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# Origin of Magnetic Fields in Astrophysics
## (Turbulent "Dynamo" Mechanisms)

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We consider the generation of magnetic fields under astrophysical conditions. Principal attention is paid to "dynamo" mechanisms, i.e., mechanisms in which the energy of the magnetic field is drawn from the kinetic energy of plasma motion. The important role played by turbulent dynamo mechanisms is emphasized. The dynamo problem itself is divided into two: 1) generation of regular magnetic fields (i.e., the dynamo mechanism of fields having scales commensurate with the cosmic object itself by turbulent pulsations); 2) generation of random fields. A review is presented of the existing theories for the generation of regular fields (reference is made, in particular, to the work of Steenbeck and co-workers and of Parker), and a generalization of the existing results to include large magnetic Reynolds number $Re_m$, characteristic of cosmic plasma, is indicated. Astrophysical examples are given. The existing theories in the dynamics of random fields are also reviewed. Results are presented on the turbulent dynamo in the presence of acoustic turbulence. Analogies with the question of excitation of vortices in a field of acoustic turbulence is indicated. The question of the turbulent dynamo in the field of "Kolmogorov" turbulence is discussed. Finally, an essential problem is that of the steady-state field produced by a nonlinear effect. It is shown that in the presence of a non-weak field there appears a gyrotropy acting in opposition to the action of the usual rotational gyrotropy. This observed gyrotropy is called magnetic and can cause nonlinear stabilization of the magnetic field.

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## I. INTRODUCTION

MAGNETIC fields in planets, stars, galaxies, quasars, and intergalactic space constitute a phenomenon of tremendous scale and significance. The appearance of high-energy particles such as protons, nuclei, and electrons is closely connected with the magnetic field. The motion of these particles in the magnetic field produces the electromagnetic synchrotron radiation. The proof of the synchrotron nature of the radiation of the Crab nebula[1,2] was a most important stage in the development of modern astrophysics. This discovery was followed by recognition of the very important role played by the electromagnetic field in a large number of astronomical phenomena. In many cases, the magnetic field plays the primary role in the dynamics of the astrophysical processes themselves. This raises the question of the origin of such fields. The usual plasma excitation mechanisms, such as the thermal mechanism, which will also be discussed briefly below, result as a rule in only very weak currents and magnetic fields. On the other hand, many observed processes have sufficiently high energies (gravitational or kinetic) so that if the mechanisms of the "conversion" of this energy into magnetic energy could be understood, then the observed values of the magnetic fields could be explained. Thus, for example, in the gas filling our galaxy, the kinetic energy of the gas, the kinetic energy of the cosmic rays, and the energy of the magnetic field are of the same order, as emphasized by V. L. Ginzburg.

If we have in mind not regular but random fields, then the most general considerations make it plausible to assume equipartition of the energy among the various forms. Such general statements, however, call for natural caution. Statements pertaining to the complete and true thermodynamic equilibrium are undoubtedly correct, but are absolutely of no interest: the reservoir of degrees of freedom with maximum wave vectors, i.e., essentially the motion of individual particles in the field of equilibrium "black body" radiation, suppresses all the macroscopic degrees of freedom.

Macroscopic motions and fields, whether ordered or statistical (such as turbulent fields), are of interest to the extent to which they are nonthermodynamic, and consequently are not in equilibrium. In such a situation we are dealing with the kinetics of energy transfer from one form to another and from long to short waves; there is no simple thermodynamic equipartition. We can mention universally known cases of violation of equipartition. For example, in a turbulent stream the longitudinal (acoustic) motions are weaker than the transverse ones by a factor $M^5$, where M is the Mach number. From the thermodynamic point of view, the electric field does not differ from the magnetic one, but in magnetohydrody-

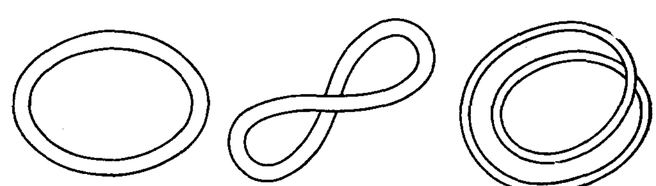


FIG. 1

namic processes the energy of the electric field is lower than that of the magnetic field by a factor of at least $(v/c)^2$ (v is the velocity of motion). These examples show that a superficial application of the idea of equipartition is not permissible without a concrete analysis of the problem.

In cosmic plasma there usually take place various hydrodynamic motions whose energy is not low. It is they which offer promise of serving as a possible source of the enhancement of magnetic fields. Such a mechanism is usually called the "dynamo" mechanism. In the "dynamo theories" to which the present review is mainly devoted, the most important fact is the freezing-in of the magnetic field. The large conductivity of the plasma together with the large spatial scale of the phenomena cause the ohmic resistance to play no role whatever (the dimensionless magnetic Reynolds number $\mathrm{Re}_m$ is large). A decisive role is played by the inductance, as a result of which we get conservation of the magnetic flux through each contour that moves together with the plasma, i.e., consisting all the time of the same particles. It is universally known that in this case the damping time of the magnetic field in an immobile medium of astronomical size is gigantic and is absurdly large even in the astronomical time scale. The burning question in the theory of generation is whether the growth time of macroscopically ordered fields is just as large. An affirmative answer would mean that it is impossible in practice to generate fields.

The dynamo theory has been in existence about fifty years, but until recently there was no distinct separation into fast and slow dynamos, in other words, into generation with and without frozen-in fields. An example of a slow dynamo is any generator in an electric station or a magnetohydrodynamic generator. In the slow dynamo, the growth of the field is connected with its penetration into matter, and is therefore of no interest in astrophysics, where the mechanism of the fast dynamo effect is necessary. It has turned out that field generation under cosmic conditions is not as simple to realize as in a laboratory. During the initial stage of the development of the dynamo theory, only negative results were obtained. It was shown[3-5] that motions having a high degree of symmetry (two-dimensional, axisymmetrical, centrally-symmetrical) are incapable of generating a field. This circumstance greatly complicates the problem, which has not yet been solved in general form. On the other hand, using the concept of the freezing-in of the magnetic field, one can point to a concrete example of a fast dynamo mechanism. Let us imagine a conducting liquid torus in which there exists an initial toroidal (parallel to the equator plane) magnetic field (Fig. 1a). Further, it is easy to imagine motion that transforms the torus into a "figure 8" (Fig. 1b). The next stage is congruence of the circles of the figure-8 (Fig. 1c). It is seen from Fig. 1c that the magnetic field flux has doubled. If we repeat such a motion n times, we obtain an enhancement by a factor $2^n$; thus, the flux increases exponentially. It should be noted, however, that the field does not simply double, for in addition to the toroidal components there appear also "extra fields," which can be annihilated by finite diffusion. Thus, without forgoing the ideal freezing-in, we can obtain an unlimited increase of the ordered flux. It is clear also that the velocity field in this example does not have cylindrical symmetry on the whole.

Together with the increase of the flux, there occurs a certain change in the subtler topological properties of the current lines. In this sense, certain violation of the frozen-in property is nevertheless necessary in order to reproduce exactly all the details of the field upon amplification. The velocity field is stationary when averaged over the cycle, but not as each instant of time. The need for finite diffusion for the dynamo was indicated already by Elsasser.[6] A mechanism analogous to that described above was indicated by Davis.[7]

The feasibility, in principle, of a dynamo for helical motions was rigorously demonstrated analytically by Lortz.[8] The doubts concerning the dynamo mechanism in astrophysics was dispelled after Parker[9] and Elsasser[6] developed the theory of the solar cycle as an oscillatory dynamo. We shall not describe their mechanism, since it is widely known (see, e.g., [10]). We note only that according to this mechanism a toroidal field is generated from a poloidal one (whose force lines lie in meridianal planes) with the aid of differential rotation. The convective motions produce "loops," and in the presence of a Coriolis force the loops rotate; it is easy to understand that if the loops are rotated through approximately 90° and then coalesce as a result of the finite electric conductivity, then a new field is produced in them, either parallel to the initial poloidal field (in which case enhancement of the field takes place), or antiparallel (and then the fields reverse sign). It is easy to see the similarity between this mechanism and the example given above.

At the present time, great promise is held by statistical mechanisms, i.e., the turbulent dynamo. Of course, no general theory of turbulence in the presence of magnetic fields exists at present: the point is that a decisive influence was exerted on the establishment and development of the theory of ordinary hydrodynamic turbulence by an abundance of experimental data. As to laboratory experiments on magnetohydrodynamic turbulence, it is difficult to create conditions with $\mathrm{Re}_m \gg 1$ (which is characteristic of cosmic plasma), while the interpretation of the observations of cosmic plasma frequently is itself in need of theoretical premises. The dynamo problem is usually formulated as a kinematic one, i.e., the velocity field is specified and the reaction of the magnetic field on the magnetic motion is disregarded. In this formulation, the problem can be divided into two:

1. The interaction of large-scale magnetic fields with the turbulence (this includes turbulent diffusion and turbulent generation of the overall fields of stars and planets and of the overall field of the galaxy by turbulent mechanisms).

2. Interaction of small-scale pulsational magnetic fields with the turbulence (this includes the possible excitation of random magnetic fields of the galaxy, which play an important role in the acceleration of cosmic rays and the polarization of the interstellar medium, random interplanetary magnetic fields of solar wind, which are important for the explanation of diffusion and isotropization of the cosmic rays, and random solar and stellar magnetic fields which play a major role in dynamics of solar and stellar processes).

Some understanding of problem 1 has been reached by now, starting with the work of Steenbeck and Krause.[11] It has become known that simple isotropic or anisotropic turbulence causes only turbulent diffusion of the magnetic field. Only gyrotropic turbulence (which is non-invariant under reflection) in which either right-hand or left-hand helical motion predominates is capable of operating as a field generator. Steenbeck and Krause have demonstrated this for $Re_m \ll 1$; one of us[12] generalized this result to the case $Re_m \gg 1$, which is realistic in cosmic electrodynamics. The problem of Steenbeck and Krause is that of a slow dynamo in which the field growth increment depends on the electric conductivity $\sigma$; the problem described in [12] is that of the fast dynamo in which the increment depends only on the turbulent (gross) characteristics.

Notice that all the dynamo mechanisms (the one indicated in Fig. 1, the Parker dynamo, and the Steenbeck and Krause dynamo) reduce to one and the same mechanism wherein the loop breaks away from the main field and is rotated in such a way that it becomes parallel to the initial field; it is thus possible that all the generation mechanisms reduce to the "figure 8" of Fig. 1.

We point out that introduction of the concept of turbulent resistance has made the problem of fast or slow dynamo less acute: if the turbulence time is substituted in the expression for the damping time, then the damping period (meaning also the growth time) turns out to differ little from the period of the hydrodynamic motions.

