NATURAL FLUCTUATIONS IN LASERS

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A review is presented of theoretical and experimental results of an investigation of natural fluctuations in gas and solid-state lasers. All the known theoretical results are obtained by a single method based on the equations of the quasiclassical laser theory. In the calculation of the fluctuations, the equations for the amplitudes and phases of the field are regarded as Langevin equations with suitably introduced fluctuation sources. The sources of the thermal fluctuations of the resonator are determined by the Kallen-Welton formula. The sources of the non-equilibrium fluctuations of the polarization of the working medium are calculated on the basis of the equations for the density-matrix elements. The fluctuations of the amplitudes and phases in linear and ring lasers are considered under arbitrary pump-to-threshold ratios. The natural line width of the laser radiation is calculated. The coupling of opposing waves in a ring laser, due to scattering by the mirrors, is taken into account. The maximum sensitivity of a laser gyroscope is estimated. The results of the theory are compared with the experimental data. A brief description of the status of the problem of calculating the natural fluctuations in lasers and an estimate of the possibilities of the quasiclassical method of calculating fluctuations are given in the conclusion.

CONTENTS

1. INTRODUCTION

THE investigation of fluctuations in lasers is necessary in order to estimate the stability of their radiation, and to determine the maximum capabilities of laser devices, for example the maximum sensitivity of an optical gyroscope. It is customary to distinguish between technical and natural fluctuations in lasers. Technical fluctuations are due to instability of the resonator and pump parameters. Natural fluctuations are due to the atomic structure of the working medium and of the resonator and to the quantum character of the radiation. Technical fluctuations can be greatly decreased by improving the apparatus, whereas natural fluctuations are independent of the apparatus. Technical fluctuations are much slower (their spectral width is of the order of 10^3–10^4 Hz) than the natural ones. This makes it possible to separate weak natural fluctuations against the background of the stronger technical fluctuations.

Considerable experimental and theoretical material on natural fluctuations in lasers has been accumulated by now. The lack of a unified approach and the complexity of the theoretical calculations prevent the experimenters and theoreticians from using the available material unless they are specialists in this field. This has necessitated a review of the results obtained by one of the simplest possible methods and a comparison of these results with calculations performed by other more complicated methods. By the same token it is possible to understand to some degree to what extent it is necessary to use more accurate but much more cumbersome calculation methods.

The simplest approach, from our point of view, is to start with equations that serve as the basis for the quasiclassical laser theory. In the calculation of fluctuations, the deviation from Lamb's quasiclassical theory lies in the fact that the initial equations are treated as Langevin equations with suitably introduced fluctuation sources. Before we proceed to a consistent exposition of the basic material, let us indicate a number of experimental and theoretical investigations devoted to natural fluctuations of the radiation of gas and solid-state lasers.

The first attempts to measure the natural line width of an He-Ne laser were made by Javan, Ballik, and Bond[1] and Jaseja, Javan, and Townes[2].

A theoretical estimate of the line width was carried out in accordance with the formula of Shawlow and

Townes[16]

$$\Delta\omega = \hbar\omega_0 (\Delta\omega_p)^2/2P\,^*). \tag{1.1}$$

At P = 1 MW and at resonator losses on the order of 1%, the line width is $\Delta\omega/2\pi \approx 10^{-3}$ Hz. It was established in[2] that the line width at these values does not exceed 20 Hz. According to the measurements of Leikin et al.[3], the line width of the He-Ne laser is of the order of 3.5 Hz. This means that the authors of the cited papers measured not the natural line width but the technical one.

Zaĭtsev and Stepanov[4] measured the natural line width far from the excitation threshold, at different values of the power, with sufficient accuracy. The line width near the excitation threshold was measured by Siegman and Arrathgon[5]. The experimental results of[4,5] give a rather complete idea of the dependence of the natural line width of the laser radiation on the power. In[4,5], the line width was determined by measuring the spectrum of the difference frequency of two lasers, one of which was taken to be the standard. Egorov[7] proposed a different method of measuring the line width, namely the method of intermode beats. The results of[4,5] will be considered in greater detail later in the article. The line width of solid-state lasers has apparently not yet been measured.

The measurement of the natural fluctuations of the radiation intensity is the subject of[8-15]. Freed and Haus[8] obtained the spectral densities of the intensity fluctuations at different values of the average power. For the region above the lasing threshold, a more detailed investigation of the intensity fluctuations was carried out by Andronova and Zaĭtsev[9-11]. Zaĭtsev[9] investigated also the intensity fluctuations of the opposing waves in a ring laser. He measured, in particular, the spectral density of the correlation coefficient of the opposing-wave intensities.

The results of a number of measurements of the amplitude fluctuations in gas lasers are given in the review articles of Smith and Armstrong[12]. Arecchi et al.[6,13] investigated the statistics of the photons near the generation thresholds in the stationary regime, and determined the width of the spectrum of the amplitude fluctuations near the threshold. Analogous investigations were carried out in[14,15] for the transient regime.

Let us indicate the main theoretical calculations of the natural fluctuations in lasers. In the first papers by Shawlow and Townes[16] and by Lamb[17] on the theory of the natural line width of laser radiation, only thermal fluctuations of the resonator were taken into account. The results of[16,17] differed by a factor of 2. Lamb's formula agreed with the results of subsequent calculations.

Simultaneous allowance for both the thermal and the polarization noise in the calculation of the line width was apparently made first by Haken[18], Haken et al.[19], and Lax[20]. The result of the calculation can be represented in the form

$$\Delta\omega = \frac{\hbar\omega_0 (\Delta\omega_p)^2}{2P}\left(\bar{n} + \frac{1}{2} + \frac{1}{2}\,\frac{\rho_a + \rho_b}{\rho_a - \rho_b}\right). \tag{1.2}$$

*A list of symbols is given at the end of the article.

The first two terms, which contain $\bar{n} + (1/2)$, take into account the contribution due to the equilibrium thermal fluctuations in the free resonator. The last term takes into account the contribution of the non-equilibrium fluctuations of the polarization of the working medium with allowance for the zero-point oscillations.

Formula (1.2) can be rewritten in the form

$$\Delta\omega = \frac{\hbar\omega (\Delta\omega_p)^2}{2P}\left(\bar{n} + \frac{\rho_a}{\rho_a - \rho_b}\right).$$

This shows that the summary contribution of the zero-point oscillations to the line width is equal to zero. Indeed, the quantity $\bar{n}$ is the average number of thermal-radiation photons, and $\rho_a/(\rho_a - \rho_b)$ is that part of the polarization fluctuations which is due to the spontaneous emission.

Formula (1.2) in an arbitrary field is valid only for the case of immobile atoms in the regime of one traveling wave. For the case of moving atoms, it is valid only in a weak field, when the field can be neglected in the calculation of the polarization noise. In a strong field, the result of the calculation of the emission line width *depends on the form of the field in the resonator* (traveling wave, standing wave, opposing waves) and on the character of the thermal motion of the atoms; formula (1.2) is therefore no longer sufficient.

All papers on the theory of natural fluctuations in lasers can be broken up into two groups that differ from each other in the approach used to the solution of the problem. The first group includes papers in which the initial equations constitute a system of operator equations for the density matrix elements and the field. Fluctuation sources, whose intensity is calculated in one manner or another, are introduced into these equations in a suitable manner. The result is either a system of Langevin equations or the corresponding Fokker-Planck equation. Since the average number of photons in the resonator turns out to be quite appreciable even at the generation threshold (according to Arecchi[6], it is of the order of 4000), it is possible to use for the field the classical Maxwell equations with random sources.

Such an approach, with one modification or another, was used by Lamb[17], Haken[18], Haken et al.[19], Lax[20], Sauerman[21], Haus[22], Fleck[23], Bernshtein, Andronova, and Zaĭtsev[24], Risken[25], the present authors[26-30], and others.

A different approach is used in the second group of papers, and is based on the approximate solution of the equations of the first and second distribution functions of the atomic and field variables. This group includes the papers of Glauber[31], Korenman[32], Lamb and Scully[33], Fleck[34], Willis[35], Brunner[36], Kazantsev and Surdutovich[37], Weidlich[38], and a number of others.

In many of the papers listed here, the calculations are valid only for the case of weak fields. In most papers, no account was taken of the motion of the active atoms, and the difference between the fluctuations in the traveling- and standing-wave modes is not brought out. Allowance for these phenomena is quite important and leads to a number of new interesting effects.

We note also a cycle of investigations by Risken[25,60], who calculated the field amplitude and phase fluctuations in a region close to the excitation threshold. Risken set up a Fokker-Planck equation, solved it, and obtained the spectral characteristics of the fluctuations near the excitation threshold. Such a problem is mathematically quite cumbersome, since it becomes necessary to find a nonstationary solution of the Fokker-Planck equation. In[25], and also in[39], it was shown theoretically that the spectrum of the amplitude fluctuations has a near-Lorentz shape even near the generation threshold. This result is quite useful, for if we assume a certain spectrum shape beforehand, then we can calculate its parameters by starting from the stationary solution of the Fokker-Planck equation, which is much easier to obtain.

A number of results for a ring laser were obtained by Smirnov and Zhelnov[40], and by Belenov[41], but only for the weak-field case. The fluctuations were calculated in[40] by a quantum approach similar to that used by Kazantsev and Surdutovich[37]. In[41], the fluctuation sources were introduced into the wave equation for the field in the form of specified external forces. The intensity of these sources was determined in the same manner as in[24].

2. SOURCES OF FLUCTUATIONS IN THE EQUATIONS FOR THE AMPLITUDES AND PHASES IN LINEAR AND RING LASERS

The fluctuations can be calculated on the basis of the system of equations for the field and the density matrix elements of the working levels $\rho_a(v)$, $\rho_b(v)$, $\rho_{ab}(v)$, and $\rho_{ba}(v)$:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\rho_a = \frac{ie}{\hbar}\left(r_{ab}\rho_{ba} - \rho_{ab}r_{ba}\right)E - \gamma_a\left(\rho_a - \rho_a^0\right), \tag{2.1}$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\rho_b = -\frac{ie}{\hbar}\left(r_{ab}\rho_{ba} - \rho_{ab}r_{ba}\right)E - \gamma_b\left(\rho_b - \rho_b^0\right), \tag{2.2}$$

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r} + i\omega_{ab} + \gamma_{ab}\right)\rho_{ab} = \frac{ie}{\hbar}r_{ab}\left(\rho_b - \rho_a\right)E, \tag{2.3}$$

$$\rho_{ba} = \rho_{ab}^{*}, \tag{2.4}$$

$$\frac{\partial^2 E}{\partial t^2} + \frac{\omega_0}{Q}\frac{\partial E}{\partial t} - c^2\Delta E = -4\pi\frac{\partial^2 P}{\partial t^2} + \omega_0^2 E^{(T)}. \tag{2.5}$$

The polarization vector P(r, t) is connected with the density matrix elements by the well-known relation

$$P(r,\ t) = en\int\left(r_{ba}\rho_{ab} + r_{ab}\rho_{ba}\right)dv. \tag{2.6}$$

Equations (2.1)—(2.5) contain four dissipative coefficients: γ_a, γ_b, γ_{ab}, and ω_0/Q, which are assumed known within the framework of the description considered here. The quantities γ_a, γ_b, and γ_{ab} depend on the pressure. The experimentally obtained plots of γ_a, γ_b, and γ_{ab} against the pressure are given in the paper by Fork and Pollak[42] (Fig. 1). For this reason, the initial equations (2.1)—(2.5) are semiphenomenological.

Within the framework of the initial equations, the influence of the collisions is taken into account only via the coefficients γ_a, γ_b, and γ_{ab}. A more detailed account of the influence of collisions in lasers is given in the papers of Rautian[55], Lamb et al.[65], and Stenholm[66].

