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Methodological Notes

TWO PARADOXES OF THE SPECIAL THEORY OF RELATIVITY

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This was sometime a paradox,
but now the time gives it proof.

W. Shakespeare

In the history of any science, along with the fundamental problems which determine the essential advance of the science at times, problems, or questions arise which are by no means of a front rank or fundamental nature. At first no attention is simply paid to these problems, but on one fine day they unexpectedly elicit interest and a series of papers appears; different authors give different answers, and sometimes these answers turn out to be diametrically opposite. This evokes considerable surprise, since questions of this kind have long ago fallen behind the advancing frontier of the science and lie in the domain where, in the opinion of many, everything is clear. Thus, peculiar paradoxes arise. As is the case in any paradox, the decipherment and the solution lie in the elucidation of the incorrect statement of the problem. Two problems of this type which have given birth to quite a number of articles in different physics journals (the number of such articles continues to increase) will be dealt with in the present article.

1. THE APPARENT SHAPE OF RAPIDLY MOVING BODIES*

In 1892 Lorentz formulated an unexpected assertion which enabled him to explain the result of the Michelson experiment. Lorentz supposed that all bodies moving with respect to a universal ether, which was regarded as a kind of a medium, undergo a contraction in the direction of motion. An analogous statement is also due to Fitzgerald, so that in the literature there are frequent references to the Lorentz-Fitzgerald contraction (concerning the role played by Fitzgerald cf.,[1]).

After the special theory of relativity was formulated the contraction of the length of a scale in the direction of its motion became a direct consequence of the Einstein postulates; in particular, it is an elementary consequence of the Lorentz transformation: the contraction is observed when one measures the length of a scale moving with respect to an observer who carries out a measurement of the length of the scale.

The first paper of Einstein contains in connection with this the following statement: "... A body, which in a state of rest has the shape of a sphere, is seen from a reference system at rest to take on the shape of an ellipsoid with semiaxes $R(1-\beta^2)^{1/2}$, R, and R when it is in motion ([2], p. 18). Apparently Einstein has in mind a Lorentz contraction in the direction of motion and constancy of transverse dimensions. Six years later, in a polemic with Varichak, Einstein answers two questions at the same time (cf.,[2], p. 187).

1. "The question of whether the Lorentz contraction is real or not has no meaning. The contraction is not real, since it does not exist for an observer who moves together with the body; however, it is real, since it can be in principle proven by physical means for an observer not co-moving with the body."

2. "We obtain in a reference system K the shape of a body moving with respect to this system by determining the points in the system K with which at a given instant of time t material points of the moving body coincide. Since the concept of simultaneity utilized in carrying this out is so defined that on the basis of this definition a demonstration of simultaneity by an experimental method is in principle possible, the Lorentz contraction is also in principle observable." This, of course, exhausts that aspect of the problem which is based on principle.

But returning to the problem of the measurement of the Lorentz contraction of a rod, it must be said that if the measurement of the length is carried out in accordance with the rules of the theory, i.e., if one records the coordinates of both ends of the moving rod simultaneously in the system with respect to which the length is being measured (for this two observers or two pieces of apparatus are required situated at two points of the reference system) then the expected result of the experiment (contraction) is not in doubt.

But now fifty years after the creation of the theory of relativity a somewhat different question has arisen: could one discover the Lorentz contraction by photographing a rapidly moving body or visually observing it? Of course, we are speaking only of thought experiments. Therefore, setting aside the physiology of vision, one need not make a distinction between visual observation and photography. The problem posed above, of course, does not touch upon the principles of the theory, but it is useful to have a clear and unambiguous answer to this problem because a photograph, which could qualitatively and quantitatively demonstrate the Lorentz contraction, would be a direct proof of the

*Throughout we are dealing with relative motion with relativistic velocities, i.e., with velocities of the order of the velocity of light in vacuo. All the systems of reference under discussion are inertial.

reality of the contraction (in the sense of which Einstein spoke; cf., above). However, the answer to the question posed above has turned out to be not so simple, while the direct proof of the contraction of time intervals between events has already been known for a long time: the increase in the lifetime of unstable particles (for example, of π and μ mesons) in a system with respect to which they are in motion has been established experimentally.

