

Methodological Notes

THE GIBBS PARADOX

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1. The Gibbs paradox has for many decades been the object of great attention of physicists as well as philosophers. Unfortunately, owing to a number of historical and psychological reasons, an extraordinary confusion has arisen in the extensive literature dealing with the paradox; there exists a multiplicity of view-points even in respect of the definition of the content of the paradox itself. There is an even greater lack of coordination in the attempts to solve it. Therefore, in order to understand the essence of the matter and separate out the major points, it is necessary to begin with a brief exposition of well-known things. We shall restrict ourselves in this to the analysis of only the thermodynamic side of the question.

Tradition—not too well-founded, as will be seen below—associates the Gibbs paradox with peculiar properties of entropy. If we are interested in only changes in the volume, i.e., in isothermal processes, then the entropy of a rarefied ideal gas, consisting of atoms of one type, can be written in the form

$$S = kN \ln V + S_0, \tag{1}$$

where N is the number of atoms and k is the Boltzmann constant. It is important to emphasize that the quantity S_0 is a constant only for a given working medium; on changing to another working medium, it may change.

How do we find the form of the function $S_0 = S_0(N)$? For this purpose, let us divide the volume V into two parts V_1 and $V_2 = V - V_1$, containing N_1 and N_2 ($N = N_1 + N_2$) molecules, respectively. From the thermodynamic meaning of the concept of entropy arises its additivity with respect to the indicated division, i.e.,

$$S(N_1 + N_2, V_1 + V_2) = S(N_1, V_1) + S(N_2, V_2). \tag{2}$$

This equality should be satisfied provided $N_1/V_1 = N_2/V_2$. It is easy to derive from the expression (2) a general expression for the entropy.

$$S(N, V) = Nf(V/N), \tag{3}$$

where f is an arbitrary function.

In the special case of a rarefied ideal gas, which is of interest to us, $f(z) = k \ln z$ and (3) becomes

$$S = kN \ln(V/N), \tag{4}$$

i.e.

$$S_0 = -kN \ln N.$$

Let us assume that the volume V contains a mixture of M and N atoms of the two different gases A and B. It

is well known* that such a mixture can be separated into its component gases with the aid of a reversible adiabatic process involving the participation of a semi-permeable membrane. The temperature of each of the gases will not change in the process and each will occupy the same volume V as before. It follows from this that the entropy of the mixture is equal to the sum of the entropies of the gases A and B:

$$S = kM \ln(V/M) + kN \ln(V/N). \tag{5}$$

We can now formulate the essence of the Gibbs paradox. Let us imagine two equal volumes V separated by an impermeable membrane; the volumes are filled with different gases A and B, the temperatures and pressures of the gases are equal. Then the entropy of the whole system

$$S' = S'_A + S'_B = 2kN \ln(V/N), \tag{6}$$

where N is the number of atoms of each of the gases. The removal of the partition results in an increase in the entropy due to irreversible diffusion. After equilibrium has been established, each component occupies a volume $2V$, its entropy

$$S''_A = S''_B = kN \ln(2V/N),$$

and the total entropy

$$S'' = S''_A + S''_B = 2kN \ln(2V/N). \tag{6'}$$

The increase in the entropy

$$\Delta S = 2kN \ln 2 \tag{7}$$

does not depend on the nature of the mixed ideal gases.

On the other hand, if both volumes are occupied by one and the same gas, then the withdrawal of the partition does not change the thermodynamic state of the system and should not give rise to a change in entropy. Using the expression (4), we can easily verify that this is in fact what one obtains. We should here take into account the fact that without the partition the number of atoms is $2N$ and the volume is $2V$.

Thus, no matter how similar two different gases are in their properties, their total entropy increases by one and the same value (7) when they are mixed, whereas for identical gases there is no increase in the entropy. It is this discontinuity ΔS in behavior on going from similar to identical gases that constitutes the Gibbs paradox†.

It should be noted that the entropy of an ideal gas is sometimes (see, for example, [3,5]), without any foundation, written in the form

*See, for example, [3], Sec. 16.

†The indicated formulation can, for example, be found in the works of Schrödinger [1] and Tamm [4]. It is, however, in essence, given by Gibbs [2].

