value. As a result the particles are lost. Under some conditions the lifetime of a beam in a storage ring is determined by just this effect, which is sometimes called the Touschek or ADA effect. Internal scattering effects depend on the particle polarization, since the electron-electron scattering cross section at the large angles which determine the internal scattering effect depends substantially on electron polarization, and in particular the cross section for scattering of identically and completely polarized electrons by an angle $\pi/2$ goes to zero in the nonrelativistic limit. This dependence of the internal scattering effect on polarization can in principle be used to measure the polarization of electrons in a storage ring:

a) by analysis of the dependence of the lifetime (for the condition that it is determined by internal scattering effects) on polarization;

b) by analysis of the dependence of pairs of particles knocked out of the beam on polarization.

Let us find the lifetime of a beam of polarized particles against internal scattering effects. Here, as usually,[20-22] we will assume that:

1) all particles in the beam have the same energy (the spread in energy is appreciably less than the permissible value);

2) $\delta q_z \ll \delta q$ (δq_z and δq are the mean-square momenta of the vertical and radial oscillations, respectively).

In considering the general case in which the energy of the transverse oscillations is relativistic, it is impossible to use in the calculations the so-called small angle approximation (in which only the $\sin^{-4}\vartheta$ terms are retained), since it is not sufficiently accurate (10—30%), and the polarization-dependent terms have a structure which cannot be determined in this approximation.

We list below the results of calculations carried out by a well known method[21,22] for the coefficient α which determines the beam lifetime τ (τ is the time in which the number of particles decreases by a factor of two):

$$1/\tau = \alpha N_0, \tag{5.9}$$

where N_0 is the initial number of particles in the beam.

For a rectangular distribution of radial momenta of the electrons in the beam we have

$$\alpha = \frac{2\pi r_0^2 c}{V(\Delta p)^2 \sqrt{3}\,\delta q}\left\{2\sqrt{1+(\delta p)^2} - \frac{23}{4} + \ln\frac{2}{\eta} + \ln\frac{\delta p}{1+\sqrt{1+(\delta p)^2}} + \frac{2}{\delta p}\ln\left(\delta p + \sqrt{1+(\delta p)^2}\right) - \frac{\zeta_1\zeta_2}{4} + \frac{1}{y}(3+\zeta_1\zeta_2)\right\}, \tag{5.10}$$

where V is the volume of the beam in the laboratory system, Δp is the maximum permissible deviation of momentum from the equilibrium value in the laboratory system, δp is the *maximum momentum of the distribution*, which is related to the mean-square value by $\delta q = \delta p/\sqrt{3}$, $\eta = \Delta p/\epsilon$, ϵ is the electron energy in the laboratroy system, and $y = \delta p/\eta$; we have used the system of units $m = 1$.

In derivation of this formula we have systematically expanded all quantities in powers of $1/\epsilon^2$, η^2, $1/y^2$, and have retained only the leading terms of the expansion. Under actual conditions $\eta \sim 10^{-2}$, y varies over the range $10-10^3$ when the electron energy is 10^2-10^3. In the nonrelativistic approximation $\delta p \ll 1$ we obtain

$$\alpha_N = \frac{2\pi r_0^2 c}{V(\Delta p)^2\sqrt{3}\,\delta q}\left[\ln y - \frac{7}{4} - \frac{\zeta_1\zeta_2}{4} + \frac{1}{y}(3+\zeta_1\zeta_2)\right]. \tag{5.11}$$

In the ultrarelativistic limit

$$\alpha_R = \frac{2\pi r_0^2 c}{V(\Delta p)^2\sqrt{3}\,\delta q}\left[2\sqrt{3}\,\delta q - \frac{23}{4} + \ln\frac{2}{\eta} - \frac{\zeta_1\zeta_2}{4}\right]. \tag{5.12}$$

For a Gaussian distribution of radial momenta of the electrons we have

$$\alpha = \frac{2\sqrt{\pi}\, r_0^2 c}{V(\Delta p)^2\delta q}\left\{\ln\frac{2}{\eta} - \frac{7}{4} - \frac{\zeta_1\zeta_2}{4} + 2\sqrt{\pi}\,\delta q e^{1/(\delta q)^2}\left(1+\frac{1}{2(\delta q)^2}\right)(1-\Phi(1/\delta q)) - \sqrt{\pi}\int_0^{1/\delta q} e^{x^2}(1-\Phi(x))\,dx\right\}, \tag{5.13}$$

where $\Phi(x)$ is the probability integral.

In the nonrelativistic approximation $\delta q \ll 1$ we have

$$\alpha_N = \frac{2\sqrt{\pi}\, r_0^2 c}{V(\Delta p)^2\delta q}\left\{\ln y - \frac{3+2C}{4} - \frac{\zeta_1\zeta_2}{4}\right\}, \tag{5.14}$$

where C is Euler's constant, $C = 0.577...$

In the ultrarelativistic limit

$$\alpha_R = \frac{2\sqrt{\pi}\, r_0^2 c}{V(\Delta p)^2\delta q}\left\{2\sqrt{\pi}\,\delta q - \frac{23}{4} + \ln\frac{2}{\eta} - \frac{\zeta_1\zeta_2}{4}\right\}. \tag{5.15}$$

The value of α depends weakly on the shape of the momentum distribution, especially in the ultrarelativistic limit (and also in the intermediate region[22]); for this reason it is possible to use for estimates the simpler formulas for a rectangular distribution. This is due to the fact that the main contribution to internal scattering effects is from small scattering angles and the low-velocity region in the distribution. From the same circumstance it follows that the relative contribution of the constants in α (including those dependent on polarization) remains appreciable up to $\delta q \approx 1$, and only when the momentum of the majority of electrons becomes relativistic, $\delta q \gg 1$, does this contribution drop substantially.