Gyrotropic turbulence can be an effective generator of magnetic fields in rapidly rotating objects. The growth increment is in this case quite large and is smaller than the rotation frequency by only a few times. In other cases the sole action of the gyrotropic turbulence (called by Steenbeck the $\alpha$ effect) is insufficient, and must be aided by regular motions, for example differential rotation, acting "in the same direction" as the $\alpha$ effect. Such a situation obtains in our galaxy. On the other hand, if the differential rotation acts in opposition to the $\alpha$ effect, i.e., if the field generated from the poloidal field is toroidal and antiparallel to the toroidal field generating the $\alpha$ effect, then a vibrational dynamo cycle is obtained (the solar cycle is an example). It is interesting to note that the differential rotation itself produces only a growth of the field that is linear in time (if no diffusion is taken into account), i.e., a rather slow growth. When the $\alpha$ effect is added, the field growth is exponential, and the increment is the geometric mean of the rotation frequency and the $\alpha$-effect increment.

There are also other interesting aspects of problem 1. Many stars have convective cores or convective envelopes, and their remainder is immobile, i.e., the turbulence is inhomogeneous. This raises the question of how the total field of the star behaves. If it attenuates effectively in the convective core, then it becomes by the same token rapidly forced out into the immobile regions. The force lines "go around" the convective core. If this is so, then the core behaves like a diamagnet, and it is possible to develop a "macroscopic" electrodynamic theory, i.e., electrodynamics of large-scale fields, with the magnetic permeability $\mu$ dependent on the turbulent characteristics. The solution of problems such as 1 is facilitated by the fact that such problems have a physically small parameter $l/L$, where $l$ is the correlation length and L is the scale of the field, although no expansion is carried out in terms of this parameter. It is interesting to note that the question of forcing out the field from the turbulent core can be solved in the two-dimensional case, which includes the cases of diamagnetism and of turbulently-accelerated attenuation of the field.

Finally, an important question is that of the steady-state field, which obviously depends on the nonlinearity of the effect. In fact, in the kinematic problem, i.e., when the characteristics (statistical or regular) of the velocity field are specified, the dynamo results in an unbounded (exponential, see above) enhancement of the field, which naturally has no physical meaning. It is also natural to assume that the energy of the steady-state field does not exceed the kinetic energy. But this is the upper limit. Can stabilization occur in a weakly-linear regime, i.e., can the energy of the steady-state magnetic field be much smaller than the kinetic energy? It turns out that such a situation is possible. The gist of the situation is that the reaction affects primarily the gyrotropy itself, and not the average energy of the turbulence. But a small part of the entire random motion can be gyrotropic.

We proceed to problem 2, the interaction of small-scale pulsational magnetic fields with the turbulence. This question was discussed already in the papers of Batchelor[13] and Biermann and Schluter.[14] Batchelor called attention to the analogy between the equation for the magnetic field H and curl v, and arrived at the conclusion that when $\nu/\nu_m > 1$ ($\nu$ and $\nu_m$ are the kinematic and magnetic viscosities, respectively) the pulsational fields increase. It was subsequently noted by many workers that these equations are not physically analogous, since the velocity field (and by the same token also the field curl v) is maintained by external sources, but the field H is not.[4,15] It is not surprising that Batchelor's successors (Saffman,[16,17] Moffatt[18]) reached the opposite conclusion, that a turbulent dynamo is impossible in the inertial subregion (by analyzing the same equations for H and curl v!). Wherein does the difficulty of the problem lie?

We turn to the picture of motion with frozen-in fields. It is known that in a turbulent stream the distance between neighboring particles increases on the average, so that the force lines become entangled. One can therefore expect the mean-squared magnetic field to increase with time. However, the entanglement of the force lines is accompanied by a decrease in the scales of the fields, i.e., the "transfer" energy into the region of large wave numbers, which is usual for hydrodynamic turbulence, takes place. This means that there is a danger that the enhancement of the field occurs only as a result of breaking up the scales.

Exact investigations of idealized cases (axially symmetrical motion and two-dimensional motion) have confirmed these dangers. In the indicated degenerate cases, the growth of the field is indeed connected with a decrease in scale, the vector potential does not increase, and no exponential growth of the field is possible. In a real three-dimensional case, the investigation is made difficult by the fact that there is no small parameter in the problem, and the rates of the aforementioned two competing processes are of the same order of magnitude. For this reason, heuristic theories cannot give a final answer to the question whether a turbulent dynamo is possible. A significant contribution to the analysis of these questions was made by Kazantsev.[28]

It is possible to make progress in this problem by considering acoustic turbulence, namely an ensemble of interacting acoustic waves. The foregoing competing processes occur here, too, and their rates are approximately equal. On the other hand, a small parameter $v/\lambda\omega$ appears (v is the amplitude of the velocity, $\lambda$ is the wavelength, and $\omega$ is the frequency), and by expansion in terms of this parameter it is possible to obtain correctly an equation for the spectral function of the magnetic-field fluctuations. It turns out that exponentially increasing solutions do exist, i.e., the dynamo does take place. Finally, we shall advance later on arguments favoring the turbulent dynamo in ordinary hydrodynamic turbulence.

## II. REGULAR (LARGE-SCALE) MAGNETIC FIELDS

1. Origin of "priming" magnetic fields. Of course, one can assume that there exists a certain metagalactic field and that the galactic fields become enhanced in comparison with the primordial one as a result of condensation of the galaxies themselves, and the stellar fields are enhanced as a result of condensation of the stars.[19,65] To be sure, the need for dynamo theories still remains, since, first, alternating magnetic fields are observed at the stars and, second, turbulent diffusion of the magnetic field is not so slow and is frequently appreciable over cosmological times.

We shall follow an alternative approach, and assume that the field is excited by certain priming mechanisms, after which it is enhanced by the dynamo mechanism.

The first to propose a mechanism for the excitation of the priming field were Biermann and Schlüter.[14] The mechanism recalls the thermoeffect: in the presence of a pressure gradient, it is easier for electrons to leave places where the pressure is higher than for ions, and this gives rise to a current. A magnetic field is excited if curl $\rho^{-1}\nabla p \neq 0$, and in this case the electric field contains a non-potential component. This condition can be satisfied in rotating bodies; for stars, the priming fields are as a rule small. For example, for the sun $H \approx 10^{-5}$ G. Primary generation in stars was considered also by Drobyshevskiĭ.[66] Harrison[20] proposed a mechanism for the generation of priming fields in protogalaxies in an expanding universe, in the period when the density of the radiation was much higher than the density of matter. It is assumed here that the protogalaxies have an initial rotation. Qualitatively, Harrison's mechanism can be explained in the following manner. Let us imagine a spherical uniformly-rotating region of radius r; the radiation density is $\rho_\gamma$ and the density of matter is $\rho$ (ions and nonrelativistic electrons). As this vortex expands we have $\rho r^3$ = const and $\rho_\gamma r^4$ = const; consequently, if the angular momentum is conserved we have $\omega \sim r^{-2}$ and $\omega_\gamma \sim r^{-1}$ ($\omega$ are the angular velocities), so that if no account is taken of the interaction between the radiation and the matter, the ions rotate more slowly that the radiation. The electrons are drawn by the radiation, so that the cross section of the Thomson scattering is not small; a current is thus produced—the electrons rotate more rapidly than the ions. More accurately speaking, they "would like" to rotate more rapidly, but generate in this case a magnetic field such that the induced emf offsets the inertia of the ions; from this we can easily obtain the value of the field. The magnetic field generated thereby can be estimated at

$$B = -2\,(m_H c/2)\,\omega = -2\cdot 10^{-4}\omega\ \text{G}$$

Generation by this method stops when the friction between the ions and the electrons becomes large enough. This occurs at the end of the radiation period, and the final field is estimated by Harrison at $\sim 10^{-18}$ G.

A somewhat different mechanism for the last stage is proposed by Mishustin and Ruzmaĭkin.[21] During a period close to our own epoch, the interaction with the radiation can be neglected in first-order approximation, and the electrons and ions rotate together relative to the radiation. The collisions between the protons and the neutrals causes the former to rotate with the same angular velocity as the neutral matter. The electrons, on the other hand, interact with the radiation (homogeneous background) much more effectively than with the neutrals. The radiation slows down the rotation of the electrons. The resultant emf is compensated by the induced field.

The presence of two mechanisms is useful, since it is not clear to this day when the rotation of the galaxies, which is undoubtedly observed at the present time, set in. According to the vortex theory developed by Chernin and Ozernoĭ[22] and by others, the rotation took place already in the earlier pre-galactic stage to which the Harrison mechanism pertains. One can however, advance the hypothesis that the rotation developed as a result of gravitational instabilities and density perturbations. In this theory, the rotation occurs later[37] and then the mechanism of Mishustin and Ruzmaĭkin is more suitable.

2. The symmetrical problem. We illustrate below the symmetrical (two-dimensional) problem using the flat case as an example, and following Zel'dovich.[4] Thus, let $v_z = 0$ and let the gradients in the z direction be equal to zero, $\partial/\partial z = 0$. Then the magnetic field satisfies the equations

$$\frac{\partial H_z}{\partial t} + \mathbf{v}\nabla H_z = \nu_m \Delta H_z, \tag{1}$$

$$\frac{\partial A}{\partial t} + \mathbf{v}\nabla A = \nu_m \Delta A, \qquad H_x = \frac{\partial A}{\partial y}, \quad H_y = -\frac{\partial A}{\partial x}, \quad \operatorname{div}\mathbf{v} = 0. \tag{2}$$

The problem has broken up into two, one concerning the $H_z$ component of the field and the other concerning $H_x$ and $H_y$, which are expressed in terms of the $A_z$-component of the vector potential, henceforth denoted for brevity by A (without a subscript), A = A(x, y). Multiplying (1) by $H_z$ and (2) by A, and integrating over the entire (x, y) plane, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int H_z^2\,dx\,dy = -\nu_m \int (\nabla H_z)^2\,dx\,dy, \tag{3}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\int A^2\,dx\,dy = -\nu_m \int (H_x^2 + H_y^2)\,dx\,dy. \tag{4}$$

From (3) we see immediately that $H_z \to 0$ as $t \to \infty$. Integrating (4) with respect to t, we obtain

$$\int A^2(x, y, 0)\,dx\,dy - \int A^2(x, y, t)\,dx\,dy = 2\nu_m \int_0^t (H_x^2 + H_y^2)\,dx\,dy\,dt'.$$

The expression on the right is a monotonic function of t (an integral of a positive function), bounded by the quantity $\int A^2(x, y, 0)\,dx\,dy$.

Consequently, the integral converges as $t \to \infty$; it follows therefore that $(H_x^2 + H_y^2) \to 0$ as $t \to \infty$. We have thus obtained the well known result that a two-dimensional dynamo is impossible. We note that the conclusion can be generalized to include the case when $v_z \neq 0$, $v_z = v_z(x, y)$—the problem remains two-dimensional. Equations of the type (3) and (4) for the axially-symmetrical case were written out by Braginskiĭ.[5] In addition to these negative results, positive results concerning large-scale fields were also obtained. So far in this section we did not deal with turbulence at all, and the only requirement that v must satisfy is that it decrease at infinity.