The word "observation" in the article of Einstein quoted above could be interpreted as visual observation or, perhaps, as photography. Such an interpretation, apparently, is the one that led to the unanimous conviction that by observing (photographing) a moving sphere we would observe an ellipsoid in the photograph. For quite a long time the fact was not realized that a determination of the shape and the dimensions of a body as a simultaneous position of all the points of its surface, and the image of the body obtained on a photograph is, generally speaking, by no means the same thing. Here we should note two points. Suppose that we are making a photograph with an infinitesimally short exposure time. Then the plate will be reached by rays which arrived at the objective lens simultaneously. But if the different points of the body are situated at different distances from the objective lens, then the rays emitted from these points—owing to the finite velocity of propagation of light—require different times to reach the objective lens. Consequently, if the body is emitting light continuously, then the plate will be reached at the same instant by rays emitted by different points of the body at different instants of time. For a body which is at rest with respect to the camera this circumstance will not affect the image obtained. In the case of a moving body the image obtained will differ from the image obtained in photographing a stationary body. This effect is due simply to the finite velocity of the propagation of light and has no relation to the Lorentz contraction. This is the first point. The second point consists of the fact that when one speaks of the apparent shape of an object one usually has in mind the image obtained on the plane of a photographic plate or (with certain limitations) on the retina of the eye. But such an image represents the projection of the body on a plane. If we return to the problem of photographing the body whose Lorentz contraction we wish to observe, then it is required to "capture" this contraction on a two-dimensional projection of the body. The points mentioned above demonstrate the sources of the non-unique interpretation of the image obtained on the photograph. First of all, it is clear that from a single photograph it would be impossible to determine anything at all without having additional information. For example, having a single photograph of a rod (one-dimensional body) moving against a homogeneous background we can say nothing about its length, and on the basis of a single photograph of a three-dimensional extended body it is impossible to reproduce its shape. It should be emphasized that the second point also is not related to "relativistic effects"; however, they were noted in connection with the discussion of the outward appearance of a rapidly moving body, the greatest interest being aroused by the question as to the manner in which the Lorentz contraction affects the apparent shape.

The discussion began with the publication in 1959 of the article by Penrose "The Apparent Shape of a Relativistically Moving Body"[3]. Penrose's article was by no means trivial; it investigated for the first time the conformal properties of the Lorentz transformation. In particular, in this paper it was shown that a moving sphere will not differ from a stationary one in terms of its two-dimensional projection on a photographic plate, or, more accurately, in terms of the shape of its contour. The physical explanation of this result follows from the article by Terrell[4]. Terrell solves the problem quite radically, as is evidenced by the title of his article "Invisibility of the Lorentz Contraction." It is just after these articles that the problem arose as to whether one can in general by any method whatsoever observe or photograph the change in the dimensions of a body as a result of the Lorentz contraction*. In order to avoid misunderstandings we repeat that the physiology of vision makes the photographing of a body and its visual observation essentially different procedures. Speaking of visual observation, we shall have in mind an eye possessing ideal properties close to the properties of a photographic plate.

We explain Terrell's result. If one observes a moving body with a luminous surface, the photographic plate records in the case of an infinitesimally short exposure time the simultaneous signals (photons) emitted by different points on the surface of the body. Since different points on the surface of the body, generally speaking, are situated at different distances from the photographic plate, the plate records the positions they had at different instants of time. The fact that the plate records or the observer "sees" at a given instant of time different portions of the surface of the moving body in those positions which they occupied at different times, leads to a curious result which can be illustrated by a simple example.

We imagine a luminous cube moving along a straight line parallel to one of its edges flies past a photographic camera (or an observer). The photography or the observation occurs at the moment when the center of the cube reaches the normal drawn from the point where the photographic camera is situated to the direction of motion. Of course we must know beforehand that the moving body has the shape of a cube in its own reference system.

At a given instant of time the plate will be reached by all photons emitted simultaneously in the reference system of the plate along the line AD, and the photons emitted by the point B earlier by a time interval l/c (l is the length of the cube edge). But at this instant the point B was situated in the position B′. The simultaneous determination of the positions of points A and D in the reference system of the plate leads, in accordance with the usual rule for the measurement of length, to the Lorentz contraction: $l' = l(1 - \beta^2)^{1/2}$. On the other hand $BB' = (l/c)v = \beta l$.

From Figs. 1b and c one can deduce that the picture that would be seen by a stationary (idealized) observer in observing a moving cube coincides with that observed when a stationary cube is rotated through an angle φ.

