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DUALITY AND THE VENEZIANO MODEL

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CONTENTS

FOR the past few years the interest of physicists who are involved with the theory of elementary particles has been attracted by the model proposed by Gabriele Veneziano[1]. In a mathematically amazingly simple form this model reflects the concepts which have been developed over the past few years regarding the character of the strong interactions of elementary particles. Although in its primitive form the model runs into considerable difficulties, due mainly to attempts to take unitarity into account, the many positive characteristics of the model lie at the basis of the conviction that its further development is full of promises.

In this brief review it is impossible to touch upon all the directions related to the Veneziano model.* We discuss the fundamental and characteristic features of the Veneziano model, as well as its consequences and the degree of agreement with experimental data.

Before describing the model one must discuss the experimental and theoretical concepts which have led to its introduction. We shall pay particular attention to the concept of "duality"[2,3], which immediately preceded the Veneziano model and was to a large degree responsible for it. At the same time, the mathematical and physical meaning of the duality concept was clarified through the study of the Veneziano model.

I. THE RESONANCE SPECTRUM AND REGGE POLES

1. The Resonances

In the region of not very high energies the physics of strong interactions is resonance physics. A characteristic feature of practically all experiments carried out with elementary particle accelerators is the presence of resonance peaks in the effective mass spectrum of the particles formed in strong interactions (i.e., in the quantity $M = [(E_1 + E_2 + \ldots + E_n)^2 - (p_1 + p_2 + \ldots + p_n)^2]^{1/2}$). These resonances correspond to unstable particles which decay during nuclear time intervals $\tau = (1/\Gamma) \sim 10^{-24}$ sec, where Γ is the width (in energy) of the resonance†. At the present time approximately 50 boson resonances and about as many baryon resonances are known, and their list seems far from being completed[4]. The resonances, as well as the particles which are stable under strong interactions (i.e., the proton p, the neutron n, the pion π, the kaon K, etc.) are characterized by definite values of conserved quantum numbers like the baryon number, strangeness (hypercharge), parity, spin, isospin, etc. (cf., e.g., the review by Gell-Mann, Rosenfeld, and Chew[5]). The spectrum of known particles and resonances exhibits definite regularities, which lead to a simple classification of the resonances. For the sequel it will be essential to note that:

a) Among the well-established particles and resonances there are no so-called "exotic" states, i.e., boson resonances with electric charge $|Q| > 1$ and baryon resonances with positive strangeness and charge $|Q| > 2$. In group-theoretical language this signifies that the particles are classified according to the simplest representations of the SU(3) group: the singlets and octets for the bosons, and singlets, octets and decimets (decuplets) for the baryons (for the SU(3) classification of elementary particles cf., e.g., the review papers[6-8]).

b) If one constructs a graph for particles differing in their spin values, but having the other internal quantum numbers identical, such that the horizontal axis is calibrated in values of the square of the mass M^2 and the vertical axis in values of the spin J (the Chew-Frautschi plot[9]), then the lines which join the points representing the particles turn out to be practically straight (Fig. 1, 2). Such a line is called a Regge (pole) trajectory.* Poles with noninteger (and in general complex) values of the angular momentum J have first been introduced by Regge[10] in the context of nonrelativistic quantum theory of scattering and then generalized to the case of relativistic scattering[9,11]; they are a natural extension of the concept of resonances with integer values of J to noninteger spin values. In order to clarify the relation between Regge poles and resonant states we recall the description of integer spin resonances in the theory of elementary particles. We shall discuss the scattering process $a + b \to c + d$ of spinless particles in terms of the amplitude $A(W, \theta)$, where W is the center-of-mass energy of the particles, θ is the scattering

*The list of references contains only a small fraction (~5%) of the huge number of papers dedicated to this problem which have appeared over the past three years.

†We use a system of units with $\hbar = c = 1$.

*In reality, in relativistic theories a Regge trajectory joins only resonances with even (or odd) values of J for bosons, or of the quantity J–½ for baryons.

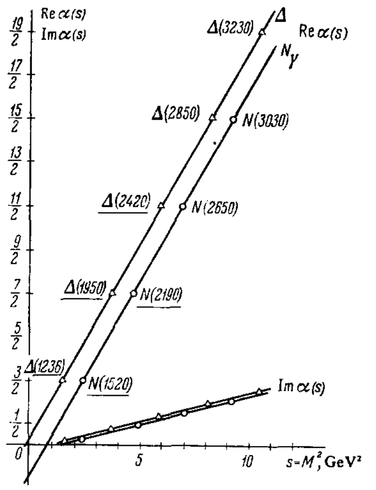


FIG. 1. The baryon trajectories Δ and N in the Chew-Frautschi plot. The triangles correspond to the Δ-trajectory, the circle to the N_γ-trajectory. The two upper lines represent Re α, the two lower ones represent Imα. The resonances, for which the quantum numbers are well established [4], have been underlined.

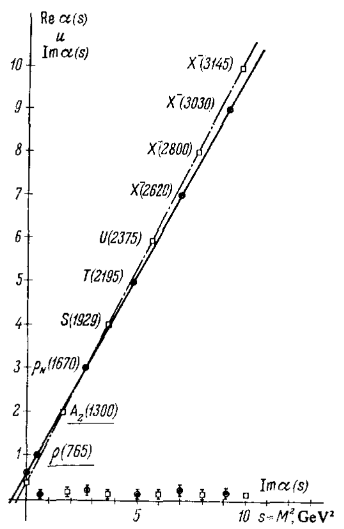


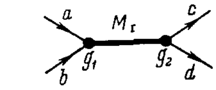
FIG. 2. The boson trajectories: O denotes the ρ-trajectory, □ denotes the R-trajectory.
The upper lines represent Re α, the lower points represent Im α.

