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MULTIPLE GENERATION OF HADRONS AND STATISTICAL THEORY

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CONTENTS

"There is still much to be said in C major."
A. Schoenberg*

1. INTRODUCTION

a. Purpose of Article

THE purpose of this article is to rehabilitate the statistical theory of multiple generation as a significant albeit particular element in the theory of collisions of strongly-interacting particles, and to show that it answers a number of urgent questions.

For many years, the attention of the theoreticians was concentrated on binary reactions of the type $A + B \rightarrow A' + B'$, where A′ and B′ can be decaying resonances as well as hyperons. At high energies, however, most collisions constitute truly multiple processes with $n \gtrsim 5$ final particles, and it is precisely their investigations, which for a long time was in the domain of only cosmic-ray physics, which now comes to the forefront also at accelerator energies. We encounter here qualitatively different distinguishing features, and it is natural to recall the statistical theory.

This theory was developed in the papers of Heisenberg,[1] Wataghin,[2] Fermi,[3] Pomeranchuk,[4] and Landau[5] approximately twenty (and more) years ago, when hardly anything was known of the mechanism of strong interactions. It lost its preeminence when it was demonstrated experimentally (in cosmic rays, and ten years later also with accelerators) that the main hypothesis of the theory is not satisfied in the overwhelming majority of cases, namely, the single compound system or the single "melting pot," from which all the final particles should emerge in accordance with this theory, is not produced.

We shall attempt to show that, in spite of this, it is precisely through a better understanding of the mechanism of collision that the statistical theory turns out to be correct and fruitful, although it fails to explain the entire phenomenon. Moreover, it raises also certain fundamental problems (see Chap. 6). All this, however, is subject to two essential conditions:

1. It is necessary to choose carefully an object satisfying the main requirement of the theory, namely a statistical system. It turns out that such objects do exist. Frequently they are merely subsystems of the entire collision-produced system which cannot be described statistically as a whole.

2. It is necessary to use not the initial theory variant proposed by Fermi,[3] but a variant developed by Pomeranchuk[4] (who indeed pointed out this internal contradiction) and free of the internal contradiction of the theory. This approach was attempted in several papers,[6–9] but was subsequently essentially forgotten.† Recently, however, it was demonstrated that it explains splendidly the generation of heavy particles (up to $\widetilde{He}_3$), and its principles were therefore reviewed anew. The difference from the Fermi variant is radical.

A statistical theory "with Lorentz-invariant phase volume" has recently become popular. It is used at accelerator energies, and the shortcomings of the Fermi model are compensated for by selecting the values of a number of arbitrary parameters (see the Appendix for details).

It may seem strange and "old-fashioned" to turn to such a simple, crude, and even naive theory in the age of electronic computers, which make it possible to process multi-particle diagrams of almost any complexity. We shall attempt to show, however, that the physically lucid statistical theory has tremendous advantages, primarily because it is adequate for the problem, since it separates and takes into account the principal physi-

*Arnold Schoenberg–Composer, founder and "God" of the atonal and 12-tone music. This statement was made in the last years of his life.

†The latest reviews of statistical theory date back apparently to 1961 [8-11,16]. Kretzschmar's review [8b] is unique, in particular, in that it devotes considerable space to Pomeranchuk's work. It is emphasized here (as in [6,7,9]) that many discrepancies between the Fermi theory and experiment (the exceedingly large transverse momenta, the excess of heavy and strange particles) are eliminated if the decay temperature is assumed to be low and the volume large, as called for by Pomeranchuk's statistical theory.

cal properties of the process. On the other hand, the rigor of, say, the diagram method is frequently illusory, since it is usually based on an arbitrary choice of the "most essential" diagrams, and this leads to errors. It is not by accident that numerous diagram calculations of the generation of heavy pairs have given quite incorrect results, while the statistical theory has predicted the phenomenon accurately (see Sec. 4c below).

b. History of Problem

Let us recall, however, the ancient and instructive history of the entire subject. In 1936 no one thought as yet of multiple meson generations. The Auger electron-photon showers, now called extended air showers, were discovered, but the cascade theory of Bhabha and Heitler did not exist as yet. It seemed possible, and even more probable, that all the shower electrons are produced at very high energy in a single act. In particular, as indicated by Heisenberg,[1] such a possibility was uncovered by the long discarded variant of the theory of weak interactions (Konopinski and Uhlenbeck), in which the interaction Lagrangian contained higher derivatives of the ψ operators with respect to time. Each derivative adds one more energy multiplier. Hence the strong growth of the interaction with increasing energy. This means that perturbation theory should no longer hold at high energy and multiple generation should arise.* Heisenberg indicated that as they interact with one another, the final particles enter in thermodynamic equilibrium, and their energy has a Planck distribution about some temperature. The entire quasiclassical electron cloud expands and multiples, and thus will cool down to a certain critical temperature at which multiplication stops. Once the Auger shower was interpreted as being a cascade process, this idea of Heisenberg was completely forgotten.

This was followed by accumulation of data on strongly-interacting mesons in cosmic rays. Until Lattes, Occhialini, and Powell discovered pions, such data were not very definite. This did not prevent Wataghin from noting[2] that in multiple generation (of n particles) the principal role in the quantum-mechanical expression for the probability of the process

$$dW_n = \frac{2\pi}{\hbar} |M_n|^2 \prod_{i=1}^{n} \left(\frac{V dp_i}{(2\pi)^3}\right) \delta\left(E - \sum_{1}^{n} E_i\right) \qquad (1)$$

is played not by the matrix element M_n but by the statistical weight of the final state, which depends strongly on the number of particles and on the momenta. The relative probabilities of the different channels should therefore be determined from statistical considerations.

This main idea of the statistical theory was subsequently advanced also by Fermi,[3] and was developed by him in detail. Its entire history is usually said to begin with him.

The idea that the probability of a state is determined by the phase volume is the basis of the microcanonical distribution in classical statistical mechanics. It is therefore natural here and even unavoidable to use thermodynamics. Thermodynamics and even hydrodynamics (see below) "inside the nucleon" seem of course to be a paradox. Thermodynamics, however, is a fully justified consequence of the quasiclassical character of the system, provided only that a single system is formed, as is indeed the main hypothesis of the theory.

*This theory, as well as the experiments "corroborating" it, turned out to be incorrect.

Fermi formulated a very lucid picture. The main premise in his model is that in the collision of two fast nucleons (and similarly of pions) having Lorentz-contracted volumes*

$$V_F = V_0 \frac{M}{E_c}, \qquad (2)$$

$$V_0 \sim \frac{4\pi}{3} \frac{1}{\mu^3} \qquad (2a)$$

(M is the nucleon mass, μ the pion mass, and E_c the energy of one nucleon in the c.m.s.); these nucleons overlap, interact completely, and release their entire energy in the volume V_F (2) (in the c.m.s.). In this volume, thermodynamic equilibrium is immediately established between the degrees of freedom of all the generated particles. For a given total energy of the system

$$W = 2E_c; \qquad (3)$$

we can therefore determine directly from the thermodynamic formulas the composition, multiplicity, and energies of the produced and isotropically-spreading particles.

Fermi's theory was used extensively to explain the experimental data. Strange as it may seem, the results turned out at first to be fairly good for small multiplicities (one or two new particles) and for the low energies then obtainable with accelerators ($E_{lab} \sim 2-3$ GeV). Particularly good results were obtained when S. Z. Belen'kiĭ and A. I. Nikishov calculated[16,17] that the final nucleons can be emitted in the isobar state ($^3/_2$, $^3/_2$) and only then, after decaying, can they produce a pion. Sharp discrepancies were observed, however, on going to larger E_{lab}. We now know the main reason. Indeed, in the overwhelming majority of the collisions the main hypothesis is far from correct, since no single compound system is produced: the incident particle usually jumps through forward, giving up only a fraction of its energy to the production of new particles ("leading particle").

Of course, this can be only one of the causes. Thus, a single system is apparently produced in $N\tilde{N}$ annihilation, and the spreading in the c.m.s. is isotropic, but the particle composition predicted by the Fermi theory is quite incorrect: the K-particle content actually amounts to $\sim 3-7\%$, as against the predicted $\sim 30\%$.

But there is also another cause, namely the internal contradiction of Fermi's two premises: if the primary hadrons interact so strongly that they are completely stopped and give up their energy to the pionic degrees of freedom after covering only a path on the order of $M/\mu E_c$, equal to the thickness of their overlap region (Fig. 1b), then it is inconceivable that the pions produced in the volume V_F (2) would move apart without interacting. It is more likely that they continue to interact and experience mutual transformations until the distance between them increases to a value of the order of the radius $1/\mu$ of the forces, so that when n particles are generated the entire system occupies a volume

$$V_P = nV_0 \qquad (4)$$

(Fig. 1c).

*Here and throughout $\hbar = c = 1$, μ is the pion mass, $\mu \approx 0.138$ GeV.

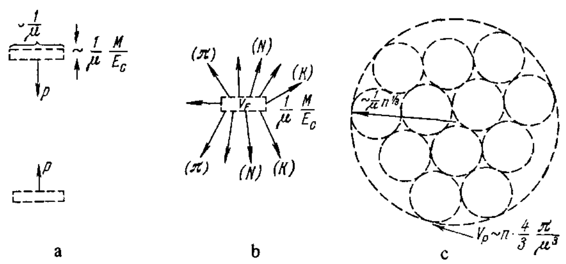


FIG. 1. Collision picture as interpreted by Fermi and by Pomeranchuk. a) Prior to collision, b) spreading of statistical system after collision as interpreted by Fermi, c) final state of the system prior to decay into individual particles in accordance as interpreted by Pomeranchuk.

This circumstance was pointed out, as already mentioned, by Pomeranchuk.[4] The resultant multiplicity was entirely different. In fact, if the thermodynamics of black-body radiation is applied to a statistical system, as was done by Fermi, then in the system of units in which $\hbar = c = k = 1$ (k is Boltzmann's constant) we have for the energy density ϵ, the temperature T, the entropy density s, and the total entropy $S = sV$, which is proportional to the number of particles n (we assume only that the pions have three internal degrees of freedom, and not two as the photon):

$$\varepsilon = \frac{W}{V} = \text{const}\cdot T^4, \quad \text{const} = \frac{\pi^2}{10} \approx 0.99; \tag{5a}$$

$$s = \text{const}\cdot T^3, \quad \text{const} = \frac{4\pi^2}{30} \approx 1.32; \tag{5b}$$

$$S = \text{const}\cdot n, \quad \text{const} = \frac{2\pi^4}{45\zeta(3)} \approx 3.61. \tag{5c}$$

Hence

$$n \sim W^{3/4} V^{1/4}. \tag{6}$$

According to (2) and (3) we have in Fermi's theory

$$n \equiv n_F = 3.8 \left(\frac{E_c}{M}\right)^{1/2} \approx 3.2 \left(\frac{E_L}{M}\right)^{1/4}, \tag{7}$$

but in Pomeranchuk's theory, in accordance with (3) and (4),

$$n \equiv n_P = 5.7 \frac{E_c}{M} \approx 4.0 \left(\frac{E_L}{M}\right)^{1/2} \tag{8}$$

(here E_L is the laboratory energy of the incident nucleon).

Thus, in Pomeranchuk's model the multiplicity is entirely different, and is furthermore larger. We note that in the Fermi theory the system-decay temperature (which determines the average energy of the generated particles) is large at high energies, as follows from (5a), (2), and (3), and increases with energy:

$$T_F \approx 1.2 \left(\frac{E_L}{M}\right)^{1/4} \mu \gg \mu, \tag{9}$$

while in the Pomeranchuk theory, according to (4), it is constant and small:

$$T_P \sim \mu. \tag{10}$$

Actually, the temperature should correspond to a state in which the colliding pions can no longer generate new particles. This occurs when the pions become nonrelativistic. This explains also the larger multiplicity.

Even before Pomeranchuk's paper was published, Landau noted that Pomeranchuk's calculation was insufficient. If the particles continue to interact and experience transformations as they move apart, and if their number is large, then they are further accelerated by the macroscopic pressure in the system, so that the process must be treated hydrodynamically (since the condition for applicability of hydrodynamics is the same as for thermodynamics, namely smallness of the mean free path); of course, this should be relativistic hydrodynamics. Accordingly, Landau developed an extremely brilliant theory.[5,18] E. M. Lifshitz[19] quotes Landau as stating that none of his efforts required as much labor as this theory.

The theory was subsequently developed mathematically by a number of workers. In addition, Japanese theoreticians have shown how the equations of relativistic hydrodynamics are obtained from quantum field theory at large occupation numbers. The result was a highly orderly theory with definite predictions. It was expected to be valid at $E_L \gtrsim 10^{12}$ eV, when n is very large.

Landau accepted the main hypothesis of statistical theory, namely the formation of a common system in thermodynamic (in our case, hydrodynamic) quasi-equilibrium in the volume $V = F_F$ (see (2)) in which the colliding nucleons overlap. He assumed furthermore that only during this stage are shock waves produced and the entropy increases. The subsequent expansion is isentropic. Therefore the entropy, and consequently the final average multiplicity, assume a steady state already in this small volume V_F. As a result, the multiplicity in Landau's theory coincides with the multiplicity in the Fermi model (7):

$$n_L \sim n_F \sim \left(\frac{E_L}{M}\right)^{1/4}. \tag{11}$$

The final decay of the hydrodynamic system into individual particles occurs when a given element of the system expands and cools down to a certain temperature $T = T_c$ at which the particles cease to interact and generate new particles. Consequently, this critical temperature is the same as given by Pomeranchuk:

$$T_L \sim T_P \sim \mu. \tag{12}$$

As in Pomeranchuk's model, it determines the final composition of the generated particles.[18]

It became clear subsequently, however, that the formula (11) for the multiplicity cannot be regarded as reliable. The point is that the viscosity was incorrectly estimated in [18b], where the estimate actually pertained only to the final stage. If the role of the viscosity is estimated by quantum-field methods[20] or from dimensionality considerations,[21] then it can be readily seen that the viscosity, and consequently the increase in entropy, continues to play a role long after the system has expanded beyond the limits of the initial volume V_F. As a result, for the same equation of state as used by Landau, we can obtain in place of (11)[21]

$$n \sim \left(\frac{E_L}{M}\right)^{1/3}. \tag{13}$$

Landau's theory is undoubtedly a brilliant development. If we disregard some of its particular assumptions (the

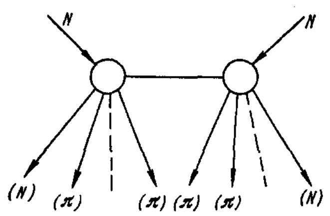


FIG. 2. The peripheral process. The particle exchanged is a pion or a reggeon.

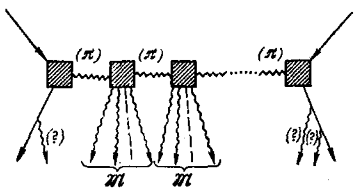


FIG. 3. Multiperipheral process. The particle exchanged is a pion. The irreducible vertices are denoted by shaded squares. The nucleons can be excited in the final state, in which case they decay and emit mesons.

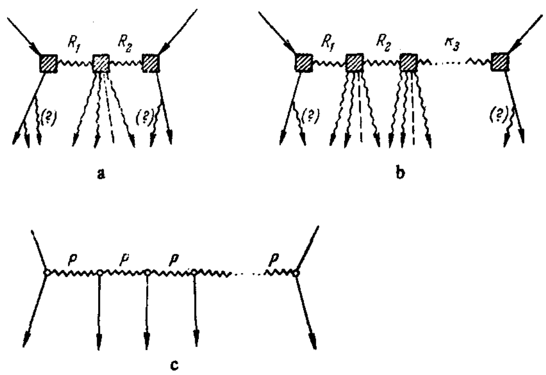


FIG. 4. Formation of clusters or fireballs in reggeon exchange. R_1, R_2 . . . —Regge trajectories (not necessarily vacuum). a) One intermediate cluster; b) multi-Regge process; c) multi-Regge process without formation of clusters and with exchange of vacuum pole (P—Pomeranchuk pole).

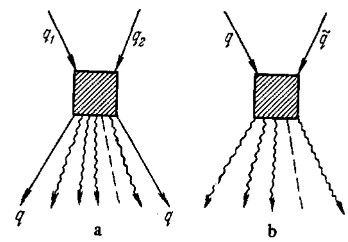


FIG. 5. a) Central collision of two hadrons q_1 and q_2; b) annihilation of a pair of hadrons q and q̃.

choice of the equation of state, etc.), it makes use of only one fundamental hypothesis, namely that a single initial system is produced when high-energy hadrons collide; in all other respects it is exceedingly consistent and well founded.

c. Basic Mechanisms of Inelastic Collisions

Is this basic hypothesis of the theory realized at $E_L \gtrsim 10^{12}$ eV in at least some collisions? This still remains unclear. To the contrary, experiment as well as the theoretical approaches point more and more to the predominance of peripheral (at $E_L \sim 10$–100 GeV) (Fig. 2), and even multiperipheral[22] processes (at still higher energies). Thus, a quantum field theory based on the Bethe-Salpeter equation[23] shows that at very high energies the inelastic collision process is described mainly by Feynman diagrams of the type of Fig. 3, where one-pion exchange connects the irreducible parts (rectangles) having an invariant mass $\mathfrak{M} = \sqrt{s_0} \sim 2$–$4$ GeV. With increasing primary energy, the number of irreducible centers increases logarithmically, but not their average energy—this ensures precisely the correct asymptotic behavior of the total and elastic cross sections and the logarithmic growth of the total multiplicity. All the properties of these irreducible centers—clusters of nuclear matter decaying into final pions—correspond to the properties of fireballs, the existence of which is firmly evidenced by cosmic-ray experiments at $E_L \leq 10^{12}$–10^{13} eV (two fireballs)[24,25,26,116] and at $E_L \sim 10^{11}$–10^{12} eV[27,28] (one fireball), see the review[29] (although these conclusions are still vulnerable and encounter criticism).

An attempt to reduce the experimental data at $E_L \sim 20$–30 GeV in accordance with the phenomenological Regge-pole scheme[30] also leads, with properly chosen numerous arbitrary constants, to a scheme of the type of Fig. 3, but with exchange of reggeized mesons—Regge poles R_i, corresponding mainly to non-vacuum trajectories (Fig. 4).[31] It is important that here, too, it is necessary to introduce clusters, i.e., fireball-type accumulations of generated pions, which cannot be reduced to a system of particles that exchange individual poles peripherally. Van Hove[32] defines a cluster as an aggregate of final particles such that in the rest system of this aggregate the transverse and longitudinal momenta of the individual particle (relative to the collision axis) are of the same order of magnitude. This definition holds also for the fireball. At higher energies, the scheme of Fig. 4a goes over into the multireggeon scheme of Fig. 4b, which differs from the multiperipheral[22] or multifireball[23] scheme of Fig. 3 only that in the Regge approach the exchanged pions are "reggeized"—replaced by pion Regge poles. Thus, the Regge-pole approach to the problem of multiple generation at high energies, initially[30] based on a multiperipheral chain with exchange of only vacuum pole P and with generation of only one or two particles in each vertex (Fig. 4c), evolved gradually (to take into account the decisive contribution of non-vacuum poles and the introduction of fireball-type clusters in the vertices[31]). It differs now from the multiperipheral scheme[22,23] only in the absence of the dynamic equation that follows from the quantum field theory (the Bethe-Salpeter equation), and also in that the pions are replaced by pion trajectories. The latter difference is immaterial for most conclusions (see [33]).

None of these schemes admit of a direct application of the Landau theory,* for which it is necessary to have

*In parallel with Landau's hydrodynamic theory, Heisenberg [34] developed a hydrodynamic theory which is less detailed and is seemingly different in principle. G. A. Milekhin [35] has shown, however, that the two theories are in principle the same, and differ only in the choice of the equation of state.

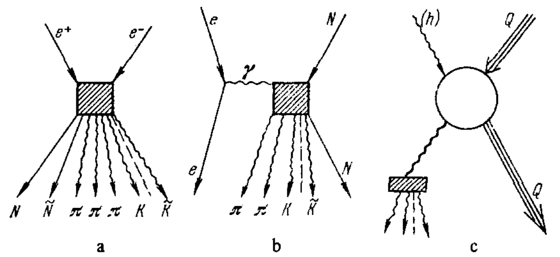


FIG. 6. Different processes of formation of a statistically decaying subsystem. a) Annihilation of a pair of leptons ($e\tilde{e}$ or $\mu\tilde{\mu}$) into hadrons; b) electromagnetic "deep inelasticity" process in the collision of a lepton with a nucleon; c) diffraction excitation of a hadron h (pion, nucleon, etc.) into a state that decays statistically into hadrons (Q—hadron or spectator nucleus).

nonperipheral central collisions of the type of Fig. 5* or the formation of very large clusters (with mass $\mathfrak{M} \gtrsim 20$ GeV). The hydrodynamic theory can be applied to the dispersal of each such cluster. At low energies ($E_L \sim 10$–20 GeV), the central collisions are present in some fraction of the cases (see Chap. 2 below). It is unknown, however, whether they survive at $E_L \gtrsim 10^{12}$ eV, where the Landau theory would be applicable.†

All this became clear only in recent years. Publication of Landau's theory, however, made such a strong impression that Pomeranchuk's work[4] was left in the shadow. Pomeranchuk himself did not like to recall it, and it was almost completely forgotten (but see [6–9]).

