Use of Functional Integrals in Quantum Mechanics and Field Theory

D. I. Blokhintsev and B. M. Barbashov
Joint Institute for Nuclear Research, Dubna
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Physicists are becoming more interested in solving the equations of quantum theory without using methods requiring expansion in powers of the interaction constant. The perturbation-theory method, which gave results that agree splendidly with experiment in quantum electrodynamics, turned to be inapplicable in strong-interaction theory. One of the methods in which a radical attempt is made to go beyond the framework of perturbation theory, is the method of functional integration in quantum theory, first proposed by Feynman. The present review, which is devoted to this method, introduces in lucid fashion the concepts of functional integrations and then explains some applications of this method in quantum field theory. Much attention is paid to the use of functional integrals in infrared and high-energy asymptotic relations in field theory. The review does not claim an exposition of the mathematical difficulties connected with the concept of functional integral, and focuses attention to certain successes in its use in quantum physics.

CONTENTS

1. INTRODUCTION

THE development of an invariant perturbation theory and of a suitable renormalization method has made it possible to construct for quantum electromagnetic processes a quantitative theory that agrees with experiment and permits, in principle, a calculation of physical quantities with accuracy of arbitrary order in the constant $e^2/\hbar c = 1/137$. The problem of strong interaction, however, and the study of processes with elementary high-energy particles, as well as problems in the mathematical structure of quantum field theory (QFT) have called for the development of new methods that are not connected with perturbation. Much progress was made in this direction, primarily as a result of the development of an axiomatic approach to QFT. One of the undisputed attainments of such an approach was the discovery and proof of the dispersion relations, and also the determination of a number of important physical properties of the reaction amplitudes of strongly interacting particles on the basis of an analytic structure established for these amplitudes within the framework of the axiomatic approach.

In spite of its success, there still remains in the axiomatic formulation of QTF the fundamental question of the construction of nontrivial QFT models formulated in the language of field operators and satisfying all the requirements of the axiomatic approach.

At the present time, the physical quantities obtained by perturbation theory remain the only objects with which one can verify or guess at the general physical properties of the theory of strong interactions, such as the analytic and asymptotic properties of the amplitudes, etc. Thus, the perturbation-theory methods remain for the time being the "workshop" in which the theoretical physicists test new methods or conclusions of the axiomatic approach.

Great interest attaches in this connection to the finding of exactly-solvable model examples in QFT or, more importantly, to the development of approximate methods for solving the equations of QFT without using expansions in powers of the coupling constant. One of these methods, which are radical attempts to go beyond the framework of perturbation theory, is the method of functional integration, or in other words the "method of path integrals," first proposed by Feynman. This method is based on the representation of microparticle motion as a sequence of quantum transitions over unobservable trajectories. This representation of quantum mechanics is equivalent from the fundamental point of view to the ordinary representation, but the physical clarity and the brilliance of the mathematical formulation of the main problem of quantum theory, that of calculating the probability amplitudes for quantum transitions, which are typical of this new method, have attracted the attention of many researches.

The purpose of the present review is to describe briefly the gist of this insufficiently popular method and to consider important physical results that have been obtained in this manner in QFT. Like any method that claims to solve the problems of quantum theory, the functional-integral method is not universal and has its own problems and difficulties. These difficulties are connected, first, with the solution of the equations of quantum particles in an arbitrary external field, and second, with the functional averaging of these solutions over the external fields with an appropriate weight functional. Both the first and the second problems are mathematically very complicated. The point is that in mathematics we still do not have a well developed theory or technique for functional integration; the only integrals that lend themselves to calculation are Gaussian integrals or those that can be reduced to them by replacing the functional argument.

We confine ourselves in this review to physical results, and leave aside the question of rigorous mathematical justification of the devices used in the described method. A reader interest in the mathematical aspect of the problem of integration in functional spaces is referred to the mathematical reviews of Gel'fand and Yaglom[1] or Koval'chik[2] and to Kac's book[3].

2. MARKOV CHAINS IN QUANTUM MECHANICS

In 1948 Feynman published his well known article[4] in which he proposed a new formulation of nonrelativistic quantum mechanics (QM). Unlike the Schrödinger form of QM, where the main object is a wave function $\psi(x, t)$ that satisfies Schrödinger's equation, in the new formulation this object has become the propagator $K(x', t'; x^0, t^0)$ of the wave function. Knowledge of this propagator makes it possible to determine $\psi(x', t')$ at any instant of time t' from the initial value $\psi(x^0, t^0)$:

$$\psi(x', t') = \int K(x', t'; x^0, t^0)\,\psi(x^0, t^0)\,d^3x^0. \qquad (1)$$

It is seen from (1) that if the particle was at the point x^0 at the instant t^0 [meaning that $\psi(x^0 - x, t^0) = \delta(x^0 - x)$], then the wave function $\psi(x', t')$ is simply equal to the propagator $K(x', t'; x^0, t^0)$. Consequently, the propagator can be regarded as the probability amplitude for the transition of the particle from the point x^0 at which it was located at the instant t^0 to the point x' at the instant t', and this probability $P(x', t'; x^0, t^0)$, according to the fundamental principle of QM, is equal to

$$P(x', t'; x^0, t^0) = |\psi(x', t')|^2 = |K(x', t'; t^0, x^0)|^2. \qquad (2)$$

On the other hand, from the very definition of the propagator it follows that

$$K(x', t'; x^0, t^0) = \int K(x', t'; x'', t'')\,K(x'', t''; x^0, t^0)\,dx''. \qquad (3)$$

Thus, the amplitude of the transition from the point (x^0, t^0) to the point (x', t') can be regarded as the result of a transition of the particle from the point (x^0, t^0) to any intermediate point x'' at a certain instant t'', followed by a transition from the point x'' to the point x' at the instant of time t'. Figure 1 illustrates the foregoing. Insofar as we know, this important property of the propagator was first noted by Dirac.

We note that in classical theory, for example in the theory of Brownian motion, we would have in (2) for the transition probability

$$P(x', t'; x^0, t^0) = \int P(x', t'; x'', t'')\,P(x'', t''; x^0, t^0)\,dx'', \qquad (4)$$

i.e., the probability of transition from the point x^0 to the point x' in the time $t - t^0$ is equal to the probabilities that the particles will fall on any intermediate point x' within a time $t'' - t^0$ (it is clear that $t' > t'' > t^0$), and they will "finally" land at the point x' from the point x'' after a time $t' - t''$.

Breaking up the time intervals and introducing new intermediate points, we obtain from (4)

$$P(x', t'; x^0, t^0) = \int_1 \ldots \int_n P(x', t'; x_n, t_n)\,dx_n\,P(x_n, t_n; x_{n-1}, t_{n-1}) \ldots P(x_2, t_2; x_1, t_1)\,dx_1\,P(x_1, t_1; x^0, t^0), \qquad (5)$$

where the transition from the point (x^0, t^0) to the point (x', t') is regarded as a result of similar transitions via a sequence of intermediate points (x_1, t_1), (x_2, t_2), ..., (x_n, t_n).

Such a sequence is called a Markov chain. In quantum theory we can obtain from expression (3) a formula analogous to (5) for the propagator, namely,

$$= \int_1 \ldots \int_n K(x', t'; x_n, t_n)\,dx_n\,K(x_n, t_n; x_{n-1}, t_{n-1})\,dx_{n-1} \ldots K(x_2, t_2; x_1, t_1)\,dx_1\,K(x_1, t_1; x^0, t^0). \qquad (6)$$

Thus, in QM the Markov chain is made up not of probabilities of probability amplitudes (this feature of quantum theory ("interference of probabilities") distinguishes it radically from the classical theories. We note that various attempts to base QM on classical mechanics fail precisely in this respect. Figure 2 shows one of the "trajectories" of the particle from the point (x^0, t^0) to the point (x', t'); the times t_j are chosen to be equidistant, so that $t_j = \Delta t \cdot j$ $(j = 1, 2, \ldots, n)$.

The word "trajectory" is used in quotation marks, for in accordance with the very meaning of the Markov chain each segment $(x_j, t_j; x_{j-1}, t_{j-1})$ can be broken up into smaller broken segments. Thus, our "trajectory" has no derivative, just as a particle executing Brownian motion has no derivative.