angle in the same reference system. The square of the absolute value of the amplitude determines the probability of the process under discussion. If there is a resonance of mass M_r, spin J and width Γ, then for W values close to M_r the amplitude has the usual Breit-Wigner form

$$A(W,\ \theta)=(2J+1)\,A_J(W)\,P_J(\cos\theta)=\frac{(2J+1)\,g_1g_2P_J(\cos\theta)}{(M_r^2-W^2)-i\Gamma M_r}\approx$$
$$\approx\frac{2J+1}{2M_r}\,\frac{g_1g_2P_J(\cos\theta)}{(M_p-W)-(i\Gamma/2)}\,; \qquad (1)$$

here $P_J(\cos\theta)$ is a Legendre polynomial, which originates from the fact that scattering occurs only in a state with orbital angular momentum $l = J$. Thus, the amplitude $A(W, \theta)$ (as well as the partial wave amplitude $A_J(W)$) has a pole in the variable W at the point $W = M_r - i(\Gamma/2)$ which is removed from the real axis by a distance determined by the width of the resonance. Usually,

FIG. 3. A diagram representing resonant scattering.



the resonances occurring in systems of strongly interacting particles (hadrons) have widths Γ ~ 100 MeV. One can associate a diagram (Fig. 3) with the expression (1) for $A(W, \theta)$, with the interpretation that the colliding particles a and b first form a resonance in the intermediate state, which latter decays into the particles c and d. The probability amplitude for the process $a + b \to M_r$ is characterized by the quantity g_1, and the probability for the decay $M_r \to c + d$ is characterized by the constant g_2; these constants are called "vertices" or coupling constants.

2. Regge Poles

Before describing the Regge poles we dwell briefly on the kinematics of the process discussed above and on the concept of crossing symmetry in relativistic quantum theory.

In the relativistic theory it is convenient to describe the amplitude in terms of invariant variables, which can be formed from the four momenta p_i of the particles participating in the reaction. One usually[12] chooses the following invariants:

$$s=(p_a+p_b)^2,\quad t=(p_a-p_c)^2,\quad u=(p_a-p_d)^2. \qquad (2)$$

Conservation of the total 4-momentum in the reaction, i.e., the condition $p_a + p_b = p_c + p_d$, implies that the three quantities s, t, and u are not independent and are related by the simple equation*

$$s+t+u=m_a^2+m_b^2+m_c^2+m_d^2. \qquad (3)$$

It is easy to derive a relation between the invariant variables and the center-of-mass quantities W, θ[12]:

$$s=W^2,\quad t=-\varepsilon_a\varepsilon_c+m_a^2+m_c^2+2\,|\mathbf{p}_a|\,|\mathbf{p}_c|\cos\theta, \qquad (4)$$

where ϵ_i and $\mathbf{p}_i$ denote respectively the energy and momentum of the i-th particle in the center-of-mass system. These can be expressed in terms of the quantity s and the particle masses. When all four masses are equal to m, we have:

$$t=-2\mathbf{p}^2(1-\cos\theta)=-\mathbf{q}^2,\quad \mathbf{p}^2=\left(\frac{s}{4}-m^2\right).$$

In relativistic quantum theory the absorption of a particle of 4-momentum $(-p_i)$, i.e., with negative energy, is equivalent to the emission of an antiparticle of 4-momentum p_i. Therefore the same amplitude $A(s, t, u)$ describes the scattering process $a + b \to c + d$ as well as the annihilation processes $a + \bar{c} \to \bar{b} + d$ and $a + \bar{d} \to c + \bar{b}$ (Fig. 4). However, the ranges of variation of the invariant quantities s, t, and u are different for the three processes. Indeed, in the process $a + b \to c + d$ (the so-called s-channel) the physical region corresponds to $s > 0$, $t < 0$, $u < 0$ ($|\cos\theta| < 1$), whereas for the reaction $a + \bar{c} \to \bar{b} + d$ (the t-channel) the physical region corresponds to the values $t > 0$, $s < 0$, $u < 0$, and for $a + \bar{d} \to c + \bar{b}$ (the u-channel) $u > 0$, $s < 0$, $t < 0$. The assertion that a single analytic function $A(s, t, u)$ des-

*We have chosen the metric so that $p_i^2 = m_i^2$.

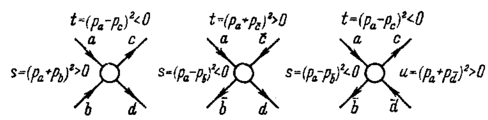


FIG. 4. The processes $a + b \to c + d$, $a + \bar{c} \to \bar{b} + d$, $a + \bar{d} \to c + \bar{b}$ which are described by the same amplitude A(s, t, u).
The values of the variables s, t, u corresponding to physical regions are also indicated.

cribes in different regions of variation of the variables all three processes is called crossing[5,12]. If one of the particles participating in the reaction, e.g., b, is the antiparticle of one of the others, say c (or d), i.e., $b = \bar{c}$ (or $b = \bar{d}$), then the s- and t-channels of the reaction (or the s- and u-channels) are identical and the amplitudes A(s, t, u) must be symmetric under the substitution $s \rightleftarrows t$ (or $s \rightleftarrows u$). This property of the amplitude is called crossing symmetry.

Analyticity and crossing are the fundamental properties of relativistic scattering amplitudes. They are essential in the construction of the Regge pole model. In the framework of this model the amplitude of the process $a + b \to c + d$ at very high energies $s \gg m^2$ and fixed momentum transfer $t = -q^2 \ll s$ is determined by the poles of the partial wave amplitudes $A_J(t)$ of the crossed t-channel (i.e., of the reaction $a + \bar{c} \to \bar{b} + d$) in the complex angular momentum plane J, and can be expressed in the form[11]

$$A(s,\ t) \approx \sum_i \bar{\beta}_i(t)\, \xi(\alpha_i(t))\, P_{\alpha_i}(z_t) \approx \sum_i \beta_i(t)\, \xi(\alpha_i(t)) \left(\frac{s}{s_0}\right)^{\alpha_i(t)} = \sum_i \frac{\beta_i(t)}{\sin \pi\alpha_i(t)} \left[\left(\frac{s}{s_0}\right)^{\alpha_i(t)} \pm \left(\frac{u}{u_0}\right)^{\alpha_i(t)}\right], \tag{5}$$

where $\alpha_i(t)$ is the "trajectory" of the i-th Regge pole (or Regge trajectory), $\beta(t)$ and $\beta(t)$ are the residues of the pole—in general unknown functions of t; s_0 and u_0 are constants having the dimensions [GeV^2], introduced to make the expressions dimensionless, $\xi(\alpha_i(t)) = (1 \pm e^{-i\pi\alpha_i(t)})/\sin \pi\alpha_i(t)$ is the so-called signature factor, which has its origin in the need for considering separately two functions $A_J^{\pm}(t)$ in the relativistic theory, corresponding to $(-1)^J = \pm 1$, respectively. Therefore the Regge poles of the amplitudes $A_J^{\pm}(t)$ are characterized by a new quantum number: the signature. To poles with signature $\sigma = \pm$ corresponds the appropriate sign in Eq. (5), and the amplitude $A_i(s, t)$ exhibits a definite symmetry under the substitution $s \rightleftarrows u$.