*Soviet readers are acquainted with an exposition of Terrell's results in Weisskopf's article [5], a translation of which was published in Usp. Fiz. Nauk 84, 183 (1964).

The angle is determined by the relation $\sin\varphi = \beta$. This is a particular case of a more general result due to Terrell: every three-dimensional moving body is seen at a given instant as having been rotated. The angle of rotation for the situation shown in Fig. 1 is determined by the equation $\varphi = \arcsin\beta$. But if the cube is situated with respect to the observer in such a way that at best it would be seen at an angle ϑ' with respect to the x′ axis, then the angle of rotation will be different. If the cube is sufficiently far away from the observer, then the light coming from it can be assumed to be a parallel beam. When this beam is observed in the system K, then for an observer in K it is propagated at an angle ϑ to the x axis, while the angles ϑ and ϑ' are related by the equation

$$\cos\vartheta = (\cos\vartheta' + \beta)/(1 + \beta\cos\vartheta').$$

The change in the direction of the front of a plane wave in going over from one coordinate system to another (which are in relative motion with respect to one another) is none other than the aberration of light. As far as the image obtained on a plate (or an idealized visible image) is concerned, it corresponds to a cube (observed in K at an angle ϑ) rotated through an angle $\vartheta - \vartheta'$. Now it is no longer difficult to understand Penrose's result: a rotation of the sphere does not alter the shape of its contour. The central point in the investigation of Terrell is that in fact he investigated for the first time the visible shape of a three-dimensional body.

Thus, the combination of a "contraction" with the finiteness of the velocity of light can lead to an apparent rotation. Therefore the question arose whether it is possible in general to distinguish from the visible picture[6] between a contraction and a rotation. Such a statement of the problem is simply incorrect. The reconstruction of a three-dimensional body from a plane photograph requires additional information, and this circumstance has no relation whatever to the Lorentz contraction.

Returning to the example involving a cube, it is evident that knowing how the cube moves one can always establish by a "direct observation" or by photography that it is in fact a contraction and not a rotation that takes place. For this one must simply have two observers or two photographs made from two positions situated along normals to two perpendicular edges of the cube (parallel to the motion). If the change in shape of the cube on the photographs is to be interpreted as a rotation, two different axes of rotation will be found. But both observers will interpret the observed picture without contradiction as a contraction of the dimensions in the direction of motion.

And still, is it possible to photograph a body which has undergone a Lorentz contraction? As we have demonstrated, observation of moving three-dimensional bodies presents definite difficulties in the interpretation of the photograph obtained. But Lorentz contraction can be established by observing a one-dimensional object, and one can make use of this. The contraction will become obvious and evident if one compares the length of a moving one-dimensional rod with its proper length. In the already mentioned Einstein's argument concerning a sphere the role of a comparison scale was played by the diameter of the sphere perpendicular to the direction of motion. It would be very convincing to photograph a moving rod against the background of its proper length marked off in the observer's system.

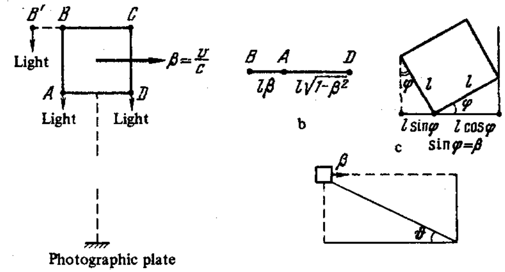

FIG. 1. Visual observation of a cube flying past an observer. a) Relative situation of the observer and the cube for $\vartheta = 0$; b) The visible picture of a cube flying past; c) a possible interpretation of the visible picture by one observer: rotation of the cube through an angle $\varphi = \arcsin\beta$; d) observation of a cube flying past made at an angle ϑ.

For this the observer in K (the rod is at rest in K′) must know beforehand that the rod is moving along a given direction, and the proper length of the rod. Then in his own system K he constructs a replica of the moving rod and photographs the moving rod against the background of its proper length. Before we discuss how it is possible to realize this—even by means of a thought experiment—we note that we make use of yet another assumption.