We will estimate the contribution of the polarization-dependent terms for the VÉPP-2 installation at Novosibirsk for an energy $\epsilon = 700$ MeV, at which the characteristic time of radiative polarization is about 30 min (see Sec. 3), $\eta \approx 10^{-2}$, $\delta q = 1$ ($\delta p = \sqrt{3}$). Then the relative contribution of polarization-dependent terms to α for complete polarization of the electrons (Eq. (5.13)) is about 6%.

5.3. Measurement of Polarization by Means of Compton Scattering[23]

In Compton scattering of circularly polarized photons by transversely polarized high-energy electrons, terms in the cross section arise which depend on the electron polarization vector. In head-on collisions of laser photons (with energy ω_1) with high-energy electrons, the final photons are emitted mainly in a narrow cone with an angle $\sim 1/\gamma$ relative to the initial electron direction and have an energy

$$\omega_2 = \frac{2\epsilon\lambda}{1+n^2+2\lambda}, \tag{5.16}$$

where $\lambda = 2\omega_1\epsilon/m^2$, and the photon emission angle is $\vartheta = n/\gamma \ll 1$. To lowest order in e^2 the cross section has the form (see refs. 24 and 25)

$$d\sigma = d\sigma_0 + d\sigma_1 \xi_2 |\zeta_1| \sin\varphi, \tag{5.17}$$

where $d\sigma_0$ is the cross section for unpolarized particles, ξ_2 is the degree of circular polarization of the photons, and φ is the angle between the plane perpendicular to the vector $\boldsymbol{\zeta}_1$ and the scattering plane. We note that the correlation term in (5.17) of the form $\xi_2(\boldsymbol{\zeta}_1\mathbf{k}_2)$ is the only possible term from considerations of P and T invariance. The azimuthal asymmetry coefficient has the form

$$\mathcal{F} = \frac{d\sigma_1}{d\sigma_0} = -\frac{2\lambda n(1+n)^2}{2\lambda^2(1+n^2)+(1+n^2+2\lambda)(1+n^4)}; \tag{5.18}$$

$\mathcal{F}$ reaches an extremum $\mathcal{F}_{ex} \approx -1/3$ for $\lambda \approx 1$, $n \approx 1$. For storage rings presently in existence and lasers, $\lambda \ll 1$; then

$$d\sigma_0 = \frac{4r_0^2(1+n^4)\,n\,dn\,d\varphi}{(1+n^2)^4}, \quad d\sigma_1 = \frac{8r_0^2\lambda n^2\,dn\,d\varphi}{(1+n^2)^4}. \tag{5.19}$$

The maximal value of the asymmetry coefficient $\mathcal{F}_{max}$ is reached for $n = 0.76$ and is $\mathcal{F}_{max} = -1.14\lambda$. The asymmetry coefficient for cross sections integrated over the scattering angle $0 \le \vartheta \le \vartheta_0 = n_0/\gamma$ is $\mathcal{F}_0 = -0.8\lambda$ for $n_0 = 2$ and $\mathcal{F}_0 = -0.6\gamma$ for $n_0 \gg 1$. Therefore it is necessary to use the shortest-wavelength photon sources possible. The effect of asymmetry in Eq. (5.17) is maximal for $\varphi = \pm\pi/2$, i.e., when the vector $\boldsymbol{\zeta}_1$ lies in the scattering plane, so that for $\xi_2(\mathbf{k}_2 \cdot \boldsymbol{\zeta}_1) < 0$ the cross section is maximal and for $\xi_2(\mathbf{k}_2 \cdot \boldsymbol{\zeta}_1) > 0$ the cross section is minimal.

If we use a krypton laser (photon energy $\omega_1 = 3.5$ eV[26]) as the photon source, then $\lambda \approx 0.09$ for $\epsilon = 3.5$ BeV, so that the up-down polarization asymmetry for $n_0 = 2$ reaches $\sim 14\%$. For a laser power of 1 watt, a number of electrons in the storage ring $N_e = 10^{11}$, a beam cross-section area $s = 10^{-2}$ cm^2, and $\Delta\varphi \sim 0.1$, the number of final photons is $\sim 10^4$ sec^{-1}.