In Eq. (5) use has been made of the fact that $u \approx -s$ in the region of s and t under consideration and that for the cosine of the scattering angle in the center-of-mass system of the reaction $a + \bar{c} \to \bar{b} + d$ we have $\cos\theta_t \equiv z_t \approx s/(2|p_a||p_{\bar{b}}|) \gg 1$; thus $P_{\alpha(t)}(z_t) \sim z_t^{\alpha(t)} \approx s^{\alpha(t)}$. In the region of negative $t = -q^2$, i.e., in the physical region of the s-channel, the amplitude (5) describes scattering of the particles at high energies. For $t > 0$ (the unphysical region), whenever $\alpha_i(t)$ passes through an integer n, which is even or odd according to the signature of the amplitude, the expression (5) corresponds to resonant scattering in the t-channel, described by Eq. (1) with the replacement of $W^2 = s$ by t and $\cos\theta_s \equiv z_s$ by $\cos\theta_t \equiv z_t$. Indeed, for t close to t_n, $\alpha(t)$ can be represented in the form

$$\alpha_i(t) = n + \alpha_i'(t_n)(t - t_n) + i\,\mathrm{Im}\,\alpha_i(t_n). \tag{6}$$

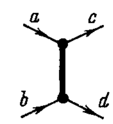


FIG. 5. Diagram corresponding to a resonance in the t-channel.

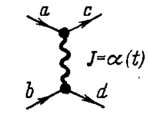


FIG. 6. Diagram corresponding to the Regge pole α(t) in the t-channel.

We have taken into account the fact that for $t > 4\mu^2$ (the threshold for production of particles) the trajectory $\alpha_i(t)$ has an imaginary part. If $\mathrm{Im}\,\alpha_i(t_n) \ll 1$ (which shall be seen below to agree with the experimental situation), then (5) takes near the point t_n the form

$$A(s,\ t) \approx \frac{2\bar{\beta}(-1)^{n+1} P_n(z_t)}{\pi\alpha_i'(t_n)\left[(t_n - t) - i\,\mathrm{Im}\,\alpha_i(t_n)/\alpha_i'(t_n)\right]}, \tag{7}$$

corresponding to a resonance in the reaction $a + \bar{c} \to \bar{b} + d$ (Fig. 5) with angular momentum (spin) J = n, mass $M_n^2 = t_n$ and width $\Gamma_n = \mathrm{Im}\,\alpha_i(t_n)/\alpha_i'(t_n) M_n$.

Consequently, the contribution of a Regge pole to the scattering amplitude (5) describes in a unified analytic form all the t-channel resonances situated on the trajectory $\alpha(t)$, and we shall describe it by the diagram of Fig. 6, where the wavy line describes a state of non-integer (in general even complex) angular momentum $J = \alpha(t)$ (this is sometimes described as a "reggeon exchange"). We note that owing to the presence of the factor $(1 \pm e^{-i\pi\alpha(t)})$ in Eq. (5) the resonance denominator occurs only at even integer values of $\alpha(t)$ for trajectories with positive signature and at odd integer $\alpha(t)$ for trajectories of negative signature.

Thus, the Regge pole model makes use of the analyticity and crossing properties of the scattering amplitudes to establish a close relation between scattering at high energies and the particle and resonance spectrum. Information about the behavior of the Regge trajectories can be derived in the region $t < 0$ from data on high-energy scattering reactions, and in the region $t > 0$ from the data on masses, widths and quantum numbers for the resonances. Figures 1 and 2 illustrate the best known baryon and boson trajectories (for more detailed information on the properties of known Regge trajectories cf., e.g., the reviews[13,14]).

The following facts call themselves to our attention:

a) With good accuracy, all trajectories are straight lines with approximately the same slope $\alpha' \approx 0.9$ GeV^{-2}.

b) The ρ-trajectory (on which the ρ-resonance with mass $m_\rho \approx 750$ MeV and quantum numbers $J^P = 1^-$ is situated, and the R trajectory (which passes through the resonance A_2 with $m_{A_2} \approx 1300$ MeV and $J^P = 2^+$), which have opposite signature, almost coincide.

c) $\mathrm{Im}\,\alpha_i(t) \ll \mathrm{Re}\,\alpha_i(t)$; thus $\mathrm{Im}\,\alpha_\Delta \approx \mathrm{Im}\,\alpha_{N_\gamma} \approx (1/7)\,\mathrm{Re}\,\alpha_{\Delta, N_\gamma}$ (the Δ trajectory carries the nucleon isobar Δ_{33} of mass $m_{\Delta_{33}} = 1236$ MeV and $J^P = 3/2^+$; the first isobar on the N_γ trajectory is N′ with $m_{N'} = 1520$ MeV and $J^P = 3/2^-$)*.

*The fermion Regge trajectories corresponding to isobars with isospin I = 3/2 are usually denoted by Δ, and the trajectories with I = 1/2 are denoted by N. The best known among these are Δ_{33}, N_γ, and N_α which passes through the nucleon $m_p = 939$ MeV, $J^P = ½^+$.

d) Im α_Δ, N_γ also deviates only little from a linear behavior[14] (the situation regarding the ρ and R trajectories is less clear, since the widths of the higher-lying resonances are not well determined).