Now we see, however, that this was incorrect. Pomeranchuk's scheme should be regarded as a valid limiting case, at low energies, of the same approach that leads at high energies to the Landau theory. If we apply the formulas of Landau's theory the case when the "macroscopic" hydrodynamic velocity does not have time to grow to a value close to the speed of light (even by the instant of decay into individual particles), we obtain the (rather crude) condition*

$$n \leqslant 10. \tag{14}$$

Thus, if the statistical system is small, there is still no need to take into account in it the hydrodynamic motion of the individual sections (and there is also no noticeable Lorentz contraction of each of the final particles). The stage of isentropic expansion cannot be separated in this case from the initial dissipative stage, and neither (11) nor (13) can be regarded as a valid formula for the multiplicity. Thus, if relations (14) holds, all the conditions for the applicability of Pomeranchuk's statistical model are satisfied.

At the same time, the number $n \sim 10$ is already large enough for the statistical laws to manifest themselves (with an appropriate accuracy, say on the order of $1/\sqrt{n}$).†

d. Conclusion for Further Consideration

The entire sequel is based on the existence of a number of objects that satisfy the conditions of applicability of the statistical theory. According to (3), (8), and (14) these should be statistical systems with total energy (in their rest system) $W \lesssim nW/3 \sim 3$–6 GeV. Such objects are (a more detailed discussion follows):

1. Possible central collisions of the type of Fig. 5 at $E_L \sim 5$–25 GeV (Fig. 5a).
2. The object produced by $N\tilde{N}$ annihilation at rest or at $E_L \sim 5$–25 GeV (Fig. 5b).
3. A fireball or cluster with mass of the same order in the "statistically-peripheral" collision process of the type of Figs. 3 and 4.
4. A strongly excited system produced in any hadron vertex, for example in the vertex of the simple peripheral process of Fig. 2 or in the lepton annihilation of Fig. 6a, in an electromagnetic process with "deep inelasticity" of Fig. 6b, in elastic scattering of a hadron ("diffraction generation"), Fig. 6c, etc.

We shall see that actually many characteristic features of these systems can be explained naturally within the framework of the described model.

We use all the relations of the statistical theory in their literal sense and consider, in particular, the ex-

*The question of what distinguishes central collisions (Fig. 5) from peripheral ones (Figs. 2–4) is quite confused. It is frequently assumed intuitively that in central collisions the multiplicity is larger than in peripheral ones, the impact parameter is smaller, and accordingly the angular momenta l are smaller (isotropy of the expansion). As seen already from the foregoing, we choose a different attribute as the basis, viz., in a central collision all the particles form in the final state a single system of the type of the Van-Hove cluster, in which no particle is singled out kinematically (for details see the start of Chap. 2 below). Although this corresponds as a rule to the just-mentioned segregation of the collisions by the effective values of l and simultaneously by multiplicities, there is no uniqueness here [36]. In particular, for the diagram of Fig. 4c, the larger the multiplicity (number of vertices), the larger the effective l, and the less isotropic the expansion. Such a diagram, however, is realized in a negligible fraction of the collisions. On the other hand, the fireball or the cluster is a result of a central collision of particles, possibly virtual ones. The following definition is therefore also possible: a central collision is one resulting in a single compound system say a single cluster. (In the case of ultrahigh cluster energy, however, the hydrodynamic dispersal can result in $p_\parallel \gg p_\perp$.) This criterion, which we shall indeed use, is defined more accurately in Chap. 2.

†Relatively recently, Hagedorn [37] developed in a series of papers a theory of the hydrodynamic type, differing from Landau's (and Heisenberg's) theory in that it takes non-central and peripheral collisions into account phenomenologically. In this theory, particle generation is calculated by means of statistical formulas for each given element of the overlapping volumes of the colliding hadrons. On the other hand, the c.m.s. velocity of the entire element is described by a velocity-distribution function chosen to agree with experiment. The author believes that the temperature of each element does not exceed a certain limit T_0, and that the generated particles move apart without interacting. Comparison with experiment yields $T_0 \sim \mu$. It is usually seen that the result of such a calculation is exactly the same as when the Pomeranchuk statistical theory is used: the initial temperature of the element can be high, but the interaction and cooling upon expansion cause the spectrum and composition of the particles to be determined by the final temperature $T_c \sim T_0 \sim \mu$. It is therefore not surprising that the composition, the distribution with respect to $p_\perp$, and other characteristics of the process turn out to be the same as in our analysis. Furthermore, allowance for the motion of the statistical subsystems along the collision axis yields additional results. They are obtained if two unknown distribution functions of the "hydrodynamic" velocities are successfully chosen and constitute a phenomenological element of the theory.

*This condition was written out for hadron-hadron collisions. In hadron-nucleus or nucleus-nucleus collisions at such low hydrodynamic velocities, the multiplicity is much larger and the statistical theory is valid for larger n.

†The question of the accuracy of the statistical formulas for limited n becomes clear if the statistical weight calculated by the Bose and Fermi statistics formula is compared with the exact formulas. The most general calculations known for them are those of I. L. Rozental' and V. M. Maksimenko [111] (see also [17,67]).

pansion of the system in time and in space. The principal justification for such an approach is that a system with a large number of degrees of freedom is quasiclassical. Its seeming naivete and crudeness are offset by its adequacy and by the fact that the theory contains only one indeterminate parameter, chosen from comparison with experiment—the decay temperature T_c, the order of magnitude of which is furthermore known beforehand (Eq. (10)).

At the same time, as we shall see in Chap. 6, by investigating the statistical model we meet head-on the principal question of the role of the statistical and dynamic principles in the elementary act. It is not excluded that the statistical treatment touches upon fundamental problems of quantum field theory.

2. FORMATION OF STATISTICAL SYSTEM IN THE PROCESS OF FAST-HADRON COLLISION

a. Separation of the Statistical Subsystem

As we have already emphasized, the success of statistical theory is determined by the correct choice of the object satisfying the conditions under which the theory is valid. We must therefore first discuss the data on the generation mechanisms.

It is apparently widely admitted that these mechanisms are manifold (see, e.g., the review [38], in which it is emphasized that many experimental data demonstrate the superposition of distributions of different types, corresponding to collisions with different inelasticity coefficients). The statistical theory considers only the decay of the resultant statistical system, the probability (cross section) of whose production lies outside the competence of the statistical theory and should be estimated by other methods, at the present primarily from experiment (such a "statistically-peripheral" approach was used already long ago in conjuction with one-meson exchange[39-41]).

A characteristic attribute of a statistical system is that the particles contained in it exchange large 4-momenta k. More accurately, for the particles i and j, the Mandelstam variables s_{ij} and $-t_{ij}$ are of the same order (the momentum transfer is of the order of the particle energy):

$$s_{ij} \sim -t_{ij} \equiv k_{ij}^2 \equiv \Delta_{ij}^2 \qquad (15)$$

(we also show the symbols k and Δ which are usually employed in such problems in place of t), whereas in peripheral collisions $s \gg -t$. Accordingly, the particles move away from the statistical system isotropically (or approximately isotropically), whereas in peripheral collisions they are collimated. The criterion (15) is the basis of the method proposed by Dremin[42] for identifying the type of the process. Namely, after measuring for each individual collision the momenta of all the emitted n particles, arranged in a known sequence, i = 1, 2, . . . , n, one calculates the 4-momentum $k_{i,i+1}$ transferred from the first i particles to the remaining n − i particles. The values of $k_{i,i+1}^2$ are then plotted against i and a continuous curve is drawn through the points. In those places where $k_{i,i+1}^2$ is small, peripheral interaction takes place, as expected. Figure 7 shows an example[43] illustrating the method of setting a definite Feynman diagram in correspond-

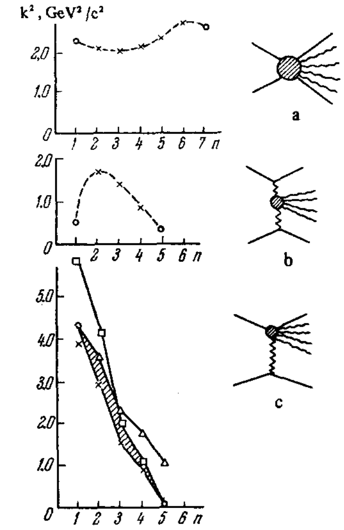


FIG. 7. Separation of peripheral interactions (small k²) in individual acts of multiple generation and assignment of a definite Feynman diagram to the given act in accordance with [42,43]. Shaded circle—irreducible vertex (cluster, fireball).

Table I. Distribution of the ratios $|s_{i,i+1}/t_{i,i+1}|$ in pN collisions with $E_L \sim 200$ GeV[44]

$s_{i,i+1}/t_{i,i+1}$	0 – 0.5	0.5 – 1.0	1.0 – 2.0	> 2.0
Fraction of cases, %	61	11	11	17

ence with the given collision.* Table I gives the results of analogous measurements of $k_{i,i+1}^2$ and $s_{i,i+1} = -(p_i + p_{i+1})^2$, where p_i and p_{i+1} are the momenta of the i-th and (i + 1)-st particles for pN collisions at ~200 GeV (in cosmic rays).[44] We see that a peripheral interaction of the particles (for which we assume arbitrarily, by way of an example, the criterion $|s_{i,i+1}t_{i,i+1}| > 2.0$) can occur only in 17% of the cases, while the bulk of the particles interact nonperipherally, and the particles are generated in the form of a statistical system (cluster) customarily called fireball.

The number of measurements of this type, however, is very small and the presence of a statistical system is frequently established in accordance with simpler attributes, primarily by finding for the aggregate of particles a reference frame in which the particles move apart isotropically or almost isotropically (by virtue of which they satisfy the Van Hove definition of a cluster).

Figure 8 shows very instructive results of a study of the π^+p collision process at E_L = 8 GeV.[45]

The longitudinal and transverse momentum components $p_\parallel$ and $p_\perp$ of the individual generated pions are shown in the common c.m.s. separately for different multiplicities. We see that at low multiplicities (1–2 generated particles) the new particles move in the same direction as the primary ones. But with increasing n, the distribution becomes more and more isotropic, with $p_\perp \sim p_\parallel \sim 0.4$ GeV/c.

According to Van Hove's definition of a cluster (see Sec. 1c above),[32] the aggregate of these particles can be regarded as a cluster at rest in the common c.m.s. Such clusters made up of a small number of particles

*Of course, owing to fluctuations (the number of particles is small), this method does not ensure a reliable correspondence. An improved correlation method of separating fireball-type clusters was recently proposed and used in [108].

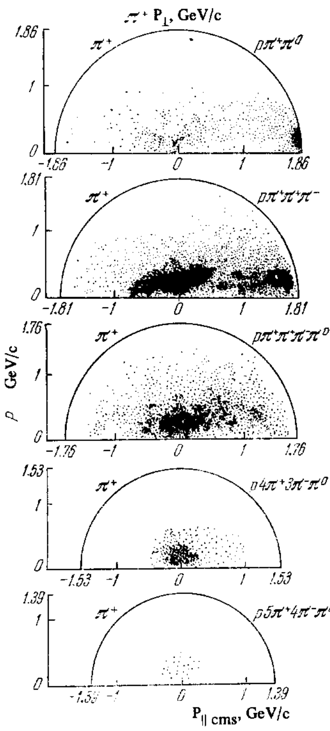


FIG. 8. Distribution of the final particles in the c.m.s. with respect to the longitudinal and transverse momenta $p_{\parallel}$ and $p_{\perp}$ (relative to the collision axis) vs. the multiplicity n_{π} in π^+p collisions (E_L = 8 GeV) [45]. Each point is an individual particle. We see that at small n_{π} the particles are collimated, but at $n_{\pi} \gtrsim \langle n_{\pi}\rangle$ (here $\langle n_{\pi}\rangle \approx 6$) the particles move apart with $p_{\parallel} \sim p_{\perp}$, as should be the case for a statistical system at rest in the c.m.s.

can also be expected whenever the final number of particles is small, n = 3–4. For these cases, Van Hove developed an interesting method of analyzing the experimental data.[32] This method was already applied successfully to the reactions $\pi + p \to 2\pi + N$ and $\pi + p \to 3\pi + N$, and also to $K + p \to K + \pi + N$. The first results show that processes with diffraction formation of small clusters (2π and 3π) play a predominant role. We are primarily interested, however, in much larger n. For these, the method of [32] apparently becomes unsurmountably complicated, and a conclusion that the diffraction mechanism of cluster formation plays a predominant role would be premature.

Thus, starting with a medium multiplicity (for Fig. 8 this is $\langle n_{\pi}\rangle \sim 6$), we can regard all or almost all the generated pions as a statistical system at rest in the

Analogous results were obtained also in other investigations where, to be sure, the measurement of the individual kinematic characteristics was not as thorough: in pN collisions (at E_L = 24 and 21 GeV),[46,47] πN collisions,[58] etc. An analysis of accelerator data in [48] led to the same conclusions; see also the reviews,[29,65] as well as Figs. 10 and 19 and their analysis below.

Thus, if we are interested in the predominant medium and large multiplicities (and not the rare cases of the generation of one to three particles, on which the attention was focused for many years in the comparison of the theory, say the Regge-pole model, with experiment), then we can apparently state that the characteristic process in hadron collisions is the formation of a statistical system, usually detached from the primary particles that lose a small fraction of their energy, and that at accelerator energies this system is at rest in the common c.m.s. This is one of the most important results of the experiments of recent years.

At $E_L \gtrsim 100$ GeV, we can speak of a fireball with mass $\mathfrak{M} \sim 2$–4 GeV, and at lower energies, of an "immature" fireball, a cluster.

b. Average Energy of Statistical Subsystem Formed in Peripheral Interactions

Experiment shows that in pN collisions at $E_L \sim 20$–30 GeV[38,49] (and apparently also for $E_L \sim 200$ GeV[28]) the average energy W of such a system in its c.m.s. is of the order of $0.3\sqrt{s}$, where s is the invariant energy of the colliding particles.

If we introduce the coefficient K of inelasticity relative to the energy transferred to this system, we can write (using the obvious notation)

$$W \approx \langle E_{\text{stat. syst. c.m.s.}}\rangle = \langle K\rangle \sqrt{s} \approx 0.3\sqrt{s}. \qquad (16)$$

However, the distribution of the values of K is smeared out to an unusual degree (see, e.g., [28,38,50]).

The inelasticity coefficient can be defined in different ways. Thus, if we take it to mean the entire energy $K_0\sqrt{s}$ consumed by the newly produced particles (including, e.g., the fast pion produced by the decay of the nucleon that is excited and moves on forward), then $K_0 \leq K$. We can apparently assume $\langle K_0\rangle \approx 0.4$.

It is necessary, however, to introduce two essential complications into this picture. First, the fireball still moves in the c.m.s., and as a unit it acquires not only a transverse momentum with $p_{\perp}^{\text{f.b.}} \sim 1$ GeV,[51] but also a longitudinal motion in the common c.m.s., with a Lorentz factor γ that increases with the energy. Nevertheless, at moderate energies we have $\bar{\gamma} \approx 1$. Only at $E_L \sim 200$ GeV do we get apparently $\bar{\gamma} \approx 1.2$, and at $E_L \sim 1000$ GeV we have $\bar{\gamma} \approx 1.5$[28,52] (these estimates were made for cosmic rays, where the measurements are very difficult, and the figures presented should be regarded as approximate). Owing to this motion, the value $E_{\text{stat.syst.c.m.s.}}$ given by formula (16) coincides with the mass $\mathfrak{M}$ of the statistical system only at not too high an energy (at $E_L \lesssim 50$ GeV), and we can still assume that

$$\mathfrak{M} = K\sqrt{s}, \qquad (17)$$

$$\langle K\rangle \sim 0.3. \qquad (18)$$

Second, and this is more important, there is always an admixture of particles generated in the decay of the primary particle which is excited during the collision process and moves on ahead. At low multiplicities they predominate (cf. Fig. 8). At medium and large multiplicities their relative contribution can be small. On the one hand, however, it can be different in πN and NN collisions. On the other hand, it is precisely these particles that have the highest energy in the laboratory system and can play the principal role in the corresponding experiments.

It is important to emphasize that if the excitation energy of this particle reaches several GeV, then its decay can also be considered statistically. It does not differ appreciably from the fireball; for example, if we deal with nucleon excitation, then the baryon number is equal to unity and not to zero as in the case of the clusters of Fig. 8.

Excitation can be via exchange of a pion or meson pole, as in Figs. 2–4, or else by diffraction, with elastic scattering (Fig. 6c) (e.g., but not necessarily, with exchange of a vacuum pole). Figure 4c, in essence, also describes diffraction excitation of a special type, which at one time greatly attracted the theoreticians' attention, but which plays a minor role on the whole; according to some estimates it accounts for about 1% of all the inelastic collisions.

In diffraction excitation of an incoming particle of mass m and energy E_L to a state with mass $\mathfrak{M}$, there is a certain limitation on the longitudinal momentum transfer $q_{\parallel}$, namely $q_{\parallel}\mu^{-1} < 1$.[54] It can be written in the form[55]

$$q_{\parallel} = \frac{\mathfrak{M}^2 - m^2}{2E_L} < \mu. \qquad (19)$$

Therefore at an energy E_L an excitation is possible to a state with $\mathfrak{M}$ equal to (assuming that $\mathfrak{M}^2 \gg m^2$)

$$\mathfrak{M} \leqslant \sqrt{2\mu E_L} \approx 0{,}5\sqrt{E_{L,\text{GeV}}}, \qquad (20)$$

where $E_{L,\text{GeV}}$ is in GeV. Consequently, $\mathfrak{M} \sim 2$ GeV is possible at $E_L \sim 16$ GeV and $\mathfrak{M} \sim 4$ GeV at $E_L \sim 70$ GeV. Indeed, in $\pi^- N$ collisions at 16 GeV, the diffraction splittings $\pi \to 3\pi$ and $\pi \to 5\pi$ were observed, and in the latter case it was found, in accord with (20), that $\mathfrak{M} = 1.8$–1.9 GeV.[56] Such a system has in the main the same properties as the statistical system produced in $N\bar{N}$ annihilation at rest (Fig. 5b), where $\mathfrak{M} = 1.88$ GeV. In particular, one can expect $K\bar{K}$ pairs to be present here in $\sim 5\%$ of the cases, as is the situation in annihilation (see Sec. 5e below) (there are still no experimental data on this question). At $E_L \sim 70$ GeV, diffraction excitation in πN collision, as we see, can give a statistical system of the same mass as peripheral production of a fireball.

As already mentioned, an analysis of processes with few particles (the number of newly produced pions is 1–2, $E_L \sim 10$–16 GeV) by the Van Hove method[32] leads to the conclusion that a predominant role is played by diffraction excitation of the incident particle to the "cluster state." We see that at large E_L this process can lead to the formation of large clusters, on the order of the fireball mass, $n_\pi \sim 8$. Indeed, the cross section for the diffraction dissociation of a pion into three and five pions increases rapidly with energy.[57] This increase can be attributed to the increase of the phase volume for a cluster of mass $\mathfrak{M} \sim 2$ GeV. Nevertheless, this process is not the principal channel of multiple generation at $n \sim \langle n \rangle \gg 1$.*

c. Relative Contribution of Particles from a Statistical System (Fireball) at Rest in the c.m.s. and from Excitation of the Incident Particle

This contribution has not yet been determined. It is clear from Fig. 8 that in πN collisions at 8 GeV, when $n < \langle n \rangle$, this contribution is small. But in the same processes at $E_L \sim 60$ GeV, as shown in [58], from which Fig. 9 is borrowed, the angular distribution of relativistic charged particles in the c.m.s. (583 collisions gathered together), is anisotropic even after eliminating the diffraction-generated groups of three and groups of five. This distribution can be broken up into three parts:

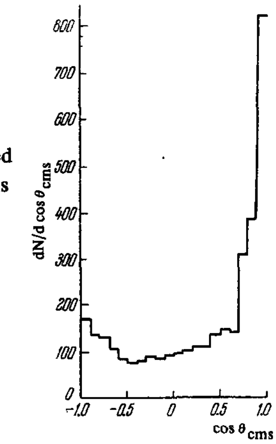


FIG. 9. Angular distribution of charged particles in the c.m.s. (emission angle θ_{cms} relative to the momentum of the primary pion in $\pi^- N$ collisions at $E_L = 60$ GeV; summary plot for 583 events) [58].