5.4. Scattering In a Polarized Electron Target and Measurement of the Polarization

An azimuthal asymmetry also exists in the cross section for scattering of transversely polarized fast electrons by a polarized electron target[27]:

$$d\sigma = d\sigma_0 + d\sigma_1 |\zeta_1| |\zeta_2| \cos(2\varphi + \varphi_1), \quad d\sigma_1 = (r_0^2/2\gamma)\, d\Omega_c, \qquad (5.20)$$

where $d\sigma_0$ is the Møller cross section, $|\boldsymbol{\zeta}_2|$ is the degree of polarization of the electron target, and the angle φ is defined as in Eq. (5.17). The vector $\boldsymbol{\zeta}_2$ is chosen in the plane perpendicular to the momentum vector of the initial electron (then the asymmetry is maximal), φ_1 is the angle between the vectors $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$, and $d\Omega_c$ is the element of solid angle in the center-of-mass system. The greatest asymmetry occurs for $2\varphi + \varphi_1 = 0$ or π. For example, for $\boldsymbol{\zeta}_1 \parallel \boldsymbol{\zeta}_2$ ($\varphi_1 = 0$) this corresponds to $\varphi = 0$ (scattering plane perpendicular to the vector $\boldsymbol{\zeta}_1$) and $\varphi = \pi/2$ (the vector $\boldsymbol{\zeta}_1$ lies in the scattering plane). The asymmetry coefficient $\mathscr{P} = d\sigma_1/d\sigma_0$ is maximal for a scattering angle $\vartheta = \sqrt{2/\gamma}$ (which corresponds to a c.m.s. scattering angle $\vartheta_c = \pi/2$) and is given by $\mathscr{P}_{max} = 0.11$. For cross sections integrated over scattering angle,

$$\sqrt{2/\gamma}\,\mathrm{tg}\,\vartheta_c/2 \leqslant \vartheta \leqslant \sqrt{2/\gamma}\,\mathrm{ctg}\,\vartheta_c/2,$$

the asymmetry coefficient is

$$\mathscr{P}_0 = \frac{\sin^2\vartheta_c}{8 + \sin^2\vartheta_c}. \qquad (5.21)$$

For $\vartheta_c = 75°$ $\mathscr{P}_0 = 0.1$. The low value of the asymmetry coefficient does not permit use as a target of magnetized ferromagnetic materials, where $|\boldsymbol{\zeta}_2| < 0.09$, so that the total up-down asymmetry is $<2\%$. Obviously it is desirable to use as a target atomic beams in which the electron polarization can be raised to $|\boldsymbol{\zeta}_2| \sim 1$ and the total asymmetry for $\vartheta_c = 75°$ reaches 20%. For known densities of polarized atomic beams ($n \sim 10^{11}$ cm^{-3}) for $\epsilon = 700$ MeV, $N_e \sim 10^{11}$, a size of the interaction region ~ 1 cm, $\vartheta_c = 75°$, and $\Delta\varphi \sim 0.1$, the number of scattered electrons is ~ 10 sec^{-1}.

The methods described above, from our point of view, are most promising for determination of the transverse polarization of high-energy electrons in a storage ring. It should be noted that the relative contribution of terms depending on the electron polarization increases with energy for Compton scattering of laser photons (so that the method is suitable at an energy of several BeV) and decreases with energy for internal scattering effects (so that the method is suitable at an energy of several hundred MeV), and is independent of energy for scattering by an electron target (i.e., this method is applicable for any energy if the number of events is sufficiently great).

5.5. Other Methods

We will discuss other methods of determining the transverse polarization of high-energy electrons:

a) The cross section for scattering of transversely polarized electrons by a polarized nuclear target, with an accuracy to terms of order $1/\gamma$, does not depend on the electron polarization, a consequence of helicity conservation (see, for example, ref. 28).

b) The degree of circular polarization of a bremsstrahlung photon in electron scattering by a Coulomb field depends on the electron polarization; for the cross section integrated over final-electron emission angle, the degree of circular polarization of the photon for transversely polarized initial fast electrons under optimal conditions does not exceed 10%;[29] in addition, the photon polarization measurement which is necessary in this method is in itself a rather complex problem; the bremsstrahlung cross section summed over final-particle polarization with inclusion of all Coulomb corrections with an accuracy to $1/\gamma$ terms has the same structure as the Born cross section, and consequently, does not depend on electron polarization.[29]

c) Quantum corrections to the intensity of synchrotron radiation, which depend on electron polarization, are of order $\chi = (H/H_0)\gamma$ and are extremely small.

Thus, the methods listed in this section are not very suitable for determination of the transverse polarization of electrons.

For determination of the transverse polarization of electrons, a method with conversion of transverse polarization to longitudinal polarization is also promising*. This can be accomplished, for example, as the result of precession of the electron spin relative to the velocity in a magnetic field perpendicular to the spin vector and the momentum. The rotation angle of the electron in this field, in which the transverse polarization is transformed to longitudinal, can be found from Eq. (B.6); it is

$$\Delta\varphi = \frac{\pi}{2\gamma\eta}. \qquad (5.22)$$

The same result is obtained if, instead of a magnetic field, we use an electric field directed along the spin vector with the condition that $\eta\gamma^2 \gg 1$. Measurement of the longitudinal polarization obtained can be easily carried out, for example, in experiments on scattering in a polarized electron target (the contribution of polarization-dependent terms is ~ 1) or in experiments on scattering by a polarized proton target.