One cannot escape the impression that a purely linear approximation for the Regge trajectories is a good one. Therefore one often assumes that the trajectories of all Regge poles (excluding the Pomeranchuk pole, vide infra) can be written in the form

$$\alpha_i(x) = \alpha_i(0) + \alpha_i' x. \tag{8}$$

It is assumed that this form of the Regge trajectories is valid for all values of x, including $x \to \infty$, i.e., that the trajectories are infinitely rising. This is one of the basic assumptions of the models we shall discuss in the sequel. On such rising trajectories there is an infinite number of resonances, for which the spins are linear in the squares of their masses:

$$J_n = \alpha_i(0) + \alpha_i' M_n^2. \tag{9}$$

Since the imaginary part of the trajectories is neglected in the approximation under discussion, and since the imaginary part is proportional to the width of the resonances, all the resonances are infinitesimally narrow. It is clear that such a picture can only be a rough approximation to reality.

II. DUALITY

1. Definition of Duality

We consider the consequences of a simple picture of the interaction, wherein all the reaction channels (s, t, u) contain only resonances (poles) situated on linearly rising Regge trajectories. In this approximation the scattering amplitude A(s, t) can be represented either as a sum of resonances in the s- and u-channels, or as a sum over all the Regge poles in the t-channel, i.e., over all the resonances of the t-channel. This property of the amplitude is what is called "duality." We illustrate it using as an example the amplitude for the process $K^-\pi^+ \to K^-\pi^+$, which has no known u-channel resonances (i.e., in the reaction $K^-\pi^- \to K^-\pi^-$: such resonances would have Q = 2 and would be exotic). By assumption the s-channel contains an infinite number of very narrow resonances (poles in the limit $\Gamma \to 0$), and the amplitude of the process can be written in the form of a sum over all such resonances (the amplitude has no other singularities in the complex s-plane):

$$A(s, t) = \sum_n \frac{C_n^{(s)}(t)}{s - s_n} = \sum_n \frac{R_n(t)}{n - \alpha_s(s)}. \tag{10}$$

According to Eq. (1) the residues of these resonances are functions of $\cos\theta$ or t. In Eq. (10) we have utilized the linear relation between s and the trajectory $\alpha(s)$ on which all s-channel resonances are situated. On the other hand A(s, t) can be expressed as a sum over all the resonances in the t-channel, and according to what was said above, as a sum of all the Regge poles:

$$A(s, t) = \sum_k \frac{\overline{R}_k(s)}{k - \alpha_t(t)} = \sum_i \overline{\beta}_i(t)\, \xi_i(\alpha_i(t))\, P_{\alpha_i}(z_t). \tag{11}$$

Therefore, we can write finally the duality relation in the form

$$\sum_n \frac{R_n(t)}{n - \alpha_s(s)} = \sum_i \overline{\beta}_i(t)\, \xi_i(\alpha_i(t))\, P_{\alpha_i}(z_t) = \sum_k \frac{R_k(s)}{k - \alpha_t(t)}. \tag{12}$$

The duality relation is illustrated graphically in Fig. 7. This is the so-called strong, or local, form of duality. We note that for real s the imaginary part of the amplitude in Eqs. (10) and (12) has its origin in the infinitesimal imaginary terms in the resonance denominators. Since $\mathrm{Im}(1/(s_n - s - i\epsilon)) = \pi\delta(s - s_n)$ for $\epsilon \to 0$ and $\epsilon > 0$, in the limit of vanishing widths of the resonances the imaginary part of the amplitude has the form of a superposition of delta functions. In the case where a process can have resonances not only in the s-channel, but also in the u-channel, the left-hand side of (12) must also take into account the u-channel resonances. In the physical region of the s-channel these resonances contribute only to the real part of the amplitude. The imaginary part of the amplitude is always determined by the resonances in "its own" channel (in the case under discussion, the s-channel). This corresponds to the usual physical conceptions based on the unitarity condition, according to which the imaginary part of an amplitude in the physical region of the s-channel owes its existence to the existence of real intermediate states in the s-channel. If one integrates the imaginary part of (12) over s from zero to some value $s_1 > m^2$, we obtain the weak, or global, form of the duality relation

$$\int_0^{s_1} \mathrm{Im}\, A_{\mathrm{res}}^s(s', t)\, ds' = \sum_i \frac{\beta_i(t)}{\alpha_i(t)+1} \left(\frac{s_1}{s_0}\right)^{\alpha_i(t)+1}, \tag{13}$$

which corresponds in form to the finite energy sum rules[15] and can be derived (for $s_1 \gg m^2$) by means of dispersion relations based on the weaker assumptions that the asymptotic behavior of the amplitude is determined by the Regge poles and that the principal role in the s-channel is played by the resonances. Of course, Eq. (13) contains far less information than Eq. (12).

Duality relations analogous to Eq. (12) can also be derived for particle production amplitudes. Thus, for the three-particle production amplitude the duality relation is graphically illustrated in Fig. 8.

Equations (12) and (13) mean, in particular, that the residues $C_n^{(s)}(t)$ of the s-channel resonances are not arbitrary, but have to be selected in such a manner that the asymptotic behavior (5) is satisfied, corresponding to the "exchange" of Regge poles in the t-channel. Thus, duality, which is a consequence of the assumption

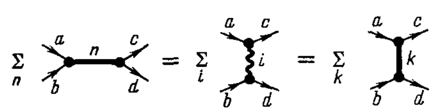


FIG. 7. Diagrammatic form of the duality relation for the transition amplitude of two particles into two particles.

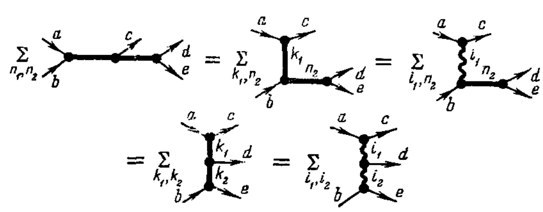


FIG. 8. Diagrammatic representation of the transition amplitude of two particles into three particles.

of linearly rising trajectories, ties closely together the resonances in the direct (s) channel and the Regge poles in the crossed (t) channel of a given reaction.