We call attention to the fact that (1) and (2) are analogous to the equation for a scalar impurity (smoke density, temperature) in an incompressible liquid. Now the results obtained above are clear. The equation for $H_z$ is particularly simple: $H_z$ does not increase and $H_z \to 0$ as $t \to \infty$. We therefore put $H_z = 0$. Then the equation for the vector potential is analogous to the equation for the temperature, and the absolute value of H is equal to the gradient: $H^2 = |\nabla A|^2$. In the presence of turbulence, the average temperature becomes smoothed out with the turbulent coefficient of the temperature conductivity. During the first stage, however, as a result of the turbulent motion, elements of the liquid having different temperatures appear together. The mean-squared gradient first increases (and corresponds to the mean-squared H), but the scale decreases; ultimately the gradient also decreases to zero.

Positive results are obtained by considering averaging over the time (or over the ensemble), which we denote by the angle brackets $\langle \ldots \rangle$.

We obtain a meaningful result if we assume that the intensity of the turbulent pulsations depends on the coordinates, i.e., the turbulence is inhomogeneous. The coefficient of turbulent temperature conductivity $\chi$ depends on the coordinates, and the equation for the "large-scale" temperature takes the form

$$\frac{\partial A}{\partial t} = \operatorname{div} \chi \nabla A.$$

Hence

$$\frac{\partial \langle H_x \rangle}{\partial t} = \frac{\partial}{\partial y} \operatorname{div} \chi \nabla \langle A \rangle, \qquad \frac{\partial}{\partial t} \langle H_y \rangle = -\frac{\partial}{\partial x} \operatorname{div} \chi \nabla \langle A \rangle,$$
$$\frac{\partial \langle \mathbf{H} \rangle}{\partial t} = -\operatorname{rot} \operatorname{rot} \chi \langle \mathbf{H} \rangle; \tag{5}$$

here $\langle \mathbf{H} \rangle$ has the components $\{\langle H_x \rangle, \langle H_y \rangle, 0\}$. We assume that the turbulence ends somewhere (convective core), and it is inhomogeneous in the region where the turbulence does take place. In other words, let $\chi = \chi_0$ when $\mathbf{r} \in Q$ and $\chi = \nu_m$ when $\mathbf{r} \bar{\in} Q$. By the same token, we neglect the thickness of the boundary layer. Using the equation div $\langle \mathbf{H} \rangle = 0$, we obtain the following boundary conditions:

$$\langle \mathbf{H}_{n_1} \rangle = \langle \mathbf{H}_{n_2} \rangle,$$
$$\chi_c \langle \mathbf{H}_{t_1} \rangle = \nu_m \langle \mathbf{H}_{t_2} \rangle, \quad \chi_0 \langle \operatorname{rot}_{t_1} \mathbf{H} \rangle = \nu_m \langle \operatorname{rot}_{t_2} \mathbf{H} \rangle; \tag{6}$$

the subscripts $n_1$, $n_2$, $t_1$, and $t_2$ correspond here to the normal and tangential components of the fields on the boundary, and the numbers 1 and 2 pertain to the internal and external sides of the surface containing Q. The boundary conditions (6) correspond to a diamagnet with magnetic permeability

$$\mu \sim \nu_m/\chi_0 \sim 1/\mathrm{Re}_m, \quad \mu \ll 1.$$

Thus, the field is not completely forced out of the turbulent region, as assumed by Landau and Lifshitz.[23] We recall that in [23] it was assumed that the turbulent region behaves like a superconductor. After the field is forced out of the turbulent region, a certain quasistationary state is established and corresponds to external non-turbulent fields which we assume specified.

We shall assume that the dimension of the non-turbulent part of the plasma is at least not smaller than that of the turbulent one; we denote it by L. Then the field attenuation time is $t_0 = L^2/\nu_m$, since both the field and the currents are forced out from the region where the dissipation is large.

What is the relation between the energies of the large-scale field and the pulsation fields? We denote by the symbol $\overline{N}$ the average dissipation of the "temperature" inhomogeneities $d\overline{A^2}/dt$; from (5) we can obtain the energy that is drawn from the inhomogeneities with the largest scales, hence $N = \chi A_0^2/L^2$, where $A_0$ is the large-scale component of the temperature.

On the other hand, using (4), we obtain

$$\overline{N} = \chi A_0^2/L^2 = \nu_m \overline{H^2},$$
$$\overline{H^2} = \mathrm{Re}_m H_0^2;$$

$H_0$ is the large-scale component of the magnetic field. We express the pulsational fields in the turbulent region in terms of the external field: since the average field in Q is smaller by a factor $\mathrm{Re}_m$ than the external field $H_0$, we have $H_{\mathrm{int}}^2 = H_{0\mathrm{ext}}^2 \mathrm{Re}_m^{-1}$. All this pertains to the $H_x$ and $H_y$ components; the reader can easily consider the case $H_z \neq 0$ by himself. We note also in this connection the paper by Weiss,[24] who specified the velocity field in the form of a solitary two-dimensional vortex, i.e., the velocity vanished outside a certain volume. By numerically integrating the induction equation he was able to find $\mathbf{H}(\mathbf{r}, t)$. He also found that a field that is homogeneous at the initial instant of time is forced out of the volume. To be sure, Weiss's velocity field is not a random function, so that one or several two-dimensional vortices are too crude a model of turbulence.

3. Generation of regular field by gyrotropic turbulence. At first glance it seems that random motions cannot generate a regular field in any way. Of course, if the initial magnetic field is random, then random motions do not produce a regular field. The merit of the work of Steenbeck and Krause lies in the fact that they have shown that gyrotropic turbulence, i.e., turbulence in which v and rot v correlate

$$\langle \mathbf{v} \operatorname{rot} \mathbf{v} \rangle \neq 0, \tag{7}$$

is capable of amplifying an initially weak field. In essence, Parker and Elsasser proposed a qualitative dynamo model—in the language of frozen-in fields, convective elements, and rotating magnetic loops; Steenbeck and Krause gave a mathematical approach to the same processes in the language of averagings and correlation tensors. To be sure, the results of their approach pertain to weakly frozen-in fields: $\mathrm{Re}_m < 1$. We note that a more accurate derivation of the generation equation was given by Moffatt.[25] When (7) is satisfied, the correlation tensor of the isotropic velocity field is of the form

$$B_{ij}(\mathbf{x}, \mathbf{x}', t, t') = \langle v_i(\mathbf{x}, t) v_j(\mathbf{x}', t')\rangle = A(r, |t-t'|)\delta_{ij} + B(r, |t-t'|) r_i r_j + C(r, |t-t'|)\varepsilon_{ijl} r_l \quad (8)$$

(see [26]). It was assumed in [27] that the large-scale field is homogeneous and the pulsations are small ($\mathbf{H} = \mathbf{B} + \mathbf{h}_1$, $\langle \mathbf{H}\rangle = \mathbf{B}$, $\mathbf{h}_1 \ll \mathbf{B}$). Then the induction equation

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}[\mathbf{vH}] + \nu_m \Delta \mathbf{H} \quad (9)^*$$

can be simplified as follows:

$$\frac{\partial \mathbf{h}_1}{\partial t} = \operatorname{rot}[\mathbf{vB}] + \nu_m \Delta \mathbf{h}_1. \quad (10)$$

With the aid of Green's function, the heat-conduction equation (10) can be solved with respect to $\mathbf{h}_1$; further, the obtained value of $\mathbf{h}_1$ can be used to obtain the quadratic correction, namely, we can calculate $\langle \mathbf{v} \times \mathbf{h}_1\rangle$, which is expressed in terms of $B_{ij}$. We then obtain $\langle \mathbf{v} \times \mathbf{h}_1\rangle = \alpha\mathbf{B}$; $\alpha$ is expressed in terms of C (r, s) from (8). If we now take into account the weak dependence of B on the coordinates, we obtain the generation equation

$$\frac{\partial \mathbf{B}}{\partial t} = \alpha \operatorname{rot} \mathbf{B} + \nu_m \Delta \mathbf{B}. \quad (11)$$

Using (11), it can be easily shown that the generation of the field does indeed take place. In fact, if B contains at t = 0 only a poloidal part, then in the linear approximation in t we get from (11) $\mathbf{B}^{(1)} = \alpha t \operatorname{curl} \mathbf{B}^{(0)}$ (curl $\mathbf{B}^{(0)}$ is the toroidal field); in the approximation quadratic in t we get $\mathbf{B}^{(2)} = \alpha^2 t^2 \operatorname{curl}\operatorname{curl} \mathbf{B}^{(0)}$ (the field $\mathbf{B}^{(2)}$ correlates with $\mathbf{B}^{(0)}$, i.e., it is poloidal). Thus, enhancement of the field is obtained in second order.

In astrophysics the field is practically always frozen into matter, $\mathrm{Re}_m \gg 1$. The pulsational fields are strongly entangled and $h \gg B$, where h is the intensity of the pulsational field: $\mathbf{H} = \mathbf{B} + \mathbf{h}$, $\langle \mathbf{H}\rangle = \mathbf{B}$. Thus, the perturbations are not small; can we use perturbation theory? It turns out that it is possible to develop a theory analogous to the theory of strong perturbations in quantum electrodynamics.[12] To this end it is convenient to change over to Fourier space. We use the Fourier integral (for an unbounded homogeneous turbulence this will be the Fourier-Stieltjes integral) and the Fourier representation of (9):

$$\left.\begin{aligned} \mathbf{H}(\mathbf{r}, t) &= \int \mathbf{H}(\mathbf{k}, t)\exp\{i\mathbf{kr}\}\,d\mathbf{k},\\ \mathbf{v}(\mathbf{r}, t) &= \int \mathbf{u}(\mathbf{k}, t)\exp\{i\mathbf{kr}\}\,d\mathbf{k}, \end{aligned}\right\} \quad (12)$$

$$\mathbf{H}(\mathbf{k}, t) = \mathbf{H}^*(-\mathbf{k}, t),\ \mathbf{u}(\mathbf{k}, t) = \mathbf{u}^*(-\mathbf{k}, t),\ \langle \mathbf{u}\rangle = 0;$$

$$\mathbf{H}(\mathbf{k}, t) = \mathbf{H}(\mathbf{k}, 0)\exp(-\nu_m k^2 t) + i\int_0^t dt_1 \exp[-\nu_m k^2(t-t_1)]\int d\mathbf{k}_1 [\mathbf{k}[\mathbf{u}(\mathbf{k}-\mathbf{k}_1, t_1)\mathbf{H}(\mathbf{k}_1, t_1)]].$$

The convenience of the representation (12) lies in the fact that we are dealing already with an integral equation for which it is easy to write down an iteration series:

$$\left.\begin{aligned} \mathbf{H}(\mathbf{k}, t) &= \sum_{n=0}^{\infty} \mathbf{H}^{(n)}(\mathbf{k}, t),\quad \mathbf{H}^{(0)} = \mathbf{H}(\mathbf{k}, 0)\exp(-\nu_m k^2 t)\\ \mathbf{H}^{(n+1)} &= i\int_0^t \exp[-\nu_m k^2(t-t_1)]\,dt_1 \int [\mathbf{k}[\mathbf{u}(\mathbf{k}-\mathbf{k}_1, t_1)\mathbf{H}^{(n)}(\mathbf{k}_1, t_1)]]\,d\mathbf{k}_1. \end{aligned}\right\} \quad (13)$$

When $\mathrm{Re}_m \ll 1$, the series (13) can be terminated, say, with the second term—the perturbations are small in comparison with $\mathbf{H}^{(0)}$, and we obtain Steenbeck's result. At $\mathrm{Re}_m \gg 1$, an estimate shows that $\mathbf{H}^{(0)} \approx \mathbf{H}^{(1)} \approx \mathbf{H}^{(2)}$ etc., so that the series cannot be terminated. We are interested in the large-scale component, so that (14) must be averaged term by term.