6. AN EXPERIMENT ON STUDY OF RADIATIVE POLARIZATION

An experimental study of the radiative polarization of electrons has been carried out recently in the storage ring VÉPP-2 in Novosibirsk.[32] The polarization measurement was accomplished by the method described in Section 5.2 which utilizes the dependence of internal

*See also Appendix C.

scattering effects on the polarization of the electrons in the beam.[19] For the energy chosen (ϵ = 650 MeV) the polarization time is $T \approx 50$ min and the theoretical degree of polarization (3.27) during the experiment is $|\zeta_3(2T)| \approx 0.80$. In this experiment it is extremely important to exclude the effect of depolarizing factors (see Section 4). For this purpose it is necessary first of all to be sufficiently far from spin resonances (4.11). The result of a specific analysis of resonance harmonics for the VÉPP-2 storage ring is shown in Fig. 4, in which the frequencies are given in units of ω_0 ($\nu = \eta\gamma = (g-2/2)\gamma$, $\nu_z = \omega_z/\omega_0 = 0.8093$, $\nu_x = \omega_x/\omega_0 = 0.7614$) and the order of the resonance is indicated.[32] The depolarization time (height of the resonance) in Fig. 4 was estimated (see Sec. 4.1b) on the assumption that dynamical mixing occurs as the consequence of the spread in amplitude of betatron oscillations, and that the stochastic mechanism is radiative damping. It is clear that, if ω_n (4.17) satisfies $\omega_n \gg \tau_0^{-1}$, then $\tau_{\text{dep}}^{-1} \sim \tau_0^{-1}$ (compare Eq. (4.50)); and if $\omega_n \ll \tau_0^{-1}$, then $\tau_{\text{dep}}^{-1} \sim \omega_n(\omega_n\tau_0)$. This situation is described by the interpolation formula

$$\tau_{\text{dep}}^{-1} = \tau_0^{-1}\frac{\omega_n^2}{\omega_n^2+\tau_0^{-2}} \tag{6.1}$$

The widths of the resonances in Fig. 4 (the "wings" of the resonances) were evaluated on the basis of (4.50). In Fig. 4 we have also shown the radiative polarization times T and the radiative damping time τ_0. It can be seen from Fig. 4 that in the region below ϵ = 500 MeV the electron beam is practically unpolarized. We note that with increasing energy 1/T increases rapidly, and for higher energies we can expect that depolarizing factors will play a smaller role. In accordance with an analysis which was made, the working point was chosen at an energy ϵ = 650 MeV. When depolarizing effects (4.53) are taken into account, the expected degree of radiative polarization is $|\zeta_3^r(2T)| \approx 0.66$.

The measurements were made in the following way. The electron beam in the storage ring was polarized for

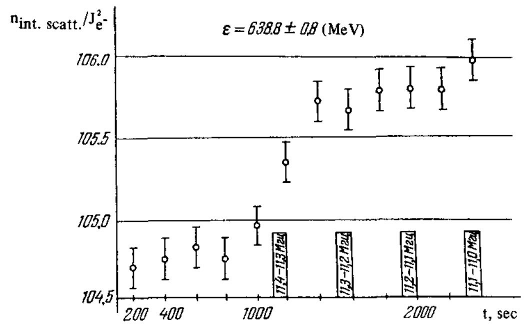

FIG. 5

a time $t \sim 2T$, and the particles leaving the beam as a consequence of internal scattering effects were recorded by two counters. Then the beam was depolarized by application of an external longitudinal field (Sec. 4.1e) for time (4.42) $\tau_{\text{dep}} \sim 100$ sec. In this case according to (5.13) the rate of departure of particles from the beam increases (i.e., the number of counts in the counters increases). Figure 5 shows the experimental results for an energy ϵ = 638.8 ± 0.8 MeV, where a jump can be seen in the counting rate (normalized to the square of the current), occurring at the turning on of the depolarizing field*. From the size of the jump we can deduce the following value of the degree of polarization of the electron beam:

$$|\zeta_3^{\text{exp}}(2T)| = 0.52 \pm 0.13, \tag{6.2}$$

which is consistent with the expected value of the degree of polarization given above with inclusion of depolarizing effects $|\zeta_3^r(2T)| \approx 0.66$, although it is somewhat smaller. Thus, we have obtained the first experimental indications of the existence of the radiative polarization effect. Naturally, further experimental investigation of this effect is desirable, in particular, removal of the influence of depolarizing effects.

In conclusion the author takes pleasure in expressing his sincere gratitude to V. M. Katkov and A. N. Skrinskiĭ for numerous discussions of the radiative polarization problem and for valuable observations. The author is grateful to V. M. Strakhovenko for discussions, and to Ya. S. Derbenev, A. M. Kondratenko, A. N. Skrinskiĭ, G. M. Tumaĭkin, and Yu. M. Shatunov for making available their results prior to publication.