2. Daughter Trajectories

Let us consider in more detail the properties of Eq. (12). It is very important that (12) can be satisfied only because the numbers of resonances and Regge poles in the two are infinite. Indeed, a finite number of terms in the sum over i cannot lead to divergences for $s \rightarrow s_n$, since each term exhibits a power-law dependence on s. Therefore, the poles at the points s_n can appear only as a result of a divergence of the series in i at these points. And conversely, only an analytic continuation of the infinite series with respect to n can lead to an asymptotic behavior $(s/s_0)^{\alpha(t)}$ in the region $\alpha(t) > 0$ for $s \rightarrow \infty$. The fact that the number of resonances is infinite is natural, since the Regge trajectories rise indefinitely. However, an infinite number of Regge trajectories situated in the J-plane to the left of the principal trajectory $\alpha(t)$ is a new circumstance for us. The occurrence of such trajectories, which are usually designated as "daughters," is a general feature of amplitudes in the Regge pole model. Even before the concept of duality was introduced it was found[16] that in order to ensure that the analytic properties of the scattering amplitude hold at $t = 0$ it is necessary that in addition to the main trajectory $\alpha(t)$ there exist "daughter" trajectories situated for $t = 0$ at integer intervals from it. In order to ensure duality (12) it turns out that it is necessary for this property to hold for all values of t, i.e., that the daughter trajectories be strictly parallel to the main trajectory $\alpha(t)$. The importance of the "daughter" trajectories for the construction of a dual amplitude can also be understood if one makes use of the following simple physical considerations. For large values of s the amplitude has the Regge form (5) and it is known[11] that in the s-channel it is described by the partial wave amplitudes $A_l(s)$, where the important values of the angular momentum are $l \sim |p_a|R \sim (s\alpha' \ln(s/s_0))^{1/2}$; for large values of l the A_l are exponentially small. At the same time, the resonances in the s-channel, which are situated on the main trajectory, will have for the same s values spins $J = \alpha(s) \approx \alpha' s$, and consequently cannot guarantee the required asymptotic regime. The asymptotic behavior (5) is realized only due to the presence of daughter trajectories, which carry the resonances with all spins J from 0 to $J_{max} \sim \alpha' s$, and in particular with $J \sim (s\alpha' \ln(s/s_0))^{1/2}$. Thus, the existence of daughter trajectories (in addition to the main trajectory), parallel to the main trajectory for all t(s), is a necessary condition for the existence of dual models.

3. The Interference Model

There arises a natural question: is duality preserved if one recognizes that in reality the scattering amplitudes have not only poles, but also branch points corresponding to the thresholds for the production of real particles, and when the resonances have finite widths? In this case duality cannot be proved. Moreover, in the presence of branch cuts one can construct examples[17] of the so-called "interference" model, where the amplitudes are represented in the form

$$A(s, t) = G_s(s, t) + G_t(s, t), \qquad (14)$$

where the function G_s has only resonances in the s-channel and is an entire function of the variable t, and the function G_t has only resonances in the t-channel and is an entire function of the variable s, i.e., the amplitude is a sum of resonances in the s-channel and Regge poles in the t-channel. By its character such a model is the opposite of the dual model. Therefore the problem which of the two models, dual or interference, are a better approximation to reality, can at present be resolved only experimentally.

4. The Pomeranchuk Trajectory

A special role in the Regge pole model is played by the Pomeranchuk pole and its trajectory, which passes through the point $J = 1$ at $t = 0$. This pole was introduced into the theory in order to produce the experimentally observed constancy of the total interaction cross sections in the high-energy region, and at $t = 0$ its trajectory is the Regge trajectory situated farthest to the right in the complex J-plane[11]. We show that this pole does not fit manifestly into the dual model[18]. Indeed, the Pomeranchuk pole has a positive signature*, i.e., according to Eq. (5) its contribution to A(s, t) is symmetric under the interchange of s and u, and therefore it leads to identical (and for $t = 0$, purely imaginary) amplitudes for the scattering of particles and of antiparticles. Thus, the contribution of the Pomeranchuk trajectory to A(s, t) is independent of the s-channel quantum numbers: the baryon number (the contribution is the same for pp- and $p\bar{p}$-scattering), strangeness (K^+p and K^-p), etc., i.e., they do not depend on whether or not there are resonances in the s-channel. Consequently, the Pomeranchuk trajectory is not related to s-channel resonances, and cannot satisfy the duality relation (12). At the same time, the Pomeranchuk pole that describes diffraction scattering seems to be related with the production of a large number of real particles at high energies, i.e., with cuts in the s-plane. Therefore, the dual approach is usually applied to all Regge poles with the exception of the vacuum (Pomeranchuk) pole, i.e., to the trajectories ρ, R, ω, f, Δ, N, etc. which are known to be almost rectilinear†.

5. Duality and Experiment

We now consider the total cross sections for the interactions of different particles. The unitarity condition relates the total cross section to the imaginary part of the elastic scattering amplitude in the forward direction via the optical theorem:

$$\sigma_{tot}(s) = \frac{\operatorname{Im} A(s, 0)}{s}. \qquad (15)$$

In the reactions of elastic pp-, pn-, K^+p-, and K^-n-scattering (in distinction from $\bar{p}p$-, $\bar{p}n$-, K^-p- and K^-n-scattering) there are no resonances in the s-channel, and therefore the total contribution to the imaginary

*If the Pomeranchuk trajectory had a negative signature, the total interaction cross sections of particles and antiparticles would have opposite signs, which is absurd.

†At present it is not known whether there are any resonances on the vacuum (Pomeranchuk) trajectory.