It is not possible to take two identical scales checked out in one system and then transfer one of them into a moving reference system, because then there can always arise the question of the change in the length of the scale as a result of it being accelerated. But it is possible to obtain identical scales in systems in a state of relative motion even without a transfer of scales. One should merely utilize quantum ideas regarding the identical nature of microparticles. We assume that the wavelength emitted by atoms of a given kind, say by cadmium atoms, in any system in which they are at rest or, more accurately, where they move with nonrelativistic velocities, is always the same. This means that in any arbitrary inertial system one can choose identical lengths as scales. This naturally also applies to time scales. Thus, if one wishes, one can provide all inertial systems with rods of strictly identical proper length.

The simplest arrangement for photographing a rod undergoing a Lorentz contraction might be the following (Fig. 2). The rod is parallel to the x axis and moves along that axis. The observer is situated along a normal to the x axis, and this normal passes through the middle of the rods replica which is at rest in the system K. When the middle of the moving self-luminous rod coincides with the normal, a mechanism is activated which opens the shutter of the camera at the instant when it is reached by light emitted by points of the rod at the instant when its middle coincides with the normal. It is of course possible to photograph the stationary replica whenever one wishes. A more detailed discussion of this question is given in[7]. It is shown there, in particular, that one can photograph, say, a meter rod moving with a relativistic velocity as being contained in a match box if the camera is at rest in the system K of the match box. In the same article there is given a discussion of the interesting question (if one does not for-

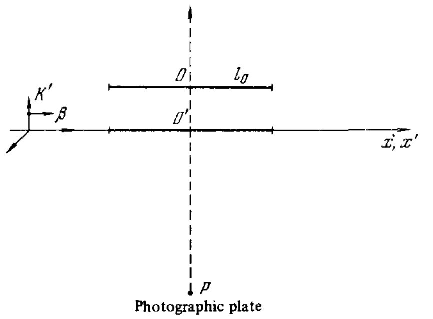

FIG. 2. A scheme which in principle permits making a photograph of the Lorentz contraction of a moving rod. When the midpoint of the rod O′ turns out to lie on the line PO a device is activated which opens (momentarily) the shutter in P in such a manner that it accepts rays emitted by the points of the rod at the instant when the point O′ crosses the line PO.

get the equivalence of systems) as to what will be found on a photograph made at exactly the same point and at exactly the same instant where the camera in K was situated by a camera from the rod's system K′. It turns out that in this photograph the rod no longer fits inside the box. This is due to a distortion of the length as a result of the oblique perspective. Nevertheless, in any reference system will contain one point, a photograph taken from which at an appropriate moment shows that the rod fits completely inside the box.

Copious literature is devoted to the visible shape of a moving body with various possibilities for the relative position of bodies and the camera lens and of the nature of illumination of the body. The most recent review article "The Apparent Shape of Rapidly Moving Objects According to the Theory of Relativity" is by McGill[8] (the same article also contains a list of references among which we particularly note[17]). In this article, in addition to a detailed discussion of the question of measuring and photographing the length of a moving rod, analytic methods are described for the construction of the visible surface of moving bodies, and possibilities of stereoscopic photography and of photography when the object is illuminated by an instantaneous flash are analyzed. From the more recent articles devoted to the shape of moving bodies we call attention to[9] in which a discussion is given of the apparent shape of a moving vertical straight line when observation is carried out along a normal to the straight line. The apparent picture is obtained as the geometrical locus of points from which light reaches the point of observation simultaneously. This picture changes as the straight line moves. In[10] the same method, utilizing digital computers, is employed to investigate the apparent pictures for the following cases: a) for the celestial sphere with certain constellations; b) for a sphere on which circles of parallels and meridians are drawn if its center passes at a distance from the observer equal to the sphere diameter; c) the motion of a number of cubes. We reproduce two diagrams from this paper. Figure 3 shows the change in the visible picture of the celestial sphere for a moving observer (cf., also[18]). Figure 4 shows the change in the visible picture of the surface of a sphere on which meridians and parallels have been drawn as the velocity of its motion is increased. The cause of the distortion of the surface of a body when it