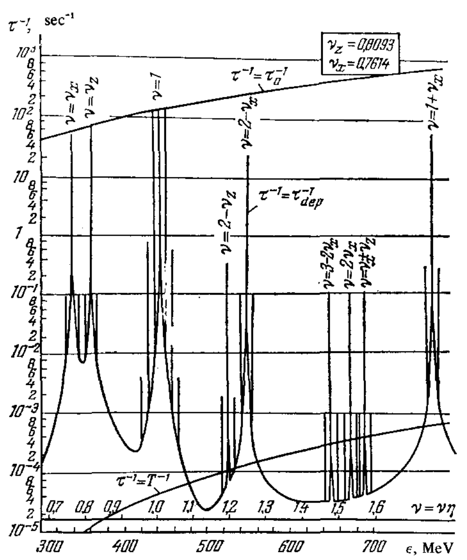

FIG. 4

APPENDICES

A. CALCULATION OF $\langle t_0|\text{Re } T_2|t_0\rangle$

By definition $\langle t_0|T_2|t_0\rangle$ is a scattering matrix element of second order in the coupling constant e, taken between single-particle states, i.e., the contribution of the self-energy diagram. This contribution has been found (for arbitrary χ) in ref. 8. Taking into account that in Sec-

*In this fact we have a method for extremely accurate absolute measurements of the electron energy in the storage ring, since the (g−2)/2 factor of the electron is known very accurately. With the depolarization method used, the accuracy is determined by the band width of the frequency modulation $\delta\omega$ (4.36).

tion 3 we use a state vector which is a two-component spinor, i.e., $|t_0\rangle = \varphi|i\rangle$, we have, using Eq. (2) of ref. 8,

$$\langle t_0 | T_2 | t_0\rangle = T_{qq}^{(2)} = \int dt\,(T_1^{(2)} + T_2^{(2)})\,(\zeta\,[\mathbf{vs}]) \tag{A.1}$$

$$= i \int dt\,\langle i\,|\,\varphi^{+}(D_0 + i\boldsymbol{\sigma}\mathbf{D})\,\varphi\,|\,i\rangle,$$

where $\mathbf{s} = \dot{\mathbf{v}}/|\dot{\mathbf{v}}|$,

$$T_2^{(2)} = -\frac{\alpha m^2}{2\pi\varepsilon}\int_0^\infty \frac{u\,du}{(1+u)^3}\left[L_{1/3}\left(\frac{2u}{3\chi}\right) + \frac{i}{\sqrt{3}}K_{1/3}\left(\frac{2u}{3\chi}\right)\right], \tag{A.2}$$

$$L_{1/3}(z) = \int_0^\infty dx \sin\frac{3z}{2}\left(x + \frac{x^3}{x}\right);$$

$K_{1/3}(z)$ is the Macdonald function. A unique relation exists between the terms $T_1^{(2)}$ and D_0 and between $T_2^{(2)}$ and $\mathbf{D}$; therefore from $T_2^{(2)}$ we can also determine $\mathbf{D}$ (the term $T_1^{(2)}$ (D_0) does not contribute to Eq. (3.20) and therefore is not given here). If we take into account that

$$(\zeta\,[\mathbf{vs}]) = -\frac{e\tilde{F}_{\mu\nu}p^{\mu}s^{\nu}}{\sqrt{|eF_{\lambda\sigma}p^{\sigma}|^2}} = -\frac{2\mu_0(\zeta\mathbf{H}_R)}{m\chi}, \tag{A.3}$$

where s^{ν} is the polarization 4-vector, $\mu_0 = e/2m$, $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, and $F^{\alpha\beta}$ is the external field tensor, then $\mathbf{H}_R$, the magnetic field in the rest system of the particle, is found to be

$$\mathbf{H}_R = \gamma\left[\mathbf{H} - \frac{\mathbf{v}\,(\mathbf{vH})}{1 + 1/\gamma} - [\mathbf{vE}]\right]. \tag{A.4}$$

From (A.1)–(A.3) we have

$$\langle i\,|\,\varphi^{+}\,\mathrm{Re}\,\mathbf{D}\varphi\,|\,i\rangle = -\frac{\mu'}{\gamma}\mathbf{H}_R,\quad \frac{\mu'}{\mu_0} = \frac{\alpha}{2\pi}\,\frac{2}{\chi}\int_0^\infty \frac{u\,du}{(1+u)^3}L_{1/3}\left(\frac{2u}{3\chi}\right). \tag{A.5}$$

The quantity μ' is the anomalous magnetic moment of the electron to order e^2 in the interaction with the radiation field and with inclusion of all orders in the external field.[8] For $\chi \ll 1$, $\mu'/\mu_0 = (\alpha/2\pi)(1 - 12\chi^2 \ln\chi + \ldots)$. If we limit ourselves to an accuracy $\mu'/\mu_0 = \alpha/2\pi$, then the result (A.5) can be obtained also by means of Schwinger's results,[30] as was done in ref. 10.