We shall henceforth assume that the velocity probability distribution is Gaussian, i.e., the odd moments are equal to zero and the even ones are expressed in terms of the second moments (spectral tensor). We can now use a diagram technique. Figure 2 shows an example of a fourth-order diagram corresponding to $\langle \mathbf{H}^{(n)}\rangle$. The circles correspond to $\mathbf{H}(\mathbf{k}, 0)\exp(-\nu_m k^2 t)$. The points correspond to the variables of integration with respect to time $t_n$; if it follows from the integration limits that $t_n \le t_m$, then the point $t_n$ is placed to the left of $t_m$. The straight diagrams are the time axes; zero is on the left, t on the right, and the dashed lines correspond to $B_{ij}(\mathbf{r}, t_p - t_m)$. We carry out a partial summation: we retain only diagrams of the type of Fig. 2a.

In the diagrams of Fig. 3, the dashed lines join only neighboring points. If we sum the diagrams of Fig. 3 (we can use a recurrence formula for this purpose) we obtain

$$\mathbf{B}(\mathbf{k}, t) = \exp[-(v_0 + \nu_m)k^2 t]\{\mathbf{B}(\mathbf{k}, 0)\operatorname{ch}(k\alpha t) + i[\mathbf{kB}(\mathbf{k}, 0)]k^{-1}\operatorname{sh}(k\alpha t)\}. \quad (14)$$

The values of $v_0$ and $\alpha$ will be written out later. Such a partial summation (selective summation) can be substantiated for a "white noise" turbulence model—δ-like correlation with respect to time:

$$B_{ij}(\mathbf{r}, t-t') = P_{ij}(\mathbf{r})\,\delta(t-t'). \quad (15)$$

When (15) is satisfied, the diagrams type 2b and 2c are equal to zero and only the diagrams of Fig. 3 remain. Such a model was proposed by Kazantsev.[28] The δ correlation means that the correlation time τ is neglected; this is correct in the present problem, since τ is much smaller than the period of variation of the large-scale field. The dynamics of the turbulence is specified by the form of the spectral tensor $T_{ij}$:

$$\langle u_i(\mathbf{k}, t) u_j^*(\mathbf{k}', t')\rangle = T_{ij}(\mathbf{k}, |t-t'|)\,\delta(\mathbf{k}-\mathbf{k}') = \delta(\mathbf{k}-\mathbf{k}')\{A(k, |t-t'|)[\delta_{ij} - (k_i k_j/k)] + iC(k)\varepsilon_{ijl}k_l\}.$$

Here

$$v_0 = \frac{1}{3}\int A(k, s)\,dk\,ds,\quad \alpha = \frac{1}{3}\int C(k, s)\,k^2\,dk\,ds. \quad (16)$$

Using (14), we easily obtain an equation for B (r, t):

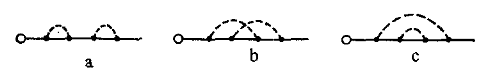


FIG. 2


FIG. 3

*[vH] ≡ v × H

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot} \alpha \mathbf{B} + (v_0 + \nu_m) \Delta \mathbf{B}. \qquad (17)$$

4. Properties of the equation of turbulent degeneration. At C = 0, when there is no gyrotropy, we obtain turbulent diffusion of the field

$$v_0 = \tau \langle v^2 \rangle / 3, \quad v_0 \gg \nu_m.$$

Turbulent diffusion was usually obtained from dimensionality considerations. Equation (17) differs from (11) in the constants. $\nu_m$ in (17) can be neglected, and the constants depend only on the turbulent characteristics. If (17) is solved for eigenfunctions in an unbounded space, then it is easy to show that the eigenfunctions are given by

$$B_1 = \cos(\mathbf{nr}), \quad B_2 = \sin(\mathbf{nr}), \quad B_3 = 0; \qquad (18)$$

here **n** is directed along the third axis. Of course, the helical field (18) can be turned through any angle, and then its form in the previous system of coordinates becomes somewhat more complicated.

Expansion in terms of the functions (18) is equivalent to expansion in a Fourier integral. The fields (18) are force-free and have a helical character. In the case of a body of limited volume, these functions therefore do not satisfy the boundary conditions, which call for continuity at the interface with the vacuum, and are therefore not suitable as eigenfunctions. Substituting the eigenfunctions in (17), we obtain the connection between the field growth increment $\gamma$ and the wave vector **n**:

$$\gamma_i = -\alpha n_i - v_0 n_i^2, \qquad i = 1, 2, 3.$$

We see therefore that the dynamo will take place if

$$-\alpha n_i > v_0 n_i^2.$$

It is easy to obtain the maximum increment $\gamma_{max} = \alpha^2/4v_0$; accordingly, $n_i^{max} = -\alpha/2v_0$. Consequently, fields with a scale $2v_0/\alpha$ will increase more rapidly than all others. The choice of a right-hand or left-hand helical field depends on the value of $\alpha$. If a right-hand field increases exponentially, then a left-hand field attenuates exponentially.

5. Astrophysical examples. Let us first estimate $\alpha$. The exact value of $\alpha$ can be obtained by using the Navier-Stokes equation with allowance for the Coriolis force. For the estimate it suffices to use qualitative considerations. It follows from (16) that $\alpha = -\langle \mathbf{v} \operatorname{curl} \mathbf{v} \rangle \tau/3$. We recall that the quantity $\langle \mathbf{v} \operatorname{curl} \mathbf{v} \rangle$ is determined by the action of the Coriolis acceleration $2\mathbf{v}_1 \times \omega$ on that part of the velocity $\mathbf{v}_1$ which is determined by the density gradient (by the change of the volume of the convective or turbulent element) and with the associated rotation of the volume element. Hence

$$\alpha = \frac{1}{3} \frac{\langle v^2 \rangle}{l} \frac{\omega}{1/\tau} \frac{l}{L_\rho} = \frac{1}{3} \langle v^2 \rangle \frac{\tau^2 \omega}{L_\rho} = \frac{1}{3} \frac{l^2}{L_\rho} \omega;$$

$L_\rho$ is the scale of density variation; we have taken into account the fact that $\tau = l/v$. We now write down the generation conditions. It follows from (14) that generation occurs at small k, $k < k_m$:

$$k_m = \alpha/(v_0 + \nu_m).$$

Since we are dealing with generation in a body of finite dimensions, the growth of **B**(k, t) at $k < k_m$ may not have a physical meaning if the dimensions of the body are not large enough. The dimension L of the body should exceed a certain critical value $L_c = 2\pi/k_m$. Thus, it is necessary to satisfy the condition

$$\alpha > 2\pi v_0/L. \qquad (19)$$

The growth increment of the field of largest dimension is then

$$\gamma = \alpha/L.$$

There is one more requirement

$$\gamma < v/l. \qquad (20)$$

This is the condition for the applicability of the theory, namely, the increment must be smaller than the correlation time.

Let us consider by way of an example a sero-star of the principal sequence,[29 30] $M = 30M_\odot$, $T_e = 40\,000°K$, and $R = 6.6R_\odot$. The parameters of the star are taken from [31]. The rotation speeds of star surfaces reach $v_r = 250$ km/hr, hence $\omega = 5.4 \times 10^{-5}$ sec$^{-1}$. A convection velocity $v = 2 \times 10^5$ cm/sec was obtained in [29]. We assume $L \approx L_\rho$, and then $\alpha = 7.5 \times 10^5$ cm/sec, $vl/L = 2 \times 10^4$ cm/sec, and $vL/l = 10v = 2 \times 10^6$ cm/sec. Thus, conditions (19) and (20) are satisfied and a magnetic field is generated in the convective core within a characteristic time

$$T = (l/l)\,\omega^{-1} = 10/\omega = 2 \cdot 10^5 \text{ sec} = 6 \cdot 10^{-3} \text{ years}.$$

Of course, it can be assumed that the generation is very rapid. We note that since all the harmonics for which the condition (19) is satisfied are excited, the fields with the largest scale will be of the dipole type; in addition, it follows from (14) that a toroidal component is also excited.

The field may emerge to the surface of the star from the convective core as a result of meridional circulation, which is sufficiently effective here. The circulation velocity is[32]

$$v_c = v_r^2 L R^3 / G^2 M^3 \approx 3 \cdot 10^{-2} \text{ cm/sec}$$

The time of emergence of the field is $\tau_g \sim R/v_c = 5 \times 10^5$ years, which is one tenth the lifetime of a star with mass $30M_\odot$ in the principal sequence.

Another interesting example is the generation of a magnetic field in the interstellar gas of the galaxy. We have in mind here the regular component, the scale of which is comparable with the dimensions of the galactic disk. In this case, the self-excitation criterion (19) is not satisfied. Generation may nevertheless occur as a result of the presence of a differential rotation. The latter accelerates the generation.

The first attempt to explain the field of the galaxy was made in a note by Fitremann and Frisch.[33] No account was taken there, however, of the differential rotation and the value of the gyrotropy coefficient $\alpha$ was obtained from dimensionality considerations. This question was investigated recently by Parker[34,35] and by Vainshtein and Ruzmaĭkin.[36,37]

Let us examine, following [36], generation in a disk. We introduce a cylindrical coordinate system $(r, \varphi, z)$ with a z axis parallel to the disk axis; we consider the axially-symmetrical problem, $\partial/\partial\varphi \equiv 0$. Further, differential rotation with $v_\varphi = \omega(r) r$ is present.