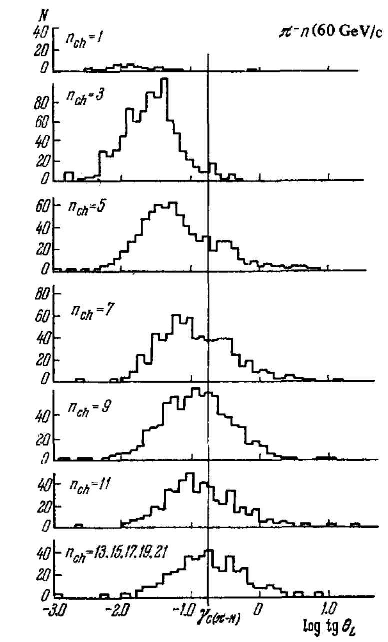


FIG. 10. Angular distribution of outgoing particles (relative to log tan θ_L, where θ_L is the particle emission angle in the laboratory frame) for $\pi^- N$ collisions, $E_L = 60$ GeV, at different multiplicities n_{ch} [58]. $\gamma_c(\pi^- N)$ indicates the Lorentz factor of the c.m.s. It is seen that the particles are produced mainly in the cluster, and the velocity of the cluster in the c.m.s. decreases with increasing n_{ch}.

a) isotropic part (approximately half the particles), which can be ascribed to the decay of the statistical system at rest in the c.m.s.; b) a large forward peak, which can be ascribed to excitation and decay of the incoming pion, and c) a small backward peak, which can be similarly ascribed to the decay of the nucleon excited by collision. It should be assumed in such an interpretation that the probability of excitation of the vertex in peripheral πN collision is large.

Even more instructive in this sense is Fig. 10, where the angular distribution of the particles in $\pi^- n$ collision at $E_L = 60$ GeV is plotted against log tan θ_L (θ_L is the emission angle in the laboratory system) separately for different numbers n_{ch} of the charged particles. In such a scale, a Gaussian-type curve denotes isotropic dispersal of the particles in a certain reference frame. The position of the maximum of the curve, $\theta_{L,\max}$, gives the value of the Lorentz factor of this system, γ_0, in the lab-

*Incidentally, the opposite view, not confirmed by experiment, has also been advanced, but it is difficult to agree with it. It could be based only on predominance of the role of Pomeranchuk-pole exchange not only on for elastic but also for inelastic collisions. It has been shown long ago, however, [113,114] that at the actually existing multiplicities (and accordingly values of $\mathfrak{M}$), the value of $\cos\theta_t$ in the transverse channel is far from large (it is of the order of unity), the condition for the Regge asymptotic behavior is not satisfied, and the hierarchy of the trajectories is completely disrupted as $s \to \infty$: the contribution of the vacuum pole is by far not the largest.

oratory frame, $\gamma_0 = -\log\tan\theta_{L,max}$. The straight line $\gamma_{c(\pi-N)}$ shows the value of the Lorentz factor of the c.m.s. We see that even at $n_{ch} = 3$ the expansion is approximately isotropic in a system moving forward with considerable velocity in the c.m.s. ($\gamma_0 \neq \gamma_{c(\pi-N)}$). This agrees fully with the conclusion, based on the Van Hove method, that the cluster production by diffraction excitation of the incoming pion predominates at such small values of n_{ch}. With increasing n_{ch}, an increase takes place in the contribution of the second maximum (which is also, at least approximately, of the isotropic type), which describes the expansion of a system at rest in the c.m.s. (its γ_0 coincides with $\gamma_{c(\pi-N)}$). At the average multiplicity for these collisions, $\langle n_{ch}\rangle = 6.24 \pm 0.15$ (see the curve for $n_{ch} = 7$), this contribution is already decisive. At $n_{ch} \gtrsim 13$, expansion is almost symmetrical in the c.m.s.

It is easy to see that this limiting case corresponds precisely to a fully central collision. In fact, here $\sqrt{s} \sim 10.5$ GeV, and therefore in a central collision we are left with $\sqrt{s} - m_N \sim 9.6$ GeV for pion generation (taking into account the rest energy of one nucleon). According to (32) (see below) in the statistical theory considered by us, this makes it possible to obtain on the average $\langle n_\pi\rangle \sim 9.6/0.43 \approx 22$ pions, i.e., ~14 charged pions, which agrees with $n_{ch} \sim 13$. In other words, this is the case when the initial particles lose practically all the kinetic energy and become part of the statistical system.

In NN collisions the probability of an equally high excitation of the incoming particles is apparently not so large (this is seen already from the fact that the area under the rear peak—from the excited nucleon—on Fig. 9 is smaller by a factor 5–6 than that under the front peak, which we ascribe to pion excitation). The same can be seen also in Fig. 11, which shows the momentum distribution of the particles generated in pN collisions at $E_L \sim 200$ GeV[27] (it is constructed in a reference frame in which the total momentum of the forward-moving particles is equal to the total momentum of the backward-moving particles; this system does not differ greatly from the c.m.s.). We see that the overwhelming majority of the particles are described by a Planck distribution (continuous curve) at $T \sim \mu$ (more accurately, $T = 0.8\mu$ with accuracy $\pm 0.2\mu$), and thus can be ascribed to fireball decay. There are few fast particles outside this curve.

Koshiba[59] finds that at $E_L \sim 10^3$ GeV the excitation of the nucleon to a state with mass ~2 GeV (he called this the "Aleph" state) occurs in each collision. This does not contradict the statements made above.

d. Central Collisions and Their Fraction

The values of the inelasticity coefficient K are continuously distributed. Therefore central collisions (Fig. 5a) in which the initial particles lose the greater part of their energy and are energywise on a par with the generated particles cannot be separated strictly from the peripheral ones. Such a separation can be carried out approximately, however. Moreover, we shall show that it is quite distinctly observed in experiment.

The criterion for the separation of central collisions can be based precisely on the fact that in this case the nucleons constitute part of a statistical system that is at rest in the c.m.s. at accelerator energies, so that their average kinetic energy after the collision is (in the c.m.s.) 3T/2, and since $T \sim \mu$, it follows that $3T/2 \approx 0.2$ GeV. Consequently, the average fraction K_{kin} of the kinetic energy lost by the nucleon satisfies the relation

$$(1-K_{kin})(E_c-m_N) \approx \frac{3}{2}T, \qquad (21)$$

$$K_{kin} \approx 1-\frac{3}{2}T/(E_c-m_N), \qquad (21a)$$

where $E_c = \sqrt{s}/2$ is the energy prior to the collision. Inasmuch as $K_0E_c = K_{kin}(E_c - m_N)$, the total inelasticity coefficient K_0 is connected with K_{kin} by the relation $K_0 = K_{kin}[1 - (m_N/E_c)]$. For central collisions

$$K_0 \gtrsim 1-\frac{m_N+\frac{3}{2}T}{E_c}. \qquad (21b)$$

The distribution of the protons from pp collisions with $E_L = 12.5$ GeV ($E_c \approx 2.5$ GeV) with respect to the transverse momenta $p_\perp$ was investigated in [62] separately for different K_{kin}. In this case, for simple collisions, we should have $K_{kin} \gtrsim 0.87$ in accordance with (21a) (and $K_0 \gtrsim 0.55$ according to (21b)). Actually the experimental distribution with respect to $p_\perp$ (see Fig. 14) for such protons (and slower ones) turned out to differ considerably from the distribution of protons with smaller K_{kin} (steeper). As will be shown in Sec. 4e, it agrees exactly with the prediction of the statistical theory at $T \approx 0.92\mu$. Therefore the fraction of such collisions can be assumed to be also the fraction of the central collisions. According to Fig. 14 (which is a reproduction of Fig. 2 of [62]), this fraction ξ_{centr} can be roughly estimated at 0.2. We shall assume quite roughly that

$$\xi^{pp}_{centr} \sim 0.1 \qquad (E_L = 12.5 \text{ GeV}). \qquad (22)$$

Older and less reliable estimates of the fraction of such collisions at $E_L \lesssim 20$–30 GeV also gave figures on the order of 0.1 (in pN collisions; see, e.g., [60,61]).

Considerations have been advanced,[63] according to which this fraction should decrease with energy. Unfortunately, there are no experimental data at high energies (although it is probably no longer difficult to obtain them by performing the same measurements as in [62], and employing the analysis presented above).

No similar estimate has been made as yet for ξ_{centr} in πN collisions. A rather indirect estimate can be obtained from [58], where it is shown that events with very large multiplicity, $n_s > 10$ (whereas $\langle n_s\rangle \approx 5$) constitute about 9% of all the events. If we separate the isotropic part of this distribution and ascribe it to central collisions, then, since it amounts to about half of all the particles, we can conclude that the probability of formation of a unified statistical system for all particles (i.e., of central collision) does not exceed 5–10% also for πN collisions (in spite of the older estimate,[60] where the figure ~50% was indicated).

The experimental data described above point to objects which we have interpreted as statistical systems. However, only by applying the statistical theory to them and observing that the various quantitative characteristics prescribed by it agree with experiment will it be possible to verify the correctness of this interpretation and at the same time to verify the validity and fruitfulness of the statistical theory. We now proceed to this task.

3. FUNDAMENTAL FORMULAS OF THE STATISTICAL THEORY OF MULTIPLE GENERATION

We already mentioned that an attempt was made to use Pomeranchuk's version when the discrepancy between the Fermi theory and experiment was observed. Whereas for generation in the collision of two particles this discrepancy could still be attributed to the peripheral character of the interaction, nucleon annihilation is a genuine case of the statistical theory.[17,115] But even here the Fermi theory gave too large an admixture of heavy particles, too low a pion multiplicity, and too high a pion energy. All this was eliminated by choosing arbitrarily $V = (10-15)V_F$ instead of $V = V_F$ (2). This actually meant a transition (for the given particular process) to a model of the Pomeranchuk type, which was also applied in [6-9] to annihilation explicitly, and furthermore successfully (although not always consistently).

The theory proposes that when the statistical system expands (Fig. 1c) the thermodynamic equilibrium is conserved at each instant of time. At the instant of separation, the particles no longer interact and their momentum distribution at the temperature T and a total volume V is determined by the formulas of the Bose and Fermi statistics. For particles of sort i we have

$$dn_i(\mathbf{p}) = \frac{g_i}{(2\pi)^3} V \frac{d\mathbf{p}}{e^{z_i\sqrt{1+x^2}-\mu_i/T} \pm 1}, \quad z_i = \frac{m_i}{T}, \quad x = \frac{|\mathbf{p}|}{m_i}, \tag{23}$$

where m_i is the mass, g_i the internal (spin and isotopic) statistical weight of the particles, and μ_i the chemical potential. The latter is determined by specifying the difference between the particle and antiparticle numbers initially. We shall assume it to be zero, and take the conservation of this difference into account only in the factors g_i (see [18] for the case $\mu_i \neq 0$).

The total number of Bose and Fermi particles of a given sort (indices B and F) is

$$n_{F,i} = \frac{g_i}{2\pi^2} VT^3 F_F(z_i), \quad F_F(z_i) = z_i^3 \int_0^\infty \frac{x^2\,dx}{e^{z_i\sqrt{1+x^2}}+1}, \tag{24}$$

$$n_{B,i} = \frac{g_i}{2\pi^2} VT^3 F_B(z_i), \quad F_B(z_i) = z_i^3 \int_0^\infty \frac{x^2\,dx}{e^{z_i\sqrt{1+x^2}}-1}, \tag{24a}$$

$$z_i = m_i/T.$$

Analogously, the total energy of all particles of a given sort is

$$E_{F,i} = \frac{g_i}{2\pi^2} VT^4 \Phi_F(z_i), \quad \Phi_F(z_i) = z_i^4 \int_0^\infty \frac{x^2\sqrt{1+x^2}\,dx}{e^{z_i\sqrt{1+x^2}}+1}. \tag{25}$$

$$E_{B,i} = \frac{g_i}{2\pi^2} VT^4 \Phi_B(z_i), \quad \Phi_B(z_i) = z_i^4 \int_0^\infty \frac{x^2\sqrt{1+x^2}\,dx}{e^{z_i\sqrt{1+x^2}}-1}. \tag{25a}$$

The total energy of the statistical system will be denoted by W:

$$W = \sum_i (E_{F,i} + E_{B,i}). \tag{25b}$$

In the limiting cases we have:

$$T \ll m_i, \quad z_i \gg 1: \quad \Phi_F = \Phi_B = zF_F = zF_B = \left(\frac{\pi}{2} z^5\right)^{1/2} e^{-z}, \tag{26}$$

$$T \gg m_i, \quad z_i \ll 1: \quad F_F = 1.80, \quad F_B = 2.40, \quad \Phi_F = 5.68, \quad \Phi_B = 6.49. \tag{26a}$$

$$T = m_i, \quad z_i = 1: \quad F_F = 1.52, \quad F_B = 1.78, \quad \Phi_F = 5.25, \quad \Phi_B = 5.78. \tag{26b}$$

More detailed tables can be found in [18].

Following Pomeranchuk, we assume for the volume V at a given total multiplicity of the final state n (n is the sum of all the $n_{B,i}$ and $n_{F,i}$)

$$V = nV_0, \quad n = \sum_i (n_{B,i} + n_{F,i}), \tag{27}$$

where V_0 is of the order of the volume of one particle (2a). To be more accurate, however, we introduce here the only indeterminate parameter of the theory, α, obtained by comparison with experiment, putting, unlike in (2a),

$$V_0 = \alpha \cdot \frac{4}{3} \frac{\pi}{\mu^3}. \tag{28}$$

It is known, in any case, that α should be of the order of unity. This is what it turns out to be indeed.

The characteristics which we shall consider for the multiple generation process are the following: a) the decay temperature T_c, b) the average particle energy $\langle \epsilon \rangle$, c) the average multiplicity $\langle n \rangle$, d) the composition of the generated particles, e) the average transverse momentum $\langle p_\perp \rangle$, its distribution $N(p_\perp)/dp_\perp$, and the dependence of $\langle p_\perp \rangle$ on the multiplicity and mass of the particles, f) estimate for the longitudinal momentum $p_\parallel$.

We shall then consider in Chap. 5 the process of multiple generation and the closely related large-angle statistical elastic scattering.

In the comparison with experiment we shall assume everywhere that the numbers of the generated π^+, π^-, and π^0 mesons are equal:

$$n_{\pi^+} = n_{\pi^-} = n_{\pi^0} = \frac{1}{3} n_\pi. \tag{29}$$

This is an exceedingly crude assumption. First, it is known that pions are generated not only directly, but also via decay of ρ, ω, and η mesons. Thus, according to [64], at least one of these resonances is always present in $N\bar{N}$ annihilation. Second, even in pion generation it is necessary to take into account the real values of the Clebsch-Gordan coefficients. In general, according to experiment, if the multiplicity is not very large in $N\bar{N}$ annihilation, the result depends strongly on the spin of the entire system.[64] We are interested, however, in typical—large—multiplicities, and what is most important, the initial isotopic spin and spin are as a rule undetermined and it is actually necessary to average over them.

We shall assume that the decay is isotropic. Of course, this assumption is also very crude. Thus, in decay into two particles, allowance for the angular momentum of the system leads to an angular distribution of the type $(\sin\theta)^{-1}$ in the c.m.s. (see [61], and also [93,94]). A stronger angular dependence was also proposed.[59] It must therefore not be assumed in the analysis of the experiment that the presence of a weak deviation from isotropy (e.g., of the type $\sin\theta)^{-1}$) is evidence against the presence of a statistical system or subsystem.

As the theory is developed, the corresponding refinements with respect to these two items will be easily introduced. Our purpose here is to explain the principal aspects, without cluttering up the exposition with details. As will be seen below, the results are positive even with such an approach.

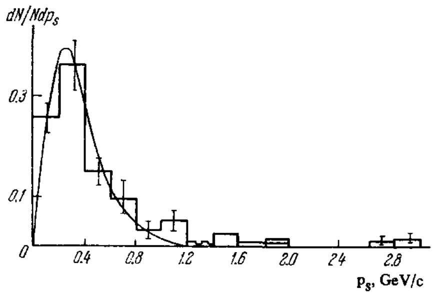


FIG. 11. Distribution of the generated charged particles with respect to the momentum p_s in pN collisions at $E_L \sim 200$ GeV (cosmic rays) [27]. Solid curve—Bose distribution for the temperature $T = 0.8\mu$ (p_s is the momentum in the symmetry system).

4. CHARACTERISTICS OF THE MULTIPLE GENERATION PROCESS (COMPARISON WITH EXPERIMENT)

From our point of view, the most important property of the decay of the system is the smallness of the decay temperature $T_c \equiv T_P \sim \mu$ (see (10)). It is clear even from this that particles heavier than the pion will enter as admixtures in an exponentially small proportion (for details see Sec. 4b). It can therefore be assumed that $\langle n \rangle \approx \langle n_\pi \rangle$ and $W \approx E_\pi$, where n_π is the number of all the pions and E_π is their total energy. Substituting (28) in (27), and then substituting the result in (24a) and replacing here n by $\langle n_\pi \rangle$, we obtain an equation for the decay temperature T_c (we have already put $g_\pi = 3$):

$$\alpha \cdot \frac{2}{\pi} F_B(z_c) = z_c^3, \quad z_c = \frac{\mu}{T_c}. \tag{30}$$

Under the simplest assumption, $\alpha = 1$, a graphic solution of this equation yields

$$z_c = 1.03, \quad T_c = 0.97\mu \quad (\alpha = 1). \tag{31}$$

F_B is a slow function. In practice therefore, in accordance with (30), we have $z_c \sim \alpha^{1/3}$. At $\alpha = 0.5$ and $\alpha = 2.0$ we have respectively $T_c = 1.19\mu$ and $T_c = 0.79\mu$. We shall use as a rule the value $\alpha = 1$, sometimes discussing in the comparison with experiment the possibility of other values of α.

a. Average Pion Energy $\langle \epsilon_\pi \rangle$

Dividing the total pion energy $E_B \equiv E_\pi$ (25a), which we assume to be approximately equal to the entire energy E (25b) of the statistical system, by their average number $n_B \approx \langle n_\pi \rangle$ (24a) we obtain (as before, $\alpha = 1$)

$$\langle \varepsilon_\pi \rangle = T_c \frac{\Phi_B(z_c)}{F_B(z_c)} \approx 0.43 \text{ GeV} \tag{32}$$

($\langle \epsilon_\pi \rangle \sim \alpha^{-1/3}$ at $\alpha \neq 1$).

This result agrees well with experiment.

Unfortunately, comparison of the theoretical figure (32) with the customarily cited experimental data is not always convincing, since the few non-statistically generated high-energy pions make an appreciable contribution when the mean value is determined in the experiment (see the tail of large p in Fig. 11). Nevertheless, if we turn, for example, to the experimental data gathered in the review,[38] we obtain the following:

For NN collisions, from the results of 23 investigations in which $E_L \geq 10$ GeV, we get

$$\langle \varepsilon_\pi \rangle_{\text{exp}} = 0.46 \text{ GeV} \quad (NN,\ E_L > 10 \text{ GeV}).$$

For πN collisions with $E_L \geq 4.5$ GeV, the average of 17 investigations yields

$$\langle \varepsilon_\pi \rangle_{\text{exp}} = 0.54 \text{ GeV} \quad (\pi N,\ E_L \geqslant 4.5 \text{ GeV}).$$

It is not surprising that this figure is large. In Sec. 2c we have emphasized that it is precisely in πN collisions that the contribution of fast non-statistical particles is particularly large. This is all the more true at such a low energy.

On the other hand for $N\bar{N}$ annihilation at rest, when a statistical system in the purest form is obtained, all 15 investigations yield

$$\langle \varepsilon_\pi \rangle_{\text{exp}} = 0.41 \text{ GeV} \quad (N\bar{N}\text{-annihilation}),$$

in splendid agreement with (32).

The distribution over ϵ and over the momenta $p = \sqrt{(\epsilon^2 - \mu^2)}$ is given by the Planck formula (23).

Thus, the average decay temperature or the average pion energy in the c.m.s. of the statistical system does not depend in any way on the initial energy. This important result is corroborated by experiment (in the common c.m.s. of the collision, the particle energy depends on the motion of the entire cluster, but at $E_L \lesssim 60$ GeV this motion apparently does not manifest itself as yet). Namely, as we shall show, the value of T determines the composition of the generated particles (Sec. 4c) and their distribution with respect to the transverse momenta (Sec. 4d). They turn out to be independent of W.

Let us add that in cosmic rays the average energy of the pions produced in fireballs is always estimated at ~0.5 GeV,[24-29] and the distribution over p always turns out to be mainly in accord with (23) at $T \sim \mu$, as is seen, for example, from Fig. 11.

b. Average Multiplicity $\langle n_\pi \rangle$

Dividing $E_\pi \approx W$ by $\langle \epsilon_\pi \rangle$ (32) we obtain

$$\langle n_\pi \rangle = \frac{W}{T_c} \frac{F_B(z_c)}{\Phi_B(z_c)} \approx 2.3 W_{\text{GeV}} \tag{33}$$

(W_{GeV} is measured in GeV; $\langle n_\pi \rangle \sim \alpha^{1/3}$).

When comparing this formula with experiment (see, e.g.,[64]) it is necessary to distinguish carefully between three types of processes.