B. EQUATION FOR SPIN MOTION IN AN EXTERNAL FIELD

In Sec. 3 we obtained in terms of quantum electrodynamics the equation of motion of the electron spin (the average value of the spin operator in the rest system). If we retain in this equation terms of order $\hbar^0(\chi^0)$ (i.e., if we omit damping terms), this equation goes over to the BMT equation. The latter does not contain Planck's constant $\hbar$ and can be obtained on the basis of purely classical considerations as the direct relativistic generalization of the equations of motion of the mechanical (spin) moment (for a given gyromagnetic ratio $\mu = (ge/2m)(\zeta\hbar/2)$) in a magnetic field in the rest system of the particle:

$$\dot{\mathbf{J}} = [\boldsymbol{\mu}\mathbf{H}]. \tag{B.1}$$

If we set $\mathbf{J} = (1/2)\zeta\hbar$, then

$$\frac{d\zeta}{dt} = \frac{eg}{2m}[\zeta\mathbf{H}]. \tag{B.2}$$

We will introduce the 4-vector s^{μ} such that in the electron rest system $s = (0, \zeta)$. Obviously, if u^{μ} is the velocity 4-vector $u = (\gamma, \gamma\mathbf{v})$, then $(su) = 0$. Hence it follows that $s_0 = (\mathbf{s}\cdot\mathbf{v})$ and in the rest system

$$\frac{ds_0}{dt} = (\zeta\dot{\mathbf{v}}). \tag{B.3}$$

The relativistic generalization of (B.2) and (B.3) is the BMT equation[7]

$$\frac{ds^{\mu}}{d\tau} = \frac{e}{m}\left\{\frac{g}{2}F^{\mu\nu}s_{\nu} + \left(\frac{g-2}{2}\right)u^{\mu}(F^{\lambda\nu}s_{\lambda}u_{\nu})\right\}; \tag{B.4}$$

here τ is the proper time. Taking into account that

$$\mathbf{s} = \zeta + \frac{\mathbf{p}\,(\mathbf{p}\zeta)}{m\,(\varepsilon + m)},\quad s_0 = \frac{(\mathbf{p}\zeta)}{m}, \tag{B.5}$$

we can convert from the equation for s (B.4) to an equation for ζ. We finally obtain (3.23) without damping terms* (see (3.19) and (3.22)):

$$\frac{d\zeta}{dt} = [\zeta\mathbf{F}],\quad \mathbf{F} = \frac{e}{\varepsilon}\left[\left(\frac{g-2}{2}\right)\mathbf{H}_R + \mathbf{H}_E\right]. \tag{B.6}$$

Let us discuss the right-hand part of the BMT equation in the form of (B.6). The magnetic moment is acted on by the magnetic field $\mathbf{H}_R$ in the rest system of the particle, if the fields are $\mathbf{H}$ and $\mathbf{E}$ in the laboratory system. However, it is necessary to keep in mind that increment in the spin vector $\Delta\zeta = (d\zeta/dt)\Delta t$ consists of parts of which one is due to rotation in the field $\mathbf{H}_R$ and the second is kinematical and is due to rotation of the spin as the result of the fact that the electron motion in the external field is accelerated (in other words, as a consequence of the fact that the rest systems at the moments of time t and t + Δt are different and that one is rotated relative to the other†). The latter increment is easy to calculate by means of the Lorentz transformation equation

$$(\Delta\zeta)_{\mathrm{kin}} = \gamma^2\,\frac{[\zeta\,[\mathbf{v}\Delta\mathbf{v}]]}{1+\gamma}. \tag{B.7}$$

The total change of ζ (compare (B.6)) is

$$\frac{d\zeta}{dt} = \frac{e}{\varepsilon}\,\frac{g}{2}[\zeta\mathbf{H}_R] + \gamma^2\frac{[\zeta\,[\mathbf{v}\dot{\mathbf{v}}]]}{1+\gamma} = \tag{B.8}$$

$$= \frac{e}{\varepsilon}\left[\zeta\left[\left(\frac{g}{2} - 1\right)\mathbf{H}_R + \mathbf{H}_R + \frac{\gamma^2}{1+\gamma}([\mathbf{v}\,[\mathbf{vH}]] + [\mathbf{vE}])\right]\right] = [\zeta\mathbf{F}].$$

Hence it is evident that the field $\mathbf{H}_R$ actually acts only on the anomalous part of the moment, while the effective field $\mathbf{H}_E$ (B.6), (3.22), which can be considered as acting on the intrinsic moment of the spinor particle, turns out to be strongly attenuated in comparison with $\mathbf{H}_R$ at high energies ($\gamma \gg 1$). Just for this reason, although the anomalous magnetic moment of the electron (in units of $e\hbar/2mc$) is extremely small, $(g-2)/2 = \alpha/2\pi + \ldots$, the terms with it are extremely important, since they contain an additional power of γ.

Equation (B.6) (see (3.10) and (3.22)) involves the fields $\mathbf{H}$ and $\mathbf{E}$ and the particle velocity, which is determined by these same fields. Therefore the form of (B.6) is not always convenient. It turns out to be useful to write Eq. (B.6) in such a way that only the independent variables occur as coefficients in the right-hand part. There are several sets of independent quantities; we

*If a vector $\mathbf{A}$ does not change with respect to some rotating system, and its change with respect to a fixed system is due only to the rotation, then $\dot{\mathbf{A}} = \boldsymbol{\Omega}\times\mathbf{A}$, where $\boldsymbol{\Omega}$ is the angular velocity (frequency) of rotation. Equation B.6) is of just this type. It is clear that $(\dot{\zeta}\zeta) = 0$, ζ^2 = const for $\mathbf{F}$ = const, $\mathbf{F}\times\zeta$ = const, i.e., the vector ζ precesses about the axis $\mathbf{F}$ with a frequency $|\mathbf{F}|$.