The method of functional integration (or "path integrals") is based on the assumption that the phase of the propagator $K(x_{j+1}, t_{j+1}; x_j, t_j)$ equal to the probability amplitude of a transition from the point (x_j, t_j) to the point x_{j+1} at the neighboring instant of time $t_{j+1} = t_j + \Delta t$ ($\Delta t > 0$ and is small) is determined by the classical action $W[x_{j+1}(t), x_j(t)]$, equal to

$$W[x_{j+1}, t_{j+1}; x_j, t_j] = \int_{t_j}^{t_{j+1}} \mathcal{L}(\dot{x}(t), x(t))\,dt, \qquad (7)$$

where $\mathcal{L}(\dot{x}, x)$ is the Lagrangian of the classical system, written out here for a system with one degree of freedom. In greater detail, we have in the nonrelativistic case

$$\mathcal{L}(\dot{x}, x) = {}^1/_2 m\dot{x}^2(t) - U(x(t)),$$

where m is the particle mass, $\dot{x}(t)$ its velocity, and U the potential energy, which depends on the particle coordinate x(t). We replace the x(t) in the small time interval Δt by a segment of the broken line $\tilde{x}(t)$ passing through the points $x(t_{j+1}) = x_{j+1}$ and $x(t_j) = x_j$. Then $\dot{x}(t) \approx (x_{j+1} - x_j)/\Delta t$ and we obtain in lieu of (7)

$$W[x_{j+1}, t_{j+1}; x_j, t_j] = [m(x_{j+1} - x_j)^2/2\Delta t] - U(x_j)\,\Delta t.$$

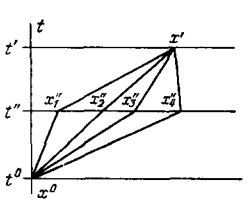


FIG. 1

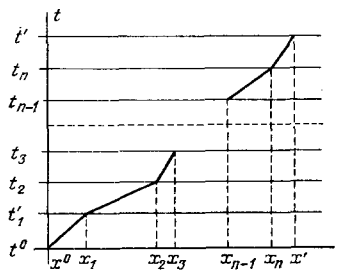


FIG. 2

The Feynman quantization postulate consists of the assumption that

$$K(x_{j+1}, t_{j+1}; x_j, t_j) = (m/2\pi i\hbar\,\Delta t)^{1/2} \exp\{(i/\hbar)\{[m(x_{j+1}-x_j)^2/2\Delta t^2] - U(x_j)\}\Delta t\}, \quad (8)$$

and the factor preceding the exponential is chosen such that

$$K(x_{j+1}, t_{j+1}; x_j, t_j)_{t_{j+1}\to t_j \atop \Delta t\to 0} = \delta(x_{j+1}-x_j).$$

To obtain the propagator pertaining to the final time interval, we construct a Markov chain in accordance with formula (6):

$$K(x', t'; x^0, t^0) = \int K(x', t'; x_{n-1}, t_{n-1})\,dx_{n-1} \int K(x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2})\,dx_{n-2} \ldots \int K(x_2, t_2; x_1, t_1)\,dx_1\,K(x_1, t_1; x^0, t^0), \quad (9)$$

where $x' = x_n$, $t' = t_n$, $x(t^0) = x^0$, ..., and assume that $t_j = t_0 + j\Delta t$, $\Delta t > 0$, $j = 0, 1, 2, \ldots, n$, so that $t' = t_n > t_{n-1} > \ldots > t_{n-r} > \ldots > t_1 > t_0$.

We now take the limit as $\Delta t \to 0$ and $n \to \infty$ (at fixed $t_n = t'$ and t_0). Then each of the propagators in (9) can be represented in the form (8), and we obtain

$$K(x', t'; x^0, t^0) = \lim_{\substack{n\to\infty,\\ \Delta t\to 0}} \left[(m/2\pi i\hbar\,\Delta t)^{n/2} \int dx_1 \int dx_2 \ldots \int dx_{n-1}\right] = \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{n}\left[\frac{m}{2}\frac{(x_{j+1}-x_j)^2}{\Delta t} - U(x_j)\,\Delta t\right]\right\} = \int \delta x \exp\left(\frac{i}{\hbar}W\{x(t)\}\right), \quad (10)$$

where $\delta x = (m/2\pi i\hbar\Delta t)^{n/2}\,dx_1\,dx_2 \ldots dx_{n-1}$, and

$$W\{x(t)\} = \int_{t^0}^{t'} \mathcal{L}(\dot{x}(t), x(t))\,dt, \quad (11)$$

and the integral is taken over all the possible paths joining the points (x', t') and (x^0, t^0). Out of these trajectories only one, namely that corresponding to the minimum of the action function

$$\delta W\{x(t)\} = 0, \quad (12)$$

is the classical trajectory of a particle moving from (x^0, t^0) to (x', t'). From (12) we obtain the equations of motion of the particle in Lagrangian form.

The integral (10), which is the limit of a sequence of n-tuple integrals, is a functional integral or path integral. Its existence in the sense of the indicated limit was proved in the mathematical literature [5] for a broad class of potentials U(x).

As is well known, $K(x', t'; x^0, t^0)$ can be expressed in terms of the eigenfunctions $\varphi_n(x)$ and eigenvalues E_n of the energy operator $\hat{\mathcal{H}}$

$$K(x', t'; x^0, t^0) = \sum_n \varphi_n(x')\,\varphi_n^*(x^0)\,e^{-iE_n(t'-t^0)/\hbar}. \quad (13)$$

Feynman gives for this quantity a new definition, which is simultaneously his quantization postulate.

It is important to note that in order to construct the quantum quantity K we must know the Lagrangian $\mathcal{L}$ of the classical system and not the Hamiltonian $\mathcal{H}$ as in the Schrödinger scheme. In addition, there is no canonical quantization postulate that calls for replacement of the classical c-number quantities by the operators $\hat{x}$ and $\hat{p}$.

We wish to dwell further on the Hamiltonian form of the Feynman integral. It is shown in [6] that instead of the path integral (10) in configuration space $\{x(t)\}$ it is more convenient in some cases to consider an expression for $K(x', t'; x^0, t^0)$ in terms of a functional integral containing the Hamiltonian $\mathcal{H}(x, p)$, where the integration is carried out along the trajectories in phase space $\{x(t), p(t)\}$:

$$K(x', t'; x^0, t^0) = C \int \exp\left\{\frac{i}{\hbar}\int_{t^0}^{t'} [p(t)\,\dot{x}(t) - \mathcal{H}(x(t), p(t))]\,dt\right\} \delta^3x\,\delta^3p. \quad (14)$$

For the simplest Hamiltonian $\mathcal{H} = (p^2/2m) + U(x)$ (and in general for $\mathcal{H}$ that depends quadratically on p) it is easy to prove the equivalence of (10) and (14). Indeed, the argument of the exponential (14) contains again the action integral expressed in terms of the Hamiltonian function. Breaking up, as before, the segment $[t^0, t']$ into n equal intervals Δt and approximating the functions x(t) and p(t) by piecewise linear and piecewise constant functions, we can rewrite accordingly (14) as the limit of an (2n + 1)-tuple integral (the values of the momenta p_1 and p_n on the ends of the trajectories are not fixed; the integrations are carried out with respect to them, too):

$$\lim_{n\to\infty}\left[(1/2\pi\hbar)^{3n}\int_{1\ldots n}\ldots\int d^3p_1 d^3p_2 \ldots d^3p_n\,d^3x_1 \ldots d^3x_{n-1} \times \exp\left\{(i/\hbar)\sum_{j=1}[p_j(x_j - x_{j-1}) - (p_j^2/2m)\,\Delta t - U(x_j)\,\Delta t]\right\}\right] = K(x', t'; x^0, t^0). \quad (15)$$

We now integrate with respect of all p_j; this is easily done by noting the following:

$$p_j\Delta x_j - \frac{p_j^2}{2m}\Delta t = -\frac{\Delta t}{2m}\left(p_j - \frac{m}{\Delta t}\Delta x_j\right)^2 + \frac{m}{2}\left(\frac{\Delta x_j}{\Delta t}\right)^2,$$

where $\Delta x_j = x_j - x_{j-1}$;

$$\int d^3p_j \exp\left[-\frac{i}{\hbar}\frac{\Delta t}{2m}\left(p_j - \frac{m}{\Delta t}\Delta x_j\right)^2\right] = \left(\frac{2\pi\hbar m}{i\Delta t}\right)^{3/2}.$$

As a result we find that both definitions of K coincide

$$\lim_{n\to\infty}\left[\left(\frac{1}{2\pi\hbar}\right)^{3n}\int_{1\ldots n}\ldots\int d^3p_1 \ldots d^3p_n\,d^3x_1 \ldots d^3x_{n-1} \times \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{n}\left[p_j(x_j - x_{j-1}) - \frac{p_j^2}{2m}\Delta t - U(x_j)\,\Delta t\right]\right\}\right] = K(x', t'; x^0, t^0)$$

$$= \lim_{n\to\infty}\left[\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{3n/2}\int d^3x_1 \ldots \int d^3x_{n-1} \times \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{n}\left[\frac{m}{2}\frac{\Delta x_j^2}{\Delta t} - U(x_j)\,\Delta t\right]\right\}\right]$$

$$= C\int \exp\left\{\frac{i}{\hbar}\int_{t^0}^{t'}\left[\frac{m}{2}\dot{x}^2(t) - U(x(t))\right]dt\right\}\delta x.$$

3. THE SCHRÖDINGER EQUATION

The Schrödinger equation for $\psi(x', t')$ in the Lagrangian and Hamiltonian forms of $K(x', t'; x^0, t^0)$ are derived in the following manner. We consider Eq. (1), in which the times t^0 and t' differ by an infinitesimally small amount Δt, $t' = t^0 + \Delta t$; then

$$\psi(x', t^0 + \Delta t) = \int K(x', t^0 + \Delta t; x^0, t^0)\,\psi(x^0, t^0)\,d^3x^0. \quad (16)$$

We take for $K(x', t^0 + \Delta t; x^0, t^0)$ the Lagrangian form (8):

$$K(x', t^0 + \Delta t; x^0, t^0) = \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{3/2}\exp\left\{\frac{i}{\hbar}\Delta t\left[\frac{m}{2}\frac{(x'-x^0)^2}{\Delta t^2} - U(x')\right]\right\};$$

expanding in terms of the small quantity Δt in (16), we

have

$$\psi(x', t^0)+\Delta t\frac{\partial\psi(x', t^0)}{\partial t^0}=\left[1-\frac{i}{\hbar}\Delta t U(x)\right]\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{3/2}\times\int\exp\left[\frac{i}{\hbar}\frac{m}{2}\Delta t\frac{(x'-x^0)^2}{\Delta t^2}\right]\psi(x^0, t^0)\,d^3x^0. \quad (17)$$