1) The simplest and, as always, the purest case is annihilation of a particle pair. We can expect the entire energy to go over here into the statistical system, so that $\sqrt{s} = W$ (see Fig. 5b). The same should hold also for hadronic annihilation of leptons (see Fig. 6). Of course, besides the scheme of Fig. 5b, a more complicated process is also possible, when several particles are also generated outside the statistical system, as in Fig. 12. Such additional particles will be collimated near the direction of the particle that generates them. The angular distribution of all the particles turns out to be non-isotropic, and the description of the process is essentially of the type of Fig. 4a (with a single cluster), i.e., close to multiperipheral. Indeed, in [64C] such a non-isotropic contribution to the distribution has been observed in the processes $\bar{p}p \to K\bar{K} + n_\pi \pi$ at $E_L = 5.7$ GeV. From Fig. 6 of [64C] we can see that this contribution covers about 25% of all kaons and less than 15% of all pions. The multiperipheral theory without clusters (Fig. 4c) describes well the angular distribution at small multiplicities ($n_\pi = 1-3$) and poorly at medium

and large multiplicities ($\langle n_\pi \rangle \approx 5$), which is understandable.

Disregarding these details, however, we assume here, as a simplification, that the process follows the scheme of Fig. 5b, so that $W = \sqrt{s}$. Then*

$$\langle n_\pi \rangle = 2.3 W_{\mathrm{GeV}} = 4.6 E_{c,\mathrm{GeV}} \equiv 2.3 \sqrt{s_{\mathrm{GeV}}}. \qquad (34)$$

In the case of pp annihilation, experiment at $2.2 < 2E_c < 3.4$ GeV gives, according to [64a], the empirical formula

$$\langle n_\pi \rangle = 2.3 W_{\mathrm{GeV}}^{0.98}, \qquad (34a)$$

which agrees with the theory at $\alpha = 1$ with unlikely accuracy. To be sure, on the other hand, another empirical formula was also cited for the multiplicity of charged pions (at $2E_c \leq 3.5$ GeV):[64d]

$$\langle n_{\pi\pm} \rangle = (1.64 \pm 0.25) + (0.77 \pm 0.15) W_{\mathrm{GeV}}. \qquad (35)$$

It is outwardly strongly different from (34) and (34a). However, assuming that $\langle n_\pi \rangle = {}^3/_2 \langle n_{\pi\pm} \rangle$, we can verify that at $W < 2.5$ GeV the numerical agreement remains good, and at $W = 3.5$ GeV the discrepancy does not exceed ~25%, i.e., it remains of the same order as the error of either the theory or the experiment.

However, the last summary of all the data on the energy dependence of $\langle n_\pi \rangle$[116] leads to a systematic lowering of $\langle n_\pi \rangle$ in comparison with (34) at large E_L. The agreement can be restored by putting $T_c \sim (1.2-1.4)\mu$. But it is possible that this is simply the result of the increase of the contribution of the fast non-statistical pions (Fig. 12), so that less than $2E_c$ remains as the energy share of the statistical system.

2) In the comparison with experiment, a decisive factor for pN and πN collisions is the fraction of the total energy going over into the statistical subsystem, i.e., K. It would be necessary to consider separately the experiments for different K. Unfortunately, there are no corresponding data in the literature, and we are forced to consider very roughly average values. Namely, to take in some manner the few non-statistical particles into account, we put $K = \langle K_0 \rangle = 0.4$. Substituting (17) in (33) we obtain for the collisions

$$\langle n_\pi \rangle = 1.8 E_c = 1.3 \sqrt{(E_L + M) M} = \begin{cases} 5.8\,(E_L = 20 \text{ GeV}), \\ 7.0\,(E_L = 30 \text{ GeV}). \end{cases} \qquad (36)$$

where again E_c, E_L, and M are in GeV. According to (29), we can write

$$\langle n_{\pi\pm} \rangle = \frac{2}{3} \langle n_\pi \rangle = 0.61 \sqrt{s} \approx 0.86 \sqrt{E_L/M}, \qquad (36a)$$

where s is in $(\mathrm{GeV})^2$.

Under the crude assumption (29), the number of charged pions, $\langle n_{\pi\pm} \rangle$, in NN collisions therefore amounts to 3.9 and 4.7 at 20 and 30 GeV, respectively. Experiment[49,65] yields 4 and 4.6, respectively. Exact coincidence, of course, would be deceiving if for no other reason than that the averages of the experimental data include cases of very small multiplicity, not described by the statistical model (although their number is relatively small), and also because the use of a single $\langle K_0 \rangle$ for all particles makes the result highly approximate. Finally, the experimentally measured number of charged particles in peripheral collisions includes the "leading" particles that move on forward (see the next footnote below).

*We neglect the difference between the isotopic and statistical weights for $\bar{p}p$ and $\bar{n}n$ annihilation, on the one hand, and $n\bar{p}$ and $p\bar{n}$ annihilation on the other. As shown in [17], this difference is not large for $\langle n \rangle$.

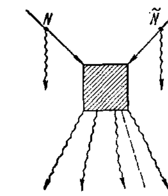


FIG. 12. Possible non-statistical contribution to the annihilation process $\bar{N}N \rightarrow$ hadrons.

The dependence $\langle n_\pi \rangle$ on E_L (36) contradicts at first glance the usually quoted relations (for experiments at high energies) $\langle n_\pi \rangle \sim E_L^{1/4}$ and $\langle n_\pi \rangle \sim \ln E_L$. Actually, however, there is no contradiction. We encounter here a fine point, namely that formula (36), like the entire statistical theory, is valid only for small n_π—according to (14) and (36) at $E_L < 60$ GeV.

At $E_L \gtrsim 50-100$ GeV, and especially in the generation of completely formed fireballs, their kinetic energy takes up an ever increasing fraction of the total energy. Thus, there are indications that for two fireballs their Lorentz factor $\bar{\gamma}$ in the common c.m.s. increases like $E_c^{1/2}$.[24-29,52] At $E_L \sim 300$ GeV,[28] in particular, measurements yield $\bar{\gamma} \sim 1.3$. Therefore the total energy of two fireballs in the common c.m.s., equal to $\langle K \rangle \cdot 2E_c$, is

$$2\mathfrak{M}_{f.b}\bar{\gamma} \approx 2\langle \epsilon_\pi \rangle \langle n_\pi \rangle E_c^{1/2} \approx 2\langle K \rangle E_c. \qquad (37)$$

Consequently, if we assume, as usual, that $\mathfrak{M}_{f.b}$, $\langle K \rangle$, and $\langle \epsilon_\pi \rangle$ (32) are independent of the energy, then we obtain in accordance with the experimental results

$$\mathfrak{M}_{f.b} \sim \langle n_\pi \rangle \sim E_c^{1/2} \sim E_L^{1/4}. \qquad (38)$$

These considerations were advanced at the very outset in the study of the kinematic characteristics of the fireball model.[26,66] On the other hand, if $\bar{\gamma}$ or $\langle K \rangle$ depend in a different fashion on the energy, then the law for the multiplicity will also be different. In addition, a role is played by the dependence of the number of fireballs on the energy. In quantum field theory[22,23] it is logarithmic. Therefore $n_\pi \sim n_0 \log E_L$, where n_0 is the average number of particles from one fireball.

Thus, the dependence of $\langle n_\pi \rangle$ on E_L has one form (36) so long as the cluster remains single and is at rest in the c.m.s., and an entirely different form at high energies, when its motion comes into play and new clusters or fireballs begin to appear. Experiment confirms this.

Figure 13a, where the experimental data for πN collisions were obtained from [65], shows a plot of (36a). We see that its agreement with experiment is quite satisfactory.*

Empirically, it is precisely such a breakdown of the single experimental $\langle n(E_L) \rangle$ curve into two, $\langle n \rangle \sim E_L^{1/2}$

*More accurately, the plotted quantity, according to (36a), is $n_s = N_{\pi}\pm + 1.0 = 0.61\sqrt{s} + 1.0$: the total number of charged particles $\langle n_s \rangle$ exceeds the number of generated charged pions $n_{\pi\pm}$ by a certain amount ν, which is approximately equal to unit, since the primary charged particle, roughly speaking, retains its charge in half the cases. The dashed line gives (36a) without this correction. Since the target nucleon is frequently nonrelativistic, a value $\nu = 1/2$ might be closer to the truth. The corresponding curves are not drawn on Figs. 13a and 13b.

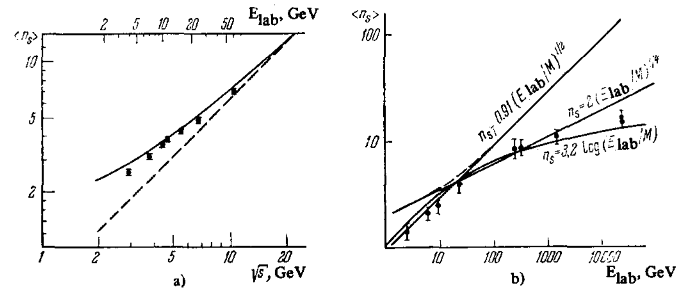


FIG. 13. Charged-particle multiplicity n_s vs. the invariant collision energy $\sqrt{s}$. a) For πN collisions from the data of [65] and from formula (36a). The solid curve, $\langle n_{\pi\pm}\rangle + 1$, takes into account the contribution of the initial particles, which experience charge exchange with a probability 50%; the dashed curve, $\langle n_{\pi\pm}\rangle$, does not take these particles into account. b) The same for NN collisions from the data of [79] for $\langle n_s\rangle$ as a function of E_L. Theory—(36a), with correction for the contribution of the primary particles—dashed (the empirical formulas from Fig. 27 of [79]).

for $\langle n\rangle \lesssim 10$ and $\langle n\rangle \sim E_L^{1/4}$ at larger multiplicities, is suggested also by the curve of Fig. 13b (Fig. 27 of the book [79]). We see that even the numerical coefficient 0.91 in the empirical formula written in the figure agrees with the theoretical value 0.86 in (36a). We note that a similar breakdown into two curves with different multiplicities is given also by Hayakawa (see Fig. 3.19 in [67]).

Thus, different multiplicities are obtained in annihilation and in collisions, and also in collisions at different values of E_L, owing to the differences in the mechanism whereby the statistical system is produced and moves in the c.m.s. If these differences are taken into account, then the multiplicity can be explained satisfactorily by a single statistical theory and for a single value of the decay temperature.

Recently Bjorken and Brodsky[68] applied an essentially statistical model also to the case of e^+e^- annihilation into hadrons, i.e., to the process of Fig. 6a. Usung a somewhat different and more quantum-mechanical calculation method (which has made it possible, in particular, to calculate the partial cross sections for annihilation into a state with given n), they obtained for $\langle\epsilon_\pi\rangle$ a value of 0.375 GeV (as against 0.43 GeV in (32)), and accordingly for the multiplicity of the charged pions

$$n_{\pi\pm} = \frac{2}{3}\,\frac{\sqrt{s}}{0.375\ \text{GeV}} \approx 1.8W, \tag{39}$$

instead of the 1.5W which follows from (33).

Caution must be exercised, however, when the theory is applied to annihilation of a pair of high-energy particles, when W exceeds the energy of the observed fireballs. In fact, there is no assurance that a single system that decays into pions is actually formed at arbitrary large s. If this is so, then we arrive ultimately at the case described by the hydrodynamic theory, which would be of great interest. Within the framework of quantum-field theory,[23] however, it is found that here, too, formation of a second fireball begins when W exceeds 3–5 GeV, i.e., the process is multiperipheral and the energy of the statistical system cannot be very large.

c. Makeup of Generated Particles

When it comes to the makeup of the particles, the statistical theory, and precisely the variant considered here, leads to remarkable success. At the same time, the initial Fermi theory has turned out to be utterly unsuitable.

The point is that according to the Fermi variant the temperature that determines the thermodynamic equilibrium is very high, $T_F \gg \mu$ (see (9)). From (7) and (8) it follows that

$$T_F \approx 0.4\langle n_\pi\rangle \mu, \tag{40}$$

i.e., this temperature is of the order of the kaon mass already at $E_L \sim 20$–30 GeV and $\langle n_\pi\rangle \sim 6$–$7$. Consequently, the numbers of the kaons and pions should be of the same order, whereas in experiment they differ by ~ 10 times. A similar discrepancy occurs also in $N\bar{N}$ annihilation (as already mentioned in Chap. 1) even when the Clebsch-Gordan coefficients are accurately taken into account.[17] At still higher multiplicities (and energies), $N\bar{N}$ pairs should predominate. This deviates greatly from the real situation. Attempts to use SU(3)-symmetry considerations and the quark hypothesis to determine the makeup of the particles in the Fermi statistical theory do not result in any improvement. Thus, the authors of [69] reached the conclusion that in π^-p collisions different final states of the statistical system (πN, KΣ, KΛ, ηN, etc.) should have probabilities of the same order. This greatly contradicts both the experimental data and the statistical theory considered by us.

At the same time, the characteristic feature of the model under consideration is a relatively low and energy-independent equilibrium temperature of the decay, $T_c \equiv T_P \sim \mu$ (31). Consequently, the admixture of particles heavier than pions is quite small at all energies E_c.

The first to calculate the composition of the generated particles in statistical theory at such low temperatures was S. Z. Belen'kiĭ[18a], who especially emphasized the fact that the study of the composition is the best method of determining the decay temperature. Even formulas (24) and (24a) give for the ratio of the average number of kaons of all four kinds (K^+, K^-, K^0, $\bar{K}^0$) and pions of all three charges $\langle n_\pi\rangle$ (using for $\langle n_K\rangle$ formula (26), owing to the smallness of $T_c \ll m_K$)

$$\frac{\langle n_K\rangle}{\langle n_\pi\rangle} = \frac{g_K}{g_\pi}\,\frac{F_B(m_K/T_c)}{F_B(m_\pi/T_c)} \approx \frac{4}{3}\,\frac{1}{F_B(1)}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{m_K}{\mu}\right)^{3/2} e^{-\frac{m_K}{\mu}} \approx 0.17. \tag{41}$$

Even this calculation, however, is inaccurate. Statistical formulas that neglect the conservation of the strangeness S, of the baryon number B, etc., can be correct only if the total number of particles of each sort is large, and according to (41) $\langle n_K\rangle$ is small at $\langle n_\pi\rangle \sim 10$. The theory must therefore be improved.[12,14]

Assume that we are dealing first with a system with S = B = 0.

We are interested in the rare possibility of generation of a pair of particles q and $\tilde{q}$ of mass m_q in the presence of a large number of pions ($n_\pi \gg 1$). The statistical weight of such a system with total energy W is the product of the statistical weights $d\rho_q(E_q)\,d\rho_{\tilde{q}}(E_{\tilde{q}})$ of the particles q and $\tilde{q}$ with momenta p_q and $p_{\tilde{q}}$ and with energies E_q and $E_{\tilde{q}}$, and the statistical weight $\rho^{(\pi)}(W - E_q - E_{\tilde{q}})$ of all the pions with energy $W - E_q - E_{\tilde{q}}$:

$$\begin{gathered} d\rho_{q\tilde{q}}^{(\pi)} = d\rho_q(E_q)\,d\rho_{\tilde{q}}(E_{\tilde{q}})\,\rho^{(\pi)}(W - E_q - E_{\tilde{q}}),\\ d\rho_q = (2\pi)^{-3} g_q V\,d\mathbf{p}_q,\quad d\rho_{\tilde{q}} = (2\pi)^{-3} g_{\tilde{q}} V\,d\mathbf{p}_{\tilde{q}}. \end{gathered} \tag{42}$$

Here q and $\tilde{q}$ are nonrelativistic particles (at $T_c \ll m_q$). Therefore

$$E_q = m_q + \frac{p_q^2}{2m_q}, \quad E_{\tilde{q}} = m_q + \frac{p_{\tilde{q}}^2}{2m_q}.$$

But this process is a rare event. It is much more probable that the system decays only into pions—the statistical weight of this state is $\rho^{(\pi)}(W)$. Therefore the probability of pair generation is the ratio of $\rho_{q\tilde{q}}^{(\pi)}$ to the sum of $\rho_{q\tilde{q}}^{(\pi)}$ and $\rho^{(\pi)}$, or, in practice, simply to $\rho^{(\pi)}$. We write down further, approximately,

$$\rho^{(\pi)}(W) = \rho^{(\pi)}(W - E_q - E_{\tilde{q}})\rho^{(\pi)}(E_q + E_{\tilde{q}}).$$

Since we are dealing with relatively heavy particles, and since $T \sim \mu$, it follows that $\rho^{(\pi)}(E_q + E_{\tilde{q}})$ is the number of states of a large number of pions, and we can use for it the thermodynamic approximation $\rho^{(\pi)}(E) = \exp(S(E))$, where $S(E) = \int dE/T$ is the entropy. In our theory the temperature does not depend on either the total energy or the volume. Therefore $S(E) = E/T$.

Gathering all these formulas together, we obtain the pair generation probability $dw = d\rho_{q\tilde{q}}^{(\pi)}/\rho^{(\pi)}$:

$$dw_{m_q\mathbf{p}_q,\, m_{\tilde{q}}\mathbf{p}_{\tilde{q}}} = (2\pi)^{-6} g_q g_{\tilde{q}} V^2\, d\mathbf{p}_q\, d\mathbf{p}_{\tilde{q}} \exp\left\{-\frac{2m_q}{T} - \frac{p_q^2 + p_{\tilde{q}}^2}{2m_q T}\right\}. \tag{43}$$

It is easy to see that for the more general case of generation of n_h heavy particles with masses m_1, $m_2, \ldots$, the foregoing reasoning gives, if we integrate with respect to the momenta,

$$w_{m_1, \ldots, m_{n_h}} = \left(\frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{V}{V_0}\right)^{n_h} \left(\frac{T}{\mu}\right)^{\frac{3}{2}n_h} \left(\frac{m_1 \ldots m_{n_h}}{\mu^{n_h}}\right)^{\frac{3}{2}} e^{-\frac{m_1 + m_2 + \ldots + m_{n_h}}{T}} g_1 g_2 \ldots g_{n_h}. \tag{44}$$

Actually, however, among the states taken into account in the product of the internal statistical weights $g_1 g_2 \ldots$ there may be encountered identical states (e.g., the first particle is a proton and the second is a neutron, or vice-versa), which must be taken into account only once. Therefore it is best to write the general internal weight in the form $g_{q_1 q_2 \ldots q_{n_h}}$.

We have considered here the case of a statistical system with S = B = 0, which most frequently decays only into pions. But in the case of a central collision, for example pp or K^-p, the purely pionic state is impossible: the probability of the rare state $m_{q_1}, m_{q_2}, \ldots$ with the statistical weight $\rho_{m_1, m_2}, \ldots$ is obtained in this case by dividing not by the statistical weight $\rho^{(\pi)}$, but by the summary statistical weight of the main decay channels. For pp collisions, this weight is determined with good accuracy by the N + N + pions channel:

$$\begin{aligned}\rho_{NN}^{(\pi)} &= \int d\rho_{N_1}\, d\rho_{N_2} \rho^{(\pi)}(W - E_{N_1} - E_{N_2}) \\ &= g_{N_1 N_2}\left(\frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{V}{V_0}\right)^2 \left(\frac{T}{\mu}\right)^3 \left(\frac{m_N}{\mu}\right)^3 e^{\frac{W - m_{N_1} - m_{N_2}}{T}},\end{aligned} \tag{45a}$$

and for K^-p it is determined by the sum of the channels Λ + pions, Σ + pions, and K + N + pions:

$$\begin{aligned}\rho_{\Lambda}^{(\pi)} + \rho_{\Sigma}^{(\pi)} + \rho_{KN}^{(\pi)} &= e^{W/T}\frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{V}{V_0}\left(\frac{T}{\mu}\right)^{3/2}\left\{\left(\frac{m_\Lambda}{\mu}\right)^{3/2} g_\Lambda e^{-m_\Lambda/T}\right. \\ &\left.+ \left(\frac{m_\Sigma}{\mu}\right)^{3/2} g_\Sigma e^{-m_\Sigma/T} + \frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{V}{V_0}\left(\frac{T}{\mu}\right)^{3/2}\left(\frac{m_N m_K}{\mu^2}\right)^{3/2} g_{NK} e^{-\frac{m_N + m_K}{T}}\right\}\end{aligned} \tag{45b}$$

(at $T \sim \mu$, the terms in the sum are of the same order, while the remaining channels make a small contribution). Therefore the probability of a state with n_h particles $m_1, \ldots, m_{n_h}$, for example in a central collision of two nucleons, is equal to the ratio of expressions (44) and (45a):

$$\begin{aligned}w_{m_1, \ldots, m_{n_N}} &= \left(\frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{V}{V_0}\right)^{n_h - 2}\left(\frac{T}{\mu}\right)^{3/2(n_h - 2)}\frac{g_{q_1 q_2 \ldots q_{n_h}}}{g_{q_1^0 q_2^0}} \\ &\times \left(\frac{m_1 m_2 \ldots m_{n_h}}{m_N^2 \mu^{n_h - 2}}\right)^{3/2} e^{-\frac{m_1 + \ldots + m_{n_h} - 2m_N}{T}}.\end{aligned} \tag{46}$$

Since $V/V_0 = n$ (in a purely pionic system $n = n_\pi$), it is convenient to replace V/V_0 in these formulas by the observed average number of pions. These formulas show that the probability of generation of heavy particles decreases very sharply, exponentially, with increasing particle mass. It is very important quantitatively that the exponential contains a sum of the masses of all the heavy particles (e.g., this gives rise to the coefficient 2 in formula (43)), something not found in the earlier investigations.