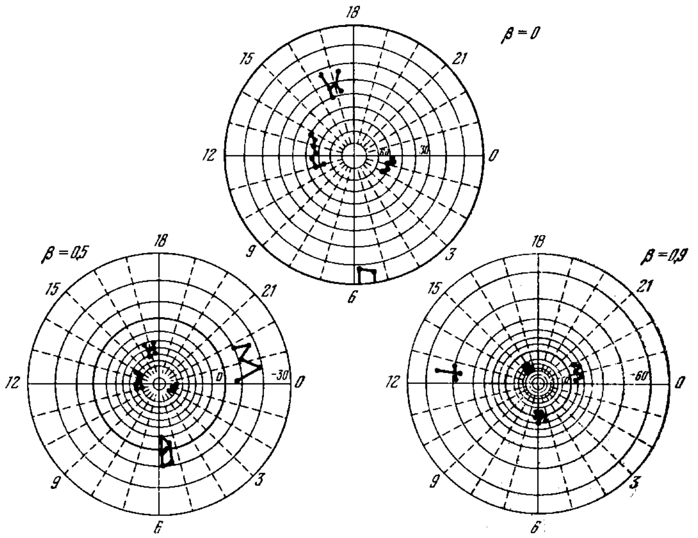

FIG. 3. A picture of the northern celestial hemisphere, as seen by an observer situated at the centre and moving towards the north celestial pole. An observer at rest sees constellations shown in the figure: the Giant Dipper, Cassiopeia, and Hercules. As the speed increases the field of view already begins to contain Orion, a part of the constellation of Aquarius and the Southern Cross.

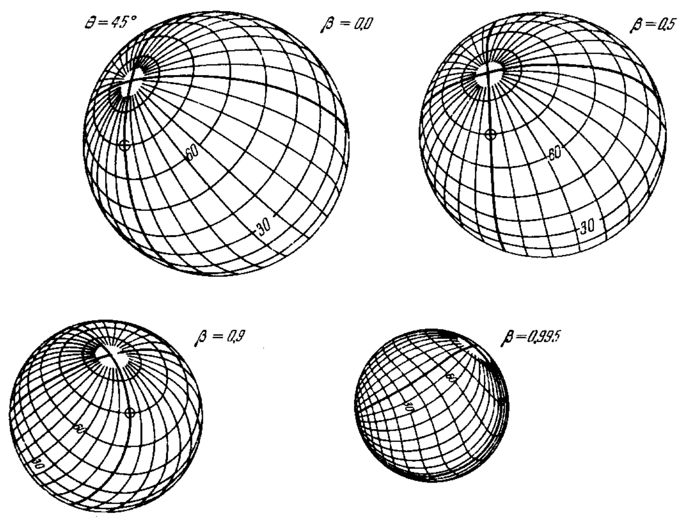

FIG. 4. The visible picture of a sphere approaching an observer with different velocities. The centre of the sphere moves along a straight line which passes at a distance from the observer equal to one sphere diameter. The direction from the point marked on the diagram by a circle towards the observer makes an angle $\vartheta = 45°$ with the direction of motion. The sphere is so rotated that one of its poles is visible.

is observed visually is obvious. If the body is observed in a finite solid angle, then the angle of aberration corresponding to different parts of the surface of the body is different and the surface of the body appears to be distorted.

The finite velocity of propagation of light can lead, in the observation of celestial objects, to the fact that the apparent velocity of motion of cosmic objects, for example of the shell after an explosion of a celestial object, can turn out to be greater than the velocity of light[11].

Thus, the determination of the true shape of a moving object requires, besides a photograph, also additional information in order to make the interpretation unique irrespectively of any relativistic effects. However, we note in addition that an observer capable of reasoning physically can not agree in any event with the result that the moving object rotates*. From Fig. 1d it follows that the angular velocity $d\vartheta/dt$ of rotation of the cube, and consequently also ϑ', varies with time and moreover non-uniformly. But if the object turns, and moreover with a variable angular velocity, then a stationary observer would argue that it must be acted upon by a variable torque. But from where would come the forces acting on a freely moving body if the whole discussion is carried out in an inertial frame of reference? Therefore an observer would have to recognize that an explanation of the apparent shape of the body by means of a rotation is simply wrong. Consequently, the statement of the problem in the form "contraction or rotation" is rather a logical trap than a physical question.

It is essential that the example discussed by Penrose is free from the difficulty indicated above—the rotation of a homogeneous sphere is unobservable. If one marks the sphere by drawing on it for example parallels and meridians, then the observer will immediately convince himself that there is no rotation of any kind[7]. Therefore in stating paradoxes and in discussing them one should not deprive the observer of reason.