We introduce the substitution $\xi = x' - x^0$; then

$$\psi(x^0, t^0)=\psi(x', t^0)-\xi\nabla\psi(x', t^0)+(\xi^2/2)\nabla^2\psi(x', t^0)+\dots \quad (18)$$

Integrating with respect to ξ in (17) with allowance for the substitution, we note that the succeeding terms of the expansion in (18) give higher orders in Δt. Indeed,

$$\int d^3\xi\exp[(im/2\hbar\,\Delta t)\,\xi^2]=(2\pi\hbar i\,\Delta t/m)^{3/2},$$

$$\int d^3\xi\xi\exp[(im/2\hbar\,\Delta t)\,\xi^2]=0,$$

$$\int d^3\xi\xi^2\exp[(im/2\hbar\,\Delta t)\,\xi^2]=(2\pi\hbar i\,\Delta t/m)^{5/2}.$$

As a result we obtain Schrödinger's equation

$$\frac{\hbar}{i}\frac{\partial\psi(x', t^0)}{\partial t^0}=\left[\frac{\hbar^2}{2m}\nabla^2-U(x')\right]\psi(x', t^0). \quad (19)$$

In the Hamiltonian formulation we have for the case when $t' = t^0 + \Delta t$

$$\psi(x', t^0+\Delta t)=\int d^3x^0\int\frac{d^3p}{(2\pi\hbar)^3}\exp\left\{\frac{i}{\hbar}\left[p(x'-x^0)-\frac{p^2\Delta t}{2m}-\Delta t\cdot U(x')\right]\right\}\psi(x^0, t^0), \quad (20)$$

and, taking into account the smallness of Δt, we can integrate with respect to p not exactly, as in (11), but by expanding in terms of Δt; then

$$\left(\frac{1}{2\pi\hbar}\right)^3\int d^3p e^{(i/\hbar)p(x'-x^0)}\left\{1-\Delta t\frac{i}{\hbar}\left[\frac{p^2}{2m}+U(x')\right]\right\}=\delta^3(x'-x^0)+\Delta t\frac{i}{\hbar}\left[\frac{\hbar^2}{2m}\nabla^2-U(x')\right]\delta^3(x'-x^0). \quad (20')$$

Substituting this expression for $K(x', t^0 + \Delta t; x^0, t^0)$ in (20) and integrating with respect to x^0, we arrive again at Schrödinger's equation:

$$\psi(x', t^0+\Delta t)=\psi(x', t^0)+\Delta t\,(i/\hbar)\,[(\hbar^2/2m)\nabla^2-U(x')]\,\psi(x', t^0),$$

from which we get (19).

The scheme of quantization with the aid of functional integrals makes it necessary to consider objects whose theory has not been sufficiently well developed from the mathematical point of view, and calls for the introduction of new mathematical concepts, but it does have a number of advantages, as already noted, over the canonical quantization procedure. One of them is that in the usual scheme we must specify the rules for the arrangement of the non-commuting quantities, but in our case we do not have this uncertainty, since the sequence of p and x in the functional itself is immaterial, inasmuch as these are c-number functions; this feature of the scheme is apparently important when it comes to quantizing nonlinear systems[7,8].

It should be noted in this connection that Berezin pointed out in a recent paper[9] an important feature of functional integrals in (p, x) phase space. He cites a number of examples demonstrating that the limits of finite-dimensional integrals [cf. formula (15)] approximating the functional integral (14) can tend to different values as $n \to \infty$, depending on the method used to break up p(t) and x(t) into discrete values p_k and x_k, and indicated a connection of the problem of placing the operators $\hat{p}$ and $\hat{x}$ in ordinary quantum theory with the method of constructing the approximating finite-dimensional integrals. Thus, the question whether the problem of the placement of the operators exists within the framework of the functional method remains open, at any rate for integrals in phase space (p, x).

4. TRANSFORMATIONS OF FUNCTIONAL ARGUMENTS

As seen from the preceding formulas, the functional integrals were expressed in the Cartesian coordinates $x_j(t)$ and $p_j(t)$, but to extend the practical applicability of this method it is necessary to express such quantities in curvilinear coordinates, primarily spherical ones, which are important for quantum problems with spherical symmetry. The correct form of the integral in polar coordinates was derived in[10]; Edwards and Gulyaev[11] pointed out the difficulties of this problem. Indeed, whereas the correspondence $p_j \to i\partial/\partial x_j$ holds for canonical quantization in Cartesian coordinates, it does not hold in polar coordinates. Peak and Inomata[10] consider a particle of mass m in a central field $U(|\mathbf{x}|)$. Within the framework of the Lagrangian formulation, for a small interval $\Delta t = t_j - t_{j-1}$ and closely-lying coordinates x_j and x_{j-1}, the action function (7) can be expressed in the polar coordinates

$$\mathbf{x}_j=\{r_j\sin\theta_j\sin\varphi_j,\ r_j\sin\theta_j\cos\varphi_j,\ r_j\cos\theta_j\} \quad (21)$$

in the form

$$W(x_j, x_{j-1})=(m/2)\,[(r_j^2+r_{j-1}^2)/\Delta t]-m(r_jr_{j-1}/\Delta t)\cos\Theta_j-\Delta t\cdot U(r_j); \quad (22)$$

we have taken into account here the fact that

$$\cos\Theta_j=\cos\theta_j\cos\theta_{j-1}+\sin\theta_j\sin\theta_{j-1}\cos(\varphi_j-\varphi_{j-1}),$$

$$(x_j-x_{j-1})^2=r_j^2+r_{j-1}^2-2r_jr_{j-1}\cos\Theta_j.$$

If we use an expansion in Legendre polynomials

$$e^{u\cos\Theta}=\left(\frac{\pi}{2u}\right)^{1/2}\sum_{l=0}^{\infty}(2l+1)\,P_l(\cos\Theta)\,I_{l+\frac{1}{2}}(u),$$

where $I_{l+1/2}$ is a Bessel function than the quantity $\exp[(i/\hbar)\sum_j W_j]$ in (10) can be represented in the form

$$\exp\left[\frac{i}{\hbar}\sum_j W(x_j, x_{j-1})\right]=\prod_{j=1}^{n}\sum_{l_j=0}^{\infty}(2l_j+1)\,P_{l_j}(\cos\Theta_j)\,R_{l_j}(r_j; r_{j-1});$$

here R_{l_j} is the radial part of the expression

$$R_{l_j}(r_j; r_{j-1})=\left(\frac{i\pi\Delta t}{2mr_jr_{j-1}}\right)^{1/2}\exp\left\{\frac{i}{\hbar}\left[\frac{r_j^2+r_{j-1}^2}{\Delta t}-\Delta tU(r_j)\right]\right\}I_{l_j+\frac{1}{2}}\left(\frac{m\hbar}{i\Delta t}r_jr_{j-1}\right).$$

Interchanging the order of summation and multiplication in this formula, we obtain for the function $K(x', t'; x^0, t^0)$

$$K(x', t'; x^0, t^0)=\lim_{n\to\infty}\left\{\left(\frac{m}{2\pi i\hbar\,\Delta t}\right)^{3n/2}\sum_{l_1, l_2, \dots, l_n}^{\infty}\int\prod_{j=1}^{n}[(2l_j+1)\,P_{l_j}(\cos\Theta_j)\,R_{l_j}(r_j, r_{j-1})]\times\prod_{t, i=1}^{n-1}r_i^2(t)\,dr_i(t)\sin\theta_i(t)\,d\theta_i(t)\,d\varphi_i(t)\right\}.$$

It turns out further that for the considered angular dependence (22) it is possible to integrate exactly over all the angle variables; to this end it suffices to take into account only the following equations:

$$P_{l_j}(\cos\Theta_j)=\frac{4\pi}{2l_j+1}\sum_{n_j=-l_j}^{l_j}Y_{l_j}^{n_j*}(\theta_j,\varphi_j)\,Y_{l_j}^{n_j}(\theta_{j-1},\varphi_{j-1}),$$

where Y_l^n are spherical harmonics and

$$\int\int Y_l^{n*}(\theta,\varphi)\,Y_{l'}^{n'}(\theta,\varphi)\,d\varphi\sin\theta\,d\theta=\delta_{ll'}\delta_{nn'}. \tag{23}$$

Therefore

$$\int\cdots\int\prod_{j=1}^{n}[(2l_j+1)\,P_{l_j}(\cos\Theta_j)]\prod_{i=1}^{n-1}\sin\theta_i\,d\theta_i\,d\varphi_i$$
$$=(4\pi)^n\,\delta_{ll_n}\prod_{j=1}^{n}\delta_{l_{j+1}l_n}\sum_{n=-l}^{l}Y_l^{n*}(\theta',\varphi')\,Y_l^n(\theta^0,\varphi^0).$$
$$x'=x'(r',\theta',\varphi'),\ x^0=x^0(r^0,\theta^0,\varphi^0).$$

As a result, the radial and angular contributions to the propagator K separate for each quantum number l

$$K(x',t';x^0,t^0)=\sum_{l=0}^{\infty}\sum_{n=-l}^{l}K_l(r',t';r^0,t^0)\,Y_l^{n*}(\theta',\varphi')\,Y_l^n(\theta^0,\varphi^0). \tag{24}$$

The radial part of the propagator for the l-wave is given by a functional integral with respect to the variable r(t) only:

$$K_l(r',t';r^0,t^0)=\lim_{n\to\infty}\left[(4\pi^n)\left(\frac{m}{2\pi i\hbar\,\Delta t}\right)^{3n/2}\int\prod_{j=1}^{n}R_l(r_j;r_{j-1})\prod_{i=1}^{n-1}r_i^2\,dr_i\right], \tag{25}$$

and its calculation is already determined by the concrete form of the potential U(r).