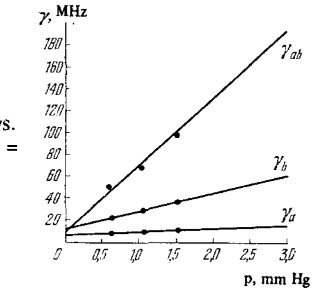

FIG. 1. Plots of γ_a, γ_b, and γ_{ab} vs. helium pressure. μ = 215 MHz, $\sqrt{2}ku$ = 1000 MHz.

We note one more limitation contained in the initial equations. In the equation for ρ_b, no account is taken of the increase of ρ_b as a result of the spontaneous transition from the level a, i.e., no account is taken of the additional term $A_b^a\rho_b$, where A_b^a is the Einstein coefficient. Usually $A_b^a \ll \gamma_a$, γ_b, so that the role of this term can be neglected.

Field fluctuations in lasers are due to two causes. The first is connected with thermal fluctuations of the field in the empty resonator. To take these fluctuations into account, a random source $E^{(T)}$ is introduced in the field equation (2.5).

The spectral density of the random thermal source $E^{(T)}$ is determined on the basis of the Kallen-Welton formula[43,44]:

$$\left(E^{(T)2}\right)_\omega = \frac{8\pi\hbar\,\Delta\omega_r}{\omega_0 V}\left(\bar{n} + \frac{1}{2}\right). \tag{2.7}$$

The second source of the field fluctuations is the polarization noise. The polarization vector P in (2.5) can be represented in the form of a sum of two parts:

$$P = P^{(\text{инд})} + \delta P. \tag{2.8}$$

In this expression $P^{(\text{ind})}(E)$ is the induced part of the polarization or the response of the system to the total field $E = \langle E\rangle + \delta E$, and δP is the polarization fluctuation due to the atomic structure of the working medium.

The concrete expression for the spectral density of the polarization noise depends on the form of the field, i.e., on the operating conditions of the laser. We present below results for three regimes: the traveling-wave regime, which is realized in ring lasers when one of the opposing waves is suppressed, the standing-wave regime in a linear laser, and the regime of two opposing waves in a ring laser.

The amplitude and phase fluctuations of the laser radiation depend significantly on the ratio of the characteristic temporal parameters of the working medium and of the field in the empty resonator. Three cases can be separated:

1) Gas laser, $\gamma_a \sim \gamma_b \sim \gamma_{ab} \gg \Delta\omega_r$.
2) Molecular generator, $\gamma_a \sim \gamma_b \sim \gamma_{ab} \ll \Delta\omega_r$.
3) Solid-state laser, $\gamma_a \sim \gamma_b \ll \Delta\omega_r \ll \gamma_{ab}$.

In a gas laser, the polarization becomes established much more rapidly than the field. As a result, the field can be regarded as constant during the time of establishment of the polarization. Let us consider this case first. Other cases will be considered in Chap. 6.

When the field is specified in the form of two opposing waves

$$E = \frac{1}{2}\left(E_1 e^{-i(\omega_0 t - k_0 r + \varphi_1)} + E_2 e^{-i(\omega_0 t + k_0 r + \varphi_2)} + \text{c.c.}\right) \tag{2.9}$$

we obtain from the field equation (2.5) the following equations for the amplitudes and phases (without allowance for the coupling via the scattering):

$$\frac{dE_{1,2}}{dt} = -\frac{\omega_0}{2}\left(4\pi\varkappa''_{1,2} + \frac{1}{Q_{1,2}}\right) E_{1,2} + \omega_0 \xi_{a1,2}(t), \tag{2.10}$$

$$\frac{d\varphi_{1,2}}{dt} = -\frac{\omega_0}{2}\cdot 4\pi\varkappa'_{1,2} + \frac{\omega_0}{E_{1,2}}\xi_{\Phi 1,2}(t); \tag{2.11}$$

here κ' and κ'' are the real and imaginary parts of the complex polarizability, determined from the relation

$$P_{1,2}^{(\text{ind})} = (\varkappa'_{1,2} + i\varkappa''_{1,2}) E_{1,2}; \tag{2.12}$$

$\xi_{a_{1,2}}$ and $\xi_{\text{ph}_{1,2}}$ are the sources of the fluctuations of the amplitude and phase. They are defined by the expressions

$$\begin{aligned} \xi_{a1,2} &= -\frac{1}{V}\int (4\pi \mathbf{e}\,\delta\mathbf{P} + \mathbf{e}\mathbf{E}^{(T)}) \sin(\omega_0 t \mp k_0 r + \varphi)\, dr, \\ \xi_{\text{ph}\,1,2} &= -\frac{1}{V}\int (4\pi \mathbf{e}\,\delta\mathbf{P} + \mathbf{e}\mathbf{E}^{(T)}) \cos(\omega_0 t \mp k_0 r + \varphi)\, dr. \end{aligned} \tag{2.13}$$

The upper sign corresponds to the first wave and the lower to the second, and e is a unit vector directed along the vector **E**. In (2.13),

$$\delta\mathbf{P} = en\int (r_{ba}\delta\rho_{ab} + r_{ab}\delta\rho_{ba})\, dv. \tag{2.14}$$

No account is taken of the coordinate dependence in eqs. (2.10) and (2.11) for the amplitudes and phases. Such an approximation is justified when the quasistationary condition is satisfied, whereby the characteristic time $Q/\omega_0 \sim 10^{-6}$ sec greatly exceeds $L/C \sim 10^{-8}$ sec—the time of travel of the wave in the resonator.

In accordance with (2.5), (2.8), and (2.13), we represent $\xi_{a_{1,2}}$ and $\xi_{\text{ph}_{1,2}}$ in the form of sums of two parts:

$$\xi_a = \xi_a^{(T)} + \xi_a^{(\Pi)}, \quad \xi_{\text{ph}} = \xi_{\text{ph}}^{(T)} + \xi_{\text{ph}}^{(\Pi)};$$

here $\xi_{a,\text{ph}}^{(T)}$ are the parts due to the thermal noise of the free resonator, and $a\xi_{a,\text{ph}}^{(P)}$ are the polarization parts.

In a gas laser, the conditions $\Delta\omega$, $\Delta\omega_a$, $\Delta\omega_r \ll ku$ are satisfied ($\Delta\omega_a$ is the width of the amplitude-fluctuation spectrum and $\Delta\omega$ is the width of the laser emission spectrum). It suffices therefore to know the spectral densities of the thermal and polarization noises at zero frequency. It is assumed here implicitly that the spectral density of the fluctuation noises decreases monotonically with increasing ω. In the presence of a field, the source fluctuation spectrum, generally speaking, is not monotonic. Using immobile atoms as an example, it can be shown that this circumstance is not essential.

From (2.7) and from the definitions of $\xi_{a,\text{ph}}^{(T)}$ it follows that the spectral density of the thermal parts of the fluctuation sources are given at zero frequency by

$$(\xi_{a1,2}^{(T)2})_0 = (\xi_{\text{ph}1,2}^{(T)2})_0 = \frac{4\pi\hbar\,\Delta\omega_r}{V\omega_0}\left(\bar{n} + \frac{1}{2}\right),$$

$$(\xi_{a1}^{(T)}\xi_{a2}^{(T)})_0 = (\xi_{\text{ph}1}^{(T)}\xi_{\text{ph}2}^{(T)})_0 = (\xi_{a1,2}^{(T)}\xi_{\text{ph}1,2}^{(T)})_0 = 0. \tag{2.15}$$

To calculate the spectral densities of the sources of the polarization noise, we represent the density matrix elements in the initial equations in the form of sums of induced and fluctuation parts.

The equations for the induced parts coincide with the initial equations (2.1)—(2.4). The equations for the fluctuation parts differ from (2.1)—(2.4) in that there are no terms with the functions $\rho_a^{(0)}$ and $\rho_b^{(0)}$.

The system of equations for the induced and fluctuation parts is solved under the assumption that the diagonal density matrix elements do not depend on the coordinates and depend slowly on the time, while the off-diagonal elements are represented by sums of two opposing waves.

In the case of a weak field, such an approximation for a gas laser with inhomogeneous line broadening can be rigorously justified. For a strong field at arbitrary parameters in the regime of two opposing waves, this approximation may turn out to be incorrect.

For the induced part of the polarization, the solution with allowance for all the spatial harmonics was obtained in[45,46]. It is shown in[46] that the contribution of the second and higher spatial harmonics is proportional to the parameter $(\gamma^2/\gamma_{ab}^2)aE^2$.

For a laser with homogeneous line broadening, under the condition $ku \ll \gamma_{ab}$, the role of the spatial modulation turns out to be noticeable. This is due to the fact that in this case the presence of thermal motion does not lead to a noticeable smoothing of the standing wave. In the approximation $ku \ll \gamma_{ab}$ it is possible to obtain an exact expression for the polarization[45-47].

We consider first the case of inhomogeneous line broadening. At not too low pressures, the parameter γ^2/γ_{ab}^2 for an He-Ne laser is small. This is seen from Fig. 1. Thus, for example, for a mixture pressure 2.5 mm Hg we have $\gamma^2/\gamma_{ab}^2 \approx 0.1$. The results considered here correspond to the zeroth approximation in terms of this parameter*. In this approximation we put

$$\rho_{ab} = \tilde{\rho}_{ab}^{(1)}(t)\, e^{-i(\omega_0 t - k_0 r + \varphi)} + \tilde{\rho}_{ab}^{(2)}(t) e^{-i(\omega_0 t + k_0 r + \varphi)}. \tag{2.16}$$

We now write expressions for the polarization-noise source intensities in terms of the spectral densities of the fluctuations $\delta\rho_{ab}$. Substituting (2.14) in (2.13) and representing $\delta\rho_{ab}$ in the form (2.16), we obtain

$$\left.\begin{aligned} (\xi_{ai}^{(P)}\xi_{aj}^{(P)})_{\omega=0} &= \frac{8\pi^2 e^2 n^2 |r_{ab}|^2}{3}\int [(\delta\tilde{\rho}_{ab}^{(i)}\delta\tilde{\rho}_{ba}^{(j)})_{\omega=0} \\ &\qquad - \frac{(r_{ab})^2}{|r_{ab}|^2}(\delta\tilde{\rho}_{ba}^{(i)}\delta\tilde{\rho}_{ba}^{(j)})_{\omega=0}]\, dv\, dv', \\ (\xi_{\text{ph}i}^{(P)}\xi_{\text{ph}j}^{(P)})_{\omega=0} &= \frac{8\pi^2 e^2 n^2 |r_{ab}|^2}{3}\int [(\delta\tilde{\rho}_{ab}^{(i)}\delta\tilde{\rho}_{ba}^{(j)})_{\omega=0} \\ &\qquad + \frac{(r_{ab})^2}{|r_{ab}|^2}(\delta\tilde{\rho}_{ba}^{(i)}\delta\tilde{\rho}_{ba}^{(j)})_{\omega=0}]\, dv\, dv', \\ (\xi_{\text{ph}i}^{(P)}\xi_{aj}^{(P)})_0 &= (\xi_{ai}^{(\Pi)}\xi_{\text{ph}j}^{(\Pi)})_0 = 0; \end{aligned}\right\}$$

*For the regime with one traveling wave, the results presented below are valid for an arbitrary ratio of the parameters. We note that for the regime of two opposing waves, at large detunings relative to the center of the Doppler line, i.e., at $\mu/\gamma_{ab} \gg 1$, the limitation on the parameters γ_a, γ_b, and γ_{ab} also becomes immaterial.

here i, j = 1, 2. Thus, the problem of finding the spectral density of the polarization noise reduces to a determination of the spectral densities of the fluctuations $\delta\tilde{\rho}_{ab}^{(1,2)}$. We shall need later on expressions for the induced parts of the density matrix elements. They are given in[26b].