*It should be especially emphasized that the presence of N under the logarithm sign plays a fundamental role in all the cases when processes with a varying number of particles are considered (see in this connection Chap. 8 of [1], as well as the original work of Gibbs [2], p. 206). Strictly speaking, we must add to (4) a term βN with the coefficient β not depending on N . This term does not play any role below and we drop it.

$$S = kN \ln V,$$

i.e., the term $S_0 = -kN \ln N$ is dropped. Then, the removal of the partition leads to the increase (7) in the entropy not only for different, but also for identical gases, although the thermodynamic state of the system in the latter case does not change. It is precisely this result that is called the Gibbs paradox in the cited works.

It seems to us that the question here is not one of a paradox, but rather one of a misunderstanding which can be resolved by using the correct expression (4) for the entropy.

2. Some physicists thought that the Gibbs paradox was not resolvable in the framework of thermodynamics. Later, another point of view appeared in the literature*. In our opinion, there is a grain of truth in it, and we shall discuss it in a somewhat greater detail than has hitherto been done.

The Gibbs paradox amounts to the discontinuity in behavior of the quantity ΔS during a continuous approach of certain parameters characterizing mixed gases. An important question, however, arises: Is such a continuous approach actually possible; does it not contradict the laws of physics? If the answer is yes, the paradox remains. If it is no, i.e., if the differences between gases can only vary discretely, the paradox disappears: there is nothing surprising about the fact that during a discontinuous variation of the parameters, the properties of the mixture also vary discontinuously.

The point of view being discussed is based on the assertion that it is the second type of situation that obtains in reality. Indeed, when we speak about different gases, we usually imply that their atoms differ from each other in some discrete and conserved quantum number (charge, number of nucleons, etc.). Under these conditions the parameters, defining the difference between the gases, cannot vary continuously. The indicated difference in the discrete quantum numbers may turn out to be in some respects unimportant, but it is precisely this difference, in the sense of interest to us now, that is of fundamental importance. Indeed, the definition of the concept of entropy itself presupposes the possibility of a reversible separation and mixing of gases, and this possibility may be related only to those properties of gases in respect of which they differ from each other.

The expounded approach to the Gibbs paradox is limited to only those cases when the mixed gases cannot change into one another, and the parameters determining the difference between them vary discretely. If these parameters could vary continuously, we should have had to consider the discontinuity in behavior ΔS as indeed paradoxical. It is important to emphasize that such a behavior of the "difference parameters" is, physically, fully realizable, but, as will now be demonstrated, the quantity ΔS then also varies continuously and does not undergo any discontinuity in the transition from similar to identical gases. The indicated circumstance is, in our opinion, the final solution to the paradox. The discussion of the causes which make (or do not make) the jump ΔS paradoxical is replaced by an assertion about the absence of any jump^[7,8].

Let us begin with the simplest case. Let us suppose that each of the considered volumes V is occupied by different mixtures of the gases A and B, the mixtures differing from each other only in the relative concentrations (one volume contains M_1 atoms of the gas A and $N_1 = N - M_1$ atoms of B, the other volume— M_2 atoms of A and $N_2 = N - M_2$ atoms of B). It is clear that from the thermodynamic point of view, both these mixtures should be considered as different—in their properties—working media. It is also clear that the degree of likeness between them depends only on the ratio of the concentrations and can be continuously varied. The maximum difference corresponds to the case when one of the volumes is occupied by the pure gas A, and the other volume—by B. The absence of a difference (coincidence of all the properties) is attained when the relative concentrations of A and B in the two volumes are equal.

In the general case, the removal of the partition leads to an additional intermixing and increase in entropy. Applying the relation (5) first to two volumes V , and second to one volume $2V$, we obtain

$$\Delta S = k(N_1 + N_2) \ln \left(\frac{2V}{N_1 + N_2} \right) + k(M_1 + M_2) \ln \left(\frac{2V}{M_1 + M_2} \right) - [kN_1 \ln(V/N_1) + kM_1 \ln(V/M_1) + kN_2 \ln(V/N_2) + kM_2 \ln(V/M_2)]. \quad (8)$$

As can easily be seen, the quantity ΔS approaches zero continuously as $N_1 \rightarrow N_2$ and $M_1 \rightarrow M_2$; if $M_1 = N_2 = 0$, or $M_2 = N_1 = 0$, then,

$$\Delta S = 2kN \ln 2.$$

For arbitrary concentrations

$$0 \leq \Delta S \leq 2kN \ln 2. \quad (9)$$

Let us turn to the more interesting situation, when the atoms of the gas have some internal degree of freedom. We shall, for definiteness, speak of spin, although all the results that follow have a general meaning. Let the spin $j = 1/2$, which corresponds, in particular, to a ^3He gas.