Thus, the feasibility of obtaining functional quadratures in terms of the angle variables with respect to $\theta(t)$ and $\varphi(t)$ for a given W in the spherically-symmetrical problem enables us to write for K a functional integral only with respect to the radial variable r(t) in the form (24) and (25).

For the Hamiltonian form of the integral in spherical coordinates we can write an analogous expression[10]:

$$K(x',t';x^0,t^0)=\lim_{n\to\infty}\left\{\left(\frac{1}{2\pi\hbar}\right)^{3n}\int\exp\left[\frac{i}{\hbar}\int_{t^0}^{t'}(p_r\dot r+p_\theta\dot\theta+p_\varphi\dot\varphi-\mathscr{H})\,dt\right]\right.$$
$$\left.\times\prod_{j=1}^{n}(dp_{r_j}\,dp_{\theta_j}\,dp_{\varphi_j})\prod_{i=1}^{n-1}(r_i^2\sin\theta_i\,dr_i\,d\theta_i\,d\varphi_i)\right\},$$

and after integrating with respect to dp_{θ_j}, dp_{φ_j}, $d\theta_j$, and $d\varphi_j$ we obtain as a result formula (24) with a radial function

$$R_l(r',t';r^0,t^0)=\frac{1}{r'r^0}\lim_{n\to\infty}\left\{\left(\frac{m\hbar}{2\pi i\Delta t}\right)^{3n/2}\right.$$
$$\left.\times\int\exp\left[\frac{i}{\hbar}\int_{t^0}^{t'}(p\dot r-\mathscr{H}_l)\,dt\right]\prod_{j=1}^{n}dp_j\prod_{i=1}^{n}r_i^2\,dr_i\right\},$$

where

$$\mathscr{H}_l(r_j,p_j)=(2m)^{-1}\{p_j^2+[l(l+1)/r_j^2]\}+U(r_j).$$

If, for example, we consider a particle in an attracting potential $U(r)=k^2/r^2$, then the integration over all r_j in (25) is possible, (see[10]), and we then have the closed expression

$$K(x',t';x^0,t^0)=[m/i(t'-t^0)(r'r^0)^{1/2}]\exp[im(r'^2+r^{02})/2(t'-t^0)]$$
$$\times\sum_{l=0}^{\infty}\sum_{n=-l}^{l}I_{\lambda(l)}(mr'r^0/i(t'-t^0))\,Y_l^{n*}(\theta',\varphi')\,Y_l^n(\theta^0,\varphi^0),$$
$$\lambda(l)=[(l+1/2)^2+k^2]^{1/2}.$$

We note in conclusion that, unfortunately, this brilliant method of integrating with respect to the angle variables in the functional integral is not effective for more complicated functionals encountered in field theory. For example, for the integral[12]

$$\int\exp\left[\frac{i}{\hbar}\int_{t^0}^{t'}D(t_1-t_2)\,\mathbf{x}(t_1)\,\mathbf{x}(t_2)\,dt_1\,dt_2+i\frac{g}{\hbar}\int_{t^0}^{t'}U(\mathbf{x}^2)\,dt\right]\delta\mathbf{x}$$

introduction of the polar coordinates r, θ, and φ:

$$\int\exp\left[\frac{i}{\hbar}\int_{t^0}^{t'}\int D(t_1-t_2)\,r(t_1)\,r(t_2)\cos\theta(t_1,t_2)\,dt_1\,dt_2\right.$$
$$\left.+i\frac{g}{\hbar}\int_{t^0}^{t'}U(r^2(t))\,dt\right]\prod_{i=1}^{n}d\varphi_i r_i^2\,dr_i\sin\theta_i\,d\theta_i,$$

where

$$\cos\theta(t_1,t_2)=\cos\theta(t_1)\cos\theta(t_2)+\sin\theta(t_1)\sin\theta(t_2)\cos(\varphi(t_1)-\varphi(t_2)),$$

leads, upon expansion in spherical harmonics, to the expression

$$\exp\left(\frac{i}{\hbar}\sum_{j,k=1}^{n}D_{jk}r_jr_k\cos\theta_{jk}\right)=\sum_{j,k}^{n}4\pi\left(\frac{\pi\hbar}{iD_{jk}r_jr_k}\right)^{1/2}$$
$$\times\sum_{l_{jk}=0}^{\infty}I_{l_{jk}+\frac12}\left(\frac{i}{\hbar}D_{jk}r_jr_k\right)\sum_{n_{jk}=-l_{jk}}^{l_{jk}}Y_{l_{jk}}^{n_{jk}*}(\theta_j,\varphi_j)\,Y_{l_{jk}}^{n_{j,k}}(\theta_k,\varphi_k), \tag{26}$$

which is extremely difficult to integrate with respect to θ_j and φ_j, for at each fixed angle θ_j or φ_j we integrate a product of n factors $Y_{l_{jk}}^{n_{jk}}(\theta_j,\varphi_j)$, and formula (23) does not hold. The reason is that the quadratic form in the argument of the exponential in (19) in $\mathbf{x}(t)$ depends on two times t_1 and t_2, and for one fixed $t_1=t_j$ we are left with a sum in the exponential or with a product of factors in $t_2=t_k$ in (26).

The connection between the canonical transformations in classical and quantum mechanics is more lucid in the functional notation.

Let us consider the simple but important particular case of the canonical transformation[6]

$$x\to p,\quad p\to -x.$$

The expression (15) for the propagator then takes the form

$$K(p',t';p^0,t^0)=\lim_{n\to\infty}\left\{\frac{1}{(2\pi\hbar)^{3n}}\int\cdots\int d^3x_1\cdots d^3x_n\,d^3p_1\cdots d^3p_n\right.$$
$$\left.\times\exp\left[-\frac{i}{\hbar}\left(\int x\,dp+\int\mathscr{H}\,dt\right)\right]\right\}.$$

If, as before, we put $\mathscr{H}=(p^2/2m)+U(x)$ and consider the case when $t'=t^0+\Delta t$, then, proceeding as in the derivation of (20′), we obtain

$$K(p',t^0+\Delta t;p^0,t^0)$$
$$=\frac{1}{(2\pi\hbar)^3}\int d^3x\exp\left\{-\frac{i}{\hbar}x(p'-p^0)-\frac{i}{\hbar}\Delta t\left[\frac{p^2}{2m}+U(x)\right]\right\}.$$

Further, assuming that the potential U(x) can be expanded in the series $U(x)=U_0+U_1x+U_2x^2+\ldots$, we obtain

$$K(p',t^0+\Delta t;p^0,t^0)=\delta^3(p'-p^0)$$
$$-\frac{i}{\hbar}\Delta t\left[\frac{p^2}{2m}+U_0+U_1\left(i\hbar\frac{\partial}{\partial p'}\right)+U_2\left(i\hbar\frac{\partial}{\partial p'}\right)^2+\ldots\right]\delta(p'-p^0).$$

From this follows the Schrödinger equation for the function $\psi(p',t^0)$, but in the momentum representation).

There are published proofs[6,8] that linear canonical transformations in classical mechanics correspond to unitary transformation in the Feynman formulation of quantum mechanics. A study of nonlinear canonical

transformation within the framework of the functional formulation raises difficulties, since mathematical procedures of nonlinear transformations for functional integrals have not yet been fully developed (see[1,2]).

Fadeev has recently pointed out[8] an important application of the Feynman method of quantization with the aid of a functional integral in Hamiltonian form in situations when the canonical method encounters difficulties.

The point is that a classical system can have a Lagrangian $\mathscr{L}(\dot{q}, q)$ for which the relations

$$p_j = \frac{\partial \mathscr{L}}{\partial \dot{q}_j} \qquad (j = 1, 2, \ldots, n)$$

cannot be solved with respect to $\dot{q}_j$, i.e., we cannot express $\dot{q}$ in terms of p and q, as is required to write down to Hamiltonian $\mathscr{H}(q, p)$. Such Lagrangians are called singular in[8]. The most interesting examples of systems with singular Lagrangians is obtained from field theory for gauge-invariant fields (electromagnetic field, Yang-Mills field, gravitational field). All are expressed by singular Lagrangians with additional conditions or constraints imposed on the canonical variables. In the n-dimensional classical-mechanics case, the canonical variables $(q_1, q_2, \ldots, q_n)$ and $(p_1, p_2, \ldots, p_n)$ therefore do not run through all the 2n-dimensional phase space, owing to the constraints

$$\varphi_\alpha(q, p) = 0 \quad (\alpha = 1, 2, \ldots, m), \quad m < n.$$

The quantization of such systems in the canonical scheme has been the subject of a paper by Dirac[13]. His schemes contains a number of difficulties, mainly with the placement or, in Feynman's terminology, ordering of the operator multipliers.