We shall assume that $\partial/\partial z \gg \partial/\partial r$; this is connected with the fact that the height of the disk is much smaller than the radius. Then the generation equations take the form

$$\left.\begin{aligned} \frac{\partial B_\varphi}{\partial t} &= r\frac{\partial \omega}{\partial r} B_r + v_0 \frac{\partial^2 B_\varphi}{\partial z^2}, \\ \frac{\partial B_r}{\partial t} &= -\frac{\partial}{\partial z}\alpha B_\varphi + v_0 \frac{\partial^2 B_r}{\partial z^2}, \\ \frac{\partial B_z}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r} r\alpha B_\varphi + v_0 \frac{\partial^2 B_z}{\partial z^2}. \end{aligned}\right| \tag{21}$$

In the first equation of (21) we have already taken into account the fact that differential rotation generates a toroidal field much more effectively than the $\alpha$ effect, which we have neglected. Substituting $B_r$ from the first equation of (21) in the second, we obtain an equation for $B_\varphi$:

$$\frac{\partial^2 B_\varphi}{\partial t^2} - 2v_0 \frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2} B_\varphi + \frac{\partial}{\partial z}\alpha r B_\varphi \frac{\partial \omega}{\partial r} + v_0^2 \frac{\partial^4}{\partial z^4} B_\varphi = 0. \tag{22}$$

An estimate shows that the last term in the left-hand side of (22) is small, and we shall neglect it. We now turn to the eigenvalue problem: $B_\varphi = N(z)\exp(\gamma t)$;

$$\gamma^2 N - 2v_0\gamma N'' + r\frac{\partial \omega}{\partial r}(\alpha N)' = 0. \tag{23}$$

The boundary conditions are N = 0 on the boundary, as follows from the axial symmetry and the condition for matching with the vacuum, $\mathrm{curl}_n\, \mathbf{B} = 0$. Indeed, on the "bases" of the disk we have $r^{-1}(\partial r B_\varphi/\partial r) = 0$, hence $B_\varphi = 0$; on the "lateral surfaces" $\partial B_\varphi/\partial z = 0$, from which $B_\varphi = 0$ when the boundary conditions are taken into account. The existence of eigenfunctions of (23) is not subject to any doubt, since (23) can be reduced to a Schrödinger equation without the time. If dissipation can be neglected, then

$$\gamma = \left(-r\frac{\partial \omega}{\partial r}\frac{\partial \alpha}{\partial z}\right)^{1/2}.$$

Since the Coriolis force and $\alpha$ reverse sign at the half-thickness of the disk, we assume that $\alpha = \alpha_0 z$ (the origin is placed at the center of the disk). We now estimate the increment. As is known from [38], the linear velocity of the galaxy rotation is practically constant (with the exception of the core), $v_\varphi \approx 200$ km/sec, and the turbulent velocity (the random motion of the clouds) is $\sim 5$ km/sec, $l \approx 100$ parsec, the half-thickness of the disk is $z_0 \approx 400$ parsec, and

$$\gamma^{-1} \sim 2\cdot 10^{-7} r \text{ sec}. \tag{24}$$

At r = 10 kiloparsec (the vicinity of the sun) we have $\gamma^{-1} = 2\times 10^8$ yrs. Parker[34] assumed that $\alpha$ changes jumpwise on going through the equator and is constant at $z > 0$ and $z < 0$. He then obtained a fourth-order equation with constant coefficients. The numerical increment obtained by him was of the order of (24).

Interest attaches also to numerous theories explaining the solar cycle. These include the works of Steenbeck and Krause,[39] Vaĭnshtein,[40] and also those in which an ensemble of convective cells acted upon by a Coriolis force plays an important role, even though no turbulence is used and the dynamo is laminar.[41,42,43–44,45–46] We note also that the $\alpha$ effect itself has already been experimentally confirmed.[47–48]

6. Diamagnetism of a turbulent liquid. Boundary conditions. As already indicated in the two-dimensional problem, the inhomogeneous turbulence exhibits diamagnetic properties. We proceed to the general three-dimensional case.[49] Let the mean-squared velocity depend on the coordinates, and then v can be represented in the following form:

$$\mathbf{v}(\mathbf{x}, t) = f(\mathbf{x})\,\mathbf{u}(\mathbf{x}, t), \quad \langle u^2(\mathbf{x}, t)\rangle = 1.$$

We assume that f(x), which determines the inhomogeneity, depends little on the coordinates, or more accurately, varies little over the correlation length $l$. We can then expect the velocity field to be already not only inhomogeneous but also anisotropic: the predominant direction is parallel to $\nabla f$. Thus, we have a small parameter, the ratio of $l$ to the inhomogeneity scale, in terms of which we can expand the correlation tensor in order to simplify it. We shall not present here the intermediate steps, referring the reader for details to [49], where an expression is derived for the spectral tensor with allowance for the conditions imposed on it (see, e.g., [50]). We write out the spectral tensor:

$$\begin{gathered} \langle u_i(\mathbf{k}_1)\, u_j(\mathbf{k}_2)\rangle = \varphi(\mathbf{k}_1 + \mathbf{k}_2)\Big\{A(k_2)\,[k_{2i}k_{1j} - (\mathbf{k}_1\mathbf{k}_2)\,\delta_{ij}] \\ + \frac{dA(k_2)}{dk_2}\frac{1}{2k_2}[k_2^2 + (\mathbf{k}_1\mathbf{k}_2)]\,(k_{2i}k_{2j} - k_2^2\delta_{ij})\Big\}, \\ \varphi(\mathbf{x}) = f^2(\mathbf{x}) = \int \varphi(\mathbf{k})\exp(i\mathbf{k}\mathbf{x})\,d\mathbf{k}. \end{gathered} \tag{25}$$

Naturally, the spectral tensor of homogeneous turbulence should follow from (25) as a limiting case. This is actually obtained, provided we put

$$\varphi(\mathbf{k}_1 + \mathbf{k}_2) = \langle v^2\rangle\,\delta(\mathbf{k}_1 + \mathbf{k}_2).$$

It is now again necessary to average the induction equation (9). Since $\langle \mathrm{curl}\,[\mathbf{v}\times\mathbf{H}]\rangle = \mathrm{curl}\,\langle \mathbf{v}\times\mathbf{H}\rangle$, we calculate $\langle \mathbf{v}\times\mathbf{h}\rangle$. To this end we use again the series (13) and the diagrams of Figs. 2 and 3. This time, however, the integration of the multiple integrals is made difficult by the fact that we have the function $\varphi$ instead of the $\delta$-function. Changing variables, we can integrate the expressions in such a way that $\varphi$ enters in the final expression; this is natural, since $\varphi$ describes the inhomogeneity of the turbulence. We obtain ultimately

$$\begin{gathered} \langle[\mathbf{v}\mathbf{H}]\rangle = -\chi\,\mathrm{rot}\,\mathbf{B} - (1/2)\,[\nabla\chi\mathbf{B}], \\ \frac{\partial \mathbf{B}}{\partial t} = -\mathrm{rot}\,\nu_m\left(1 + \frac{\chi}{\nu_m}\right)^{1/2}\mathrm{rot}\left(1 + \frac{\chi}{\nu_m}\right)^{1/2}\mathbf{B}. \end{gathered} \tag{26}$$

Using (26), it is now easy to average Maxwell's equation and obtain "macroscopic" Maxwell's equations, in a method similar to that used in the electrodynamics of continuous media. We write out these equations (we denote the electric field by **e**, and the average field is $\langle \mathbf{e}\rangle = \mathbf{E}$):

$$\begin{gathered} \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\mathrm{rot}\,\mathbf{E}, \quad \mathrm{div}\,\mathbf{E} = 4\pi\rho, \quad \mathrm{div}\,\mathbf{B} = 0, \\ \mathrm{rot}\,\mathbf{H} = (4\pi/c)\,\mathbf{j}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \mu = [1 + (\chi/\nu_m)]^{-1/2}. \end{gathered} \tag{27}$$

Ohm's law takes the form

$$\begin{gathered} \mathbf{j} = \sigma_{\mathrm{eff}}\,\mathbf{E}, \\ \sigma_{\mathrm{eff}} = \sigma/[1 + (\chi/\nu_m)]^{1/2}. \end{gathered} \tag{28}$$

Thus, (27) and (28) describe a conductor with an inhomogeneous electric conductivity and an inhomogeneous magnetic permeability, and since $\mathrm{Re}_m \gg 1$, we get

$$\sigma_{\mathrm{eff}} \ll \sigma, \quad \nu_{\mathrm{eff}} \gg \nu_m, \quad \mu \ll 1.$$

The results derived in this section are important primarily for the formulation of the boundary conditions. As already stated, the turbulent convection does not involve the entire star, but only a convective zone or else a convective envelope (e.g., the sun). The question of how the presence of a convective core influences the field of the entire star was raised at one time (see [51]).

Spitzer[52] advanced the hypothesis that this influence is not very strong, since the field and the currents are "crowded out" from the convective core and the total field dissipation is small. We shall examine below whether this hypothesis is confirmed. On the sun, this effect can lead to a crowding out of the magnetic fields into the sub-convective zone (where their intensity should be much larger than in the convective zone!), and this may turn out to be a very important factor in the theory of the solar cycle.

We proceed to formulate the boundary-value problem. Assume that $\chi = \chi_0$ at $r \in Q$ and $\chi = 0$ at $r \bar{\in} Q$. We neglect the thickness of the boundary layer. As is well known, the boundary conditions in electrodynamics are obtained from Eqs. (27) by integrating them over the boundary layer and letting the thickness of the layer go to zero. For the electric field we have

$$\mathbf{E}_{t_1} = \mathbf{E}_{t_2}, \quad \mathbf{E}_{n_1} = \mathbf{E}_{n_2}[1+(\chi/\nu_m)]^{1/2}. \tag{29}$$

For the magnetic fields

$$\mathbf{B}_{t_1} = \mathbf{B}_{t_2}[1+(\chi/\nu_m)]^{1/2}, \quad \mathbf{B}_{n_1} = \mathbf{B}_{n_2}. \tag{30}$$

For the currents

$$\mathrm{rot}_{t_1}\mathbf{B} = \mathrm{rot}_{t_2}\mathbf{B}[1+(\chi/\nu_m)]^{-1},$$
$$\mathrm{rot}_{n_1}\mathbf{B} = \mathrm{rot}_{n_2}\mathbf{B}[1+(\chi/\nu_m)]^{-1/2}.$$

It is easy to calculate also the surface currents flowing on the surface of any diamagnet.

It is of interest to repeat all the derivations of the present section for two-dimensional fields. This makes it possible to verify the method of the present section, since in Sec. 2 the results were obtained by an entirely different method. A calculation performed by the method of the present section duplicates fully the conclusions of Sec. 2.

Let us discuss now the characteristic field attenuation times, going over by the same token to the Spitzer hypothesis. Let the dimension of Q be $L_1$ and the dimension of the non-turbulent part of the liquid be L, with $L \geqslant L_1$. Then the time of crowding out the field is determined by its attenuation in Q:

$$t_1 = L_1^2/(\chi+\nu_m).$$

The field attenuation problem is formulated as an eigenfunction problem; to find the smallest eigenvalue corresponding to the attenuation decrement of the entire field we use for **E** an equation that holds in all of space:

$$\frac{1}{\nu_m[1+(\chi/\nu_m)]^{1/2}}\frac{\partial \mathbf{E}}{\partial t} = -\mathrm{rot}\left(1+\frac{\chi}{\nu_m}\right)^{1/2}\cdot \mathrm{rot}\,\mathbf{E}. \tag{31}$$

In vacuum we have $\nu_m = \infty$ and (31) goes over into $\mathrm{curl}^2\,\mathbf{E} = 0$—a current-free field.