At the same time the question of rotation of a moving body and of an equation describing such a rotation is far from being simple. Euler's equation is known in a system in which the center of inertia of the body is at rest. One can write down Euler's equations if the body moves with a nonrelativistic velocity. But an extension of the very same arguments to relativistic velocities leads to new paradoxes one of which will be stated below.

2. TRANSFORMATION OF FORCES AND TORQUES IN A STATE OF EQUILIBRIUM ON GOING FROM ONE REFERENCE SYSTEM TO ANOTHER

Although the transformation law for the vector of a three-dimensional force follows directly from the definition of the Minkowski four-dimensional force, this law has recently become the subject of discussion[12-14]. The discussion arose in connection with the following example. Let a rectangular plane frame ABCD be at rest in a system K^0 (the proper reference system for the problem) and let an elastic thread be stretched along the diagonal AC and pull in two directions a sphere whose rest mass is equal to m (Fig. 5a). In the system K^0 the direction of the thread is determined from the triangle ABC. If one introduces the notation AB = a_0, and BC = b_0, then $\tan \alpha_0 = b_0/a_0$. In the K^0 system the elastic forces are directed along the thread and therefore one can also write that

$$\operatorname{tg} \alpha_0 = b_0/a_0 = F^0_{1x}/F^0_{1y}, \tag{1}$$

Where $\mathbf{F}_1$ denotes a force directed towards the vertex C (similar relations are also valid for $\mathbf{F}_2$).

We now go over to the system K with respect to which the system K^0 moves with the velocity $\mathbf{V}$. We assume, as usual, that the axes x^0 and x coincide, while

*It is rather strange that we have found no mention of this matter in the literature.

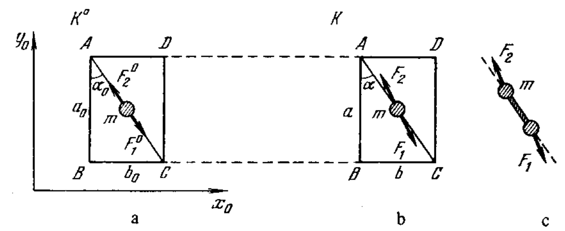

FIG. 5. A rectangular frame along the diagonals of which elastic threads are stretched which pull on the sphere m. a) The picture in the "proper reference system" K^0; b) this is the appearance of the same picture from the point of view of the system K; c) if instead of a sphere we take a dumbell, then from the point of view of K a torque acts on it.

the axes y^0, y and z^0 are respectively parallel. In accordance with the formulas for the transformation of lengths and forces we obtain if we utilize the notation* B = V/c:

$$a = a_0, \quad b = b_0 (1 - B^2)^{1/2}, \tag{2}$$

$$F_{1x} = F^0_{1x}, \quad F_{1y} = F^0_{1y} (1 - B^2)^{1/2}. \tag{3}$$

From this it can be seen that Eq. (1) is no longer valid; in the system K the angle determining the direction of the thread, and the angle determining the direction of the forces are by no means equal to one another:

$$\operatorname{tg} \alpha' = b/a = (b_0/a_0)(1 - B^2)^{1/2} = \Gamma^{-1} b_0/a_0 = \operatorname{tg} \alpha_0/\Gamma, \tag{4}$$

$$\operatorname{tg} \alpha'' = F_{1x}/F_{1y} = (F^0_{1x}/F^0_{1y})/(1 - B^2)^{1/2} = \Gamma F^0_{1x}/F^0_{1y} = \Gamma \operatorname{tg} \alpha_0. \tag{5}$$

Although the sum of the forces remains equal to zero as before, nevertheless the forces in the system K are directed at an angle to the thread (Fig. 5b). This circumstance appears at first glance to be surprising. Indeed, what will happen, for example, if one cuts the thread along the segment 2. In the K^0 system acceleration at the initial instant must be parallel to the direction of the force (this is a clearly nonrelativistic case and the usual Newton's law is entirely applicable), i.e., it is directed along the thread. In the system K it would appear that the acceleration should be directed at an angle to the thread, since the direction of the thread and the direction of the force $\mathbf{F}_1$ do not coincide. In connection with this example it was even proposed[12] to give up the rule for the transformation of forces (3). However, the paradox resolves simply: in relativistic dynamics the acceleration, generally speaking, does not coincide with the direction of the acting force and, even though the force acts at an angle to the direction of the thread the acceleration is directed along the thread. The paradox itself represents a useful illustration of the peculiarities of the relativistic equation of dynamics.