The additional conditions are taken into account in the Hamiltonian form of the functional integral in the following manner. If the relations $\varphi_\alpha(q, p) = 0$ can be solved with respect to m coordinates and expressed in the form

$$q_\alpha = q_\alpha(q^*, p)$$

(q^* are $n - m$ independent coordinates), then this calls for $\det|\partial\varphi_\alpha/\partial q| \neq 0$. It turns out furthermore that the observables are expressed not in terms of all the n momenta p_j, but only in terms of $n - m$ momenta, i.e., there exist also m relations $f_\alpha(q, p) = 0$. Therefore, after the canonical transformations in which the new momenta are $p^* = f_\alpha(q, p)$, we shall have $2(n - m)$ variables $(q_1^*, q_2^*, \ldots, q_{n-m}^*)$, $(p_1^*, p_2^*, \ldots, p_{n-m}^*)$, with the aid of which we express the functional integral:

$$K(q', t'; q^0, t^0) = \int \exp\left[\frac{i}{\hbar}\int_{t^0}^{t'}\left(\sum_{j=1}^{n-m} p_j^*\dot{q}_j^* - \mathscr{H}\right)dt\right]\prod_{t,\,j=1}^{n-m}\frac{dp_j^*(t)\,dq_j^*(t)}{(2\pi\hbar)^{n-m}}.$$

We can also express K in terms of 2n variables, with allowance for the constraints and the additional conditions, in the form of an integral with respect to all 2n variables:

$$K = \int \exp\left[\frac{i}{\hbar}\int_{t^0}^{t'}\left(\sum_{j=1}^{n} p_j\dot{q}_j - \mathscr{H}\right)dt\right] \times \prod_{t,\,\alpha=1}^{m}\delta(p_\alpha)\,\delta(q_\alpha - q(q^*, p^*))\,dp_\alpha(t)\,dq_\alpha(t)\prod_{t,\,j=1}^{n-m}\frac{dp_j^*(t)\,dq_j^*(t)}{(2\pi\hbar)^{n-m}}.$$

In concluding this chapter, we indicate one interesting numerical calculation of a functional integral, which yields the ground level of the helium atom. A method for calculating the Wiener functional integral can be found already in the paper by Gel'fand and Chentsov[14] and consists in the following. The functional integral is approximate by finitely-multiple Riemann or Stieltjes integrals, and the latter are calculated by the Monte Carlo method with a computer. This method was used to calculate the lowest polaron energy level, by evaluating 190-tuple and 280-tuple approximating integrals. The results are: 0.9912 for 300 trajectories, 0.9940 for 400 trajectories, and 0.9999 for 600 trajectories, as against the exact value 1.0000 (in the appropriate units).

Evseev[15] calculated in this manner the value of the integral

$$\int_{-\infty}^{+\infty} K(x', t'; x^0, 0)\, e^{(i/\hbar)Et'}\,dt' = \int_{-\infty}^{+\infty} dt' e^{(i/\hbar)Et'}\int e^{(i/\hbar)W[x', t', x^0]}\delta x,$$

where W is the classical action of two electrons in the field of the helium nucleus. Since, on the other hand, we have from (13)

$$\int e^{(i/\hbar)Et'}K(x', t'; x^0, 0)\,dt' = \sum_n \varphi_n(x')\,\varphi_n^*(x^0)\cdot 2\pi\delta(E_n - E),$$

an approximate calculation of K yields upon integration with respect to t maxima (in lieu of $\delta(E_n - E)$) at the points E_n. Integration in the m-tuple approximating integral with respect to x_i is carried out from $-a$ to $+a$ ($a = \hbar/me^2$ is the radius of the first Bohr orbit), since the function $\exp(iW/\hbar)$ oscillates rapidly at large x and makes a small contribution to the integral. The value obtained for the ground level of helium is 2.92 ± 0.05, which differs by 0.57% from the experimental 2.90351.

5. CONTINUAL REPRESENTATION OF THE GREEN'S FUNCTIONS IN THE THEORY OF QUANTIZED FIELDS

We turn now to quantum field theory (QFT). A fundamental role is played in QFT by the Green's functions of quantized fields, knowledge of which enables us to find the physical characteristics of the interacting fields; in particular, they make it possible to obtain also the scattering amplitudes (this question will be discussed in detail later on).

Many authors[16,17] obtained with the aid of functional integrals expressions in closed form for the Green's functions. For example, the single-particle Green's function of the fermion field is given by

$$G(x, y) = \int G(x, y|A)\,S_0(A)\exp\left[-\frac{i}{\hbar}\int\int d^4\xi\,d^4\eta D_{\mu\nu}^{-1}(\xi - \eta)\,A_\mu(\xi)\,A_\nu(\eta)\right]\delta^4 A \times\left\{\int S_0(A)\exp\left[-\frac{i}{\hbar}\int\int d^4\xi\,d^4\eta D_{\mu\nu}^{-1}(\xi - \eta)\,A_\mu(\xi)\,A_\nu(\eta)\right]\delta^4 A\right\}^{-1}; \tag{27}$$

where $\delta^4 A = \prod_x d^4 A(x)$; $G(x, y|A)$ is the Green's function of the fermion in a classical external field A_μ, and $S_0(A)$ is the S matrix averaged over the fermion vacuum, with the operators of the boson (electromagnetic) field A_μ replaced by classical field functions; $D_{\mu\nu}^{-1}$ is the reciprocal of the free propagation function of the boson

$$\int D_{\mu\varkappa}^{-1}(x - z)\,D_{\varkappa\lambda}(z - y)\,d^4 z = \delta^4(x - y)\,\delta_{\mu\lambda}.$$

The first problem arising in the calculation of (27) is the determination of the functions $G(x, y|A)$ for an arbitrary external field A_μ. For an electron-positron field,

G satisfies the Dirac equation

$$[i\gamma_\mu\partial_\mu - m + e\gamma_\nu A_\nu(x)]\, G(x, y \mid A) = -\delta^4(x-y). \quad (28)$$

Even the solution of this equation with an arbitrary field $A_\mu(x)$ is a tremendously difficult problem in mathematics. However, as first shown by Feynman[4], the solution of this equation can be formally represented also in the form of a functional integral (this procedure yields the Green's functions of the Klein-Gordan, Dirac, and Schrödinger equations[16,18,19] for an arbitrary external field. If the external fields admits of a closed solution of (28), functional quadratures can be obtained. A favorable feature in this approach is that the Green's function obtained in this manner makes it possible, if the vacuum terms in (27) are neglected (by putting $S_0(A) = 1$], to carry out functional averaging over the external fields and obtain the quantum function G(x, y) without taking into account the polarization of the fermion vacuum. It will be shown subsequently that this formalism has a fully covariant form since, in contrast to the preceding formulas, the variable singled out here is not the time t, but the proper time τ.

Let us examine this formalism using the Dirac equation (28) as an example, and let us introduce, as usual, the Green's function of the squared Dirac equation:

$$G(x, y \mid A) = [i\gamma_\mu\partial_\mu + m + e\gamma_\mu A_\mu(x)]\, \mathcal{G}(x, y \mid A).$$

We then have for $\mathcal{G}$ the equation

$$[(i\partial_\mu + eA_\mu(x))^2 - m^2 + e\sigma_{\mu\nu}\partial_\mu A_\nu(x)]\, \mathcal{G}(x, y \mid A) = -\delta^4(x-y). \quad (29)$$

Using the exponential representation of the reciprocal operator, proposed by Fock[20] and developed by Feynman[21], we represent the solution (29) in operator form:

$$\mathcal{G}(x, y \mid A) = i\int_0^\infty d\tau e^{-im^2\tau} \exp\left\{ i\int_0^\tau d\xi\, [(i\partial_\mu(\xi) + eA_\mu(\xi))^2 + e\sigma_{\mu\nu}(\xi)\,\partial_\mu(\xi)\,A_\nu(\xi)] \right\} \delta^4(x-y); \quad (30)$$

the exponential in this expression, which contains the non-commuting operators ∂_μ, A_μ and $\sigma_{\mu\nu}$, is meant here to be[21] a T_τ-exponential, where the ordering subscript τ has the meaning of the proper time divided by the mass m. All the operators in (30) are assumed to be commuting functions that depend on the parameter τ. We now carry out a functional Fourier transformation, which leads to the first degree of the differentiation operator ∂_μ in the argument of the exponential:

$$\exp\left\{ i\int_0^\tau d\xi\, [i\partial_\mu(\xi) + eA_\mu(x, \xi)]^2 \right\} = C\int \delta^4\nu \exp\left\{ -i\int_0^\tau \nu_\mu^2(\xi)\, d\xi + 2i\int_0^\tau \nu_\mu(\xi)\, [i\partial_\mu(\xi) + eA_\mu] \right\}, \quad (31)$$

where

$$\delta^4\nu = \prod_\xi d^4\nu(\xi), \quad C = 1/\exp\left[-i\int_0^\tau \nu_\mu^2(\xi)\, d\xi \right].$$

Having now the first degree of the operator ∂_μ in the argument of (31), we can use it as a shift operator in accordance with the "disentanglement" rules[21]; as a result we obtain the following expression for $\mathcal{G}(x, y|A)$:

$$\mathcal{G}(x, y \mid A) = i\int_0^\infty d\tau e^{-im^2\tau} C\int \delta^4\nu \exp\left\{ -i\int_0^\tau d\xi \left[\nu_\mu^2(\xi) - 2e\left[\nu_\mu(\xi) \right. \right. \right. \quad (32)$$

$$\left. \left. \left. + \frac{i}{2}\sigma_{\mu\nu}(\xi)\,\partial_\nu(\xi)\,A_\mu\left(x - 2\int_\xi^\tau \nu(\eta)\, d\eta\right) \right] \right] \right\} \delta^4\left(x - y - 2\int_0^\tau \nu(\eta)\, d\eta\right).$$

In this expression the operators $\sigma_{\mu\nu}(\xi)$ remain, in Feynman's terminology, not "disentangled"*, i.e., they depend on ξ as an ordering index, and the T_τ-ordering remains in force for them.

The Green's function for the Klein-Gordon equation is obtained from (32) at $\sigma_{\mu\nu} = 0$. For the integral with respect to τ to be convergent at the upper limit, it is assumed that the mass m has a negative imaginary increment $-i\epsilon$; formula (32) thus determines the causal Green's function of the Dirac or Klein-Gordon equation.