Taking the scalar product of (31) with **E** and integrating over all of space, we obtain

$$\frac{1}{2}\frac{d}{dt}\int\frac{E^2}{\nu_m[1+(\chi/\nu_m)]^{1/2}}\,dr = -\int\left(1+\frac{\chi}{\nu_m}\right)^{1/2}(\mathrm{rot}\,\mathbf{E})^2\,dr. \tag{32}$$

Using (32) and the boundary conditions (29) and (30), we obtain without difficulty the attenuation time of the entire field:

$$t_2 = L^2/\nu_m. \tag{33}$$

We recall that $t_2$ from (33) coincides with the time of attenuation of the field in a solid conductor, and we conclude therefore that the turbulent region has little effect on the attenuation of the entire field. This confirms Spitzer's hypothesis.

## III. RANDOM (SMALL-SCALE) MAGNETIC FIELDS

1. Difficulties of the problem. Whereas for regular fields one can find features common to the two-dimensional and three-dimensional problems (turbulent diffusion, diamagnetism), in the case of random fields the fact that a two-dimensional dynamo is impossible still does not prove anything in the three-dimensional case. The difficulty of solving the problem of the random turbulent dynamo is connected with the absence of a small parameter. More accurately, there is a small parameter, say $1/\mathrm{Re}_m$, but this does not yield anything, since the zeroth-approximation solution is not known. This problem is closely related to the problem of the possibility of propagation of magnetic and kinetic energy. In fact, if the turbulent dynamo does exist, i.e., if Eq. (9) results in growth of the mean-squared field, then the steady-state energy $\langle H^2\rangle/8\pi$ cannot be much larger than $\rho v^2/2$, so that (9) holds true as before. Only when the energies become comparable, v itself in (9) already depends on H (via the equation of motion), and (9) becomes nonlinear. On the other hand, in the steady-state we cannot have $\langle H^2\rangle/8\pi \gg \rho v^2/2$, since the force lines start to "disentangle themselves" and the electromagnetic forces cause the liquid to move. This can be demonstrated by using an equation of motion in which one can neglect the terms that are nonlinear in the velocity, as a result of the predominance of the magnetic forces:

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{1}{4\pi\rho}[\mathrm{rot}\,\mathbf{H}\mathbf{H}]. \tag{34}$$

We integrate (34) with respect to t, take the scalar product with v(t), average the fourth-order moment obtained on the right-hand side, and express in terms of the second moments after Gauss. We then obtain

$$\langle v^2\rangle = \frac{2}{3}\frac{1}{(4\pi\rho)^2}\int_0^t\int_0^t \langle \mathrm{rot}\,\mathbf{H}(t_1)\,\mathrm{rot}\,\mathbf{H}(t_2)\rangle\langle \mathbf{H}(t_1)\,\mathbf{H}(t_2)\rangle\,dt_1\,dt_2. \tag{35}$$

Expanding the right-hand side of (35) in powers of t, we obtain

$$\langle v^2\rangle = \frac{2}{3}\frac{1}{(4\pi\rho)^2}\langle(\mathrm{rot}\,\mathbf{H})^2\rangle\langle \mathbf{H}^2\rangle t^2.$$

Thus, the problem of the "dynamo" of the velocity field (i.e., the problem which in a certain sense is the inverse of the usual dynamo problem) can be solved in trivial fashion.

We thus have the following alternatives: either equipartition (but not necessarily over all scales) or ultimate attenuation of the fluctuations of the magnetic fields without external sources. We note the following interesting analogy. If the dynamo of random fields is actually realized, then this process recalls instability (this has already been noted by Moffatt[18] and others). In fact, if H = 0 when t = 0, then it follows from (9) that the field will no longer arise: $\mathbf{H} \equiv 0$. On the other hand, the presence of arbitrarily weak fields immediately brings the dynamo mechanism into action, and the fields will increase exponentially, i.e., a turbulent medium is unstable against introduction of weak perturbations in the form of magnetic fields. However, the analogy with the instability is in this case purely physical, and the mathematical theory of stability is not suitable for the solution of the problem.

This instability has a stochastic character, i.e., only $\langle H^2 \rangle$ becomes enhanced, whereas an individual perturbation of $H^2$ can also decrease.

We have explained that we are dealing with alternatives. Is it possible to approach in this case the problem of the turbulent dynamo from the point of view of the possibility or impossibility of equipartition? A remarkable feature of the equations of magnetohydrodynamics is that they can be written in symmetrical form with respect to v and H (see, e.g., [53]). One might be able to conclude from this that since v and H are in a certain sense on a par, equipartition does indeed take place. These attempts, however, were long ago criticized by Cowling: in fact, only the fictitious fields $v + (4\pi\rho)^{-1/2}H$ and $v - (4\pi\rho)^{-1/2}H$ are on a par, and furthermore only if $\nu = \nu_m$ and the boundary conditions are identical. The last condition is certainly not satisfied. An interesting approach was employed by Lee.[54] He noted that the equations of incompressible magnetohydrodynamics in Fourier space are similar to the Boltzmann kinetic equation, if the Fourier amplitudes are treated as coordinates in phase space. In this case $\nu = \nu_m = 0$.

Using Gibbs statistics, Lee has shown that equipartition is possible and that in this case the power spectrum of E is proportional to . . . $k^4$(!). E increases with increasing k; this is so highly improbable that a doubt immediately arises as to whether this result has any bearing at all on the theory of turbulence. A negative answer to this question is given by Kraichnan and Nagrarjan.[15] The situation in which $\nu_m = \nu = 0$ seems harmless, since $Re_m$ and $Re_e$ are much larger than unity. In fact, however, in the dynamic situation (i.e., in the usual situation), the energy is transferred into the region of large wave numbers, where $\nu_m$ and $\nu$ are significant; they are small but not equal to zero!

At the present time there is apparently no regular (non-heuristic) mathematical formalism for an exact solution of the problem. Nonetheless, certain problems that bring us closer to the solution have been considered. We note first of all the work by Kazantsev[28] and by one of us.[56] Kazantsev proposes a turbulence model that has already been used many times above; in such a model, the problem can be solved exactly. An equation was obtained for the spectral function F(k, t) of the magnetic field, and it was found that an unstable solution exists. Acoustic turbulence, which is an ensemble of interacting acoustic oscillations, is considered in [56]. It is important to bear in mind that the two aforementioned competing processes are present in such a turbulence, and the rates of the processes are of the same order of magnitude. On the other hand, in this case there is a small parameter in which it is possible to expand the induction equation and to obtain an equation for F(k, t), i.e., a regular approach is available.

2. Exact solution of the problem in a certain turbulence model. Following [28], we derive an equation for F(k, t), defined by the relation

$$\langle H_i(\mathbf{k}, t) H_j^*(\mathbf{k}', t)\rangle = F(k, t)\,[\delta_{ij} - (k_i k_j/k^2)]\,\delta(\mathbf{k}-\mathbf{k}').$$

It is necessary here to multiply the series (14) by its complex conjugate and average. The corresponding diagrams are given in [28]. It is important that when a δ-like correlation is used, only diagrams of the "ladder" type remain. As a result of their summation we obtain the equation

$$\frac{\partial F}{\partial t} + 2\chi k^2 F = \int d\mathbf{q} F(p)\, v(q) \left(k^2 - \frac{(\mathbf{kq})(\mathbf{kp})(\mathbf{pq})}{p^2 q^2}\right),$$
$$\mathbf{p} = \mathbf{k} - \mathbf{q},$$

and v(q) is defined in the following manner:

$$\langle u_i(\mathbf{k}, t)\, u_j^*(\mathbf{k}', t')\rangle = v(k)\,\delta(t-t')\,\delta(\mathbf{k}-\mathbf{k}')\,[\delta_{ij} - (k_i k_j/k^2)].$$

A similar equation was obtained by Kraichnan and Nagarajan,[15] who used a Lagrangian description of the turbulence. The problem then reduces to an eigenvalue problem and a search for increasing solutions is carried out. A positive growth increment, i.e., an unstable solution, was obtained in [28] for a non-analytic correlation function, i.e., one in which the first derivative at the point r = 0 is not equal to zero. On the other hand, in [15] they obtained numerically a growing solution for the "Kolmogorov" turbulence, namely, a certain initial function F(k, t) was specified and the Cauchy problem was solved, thus determining the behavior of F(k, t) in time. It turns out that F(k, t) increases with time. Thus, the turbulent dynamo does exist in this model.

3. Turbulent dynamo in the presence of acoustic turbulence. The hydrodynamic theory of acoustic turbulence was developed in a paper by Zakharov and Sagdeev.[55] Acoustic turbulence can occur in a given region of space if there is a flux of sound sustained by external sources through its boundary, and in addition, if there is a linear interaction between the waves and leads to randomization of the oscillations. Such a situation can occur in the solar corona. The turbulence has then mainly a potential character.

Using the results of [55], we write down the power spectrum E(k) and the time of interaction of the oscillations ("phonon lifetime") τ(k):

$$E(k) = A\langle v^2\rangle \lambda^{-1/2} k^{-3/2}, \quad A \sim 1, \qquad (36)$$
$$\tau(k) = c/E(k)\,k^2,$$

where λ is the characteristic wavelength. Formulas (36) hold true when $k > 1/\lambda$; when $k < 1/\lambda$ we have $E \to 0$.

We proceed now to the question of particle diffusion in the field of acoustic turbulence. To this end, we add a scalar admixture to the liquid; the rate of its mixing be a reflection of the particle drift velocity. Thus, we deal with the equation

$$\frac{\partial n}{\partial t} + \operatorname{div} \mathbf{v}n = \mu\Delta n.$$

Let us determine the regular component $\langle n \rangle = N$. An equation for N is obtained in exactly the same manner as the equations for B, but now there is no need for summing an infinite series, since we are dealing with oscillations and $v/\lambda\omega \ll 1$. We therefore confine ourselves to the quadratic correction. We assume that the initial perturbation of n is not correlated with v, so that it is meaningful to consider the resultant expression at t larger than the "memory" time of the system, i.e., the "phonon lifetime" τ(t) from (36).

The quadratic correction gives rise to the expression $\int_0^\infty ds \int f(q, s)\,dq$, where f(q, s) is the spectral function:

$$\langle v_i(\mathbf{x},\, t)\, v_j(\mathbf{x}+\mathbf{r},\, t+s)\rangle = \int (k_i k_j/k^2)\, f(k,\, s) \exp(i\mathbf{kr})\, dk.$$

The latter, in turn, is naturally expressed in terms of the spectral function J(k, ω): in (k, ω) space we have

$$f(k,\, s) = \int J(k,\, \omega) \exp(-i\omega s)\, d\omega.$$

It is now easy to derive an equation for N:

$$\frac{\partial N}{\partial t} = \chi_a \Delta N,$$

$$\chi_a = \frac{1}{3} \int_{-\infty}^{+\infty} ds \int f(q,\, s)\, dq\, ds = \frac{\pi}{3} \int J(k,\, 0)\, dk$$

We see thus that the quantity J(k, ω) plays a major role in diffusion at ω = 0. We shall show later on, with magnetic fields as an example, that it is also used in the calculation of the velocity at which the particles move apart.