Substituting $\tilde{\rho}_{ab}^{(1,2)}$ from[26b] into the expression for the polarization vector (2.6) and integrating over the velocities under the assumption that the line is inhomogeneously broadened and the intensities of the opposing waves differ little from each other, i.e., $a|E_1^2 - E_2^2| \ll 1$, we obtain the following values for the real and imaginary parts of the polarizability:

$$\varkappa'_{1,2} = -\frac{d}{4\pi}\frac{\mu}{\gamma_{ab}} f\{1 - F \mp f^2[1-(1-2g)F]\,a(E_1^2 - E_2^2)\}\,^{**}),$$

$$\varkappa''_{1,2} = -\frac{d}{4\pi} f\left\{1 + F\left[1 \mp 2\frac{\mu^2}{\gamma_{ab}^2}\,gf^2a(E_1^2 - E_2^2)\right\}\right. ; \tag{2.18}$$

here

$$\left.\begin{aligned} f &= \left[\frac{2}{g}\left(\sqrt{1+2gaE_0^2} - 1 + 2g + gaE_0^2\right)\right]^{-1/2}, \\ F &= (1+2gaE_0^2)^{-1/2}, \\ E_0^2 &= (E_1^2 + E_2^2)/2, \quad g = \gamma_{ab}^2/(\mu^2 + \gamma_{ab}^2). \end{aligned}\right\} \tag{2.19}$$

In the limiting cases of small and large fields, the functions f and F take the form

$$f = \frac{1}{2}\left(1 - \frac{aE_0^2}{2}\right), \qquad F = 1 - gaE_0^2, \qquad aE_0^2 \ll 1.$$
$$f = 1/\sqrt{2aE_0^2}, \qquad F = 1/\sqrt{2gaE_0^2}, \qquad gaE_0^2 \gg 1.$$

For one traveling wave, putting $E_2 = 0$ in (2.18) and (2.19), we obtain

$$\varkappa' = 0, \quad \varkappa'' = -\frac{d}{4\pi}\frac{1}{\sqrt{1+aE_1^2}}. \tag{2.20}$$

For the standing-wave regime ($E_1 = E_2 = E_0$) at $\mu = 0$ we have

$$\varkappa'_s = 0, \quad \varkappa''_s = -\frac{d}{4\pi}\frac{1}{\sqrt{1+2aE_0^2}}. \tag{2.20$'$}$$

In the case of weak fields, expressions (2.18), (2.20), and (2.20$'$) can be expanded in powers of aE_0^2:

$$\left.\begin{aligned} &\varkappa'_{1,2} = -\frac{d}{4\pi} baE_{2,1}^2, \qquad \varkappa''_{1,2} = -\frac{d}{4\pi}(1 - \alpha aE_{1,2}^2 - \beta aE_{2,1}^2), \\ &\varkappa' = 0, \qquad \varkappa'' = -\frac{d}{4\pi}(1 - \alpha aE_0^2), \\ &\varkappa'_s = 0, \qquad \varkappa''_s = -\frac{d}{4\pi}(1 - aE_0^2); \end{aligned}\right\} \tag{2.21}$$

here

$$\alpha = \frac{1}{2}, \qquad \beta = \frac{g}{2}, \qquad b = \frac{\mu}{\gamma_{ab}}\frac{g}{2}. \tag{2.22}$$

Expressions (2.21) are valid with allowance for the spatial modulation of the populations. The coefficient β is then given by

$$\beta = \frac{g}{2} - \frac{\gamma^2}{2(ku)^2}.$$

In a strong field ($aE_0^2 \gg 1$) we obtain from (2.18)–(2.20)

$$\varkappa'_{1,2} = -\frac{d}{4\pi}\frac{\mu}{\gamma_{ab}\sqrt{2aE_0^2}}, \qquad \varkappa''_{1,2} = -\frac{d}{4\pi\sqrt{2aE_0^2}},$$
$$\varkappa' = 0, \qquad \varkappa'' = -d/4\pi\sqrt{aE^2}.$$

As already noted above, for the case $ku/\gamma_{ab} \ll 1$, when the motion of the atoms can be neglected, it is possible to obtain an exact solution of the equations for the density matrix with allowance for the spatial modulation of the populations. The expressions for $\kappa'_{1,2}$ and $\kappa''_{1,2}$ are in this case

$$\varkappa'_{1,2} = (\mu/\gamma_{ab})\,\varkappa''_{1,2}, \tag{2.23}$$

$$\varkappa''_{1,2} = -\frac{e^2|r_{ab}|^2 nD^0}{6\hbar\gamma_{ab}aE_{1,2}^2}\left\{1 - \frac{1 \mp ga(E_1^2 - E_2^2)}{\sqrt{1 + 2ga(E_1^2 + E_2^2) + g^2a^2(E_1^2 - E_2^2)^2}}\right\}.$$

In the standing-wave regime we have in (2.23) $E_1 = E_2 = E_0$. The corresponding expression $\kappa''_{1,2}$ without allowance for modulation is

$$\varkappa''_{1,2} = -\frac{e^2|r_{ab}|^2 nD^0}{3\hbar\gamma_{ab}}\frac{g}{1 + g(aE_1^2 + aE_2^2)}. \tag{2.24}$$

We proceed to consider the spectral densities of the spontaneous fluctuations. They are calculated by using the equations for the functions $\delta\rho_a$, $\delta\rho_b$, and $\delta\tilde{\rho}_{ab}$. Instead of $\delta\rho_a$ and $\delta\rho_b$ it is more convenient to use the functions $\delta D = \delta\rho_a - \delta\rho_b$ and $\delta R = \delta\rho_a + \delta\rho_b$.

The corresponding equations follow from (2.1)–(2.4) and are given by

$$\left.\begin{aligned} \frac{\partial\delta D}{\partial t} &= -\frac{ie}{\hbar}[r_{ba}(\delta\tilde{\rho}_{ab}^{(1)}E_1 + \delta\tilde{\rho}_{ab}^{(2)}E_2) - \\ &\quad - r_{ab}(\delta\tilde{\rho}_{ba}^{(1)}E_1 + \delta\tilde{\rho}_{ba}^{(2)}E_2)] - \gamma_+\delta D + \gamma_-\delta R, \\ \frac{\partial\delta R}{\partial t} &= \gamma_-\delta D - \gamma_+\delta R, \\ \frac{\partial\delta\tilde{\rho}_{ab}^{(1,2)}}{\partial t} &= i(\mu \mp k_0 v + i\gamma_{ab})\,\delta\tilde{\rho}_{ab}^{(1,2)} - \\ &\qquad - \frac{ie}{2\hbar}\,\delta D r_{ab}E_{1,2}, \\ \delta\tilde{\rho}_{ba}^{(1,2)} &= (\delta\tilde{\rho}_{ab}^{(1,2)})^*. \end{aligned}\right\} \tag{2.25}$$

Let us multiply Eqs. (2.25) by $\delta\tilde{\rho}_{ba}^{(1,2)}(t')$ and average. As a result we obtain a system of homogeneous equations for the correlation functions of the argument $\tau \equiv t - t'$.

It is necessary to add to this system of equations the initial conditions, namely the value of the correlations functions at $\tau = 0$. They follow from formula (5) of Appendix 1, if we neglect the last term of this formula

$$\langle\delta D\delta\tilde{\rho}_{ba}^{(i)}\rangle_{\tau=0} = 0, \qquad \langle\delta R\delta\tilde{\rho}_{ba}^{(i)}\rangle_{\tau=0} = \frac{1}{nV}\tilde{\rho}_{ba}^{(i)}\delta(v - v'), \tag{2.26}$$
$$\langle\delta\tilde{\rho}_a^{(i)}\delta\rho_{ba}^{(j)}\rangle_{\tau=0} = \frac{\delta_{ij}\delta(v - v')}{2nV}\bar{R}, \qquad \langle\delta\tilde{\rho}_{ba}^{(i)}\delta\tilde{\rho}_{ba}^{(j)}\rangle_{\tau=0} = 0, \qquad i, j = 1, 2.$$

The system of equations for the correlation functions can be solved by using the Laplace transformation

$$(\delta\tilde{\rho}\delta\tilde{\rho})_\omega^+ = \int_0^\infty \langle\delta\tilde{\rho}\delta\tilde{\rho}\rangle_\tau e^{i\omega\tau}\,d\tau. \tag{2.27}$$

*Expressions (2.18) were obtained for a laser using a pure isotope of the active gas. It is difficult to obtain analogous expressions for a laser with a mixture of isotopes in a strong field.

The expressions for the spectral densities of the noise sources (2.17) contain the spectral densities of the fluctuations $\delta\tilde{\rho}$ at zero frequency. They are connected with the fluctuations (2.27) by the relation

$$(\delta\tilde{\rho}\,\delta\tilde{\rho})_0 = 2\,\mathrm{Re}\,(\delta\tilde{\rho}\,\delta\tilde{\rho})^{+}_{\omega=0}.$$

The simplest expressions for the spectral densities at zero frequency (intensities) of the polarization-noise sources are obtained in the regime of one traveling wave:

$$(\xi_a^{(P)2})_0 = \frac{2\pi\hbar\,\Delta\omega_r}{V\omega_0}\,\frac{R^0}{D^0}, \qquad (2.28)$$

$$(\xi_\Phi^{(P)2})_0 = \frac{2\pi\hbar\,\Delta\omega_r}{V\omega_0}\,\frac{R^0}{D^0}\,(1 + aE^2).$$

General expressions for the noise intensities were obtained in[26b]. They are given in Appendix 2.

The polarization-noise source intensities in the standing-wave regime can be obtained from the general formulas (6) and (7) of Appendix 2, by taking into account the fact that

$$\xi_{a,c} = (\xi_{a1} + \xi_{a2})/2, \qquad \xi_{ph,s} = (\xi_{ph1} + \xi_{ph2})/2$$

and consequently

$$(\xi^2_{a,s})_0 = \frac{1}{2}\,[(\xi^2_{a1,2})_0 + (\xi_{a1}\xi_{a2})_0], \qquad (\xi^2_{ph\,s})_0 = \frac{1}{2}\,[(\xi^2_{ph1,2})_0 + (\xi_{\Phi1}\xi_{\Phi2})_0].$$

Thus, the expressions for the spectral densities $(\xi^2_{a,s})_0$ and $(\xi^2_{ph,s})_0$ take into account the contribution of the amplitude and phase fluctuation sources in the equations for the opposing waves.

In particular, at $\gamma_a = \gamma_b$ we obtain

$$(\xi^{(P)2}_{a,c})_0 = \frac{\pi\hbar\,\Delta\omega_r}{V\omega_0}\,\frac{R^0}{D^0}, \qquad (2.29)$$

$$(\xi^{(P)2}_{\Phi,c})_0 = \frac{\pi\hbar\,\Delta\omega_r}{V\omega_0}\,\frac{R^0}{D^0}\left(1 + \frac{2FaE_0^2}{1+F}\right).$$

Formulas (6), (7), (2.28), and (2.29) contain, besides the explicit dependence on the field, also an implicit dependence on the field via the parameters $\Delta\omega_r$, D^0, and R^0. For a comparison with the experimental data, it is necessary to include this implicit dependence. The form of this dependence is determined by the method of varying the field. Zaĭtsev, Andronova, et al.[9-11] varied the field by varying the losses in the resonator, i.e., by varying $\Delta\omega_r$, at a constant pump current. Substituting $\Delta\omega_r = -4\pi\omega_0\kappa''(E_0)$ in formulas (2.29), (6), (7), and (2.29) we obtain the explicit dependence of the fluctuation-source intensities on the field. Plots of these dependences are shown in Fig. 2.