It is possible to obtain in similar fashion the retarded Green's function of the Schrödinger equation[19] with an arbitrary potential U(x, t):

$$G(x', t'; x^0, t^0) = i\theta(t'-t^0)\, C\int \delta^3\nu \times \exp\left[\frac{im}{2}\int_0^{t'-t^0} \nu^2(\xi)\, d\xi \right] \exp\left[-i\int_0^{t'-t^0} U\left(t'+\xi;\ \mathbf{x}^0 + 2\int_0^\xi \nu(\eta)\, d\eta\right) d\xi \right] \times \delta^{(3)}\left(\mathbf{x} - \mathbf{x}^0 - 2\int_0^{t'-t^0} \nu(\eta)\, d\eta\right). \quad (33)$$

Both expressions, (32) and (33), can be easily reduced to a Feynman path integral of e^{iW}, where W is in both cases the action integral. To this end it suffices to change over from integration with respect to $\nu_\mu(\xi)$ to the new functional variables $\dot{x}_\mu(\xi) = 2\nu_\mu(\xi)$. The determinant of this transformation is equal to unity, and we obtain for the relativistic function $\mathcal{G}$:

$$\mathcal{G}(x, y \mid A) = iC\int \delta^4 x \int_0^\infty d\tau \exp\left\{ -i\int_0^\tau d\xi\, [\dot{x}_\mu^2(\xi) - 2e\dot{x}_\mu(\xi)\, A_\mu(x) + m^2 - i\sigma_{\mu\nu}\,\partial_\nu A_\mu] \right\} \quad (34)$$

(owing to the δ^4-function in (32), we have put $x_\mu(\tau) = x$ and $x_\mu(0) = y$). The argument of the exponential in (34) contains the relativistic action function of a charged particle in a field A_μ:

$$W = \int_0^\tau [\dot{x}_\mu^2(\xi) - 2e\dot{x}_\mu A_\mu(x) - m^2]\, d\xi$$

and a spin part $i\sigma_{\mu\nu}\partial_\mu A_\mu$, which has no counterpart in the classical theory. The integral is taken over all the trajectories joining x and y. For the Schrödinger function (33) one makes the substitution $\nu(\xi) = \dot{x}(\xi)$, where ξ is already simply the time.

Examples of concrete fields $A_\mu(x)$ for which the integration with respect to ν_μ in (32) can be carried out accurately are given in[18].

6. APPROXIMATE METHODS IN FUNCTIONAL INTEGRATION

For an approximate calculation of the functional integrals in QM and QFT, many authors[4,19,23] have considered, first of all, the stationary-phase method developed in mathematics for ordinary integrals. Its applicability can be regarded as justified if $W \gg \hbar$, but there are still no estimates of this method even for the simp-

*The problem of "disentangling" Dirac matrices in the solution of the Dirac equation in an arbitrary external field was considered by Fradkin[22].

lest functional integrals[23,24]. In this method one seeks the extremal trajectories [classical, see (12)], from the condition

$$\delta W[x^{cl}]=0. \tag{35}$$

The action is expanded about these trajectories:

$$W[x]=W[x^{cl}]+\frac{1}{2}\delta^2W[x^{cl}]+\frac{1}{6}\delta^3W[x^{cl}]+\ldots,$$

where, for example,

$$\delta^2W[x^{cl}]=\int_{t^0}^{t}dt\left\{\frac{\partial^2\mathcal{L}}{\partial x^2}\Big|_{x=x^{cl}}(x-x^{cl})^2+\frac{\partial^2\mathcal{L}}{\partial x\,\partial\dot{x}}\Big|_{x=x^{cl}}(x-x^{cl})(\dot{x}-\dot{x}^{cl})\right\}$$

is a quadratic functional with respect to x(t). Equation (35) on a classical trajectory has the following form in the case of the relativistic problem (34) (without the term $\sigma_{\mu\nu}\partial_\nu A_\mu$):

$$\frac{\partial^2x^{cl}(\xi)}{\partial\xi^2}=e\frac{\partial x^{cl}(\xi)}{\partial\xi}\left[\frac{\partial A_\mu(x)}{\partial x_\nu^{cl}}-\frac{\partial A_\nu(x)}{\partial x_\mu^{cl}}\right]. \tag{36}$$

For the Schrödinger equation we have (see[19])

$$m\ddot{\mathbf{x}}^{cl}=\frac{\partial U(t^0+\xi,\,x^{cl}(\xi))}{\partial\mathbf{x}^{cl}}. \tag{37}$$

Equations (36) and (37) differ from the equations of motion of a classical particle in that at the initial instant one specifies not the coordinate and the velocity, but two values of the coordinate at the ends of the interval $[t^0, t']$:

$$\mathbf{x}_{cl}(t'-t^0)=\mathbf{x},\ \mathbf{x}^{cl}(0)=\mathbf{x}^0.$$

Unfortunately, for arbitrary fields A_μ and potentials U, which must be considered in order to construct the quantum function G(x, y) in accordance with (27), it is impossible to obtain exact solutions of (36) and (37). This quasiclassical approximation of the functional integral can therefore be of practical use in those cases when (36) and (37) can be solved by some other method outside the framework of perturbation theory.

It was pointed out in[25] that functional integrals can be calculated by the stationary-phase method in the case of quantum fields described by Lagrangians of the type $\mathcal{L}=\mathcal{L}(K, I)$, where $K=\partial_\mu\varphi\partial_\mu\varphi/2$ is the free-field Lagrangian and $I=\varphi^2/2$. Such fields were called by one of us (D.B.) essentially-nonlinear. It was shown in the cited paper that if the dimensionless quantity $M=\partial\mathcal{L}/\partial K\gg 1$ in the space-time region of practical importance, then the stationary-phase method can be used. The action $W\{\dot{\varphi}(x)\}$ can then be taken in the form

$$W\{\varphi(x)\}=W\{\varphi_{cl}(x)\}+\frac{M}{2!}Q\{\psi(x)\}+\ldots,$$

where $\varphi(x)=\varphi_{cl}(x)+\psi(x)$. The quadratic form $Q\{\psi(x)\}$ is the action for a free quantum field $\psi(x)$, but one propagating in space with a curved metric. This metric is determined by the field $\varphi(x)$ and its first derivatives. It turns out that this approximation is equivalent to introducing in the Poisson quantum brackets

$$[\psi(x),\,\psi(x')]=i\hbar^*\delta(x-x')$$

an effective Planck constant $h^*=\hbar/\overline{M}$, where $\overline{M}$ is the mean value of M, under the assumption that $\overline{M}\gg 1$. In this case the quantum fluctuations of the quantities $\dot{\psi}(x)$ and $\psi(x)$ are small and the quantum field becomes close to the classical one.

We proceed to consider another method of approximating functional integrals of a particular type. This method yields the asymptotic expression in the infrared region of quantum electrodynamics, and also allows us to study the asymptotic behavior of certain particle-scattering processes at high energies and low momentum transfers. In order to simplify the problem, we consider a simple relativistically-invariant model[18] of the interaction of two scalar fields, φ with mass μ and ψ with mass m. The interaction Lagrangian of this system is

$$\mathcal{L}_{int}=g\psi^2(x)\,\varphi(x).$$

The Green's function of the particle of field ψ in the classical external field $\varphi(x)$ satisfies the relation

$$[i^2\partial_\mu^2-m^2+g\varphi(x)]\,G(x,\,y\,|\,\varphi)=-\delta^4(x-y).$$

Repeating the procedure described in Sec. 5, we get

$$G(x,\,y\,|\,\varphi)=i\int_0^\infty d\tau e^{-im^2\tau}\exp\left\{i\int_0^\tau[i^2\partial_\mu^2+g\varphi(x,\,\xi)]\,d\xi\right\}\delta^4(x-y). \tag{38}$$

In analogy with (31), we make the transformation

$$\exp\left[i\int_0^\tau d\xi i^2\partial_\mu^2(\xi)\right]=C\int\delta^4\nu\exp\left[-i\int_0^\tau\nu_\mu^2(\xi)\,d\xi+2i\int_0^\tau\nu_\mu(\xi)\,\partial_\mu(\xi)\,d\xi\right].$$

Substituting this expression in (38) and "disentangling" the differentiation operator ∂_μ, we get

$$G(x,\,y\,|\,\varphi)=-i\int_0^\infty d\tau e^{-im^2\tau}\times C\int\delta^4\nu\exp\left\{-i\int_0^\tau\left[\nu_\mu^2(\xi)-g\varphi\left(x-2\int_\xi^\tau\nu(\eta)\,d\eta\right)\right]d\xi\right\}\times\delta^4\left(x-y-2\int_0^\tau\nu(\eta)\,d\eta\right). \tag{39}$$

We need next the Fourier transform of the G-function

$$G(p,\,q\,|\,\varphi)=\int d^4x\,d^4y e^{ipx-iqy}G(x,\,y\,|\,\varphi)=$$
$$=-i\int d^4y e^{i(p-q)y}\int_0^\infty d\tau e^{i(p^2-m^2)\tau}C\int\delta^4\omega\times$$
$$\times\exp\left\{-i\int_0^\tau\left[\omega_\mu^2-g\varphi\left(x+2p\xi+2\int_0^\xi\omega(\eta)\,d\eta\right)\right]d\xi\right\}; \tag{40}$$

we have made here a change of functional variable $\nu_\mu(\xi)=p_\mu+\omega_\mu(\xi)$; the integral with respect to d^4x is eliminated by the δ^4-function in (39). At g = 0 we get from (38) the free propagation function

$$G_0(p)=(2\pi)^4\delta^4(p-q)/(p^2-m^2+i\varepsilon),$$

since

$$C\int\delta^4\omega\exp\left[-i\int_0^\tau\omega_\mu^2(\eta)\,d\eta\right]=1.$$