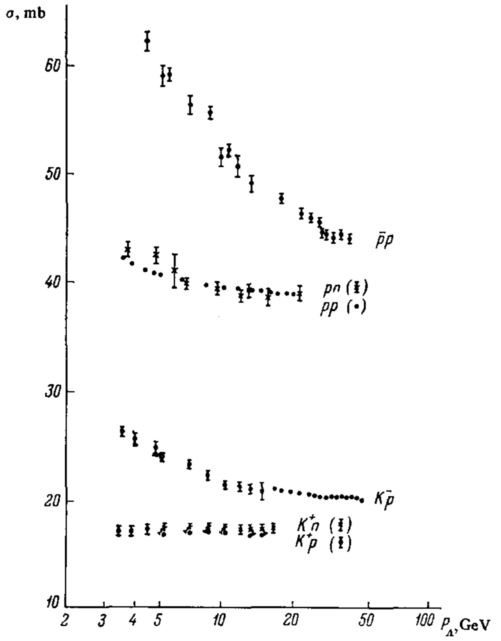


FIG. 9. Total cross sections for the interactions: K^+p [19], K^+n [20], pp [20], pn [21], K^-p [19,22] and $\bar{p}p$ [19,22].

part of the amplitude from all the Regge poles, with the exception of the Pomeranchuk pole, must vanish, according to the relation (12). Consequently, the imaginary part of the amplitudes of these reactions is, already for relatively low energies, determined only by the contribution of the Pomeranchuk (vacuum) pole, and the total cross sections must become independent of the energy and charge state of the colliding particles, i.e.,

$$\sigma_{\text{tot}}^{pp}(s)=\sigma_{\text{tot}}^{pn}(s)=\text{const},\quad \sigma_{\text{tot}}^{K^+p}(s)=\sigma_{\text{tot}}^{K^+n}(s)=\text{const}. \tag{16}$$

Experimentally, these relations are well verified (Fig. 9). From the standpoint of Regge poles the equations (16) represent the so-called "exchange degeneracy" of the various pairs of trajectories

$$\alpha_\rho=\alpha_R,\quad \alpha_\omega=\alpha_f,\quad \beta_\rho=\beta_R,\quad \beta_\omega=\beta_f. \tag{17}$$

Indeed, consider, for instance, the imaginary part of the difference of the amplitudes for pp- and pn-scattering, which at high energies is determined only by the contributions of the Regge ρ- and R- trajectories. The duality relation implies that this contribution vanishes:

$$\operatorname{Im}(A_{pp\to pp}-A_{pn\to pn})=\beta_\rho(t)\left(\frac{s}{s_0}\right)^{\alpha_\rho(t)}-\beta_R(t)\left(\frac{s}{s_0}\right)^{\alpha_R(t)}=0.$$

This leads to the equalities (17) for ρ- and R-trajectories. Similarly one can derive the relations between the ω- and f-trajectories. As noted above, such a behavior of the trajectories is in good agreement with the observed resonance spectrum (cf. Fig. 2).

III. THE VENEZIANO MODEL

1. Fundamental Properties of the Veneziano Model

An important role in the development of the dual models was played by Veneziano's paper[1], where he proposed an explicit form for the function A(s, t), exhibiting the properties of analyticity, crossing-symmetry and duality.

Let us consider the totally crossing-symmetric amplitude for the reaction $\pi^0\pi^0\to\pi^0\pi^0$. In addition to the Pomeranchuk trajectory, the f-trajectory contributes to this amplitude. The f-trajectory passes through the f-meson of mass m_f = 1250 MeV and $J^P = 2^+$, and is taken to have the linear form (8). Then all reaction channels will have resonances situated on the f-trajectory. Veneziano[1] has proposed the following representation for the amplitude of such a process:

$$V(x,\ y)=\frac{\Gamma(1-x)\,\Gamma(1-y)}{\Gamma(1-x-y)}=(1-x-y)\,B(1-x,\ 1-y), \tag{18}$$

where

$$A(s,\ t)=\frac{\beta}{\pi}\left[V(\alpha(s),\ \alpha(t))+V(\alpha(u),\ \alpha(t))+V(\alpha(s),\ \alpha(u))\right],$$

$\Gamma(x)$ is the Gamma-function, B(x, y) is Euler's beta function, and β is a constant. Let us recall some properties of the Gamma-function:

$$\Gamma(z+1)=z\Gamma(z).$$

When z is a positive integer n, $\Gamma(n)=(n-1)!$. The function $\Gamma(z)$ has simple poles for negative integral values of z and z = 0. The formula $\Gamma(1-z)=(\pi/\sin\pi z)\Gamma z$ relates the values of the function for negative and positive values of z. It shows, in particular, that at the poles $z=-n$ the residues of the gamma function are $(-1)^n/n!$. When $|z|\to\infty$ and $|\arg z|<\pi$, $\Gamma(z)\simeq(2\pi)^{1/2}\exp\{(z-\tfrac{1}{2})\ln z-z\}$ (Stirling's formula). This implies that for $|z|\to\infty$ $\Gamma(z)/\Gamma(z+a)\approx z^{-a}$.

The representation (18) of the amplitude A(s, t) has the following properties:

a) Analyticity and crossing symmetry.

b) Poles at the points $\alpha(x)=n$ (n = 1, 2, ...) coming from the poles of the gamma functions in Eq. (18). They correspond to the resonances which lie on the f-trajectory and its daughters. The residues at these poles are polynomials of degree n in the other independent variable. Indeed, $V(\alpha(s),\ \alpha(t))$ can be represented in the form

$$V(\alpha(s),\ \alpha(t))=\sum_{n=1}\frac{\Gamma(1-\alpha(t))(-1)^n}{\Gamma(1-\alpha(t)-n)\,(n-1)!\,(\alpha(s)-n)}=\sum_n\frac{C_n(t)}{\alpha(s)-n}, \tag{19}$$

where $C_n(t)$ is a polynomial of degree n in the variable t. One can write $V(\alpha(s),\ \alpha(u))$, $V(\alpha(u),\ \alpha(t))$ in a similar form. Taking into account that the cosine of the scattering angle $z=\cos\theta_s$ is, according to (6), linearly related to the variables

$$t,\ u:\ z=1+\frac{2t}{s-4\mu^2}=-1-\frac{2u}{s-4\mu^2},$$

we derive that the residue at the pole $\alpha(s)=n$, $C_n(t)+C_n(u)$, is a polynomial in the variable z:

$$C_n(t)+C_n(u)=\begin{cases}C_n'(z), & \text{n even;}\\ C_{n-1}'(z), & \text{n odd.}\end{cases}$$

Expanding $C_n'(z)$ in terms of Legendre polynomials, in order to obtain expressions for resonances with definite spins, cf. (1), we find that for $\alpha(s)=n$ there are resonances with all even spin values, ranging from 0 to n and situated on Regge trajectories. The structure of the resonances is represented in Fig. 10.

c) Regge asymptotic behavior in the whole complex plane, with the exception of the real axis. We show that for $|s|\to\infty$, $|\arg s|>0$ the amplitude (18) has the form (5), corresponding to the exchange of a Regge pole

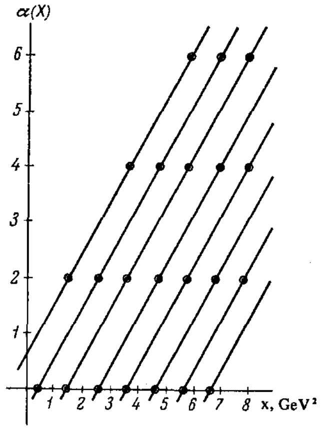


FIG. 10. The structure of the resonances for the $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude in the Veneziano model.