We can, as is well known, obtain with the aid of "optical pumping," a considerable polarization of ^3He , which is preserved for a long time (see, for example,^[9]). Let the volume V be occupied by N ^3He atoms which are completely polarized in some fixed direction, and the second volume—by the same number of ^3He atoms polarized in another direction. There is no doubt that these are two different—in their properties—gases. The degree of the difference between them depends on a continuous parameter—the angle θ between the directions of polarization. It is to be expected, therefore, that when the gases are mixed, there will be an increase ΔS in the entropy which will continuously depend on θ . We shall assume that the spin relaxation time is many times longer than the characteristic diffusion time. Then, it is clear that the gas, after mixing, will be polarized along the bisector of the angle between the directions of the initial polarizations; the degree of its polarization

$$P = \cos \theta/2. \quad (10)$$

To calculate the entropy of such a gas, we must determine the number of atoms whose spins are oriented along and opposite the direction of polarization ($N_1 = 2N(1 + p)/2$, $N_2 = 2N(1 - p)/2$), and apply the rela-

*See, for example, Sec. 13.

tion (5) to the volume $2V$. If we take into account the fact that the entropy of the system before mixing was $2kN \ln(V/N)$, then we obtain for the quantity ΔS the expression

$$\Delta S = -kN(1 + \cos(\theta/2)) \ln \left(\frac{1 + \cos(\theta/2)}{2} \right) - kN(1 - \cos(\theta/2)) \ln \left(\frac{1 - \cos(\theta/2)}{2} \right). \quad (11)$$

It is easy to see that from (11) again follows the inequality (9). The extreme values are attained for $\theta = 0$ (the initial gases are identical, the spin wave functions of the atoms coincide) and $\theta = \pi$ (the initial gases are completely distinguishable, the spin wave functions are orthogonal). If the wave functions do not coincide and are not orthogonal, the quantity ΔS takes on intermediate values, which vary continuously with $\cos(\theta/2)$.

Of fundamental importance is the fact that in the deviation of the relation (11), we considered a gas with polarization \mathbf{P} as a mixture of two gases completely polarized along and opposite the vector \mathbf{P} . The point is that if we adiabatically separate, with the aid of a semi-permeable filter, a partially polarized gas into just such components, the entropy of the gas after the separation will not change. It is this fact that gives us the right to apply formula (5) to a mixture of two unlike gases. When any other filters are used, however, an irreversible change in the spin state of the gas takes place, as a result of which the entropy increases. Formula (5) will then not yield the true value of the entropy of the gas before separation.

3. For an arbitrary statistical ensemble, a detailed justification of the procedure for calculating the entropy was given by von Neumann (see^[10], Chap. 5). According to^[10], the entropy of an ideal gas with spin density matrix $\hat{\rho}$ can be represented in the form

$$S = -kN \text{Sp} \hat{\rho} \ln \hat{\rho} + kN \ln V; \quad (12)$$

here, $\text{Sp} \hat{\rho} \ln \hat{\rho} = \sum_{i=1}^{i=2j+1} \rho_i \ln \rho_i$, where j is the spin,

and ρ_i are the eigenvalues of the density matrix $\hat{\rho}$. According to (12), the entropy S is the sum of the entropies of $(2j + 1)$ gases whose spin states are described by the eigenfunctions of the matrix $\hat{\rho}$. The von Neumann inequality

$$\text{Sp} \hat{\rho} \ln \hat{\rho} \geq \sum_i \rho_{ii} \ln \rho_{ii}, \quad (13)$$

where ρ_{ii} are the diagonal elements of the density matrix in an arbitrary representation (the equality sign corresponds to $\rho_{ii} = \rho_i$), is then valid. It follows directly from the relation (13) that the entropy of a gas does not change during its measurement only in the case when the eigenfunctions, characterizing the measuring instrument (filter), coincide with the eigenfunctions of the density matrix. If, however, such a coincidence does not exist, then the entropy will certainly increase during the measurement.