If we are not interested in the infrared region in this model at $\mu=0$, then it is proved in[28,26] that the contributions of the vacuum polarization of the field ψ can be neglected, $S_0(\varphi)=1$, and we obtain from (27) the quantum function G(p) by integrating with respect to φ in accordance with (27) and (40):

$$G(p)=$$
$$=i\int_0^\infty d\tau e^{i\tau(p^2-m^2)}C\int\delta^4\omega\exp\left\{-i\int_0^\tau d\xi\omega_\mu^2(\xi)-\frac{ig_1^2}{2}\int_0^\tau\int d\xi_1\,d\xi_2\,\Delta(\xi_1,\,\xi_2\,|\,\omega)\right\}, \tag{41}$$

where

$$\Delta(\xi_1, \xi_2 | \omega) = \int d^4kD(k) \exp\Big[-2ikp|\xi_1-\xi_2| - 2ik\int_{\xi_1}^{\xi_2}\omega(\eta)\,d\eta\Big],$$

$$D(k) = 1/(k^2+i\varepsilon),\ g_1 = g/(2\pi)^2.$$

a) Infrared asymptotic form of Green's function. It is impossible to integrate exactly with respect to $\delta\omega$ in (41), and we therefore calculate the integral approximately in the investigated infrared region (the effectively-virtual momenta k are small). To this end, we assess the role of the functional argument ω in $\Delta(\xi_1, \xi_2|\omega)$.

If we consider the perturbation-theory series, expanding (41) in powers of g_1^2, then the functional integrals can be easily evaluated, since we get expressions of the type

$$\overline{\Delta}(\xi_1, \xi_2) = C\int \delta^4\omega \exp\Big[-i\int_0^\tau \omega^2(\eta)\,d\eta\Big]\Delta(\xi_1, \xi_2|\omega) = \int d^4kD(k)\exp[-i(2pk+k^2)|\xi_1-\xi_2|]. \tag{42}$$

We see therefore that the functional argument in $\Delta(\xi_1, \xi_2|\omega)$ leads after integration to the appearance of a quadratic dependence on the virtual momenta k. Therefore, if we are interested in the low-energy region, where the k are effectively small, we can neglect in Δ the dependence on ω. Results pertaining to the infrared region were obtained by Fradkin[16] and Milekhin[27] to the same degree of an approximation, but by another method. Such an approximation, however, alters appreciably the behavior of $\overline{\Delta}(\xi_1, \xi_2)$ at large momenta k, and leads, in particular, to stronger divergences of the non-renormalizable quantities; we shall therefore not neglect this dependence, but resort to another procedure, namely approximation of the integrals with respect to ω.

To explain the meaning of this approximation, we consider the expansion, in powers of g_1^2, of the quantity

$$A = C\int \delta^4\omega \exp\Big[-i\int_0^\tau \omega_\mu^2(\xi)\,d\xi\Big]\exp\Big[-\frac{ig_1^2}{2}\int_0^\tau\int_0^\tau d\xi_1\,d\xi_2\,\Delta(\xi_1, \xi_2|\omega)\Big] = 1 - \frac{ig_1^2}{2}\int_0^\tau\int_0^\tau d\xi_1\,d\xi_2\overline{\Delta}_1(\xi_1-\xi_2) + \frac{1}{2}\Big(\frac{ig_1^2}{2}\Big)^2\int_0^\tau\cdots\int_0^\tau d\xi_1\ldots d\xi_4\overline{\Delta}_2(\xi_1, \xi_2, \xi_3, \xi_4) + \ldots; \tag{43}$$

$\overline{\Delta}_1(\xi_1, \xi_2)$ is defined in (42), and

$$\overline{\Delta}_2(\xi_1, \xi_2, \xi_3, \xi_4) = C\int \delta^4\omega\exp\Big[-i\int_0^\tau\omega^2(\eta)\,d\eta\Big]\times\Delta_1(\xi_1, \xi_2|\omega)\,\Delta_1(\xi_3, \xi_4|\omega) = \int d^4k_1\int d^4k_2 D(k_1)D(k_2)\exp\{i(k_1^2|2pk_1)|\xi_1-\xi_2| + i(k_2^2+2pk_2)|\xi_3-\xi_4| + 2ik_1k_2\theta(\xi_1, \xi_2, \xi_3, \xi_4)\},$$

where (44)

$$\theta(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1|\theta(\xi_3-\xi_1)-\theta(\xi_4-\xi_1)] + \xi_2|\theta(\xi_4-\xi_2)-\theta(\xi_3-\xi_2)] + \xi_3|\theta(\xi_1-\xi_3)-\theta(\xi_2-\xi_3)] + \xi_4|\theta(\xi_2-\xi_4)-\theta(\xi_3-\xi_4)],$$

$$\theta(x) = \begin{cases}1, & x>0,\\ 0, & x<0.\end{cases}$$

We approximate the functional $\Delta(\xi_1, \xi_2|\omega)$ in the infrared region by its mean value $\overline{\Delta}(\xi_1-\xi_2)$ defined in (42). We now rewrite (43) in the form

$$A = \exp\Big[-\frac{ig_1^2}{2}\int_0^\tau\int\overline{\Delta}(\xi_1-\xi_2)\,d\xi_1\,d\xi_2\Big]\times C\int\delta^4\omega\exp\Big\{-i\int_0^\tau\omega^2(\eta)\,d\eta - \frac{ig_1^2}{2}\int_0^\tau\int[\Delta-\overline{\Delta}]\,d\xi_1\,d\xi_2\Big\},$$

and expand the exponential under the integral sign in terms of the differences $\Delta - \overline{\Delta}$. We then obtain for A in place of the expansion (43) a new series with the exponential factor separated:

$$A = \exp\Big[-\frac{ig_1^2}{2}\int_0^\tau\int\overline{\Delta}(\xi_1\,\xi_2)\,d\xi_1\,d\xi_2\Big]\times C\int\delta^4\omega\exp\Big[-i\int_0^\tau\omega^2(\eta)\,d\eta\Big]\sum_{n=0}^\infty\Big(-\frac{ig_1^2}{2}\Big)^2(n!)^{-1}\Big[\int_0^\tau\int(\Delta-\overline{\Delta})\,d\xi_1\,d\xi_2\Big]^n. \tag{45}$$

An advantage of such an expansion, in comparison with perturbation theory, is that after separating the factor

$$\exp\Big[-i\frac{g_1^2}{2}\int_0^\tau\int\overline{\Delta}(\xi_1, \xi_2)\,d\xi_1\,d\xi_2\Big]$$

in front of the entire series and integrating with respect to ω, each term of (45) no longer contains infrared singularities, which are present only in the exponential factor.

We note that further partial summation of the series (43) is possible. To this end it is necessary to choose as the approximation of

$$\int_0^\tau\int\Delta(\xi_1, \xi_2|\omega)\,d\xi_1\,d\xi_2$$

the quantity

$$-\frac{ig_1^2}{2}\int_0^\tau\int\overline{\Delta}(\xi_1, \xi_2)\,d\xi_1\,d\xi_2 + \frac{1}{2}\Big(\frac{-ig_1^2}{2}\Big)^2\int_0^\tau\cdots\int_0^\tau d\xi_{1,2,3,4}(\overline{\Delta}_2-\overline{\Delta}_1^2). \tag{46}$$

We then have in lieu of (45), as a common factor, an exponential with (46) as the argument, and a series in which g_1^6 is the first term that follows 1. In the infrared region, however, this does not change the results, since the second term of (46) has no infrared singularities, and the first term is decisive.

If (45) is substituted in the expression for the Green's function (41), then we obtain an expansion in powers of g_1^2 with a common factor

$$\exp\Big[-\frac{ig_1^2}{2}\int_0^\tau\int\overline{\Delta}(\xi_1, \xi_2)\,d\xi_1\,d\xi_2\Big],$$

which, as shown in[18], includes all the infrared singularities of the Green's function.

Each term of the series (45) represents the difference between the expression $\overline{\Delta}(\xi_1, \ldots, \xi_n)$, obtained in perturbation theory, and a like expression in which the terms k_ik_j with $j \neq j$ is discarded. This can be seen in the g_1^4-order:

$$\frac{1}{2}\Big(\frac{ig_1^2}{2}\Big)^2\int\cdots\int d\xi_1\ldots d\xi_4[\overline{\Delta}_2(\xi_1, \xi_2, \xi_3, \xi_4) - \overline{\Delta}_1(\xi_1, \xi_2)\,\overline{\Delta}_1(\xi_3, \xi_4)],$$

by comparing (42) and (44). Thus, in the language of perturbation theory the proposed approximation of the functional integral means, that terms of the type k_ik_j are discarded from the particle propagators,

$$1/[(p+\sum_i k_i)^2 - m^2] \to 1/(2p\sum_i k_i + \sum_i k_i^2). \tag{47}$$

b) Asymptotic behavior of the scattering amplitudes at high energies and low particle scattering energies. Recently, a number of workers have stated that the approximation of the propagators (47) is valid also at high energies of the colliding particles, at momenta $|p| \gg m$. It is assumed in this case that the term $2p\sum_i k_i$ in the propagator (47) is dominant in comparison with $\sum_{i,j} k_ik_j$.