In writing down (2.28), (2.29), (6), and (7), we used the condition $\Delta\omega_r + 4\pi\omega_0\kappa'' = 0$ for the stationary generation regime. If we again substitute $-4\pi\omega_0\kappa''$ for $\Delta\omega_r$ in these formulas, we can obtain the limiting transition to the equilibrium state.

At equilibrium $E = 0$, $\kappa'' > 0$, and

$$-\frac{1}{2}\,\frac{R^0}{D^0} = \frac{1}{2} + \frac{1}{e^{\hbar\omega_{ab}/kT} - 1} \equiv \bar{n} + \frac{1}{2}.$$

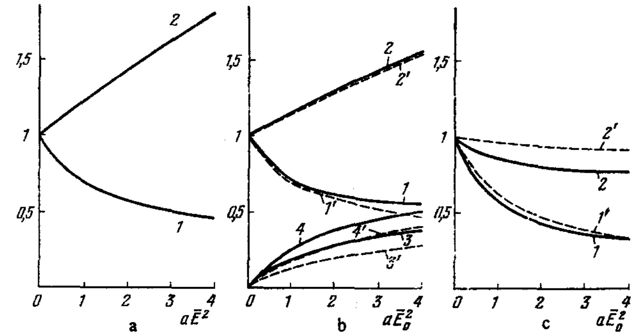

FIG. 2. Dependence of the noise-source intensity on the field amplitude. A) Traveling-wave regime: $1 - (\xi_a^2)_0/(\xi_a^2)_0|_{E=0}$, $2 - (\xi_\Phi^2)_0/(\xi_\Phi^2)_0|_{E=0}$; b) regime of two opposing waves: $1,\ 1' - (\xi^2_{1,2})_0/(\xi^2_{a1,2})_0|_{E_0=0}$, $2,\ 2' - (\xi^2_{\Phi1,2})_0/(\xi^2_{\Phi1,2})_0|_{E_0=0}$, $3,\ 3' - (\xi_{a1}\xi_{a2})_0/(\xi^2_{a1,2})_0$, $4,\ 4' - (\xi_{\Phi1}\xi_{\Phi2})_0/(\xi^2_{\Phi1,2})_0$, $1-4 - \mu = 0$, $1'-4' - \mu = \gamma_{ab}$; (c) standing-wave regime: $1,\ 1' - (\xi^2_{a,c})_0/(\xi^2_{a,c})_0|_{E_0=0}$, $2,\ 2' - (\xi^2_{\Phi,c})_0/(\xi^2_{\Phi,c})_0|_{E_0=0}$, $1,\ 2 - \mu = 0$, $1',\ 2' - \mu = \gamma_{ab}$.

We therefore obtain from (2.28)

$$(\xi_a^{(P)2})_0 = (\xi_{ph}^{(P)2})_0 = \frac{(4\pi)^2\hbar\kappa''}{V}\left(\bar{n} + \frac{1}{2}\right).$$

In the generation regime, both factors (κ'' and R^0/D^0) reverse sign.

From formulas (2.28) and (2.29) and from the general formulas (6) and (7) we see that the intensities of the noise sources which enter in the equation for the amplitudes and phases of the field differ greatly from each other at a non-zero field amplitude even when it comes to the character of the dependence on the field. Such a difference is due to the following causes. The fluctuations of the density matrix depend, in accordance with (2.25), on the instantaneous value of the average field E, and consequently are not stationary random processes. (Only the slowly varying amplitudes of these fluctuations are stationary.) Accordingly, the polarization fluctuations δP, which enter in expressions (2.13) for the noise sources, are likewise not stationary. By virtue of this, the expression $\langle \delta P(t)\,\delta P(t+\tau)\cos(\omega_0(2t+\tau) \mp 2k_0 r + \varphi)\rangle$ differs from zero. It enters in the expression for the correlation functions of the amplitude and phase noise sources with different signs, so that the intensities of these sources are different.

For the same reason, the correlations of the fluctuation sources $(\xi_{a1}\xi_{a2})_0$ and $(\xi_{ph1}\xi_{ph2})_0$ are likewise different from zero for opposing waves in a ring laser, even if no account is taken of the second spatial harmonics of the working-level populations, i.e., without allowance for the spatial modulation of the populations. Allowance for the spatial modulation can yield additional terms in the correlation functions of the sources.

We note one more important circumstance. It follows from (2.13) that the mean values of the noise sources ξ_a and ξ_{ph} are equal to zero if we can neglect the correlation of the random deviations δP and $E^{(T)}$ and the phase of the field. This takes place in a sufficiently strong field, since the change of the phase at a specified noise is reversely proportional to the field.

At the generation threshold, the mean value of the amplitude noise differs from zero and is equal to[49]

$$\bar{\xi}_a = \omega_0\,(\xi_a^2)_0/2E, \quad \text{and} \quad \bar{\xi}_{ph} = 0.$$

The bar denotes averaging over the time interval $1/\gamma_{ab} \ll \Delta t \ll 1/\Delta\omega_{ph}$.

An analysis of the obtained expressions and diagrams for the polarization-noise sources shows that at a sufficiently large frequency deviation, when $|\mu| \gg \gamma_{ab}$, the intensities of the noise sources are the same for all three regimes (the factor of 2 for the standing-wave regime is connected with the method of specifying the amplitude). At frequency deviations $|\mu| \lesssim \gamma_{ab}$, the source intensities are different for different regimes.

When the power is varied by varying the losses, in all regimes, the noise-source intensities in the equations for the amplitudes decrease with increasing field. On the other hand, the noise-source intensities in the equations for the phases increase with increasing field in the traveling-wave regime and in the regime of two opposing waves. In the standing-wave regime, on the other hand, the character of the dependence of the phase-noise intensity on the field is governed by the detuning. When

$$\mu^2/\gamma_{ab}^2 \gg \gamma_b D^0/\rho_b\gamma_+$$

the intensity of the phase noise increases with increasing field, and in the opposite case it decreases. In a strong field, the intensity of the phase-noise source tends to a constant value.

We note also that in a strong field there is an almost complete "anticorrelation" between the polarization-noise sources for opposing waves (the correlation coefficient tends to -1). This means that the sum of the sources fluctuates much less than their difference.

In the standing-wave regime, in the case of homogeneous line broadening, when $ku \ll \gamma_{ab}$ and $\gamma_{ab} = \gamma_b$, we obtain for the spectral densities of the spontaneous fluctuations of the polarization[47]

$$\begin{aligned}(\xi_{a,c}^{(P)2})_0 &= \frac{\pi\hbar\,\Delta\omega_r}{\omega_0 V}\,\frac{R^0}{D^0},\\ (\xi_{phcs}^{(P)2})_0 &= \frac{\pi\hbar\,\Delta\omega_r}{\omega_0 V}\,\frac{R^0}{D^0}\left(\frac{2aE_0^2\sqrt{1+4gaE_0^2}}{\sqrt{1+4gaE_0^2}-1}-\frac{\mu^2}{\gamma_{ab}^2}\right).\end{aligned} \tag{2.30}$$

$\Delta\omega_r = -4\pi\omega_0\kappa''$, where κ'' is determined by formula (2.23) with $E_1 = E_2 = E_0$. For the traveling-wave regime, the expressions for the noise intensity coincide with formulas (2.28).

We note that the calculation of the fluctuation intensities for a ring laser with homogeneously broadened line shows that allowance for the spatial modulation exerts a strong influence on the correlations of the fluctuation sources for opposing waves. The influence of the population modulation on the fluctuation correlation may turn out to be significant also for the case of an inhomogeneously broadened line.

We note once more that the expressions presented for the sources of the thermal and polarization noises include contributions of the zero-point fluctuations. Analogously, using initial conditions that do not include zero-point polarization fluctuations, it is possible to obtain noise sources determined only by spontaneous emission. We do not present here the corresponding expressions, since the separation of the spontaneous parts can be carried out in the final expressions.

3. AMPLITUDE AND PHASE FLUCTUATIONS IN A LINEAR GAS LASER

When considering the fluctuations in a linear laser, we specify the field in the form of a standing wave

$$E(r,\,t) = E_0\cos\left(kr-\frac{\Phi}{2}\right)e^{-i(\omega_0 t+\varphi)} + \text{к. с.} \tag{3.1}$$

This expression follows from (2.9) for opposing waves if we put $E_1 = E_2 = E_0$ and $\varphi = (\varphi_1 + \varphi_2)/2$. The quantity Φ is constant and is determined by the boundary conditions.

Thus, the description of the fluctuations in a linear laser differs from the description of the fluctuations in a ring laser in that the differences $E_1 - E_2$ and $\varphi_1 - \varphi_2$ do not fluctuate.

The equations for E and φ are (we omit the subscript "0" of E_0; E_0 will henceforth denote the field amplitude without allowance for the fluctuations)

$$\begin{aligned}\frac{dE}{dt} &= \frac{\omega_0}{2}\left(4\pi\kappa_c''+\frac{1}{Q}\right)E+\omega_0\xi_{a,s}(t),\\ \frac{d\varphi}{dt} &= -\frac{\omega_0}{2}4\pi\kappa_s'+\frac{\omega_0}{E}\xi_{ph\,s}(t).\end{aligned} \tag{3.2}$$

The quantities κ_s' and κ_s'' are determined by formulas (2.18) with $E_1 = E_2 = E_0$.

3.1. Amplitude fluctuations. In the correlation approximation, assuming $E = E_0 + \delta E$, we obtain from the first equation of (3.2) the following expressions for the spectral densities of the fluctuations of the amplitude E and of the intensity E^2:

$$(\delta E^2)_\omega = \frac{\omega_0^2(\xi_{a,s}^2)_0}{\omega^2+(\Delta\omega_a)^2}, \qquad (\delta(E^2)^2)_\omega = 4E_0^2(\delta E^2)_\omega. \tag{3.3}$$

The width $\Delta\omega_a$ of the amplitude-fluctuation spectrum is

$$\Delta\omega_a = 4\pi\omega_0\frac{\partial\kappa''}{\partial E^2}E_0^2 = \omega_0 df\,[f^2(1+F)^2+gF^3]\,aE_0^2. \tag{3.4}$$

In a weak field, with allowance for (2.21) and for the condition of stationary generation, this expression takes the form

$$\Delta\omega_a = \Delta\omega_r\eta, \tag{3.5}$$

where $\eta = Qd - 1$ is the excess of pump over threshold.

Let us compare the results of the calculation of the intensity fluctuations with the experimental data of Zaĭtsev[9]. Figure 3 (curve 1) shows the dependence of the spectrum width $\Delta\omega_a$ on the field intensity, plotted in accordance with formula (3.4) at zero detuning ($\mu = 0$). Under this condition formula (3.4) contains one unknown parameter d. However, in comparing with the experimental dependence of $\Delta\omega_a$ on the power, it is necessary to know one more parameter that characterizes the connection between E_0^2 and P.

The total laser radiation power is connected with E_0^2 by the relation

$$P = (E_0^2/4\pi)\,V\,\Delta\omega_r.$$

The width of the resonator band at constant pumping varies itself with changing power. This dependence follows from the condition of stationary solution of (3.2).

In experiment one measures not the quantity P but a fraction of this quantity P_T, determined by the trans-

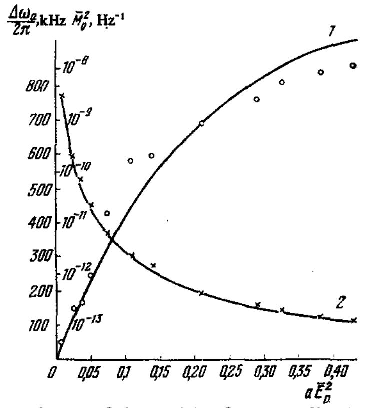

FIG. 3. Dependence of the width of the amplitude-fluctuation spectrum (1) and of the relative spectral density of the fluctuations and the intensity at zero (2) on the power at $\mu = 0$.

parency of the output mirror. The quantity P_T is proportional to E_0^2:

$$P_T = (E_0^2/4\pi)\, V\, \Delta\omega_T,$$

where $\Delta\omega_T = c(1-r)/L$, i.e., it does not depend on E_0^2.