Let a space of volume V contain N atoms of a gas in a spin state of density matrix $\hat{\rho}^{(A)}$, and let another space of the same volume, separated from the first by an impermeable membrane, contain N atoms of the same gas in a spin state of density matrix $\hat{\rho}^{(B)}$. After complete mixing, the density matrix of the gas in the volume $2V$ will have the form

$$\hat{\rho} = \frac{1}{2} (\hat{\rho}^{(A)} + \hat{\rho}^{(B)}). \quad (14)$$

Taking (12) into account, we can write the change in the entropy after mixing in the form

$$\Delta S = kN \left\{ \text{Sp} (\hat{\rho}^{(A)} + \hat{\rho}^{(B)}) \ln \left(\frac{\hat{\rho}^{(A)} + \hat{\rho}^{(B)}}{2} \right) - \text{Sp} \hat{\rho}^{(A)} \ln \hat{\rho}^{(A)} - \text{Sp} \hat{\rho}^{(B)} \ln \hat{\rho}^{(B)} \right\}^*. \quad (15)$$

Let us consider the important particular case, when A and B correspond to pure ensembles of spin wave functions $\psi^{(A)}$ and $\psi^{(B)}$. For pure states one of the eigenvalues of the density matrix is equal to unity while the rest vanish. Consequently, we should in formula (15) set $\text{Sp} \hat{\rho}^{(A)} \ln \hat{\rho}^{(A)} = \text{Sp} \hat{\rho}^{(B)} \ln \hat{\rho}^{(B)} = 0$. Solving further the standard problem of the determination of the eigenvalues of a Hermitian density matrix $\hat{\rho}$, we obtain

$$\rho_1 = \frac{1}{2} (1 + |\langle \psi^{(A)} | \psi^{(B)} \rangle|), \quad \rho_2 = \frac{1}{2} (1 - |\langle \psi^{(A)} | \psi^{(B)} \rangle|), \quad \rho_3 = \rho_4 = \dots = \rho_{2j+1} = 0. \quad (16)$$

To the eigenvalues ρ_1 and ρ_2 correspond the eigenfunctions

$$|\psi_1\rangle = \frac{|\psi^{(A)}\rangle + |\psi^{(B)}\rangle e^{i\eta}}{\sqrt{2(1 + |\langle \psi^{(A)} | \psi^{(B)} \rangle|)}}, \quad |\psi_2\rangle = \frac{|\psi^{(A)}\rangle - |\psi^{(B)}\rangle e^{i\eta}}{\sqrt{2(1 - |\langle \psi^{(A)} | \psi^{(B)} \rangle|)}}, \quad (17)$$

where

$$\eta = \arg \langle \psi^{(B)} | \psi^{(A)} \rangle.$$

Substituting (16) into (15), we arrive at the relation

$$\Delta S = -kN(1 + |\langle \psi^{(A)} | \psi^{(B)} \rangle|) \ln \left(\frac{1 + |\langle \psi^{(A)} | \psi^{(B)} \rangle|}{2} \right) - kN(1 - |\langle \psi^{(A)} | \psi^{(B)} \rangle|) \ln \left(\frac{1 - |\langle \psi^{(A)} | \psi^{(B)} \rangle|}{2} \right). \quad (18)$$

When $|\langle \psi^{(A)} | \psi^{(B)} \rangle| = 1$, the quantity $\Delta S = 0$; in the case when the wave functions are orthogonal, the quantity $\Delta S = 2kN \ln 2$. For intermediate values of $|\langle \psi^{(A)} | \psi^{(B)} \rangle|$, the inequality

$$0 < \Delta S < 2kN \ln 2.$$

is fulfilled. We emphasize that the expression (18) is valid for any value of the spin (and, in general, for internal degrees of freedom of any nature). If the spin is equal to $1/2$, then $|\langle \psi^{(A)} | \psi^{(B)} \rangle| = \cos(\theta/2)$ and the relation (18) is equivalent to (11).

Let us return to the general expression (15). It is easy to see that in the case of commuting density matrices, we obtain a formula of the type (8). For identical gases,

$$\hat{\rho}^{(A)} = \hat{\rho}^{(B)} \quad (19)$$

and the quantity $\Delta S = 0$. The condition for complete distinguishability of the gases is the fulfillment of the matrix equality

$$\hat{\rho}^{(A)} \hat{\rho}^{(B)} = 0. \quad (20)$$

In this case the change in the entropy $\Delta S = 2kN \ln 2$. It can be shown^[8], that the change in the entropy ΔS ,