†This fact is sometimes called Thomas precession.

will choose the set $\mathbf{v}$, $\dot{\mathbf{v}}$, $H_{\|}$, $\mathbf{E}_{\perp}$ ($H_{\|}$ and $\mathbf{E}_{\perp}$ are the field components along the velocity and perpendicular to the velocity).

Taking into account that

$$\mathbf{H}_{\perp}=\frac{1}{v^2}\left\{[\mathbf{vE}]-\frac{\varepsilon}{e}[\mathbf{v}\dot{\mathbf{v}}]\right\}, \tag{B.9}$$

we obtain for the BMT equation (B.6)

$$\mathbf{F}=-(1+\eta\gamma)\frac{[\mathbf{v}\dot{\mathbf{v}}]}{v^2}+\frac{2\mu}{\gamma^2 v^2}[\mathbf{vE}]+\frac{2\mu}{\gamma}\frac{(\mathbf{Hv})\mathbf{v}}{v^2}, \tag{B.10}$$

where $\mu = (g/2)(e/2m)$, $\eta = g/2 - 1$. If we introduce the unit vectors

$$\mathbf{e}_1=\frac{\mathbf{v}}{|\mathbf{v}|},\quad \mathbf{e}_2=\frac{\eta\gamma\dot{\mathbf{v}}_{\perp}-\frac{2\mu}{\gamma^2}\mathbf{E}_{\perp}}{\left|\eta\gamma\dot{\mathbf{v}}_{\perp}-\frac{2\mu}{\gamma^2}\mathbf{E}_{\perp}\right|},\quad \mathbf{e}_3=[\mathbf{e}_1\mathbf{e}_2] \tag{B.11}$$

and expand the vector $\boldsymbol{\zeta}$ in these unit vectors:

$$\boldsymbol{\zeta}=\zeta_1\mathbf{e}_1+\zeta_2\mathbf{e}_2+\zeta_3\mathbf{e}_3, \tag{B.12}$$

then the following system of equations[13] follows from Eq. (B.10):

$$\left.\begin{aligned}\frac{d\zeta_1}{dt}&=-\Omega\zeta_2,\\ \frac{d\zeta_2}{dt}&=\Omega\zeta_1+\omega\zeta_3,\\ \frac{d\zeta_3}{dt}&=-\omega\zeta_2,\end{aligned}\right| \tag{B.13}$$

where the frequencies are

$$\Omega=\frac{1}{|v|}\left|\gamma\eta\dot{\mathbf{v}}_{\perp}-\frac{2\mu}{\gamma^2}\mathbf{E}_{\perp}\right|,\quad \omega=(\dot{\mathbf{e}}_2\mathbf{e}_3)+\frac{2\mu H_{\|}}{\gamma}, \tag{B.14}$$

or

$$\frac{d\boldsymbol{\zeta}}{dt}=[\boldsymbol{\zeta}\mathbf{u}], \tag{B.15}$$

where the frequency $\mathbf{u}$ has components

$$\mathbf{u}=(\omega,\ 0,\ -\Omega). \tag{B.16}$$

Thus, in the chosen system of unit vectors the spin vector moves around the $\mathbf{e}_3$ axis with a frequency Ω and around the $\mathbf{e}_1$ axis (the direction of the velocity) with a frequency ω.

If the electric field $\mathbf{E} = 0$, then $\dot{\mathbf{v}} = \dot{\mathbf{v}}_{\perp}$ and the expressions for the frequency take the form

$$\Omega=\eta\gamma\frac{|\dot{\mathbf{v}}|}{|\mathbf{v}|},\quad \omega=\frac{(\mathbf{e}_3\ddot{\mathbf{v}})}{|\dot{\mathbf{v}}|}+\frac{2\mu H_{\|}}{\gamma}=\eta\frac{eH_{\|}}{\varepsilon}+\frac{(\dot{\mathbf{v}}\dot{\mathbf{H}})}{|\dot{\mathbf{v}}||\mathbf{H}_{\perp}|}, \tag{B.17}$$

where for time-independent fields $\dot{\mathbf{H}} = (\mathbf{v}\cdot\nabla)\mathbf{H}$, and as unit vectors we choose the vectors

$$\mathbf{e}_1=\frac{\mathbf{v}}{|\mathbf{v}|},\quad \mathbf{e}_2=\frac{\dot{\mathbf{v}}}{|\dot{\mathbf{v}}|},\quad \mathbf{e}_3=\frac{[\mathbf{v}\dot{\mathbf{v}}]}{|\mathbf{v}||\dot{\mathbf{v}}|}. \tag{B.18}$$

The first term in ω (B.17) is extremely small in comparison with Ω (their ratio is $\sim(1/\gamma)(H_{\|}/H_{\perp})$, where ordinarily $H_{\|} < H_{\perp}$). Therefore in nonuniform fields the principal role is played by the second term, which explicitly depends on the field nonuniformity. Under typical storage-ring conditions the particles execute small oscillations with a frequency ω_{0s} and amplitude a. Then

$$\frac{\omega}{\Omega}\approx\frac{(\mathbf{v}\nabla)H}{\gamma\eta\omega_0 H_{\perp}}\sim\frac{\omega_{0s}a}{\gamma\eta\omega_0 R}\ll 1. \tag{B.19}$$

This circumstance has been systematically utilized in this work.