In[9-11], the curves were plotted as functions of P. This requires additional recalculation of the experimental and theoretical data. It is more natural to plot the curves as functions of P_T. The corresponding additional information on the values of P_T were kindly supplied to us by the authors of[9-11]. The experimental points based on Fig. 3 are based on these data. The unknown parameters d and $\gamma_T = aE_0^2/P_T$ were determined from the condition that the results coincide at the two points.

The values of these parameters turned out to be

$$d = 1{,}15\cdot 10^{-8}, \quad \gamma_T = 1.16\cdot 10^{-2} \quad \mu\,\mathrm{W}^{-1}.$$

Curve 2 on Fig. 3 is a plot of the relative intensity fluctuation density at zero frequency ($M_0^2 = (\delta(E^2)^2)_0/E_0^4$) on the power P_T. We see that there is sufficiently good agreement between theory and experiment. We note that when no account is taken of the dependence of the intensity of the amplitude-fluctuation source on aE_0^2 the theoretical curve lies much higher, starting with $aE_0^2 = 0.2$.

From (3.4) and (3.3) we obtain expressions for the variances of the amplitude and of the intensity:

$$\langle\delta E^2\rangle = (\omega_0^2/2\Delta\omega_a)\,(\xi_{a,s}^2)_0, \quad \langle\delta(E^2)^2\rangle = 2\omega_0^2 E_0^2\,(\xi_{a,s}^2)_0/\Delta\omega_a. \tag{3.6}$$

Let us establish the connection between the expressions for the fluctuations of the amplitude and the experimentally measured quantities, namely the average number of photons $\langle n_{ph}\rangle$ and the variance of the number of photons $\langle\delta n_{ph}^2\rangle$.

The average number of photons is connected with the mean square of the field in the traveling-wave regime by the relation

$$\langle E^2\rangle = E_0^2 + \langle\delta E^2\rangle = (8\pi\hbar\omega_0/V)\left(\langle n_{ph}\rangle + \tfrac{1}{2}\right).$$

From this, knowing the excess above threshold and using (3.6), we can find the average number of photons.

Expressions (3.6) determine the radiation-intensity fluctuations that are directly connected with $\langle\delta n^2\rangle$. Thus, for a traveling wave

$$\langle\delta(E^2)^2\rangle = (8\pi\hbar\omega_0/V)^2\,\langle\delta n_{ph}^2\rangle.$$

The relative variance of the intensity in a weak field is determined by the expression

$$\frac{\langle\delta(E^2)^2\rangle}{E_0^4} = \frac{N_s^2}{I^2}, \qquad I = \frac{2}{1+g}\,\eta\,\frac{1}{Qd}. \tag{3.7}$$

We have introduced here the following symbol for the dimensionless noise intensity

$$N_s^2 = \frac{4}{1+g}\,\frac{\omega_0 a\,(\xi_{a,s}^2)_0}{d}. \tag{3.8}$$

From (3.7) we get the condition for the applicability of the correlation approximation

$$N_s \ll I. \tag{3.9}$$

In the zeroth approximation in the field, at parameter values $\omega_0 = 3\times 10^{15}$ sec^{-1}, V = 1 cm, $d \approx 10^{-8}$, $(a/V)(1 + R^0/D^0) = 2\times 10^3$ cgs esu, and $\mu = 0$ we obtain $N_s = 0.5\times 10^{-4}$.

It follows from (3.9) that the correlation approximation is valid if the radiation power is much larger than 1 μW.

Using formula (3.4) and (2.23) it is possible, for example, to calculate the spectral width of the amplitude fluctuations for a laser with homogeneous line broadening, when $\gamma_{ab} \gg ku$. In a weak field we obtain for $\Delta\omega_a$ the expression

$$\Delta\omega_a = 3\Delta\omega_T\, g a E_0^2.$$

If no account is taken of the spatial modulation of the populations, the coefficient 3 in this formula is replaced by 2. If we express $\Delta\omega_a$ in terms of the excess of pump over threshold, then the results coincide. Indeed, taking the modulation into account,

$$aE_0^2 = \frac{1}{4}\left(3 + 4\eta - 3\sqrt{1 + \frac{8}{9}\eta}\right).$$

from which we get $aE_0^2 = 2\eta/3$ for small η. If modulation is not taken into account, then $aE_0^2 = \eta$.

3.2. Amplitude and intensity fluctuations at the generation threshold. At the generation threshold, the condition (3.9) is not satisfied. To calculate the amplitude and intensity fluctuations in this case it is necessary to use the method of the Fokker-Planck equation[48,49]. Near the threshold, the field is weak ($aE^2 \ll 1$), therefore the function κ'' in Eq. (3.2) for E can be expanded in terms of aE^2 and only the first two terms retained. As a result Eq. (3.2) takes the form

$$\frac{dE}{dt} = \frac{\Delta\omega_p}{4}\,(1+g)\,(I - aE^2)\,E + \omega_0\xi_a(t). \tag{3.10}$$

In the stationary generation regime without allowance for the noise we have

$$aE_0^2 = I.$$

The corresponding Fokker-Planck equation for the function W(E), with allowance for the fact that $\overline{\xi_a(t)} = (\xi_a^2)_0/2E$, is

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial E}\left\{\left[\frac{\Delta\omega_r}{4}(1+g)(I-aE^2)E+\frac{\omega_0^2(\xi_a^2)_0}{2E}\right]W\right\} + \frac{\omega_0^2}{2}(\xi_a^2)_0\frac{\partial^2 W}{\partial E^2}. \quad (3.11)$$

From this we obtain the stationary distribution

$$W(E)=\sqrt{\frac{2}{\pi}}\frac{2a}{N_s}\left[1+\Phi\left(\frac{I}{N_s}\right)\right]^{-1}E\exp\left(-\frac{(aE^2-I)^2}{2N_s^2}\right). \quad (3.12)$$

From this distribution follows a general expression for the moments

$$\langle E^n\rangle=\sqrt{\frac{2}{\pi}}\left(\frac{N_s}{a}\right)^{n/2}\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\left[1+\Phi\left(\frac{I}{N_s}\right)\right]^{-1} e^{-I^2/4N_s^2}D_{-(n/2+1)}\left(-\frac{I}{N_s}\right). \quad (3.13)$$

Three limiting cases are of interest:

1) At a considerable excess over the generation threshold, when $I \gg N_s$, Eq. (3.13) leads to the formulas of the correlation theory.

2) At the generation threshold ($|I| \ll N_s$) we obtain from (3.13)

$$\langle\delta E^2\rangle=0{,}12\frac{N_s}{a}\left(1+0{,}25\frac{I}{N_s}\right),$$
$$\frac{\langle\delta E^2\rangle}{\langle E\rangle^2}=0{,}18-0{,}05\frac{I}{N_s},\quad \frac{\langle\delta(E^2)^2\rangle}{\langle E^2\rangle^2}=0{,}57-0{,}28\frac{I}{N_s}. \quad (3.14)$$

Calculation shows that formulas (3.13) and (3.14) remain unchanged in the case of immobile atoms.

3) Below the excitation threshold at $|I| \gg N_s$ and $I < 0$ we obtain from (3.13)

$$\langle E\rangle=\frac{\sqrt{\pi}N_s}{2\sqrt{a|I|}},\quad \langle E^2\rangle=\frac{N_s^2}{a|I|},\quad \langle E^4\rangle=\frac{2N_s^4}{a^2I^2}. \quad (3.15)$$

Hence

$$\langle\delta E^2\rangle=\frac{N_s^2}{aI}\frac{4-\pi}{4},\quad \langle\delta(E^2)^2\rangle=\frac{N_s^4}{a^2I^2}=\langle E^2\rangle^2. \quad (3.15')$$

The second formula of (3.15′) leads to an expression for the mean squared value of the number of photons

$$\langle\delta n_{ph}^2\rangle=\left(\langle n_{ph}\rangle+\frac{1}{2}\right)^2.$$

The exact expression for $\langle\delta n_{ph}^2\rangle$ is

$$\langle\delta n_{ph}^2\rangle=\langle n_{ph}\rangle(\langle n_{ph}\rangle+1). \quad (3.16)$$

They coincide when the number of photons is large, the only case when the semiclassical description is valid.

According to the experimental data of Arecchi, Rodari, and Sona[6], the number of photons at the threshold is 4000. Calculation shows that at $V = 0.25\ \text{cm}^3$, $a = 10^2$ cgs esu, $N_s = 10^{-4}$, and $\omega = 3\times10^{15}\ \text{sec}^{-1}$, we have $\langle n\rangle = 5\times10^3$.

Hempstead and Lax[39], and Risken[25] performed the corresponding calculations for immobile atoms. Their results coincide in form with those given above.

We now compare the theoretical and experimental data for the intensity fluctuations near the threshold. Figure 4 (curve 1) shows the dependence of the relative

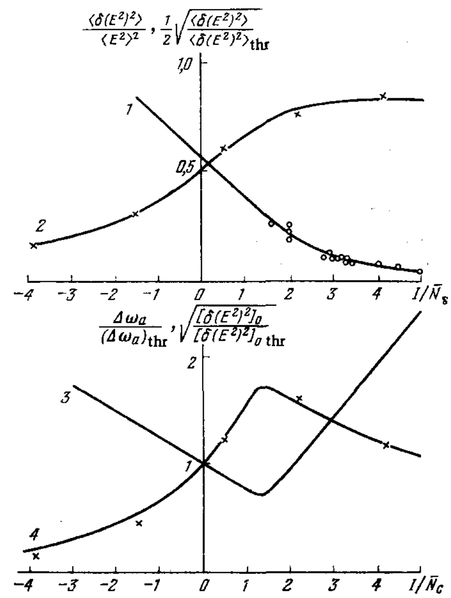

FIG. 4

$1-\frac{\langle\delta(E^2)^2\rangle}{\langle E^2\rangle^2}$; $2-\frac{1}{2}\sqrt{\frac{\langle\delta(E^2)^2\rangle}{\langle\delta(E^2)^2\rangle_{thr}}}$

$3-\frac{\Delta\omega_a}{(\Delta\omega_a)_{thr}}$; $4-\sqrt{\frac{[\delta(E^2)^2]_0}{[\delta(E^2)^2]_{thr}}}$.

variance of the intensity on the quantity I/N_s. The circles denote the experimental data of Smith and Armstrong[12]. Curve 2 shows a plot of the square root of the variance. The crosses mark the results of the experiments of Arecchi et al.[13]

To determine the spectral function of the amplitude and intensity fluctuations it is necessary to know the non-stationary solution of the Fokker-Planck equation (3.11). Such a solution was obtained in[25,39]. It turned out that the spectral line shape differs only little from a Lorentz shape near the threshold.

We present the results of the calculation of the spectrum width of the amplitude fluctuations near the threshold, obtained assuming a Lorentz line shape[27]. It follows from (3.6) that

$$\Delta\omega_a=\omega_0^2(\xi_{a,s}^2)_0/2\langle\delta E^2\rangle. \quad (3.17)$$

Hence, using (3.14), we get

$$\Delta\omega_a=(\Delta\omega_a)_{thr}\left(1-0{,}25\frac{I}{N_s}\right); \quad (3.18)$$

here

$$(\Delta\omega_a)_{thr}\approx 2\omega_0 dN_s. \quad (3.19)$$

The dependence of $\Delta\omega_a$ on I/N_s is shown in Fig. 4 (curve 3). This dependence coincides sufficiently well with that given in[57], calculated in accordance with Risken's exact theory[25].