("reggeon") $\alpha(t)$ in the t-channel. Utilizing the fact that $\Gamma(z)/\Gamma(z+a) \simeq (z)^{-a}$ for $|z| \to \infty$, $|\arg z| < \pi$, we find that for fixed t and $|s| \to \infty$

$$A(s,\ t) \approx \frac{\beta\,[\alpha(s)^{\alpha(t)} + \alpha(u)^{\alpha(t)}]}{\sin \pi\alpha(t)} \approx \frac{\beta(t)}{\sin \pi\alpha(t)} \left[\left(\frac{s}{s_0}\right)^{\alpha(t)} + \left(\frac{u}{u_0}\right)^{\alpha(t)} \right],$$

where $\beta(t) = \beta/\Gamma(\alpha(t))$, $s_0 = u_0 = (\alpha')^{-1}$.

Making use of the asymptotic behavior of the Γ-function one can also derive that the amplitude (18) does not have Regge behavior for $s \to \infty$ along the real axis:

$$A(s,\ t) = \beta(t) \left(\frac{s}{s_0}\right)^{\alpha(t)} \left[\frac{1+\cos\pi\alpha(t)}{\sin\pi\alpha(t)} + \operatorname{ctg}\pi\alpha(s) + \left(\frac{s}{s_0}\right)^{1-\alpha(t)-\alpha(u)-\alpha(s)} \frac{\Gamma(\alpha(t))}{\sin\pi\alpha(s)\,\Gamma(1-\alpha(s)-\alpha(u))} \right]. \quad (20)$$

In particular, there appears the term cot $\pi\alpha(s)$ which leads to the existence of resonances for $\alpha(s) = n$. It was natural to expect such a behavior, taking into account that $\text{Im}\ \alpha(x) = 0$ and that the resonances are infinitely narrow, i.e., poles on the real axis.

d) Duality. The partial amplitude $V(\alpha(s), \alpha(t))$ can be represented as a series over the resonances in the s-channel or the t-channel:

$$V(\alpha(s),\ \alpha(t)) = \sum_n \frac{C_n(t)}{\alpha(s)-n} = \sum_k \frac{C_k(s)}{\alpha(t)-k}.$$

Similar relations hold for the other terms of Eq. (18). In the present case the situation is somewhat more complicated than in the case of $K^-\pi^+$-scattering which we have discussed earlier, since the u-channel also contains resonances and Regge poles. However the essence of duality manifests itself clearly in the Veneziano model.

It should be remarked that the enumerated properties a)—d) do not determine the amplitude uniquely. A sum of terms of the form

$$\sum_{m,\ n,\ l} C_l^{m,\ n} \frac{\Gamma(m-\alpha(s))\,\Gamma(n-\alpha(t))}{\Gamma(l-\alpha(t)-\alpha(s))}, \quad (21)$$

where m, n, and l satisfy the conditions

$$n \geqslant 1,\quad m \geqslant 1,\quad l \geqslant \min\{m,\ n\},\quad l < m+n,\quad C_l^{m,\ n} = C_l^{n,\ m},$$

has the same properties as the expression (18). One can show[23] that for linear Regge trajectories any dual amplitude can be represented as a sum of terms of the type of (21).

2. Allowance for the Unitarity Condition

The fundamental shortcoming of the Veneziano model is the violation of the unitarity condition. This manifests itself most clearly in the fact that the resonances have zero widths, although they can decay into less massive particles (e.g., $f^0 \to 2\pi^0$). This circumstance is also related to the absence of threshold cuts in the amplitude (18), and the non-Regge behavior (20) for real s.

The simplest way to eliminate these contradictions is to introduce $\text{Im}\ \alpha(x) \neq 0$, corresponding to a finite width of the resonances situated on the trajectory $\alpha(x)$. This preserves the analyticity and crossing symmetry of the expression (18). Cuts will appear in the (s, t, u)-plane starting at the point $s = 4\mu^2$, and the resonances situated on the Regge trajectories acquire nonzero widths. In addition, if $\text{Im}\ \alpha(x) \to \infty$ for $x \to \infty$ (but in such a manner that $\text{Im}\ \alpha(x)/\text{Re}\ \alpha(x) \to 0$), the first two terms in Eqs. (18), (20) yield the correct Regge asymptotic behavior for all s, also on the real axis*:

$$A(s,\ t) = \beta(t) \left(\frac{s}{s_0}\right)^{\alpha(t)} \left[\frac{1+\cos\pi\alpha(t)}{\sin\pi\alpha(t)} - i \right] = \beta(t) \left(\frac{s}{s_0}\right)^{\alpha(t)} \xi(\alpha(t)).$$

Some difficulties arise with the third term in Eq. (18). It turns out that it decreases as $|s| \to \infty$ only for a definite character of the behavior of $\text{Im}\ \alpha(s)$ as $s \to \infty$. Thus, this term decreases exponentially with the growth of s, if $\text{Im}\ \alpha(s) \sim s/(\ln s)^\nu$, $\nu > 1$, as $s \to \infty$[24]. Such an almost linear growth of $\text{Im}\ \alpha(s)$ does not contradict the experimental data. If this is true, then the width of the resonances increases as their mass increases, and the sum of the contributions of these wide, overlapping, resonances leads to a smooth (Regge-type) behavior of the amplitude for large s.