An analysis of this approximation, within the framework of perturbation theory, has shown[30,31] that it accounts well for the asymptotic behavior of the amplitudes at high energies, $s \to \infty$, and at low momentum transfers t, $t/s \ll 1$.

As follows from the preceding section, the method of functional integration yields closed expressions for the Green's functions in the approximation of (47). We shall demonstrate in this section that this method is effective in the study of the high-energy asymptotic behavior of scattering amplitudes. Starting from the preceding considerations, we can obtain here the Glauber[32] representation, which is also called the eikonal or the geometrical-optics approximation for the scattering amplitude in field theory. The application of the geometrical-optics method to the scattering of elementary particles was developed by the present authors in earlier papers[33,34].

We begin the analysis with the scattering of a high-energy scalar particle by an external potential[35] $\varphi(x)$. We can obtain the amplitude of the process $f(p, q|\varphi)$ (p and q are the momenta of the particle before and after the scattering) if we know the Green's function $G(p, q|\varphi)$ of this particle [formula (40)]. To this end it is necessary to go over to the mass shell with respect to the external momenta in accordance with the relation

$$f(p, q|\varphi) = \lim_{p^2, q^2 \to m^2} [(p^2 - m^2)(q^2 - m^2) G(p, q|\varphi)]. \qquad (48)$$

The procedure of taking the limit in (48) consists of separating from the function $G(p, q|\varphi)$, specified by the integral (40), two pole expressions $(p^2 - m^2)^{-1}$ and $(q^2 - m^2)^{-1}$, which are two free Green's functions. This procedure was developed in detail in[29,35,36]. We present here the final result for $f(p, q|\varphi)$:

$$f(p, q|\varphi) = \int d^4x e^{i(p-q)x} g\varphi(x) \underbrace{C \int \delta^4\omega \exp\left[-i \int_{-\infty}^{+\infty} \omega^2(\eta)\, d\eta\right]}_{[\delta^4\omega]} \times \int_0^1 d\lambda \exp\left\{i\lambda g \int_{-\infty}^{+\infty} \varphi\left(x + 2\xi[p\theta(\xi) + q\theta(-\xi)] + 2\int_0^{\xi} \omega(\eta)\, d\eta\right) d\xi\right\}. \qquad (49)$$

Changing over to a new functional variable $z(\xi) = \int_0^{\xi} \omega(\eta) d\eta$, we rewrite (49) in the form of a Feynman integral along the paths $z(\xi)$:

$$f(p, q|\varphi) = g \int d^4x e^{i(p-q)x} \psi(x) \int_1^1 d\lambda \int \delta z e^{iW[z]},$$

where

$$W[Z] = \int_{-\infty}^{+\infty} d\xi \{-\dot{z}^2(\xi)^2 + \lambda g\varphi[x + 2(p\theta(\xi) + q\theta(-\xi))\xi + 2z(\xi)]\},$$

is the action function of a relativistic particle moving in a potential φ; ξ plays the role of the proper time of the particle, at $\xi = 0$ the particle is at the point x, and $z(\xi)$ is the variable describing the deviation of this particle from straight-line paths along the 4-momentum p ($\xi > 0$) before the scattering and along q ($\xi < 0$) after the scattering.

Considering the asymptotic behavior of the amplitude (49) at large momenta $p_0, q_0 \gg m$, it is natural to raise the following question: can we neglect the functional variable $z(\xi)$ in the argument φ compared with the large quantity $p\theta(\xi) + q\theta(-\xi)$, and will this be a good approximation of the integral at large p_0 and q_0?

The answer to this question is connected with the character of the function φ. If the potential φ is a smooth, non-oscillating and bounded function satisfying the conditions

$$p_0^2 \gg g \max|\varphi(x)|, \quad |\mathbf{p}| \gg \left|\frac{\partial\varphi}{\partial \mathbf{x}}\right| \Big/ |\varphi(x)|, \quad p_0 \gg \left|\frac{\partial\varphi}{\partial x_0}\right| \Big/ |\varphi(x)|, \qquad (50)$$

then it can be proved[35] that the expansion of φ in a Taylor series with respect to the functional argument in (49) and a subsequent integration with respect to ω leads in the argument of the exponential (49) to the expression

$$\varphi(x + 2\xi[p\theta(\xi) + q\theta(-\xi)]) + O(1/|p|) \qquad (51)$$

This result can be understood by reasoning as follows. The last two conditions of (50) are the conditions for the validity of the quasiclassical approximation. For an approximate calculation of the functional integral we can therefore use the stationary-phase method considered above, when the main contribution to the integral is made by the classical trajectory. Owing to the first condition of (50), the two linear paths along p when $\xi > 0$, and along q when $\xi < 0$, are also good approximations for the classical trajectory. In this case the approximation of the potential φ by its mean value in accordance with (42) is also valid (and gives a more accurate result than (51)):

$$\bar{\varphi}(x + 2\xi[p\theta(\xi) + q\theta(-\xi)]) = C \int \delta^4\omega \exp\left[-i \int \omega^2(\eta)\, d\eta\right] \varphi\left(x + 2\xi[p\theta(\xi) + q\theta(-\xi)] + 2\int_0^{\xi} \omega(\eta)\, d\eta\right).$$

The conditions (50) are the conditions for the validity of the Glauber[32] or eikonal representation of the scattering amplitude, which follows directly from (49) at small momentum transfers $|p - q|$ [the scattering angle is $\alpha \ll (pR)^{-1/2}$, where R is the effective radius of the potential]. In the case of a potential $\varphi(r)$ that does not depend in the time, we have

$$f(p, q|\varphi) = 4\pi\delta(p_0 - q_0)|p| i \int d^2x_\perp e^{i(p-q)_\perp x_\perp} [1 - e^{i\chi(x_\perp)}],$$

where

$$x_\perp = (x_1, x_2); \quad \chi(x_\perp) = \frac{g}{2|p|} \int_{-\infty}^{\infty} d\xi \varphi\left(x_\perp + \frac{p\xi}{|p|}\right).$$

The eikonal approximation obtained in this manner is valid, as shown in[35], in a wider range of angles; it has made it possible to refine the results obtained by Schiff[37] within the framework of perturbation theory. On the basis of the approach presented here, an eikonal representation was obtained in[38] for the amplitude of the scattering of a Dirac particle by an arbitrary potential.

We consider also the possible use of the approximation (45), and hence also (47), in the calculation of the asymptotic behavior of the scattering amplitude of two particles in the high-energy region at a fixed momentum transfer. This method was used in[29,30] to investigate the asymptotic behavior of elastic-scattering amplitudes, and in subsequent papers[40] to use the influence of radiation corrections to scattering diagrams.

The scheme for constructing the elastic amplitude

within the framework of the discussed approximation is as follows. Knowing G(p, q|φ), we can, neglecting the contributions from the polarization of the nucleon vacuum, $S_0(\varphi) = 1$, obtain the quantum two-particle Green's function

$$G(p_1, p_2|q_1, q_2) = C\varphi \int \delta\varphi \exp\left[-\frac{i}{2}\int D^{-1}(q)\varphi(q)\varphi(-q)\,d^4q\right] \times [G(p_1, q_1|\varphi)G(p_2, q_2|\varphi) + G(p_1, q_2|\varphi)G(p_2, q_1|\varphi)],$$

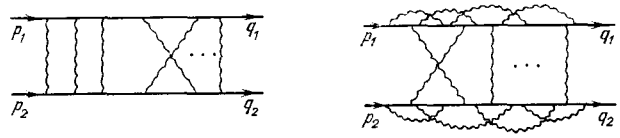
by integrating with respect to φ.

By going over to the mass shell with respect to the external momenta, we can construct the two-particle amplitude

$$(2\pi)^4\delta^4(p_1 + p_2 - q_1 - q_2)\,f(p_1, p_2; q_1, q_2) = \lim_{p_i^2, q_i^2 \to m^2} [(p_1^2 - m^2)(q_1^2 - m^2)(p_2^2 - m^2)(q_2^2 - m^2) \times G(q_1, q_2|p_1, p_2)],$$

which, unlike the single-particle amplitude (49), is expressed by a double functional integral with respect to ω_1 and ω_2 (integrals over the trajectories of the two scattered particles).

The expression for f, which is quite complicated in form (see[29,39]), includes contributions of sums of Feynman diagrams of the type indicated on the left, with diagrams of the type with radiative corrections (right):



It is shown in[31] that the approximation (47) is valid in the high-energy region $s \to \infty$ also at low momentum transfers t for expression connected with diagrams of the first type and is not true for each diagram of the second type. It is proved in[41], however, up to sixth order inclusive, that in each order of perturbation theory the sum of diagrams of the second type, but not an individual diagram, has the same asymptotic form at $s \to \infty$ and at fixed t as obtained by means of the approximation (45).

Thus, a study of the asymptotic behavior of an individual Feynman diagram may not duplicate the asymptotic behavior of a sum of diagrams of a given order, so that the devices used here for the summation and approximation of perturbation-theory series with the aid of the technique of functional integrals may turn out to be more suitable for the study of asymptotic forms in quantum field theory than the study of the asymptotic form of each diagram separately, followed by summation. We wish to emphasize here once more that it is difficult to prove mathematically the regions of applicability of the approximations made in the calculation of functional integrals[42].

From the examples given in the review of applications of the method of functional integration in quantum mechanics and quantum field theory we see that this method represents only the first steps, both mathematically and from the point of view of physical applications. However, the physical clarity and the brilliance of the formulation of the main problems of quantum field theory in this method gives grounds for hoping that it will find further development in the future.

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