We verify that in both systems the acceleration of the sphere is directed along the thread. It is convenient to write the relativistic equation of motion in the form[15]

$$m\, d\mathbf{v}/dt = \gamma^{-1} [\mathbf{F} - (\mathbf{v}/c^2)(\mathbf{F}\,\mathbf{v})];$$

here m is the rest mass, $\mathbf{F}$ is the ordinary three-dimen-

*We denote by V (and B = V/c) the relative velocity of the inertial systems, and by v (and correspondingly β = v/c) the velocity of the body with respect to K^0.

dimensional force acting on the sphere, v is the velocity of the body, $\gamma = (1 - \beta^2)^{-1/2}$, where β = v/c.

In the system K^0 at the instant t = 0 when the thread 2 is cut:

$$m\, d\mathbf{v}^0/dt = \mathbf{F}_1^0,$$

or in components:

$$m\, dv_x^0/dt = F_{1x}^0, \quad m\, dv_y^0/dt = F_{1y}^0.$$

The direction of motion at the initial instant (we divide both sides of the first equation by the second) is determined by the relation

$$dv_x^0/dv_y^0 = F_{1x}^0/F_{1y}^0 = \mathrm{tg}\, \alpha_0.$$

According to (1) this direction—the direction of acceleration—coincides with the direction of the thread, as it should. Thus, in K^0 the forces and the acceleration are parallel and the motion at the initial instant is directed along the thread.

We now go over to the system K. In this system the body is already moving with a velocity coincident with the velocity of the reference system K^0, i.e., V. Therefore $\gamma = \Gamma$ and the components of acceleration here will be written as follows:

$$m\, dv_x/dt = [F_{1x} - (V/c^2)\, F_{1x} V]/\Gamma = F_{1x}/\Gamma^3, \tag{6}$$

$$m\, dv_y/dt = F_{1y}/\Gamma; \tag{7}$$

here it has been taken into account that the velocity of the sphere coincides with that of the system K, i.e., it is equal to V and has the components (V, 0, 0); F_{1x} and F_{1y} are the components of the force in the system K*. In order to find the direction of the acceleration in K we divide (6) by (7):

$$dv_x/dv_y = (F_{1x}/F_{1y})/\Gamma^2 = \Gamma \cdot \mathrm{tg}\, \alpha_0/\Gamma^2 = \mathrm{tg}\, \alpha_0/\Gamma = \mathrm{tg}\, \alpha', \tag{8}$$

where we have utilized relation (5) in the third link in the chain of equations, and relation (4) in the last link. But from (8) it can be seen that the acceleration in K at the initial instant is also directed along the threads and no paradox arises.

However let us suppose that instead of a sphere, which is implied to point-like the threads would be pulling on a solid, for example a dumbbell. Then in the system K a couple would be acting on the ends of the dumbbell (Fig. 5c) and the dumbbell would rotate with respect to the diagonal of the frame†.

But in the proper system it is evident that the axis of the dumbbell coincides with the diagonal of the frame. Here of course we meet a paradox. And we know that the paradox arises because we have tried to describe from the point of view of the system K a phenomenon about which we know precisely how it occurs in the proper system K^0. It is clear that the error is hidden in our arguments concerning the system K.

The paradox involving the dumbbell is a variant of the well-known lever paradox[16]. We briefly recall this paradox. We assume that in K^0 there exists at rest a crank-like lever made of two rigid rods attached at the point 0 which serves as the axis of rotation of the lever. The rods are perpendicular to one another (Fig. 6).

To the end of the first rod there is applied a force $\mathbf{F}_1^0$ (the length of the rod is l_1^0), and to the end of the second rod of a length l_2^0 there is applied a force $\mathbf{F}_2^0$. It is given that the lever is in equilibrium, and this means that the torques in K^0 are equal: $F_1^0 l_1^0 = F_2^0 l_2^0$.