The approximate form of J(k, ω) is shown in Fig. 4 (the greater part of the energy, naturally, "stays" in oscillations with frequency ck). The physical meaning of J(k, 0) is clear—it is the energy of the potential component of the acoustic flow. It is clear that a non-zero frequency corresponds only to oscillations, i.e., the particle remains in place on the average; the zero frequency results in irreversible motion.

In view of the great importance of the value of J(k, 0) for the subsequent results, we shall show how it is determined.[56] Linearization of the hydrodynamic equations (first approximation) leads, as is well known, to a wave equation describing sound waves. Consequently, in the first approximation, $J_1(k, \omega) \sim \delta(\omega - ck)$ and, of course, $J_1(k, 0) = 0$. The second approximation gives already a non-zero contribution. In fact, we use the equation for the second-approximation correction to the continuity equation

$$\frac{\partial \rho_2}{\partial t} + \rho_0 \operatorname{div} \mathbf{v}_2 = -\operatorname{div} \rho_1 \mathbf{v}_1. \qquad (37)$$

Multiplying the Fourier transform of (37) by its complex conjugate and averaging the fourth-order moment that appears on the right-hand side, we can express it in terms of the second-order moments, using the random-phase approximation. Using (36), we obtain for the inertial subregion

$$J(k,\, 0) = (A^2 v^4/3\pi c^3 \lambda)\, k^{-5}, \qquad k > 1/\lambda,$$
$$\chi_a = 2\pi A^2 v \lambda M^3/3, \qquad M = v/c.$$

To obtain an equation for F(k, t) we use the same method as for the scalar admixture. Here again we confine ourselves to the correction that is quadratic in the small parameter v/(λ, ω). The resultant expression will again be considered at $t \gg \tau$, but it is important that the perturbations be small compared with F(k, 0). An estimate of the quadratic correction shows that the latter is small at $M^{-3}\lambda/v \gg t \gg \tau \approx M^{-1}\lambda/v$. The problem turns out to be self-consistent precisely because

FIG. 4

the characteristic period of the variation of the field $M^{-3}\lambda/v$ (the reciprocal increment, as will be shown below) turns out to be much larger than the correlation time $M^{-1}\lambda/v$. The equation for F is (see [56])

$$\frac{\partial F}{\partial t} + 2k^2 (\chi_a + \nu_m) E = \pi \int J(p,\, 0)\, B(q,\, t)\, \frac{(\mathbf{kp})^2 q^2 + k^2 (\mathbf{qp})^2}{p^2 q^2}\, dq. \qquad (38)$$

Following Kazantsev,[28] we can transform (38) into a Schrödinger equation; then the bound states correspond to unstable solutions. The necessary condition for the existence of bound states, namely the presence of a potential well, is satisfied when

$$\chi_a \gg \nu_m$$

or, equivalently,

$$S = M^3 \mathrm{Re}_m \gg 1. \qquad (39)$$

The stronger the inequality (39), the "deeper" the well, but this still does not mean that bound states exist when $S \gg 1$. The point is that the "mass" in the Schrödinger operator also depends on the parameters that enter in (39) (an approximate calculation of the levels in the well by the WKB method shows that there is ~1 level in the well). To determine the sufficient condition, we turn to the problem of the eigenvalues of Eq. (38) and make the substitution

$$F(k,\, t) = \Phi(k) \exp(-2Et).$$

It is known that to find the minimum value of E we can use a variational principle (see, e.g., [57]):

$$E = \frac{(\chi_a + \nu_m) \int k^2 \Phi^2\, dk - (\pi/2) \int dk\, dp\, \Phi(k)\, \Phi(q)\, J(p,\, 0)\, [(\mathbf{kp})^2 q^2 + k^2 (\mathbf{pq})^2]\, (qp)^{-2}}{\int \Phi^2\, dk},$$
$$\delta E = 0. \qquad (40)$$

It is clear that if we find a function Φ(k) such that the functional (40) becomes negative, this means that eigenfunctions with E < 0 exist, i.e., the "well" is sufficiently large. Such a function has indeed been obtained.[56] If F(k, t) is expanded in terms of the eigenfunctions, then it is clear that the first eigenfunction with E < 0 (as we have already explained) "suppresses" all the remaining eigenfunctions, and its growth increment is the largest.

Thus, acoustic turbulence is indeed unstable against fluctuations of the magnetic fields. The approximate value of the field growth increment is (see, e.g., [58])

$$\gamma = M^3 v/\lambda.$$

4. Excitation of vortices in the presence of acoustic turbulence. The problem considered above is closely related with the problem of generation of a vortical component. If a weak perturbation in the form of vortices is present in the liquid, then the velocity is represented in the form $v = v_p + v_v$, where $v_p$ and $v_v$ are the potential and vortical components, respectively, with $v_v \ll v_p$. Then the dynamic equation for curl $v_v$, linearized with respect to $v_v$, is

$$\frac{\partial \operatorname{rot} \mathbf{v}_v}{\partial t} = \operatorname{rot} [\mathbf{v}_p \operatorname{rot} \mathbf{v}_v] + \nu \Delta \operatorname{rot} \mathbf{v}_v$$

and recalls the equation for H. This analogy differs significantly from the analogy between the equation for H and for curl v in the usual turbulence, since, first, the boundary conditions are in the present case the same for curl v and H, neither field being sustained, and second, it can be assumed here that $v_p$ and curl $v_v$ are independent at t = 0. We can therefore solve for curl $v_v$ a problem of the same type as for H. In this case the quantity of greatest importance for the generation is $S_\omega = M^3 Re_m$; if $S_\omega \gg 1$, then the generation is completely analogous to the instability described above; on the other hand, if $S_\omega \ll 1$, then the vortical component $v_\omega$ increases (this is a nonlinear effect, a phenomenon known as "acoustic flow"[59]).

We confine ourselves to the case when $S_\omega \gg 1$. From qualitative considerations we can determine the steady-state energy level $v_\omega$. When the energy is increased, the "vortex-vortex" interaction comes into play, i.e., the process playing the principal role in ordinary turbulence. The equation for $v_\omega$ (of course, in a roughly approximate form), is

$$\frac{d}{dt} v_\omega^2 = \gamma v_\omega^2 - (v_\omega^3/\lambda). \qquad (41)$$

The first term in the right-hand side of (41) describes the generation of $v_\omega^2$ by the acoustics, and the second the transfer of energy into the region of larger k and attenuation. In the steady state we have

$$v_\omega \approx \lambda\gamma \approx M^3 v_u.$$

We note in conclusion that in the presence of a homogeneous magnetic field and when $S_\omega \ll 1$ (i.e., when there is no turbulent dynamo), we can obtain the fluctuation spectrum of the magnetic field in a wide range of wave numbers (in analogy with the Golitsyn spectra[60] in ordinary turbulence). The spectra were obtained in [61] and will not be discussed here.

5. Does a turbulent dynamo of random fields exist after all? This question is discussed most fully in the paper of Kraichnan and Nagarajan[15], where it is stated that no final conclusion can be drawn as yet. Let us advance some additional considerations in connection with the publication of new papers. A δ-like correlation with respect to time is proposed in [28]; the question of the feasibility of applying such a model to a "Kolmogorov" turbulence remains open, since the correlation time $\tau$ is of the same order as the growth period of the waves. On the other hand, in acoustic turbulence the diffusion term $2k^2\chi_a F$ in (38) "competes" with the right-hand side of (38), which describes the "entanglement" of the force lines; both terms are of the same order of magnitude, as in ordinary turbulence. Consequently, it is reasonable to suggest that the mechanism causing the growth of the fields prevails also in ordinary turbulence.

We can add also that the perturbation-theory series employed here many times diverges. One can therefore proceed as in Wyld's paper[62] on hydrodynamic turbulence, namely, replace the divergent series by a partial sum. In this problem we can sum only ladder diagrams in the hope that they characterize the expanded function. The result is again the Kazantsev equation, i.e., the dynamo exists.

The same equation can be easily obtained upon satisfaction of the condition $\tau \ll \lambda/v$, which is not satisfied in the "Kolmogorov" turbulence. It does occur, however, if the external force has a pulsed character. Such a turbulence is quite artificial from the point of view of hydrodynamics, and with respect to magnetic fields it behaves qualitatively in the same manner as the ordinary turbulence.

Finally, we note a paper by Thomas,[63] where this problem is simulated numerically with the one-dimensional equation as an example (this equation no longer reduces to the heat-conduction equation, and the dynamo is possible). Thomas also obtains a random-field dynamo. Taking all the foregoing inot account, we can assume the possibility of a turbulent dynamo of random fields as a working hypothesis.

## IV. NONLINEAR PROBLEM. STEADY-STATE FIELD

1. Formulation of problem. As already mentioned, if a turbulent dynamo of random fields is possible, then the approximate equality $\langle\rho v^2\rangle/2 \approx \langle H^2\rangle/8\pi$ holds; we shall henceforth assume this to be the case. As to the spectral distribution of E and F, nothing definite can be said with respect to them.

More meaningful results can be obtained by considering the turbulent dynamo of the regular field. So far, the velocity field was assumed given. When can such an assumption be made? Obviously, if the electromagnetic force in the equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}[\text{rot}\,\mathbf{H}\mathbf{H}] + \nu_m \Delta \mathbf{v} \qquad (42)$$

is sufficiently small. Comparing it with other terms of (42), we find that we should have $\rho v^2/2 \ll H^2/8\pi$ Assume now that we have a regular component: $\mathbf{H} = \mathbf{B} + h$; we assume also that the random fields are comparable in energy with $\rho\langle v^2\rangle/2$: $h^2/8\pi \approx \rho\langle v^2\rangle/2$ (owing to the turbulent dynamo of the random fields). What can be said concerning the turbulent dynamo of the regular field? It turns out that if $B^2/8\pi \ll \rho\langle v^2\rangle/2$, i.e., $\beta = 4\pi\rho v^2/B^2 \gg 1$, then the equation for $\mathbf{B}$ again takes the form (17). In fact, the δ-like correlation can be employed as before, since $\mathbf{B}$ varies slowly in comparison with the temporal correlation. In addition, the initial field can again be regarded as correlated with $\mathbf{v}$, if it is assumed that at $t = 0$ we have $\mathbf{H}(\mathbf{k}, 0)$ and $\mathbf{B}(\mathbf{k}, 0)$ and $\mathbf{h}(\mathbf{k}, 0) = 0$, and the initial field is only regular. In this case the series is summed as before. As the field becomes exponentially stronger, the turbulence parameters (the quantities $\alpha$ and $v_0$) vary slowly. If we take this circumstance also into accou nt, then, in spite of the fact that we obtain in place of (14) a somewhat more complicated expression, differentiation with respect to t again yields equation (17), in which $\alpha$ and $v_0$ already depend on t.