At $\omega_0 d/2\pi = 10^6$ Hz, $\mu = 0$, and $N_s = 10^{-4}$ we obtain from (3.19) $(\Delta\omega_a)_{thr}/2\pi \approx 200$ Hz.

Freed and Haus[8] measured the line width of the amplitude fluctuations in the regions above and below the threshold at

$$8 \leqslant I/N_s < 800, \quad -80 \leqslant I/N_s \leqslant -8.$$

The results were extrapolated to the threshold region. From the plot given in[8] it follows that $(\Delta\omega_a)_{thr}/2\pi$

$\approx$ 100 Hz. Arecchi, Rodari, and Sona[6] obtained for the spectrum width of the amplitude fluctuations, at a resonator band width $\Delta\omega_r/2\pi$ = 27 MHz, a value $(\Delta\omega_a)_{thr}/2\pi$ = 1.4 kHz at the threshold.

As already noted, the results presented here for moving atoms coincide in form with the results of calculations for immobile atoms. A review of these results is given in Risken's paper[60].

3.3. Phase and frequency fluctuations. From (3.2) we obtain in the correlation approximation an equation for the phase fluctuations

$$\frac{d\delta\varphi}{dt}=\frac{\omega_0}{E}\left[K\delta E+\xi_{\mathrm{ph},s}(t)\right]; \tag{3.20}$$

here

$$K=-4\pi\frac{\partial\varkappa'}{\partial E^2}E^2=\frac{\mu}{\gamma_{ab}}\,daE^2f\left[gF^3-f^2(1-F^2)\right].$$

The term $K\delta E$ characterizes the influence of the amplitude fluctuations on the phase and frequency fluctuations.

From (3.20) we get an expression for the spectral density of the frequency fluctuations:

$$(\delta\dot\varphi)^2_\omega=(\omega_0^2/E^2)\left[K^2(\delta E^2)_\omega+(\xi^2_{\mathrm{ph}\,s})_\omega\right]. \tag{3.21}$$

In the derivation of (3.21) we took into account the absence of correlation between the quantities $\xi_{a,s}$ and $\xi_{ph,s}$ (see (2.15) and (2.17)).

In books on statistical radiophysics[48-50] it is shown that for times greatly exceeding the noise-source correlation time, the mean squared phase shift in (3.20) is

$$\langle(\varphi(t+\tau)-\varphi(t))^2\rangle\equiv\langle\delta\varphi_\tau^2\rangle=D|\tau|. \tag{3.22}$$

The phase diffusion coefficient D is defined by the expression

$$D=(\delta\dot\varphi^2)_0.$$

For Eq. (3.20), the source correlation time is determined by the amplitude-fluctuation correlation time, and therefore the condition for the applicability of (3.22) is

$$\tau\gg 1/\Delta\omega_a.$$

Substituting in (3.21) expression (3.3) for the spectral density of the amplitude fluctuations at ω = 0, we obtain

$$D=\frac{\omega_0^2}{E^2}\left[K^2\frac{\omega_0^2}{(\Delta\omega_a)^2}(\xi^2_{a,c}{}^{s})_0+(\xi^2_{\mathrm{ph}\,s})_0\right]. \tag{3.23}$$

In a weak field,

$$K=\frac{1}{2}\frac{\mu}{\gamma_{ab}}g\,daE^2,$$

and consequently (taking (3.5) into account),

$$D=\frac{\omega_0^2}{E^2}(\xi^2_{\mathrm{ph}\,s})_0\left[1+\frac{\mu^2}{\gamma_{ab}^2}\frac{g^2}{(1+g)^2}\right]. \tag{3.24}$$

In a strong field at $aE^2\gg 1$ we have

$$D=\frac{\pi\hbar\,d\omega_0^2}{VE^2\sqrt{g}}\left(\frac{R^0}{D^0}+\frac{\gamma_-}{2\gamma_+}\right). \tag{3.25}$$

It follows from (3.24) and (3.25) that the phase diffusion coefficient decreases in inverse proportion to the square of the radiation field in the standing-wave regime, both in a weak and in a strong field.

A more exact formula for the mean-squared phase shift, which is valid also when $\tau\leq 1/\Delta\omega_a$, is

$$\langle\delta\varphi_\tau^2\rangle=D|\tau|-\frac{(\delta E^2)_0\,\omega_0^3K^2}{\Delta\omega_aE^2}\left(1-e^{-\Delta\omega_a|\tau|}\right). \tag{3.26}$$

Let us determine the form of the field spectrum in the laser and calculate the natural line width of the radiation.

From (3.1) we obtain an expression for the correlation function of the field in the laser:

$$\langle EE_\tau\rangle=\langle(\langle E\rangle+\delta E)_t(\langle E\rangle+\delta E)_{t+\tau}\cos(\omega_0t+\varphi_t)\cos(\omega_0(t+\tau)+\varphi_{t+\tau})\rangle. \tag{3.27}$$

At sufficiently large excesses of pump over threshold, when the probability distribution for the amplitude fluctuations and the phase shift can be regarded as Gaussian, we obtain from (3.27) the following approximate expression for the correlation function of the field:

$$\begin{aligned}\langle EE_\tau\rangle=&\left(E_0^2\cos\omega_0\tau-\frac{2\omega_0K\langle\delta E^2\rangle}{\Delta\omega_a}\sin\omega_0\tau\right)e^{-\langle\delta\varphi_\tau^2\rangle/2}\\&+\left(\langle\delta E^2\rangle\cos\omega_0\tau+\frac{2\omega_0K\langle\delta E^2\rangle}{\Delta\omega_a}\sin\omega_0\tau\right)e^{-\Delta\omega_a\tau-(\langle\delta\varphi_\tau^2\rangle/2)}\\&-\frac{\omega_0^2K^2\langle\delta E^2\rangle^2}{E_0^2(\Delta\omega_a)^2}\cos\omega_0\tau e^{-2\Delta\omega_a\tau-(\langle\delta\varphi_\tau^2\rangle/2)}.\end{aligned} \tag{3.28}$$

Taking the Fourier transform, at $D\ll\Delta\omega_a$, we obtain*

$$\begin{aligned}(E^2)_\omega=&\frac{E_0^2D/2}{(\omega-\omega_0)^2+(D^2/4)}\left(1+\frac{4\omega_0K\langle\delta E^2\rangle}{\Delta\omega_aE_0^2}\frac{\omega-\omega_0}{D}\right)\\&+\frac{\langle\delta E^2\rangle\left[\Delta\omega_a+(D/2)\right]}{(\omega-\omega_0)^2+\left[\Delta\omega_a+(D/2)\right]^2}\left[1-\frac{\omega_0^2K^2}{(\Delta\omega_a)^2}-\frac{2\omega_0K(\omega-\omega_0)}{(\Delta\omega_a)^2}\right]\\&-\frac{2\omega_0^2K^3\langle\delta E^2\rangle^2}{\Delta\omega_aE_0^2}\frac{1}{(\omega-\omega_0)^2+4(\Delta\omega_a)^2}.\end{aligned} \tag{3.29}$$

It follows from (3.29) that the spectrum of the generated signal in the laser is a sum of three lines. One of the lines is determined by the fluctuations of the phase shift and is a narrow line of almost Lorentz shape with width $\Delta\omega$ = D and intensity $\sim E_0^2$. The second line, due to the amplitude fluctuations and the correlation between the phase and amplitude fluctuations, has a width $\sim 2\Delta\omega_a$. This line is much broader than the first, but is much less intense. We note that the line of width $2\Delta\omega_a$ is essentially asymmetrical. The third line, even weaker, has a width $\sim 4\Delta\omega_a$. In a weak field, the largest value of the parameter $\omega_0K/\Delta\omega_a$ is 0.25, and consequently the influence of the amplitude fluctuations on the phase fluctuations is not significant. The spectrum of the field in the laser can be written, with good approximation, in the form

$$(E^2)_\omega=\frac{\langle E\rangle^2D/2}{(\omega-\omega_0)^2+(D/2)^2}+\frac{\langle\delta E^2\rangle\left[\Delta\omega_a+(D/2)\right]}{(\omega-\omega_0)^2+\left[\Delta\omega_a+(D/2)\right]^2}. \tag{3.30}$$

The dependence of the emission line width $\Delta\omega\approx D$ on aE_0^2, calculated from formula (3.23) at a zero frequency deviation from the center of the Doppler line, is shown in Fig. 5. The experimental data obtained by Zaĭtsev and Stepanov[4] are also shown. At the chosen values of the parameters, the experimental data differ from the theoretical ones by not more than 20%.

As already noted, in a weak field, and consequently also at the generation threshold, the influence of the amplitude fluctuations on the phase ones is small. As a result, even at the generation threshold, the phase shift

*An expression analogous to (3.29) was first obtained by Malakhov [63].

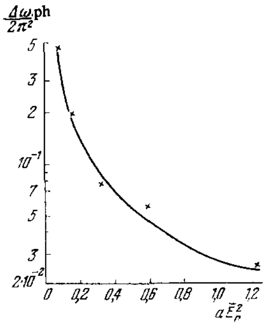

FIG. 5. Emission line width vs. signal power.

changes approximately in accordance with the diffusion law, with a diffusion coefficient

$$D = (\omega_0^2/\langle E^2\rangle)\,(\xi_{\rm ph}^2)_0. \tag{3.31}$$

The laser emission spectrum near the threshold is given approximately by formula (3.30). The width of the emission spectrum can be obtained by dividing the intensity of the spectral line by the spectral density at $\omega = \omega_0$. We then get from (3.30)

$$\Delta\omega = 4\,\frac{\langle E^2\rangle + \langle \delta E^2\rangle}{(E^2)_\omega} = \alpha\left(\frac{I}{N_s}\right) D, \tag{3.32}$$

where

$$I = \frac{2}{1+g}\left(1 - \frac{1}{Qd}\right),\quad \alpha\left(\frac{I}{N_s}\right) = 1 + \frac{\langle \delta E^2\rangle}{\langle E\rangle^2}.$$

In the case of a large excess over threshold we have $\alpha = 1$. At threshold, using (3.14), we get $\alpha = 1.18$. Below the generation threshold, $I < 0$ at $|I| \gg N_s$. From (3.32) we get

$$\alpha = 1 + \frac{4-\pi}{\pi} = \frac{4}{\pi}.$$

However, the approximations under which (3.32) was derived are no longer justified in this case. The result can be obtained directly from the equation for the field and takes the form

$$\Delta\omega_c = \omega_0\left(\frac{1}{Q} - d\right) = \Delta\omega_r\,|\eta|.$$

From a comparison with formulas (3.31) and (3.15) we see that $\Delta\omega_s = 2D$, i.e., $\alpha = 2$. The percentage values of α agree with those obtained in[18].

Let us estimate the emission line width at the generation threshold. From (3.32), (3.31), and (3.8) at zero detuning we obtain

$$\Delta\omega_c = \alpha\,\frac{\omega_0^2}{\langle E^2\rangle}\,(\xi_\Phi^2)_0 = \frac{\alpha}{2}\sqrt{\frac{\pi}{2}}\,\Delta\omega_p N_c.$$

At a resonator bandwidth $\Delta\omega_r/2\pi = 10^7$ and $N_s \sim 10^{-4}$ we obtain $\Delta\omega_s/2\pi \approx 740$ Hz.

This result agrees in order of magnitude with the spectrum-width measurements performed by Siegman and Arrathgon[5].

4. AMPLITUDE AND INTENSITY FLUCTUATIONS OF OPPOSING WAVES IN A RING LASER

There is only one known experimental investigation of ring lasers, that of Zaĭtsev[9], who measured the intensity fluctuations of each of the opposing waves and the corresponding correlation coefficients.