The equations of motion (B.13) determine the motion of the spin vector relative to a fixed system, but just this system is defined physically.

C. EVOLUTION OF SPIN VECTOR IN PERIODIC MOTION

In the case in which $\mathbf{F}(t) = \mathbf{F}(t + T)$ in Eq. (B.6), particular interest is presented by the periodic solutions of this equation, which mean that at a given point of the trajectory there exists a stable direction of polarization, which is repeated every revolution.[31] We will designate such solutions by

$$\mathbf{n}(t+T)=\mathbf{n}(t),\quad \mathbf{n}^2=1,\quad \dot{\mathbf{n}}=[\mathbf{nF}]. \tag{C.1}$$

Any solution of Eq. (B.6) can be expanded in the three orthogonal solutions $\mathbf{x}_m$ (m = 1, 2, 3):

$$\mathbf{n}(t)=\sum n_m\mathbf{x}_m(t), \tag{C.2}$$

where n_m are constants. Then the periodicity condition acquires the form

$$\sum_m n_m\mathbf{x}_m(t)=\sum_m n_m\mathbf{x}_m(t+T) \tag{C.3}$$

or

$$\sum_m(\delta_{km}-\Lambda_{km})n_k=0, \tag{C.4}$$

where

$$\Lambda_{km}=\mathbf{x}_k(t)\,\mathbf{x}_m(t+T). \tag{C.5}$$

The system of Eqs. (C.4) has solutions if

$$\det(I-\Lambda)=0. \tag{C.6}$$

This condition is always satisfied, since the constant matrix Λ corresponds to a real rotation $\Lambda^T\Lambda = I$, $\det\Lambda = 1$ (a general existence theorem exists for periodic solutions of uniform linear systems of differential equations with periodic coefficients; see, for example, Lefshetz[33]). Since any two solutions of Eq. (B.6) satisfy the condition $d(\zeta_a\zeta_b)/dt = 0$, for an arbitrary initial spin direction $(\boldsymbol{\zeta}\cdot\mathbf{n})$ = const. Hence it follows that the spin vector rotates about the periodic solution $\mathbf{n}$, which is fixed for a given azimuth, conserving its projection on the $\mathbf{n}$ direction. The general solution of Eq. (B.6) can be expanded in $\mathbf{n}$ and two vectors $\boldsymbol{\eta}$ in the plane perpendicular to $\mathbf{n}$. Let $2\pi\nu$ be the angle by which the solution transverse to $\mathbf{n}$ is rotated in a revolution; in complex form this condition is $\eta(t + T) = e^{-2\pi i\nu}\eta(t)$. Expanding $\eta(t)$ in $x_m(t)$, we have

$$\sum_k(e^{-2\pi i\nu}\delta_{km}-\Lambda_{km})\eta_k=0, \tag{C.7}$$

so that it is necessary to find the eigenvalues of the matrix

$$\det(\lambda I-\Lambda)=0,\quad \lambda=e^{-2\pi i\nu}. \tag{C.8}$$

One of them, obviously, is $\lambda = 1$, and the other two are obtained from the conditions

$$\begin{gathered}\sum_{k=1}^{3}\lambda_k=\operatorname{Sp}\Lambda,\quad \lambda_1\lambda_2\lambda_3=\det\Lambda=1,\\ \lambda_1=1,\quad \lambda_2=e^{-2\pi i\nu},\quad \lambda_3=e^{2\pi i\nu},\quad \cos 2\pi\nu=\frac{\operatorname{Sp}\Lambda-1}{2}.\end{gathered} \tag{C.9}$$

The corresponding eigen solutions $\mathbf{n}$, $\boldsymbol{\eta}$, and $\boldsymbol{\eta}^*$ are orthogonal if $\cos 2\pi\nu \neq 1$. The general solution can be written in the form

$$\boldsymbol{\zeta}(t)=\zeta_n\mathbf{n}+\frac{1}{2}(c\boldsymbol{\eta}+c^*\boldsymbol{\eta}^*), \tag{C.10}$$

where ζ_n = const, c and c* = const. In the case of resonance ($\cos 2\pi\nu = 1$) there is a degeneracy $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and all solutions are periodic.

The existence of a stable periodic solution permits any polarization to be produced at a given point of the trajectory (for example, at the point where beams collide in storage rings). For example, at high energies such that $\pi/2\eta\gamma \ll 1$, by introducing into a straight section of a storage ring a radial magnetic field in which the angle of rotation is $\pi/2\gamma\eta$ (see (5.22)), it is possible to convert a transverse polarization to longitudinal, and after the collision region to again convert it to transverse. It is true that the degree of polarization in this case will be somewhat smaller than in a uniform field, as the result of radiative polarization in the radial field.

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Translated by C. S. Robinson