*In the case of spin $1/2$,

$$\hat{\rho}^{(A)} = \frac{1}{2} (1 + \sigma \mathbf{P}^{(A)}), \quad \hat{\rho}^{(B)} = \frac{1}{2} (1 + \sigma \mathbf{P}^{(B)}), \quad \rho = \frac{1}{2} (1 + \sigma \mathbf{P}),$$

where σ is the Pauli vector operator, $\mathbf{P}^{(A)}$ and $\mathbf{P}^{(B)}$ are the polarization vectors of the gases and $\mathbf{P} = 1/2(\mathbf{P}^{(A)} + \mathbf{P}^{(B)})$. And if $|\mathbf{P}^{(A)}| = |\mathbf{P}^{(B)}| = 1$, then from the general formula (15) follows at once (11).

determined from the formula (15), always satisfies the inequality (9). The above-stated conditions for the attainment of the minimum and maximum values of ΔS are then not only sufficient, but necessary as well.

Thus, we can state that the traditional approach to the Gibbs paradox is defective, since the mixed gases are considered in all cases either as totally different, or completely identical. It is accordingly assumed that either the gases can be totally separated, or their separation is absolutely unfeasible. No intermediate situations are considered. Such a view-point does not correspond to the facts, since there exist states which only partially differ from (or coincide with) each other. In particular, if the atoms of the gases under consideration are described by superpositions with respect to some internal quantum number, then the measure of the difference is just the degree of nonorthogonality, $|\langle \psi^{(A)} | \psi^{(B)} \rangle|$. Under these conditions, "separability" of the gases also ceases to be absolute: any filter that separates out one of the states, "seizes," to an extent determined by the degree of nonorthogonality, atoms of the second type (see also^[10], Chap. 5).

Thus, the complete solution of the Gibbs paradox involves a transition from the "absolutist" view of the nature of the identity and distinguishability of gases, to a broader view, according to which these concepts reflect only the limiting cases of a general situation, when the gases under consideration are partially distinguishable and partially indistinguishable*.

4. In the majority of papers the Gibbs paradox is explicitly or implicitly connected with peculiar properties of entropy. We hold the opposite view-point, and consider that the Gibbs paradox is connected with entropy mainly through history and can in fact be formulated completely independently of this concept.

Indeed, the relation (3) derived above is obviously applicable not only to entropy, but to any other additive state function as well. In particular, we can also write for the internal energy (for fixed temperature)

$$U = N\varphi(V/N), \quad (21)$$

Therefore, if one of two equal volumes V is occupied by N atoms of an ideal gas A, and the other—by the same number of atoms of the gas B, then the internal energy of the system before mixing is

$$U' = 2N\varphi(V/N).$$

After mixing, each of the gases occupies a volume $2V$, and the total internal energy of the mixture U'' is equal to the sum of the internal energies of the components. Therefore, in the case of an isothermal mixing,

$$U'' = 2N\varphi(2V/N),$$

i.e., there is a change

*It should be noted that a similar point of view has previously been expressed in a number works of A. Landé (see, for example, [11-13]). Unfortunately, these works contain physical errors pertaining to the crux of the problem. In particular, A. Landé uses for the quantity ΔS an incorrect expression, similar in form to (18) but containing $|\langle \psi^{(A)} | \psi^{(B)} \rangle|^2$, instead of $|\langle \psi^{(A)} | \psi^{(B)} \rangle|$, and comes to the conclusion that the concept of the entropy of a mixture has no meaning if the quantities of the mixed gases are not the same, etc. In [14] the continuous behavior of ΔS is analyzed in the framework of the information interpretation of entropy.

$$\Delta U = 2N\{\varphi(2V/N) - \varphi(V/N)\}, \quad (22)$$

which is independent of the degree of similarity of the mixed gases. On the other hand, in the case of identical gases, the use of (21) leads to the equality

$$\Delta U = 2N\varphi(2V/2N) - 2N\varphi(V/N) = 0.$$

Logically, the indicated situation does not differ in any way from the Gibbs paradox, although the concept of internal energy may be formulated independent of entropy and is not directly connected with the second law of thermodynamics. We must, to be sure, note that for rarefied ideal gases the function φ introduced by us is a constant, and the jump in the value of the internal energy $\Delta U = 0$. However, this circumstance is not important in respect of what we are interested in now. If we go from rarefied ideal gases to gases which are so dense that quantum mechanical degeneracy begins to play a noticeable role (see, for example, [15], Sec. 55), then $\varphi \neq \text{const}$, and the quantity ΔU turns out to be finite*.