Since $T \sim \mu$, the probability of generation of the pair $q\tilde{q}$ decreases by ~4–5 orders of magnitude when the mass m_q of the generated particle increases by $m_N \sim 1$ GeV. Experiment (see Table I below) has confirmed this result fully.[12,13] The physical cause of such a sharp dependence is clear already from the derivation of formula (43). Instead of the pair $q\tilde{q}$, this energy can give rise to approximately $2m_q/\mu \gg 1$ pions (approximately 13 pions at q ≡ N). On the other hand, the probability of the state depends exponentially on the entropy, which is proportional to the number of particles. Consequently, the pionic state is much more probable than a state with a pair of heavy particles. Even if a $q\tilde{q}$ pair is produced at the initial instant, it has a great probability of annihilating into pions during the expansion of the cluster. This indeed gives rise to the factor $\sim 10^5$. It is partly offset by the large pre-exponential factors.

Further, at a given total mass, a channel with a large number of particles is more probable. Thus, $\tilde{d}$ can be produced not only paired with d, but also in the combination $\tilde{d}N_1N_2$, where N_1 and N_2 are two nucleons. The summary masses of the particles are here the same (accurate to within the binding energy of the deuteron, which is immaterial), but the n_h are different. Since $m_N \gg \mu$ and $V/V_0 \gg 1$, the $\tilde{d}N_1N_2$ channel is more probable, according to (44), even though the exponential in both cases is the same.

It is important that the friability of the deuteron does not play a major role. The point is that at a temperature T ~ 100 MeV the antinucleons are generated with low velocities and have time to interact (compare with the large cross section of the reaction $p + p \to d + \pi^+$ at ~100 MeV, which has been known for a long time). This friability of the system introduces a factor $\eta_{\tilde{q}} \le 1$, which can be estimated, albeit not reliably. It was estimated $\eta_d \le 1/6$ for the antideuteron in the Fermi model.[70] In our case it is apparently even closer to unity because of the large volume of the system. One can assume that $\eta_{\tilde{d}} \sim 1/2$ and (for $\widetilde{He}_3$) $\eta_{\widetilde{He}_3} \approx 1$.

It should be emphasized that this analysis is valid for a statistical system of any origin: for the annihilation of a pair of hadrons (Fig. 5) or leptons (Fig. 6a), for an electromagnetic process with "deep inelasticity"

(Fig. 6b), for peripheral collision with strong excitation of the incoming nucleon ($W \gtrsim 3$ GeV, Fig. 2), including diffraction excitation, for the formation of an individual subsystem of the fireball type (Figs. 3 and 4), etc. All that matters is the total energy of the statistical system W (and also the conserved quantum numbers—strangeness and baryon number).

However, a system with the necessary energy (to generate a given heavy pair) at a given initial energy E_L cannot be produced in any such mechanism. Thus, $W \sim \mathfrak{M}_{f.b} \sim 2$–$4$ GeV in peripheral fireball formation. Even in pp collisions with $E_L \lesssim 30$ GeV and an average inelasticity coefficient $\langle K \rangle \sim 0.3$ there is produced a system with $\langle W \rangle \sim 0.3 \times 7.6 \sim 2.3$ GeV ("almost fireball"). In such a system there can be produced $K\tilde{K}$ pairs accompanied by a large number of pions. On the other hand, when a $p\tilde{p}$ pair is produced the number of accompanying pions is very small. Generation of heavier particles ($\tilde{\Lambda}\Lambda$, $\tilde{d}d$) is possible energywise here only in collisions with large inelasticity coefficients, for example in a central collision—much rarer than the average. Consequently, the generation cross section $\sigma_{\tilde{q}}$ of the particles $\tilde{q}$ is obtained from the probability (44) or (46) by multiplying not by the total inelastic collision cross section, but by the cross section $\sigma(K > K^{(q)}_{min})$ for collisions with an inelasticity coefficient K exceeding the minimal $K^{(q)}_{min}$ at which the statistical system has sufficient energy. We denote this collision cross section, which constitutes a fraction $\xi(K > K^{(q)}_{min})$ of the total inelastic cross section, by $\sigma(K > K^{(q)}_{min})$:

$$\sigma(K > K^{(q)}_{\min}) = \xi(K > K^{(q)}_{\min})\,\sigma_{\text{inel}}, \quad \sigma_q = \langle n_q\rangle \sigma(K > K^{(q)}_{\min}). \tag{47}$$

For central collisions, in accord with (22), we have $\xi \sim 0.1$ at $E_L \sim 15$–20 GeV. Therefore the cross section (45) is small at large K_{min}. But on the other hand in these collisions the energy of the statistical subsystem is larger than in an "average" collision by a certain factor ν, with $1 < \nu < \langle K \rangle^{-1} \sim 2.5$, and accordingly V/V_0 in (43)–(45) is larger by the same factor. Thus, the factors ξ and ν act in opposite directions, and the aforementioned "friability" factor is $\eta < 1$. Combining them into a common factor

$$A_i = \nu_i^{\beta_i}\eta_i\xi_i, \tag{48}$$

where the degree β depends on the degree of V in (43)–(45), we can assume that A does not differ greatly from unity (more detailed estimates for $\tilde{d}$ and $\widetilde{He}_3$ give $A_i \sim 0.3$–5[15]). Therefore, replacing $V/V_0 = \langle n_\pi \rangle$ (and in the interval $20 \lesssim E_L \lesssim 70$ GeV we have $\langle n_\pi \rangle \sim 5$–$9$), we can use formulas (43)–(46) for rough estimates, disregarding the value of K_{min} in this process.

It is important here to take correct account of the internal statistical weights: for K^--meson generation, the states K^-K^+ and K^-K^0 are possible, so that $g_{K^-K} = 2$; for $\tilde{p}$ generation, we can have $\tilde{p}p$ and $\tilde{p}n$, i.e., taking into account two spin directions for each, $g_{\tilde{p}N} = 2 \times 4 = 8$; for $\tilde{d}$, the most probable channel is $\tilde{d}N_1N_2$, and for the number of non-identical states (e.g., the states $\tilde{d}np$ and $\tilde{d}pn$ coincide) we obtain by direct calculation $g_{\tilde{d}N_1N_2} = 30$; for $\widetilde{He}_3$ (the most probable channel is $\widetilde{He}_3N_1N_2N_3$) we have $g_{\widetilde{He}_3N_1N_2N_3} = 40$ (if we neglect the identity of the states, then $g_{\tilde{d}N_1N_2} = g_{\tilde{d}}g_{N_1}g_{N_2} = 3 \times 4 \times 4 = 48$ and $g_{\widetilde{He}_3N_1N_2N_3} = g_{\widetilde{He}_3}g_{N_1}g_{N_2}g_{N_3} = 128$.

We have chosen for comparison with experiment the predicted number of negative particles of different masses, because among the non-statistically generated particles in pp collisions (e.g., in peripheral excitation of the proton to the isobar state with subsequent decay), the π^+ and K^+ particles have an advantage. This is precisely why their number is always larger than that of π^- and K^- particles.

The foregoing results contain a main conclusion that is foreign to the Fermi model, namely, the number of heavy particles is small, and since with further increase of energy the number of fireballs increases but the temperature of the decay remains the same, the relative admixture of K^- mesons and antinucleons remains small and almost constant.

Indeed, in $\tilde{p}N$ annihilation at rest ($\langle n_\pi \rangle = 4.5$, $\langle n_{\pi^-} \rangle = 1.5$) experiment[64b] yields

$$\frac{\tilde{p}N \to \tilde{K}K + \text{pions}}{\tilde{p}N \to \text{all}} = (6.8 \pm 0.5)\cdot 10^{-2}. \tag{49}$$

Since KK means both $\tilde{K}^-K$ and $\tilde{K}^0K$, the number of K^-K pairs is half as large, meaning $\langle n_{K^-} \rangle / \langle n_{\pi^-} \rangle \sim 0.034/1.5 \approx 25\%$.[49] This agrees with the theory at $T \approx 0.91\mu$. We obtain an estimate close to this also from [64c]: at $p_L = 5.7$ GeV/c the value obtained for the ratio (49) is 0.15 ± 0.03. Since $\langle n_{\pi^-} \rangle \sim 3$ at this energy in the absence of K pairs, it follows that $\langle n_{K^-} \rangle / \langle n_{\pi^-} \rangle = 0.5 \times 0.15/3 \sim 2.5\%$.

For pp collisions at $E_L \sim 20$ GeV the data of [49] give $\langle n_{K^-} \rangle / \langle n_{\pi^-} \rangle \sim 1 \times 10^{-2}$, indicating that $T \sim 0.85\mu$.

In cosmic rays it is customary to estimate $\langle n_K \rangle / \langle n_\pi \rangle$ at approximately 10%, meaning ($\langle n_{\pi^-} \rangle = 3$) that $T \approx 0.96\mu$. An appreciable contribution is made here, however, by the decay of excited isobars (which make the main contribution according to Koshiba[59]), and a more detailed comparison is difficult.

Finally, data at $E_L \sim 70$ GeV ($\langle n_{\pi^-} \rangle \approx 3$) were obtained with the Serpukhov accelerator[72,73]. Here the antiparticles of near-maximum energy are undoubtedly generated to a considerable degree nonstatistically. Therefore for comparison with the theory it is necessary to select heavy particles with laboratory momenta in the region

$$p_L^{(q)} \sim m_q\gamma_c. \tag{50}$$

$\gamma_c \approx 6.5$ is the Lorentz factor of the c.m.s.: this is precisely the region of the heavy particles of mass m_q if they are generated with low critical energy ($\sim T \sim \mu$) in a statistical system that is at rest in the c.m.s. Indeed, the maxima of the $\tilde{p}$ and $\tilde{d}$ spectra turn out to be in the regions $p_L^{(\tilde{p})} \sim 7$–9 GeV/c and $p_L^{(\tilde{d})} \sim 13$ GeV/c. The $\widetilde{He}_3$ nuclei were accordingly sought—and found—at $p_L^{(\widetilde{He}_3)} \sim 20$ GeV/c (for pions, if we average over the momenta in the c.m.s., we have $\langle p_L^{(\pi)} \rangle = \langle \epsilon_\pi \rangle \gamma_c \sim 3$ GeV/c). Table II shows the results of observations at an angle $\theta_L = 0$ (as recalculated by Yu. D. Prokoshkin).

It should be borne in mind that the accuracy of the experiment for $\widetilde{He}_3$ is low: only five cases were registered. Furthermore, in the experiment they determined the differential cross section for definite θ_L and, what is particularly important, for definite $p_L^{(q)}$, whereas Table I

Table II. Experimental data for E_L = 70 GeV

Particles	π^-	$\tilde{p}$	$\tilde{d}$	$\widetilde{He}_3$
$p_L^{(\tilde{q})}$ GeV/c (maximum of spectrum)	~ 4	9	13	20
$d^2\sigma_q/d\Omega\, dp_L$ cm²/sr-GeV/c per aluminum nucleus at $\theta_L = 0$ (at the maximum of spectrum)	$2.5\cdot 10^{-24}$	$3\cdot 10^{-26}$	$3\cdot 10^{-30}$	$5\cdot 10^{-35}$
Ratio to cross section for π^-, $d\sigma_q/d\sigma_{\pi^-} =$ $T_c/\mu =$	1	$1.2\cdot 10^{-2}$ (1.05)	$1.2\cdot 10^{-6}$ (0.97)	$2\cdot 10^{-11}$ (0.94)
Ratio to cross section for $\tilde{p}$, $d\sigma_q/d\sigma_{\tilde{p}} =$ $T_c/\mu =$		1	$1\cdot 10^{-4}$ (0.90)	$1.7\cdot 10^{-9}$ (0.89)
Ratio to cross section for $\tilde{d}$, $d\sigma_q/d\sigma_{\tilde{d}} =$ $T_c/\mu =$			1	$1.7\cdot 10^{-5}$ (0.89)

(The numbers have been refined somewhat compared with [15])

pertains to the total cross section. Finally, in the experiment the target consisted of Al nuclei and not of nucleons, and this may be particularly important for "friable" systems ($\tilde{d}$, $\widetilde{He}_3$). On the other hand, in the theoretical Table I we have the rather indeterminate factors A_i.

Nonetheless, we shall attempt to compare these data with the theory, in spite of the risks involved.

For each experimentally obtained ratio we determine the temperature T at which this ratio agrees with Table II (each cross section is taken at the maximum of its spectrum) if all the coefficients A_i are assumed equal to unity. The thus-obtained values of T/μ are entered in Table II in parentheses. We see that

$$T \sim (0.89 - 1.05)\,\mu. \tag{51}$$

as expected.

We can compare also the absolute values of the cross section. To this end the total generation cross section $\sigma_q = \langle n_q \rangle \sigma_{inel}$ must be divided by the effective values of the solid angle in which the particles are emitted forward in the laboratory frame, $\Delta\Omega \sim 2\pi/\gamma_c^2$ (γ_c is the Lorentz factor of the c.m.s.), and also of the momentum Δp_L (for $\tilde{d}$ we put $\Delta p_L^{(\tilde{d})} \sim 8$ GeV/c), and multiply it by the "effective number of nucleons" in the Al nucleus, $N_{eff} \sim A^{2/3} = 9$. This yields, for example for $\tilde{d}$ at $T = \mu$

$$\left(\frac{d^2\sigma}{d\Omega\, dp_L}\right)_{\text{per Al nucleus}} \sim \frac{\langle n_{\tilde{d}} \rangle \sigma_{inel}}{\Delta\Omega_L \cdot \Delta p_L} N_{eff} = 2.6\cdot 10^{-30} A_{\tilde{d}}. \tag{52}$$

This does not contradict the experimental value 3×10^{-30}.

Such an analysis is valid also for hyperon generation. Thus, summary data were recently published on the generation of Ω^- particles in K^-p collisions at E_L = 5–10 GeV/c.[74] The Ω^- can be produced here in the states $\Omega^- K^+ K^+$, $\Omega^- K^+ K^0$, and $\Omega^- K^0 K^0$ (and also in heavier systems), and it is easy to verify that from energy considerations we can speak here only of central collisions, $K > 0.75$, the cross section for which is $\sigma(K > 0.75) = \xi\sigma_{K^-p,inel}$; $\xi \ll 1$. This is confirmed experimentally by the fact that the Ω^- particles are emitted isotropically in the c.m.s.

Using in this case formula (45b) instead of (45a), we can obtain a formula analogous to (46) and calculate the Ω^- generation probability, and then calculate the generation cross section σ_{Ω^-} by multiplying by $\xi\sigma_{Kp,inel}$. We find that σ_{Ω^-} is equal to 11ξ, 43ξ, and 129ξ (microbarns) at T/μ = 0.9, 1.0, and 1.1, respectively.[15] Experiment yields $\sigma_{\Omega^-} = 2.5 \pm 1.0\,\mu b$. Consequently, the theory accounts for the experiment at $T/\mu \sim 0.9$–1.0 if the probability of the central K^-p collision (at $E_L \sim 10$ GeV) has the reasonable value $\xi \sim 1/4$–$1/8$.

Summarizing, we can state that the present-day (still skimpy) experimental data on the generation of heavy particles at high energy can be readily understood from the point of view of the statistical theory, if it is assumed that the critical temperature of the decay lies in the region

$$T \sim (0.9 - 1.1)\,\mu. \tag{53}$$

It should be borne in mind here that we have discussed a crude variant of the theory, in which, in essence, the conservation of the total spin is not taken into account and the conservation of the total isospin of the system is accounted for very crudely. The theory, of course, can be improved in this respect, for example by introducing Clebsch-Gordan coefficients, as in [17], or by another approximate method.[75]

Formulas (43)–(46) are valid for the generation of all strongly interacting particles. In particular, they were initially obtained for the estimate of the generation probability of the hypothetical quarks.[13] This probability, as we see, is very low at $m_q \gtrsim 3m_N$. However, as was shown in the same reference, if the quarks do not interact with the pions very strongly then, being produced at an earlier stage (at a higher temperature), the $q\bar{q}$ pair may not have time to annihilate during the course of the expansion of the system. Therefore, no matter how paradoxical it may be, heavy particles that do not interact very strongly should be generated with a higher probability than strongly-interacting particles

Table III

Particles	$\langle p_\perp \rangle$ exp, $\pi^- p$, 10 GeV	Statistical theory $T = 0.97\mu$
π^+	0.30±0.01	0.33
K^0	0.39±0.02	0.32
p	0.44±0.05	0.44
Λ^0	0.46±0.02	0.48
$\Sigma^\pm$	0.51±0.04	0.50
Ξ	0.56±0.08	0.52
Ω^- (K^-_p)	0.53±?	0.58

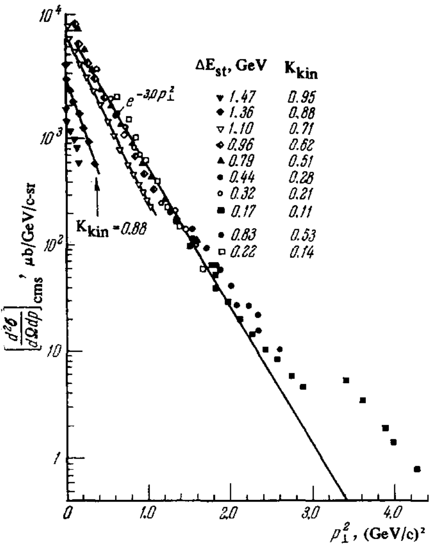


FIG. 14. Recoil-proton transverse-momentum distribution at different values of the inelasticity coefficient (pp collisions, $E_L \approx 12.5$ GeV) in the process pp → p + arbitrary particles [62].

system from the particles generated outside the subsystem: thus, for example, if the inelasticity coefficient is not close to unity, then the recoil nucleons should not be described by formulas (58)–(60), and must therefore be excluded from the experimental data when a comparison is made with the statistical theory. Conversely, at $K \sim 1$ the recoil nucleon enters in the statistical system and formulas (58)–(60) are valid. We shall nevertheless compare a rather old summary of data[85] (the figure for Ω^- is taken from [74]) with experiment at not too large values of E_L, when the contribution of the collisions with large K is large when heavy particles are generated (Table III). It is evident that the agreement is very good. The discrepancy for K^0 may be attributed precisely to the fact that their mass is small and they are generated to a considerable degree at $K \ll 1$.

There exists apparently only one experiment in which the distribution of the nucleons (protons) with respect to $p_\perp$ was measured at definite values of K. It was carried out at the Argonne Laboratory for pp collisions with $E_L \approx 12.5$ GeV.[62] It turned out (Fig. 14) that all the distributions with respect to $p_\perp$ are described at $K \leq 0.70$ by a Gaussian curve $\exp(-bp_\perp^2)$ with $b = 3.0\ (\text{GeV/c})^2$ up to $p_\perp \sim 1.4$ GeV/c (beyond which the experimental points lie above the curve). But b is much larger at $K \geq 0.88$ (namely, K = 0.88 and K = 0.94), even though the distribution has the same form (up to $p_\perp \sim 0.4$–0.65). Judging from the published plot, it can be roughly estimated at $b_{exp} \approx 4.0$–$4.4\ (\text{GeV/c})^{-2}$.

In this experiment the collision can be regarded as central if the kinetic energy of the proton is thermal at the end of the process. Since the total particle energy is initially $E_c = \frac{1}{2}\sqrt{s} = 2.50$ GeV, this will occur, according to (21a), if $K = 1 - 0.2/1.56 \approx 0.87$.

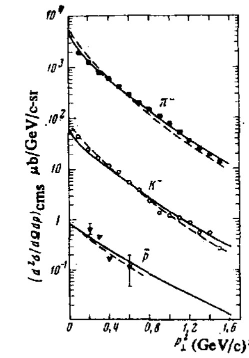


FIG. 15. Distribution of the particles π^-, K^-, and p, generated in pp; collisions, $E_L \approx 12.5$ GeV, with respect to the transverse momenta at a fixed longitudinal momentum $p_\parallel = 0.6$ GeV/c [81]. Curves–statistical theory, formula (55a) (curves normalized to the experimental data, since K is unknown); solid–at $T = \mu$, dashed–at $T = 0.9\mu$.

Thus, it is no accident that the experimental curves for K = 0.88 and 0.94 differ from the curves with smaller K, since they correspond to a central collision and agree in form with the theoretical curve (58). Comparing the theoretical value of the exponent with the experimental one $(2m_N T)^{-1} = b_{exp} \approx 4.2\ (\text{GeV/c})^{-2}$, we get $T = 0.92\mu$, which agrees with all the other estimates (see Sec. 4c, and also below).

The absolute cross section for a central collision, as can be roughly estimated from the plot, is ξ $(K > 0.88) \sim \frac{1}{5}$ to $\frac{1}{10}$ of the total inelastic cross section. This figure also agrees with other estimates (see Sec. 4c).

The curves for non-central collisions contain a contribution of very fast pions from the decay of the excited leading particle and, as can be readily understood, should have a smaller slope, in accord with experiment.