The character of the wave and fluctuation processes in a ring laser differs considerably from that in the linear laser. For example, at small deviations of the generation frequency from the center of the Doppler line, the regime of two opposing waves in a ring laser in a weak field turns out to be unstable and a transition to the regime with one traveling wave takes place[64]. In the transition regime, just as near the generation threshold, the fluctuations are not small. Consequently, it is again necessary to use the method of the Fokker-Planck equation to investigate the fluctuations.

In the correlation approximation, the equations for the amplitude fluctuations of the opposing waves follow from (2.10) and are given by

$$\frac{d\delta E_{1,2}}{dt} + A_{1,2}\delta E_{1,2} + B_{1,2}\delta E_{2,1} = \omega_0 \xi_{a1,2}(t); \tag{4.1}$$

here

$$A_{1,2} = 4\pi\omega_0\,\frac{\partial\varkappa''_{1,2}}{\partial E_{1,2}^2}\,E_{1,2}^2,\quad B_{1,2} = 4\pi\omega_0\,\frac{\partial\varkappa''_{1,2}}{\partial E_{2,1}^2}\,E_1E_2. \tag{4.2}$$

At equal values of the Q of opposing waves we get from (4.2) and (2.18)

$$E_1 = E_2 = E_0,$$

$$A_1 = A_2 = A = 2\omega_0\,df\left[(1+F)^2 + gF\left(F^2 + 4\frac{\mu^2}{\gamma_{ab}^2}f^2\right)\right]aE_0^2, \tag{4.3}$$

$$B_1 = B_2 = B = \omega_0\,df\left[f^2(1+F)^2 + gF\left(F^2 - 4\frac{\mu^2}{\gamma_{ab}^2}f^2\right)\right]aE_0^2. \tag{4.4}$$

From (4.1) we obtain the spectrum of the amplitude fluctuations for the opposing waves:

$$(\delta E_1\delta E_{1,2})_\omega = (\delta E_2\delta E_{2,1})_\omega = \frac{\omega_0^2}{2}\left[\frac{(\xi_{a1,2}^2)_0 + (\xi_{a1}\xi_{a2})_0}{\omega^2 + (A+B)^2} \pm \frac{(\xi_{a1,2}^2)_0 - (\xi_{a1}\xi_{a2})_0}{\omega^2 + (A-B)^2}\right], \tag{4.5}$$

$$(\delta E_1^2\delta E_{1,2}^2)_\omega = (\delta E_2^2\delta E_{2,1}^2)_\omega = 4E_0^2\,(\delta E_1\delta E_{1,2})_\omega. \tag{4.6}$$

Thus, the spectral densities of the amplitudes and of the intensities are sums of two Lorentz lines with widths A + B and A − B. Since the fluctuation sources are anticorrelated, i.e., $(\xi_{a1}\xi_{a2})_0 < 0$, the narrower line has a higher intensity. It follows from this, in particular, that the fluctuations of the opposing-wave amplitudes are always anticorrelated.

Formulas (4.5) and (4.6) lead to expressions for the variances of the amplitudes and intensities of the opposing waves, and also for the correlation function

$$\langle\delta(E_1^2)\,\delta(E_{1,2}^2)\rangle = \langle\delta(E_2^2)\,\delta(E_{2,1}^2)\rangle = 2\omega_0E_0^2\,\frac{(\xi_{a1}\xi_{a1,2})_0A - (\xi_{a1}\xi_{a2,1})_0B}{A^2 - B^2}. \tag{4.7}$$

From this we can obtain the condition for the applicability of the correlation approximation

$$\frac{\omega_0^2}{2E_0^2}\,\frac{(\xi_{a1,2}^2)_0A - (\xi_{a1}\xi_{a2})_0B}{A^2 - B^2} \ll 1. \tag{4.8}$$

At the stability limit of the two-wave regime we have A = B, and condition (4.8) is not satisfied.

It is shown in[64] that when account is taken of the spatial modulation of the populations at

$$aE_0^2 \gg \gamma_{ab}^2/(ku)^2 \tag{4.8$'$}$$

the region of instability of the two-wave regime vanishes, i.e., A > B in the entire range of detunings.

In a weak field we have

$$A = \Delta\omega_s\alpha aE_0^2,\qquad B = \Delta\omega_s\beta aE_0^2.$$

At not very small detunings, when $\mu > \gamma_{ab}$, the line has a near-Lorentz shape. This is confirmed by

Zaĭtsev's experimental data[9].

The relative variances of the amplitudes and intensities and the correlation coefficient of the opposing waves in a weak field are given by

$$\frac{\langle \delta E^2\rangle}{E^2}=\frac{N^2}{8(aE_0^2)^2}\frac{\alpha}{\alpha^2+\beta^2},\quad \frac{\langle \delta (E^2)^2\rangle}{E_0^4}=\frac{N^2}{2(aE_0^2)^2}\frac{\alpha}{\alpha^2-\beta^2},\tag{4.9}$$

$$\rho=\langle\delta(E_1^2)\,\delta(E_2^2)\rangle/\langle\delta(E_{1,2}^2)^2\rangle=-\beta/\alpha;\tag{4.10}$$

here

$$N=2\sqrt{(\xi_{a1,2}^2)_0\,\omega_0 a/d}.$$

Zaĭtsev[9] investigated experimentally the relative correlation of the intensity fluctuations at the frequency ω:

$$\rho_\omega=(\delta E_1^2\delta E_2^2)_\omega/(\delta E_{1,2}^2)_\omega.$$

A general expression for ρ_ω follows from (4.5). In a weak field

$$\rho_\omega=-\frac{2\alpha\beta}{\alpha^2+\beta^2}\frac{\Delta\omega_\rho^2}{\omega^2+\Delta\omega_\rho^2};\tag{4.11}$$

here

$$\Delta\omega_\rho=\Delta\omega_r aE_0^2\sqrt{\alpha^2+\beta^2},$$

$\Delta\omega_\rho$ is the width of the correlation-coefficient spectrum.

With increasing detuning, $\rho_{\omega=0}$ decreases monotonically from unity to zero. At $\mu=\gamma_{ab}$ and $\beta=0.5\alpha$ we have $\rho_{\omega=0}=-0.8$.

The width $\Delta\omega_\rho$ also decreases monotonically with increasing detuning, from $\Delta\omega_\rho=\Delta\omega_r aE^2/\sqrt{2}$ to $\Delta\omega_\rho=\Delta\omega_r aE^2/2$. Figure 6 shows the dependence of ρ_ω in accordance with formula (4.11) at $\mu=1.1\gamma_{ab}$ and $\Delta\omega_r=5.02\times10^4\ \text{sec}^{-1}$. The values of the parameters were obtained from the condition that the height ($\rho_\omega=0$) and the width ($\Delta\omega_\rho$) of the spectrum agree with experiment[9]. At high frequencies, the experimental points lie above the theoretical curve. This is due to the fact that in the derivation of (4.11) no account was taken of the contribution of the noise-source correlation $(\xi_{a1}\xi_{a2})_0$. At large ω, the value of ρ_ω tends not to zero but to $(\xi_{a1}\xi_{a2})_0/(\xi_{a1,2}^2)_0$. This makes it possible to determine from the values of ρ_ω, at large ω, the anti-correlation coefficient of the amplitude-fluctuation sources. In the case when the correlation approximation is not valid, i.e., condition (4.8) is not satisfied, the fluctuations are calculated by the Fokker-Planck method. The condition (4.8) is not satisfied in two cases, near the generation threshold ($E_0\to0$) and at the instability limit, when $A-B\to0$.

By virtue of the condition (4.8′), we can confine ourselves to the weak-field approximation. In this case Eqs. (2.10) take the form

$$\frac{dE_{1,2}}{dt}=\frac{\omega_0 d}{2}\left[(\alpha+\beta)I-\alpha aE_{1,2}^2-\beta aE_{2,1}^2\right]E_{1,2}+\omega_0\xi_{1,2};\tag{4.12}$$

here

$$I=\frac{1}{\alpha+\beta}\left(1-\frac{1}{dQ}\right).$$

The stationary solution of the Fokker-Planck equation, corresponding to the Langevin equations (4.12), is

$$W(E_1^2,E_2^2)=CE_1E_2\exp\left\{-\frac{\alpha}{N^2}\left[(aE_1^2-I)^2+(aE_2^2-I)^2+2\frac{\beta}{\alpha}(aE_1^2-I)(aE_2^2-I)\right]\right\}.\tag{4.13}$$

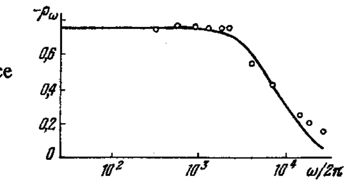

FIG. 6. Frequency dependence of the spectral density of the correlation coefficient.

The constant C is determined from the normalization condition.

Let us consider two most interesting cases.

1) Fluctuations at the generation threshold (I/N = 0) for arbitrary α and β:

$$C=8\frac{\sqrt{\alpha^2-\beta^2}}{N^2}\left(\operatorname{arctg}\frac{\sqrt{\alpha^2-\beta^2}}{\beta}\right)^{-1}a^2.\tag{4.14}$$

The corresponding expressions for the moments are

$$\langle E_{1,2}^n\rangle=Cn\frac{\Gamma(n/4)}{16(n+2)}\left(\frac{N}{a\sqrt{\alpha}}\right)^{\frac{n+4}{2}}F\left(\frac{n+2}{4};\frac12;\frac n4+\frac32;\frac{\alpha^2-\beta^2}{\alpha^2}\right),\tag{4.15}$$

$$\langle E_1^nE_2^m\rangle=C\frac{\sqrt{\pi}}{4}\frac{nm}{(n+m+2)}\left(\frac{4\alpha a^2}{N^2}\right)^{-\frac{n+m+4}{4}}\frac{\Gamma\left(\frac m2\right)\Gamma\left(\frac n2\right)}{\Gamma\left(\frac{n+m+2}{4}\right)}\times F\left(\frac{m+2}{4};\frac{n+2}{4};\frac{n+m}{4}+\frac32;\frac{\alpha^2-\beta^2}{\alpha^2}\right).\tag{4.16}$$

At the stability limit, when $\alpha-\beta=0$, we obtain from (4.15) and (4.16)

$$\langle E_{1,2}^2\rangle=\sqrt{\frac{\pi}{2}}\frac{N}{2a},\quad \langle E_1^2E_2^2\rangle=\frac13\frac{N^2}{a^2},\quad \langle(E_{1,2}^2)^2\rangle=\frac23\frac{N^2}{a^2}.\tag{4.17}$$

We present expressions for the relative variance of the intensities of the opposing waves and the correlation coefficient near the generation threshold:

$$\sigma^2=\frac{\langle(E_{1,2}^2)^2\rangle-\langle E_{1,2}^2\rangle^2}{\langle E_{1,2}^2\rangle^2}=\frac{128\alpha^2(\alpha+\beta)a^4}{\pi C^2N^4(\alpha-\beta)}\left(1-\frac{N^2\beta}{8\alpha^2a^2}C\right)-1,\tag{4.18}$$

$$\rho=\frac{\langle E_1^2E_2^2\rangle-\langle E_{1,2}^2\rangle^2}{\langle E_{1,2}^4\rangle-\langle E_{1,2}^2\rangle^2}=\frac{N^2C-8\beta a^2-\frac{\pi}{16\alpha a^2}C^2\frac{\alpha-\beta}{\alpha+\beta}N^4}{8\alpha a^2-\frac{\beta}{\alpha}CN^2-\frac{\pi}{16\alpha a^2}C^2\frac{\alpha-\beta}{\alpha+\beta}N^4}.\tag{4.19}$$

Hence

$\sigma^2=0{,}64,\ \rho=-0{,}19$ for $\mu=\gamma_{ab}$;

$\sigma^2=0{,}70,\ \rho=-0{,}22$ for $\mu=0$ $(\alpha=\beta=1/2)$.