A shortcoming of this approach is the fact that for $\text{Im}\ \alpha(x) \neq 0$ the residues at the poles $\text{Re}\ \alpha(s) = n$ are no longer polynomials in t, i.e., states with all possible orbital angular momenta now contribute to the amplitude.

However, the partial waves with $l > n$ will be considerably smaller (by a factor $\sim \text{Im}\ \alpha/(\pi\,\text{Re}\ \alpha) \sim 10^{-2}$, compared to $l \leq n$.

The introduction of $\text{Im}\ \alpha(x)$ does not solve the problem of the unitarity condition, but only removes the most flagrant contradiction. Let us show, for instance, that in elastic scattering processes unitarity is violated in the Veneziano model even for $\text{Im}\ \alpha(x) \neq 0$. The imaginary part of the amplitude in the physical region of the s-channel, is determined, as before, only by the contributions from the resonances in the s-channel (duality is preserved for $\text{Im}\ \alpha(x) \neq 0$). Therefore in elastic $\pi^+\pi^\pm$, pp- and π^+K^+-scattering, where no known resonances occur, the amplitudes will be real in the physical region. But this contradicts the theorem (15), since $\sigma_{tot}(s) \neq 0$. In the case under consideration this problem is closely associated with the question on what place there is for the Pomeranchuk pole in the framework of dual models.

Thus, at best, one can consider the Veneziano model as a first approximation to the realistic amplitude, and it becomes necessary to consider the corrections which arise from the unitarity condition. Since in relativistic quantum theory in a collision of two particles the pro-

*For $s \to \infty$ along the real axis we have cot $\pi\alpha(s) \to -i$.

duction of an arbitrary number of particles is possible, one must take into account in the unitarity condition the possibility of transitions into all multiparticle states which are permitted by the conservation laws.

3. Generalizations of the Veneziano Model to Multiparticle Processes

From this point of view, a considerable progress of the approach was the generalization of the Veneziano model to the production amplitude for an arbitrary number of particles[25]. The representations derived exhibit the properties of analyticity and crossing symmetry with respect to all the invariant variables, and have poles corresponding to the positions of the resonances lying on Regge trajectories. Whenever the energy of any pair of particles becomes large the amplitude exhibits Regge asymptotic behavior. In addition, these amplitudes automatically satisfy the duality requirement and correspond to the bootstrap principle[5]: all particles may be considered as bound states of other particles. The construction of a Veneziano amplitude for an arbitrary number of particles has raised hopes that a theory will be created where this representation will play the role of a first Born approximation and which will take into account all terms of a perturbation theory series, in the unitarity condition—both the two-particle and many particle intermediate states. In this approach the Pomeranchuk pole should appear as a result of the summation of the many-particle intermediate states in the unitarity condition. A diagram technique has been developed which is analogous to the usual Feynman technique, and which might allow, in principle, to carry through such a program[26]. However, considerable difficulties are encountered in its practical realization (cf., e.g.,[27,28]).

In order to be able to consider the Veneziano representation as a first approximation to the scattering amplitude it is necessary that it give a satisfactory description of the amplitude, i.e., that the corrections related to higher order perturbation in the Veneziano model be small.

Since this question cannot at present be decided completely on a theoretical basis, let us see what correspondence there is between the Veneziano model and the experimental data. If the model describes well the experimental situation then it is reasonable to consider it as a first Born approximation to the actual amplitudes.

4. Comparison with Experiment

The most detailed discussion of the Veneziano model as applied to scattering processes is for scattering of pseudoscalar mesons[29]. It turns out that the simplest form of the Veneziano model (of the type of Eq. (18)), which practically contains no free parameters, describes well the spectrum of known boson resonances and the widths of their decays into $\pi\pi$ or $K\bar{K}$. The only exception is the resonance ρ' of mass ~ 1250 MeV, and quantum numbers $J^P, I^G = 1^-, 1^-$, predicted by the model to lie on the daughter of the ρ-trajectory, and which up to this time was not observed experimentally. It is interesting to note that the relations between the masses and the coupling constants of the resonances that appear in this model are the same in the SU(6) group, or in the quark model[18]. In addition, the model reproduces many of the results obtained earlier in current algebra. It also turns out that in the general case of many-particle production, there is a close relationship between the quark model and Veneziano model[18]. Although quarks may not exist physically, they are nevertheless a convenient mathematical concept, which appears naturally in the construction of dual amplitudes. This latter circumstance is quite important, since it indicates that the internal symmetries of the hadrons (like SU(3), SU(6)) seem to be of dynamic origin, and could be to a large extent determined by the duality requirement.

The pion-pion scattering amplitudes obtained in the Veneziano model describe the totality of presently known data on pion-pion scattering: the pion spectra in the decays $K \to 3\pi$, $\eta \to 3\pi$, $K \to \pi\pi e\nu$, the reaction $\pi N \to N$ near threshold, as well as the same reaction at higher energies under the one-pion exchange assumption[29,31].

The situation is less clear for the processes of πN- and KN-scattering, which involve resonances (Regge poles) of the fermion type, i.e., with half-integer spins. Until now there is no model which describes well both the data on baryon resonances and the scattering at low and medium energies, as well as the data on forward and backward scattering at high energies. It is conceivable that this indicates the necessity to take into account deviations from linearity of the fermion trajectories (in distinction from the boson trajectories, the fermion trajectories may contain terms proportional to $x^{1/2}$).

The Veneziano model does not describe badly the angular and energy distributions of particles which are formed in inelastic processes, such as $K^-p \to \pi^-\pi^+\Lambda$[32], $K^+p \to K^0\pi^+p$, $K^-p \to \bar{K}^0\pi^-p$, $\pi^-p \to K^0K^-p$, etc. (for the construction of the model for these amplitudes the authors have neglected their dependence on the baryon spin, i.e., have considered the baryons as spinless particles).

5. CONCLUSION

Thus, a comparison of the Veneziano model with experiment indicates that (at least for scattering processes of pseudoscalar particles) it can be considered as a good first approximation to the scattering amplitude. The success of this model in describing the experimental data is largely related to the duality property which is proper to the Veneziano representation. Therefore one might hope that further development of models based on the duality principle will have important implications for the theory of elementary particles.

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