In the absence of other data with varying values of the inelasticity coefficient K, we can still attempt to employ the presented formulas when the heavy particle is accompanied by a large number of pions, $n_\pi \gtrsim \langle n_\pi \rangle$, assuming that K is large here. The existing measurements, on the other hand, frequently pertain to small multiplicities.

Figure 15 shows the distributions with respect to $p_\perp$ for different generated particles in pp collisions with energy $E_L = 12.2$ GeV,[81] at a fixed value $p_\parallel = 0.6$ GeV/c. We consider only the distributions for π^- and K^- since, as already mentioned, the π^+ and K^+ mesons can be generated to a much greater degree by a non-statistical mechanism. The solid lines correspond to the statistical theory (55a) for a temperature $T = \mu$ and $p_\parallel = 0.6$ GeV/c, and the dashed lines are for $T = 0.9\mu$. A separate normalization factor is chosen for each curve. It cannot be determined theoretically, since the fraction of the collisions producing the statistical system is unknown. We see that experiment can be described quite satisfactorily by the theory.*

*The authors of [82] continued the measurements up to the large $p_\perp \approx 2$ GeV/c. At $p_\perp \gtrsim 1.6$ GeV/c the results begin to deviate from the theory presented here. It must be borne in mind, however, that at $E_L = 12.2$ GeV the maximum c.m.s. energy of the statistical system is $\sqrt{s} \approx 5$ GeV. The colliding nucleons certainly carry away more than 2 GeV. Consequently, if $p_\perp = 1.6$ GeV/c, then one pion carries away ≤ 1.7 GeV and the energy left for the remaining pions is so small, that their number cannot be large. One should expect here a large contribution of non-statistical processes, and formula (55a) cannot be used.

having the same mass.*

We note that a formula that agrees in the main with (43) was obtained also by Hagedorn in his hydrodynamic theory.[37] This is not surprising, for in this theory the temperature is also low ($T_0 \sim \mu$).

In some other applications of the statistical theory, the probability of heavy-particle generation was estimated not by a formula of the type (43) but by a formula which is not suitable here, with the exponential $\exp(-m_q/T)$ (instead of $\exp(-2m_q/T)$ (43)), and furthermore in the Fermi-model variant (see, e.g., [77]), which naturally led to an unsatisfactory result.

d. Transverse Momentum

The approximate constancy of the transverse momentum $p_\perp$ of the generated particles, established experimentally in a tremendous interval of the primary energy E_L, from several GeV to $\sim 10^6$ GeV, is one of the most remarkable features of the multiple generation process. In the statistical theory it is explained without stretching any points and is a consequence of the independence of the decay temperature T_c and of the average energy of the primary energy $\langle \epsilon_\pi \rangle$ in the rest system of the cluster.† Moreover, the statistical theory explains, as we shall show, both the distribution with respect to $p_\perp$ and its dependence on the multiplicity n, as well as on the particle mass.

The distribution with respect to $p_\perp$ is obtained from (23) by introducing the longitudinal component $p_\parallel$ and the azimuthal angle φ and integrating with respect to $p_\parallel$ and φ. For pions, putting

$$x=\frac{p_\perp}{\mu},\quad y=\frac{p_\parallel}{\mu},\quad z=\frac{\mu}{T}, \tag{54}$$

we obtain a formula derived long ago:[78]

$$\frac{dn}{dx}=\frac{g_\pi}{2\pi^2}VT^3x\int_0^\infty\frac{dy}{e^{z\sqrt{1+x^2+y^2}}-1}=\mathfrak{N}x\sqrt{1+x^2}\sum_{n=1}^\infty K_1(nz\sqrt{1+x^2}), \tag{55}$$

here $\mathfrak{N}$ is a normalization factor and K_1 a cylindrical function of imaginary argument. Since $z \sim 1$ and $z\sqrt{(1+x^2)}$ exceeds unity, we can confine ourselves to the first term of the sum and replace K_1 by its asymptotic expression. Then

$$\frac{dn}{dx}\approx \text{const}\cdot x\sqrt[4]{1+x^2}\,e^{-z\sqrt{1+x^2}} \tag{56}$$

$$\approx Ax^{3/2}e^{-\frac{\mu}{T}x},\qquad A=\frac{4}{3\sqrt{\pi}}\left(\frac{\mu}{T}\right)^{5/2},\quad x=\frac{p_\perp}{\mu}. \tag{56a}$$

The last expression in (56a) is written for the region $x \gg 1$. The mean value (at $\alpha = 1$, $T = 0.97\mu$) equals, according to (56),

$$\left\langle\frac{p_\perp}{\mu}\right\rangle=2.37,\quad \langle p_\perp\rangle=0.33\ \text{GeV/c}. \tag{57}$$

The double distribution—with respect to $p_\perp$ and $p_\parallel$ (integrated only with respect to φ)—in the most essential region $\sqrt{m_i^2+p_\parallel^2+p_\perp^2} \gg T$ has the obvious form

$$\frac{dn(p_\parallel,p_\perp)}{dp_\parallel dp_\perp}=\frac{g_i}{(2\pi)^2}Vp_\perp e^{-\frac{1}{T}\sqrt{m_i^2+p_\parallel^2+p_\perp^2}}. \tag{55a}$$

Both the form of the distribution (56) (see below) and the value of $\langle p_\perp\rangle$ (57) are in splendid agreement with the experimental data in the accelerator range[49] as well as in cosmic rays at $E_L \lesssim 1000$ GeV[79] ($\langle p_\perp\rangle_{exp} \approx 0.32$–$0.38$ GeV/c*).

The approximate formula (56a) yields

$$\langle p_\perp\rangle=\frac{5}{2}T\approx 2.42\mu\approx 0.335\ \text{GeV/c}. \tag{57a}$$

On the other hand, if we deal with heavy particles with mass $m_q \gg \mu \sim T$, then we can put in (24)

$$\frac{1}{T}\sqrt{m_q^2+p_\perp^2+p_\parallel^2}\approx\frac{m_q}{T}+\frac{p_\perp^2+p_\parallel^2}{2m_qT}.$$

Neglecting unity in the denominator of (24), we obtain for both Bose and Fermi particles after integration with respect to $p_\parallel$ and φ,

$$\frac{dn_q(p_\perp)}{dp_\perp}=Bp_\perp e^{-\frac{p_\perp^2}{2m_qT}},\quad B=\frac{1}{m_qT}, \tag{58}$$

$$\langle p_\perp\rangle_q=\sqrt{\frac{\pi}{2}m_qT}. \tag{58a}$$

Analogously, if we integrate with respect to $p_\perp$ and φ, we can obtain the distribution with respect to the longitudinal momentum $p_\parallel$. We present the results only for pions, for the tail of the distribution—at $p_\parallel^2 + p_\perp^2 + \mu^2 \gg T^2$, i.e., since $\langle p_\perp\rangle \approx 2.5\,T$, for practically all p

$$\frac{dn(p_\parallel)}{dp_\parallel}=\text{const}\cdot e^{-\frac{1}{T}\sqrt{\mu^2+p_\parallel^2}}\left(1+\frac{1}{T}\sqrt{\mu^2+p_\parallel^2}\right). \tag{59}$$

Additional formulas can be found in the handbook [80].

1) Mass dependence of $\langle p_\perp\rangle$. Thus, the distributions (56) and (56a) with respect to $p_\perp$ for pions and (58) for heavy particles with $m_q \gg \mu$ are different: the former is exponential with good accuracy, and the latter is Gaussian. The ratio of the mean values of (57a) and (58a) for heavy particles of different types (K, N, Λ, Ξ, Ω^-) is equal to

$$\frac{\langle p_\perp\rangle_q}{\langle p_\perp\rangle_\pi}=\frac{\sqrt{2\pi}}{5}\sqrt{\frac{m_q}{T}}\approx 0.50\sqrt{\frac{m_q}{T}}= \tag{60}$$

$$\begin{matrix} K & N & \Lambda & \Sigma & \Xi & \Omega^- \\ =0.96 & 1.33 & 1.45 & 1.50 & 1.57 & 1.77 \end{matrix} \tag{60a}$$

It is indeed known from experiment that $\langle p_\perp\rangle$ increases with the particle mass m_q. Unfortunately, a detailed comparison with experiment is made difficult by the fact that in the experiment one cannot yet separate heavy particles generated in the statistical sub-

*The cross section for the conversion of a nucleon into three quarks as a result of diffraction splitting was recently calculated theoretically [76], and the process was considered with the aid of formulas known for diffraction splitting of the deuteron. This calculation, however, is incorrect; no account was taken of the fact that if the nucleon is excited to a state with mass $\mathfrak{M} \geqslant 3m_N$, then such a system decays into a proton and a large number of pions ($n_\pi \sim (3m_q - m_N)/\mu \gg 1$) with a much higher probability than into three heavy quarks—in agreement with the formulas of the statistical theory.

†It is obvious that it contradicts strongly the Fermi statistical theory, where T increases with energy; (see (40)).

*Frequent use is made of the empirical formula of Cocconi, Koster, and Perkins (see [67], p. 234), which is known only from a preprint (UCRL 100222, 167, 1962)

$$dn/dp_\perp \sim \text{const}\cdot p_\perp\exp(-p_\perp/p_0),$$

and describes well the experimental data for pp collisions at 23 GeV, if it is assumed that $p_0 = 0.17$ GeV/c and $\langle p_\perp\rangle = 2p_0 = 0.34$ GeV/c. We see that $\langle p_\perp\rangle$ is in splendid agreement with (57), and the difference between the functional dependence (56b) and (56) or (56a) could probably not be observed at the 1962 experimental state of the art. A comparison with new data that confirm (56) is given below.

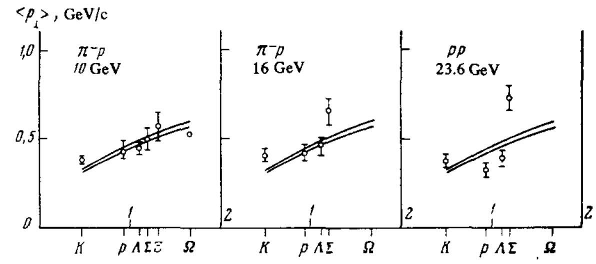


FIG. 16. Dependence of $\langle p_\perp \rangle$ on the particle mass. The experimental points are from the review [38] (Ω^- from [74]). Curves–statistical theory (58a) for $T = \mu$ and $T = 0.9\mu$.

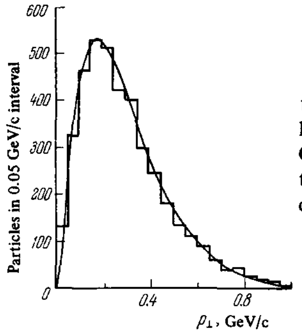


FIG. 17. Distribution with respect to $p_\perp$ of the pions generated in π^-p collisions in 10-prong events ($E_L \approx 25$ GeV). Curve–plot of the statistical-theory formula (56a) with the best choice $T_c = 0.85\mu$ (see the text of [83]).

It is curious that the π and K mesons, whose masses differ by a factor of 3.5, have accidentally close values of $\langle p_\perp \rangle$, owing to the different forms of the distributions (56) and (58).

Finally, Fig. 16 shows summary data on the values of $\langle p_\perp \rangle$, taken from the review [38], for different heavy particles. Since the results here are averaged over all the collisions, without picking out the central ones, one cannot expect good agreement with the statistical theory. Nonetheless, the discrepancy is not very large, and the growth of $\langle p_\perp \rangle$ with increasing particle mass is obvious.*

In comparison with measurements in cosmic rays and even in accelerators at high energies, it is necessary to take into account the fact that the motion of the statistical system (fireball?) as a unit in the common c.m.s. can be quite appreciable. It is superimposed on the distribution (56).

2) Dependence of $\langle p_\perp \rangle$ on the multiplicity and form of the distribution with respect to $p_\perp$. The Planck distribution with respect to $|p|$ in cosmic rays has been demonstrated quite long ago (see Fig. 11). The difficulty of measuring the particle momenta, however, has made these results not fully reliable. But now we have accurately measured distributions with respect to $p_\perp$ and $p_\parallel$ of the particles (which certainly are mainly pions) generated in π^-p collisions at $E_L = 25$ GeV[83] and in pp collisions in the E_L range from 13 to 28.5 GeV.[84] In both cases, the authors have stated that the distributions with respect to $p_\perp$ are in splendid agreement with the theoretical formula (56a), and those with respect to $p_\parallel$ agree with a formula that differs from (59) in that it has no factor $p_\parallel^{1/2}$. The data, however, are given for $p_\parallel \gg \mu$, where this difference is difficult to discern.

*The deviation from the theory is appreciable only at high energy, when the heavy particles can be generated also in central collisions, to which the theory does not apply.

Let us examine the distribution with respect to $p_\perp$. We write the experimental result in the form

$$\frac{dn(p_\perp)}{dp_\perp} = Ap_\perp^{3/2} e^{-a_\perp p_\perp}. \tag{61}$$

The agreement with this formula is demonstrated, for example, in Fig. 17, which is taken from [83] (cases with $n_{\pi\pm} = 10$ were selected).

Then a turns out to be strongly dependent on the multiplicity of the charged particles n_{ch}, which we can identify with $n_{\pi\pm} = 2n_\pi/3$. Namely, in π^-p collisions[83] and in pp collisions[84] we have (in $(\mathrm{GeV}/c)^{-1}$)

$$a_{\perp,\exp}^{\pi^-p} = 5.36 + 0.30 n_{\pi\pm}, \tag{62}$$

$$a_{\perp,\exp}^{pp} = (6.54 \pm 0.05) + (0.28 \pm 0.01) n_{\pi\pm}. \tag{62a}$$

If at first we disregard the dependence on $n_{\pi\pm}$, then we have at an average multiplicity $\langle n_{\pi\pm} \rangle = 4$–$5$ at these energies

$$a_{\perp,\exp}^{\pi^-p} \approx 6.7, \tag{63}$$

$$a_{\perp,\exp}^{pp} \approx 7.8. \tag{63a}$$

According to (56a), $a_\perp = 1/T$. Consequently, in the first case the decay temperature turns out to be 1.08μ, and in the second 0.92μ. These values are close to that determined in a perfectly independent manner from the composition of the generated particles (see (51)), thus demonstrating the theory to be highly successful.

However, the theory can do even more—it explains also the dependence of $a_\perp$ on $n_{\pi\pm}$.

What is the explanation, from the point of view of statistical theory, of the difference between the values of n_π for different cases of collisions of given particles at a given E_L? There are two explanations. First, fluctuations occur in the inelasticity coefficient K and in the quantities that are uniquely connected with it, namely the energy of the statistical system W and the average multiplicity $\langle n_\pi \rangle$. When a cluster having a given energy W expands, it can either decay until a critical temperature T_c is reached, or become "supercooled." In the former case its volume will be smaller than the average volume $\langle n_\pi \rangle V_0$ in decay, and in the latter case it will be larger. Accordingly, at $T > T_c$ the multiplicity will be smaller than the average, $n_\pi < \langle n_\pi \rangle$, and in the second case it will be larger, $n_\pi > \langle n_\pi \rangle$.

Thus, at any given K, the larger the multiplicity the smaller the decay temperature and consequently the larger $a_\perp$, as is confirmed by the experimental values (62) and (62a).

We present the calculation for a fixed value of K, meaning also fixed W.

Since at the instant of decay we have $V = n_\pi V_0$ and the temperature is T, we can write in accordance with (25a), identifying the total energy of all the pions with W,

$$W = \frac{g_\pi}{2\pi^2} n_\pi V_0 T^4 \Phi_B\left(\frac{\mu}{T}\right). \tag{64}$$

On the other hand, in the case of "normal" decay, at $T = T_c$, we have

$$W = \frac{g_\pi}{2\pi^2} \langle n_\pi \rangle V_0 T_c^4 \Phi_B\left(\frac{\mu}{T_c}\right). \tag{64a}$$

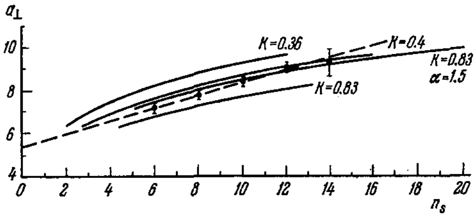


FIG. 18. Theoretical dependence of the system decay temperature on the multiplicity at different inelasticity coefficients K (solid curves; $\alpha = 1$ throughout with the exception of one curve, where $\alpha = 1.5$). The experimental points and the empirical curve (dashed) are taken from [83].

Consequently, T is expressed in terms of n_π from the equation

$$n_\pi \frac{\Phi_B(z)}{z^4} = \langle n_\pi \rangle \frac{\Phi_B(z_c)}{z_c^4}, \tag{65}$$

where $z = \mu/T$ and $z_c = \mu/T_c$.

A graphic solution of this equation yields T as a function of the ratio $n_\pi/\langle n_\pi \rangle$. For $n_\pi/\langle n_\pi\rangle = 0.50$, 1.00, and 2.00 we obtain z = 0.88, 1.03, and 1.21, respectively. Obviously, $a_\perp = 7.26z\ (\mathrm{GeV}/c)^{-1}$.

Figure 18 shows the experimental values of $a_{\perp,\mathrm{exp}}^{\pi^- p}$ and the empirical curve (62) drawn through them (dashed), taken from [83]. The solid curves show the theoretical relation under various assumptions concerning K, or equivalently, concerning $\langle n_\pi \rangle$, when the only free parameter of the theory α is assumed, as almost everywhere, to be equal to unity. In one case, for illustration, $\alpha = 1.5$ is assumed. The value K = 0.83 corresponds very roughly to a central $\pi^- p$ collision, when the proton enters the statistical system on a par with the pions.*

It is obvious that the statistical theory explains the observed dependence. The sensitivity to the value of K is low.

e. Longitudinal Momenta

Comparison with experiment was difficult here for a long time, because the experimenters did not obtain sufficiently detailed data in the region of very small $p_\parallel$ in the c.m.s., $p_\parallel \lesssim T \sim \mu$, where the particles from the statistical system should predominate. Recently, however, an important curve was finally published and gave the distribution of the negative particles (mainly, of course, π^-) with respect to $p_\parallel$ in the c.m.s. in $\pi^- p$ collisions with $E_L = 25$ GeV.[117] Figure 19 shows the experimental points, to which the following are added:

a) The distribution described by the theory (formula (59)) for statistical-system particles that are at rest in the c.m.s., at $T = \mu$ (thick curve 1) and also at $T = 0.9\mu$ and $T = 1.1\mu$ (thin curves). The only parameter chosen is the value at the maximum ($p_\parallel = 0$). We see that the statistical theory describes splendidly this main group of generated particles.

b) Curve 2, drawn "by eye," describing the small maximum at $p_\parallel \approx 3.4$ GeV/c, which corresponds clearly to elastically scattered pions (initial c.m.s. momentum 3.46 GeV/c).

*We note that in these collisions $\langle n_s \rangle \approx 5$, so that the experimental points correspond more to central collisions.

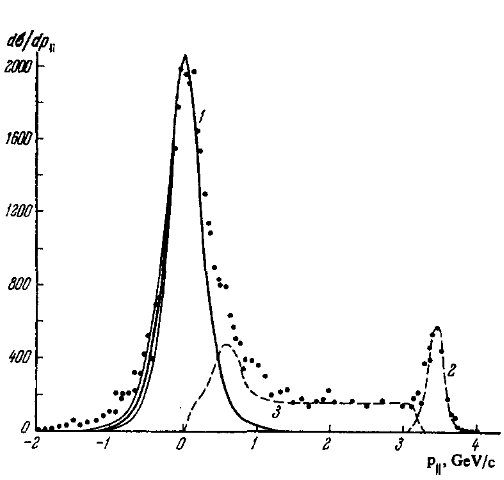


FIG. 19. Distribution with respect to $p_\parallel$ in the c.m.s. for negatively charged particles generated in $\pi^- p$ collisions at $E_L = 25$ GeV. Points—experiment [117]. Curves 1—statistical theory at $T = \mu$ (formula (59)); 2—elastic scattering (drawn "by eye"); 3—difference between experimental points and the sum of curves 1 and 2 ("nonstatistical" particles). Thin curves—theory at $T = 0.9\mu$ and $T = 1.1\mu$.

c) The excess (curve 3) of the experimental points (at $p_\parallel > 0$) over the statistical curve 1 and curve 2. It is obvious that this includes the primary pions that gave up part of their energy to the production of the statistical system, pions from the diffraction dissociation $\pi \to 3\pi$, etc. The small maximum on this curve at $p_\parallel \sim 1$ GeV/c may correspond precisely to the pions from the diffraction dissociation, when each particle retains on the average ⅓ of the primary momentum.

The total number of "statistical" particles (under the theoretical curve 1) is approximately twice as large as that of the "nonstatistical" ones (under curve 3).

The excess ("nonstatistical") particles at $p_\parallel < 0$ can be interpreted as "isobaric" particles from the collision-excited and decaying target proton.

We see thus a very good confirmation of the general picture of peripheral collisions (Fig. 3a) and of the statistical theory.