As in the case of entropy, there is no discontinuity in the variation of the internal energy as we go from identical to distinguishable gases if we introduce for the gases a continuous similarity parameter, e.g., the degree of nonorthogonality of the superpositions $|\langle \psi^{(A)} | \psi^{(B)} \rangle|$. The formula for ΔU (analogous to formula (18) for the change in entropy) will then have the form

$$\Delta U = N(1+x)\varphi\left(\frac{2V}{N(1+x)}\right) + N(1-x)\varphi\left(\frac{2V}{N(1-x)}\right) - 2N\varphi(V/N), \quad (23)$$

where

$$x = |\langle \psi^{(A)} | \psi^{(B)} \rangle|.$$

It is easy to see that when $x = 1$, the quantity $\Delta U = 0$, while when $x = 0$ the expression (23) coincides with (22). In the interval $0 < x < 1$, the quantity ΔU takes on intermediate values.

For real gases, the internal energy will depend not only on temperature (the kinetic energy of the atoms), but on the mean distance between the atoms (potential energy) as well. The latter part also includes the so-called exchange energy, connected with an additional quantum-mechanical interaction between identical atoms.

When even very similar, but still distinguishable atoms (e.g., isotopes), are mixed, the mean distance between identical atoms increases, and this leads to changes in the exchange interaction and the internal energy of the mixture. Therefore, in an adiabatic mixing of gases, which are arbitrarily close in similarity, a jump in the temperature occurs, a jump which is absent when identical gases are "mixed." Thus, we have again a paradox similar to the Gibbs paradox.

We could, if we wished, give a few more paradoxes of the same sort. The most profound of them is, in general, not connected with thermodynamics and may be formulated for a system, containing only two particles. If the particles are identical, the wave function of the system should be symmetrized; in the case of unlike particles, the wave function is arbitrary. Whence, as is well known, it is concluded that the behavior of a system of particles, which are arbitrarily close in similarity,

*In essence, precisely this phenomenon is mentioned in one of Einstein's papers (see [16], p. 488).

differs sharply from the behavior of a system of identical particles.

All the indicated paradoxes are similar to the Gibbs paradox. If the properties of the gases or particles under consideration can vary only discretely, the situation cannot be considered as paradoxical: a discrete change in causes leads to a discrete change in the effects. We would have paradoxes if the behavior varied discontinuously as the "similarity parameters" varied continuously; however, in this case the behavior also changes continuously. This was shown above for internal energy. Concerning the jump in the temperature (and pressure) when real gases are mixed, the same conclusion follows from similar arguments. It seems to us that the behavior of a system of two particles can also be analyzed in similar fashion (see^[17-19]).

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SOME DEMONSTRATIONS IN WAVE OPTICS PERFORMED WITH A GAS LASER

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We describe below a few lecture demonstrations of the principal interference and diffraction phenomena using a gas laser as the light source. The directionality of a laser radiation, its monochromaticity, and high intensity allow us to demonstrate these phenomena to a large audience.

1. DEMONSTRATIONS OF THE INTERFERENCE PHENOMENON FOR LIGHT REFLECTED FROM THE BOUNDARIES OF A PLANE-PARALLEL GLASS SHEET

We have found two ways of setting up such demonstrations. The scheme of the first method is shown in Fig. 1. A He-Ne laser 1 is placed facing the audience in the auditorium and shielded from them by a screen 2 of dimensions 30×30 cm. There is at the center of the screen an aperture large enough to pass the laser light beam. A converging lens of focal length 10 cm is

placed opposite the center of the aperture in the screen on the reverse side facing the laser.

The diverging light beam from the focus of the lens falls on a plane-parallel glass plate 4 of diameter 100 mm and thickness 15 mm situated 2 m from the screen. The non-parallelism angle between the faces of the plate is of the order of one second. Its surface is polished to within "one tenth of an interference fringe." The possibility of a fine adjustment of the plate, to set the surfaces of the plate perpendicular to the axis of the light cone incident on it, is provided for by its holder. Figure 1 shows two interfering rays produced by reflection from the front and rear surfaces of the plate. These rays may be conceived as coming from two virtual images (S_1 and S_2) of a point light source S , which is the focal point of the lens used to produce the diverging beam of rays. These virtual images are formed as a result of the reflection of light from the two surfaces of the plate, and are located on the axis of the system.