But if the same system is discussed with respect to the system K and if one defines the moment of force as the product of the force by the moment arm we arrive at a paradoxical result. The lengths contract only in the direction of motion, so that $l_1 = l_1^0(1 - B^2)^{1/2}$, while forces are transformed only in a direction perpendicular to the direction of motion; $F_1 = F_1^0(1 - B^2)^{1/2}$, but $l_2 = l_2^0$, while $F_2 = F_2^0$. Separately these formulas for the transformation of lengths and forces give rise to no doubts. But the total torque in the system K is no longer equal to zero: $F_1 l_1 - F_2 l_2 = F_1^0 l_1^0 (1 - B^2) - F_2^0 l_2^0 = -B^2 F_1^0 l_1^0 = -B^2 F_2^0 l_2^0$.

The paradox consists of the fact that although it is known from the outset that the lever is stationary, in the system K a torque acts on the lever and, consequently, the lever should turn.

Laue[16] resolved this paradox by a very ingenious method. The lever moves in the system K with the velocity V, therefore the force F_2 performs in a unit time an amount of work $F_2 V$. Thus, into the end of the lever 2 there flows an amount of energy $F_2 V$ which increases per unit time the mass at the end of the lever by Δm, such that $\Delta m = F_2 V/c^2$. The increase in momentum at the end of the lever per unit time is equal to $\Delta p = \Delta m_i V = F_2 B^2$, and consequently the change in the moment of momentum per unit time is equal to $F_2 B^2 l_2^0 = F_2^0 B^2 l_2^0$. The increase in the moment of momentum in K is exactly compensated by the torque and no rotation occurs.

The origin of the paradox is actually associated with the fact that a torque cannot be transformed by means of an independent transformation of the moment arms and the forces. A torque is a three-dimensional vector product, and its four-dimensional generalization cannot be carried out uniquely. A special feature of the problem of the lever is the fact that the total torque is de-

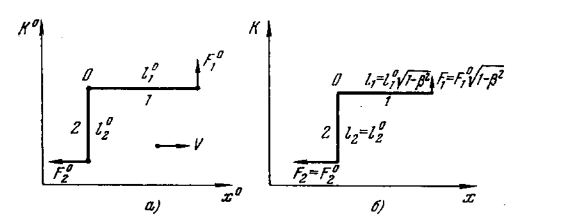

FIG. 6. The lever paradox. a) In the system K^0, where $F_2^0 l_2^0 = F_1^0 l_1^0$ the lever is in equilibrium; b) the same lever, if it is treated from the point of view of the system K: the torques acting on the arms of the lever 1 and 2 are clearly unequal. From the point of view of an observer from K: $\beta = \beta$, $\gamma = \Gamma$.

*It is not difficult to note that (6) and (7) correspond to two exceptional cases of the relativistic equation, when the force and the acceleration are parallel; the corresponding masses in this case were previously called the "transverse" and the "longitudinal" masses. At present these, generally speaking, unfortunate terms have been practically dropped, although they give not a bad impression of the tensor nature of the relation between force and acceleration in relativistic mechanics.

†Of course, one cannot say that the dumbell rotates in one system and remains at rest in the other. Indeed, let us place a glass of water near the dumbell. If the glass is broken when the dumbell turns then this fact cannot be a relative one.

termined by two forces applied at different spatial points. Relativistic mechanics always encounters difficulties when it goes over to a description of a system consisting of many bodies. In such a case calculations should always be carried out for static problems in the rest system of the medium (in our case in the system where the lever is at rest). But the transition to a reference system with respect to which the medium moves already requires transformation of quantities utilized in the theory of elasticity, and we are forced to introduce additional constants of the specific medium. Incidentally, an attempt to carry out such transformations for an elementary case is contained in[14].

We note one additional result. We assume that up to the instant t = 0 there are simply no forces acting on the lever, while at the instant t = 0 the forces $\mathbf{F}_1^0$ and $\mathbf{F}_2^0$ are "swithced on" simultaneously in K^0. At each instant of time equilibrium will hold in K^0. But in K the forces will no longer be switched on simultaneously and there will exist a time interval during which the force $\mathbf{F}_1$ is already acting, while the force $\mathbf{F}_2$ is not yet acting. Again a torque arises. The fact that it is precisely applied at different points of the body are important here (paradoxes arise, naturally, in the discussion of solid bodies) can be seen from a particularly simple example. Let a solid body of length l^0 lie on the x^0 axis in K^0. Until the instant t = 0 no forces are acting on it, while at the instant t = 0 equal but oppositely directed forces are applied on both sides. In K^0 equilibrium always exists, while in K there is a time interval during which the forces are not in equilibrium and, consequently, the body must start moving. Is this really so?

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