We note that the previously published experimental distributions for pp collisions at $E_L = 21$ GeV[84] and $\pi^- p$ collisions at $E_L = 25$ GeV[83] were described by the formula

$$\frac{dn(p_\parallel)}{dp_\parallel} = n a_\parallel e^{-a_\parallel p_\parallel}, \tag{66}$$

$$a_{\parallel,\mathrm{exp}}^{pp} = (0.76 \pm 0.03) + (0.41 \pm 0.01)\, n_{\pi\pm}\ (\mathrm{GeV}/c)^{-1}, \tag{67}$$

$$a_{\parallel,\mathrm{exp}}^{\pi^- p} = -0.23 + 0.39 n_{\pi\pm}\ (\mathrm{GeV}/c)^{-1}. \tag{68}$$

Identification with (59) yields the absurd values $T = (3-6)\mu$. Actually formulas (66)–(68) apparently describe in the main the "isobar" pions. In fact, by recalculating the isobar decay (assumed to be isotropic and statistical, with $T = \mu$ in its rest mass) to the c.m.s., we can verify that their distribution corresponds more effectively to a much higher temperature and prevails over (59) when $p_\parallel \gtrsim \mu$.

5. STATISTICAL LARGE-ANGLE SCATTERING

We proceed now to one of the unexpected accomplishments of statistical theory, namely its prediction of a special type of large-angle elastic scattering, which apparently has been confirmed experimentally.

This success is unexpected because we are dealing here with a rare phenomenon, in essence with a fluctuation. The theory might have turned out to be too crude here. This, apparently, is not the case. Equally unexpected was also the already described (Sec. 4 of Chap. 2) successful explanation of the dependence of the distribution with respect to $p_\perp$ on n (see Fig. 18). It turned to be the consequence of the fluctuations of the instant of decay into individual particles.

To be sure, in the question of statistical scattering we encounter for the first time a possible discrepancy (the absence of Ericson fluctuations). We shall show, however, that this may be, on the one hand, the consequence of simple still unaccounted-for facts, and on the other hand it leads to the most interesting and fundamental problem of the role of the statistical and fundamental principles in the elementary act (Chap. 6).

We are dealing here with the relatively rare central collisions of the type of Fig. 5, when all particles form a single system. As was discussed in detail in Chap. 4 (in connection with formula (47)), the fraction $\xi(K-1)$ of such collisions with an inelasticity coefficient $K \sim 1$ is small (see Figs. 9 and 14), and $\xi \sim 0.10$.

Among the numerous decay channels of such a system are also channels with only two final particles. In particular, the final two particles may coincide with the initial ones, and then the process will have the appearance of elastic scattering. Of course, if the total energy $W = \sqrt{s}$ is high and the average number of generated is accordingly also large, this will be a rare fluctuation. Its probability can be calculated from the general rule, as the ratio of the statistical weight of the two-particle state $\rho_2(W)$ to the summary statistical weight of all the possible final states $\sum_i \rho_i(W)$:

$$w_2 = \frac{\rho_2(W)}{\sum_i \rho_i(W)} \tag{69}$$

(we neglect for simplicity the two-particle final states in which the particles differ from the initial ones).

The foregoing possibility was indicated by Fast, Hagedorn, and Jones,[86] who also obtained a quantitative estimate.

The question arises, however, of how to calculate such a scattering against the background of ordinary scattering, and what are the characteristic features distinguishing qualitatively just this process.

Such a feature is, first, symmetry in the angular distribution with respect to $\theta_c = 90°$ in the c.m.s. for the entire statistical system. It is actually obvious that the memory of the initial direction of motion can be retained in the statistical system only because the angular momentum of the system differs from zero. It can reach a value on the order of $\theta_{max} \sim p_c\sqrt{\sigma_c}$, where $p_c \sim \sqrt{s}$ is the initial momentum of one particle in the c.m.s., σ_c is the total cross section of the central collision in question, and $\sqrt{\sigma_c}$ is its radius (the maximum impact parameter). Mutual permutation of the initial particles, however, cannot affect the properties of the intermediate system, and front-back symmetry of the scattering is essential. Since l_{max} differs from zero, the distribution need not be necessarily isotropic, but it is close to isotropy because l_{max} is not very large.

In general, other types of scattering do not have this type of symmetry (the obvious exception is the scattering of identical particles, pp → pp, etc.).

What are we to understand by "ordinary" scattering? First, diffraction or shadow scattering due to the presence of inelastic processes. Namely:

a) Diffraction peripheral scattering, which makes the main contribution, comprises, say according to quantum-field theory[22,23], shadow scattering as a result of inelastic processes of the type of Fig. 2a. From the point of view of the theory of complex orbital angular momenta, it can be described on the whole by a vacuum pole or by some other leading trajectory (an aggregate of cuts, etc.). This scattering is concentrated in the small-angle region, has a sharp forward directivity, and the cone width θ_0 decreases with increasing s, $\theta_0 \sim s^{-1/2}$ or $\theta_0 \sim (s \ln s)^{-1/2}$. It is known that a backward cone, described by the corresponding trajectory in the cross channel, is also possible, but it is in any case much weaker than the forward cone.

b) Multiple peripheral scattering in the region of intermediate angles, which also contracts into an ever-narrowing cone with increasing energy (see [87]).

c) Diffraction central scattering, which is shadow scattering due to inelastic central collisions of the type of Fig. 5. In this scattering, too, the cone contracts with increasing energy. Since, however, the central-collision radius is smaller than that of the peripheral one, this cone is broader than peripheral diffraction (item a in our list).

Appearing against the background of these types of scattering is the fundamentally different statistical scattering, which we shall now consider. It is not of shadow origin, its angular distribution depends little on the energy and is broad, and, as already mentioned, it is not described by a forward cone but, unlike all other types of diffraction scattering, is symmetrical about $\theta_c = 90°$.

It is now clear that statistical scattering is a relatively weak effect. Its total cross section constitutes only a very small part of the total cross section σ_c of the central collisions (rare decay channel), which itself is much smaller than the total cross section for peripheral collisions. It can be observed only at sufficiently high energy, when all the remaining (diffraction) contributions become compressed into a narrow forward cone (and possibly also a backward one). Then a weak contribution of statistical scattering can be observed in the region $\theta_c \sim 90°$. It must be emphasized that it can be observed, but only if its cross section decreases with energy more slowly than the tails of the narrowing diffraction scatterings decrease at a fixed angle θ_c (we shall see, incidentally, that it decreases with energy approximately exponentially). Whether this occurs or not depends on a subtle circumstance, on the ratio of these two types of scattering, which decrease rapidly with energy, in the region $\theta_c \sim 90°$.

Nevertheless, such a favorable situation apparently does take place. Figure 20 shows the experimental angular distributions of elastic np scattering from [88] for continuously increasing energies. We see that the forward cones, which extends at low energies farther than

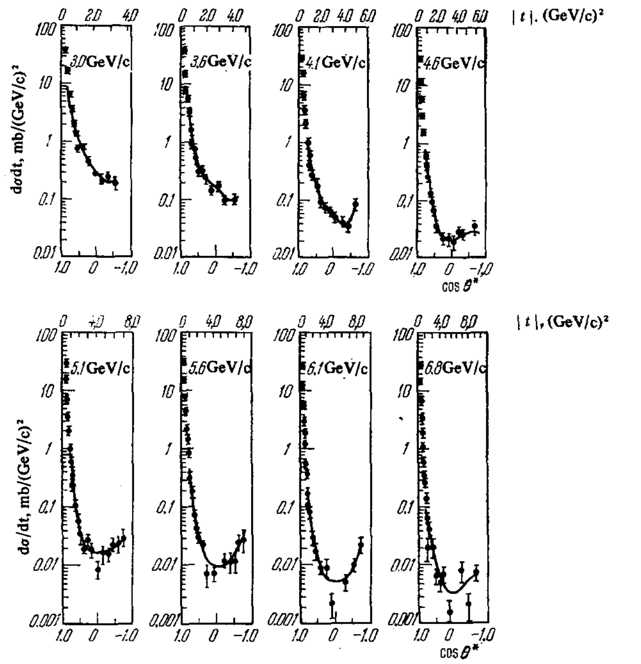


FIG. 20. Angular distribution in elastic np scattering for initial momenta that increase from 3.0 to 6.8 GeV/c. A symmetry with respect to $\theta_{cms} = 90°$ is observed upon contraction of the diffraction cone [88,89].

the angle $\theta_c = 90°$, contracts with increasing energy. Finally, at $p_L \sim 4.5$–5 GeV/c, we observe a symmetry with respect to $\theta_c = \pi/2$, of which there was no trace at lower energies.

Figure 21 shows the experimental data on π^-p and π^+p elastic scattering and illustrates the same effect.

In the case of elastic $\bar{p}p$ scattering, experiments at sufficiently high energies have not been performed, and there is no symmetry as yet. It is therefore concluded, unfortunately,[91] that the prediction of the statistical theory does not hold. This, of course, is patently premature. In this case it is also necessary to recognize that the statistical $\bar{p}p$ scattering can be weaker than statistical pp scattering (see below), and therefore symmetry about $\theta_c = \pi/2$ can be observed only at higher energies than for pp or np scattering. (It may turn out, of course, that the statistical $\bar{p}p$ scattering decreases with energy more rapidly than the tail of the diffraction scattering. Then it will be never be observed, as explained above, in contrast to the scattering of other particle pairs.)

The physical difference between statistical and diffraction scattering consists, as we see, in the fact that it proceeds via an intermediate compound state. This leads primarily to an entirely different scattering phase shift δ, namely, in the case of the compound state δ differs strongly from the phase δ_0 of the incident wave, in view of the appreciable delay of the particle emission because of the long lifetime of the compound state, $\delta - \delta_0 \gg \pi$. On the other hand, diffraction processes, and in general diffraction processes that proceed via exchange of a virtual particle or a Regge pole, are analogous to the direct processes of low-energy nuclear physics. They correspond to a phase change by an amount $\lesssim \pi$ (for more details see [13]). Statistical scattering is therefore not coherent with the remaining types of scattering, and it is necessary to add their cross sections rather than their amplitudes. Moreover, the very existence of the concept of scattering amplitude (and hence also scattering phase) for consistently considered statistical scattering is not a simple fundamental question (see Chap. 6).

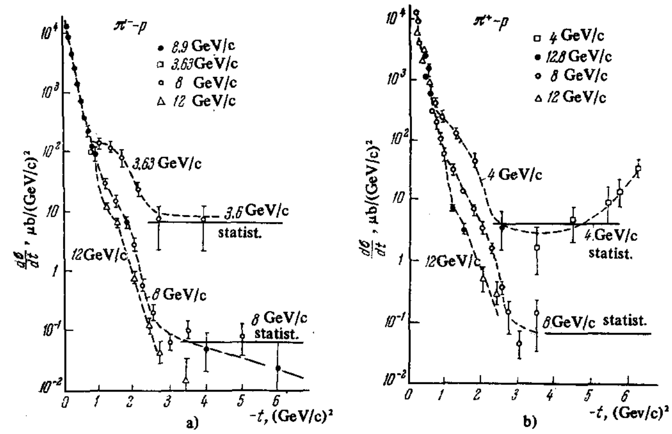


FIG. 21. π^+p and π^-p elastic scattering through large angles θ_{cms}. The predictions of the statistical theory [90] are shown in the vicinity of $\theta_{cms} = 90°$.

Going over to the theoretical formulas, we must first emphasize that, no matter how surprising this may seem, the quantitative calculation of the effect is neither simple nor unambiguous. This is undoubtedly caused to a considerable degree by the fact that we are dealing with a tremendous fluctuation, where we have two particles in place of $n \sim \langle n \rangle \gg 1$ in the final state. Its probability is exponentially small, and the customary fluctuation calculations are insufficient. There are several approaches.

In the first original papers[86] the numerator and all the essential terms (all the channels) of the denominator of (69) we calculated by the Fermi model with a computer (assuming in addition different volumes V_F for pions and for strange particles, the number of which can be large here). The result was then interpolated with the aid of the analytical expressions

$$w_2 \equiv w_{pp} = e^{-3.3(E-2m_N)}, \tag{70}$$

for pp scattering and

$$w_2 \equiv w_{\pi p} = e^{-3.17(E-1.40)}, \tag{71}$$

for πp scattering, where $E = \sqrt{s}$ is the total c.m.s. energy in GeV. $E = 2m_N$ is the maximum energy that can be consumed in the generation of new particles in pp collisions.

To obtain the total statistical-scattering cross section it is necessary to multiply w_2 by the cross section σ_{comp} for the production of a compound state (this is in essence the cross section for a central collision, $K \sim 1$), which constitutes a certain (small, cf. (22)) fraction of the total inelastic-scattering cross section σ_{inel}. This fraction is assumed in [86], on the basis of additional considerations, to depend on the energy and to equal $z^{-2}\gamma^{-2}$, where γ is the Lorentz factor of the incident particle in the c.m.s. and z ($z \sim 1$) is a numerical parameter determined from comparison with experiment,

$$\sigma_{stat} = \frac{1}{z^2\gamma^2}\sigma_{inel} w_2. \qquad (72)$$

The differential statistical-scattering cross section is obtained, assuming isotropy, by dividing by 4π.

This formula (in conjunction with (70) and (71)) was in splendid agreement with the experimental data (Fig. 22).[86]

In a subsequent theoretical paper Bialas and Weisskopf[61] (see also the paper by Cocconi[71]), starting from a thermodynamic approximation for the pion part of the system (we used this approach in Sec. 4c in the calculation of the composition of the generated particles). Namely, in NN scattering the principal channel of the denominator is the one with two nucleons and an arbitrary number of pions, and its total statistic weight in the Pomeranchuk model is given by (45a). The numerator, on the other hand, contains the statistical weight of a system containing only the two nucleons N_1 and N_2. Taking into account the energy and momentum conservation, we have (in the c.m.s.)

$$\rho_2 = g_{N_1N_2}\left(\frac{V}{(2\pi)^3}\right)^2 \int d\mathbf{p}_1\, d\mathbf{p}_2 \delta(\mathbf{p}_1 + \mathbf{p}_2)\,\delta(W - E_1 - E_2) \qquad (73)$$
$$= g_{N_1N_2}\frac{V^2}{4(2\pi)^5} W^2 \sqrt{1 - \frac{4m_N^2}{W^2}}.$$

The ratio of formulas (73) and (45a) gives exactly the analytic exponential relations (70)–(72), albeit with another pre-exponential factor. This calculation is not good, however, for two reasons.

First, the resultant ratio is not dimensionless (the same occurs in [61]). Second, formula (45a), which is based on the Pomeranchuk model, will hardly do here. In fact, we are dealing with the possibility that a compound system, which should decay into many particles after expansion to a volume $\langle n\rangle V_0 \gg V_0$, will accidentally decay only into two nucleons. It is difficult to imagine how this could occur during the final stage, when the system volume is many times larger than the aggregate volume of the two nucleons. The decay of interest to us should therefore occur during an earlier stage, when $V \lesssim 2V_0$. We put for concreteness $V = 2(4\pi/3)(1/\mu^3)\beta$, where $\beta \lesssim 1$. This means that in this case the calculation must be performed with a model having a fixed volume independent of the energy. The connection of the entropy with the energy of the system and with the temperature is then different from that in the Pomeranchuk model, where $S = W/T$. Namely,[61] from $VT^4 \approx W$ we obtain here (more accurately, for relativistic pions we have $\pi^2 VT^4/10 = W$, but $\pi^2/10 \approx 1$)

$$S = \int \frac{dW}{T} \approx \frac{4}{3} V^{1/4} W^{3/4} + \log C, \qquad (74)$$

where the constant C is determined separately, and equals, according to [61], $1/15 - 1/11$ for $0.512 \le \beta \le 1.33$. The obtained exponential dependence in the cross section is therefore entirely different[61]

$$\sigma_{stat} \sim e^{-\frac{4}{3}V^{1/4}W^{3/4}}, \quad V = 2V_0\beta. \qquad (75)$$

The parameter β is determined more precisely by comparison with experiment.

It is difficult, however, to note the difference between (70) and (75) in the energy interval for which experimental data are presently available.

We did not present the pre-exponential factor of (75) and its energy dependence. As already mentioned, it is an incorrect dimensionality. This is due to the fact that the thermodynamic approximation for the statistical weight of the pion part is dimensionless, $\sim \exp S$, while the statistical weight of the two-particle state, owing to allowance for the conservation laws (δ-function) has the dimension of $(\text{energy})^{-4}$.

By correcting this shortcoming, we can obtain[14] in the Pomeranchuk model

$$\sigma^{el}_{pp,\,stat} = \frac{\pi^2}{2}\sigma_{pp,\,comp}\left(\frac{W}{m_N}\right)^{2/3}\left(\frac{T}{m_N}\right)^{7/3}\frac{1-\frac{4m_N^2}{W^2}}{\left(1-\frac{2m_N}{W}\right)^{4/3}} e^{-\frac{W-2m_N}{T}} \qquad (76)$$

or in the model with fixed volume, which is physically more likely for the given concrete case,

$$\sigma^{el}_{pp,\,stat} = \frac{\pi^2}{2C}\sigma_{pp,\,comp}\left(\frac{W}{m_N}\right)^{5/4}\frac{1-\frac{4m_N^2}{W^2}}{\left(1-\frac{2m_N}{W}\right)^{3/4}}\frac{e^{-\frac{4}{3}V^{1/4}(W-2m_N)^{3/4}}}{(m_N V)^{7/12}}. \qquad (77)$$

No account was taken here of the isospin conservation, etc., or of the contribution of the channel with kaon generation (for details see [61]).

Figure 23 shows the energy dependences obtained in different approaches. So far, the experiments were performed actually in the region $W \sim (5.0\text{–}8.0)m_N$.

Even the last formula, however, has an unsatisfactory side to it: the entropy of the system pions in the small volume $\sim V_0$ is calculated as for a Bose gas. Yet the interaction of the pions is very large here (of course, the same occurred in Fermi's initial statistical theory of multiple generation, but this cannot serve as a justification).

Of particular interest is a comparison of pp and $\bar{p}p$ scattering. The point is that in $\bar{p}p$ collisions total annihilation is possible. Therefore a term $\rho^{(\pi)}(W)$ is added in the denominator of (69), and in the case of not very

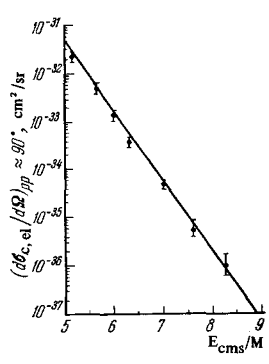
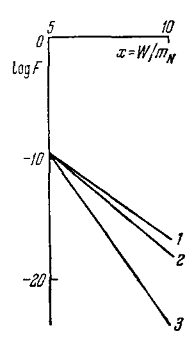


FIG. 22. pp scattering through $\approx 90°$ in the c.m.s. as a function of the energy. The continuous line is the result of the statistical theory [86] (computer calculation of the phase volumes in accord with the Fermi theory).

FIG. 23. Energy dependence of the statistical pp scattering in accordance with various models: 1–interpolation of numerical calculations by the Fermi model with a number of additional assumptions [86], which agrees well with experiment; 2–Pomeranchuk model (76) with $T = \mu$, which is physically inadequate in the given particular case; 3–model with fixed volume (77) of the type of [76], the volume being specified by the parameter $\beta = 0.5$ (a small change of β results in complete agreement with curve 1).



large W it exceeds the principal term for NN scattering $\rho^{(\pi)}_{N_1N_2}(W)$. Retaining only these two channels, we have

$$w_{2,\,\bar{p}p}=\frac{\rho_2}{\rho^{(\pi)}(W)+\rho^{(\pi)}_{\bar{N}N}(W)}, \tag{78}$$

and according to (74) we have $\rho^{(\pi)}(W) \sim \exp(4V^{1/4}W^{3/4}/3)$. Omitting from (78) the term $\rho^{(\pi)}_{\bar{N}N}$, which plays the principal role in $\sigma^{el}_{pp,stat}$, we obtain $\sigma^{el}_{\bar{p}p,\,stat}$, which differs from (77) not only in the pre-exponential factor, but also in the fact that the exponential is preceded by $W^{3/4}$ instead of $(W - 2m_N)^{3/4}$. Therefore

$$\frac{\sigma^{el}_{\bar{p}p,\,stat}}{\sigma^{el}_{pp,\,stat}}=\frac{2}{\pi^3}(m_NV)^{5/4}\left(\frac{W}{m_N}\right)^{3/4}\left(1-\frac{2m_N}{W}\right)^{3/4}e^{-\frac{4}{3}V^{1/4}W^{3/4}\left[1-\left(1-\frac{2m_N}{W}\right)^{3/4}\right]}. \tag{79}$$

Assuming that $W \gg 2m_N$, we obtain the exponential factor in (79) in the form $\exp(-2m_NV^{1/4}W^{-1/4}) \ll 1$ (similar reasoning for the Pomeranchuk model, $V \sim \langle n\rangle V_0$, yields $\exp(-2m_N/T_{eff})$; cf. [13]).