The problem thus reduces to a determination of $\alpha$ and $v_0$. At first glance it seems that the situation becomes much more complicated when the field becomes stronger. In fact, the turbulence will no longer be isotropic, and Alfven waves with entirely new interaction laws appear; the turbulent viscosity $v_0$ is significantly altered. Only one thing can be stated at once: the upper limit of the intensity $\mathbf{B}$ is determined from the equipartition condition

$$\beta_{st} = 4\pi\rho v^2/B^2 \approx 1, \qquad (43)$$

whereas $\beta \gg 1$ at $t = 0$.

It can be assumed, however, that a situation is possible in which the equipartition (43) is not reached and a stationary state in which $\beta_{st} \gg 1$ (weakly-linear regime) is established. In this case the turbulence itself changes little. In fact, in the linear problem, only the gyrotropic part of the tensor (8) with $\epsilon_{ijf}\gamma_f$ is responsible for the generation of the field; we can therefore expect the magnetic field to "suppress" precisely the gyrotropy.

We shall show below that such a weakly-linear steady state is indeed possible. We have dealt above with the gyrotropy due to the action of the Coriolis force on the turbulent element. Let us discuss the action of the electromagnetic force on this element. It can be assumed that the given turbulent element corresponds to a certain current loop acted upon, of course, by a torque, since it is located in an external magnetic field. Even if the body does not rotate, i.e., there is no Coriolis force, this torque can yield in principle $\langle \mathbf{v} \text{ curl } \mathbf{v} \rangle \neq 0$, i.e., its own $\alpha_M$. This gyrotropy will be called magnetic, to distinguish it from the Steenbeck gyrotropy which we call rotational.

2. Determination of the magnetic gyrotropy. To deduce the equation for $a_m = \langle \mathbf{v} \text{ curl } \mathbf{v} \rangle$ we must take the scalar products of (42) by curl v and of the curl of (42) by v, and add the resultant expressions. We then average over an "elementary" volume whose dimension is much larger than the correlation length, but smaller than the scale of **B**. On the left side we retain $\partial a_m/\partial t$. It is remarkable that in this case all the triple correlations with respect to velocity vanish:

$$\text{rot}\,\mathbf{v}\,[\mathbf{v}\,\text{rot}\,\mathbf{v}] = 0,$$

$$\text{rot}\,\mathbf{v}\nabla\,[(v^2/2) + (p/\rho)] = \text{div}\,\{\mathbf{v}\nabla\,[v^2/2) + (p/\rho)]\},$$

$$\mathbf{v}\,\text{rot}\,[\mathbf{v}\,\text{rot}\,\mathbf{v}] = \text{div}\,[\mathbf{v}\,[\mathbf{v}\,\text{rot}\,\mathbf{v}]].$$

The divergence vanishes after averaging over the elementary volume. We now calculate the term with the magnetic field:

$$M = (1/4\pi\rho)\,\langle \text{rot}\,\mathbf{v}\,[\text{rot}\,\mathbf{HH}]\rangle + (1/4\pi\rho)\,\langle \mathbf{v}\,\text{rot}\,[\text{rot}\,\mathbf{HH}]\rangle. \quad (44)$$

**H**(**r**, t) can be expressed in terms of **H**(**r**, 0) and in terms of **v**(**r**, t) by using the series (13). Expression (44) indeed describes the action of the electromagnetic force on the turbulent elements. Before writing down the results of the calculation of (44), we present a heuristic derivation of the result. On what should $\alpha_M$ depend? First, $\alpha_M$ is a pseudoscalar; in addition, (44) depends quadratically on **H**. The entire gyrotropy should be expressed in terms of the regular field **B**, since it is the only possible source of the magnetic gyrotropy. The only possible pseudoscalar that is quadratic in **B** is of the form **B** curl **B**. Thus, $\alpha_M \sim \mathbf{B}$ curl **B**. Let us explain the coefficients. It is necessary also to take into account somehow the fact that the magnetic field causes anisotropy of the distribution of the velocity probabilities: there appears a preferred direction parallel to **B**. It is natural to assume that the measure of the anisotropy is determined by the ratio $B^2/4\pi\rho\langle v^2\rangle$; thus,

$$M \sim (B^2/4\pi\rho\,\langle v^2\rangle)\,\mathbf{B}\,\text{rot}\,\mathbf{B}. \quad (45)$$

A numerical calculation confirms the expression (45). To calculate (44) we use the series of diagrams of Fig. 2. For small fluctuations, we confine ourselves to the first term of the series; we obtain $t \gg \tau \times \exp(-\nu_m k^2(t - t_1)) \approx 1$:

$$\begin{aligned} M &= M_a B^2 \mathbf{B}\,\text{rot}\,\mathbf{B} + (B^2/4\pi\rho)\,(A_4/3), \\ M_a &= (1/4\pi\rho)\,(1/15) \int [2C\,(k,\,s) + D\,(k,\,s)]\,dk\,ds, \\ A_4 &= \int A_1\,(k,\,s)\,k^4\,dk\,ds; \end{aligned} \quad (46)$$

we have used here the following form of the spectral tensor $T_{ij}$ (obtained in [64] assuming weak anisotropy connected with the magnetic field, as is natural when $\beta \gg 1$):

$$T_{ij} = [A\,(k,\,s) + C\,(k,\,s)\,(\mathbf{kB})^2]\,[\delta_{ij} - (k_i k_j/k^2)] + D\,(k,\,s)\,[(k_i B_j + k_j B_i)\,\mathbf{kB} - \mathbf{kB}\delta_{ij} - k^2 B_i B_j] + iA_1\,(k,\,s)\,\varepsilon_{ij}.$$

Actually, at $\text{Re}_m \gg 1$ we have $h \gg B$, and it does not suffice to take only the first term of the series into account. We shall therefore again use selective summation.[64] A calculation shows that the result of such a summation makes a small contribution to (46); this is precisely the basis for the applicability of the δ-like correlation in this model of turbulence.

3. Certain properties of magnetic gyrotropy. An important role in generation is played by the quantity $\alpha$:

$$\alpha = -\frac{1}{3}\int_{-\infty}^{+\infty} \langle \mathbf{v}\,(t)\,\text{rot}\,\mathbf{v}\,(t')\rangle\,dt' = \frac{2}{3}\int A_1\,(k,\,s)\,k^2\,dk\,ds.$$

We obtain an expression for $\alpha$ by using (46) and certain assumptions concerning the character of the turbulence:[64]

$$\begin{aligned} \alpha_M &= -(\mathbf{B}\,\text{rot}\,\mathbf{B})\,\Phi, \\ \Phi &= 2\tau n \ln (k_\nu l)/5\beta 4\pi\rho, \quad n \approx 1, \end{aligned} \quad (47)$$

where $k_\nu$ is the spectrum cutoff threshold as a result of the finite viscosity, and $l$ is the external turbulence scale.

We now proceed to determine certain properties of $\alpha_M$, which are connected with generation of the magnetic field. Assume that $\alpha = 0$, i.e., there is no rotation and there is no rotational gyrotropy. At t = 0, let $\mathbf{B} \cdot \text{curl}\,\mathbf{B} \neq 0$. From (17) it follows, neglecting diffusion, that

$$\frac{1}{2}\frac{\partial}{\partial t}B^2 = -(\mathbf{B}\,\text{rot}\,\mathbf{B})^2\,\Phi.$$

We note that $\Phi > 0$, hence

$$B^2\,(t) - B^2\,(0) = -2\int_0^t dt_1\,(\mathbf{B}\,\text{rot}\,\mathbf{B})^2\,\Phi.$$

As $t \to \infty$, the integral converges and

$$\mathbf{B}\,\text{rot}\,\mathbf{B} \to 0.$$

It is thus clear that if $\mathbf{B} \cdot \text{curl}\,\mathbf{B} \neq 0$ at t = 0, then the resultant magnetic gyrotropy is such as to cause $\mathbf{B} \cdot \text{curl}\,\mathbf{B} \to 0$, after which the gyrotropy itself vanishes in accordance with (47).

Another property of magnetic gyrotropy is that it acts in opposition to the rotational gyrotropy. In fact, in the presence of $\alpha_\omega$, if $\mathbf{B} \cdot \text{curl}\,\mathbf{B} = 0$ at t = 0, then

$$\frac{d}{dt}\int \mathbf{B}\,\text{rot}\,\mathbf{B}\,dr = \frac{1}{2}\alpha_\omega \int (\text{rot}\,\mathbf{B})^2\,dr;$$

the integration is carried out over the entire volume. We see that $\mathbf{B} \cdot \text{curl}\,\mathbf{B}$ acquires the same sign as $\alpha_\omega$; therefore, if $\alpha_\omega > 0$, then it follows from (47) that $\alpha_M < 0$, but $\alpha = \alpha_M + \alpha_\omega$, so that a stationary state sets in at $\alpha = 0$. If $\alpha_\omega < 0$, then $\alpha_M > 0$. In both cases $|\alpha| = |\alpha_\omega + \alpha_M| < |\alpha_\omega|$, i.e., the generation coefficient decreases in the presence of magnetic gyrotropy. It is precisely this last circumstance which makes possible nonlinear stabilization of the field **B** at $\beta_{st} \gg 1$, i.e., far from equipartition.

Assume that differential rotation and turbulence have been excited in a conducting liquid sphere (the model of a star), i.e., $\omega$ depends on **r**: $\omega = \omega(r)$. Such an example has been considered in Sec. 5 of Chap. V. The field growth increment is

$$\gamma = \left(-\alpha\frac{\partial\omega}{\partial r}\right)^{1/2} - \frac{\nu_0}{L^2};$$

L, the dimension of the system, should be positive. We assume that $\partial\omega/\partial r \approx -\omega/L$; in the stationary state we have $\gamma = 0$, from which we obtain $\alpha^{st} = \alpha^{st}_{M} + \alpha^{st}_{\omega}$ and $\beta_{st}$:

$$\beta^2_{st} = (2/5)\ln(k_\nu l)\,[N - (l^2/L^2)\,N^{-1}]^{-1}, \quad N = l\omega/v. \tag{48}$$

It is easy to see from (48) that if $1/N \gg l/L$, then

$$\beta^2_{st} = 2\ln(k_\nu l)/5N.$$

Usually $N < 1$, since the frequency of rotation of the star is lower than the frequency rotation of the turbulent element.

Thus, under certain perfectly realistic conditions, the stabilized field satisfies the condition $\beta_{st} \gg 1$, i.e., a weakly-linear state sets in.

We note in conclusion that although, as seen from the review, the theory of the turbulent dynamo is only in the initial development state, nontheless it helps introduce some clarity in the question of the origin of cosmic magnetic fields of tremendous scale.

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