Thus, owing to the competition of a new annihilation channel, the p̄p-scattering cross section differs from the pp-scattering cross section by an exponentially small factor. When $E_L \sim 20$–30 GeV, however, this factor is significantly suppressed by a large pre-exponential factor ($\sim 10^3$–10^4). Therefore the conclusion that the statistical p̄p scattering is much smaller than the pp scattering, deduced earlier without allowance for the pre-exponential factor,[13,92] cannot be regarded as justified.

This analysis of the statistical scattering should be supplemented. It is necessary to take into account the shadow diffraction caused by formation of the compound system. The question is considered in detail in [93,94], and only such an analysis ensures unitarity of the theory (in an earlier paper,[95] where diffraction was not taken into account, it was concluded that the statistical-scattering theory is internally contradictory; this conclusion is thus unfounded). As a result, it is possible to describe the experimental data in great detail, although this requires the introduction of new parameters chosen by comparison with experiment (the compound-system radius or the maximum effective angular momentum of the scattering, etc.). It is clear that scattering at a fixed angle, for example at $\theta_{cms} = 90°$, at energies W below a certain critical value W_{crit}, so long as the principal role is played by diffraction tail, will be entirely different from that at above-critical energy, when statistical scattering predominates. Owing to the exponential dependence of both types of scattering on W, the change of regime at $W \sim W_{crit}$ should be abrupt, like a kink or break. This effect was indeed observed.[96] Figure 24 shows the experimental points and the theoretical curve (superposition of diffraction and statistical scattering) obtained for a definite choice of additional parameters (their values turn out to be reasonable).

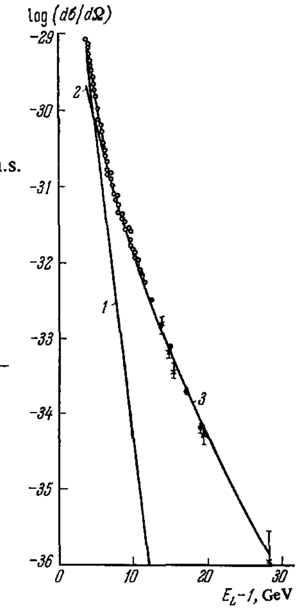


FIG. 24. pp scattering at a fixed c.m.s. angle, $\theta_{cms} = 90°$, as a function of the energy [96]. The kink at $E_L = E_{crit} \approx 6$ GeV is due, according to [93,94], to the fact that the diffraction cone contracts and the principal role at $E_L > E_{crit}$ is played by statistical scattering (70) and (72) (it is necessary to adjust the values of certain additional parameters. Points—experiment. Curves: 1—diffraction, 2—statistical scattering, 3—their sum.

This entire situation, which is quite favorable for the statistical theory, has been endangered in one aspect, namely the question of the so-called Ericson fluctuations. The question arose in analogy with another phenomenon, which also proceeds via compound-state formation, but encountered in low-energy nuclear physics. Namely, Ericson (cf., e.g., the review [97]) called attention to the following circumstance. Assume that the nucleus scatters, say, a nucleon whose energy (on the order of 5–20 MeV) is determined with an error ΔE such that

$$\Delta E \ll \Gamma, \tag{80}$$

where Γ is the width of the compound-nucleus level, and the levels of the nucleus overlap, $D \ll \Gamma$, where D is the distance between levels. Then the interference of the scattering by many levels leads to abrupt oscillations of the cross section (which frequently decreases to zero) when the incident-particle energy is changed by an amount on the order of Γ. This prediction was fully confirmed by experiment, and the phenomenon itself served as a starting point for a whole trend in nuclear physics. It might seem that such a phenomenon should occur also in statistical scattering in the GeV energy region. Of course, we do not know here the value of Γ for the compound system produced by a central collision of two fast nucleons. But if its lifetime is estimated at $\tau \sim 1/\mu$ ("natural time scale"), then the one can also estimate the required monoenergeticity of the colliding nucleons (80), which turns out to be attainable. Experiments performed at CERN with a tenfold margin of ΔE at $E_L = 16$ GeV[98] led to a definitely negative result, and no fluctuations were observed. This justified the conclusion that either the concept of statistical scattering is correct and its earlier successes were fortuitous, or for some unclear reasons the theory of Ericson fluctuations cannot be applied to our case.

Such possible reasons can undoubtedly be indicated. The analogy with the phenomena in low-energy nuclear physics cannot be regarded as convincing. Thus, the nucleon-nucleus system has at low energies clearly pronounced resonant levels. On the other hand at energies of several times 10 GeV there are no such levels. Further, in pp scattering at GeV energies one deals with decay of the compound system via a channel with exponentially low probability, whereas the number of particles in the intermediate state is indeterminate, and in low-energy nuclear physics one considers the main channels and the number of particles is conserved in all stages. Of course, it is not directly clear why the latter differences can exclude Ericson fluctuations. But there is one more fundamental and interesting possibility, which deserves a particular discussion, and to which we now proceed.

6. FUNDAMENTAL PROBLEMS OF STATISTICAL THEORY. DYNAMICS AND "TRUE STATISTICS"

The justification of the statistical treatment of multiple generation always begins, starting with the papers of Wataghin and Fermi, from the quantum-mechanical formula (1). It is therefore usually understood that the process is described by a certain amplitude and is therefore dynamic, as is always the case in quantum mechanics. Namely, if the initial wave function (or functional) is given, then the final wave function (or functional) is uniquely determined. In particular, the process is reversible, and the squares of the moduli of the direct- and inverse-transition matrix elements coincide. In such an approach the statistical theory is only an approximate method of calculating the quantum-mechanical quantities.

The consistent application of the statistical approach, however, has long ago led, in essence, outside the framework of these ideas. Thus, in the Landau hydrodynamic theory, which as logical generalization of the statistical theory, one considers the time variation of the entropy of a closed system. Yet in quantum mechanics the entropy is conserved for any closed system not subjected to the process of measurement (observation).[99] The entropy can vary only in the "truly statistical theory."

Further, an attempt to use the transition probability obtained from the statistical theory in the dispersion relations[100] led to absurd results. It turned out that to satisfy the unitarity conditions the moduli of the S-matrix elements obtained from the statistical theory must be again renormalized in the intermediate state. This operation contradicts the usual S-matrix approach and denotes the assumption that an observation act, in other words interference by an external system, takes place in the intermediate state.

All this has stimulated D. S. Chernavskiĭ [101,102] (see also [14]) to ask the following interesting question: are not the systems studied in the statistical theory of multiple generation "truly statistical"? Then they should be described not by a transition amplitude, but by a density matrix.

In classical mechanics the question of the cause of difference between statistical and dynamic systems has by now been sufficiently well clarified[103,104] by procedures pointed out qualitatively long ago (Borel and others). Simplifying the question somewhat and making it less subtle, we can state that everything depends on the dynamic stability or instability of the system. Namely, if the parameter x characterizing the state of the system (coordinate, etc.) deviates in the course of time from the "unperturbed value" x(t) by an arbitrarily large amount δ(x) that depends exponentially on t when the initial value x_0 (at the instant t = 0) changes by an amount δx_0,

$$\delta x = \delta x_0 e^{\lambda t}, \quad \lambda > 0, \tag{81}$$

then the system is unstable and becomes statistical. In fact, if the system has such a property, then an arbitrarily weak external action taking place at some instant during the evolution of the system, makes the system practically irreversible; if in the final state (at an instant $t \gg 1/\lambda$) all the particle velocities are reversed, then the system returns to the initial state x_0 only following a practically improbable event, namely if an equal but oppositely directed small perturbation (and no other, even an arbitrarily small one!) acts at precisely the required instant of time. This means that such a system cannot be isolated in practice from external actions. From the point of view of quantum mechanics this would mean that it should be described by a subsystem of a large system with which it interacts, i.e., it should be described by a density matrix.

These ideas have not yet been transferred to quantum field theory in the manner of classical mechanics (see [17] for a somewhat more detailed development of the indicated ideas in quantum mechanics, particularly the question of entropy conservation). It has therefore not been possible as yet to investigate the stability of a system containing statistical scattering of high-energy particles when many degrees of freedom are virtually excited and the number of particles is indeterminate. Instead, a model example was considered, showing how the change of purely quantitative characteristics (masses and velocities of incident particles, etc.) can cause the scattering process in nonrelativistic quantum mechanics to change from dynamic (dynamically stable) into statistical (dynamically unstable).

Namely, the authors of [10] have considered scattering of particles with mass m and momentum p by an assembly of scattering centers randomly distributed inside a volume of radius R, such that the maximum scattered angular momentum is large, $L_{\max} = pR \gg 1$. The potential V(r) as a function of the point r is a stochastic quantity with correlation radius r_0, mean value zero (for simplicity), and rms change V_0. The solution of Schrödinger's equation for this case yields an output partial-phase shift δ_l (at r = R). This phase shift is to be investigated for stability (in essence, in the sense of Lyapunov), following a small perturbation of the potential by an amount $\delta V \ll V_0$ at a certain point. If the system is dynamically stable, then δ_l changes by a small amount ξ_l on the order of $\delta V/V_0$. Actually, however, in the general case the solution takes the form

$$\xi_l(R) = \xi_l^0 e^{\lambda_l(R)}, \tag{82}$$

where $\xi_l^0 \sim \delta V/V_0$ is the small phase change due to the local perturbation of the potential, and $\lambda_l(R)$ are the eigenvalues of the problem. Their sign and magnitude at a given R (which plays the same role as the time in the classical problem referred to above) indeed determines the dynamic stability of the system. It can be stated that a role is played by a certain effective value $\overline{\lambda}(R)$, equal in the absence of spherical symmetry of the potential V(r) to

$$\overline{\lambda}(R) = \frac{mV_0}{p^2 (pr_0)^{1/2}} L_{\max}^{3/2} e^{-\frac{1}{2}\left(\frac{pr_0}{2}\right)^2}. \tag{83}$$

We see therefore that the distortion of the phase ξ_l may be the far-from-small quantity of the order of $\xi_l \sim V/V_0$. It depends exponentially on $L_{\max} = pR$ and on other parameters, and is doubly exponential in pr_0. The system is dynamically stable and insensitive to external small perturbations δV (i.e., it is dynamic), if $\lambda(R) \lesssim 1$. The system is dynamically unstable (and becomes statistical, in accordance with the foregoing) if $\lambda(R) \gg 1$. We see that the transition from one case to the other is very abrupt when p, r_0, R, m, and V_0 are changed. This is the principal and fundamental conclusion of the problem. The obtained formula can be applied to the concrete phenomena of interest to us only conditionally and approximately.

Thus, for example, if we wish to estimate the situation when a nucleon of kinetic energy ~10 MeV is scattered by a nucleus when the scattering goes through a compound state, we must imagine that the scattering centers are fixed (as is seen, for example, from Chap. 3 of [105], this does not interfere with formation of Ericson fluctuations). The average momentum of the nucleon entering such a system, $p \sim \sqrt{2mV_0}$, is $p \sim 250$ MeV/c at $V_0 \sim 30$ MeV. The correlation radius of the potential is determined by the thermal waves in the nucleus. Since the temperature here is low, $T \sim 1-5$ MeV, the correlation radius determined by these waves, $1/T \sim (30-50)/\mu$, is much larger than the radius of the nucleus. Consequently, the correlation radius should be taken to mean the radius of the nucleus, $r_0 \sim R$. Further, the effective L_{max}, as shown by experiment, is of the order of 2–3 in light nuclei and of the order of 7 in heavy ones. All this yields $pr_0 \sim pR \sim (250/140)A^{1/3} \sim 2A^{1/3} \sim L_{max}$, where A is the atomic weight of the nucleus. The exponential $\exp[-\frac{1}{2}(pr_0/2)^2]$ is consequently small, $\exp(-A^{2/3}/2) \ll 1$, and $\overline{\lambda(R)} \sim \frac{1}{2}(2-7)\exp(-A^{2/3}/2) \lesssim 1$. The system is thus dynamically stable, insensitive to external perturbations, and it is meaningful to speak of the amplitude of the process. This explains why Ericson fluctuations appear here, since they are typical of dynamic and not of statistical systems.[101,105]

It is even less legitimate to use formula (83) for statistical NN scattering at $E_L \sim 10-20$ GeV, for here we are dealing with a relativistic problem with an indeterminate number of particles. Nonetheless, the authors of [101] obtain an estimate for this case, too, purely for the sake of illustration.

In this case the intermediate state is a system of pions (and two nucleons) with particle energy on the order of the temperature T, which is much higher than in the nucleus, $T \gtrsim \mu$. The dimension of the system is of the same order, $R \sim \mu^{-1}$. L_{max} can be estimated from the c.m.s. momentum p_0 of the incident nucleon. For $E_L \sim 16$ GeV we have $p_0 \sim W/2 \sim 19\mu$ and $L_{max} = p_0R \sim 19$ (p_0 is in no way equal to p of the particle in the statistical system!). Furthermore, we can effectively put $V_0 \sim p$, and consequently also $mV_0/p^2 \sim 1$.

Thus, $\overline{\lambda(R)} \sim 19^{3/2}e^{-1/8} \sim 80$. Consequently, $\overline{\lambda(R)} \gg 1$ and the system is dynamically unstable. The phase perturbation differs tremendously from the small quantity $\delta V/V_0$, namely $\xi \sim (\delta V/V)e^{80} \sim (\delta V/V) \times 10^{35}$. It suffices to perturb the potential by an amount $\delta V \sim V_0 \times 10^{-35} \sim 10^{-27}$ eV to produce a phase change on the order of unity.

Of course, such numerical estimates are far from trustworthy in this case. What matters is another fact: owing to the high temperature produced in the system during the course of statistical generation, the correlation radius r_0 is small and the exponential factor $\exp[-\frac{1}{2}(pr_0/2)^2]$, which decreases $\overline{\lambda}$ strongly in the compound nucleus, drops out. $\overline{\lambda(R)}$ can therefore be large. Consequently, outwardly similar physical systems (the compound nucleus in low-energy nuclear physics and the compound state in statistical theory of multiple generation and scattering of relativistic particles) may have fundamentally different properties.

This example shows thus the possibility of occurrence of a relativistic many-particle system having extreme instability to a negligible change of the external conditions, the electric or magnetic field of an atom moving by, detachment of an electron from a hydrogen shell in the target, emission of soft electromagnetic radiation (a radio quantum) by a recoil proton—any one of them suffices to prevent the system produced in statistical nucleon-nucleon scattering or generally in multiple generation from being regarded as an isolated system (the number of particles is immaterial here and can be small). In this case we have a truly statistical system describable by a density matrix but not by an amplitude.

It is easy to show that in this case there should be no Ericson fluctuations in pp scattering.[14] This is the most general way of eliminating the contradiction between the statistical theory and experiment in the only point where such a contradiction may arise.

Of course, the results reported in this section should be regarded as a statement of the problem, not its solution. It is, however, obvious that even such a statement of the problem is very important for elementary-particle physics. Thus for example, if "true statistics" actually obtain in collisions, then the customarily considered amplitudes of processes for strictly defined momenta should be regarded as abstractions, and they must be replaced by certain averaged quantities, etc. It is interesting that the question of the need for averaging of this type is raised in an entirely different connection by theoreticians engaged in the problem of eliminating the fundamental difficulties of modern theory.[106,107]

7. CONCLUSION

We have attempted to show that the statistical theory of multiple generation of particles is physically adequate and of practical use if correctly employed. Of course, it cannot claim a general and complete description of the interaction process. But at yields surprisingly much as an auxiliary tool, and describes many characteristics of the process with a quantitative accuracy that is unexpected for such an approximate theory.

It is probably even indispensable in the quantum field (or in the reggeized) theory that predicts formation of clusters (fireballs) of many (5–10) particles. In fact, it is not clear how the decay of such a cluster can be presently described in any other way.

In considering various applications we encountered several times the same difficulty, namely, the experimental data do not make it possible the energy W of the statistical system to which a given experiment pertains. We were forced occasionally to resort to mean values of the inelasticity coefficient, etc. Yet, if the experimental data were to be classified in accordance with the values of K, then it would be possible to carry out a more detailed and more reliable analysis. An example is provided by the pp-scattering data shown in Fig. 14, and the associated analysis of the $p_\perp$ distribution for protons. It would be nice to be able to determine the value of W of the statistical subsystem also in other cases.

Finally, in Chap. 6 we reported briefly on results [101,102] indicating that the statistical theory can play a major fundamental role in elementary-particle physics. This possibility was actually only outlined, but it is significant and attractive enough to warrant a most intensive further study.

APPENDIX

STATISTICAL THEORY WITH LORENTZ INVARIANT PHASE SPACE

A statistical-theory variant proposed by Srivastava and Sudarshan[109] has been used in a number of papers. It reduces to replacement of dp_i in the expression for the phase space of the i-th particle by the relativistically invariant quantity dp_i/ϵ_i, where $\epsilon_i = (p_i^2 + m_i^2)^{1/2}$ is the particle energy (Lorentz-invariant phase space—LIPS). In this case the three-dimensional volume is taken into account by a factor V independent of either the multiplicity or the energy. In addition (this is widely done, but is not connected with the LIPS), different volumes V_i are chosen for different particles. In order to obtain at a fixed V the experimentally observed fraction of heavy particles, it is necessary to assume, for example, that V_K for K-particles is one-tenth that for pions. This lack of equilibrium between the K and π particles seems highly unsatisfactory to us.

The formula actually employed for the channel probability (cf., e.g., [110]) is (in the c.m.s.)

$$w = A \prod_{i=1}^{n} \left[(2s_i + 1)\, m_i \frac{V_i}{V_0} \right] R_n(W), \qquad (A.1)$$

$$R_n(W) = \frac{1}{(6\pi^2)^n} \int \delta^3\left(\sum_{i=1}^{n} p_i\right) \delta\left(W - \sum_{i=1}^{n} \varepsilon_i\right) \prod_i \frac{d^3p_i}{\varepsilon_i}; \qquad (A.2)$$

here W is the total energy of the system, s_i and m_i the spin and mass of the i-th particle, and A an indeterminate general factor.

In this case we can say that the Fermi, Pomeranchuk, and LIPS models differ in essence only in that different volumes are used:

for the Fermi model $\quad V_F = V_0 \dfrac{2M}{W}, \qquad$ (A.3a)

for the Pomeranchuk model $\quad V_P = V\langle n\rangle, \qquad$ (A.3b)

for the LIPS model $\quad \beta_i \dfrac{V_0}{\varepsilon_i}, \quad \beta_i = m_i \dfrac{V_i}{V_0}. \qquad$ (A.3c)

If we put by way of an estimate for ϵ_i in (A.3c) the average energy $W/\langle n\rangle$, then we obtain for the multiplicity $\langle n\rangle$, according to (6),

for the Fermi model $\quad \langle n_F\rangle \sim W^{1/2}, \qquad$ (A.4a)

for the Pomeranchuk model $\quad \langle n_P\rangle \sim W, \qquad$ (A.4b)

for the LIPS model $\quad \langle n_{LIPS}\rangle \sim \beta_i^{1/3} W^{2/3}, \qquad$ (A.4c)

i.e., in the LIPS model the multiplicity is intermediate between the Fermi and the Pomeranchuk models.

The application of the KIPS model to experiments yielded good results, but special assumptions had to be made concerning V_i. Namely, an analysis of experiments on the annihilation $p\bar{p} \to$ pions + kaons at $E_L \lesssim 7$ GeV[110,111] has shown that the experimentally observed values of $\langle n_\pi\rangle$ and $\langle n_K\rangle$ are obtained by assuming for pions $V_\pi \sim (4-8)V_0$ and for kaons $V_K \sim 0.1V_\pi$, i.e., $\beta_\pi \sim (4-8)m_\pi$ and $\beta_K \sim (0.4-0.8)m_K$. Since in $p\bar{p}$ annihilation in this energy region we have $\langle n_\pi\rangle \sim 4-8$ and $\langle\epsilon\rangle$ is a constant quantity, $\langle\epsilon_\pi\rangle \sim 0.4-0.5$ GeV (see (32), it follows that (A.3b) and (A.3c) coincide when $\beta_\pi \sim \langle n_\pi\rangle m_\pi$, is was indeed found in [110]. Thus, comparison with experiment makes it necessary to choose in the LIPS model the same volume as in the Pomeranchuk model. This does not suffice, however, to obtain the correct composition. It is necessary to choose very small V_K, whereas in the Pomeranchuk model the correct composition is obtained automatically.

Of course, with increasing W, if we retain approximately the same volume $V_{LIPS} \sim (4-8)V_0$, then the fraction of K particles will increase (it reaches 50% in the tables calculated in [110]), in clear contradiction to the experiment.

We note, however, that according to [110] the energy dependence of $\langle n_\pi\rangle$ in $p\bar{p}$ annihilation is in good agreement with the LIPS model. It can be verified that the prediction of formula (33) gives at the same time too high a value of $\langle n_\pi\rangle$ (by approximately $\Delta n_\pi \sim 1$). It hardly pays, however, to draw any conclusions from this and to attempt to refine the theory. In fact we do not know, for example, what contribution is made in the experiment by diagrams of the type of Fig. 12, in which the multiplicity is smaller than for Fig. 5a.

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