Thus, near the threshold, the variances of the intensities of the opposing waves and the correlation coefficient depend little on the detuning. The anti-correlation between the opposing waves near the threshold is small.

2) At the stability limit ($\alpha=\beta$) we have

$$C=\frac{4a^2}{N^2}\left\{e^{-\frac{2I^2}{N^2}}+\sqrt{2\pi}\frac{I}{N}\left[1+\Phi\left(\frac{2I}{N}\right)\right]\right\}^{-1}.\tag{4.20}$$

The corresponding expressions for the moments are

$$\left.\begin{aligned}\langle E_{1,2}^2\rangle&=\frac{I}{a}+\frac{C}{8}\sqrt{\frac{\pi}{2}}\frac{N^3}{a^3}\left[1+\Phi\left(\frac{2I}{N}\right)\right],\\ \langle E_{1,2}^4\rangle&=\frac{N^2}{a^2}\left(1+\frac43\frac{I^2}{N^2}\right)-\frac{1}{12}\frac{N^4}{a^4}Ce^{-2I^2/N^2},\\ \langle E_1^2E_2^2\rangle&=\langle E_{1,2}^4\rangle/2.\end{aligned}\right\}\tag{4.21}$$

At I = 0, these expressions coincide with (4.17). Far from the generation threshold ($I/N\gg1$) the expressions (4.21) take the form

$$\langle E_{1,2}^2\rangle=\frac{I}{a},\quad \langle(E_{1,2}^2)^2\rangle=\frac{4}{3a^2}I^2,\quad \langle E_1^2E_2^2\rangle=\frac{2}{3a^2}I^2.\tag{4.22}$$

Hence

$$\sigma^2 = 1/3, \qquad \rho = -1,$$

i.e., the opposing waves are fully anti-correlated. These results coincide with those given by Smirnov and Zhelnov[40].

The amplitude and intensity fluctuations in the regime of one traveling wave are calculated in similar fashion. In a weak field, the dependence of the amplitude and intensity fluctuations on the field coincides with the case of two opposing waves. In a strong field, the relative variances of the amplitude and intensity decrease in inverse proportion to the square of the field, whereas in the case of two opposing waves these quantities tend to a constant value. This is connected with the different field dependences of the amplitude-noise sources.

For the width of the amplitude-fluctuation spectrum in the regime of one traveling wave we obtain the expression

$$\Delta\omega_a = \frac{1}{2}\Delta\omega_r \frac{I}{1+I}.$$

Near the threshold, the results coincide with those given above for the standing-wave regime in a linear laser, the only difference being that N_s in (3.14) and (3.15) is replaced by N.

Allowance for the rotation of the ring laser and the coupling of the opposing waves through scattering by the mirrors leads to the appearance of additional terms in the equations for the amplitudes and phases of the opposing waves (2.10) and (2.11):

$$\frac{dE_{1,2}}{dt} = -\frac{\omega_0}{2}\left(4\pi\varkappa''_{1,2}+\frac{1}{Q}\right)E_{1,2} \mp \frac{\omega_0 d}{2}|m_{1,2}|E_{2,1}\sin(\Phi+\vartheta_{1,2})+\omega_0\xi_{a1,2},$$

$$\frac{d\varphi}{dt} = \pm\frac{\Omega}{2}-\frac{\omega_0}{2}4\pi\varkappa'_{1,2}-\frac{\omega_0 d}{2}|m_{1,2}|\frac{E_{2,1}}{E_{1,2}}\cos(\Phi+\vartheta_{1,2})+\frac{\omega_0}{E_{1,2}}\xi_{\mathrm{ph}\,1,2}.$$

The fluctuations were calculated with allowance for the coupling in[40,27]. It is shown in[27] that in the synchronization region, allowance for the coupling does not lead to a noticeable change in the spectrum of the amplitude fluctuations at $\omega > \sqrt{\Omega_0^2-\Omega^2}$. Calculation of the form of the spectrum of the amplitude fluctuations at $\omega < \sqrt{\Omega_0^2-\Omega^2}$ is of no interest, since technical fluctuations are quite significant in this frequency region.

The transitions between different operating modes of a ring laser under the influence of the fluctuations were calculated in[40] for a laser at rest.

5. FREQUENCY AND PHASE FLUCTUATIONS IN A RING LASER

We present the results of the calculation of the fluctuations for the region far from the generation threshold, when the correlation approximation can be used. Equations (2.11) lead to equations for the phase fluctuations of the opposing waves:

$$\frac{d\delta\varphi_{1,2}}{dt} = \frac{\omega_0}{E_0}(C\delta E_{1,2}+D\delta E_{2,1}+\xi_{\mathrm{ph}\,1,2}); \tag{5.1}$$

here

$$C = -4\pi\frac{\partial\varkappa'_{1,2}}{\partial E^2_{1,2}}E_0^2 = \frac{1}{2}\frac{\mu}{\gamma_{ab}}df\,[gF^3-f^2(3-2F(1-2g)-F^2)]\,aE_0^2,$$

$$D = -4\pi\frac{\partial\varkappa'_{1,2}}{\partial E^2_{2,1}}E_0^2 = \frac{1}{2}\frac{\mu}{\gamma_{ab}}df\,\{gF^3+f^2[1-2F(1-2g)+F^2]\}\,aE_0^2. \tag{5.2}$$

From this we obtain expressions[27] for the line width of each of the opposing waves and the line width of the beat signal $E_0\cos[(\varphi_1-\varphi_2)/2+\varphi_0]$:

$$\Delta\omega_{1,2} = (\delta\dot\varphi^2_{1,2})_{\omega=0} = \frac{\omega_0^2}{E_0^2}[(C^2+D^2)(\delta E^2_{1,2})_0+2CD(\delta E_1\delta E_2)_0+(\xi^2_{\mathrm{ph}\,1,2})_0], \tag{5.3}$$

$$\Delta\omega_\Phi = \frac{(\delta\dot\Phi^2)_0}{4} = \frac{\omega_0^2}{2E^2}[(C-D)^2[(\delta E^2_{1,2})_0-(\delta E_1\delta E_2)_0]+(\xi^2_{\mathrm{ph}\,1,2})_0-(\xi_{\mathrm{ph}1}\xi_{\mathrm{ph}2})_0]. \tag{5.4}$$

For a weak field ($aE_0^2 \ll 1$) we obtain from (5.3) and (5.4)

$$\Delta\omega_{1,2} = \frac{\Delta\omega_r N^2}{4aE_0^2}\left[1+\frac{b^2(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)^2}\right], \tag{5.5}$$

$$\Delta\omega_\Phi = \frac{\Delta\omega_r N^2}{8aE_0^2}\left[1+\frac{b^2}{(a-\beta)^2}\right]. \tag{5.6}$$

The second terms in the square brackets determine the contribution of the amplitude fluctuations. We see that when the stability region is approached, as $\alpha \to \beta$, the role of the amplitude fluctuations (according to the formulas of the correlation approximation) increases. However, on the boundary of the instability region the contribution of the amplitude fluctuations is of the order of b^2, and is consequently small, since $b^2 \ll 1$. The influence of the amplitude fluctuations was taken into account in[41], the results of which coincide with formula (5.6) without the first term.

For the regime of one traveling wave we have

$$\Delta\omega = (\omega_0^2/E_0^2)(\xi^2_\Phi)_0. \tag{5.7}$$

From this we obtain for a weak field

$$\Delta\omega = \Delta\omega_r N^2/4aE_0^2. \tag{5.8}$$

The corresponding expression for a linear laser differs from (5.8) only in the dependence on the frequency deviation.

When the coupling is taken into account, an additional term $M_{1,2}\delta\Phi$ appears in (5.1); here

$$M_{1,2} = \frac{\omega_0 d}{2}\left\{-|m_{1,2}|\sin(\Phi+\vartheta_{1,2}) \pm \frac{\omega_0}{A^2-B^2}[(CA-DB)\right.$$
$$\left.\times|m_{1,2}|\cos(\Phi+\vartheta_{1,2})+(CB-DA)|m_{2,1}|\cos(\Phi+\vartheta_{1,2})]\right\}. \tag{5.9}$$

Accordingly, the equation for the phase-difference fluctuations contains a term $M\delta\Phi$, where

$$M = M_1 - M_2 = \sqrt{\Omega_0^2-\Omega^2}. \tag{5.10}$$

The corresponding expressions for the spectral densities of the frequency fluctuations of the opposing waves and the beat frequency are[27]

$$(\delta\dot\varphi^2_{1,2})_\omega = (\delta\dot\varphi^2_{1,2})^{(0)}_\omega + \frac{M_1M_2}{M^2+\omega^2}(\delta\dot\Phi^2)^{(0)}_\omega,$$

$$(\delta\dot\Phi^2)_\omega = \frac{\omega^2}{M^2+\omega^2}(\delta\dot\Phi^2)^{(0)}_\omega. \tag{5.11}$$

The superscript (0) marks expressions without allowance for the coupling.

It follows from (5.11) that the spectral densities of the fluctuations of the opposing-wave frequencies and the beat frequency depend strongly on the magnitudes and phases of the coupling coefficients, and via M also on the position of the synchronization region inside the band (see (5.10)), i.e., on the speed of laser rotation. On the boundary of the synchronization band, when M = 0, the spectral density of the beat-frequency fluctuations does not depend on the coupling. Using the identity

$$M_1M_2 = \frac{(M_1+M_2)^2}{4} - \frac{M^2}{4},$$

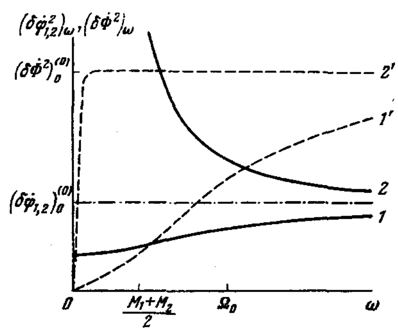

FIG. 7. Dependence of the spectral density of the frequency fluctuations with allowance for the coupling via back scattering.

we see that $M_1M_2 > 0$ at $M = 0$. Consequently the spectral density of the frequency fluctuations of each of the opposing waves increases as a result of the coupling, and tends to infinity as $\omega \to 0$.

Inside the synchronization band, when $M \neq 0$, the spectral density of the beat-frequency fluctuations tends to zero as $\omega \to 0$.

At the center of the synchronization band, at equal moduli of the coupling coefficients we have $M_1M_2 = -M^2/4$. Consequently, the second term in the first formula of (5.11) is negative. At $\omega = 0$ we get from (5.11)

$$(\delta\dot{\varphi}^2_{1,2})_0 = (\delta\dot{\varphi}^2_{1,2})^{(0)}_0 - \frac{1}{4}(\delta\dot{\Phi}^2)^{(0)}_0.$$

Plots of the spectral densities of the frequency fluctuations of the opposing waves and of the beat frequencies, calculated in accordance with formulas (5.11), are shown in Fig. 7.

In the presence of coupling, the mean-squared phase shift of the opposing waves and the phase differences do not vary in accordance with the diffusion law. Calculation yields

$$\langle\delta\Phi^2_\tau\rangle = \frac{(\delta\dot{\Phi}^2)^{(0)}_0}{M}(1 - e^{-M|\tau|}),$$

$$\langle\delta\varphi^2_{1,2\tau}\rangle = (\delta\dot{\varphi}^2_{1,2})^{(0)}_0|\tau| + \frac{M_1M_2}{M_3}(\delta\dot{\Phi}^2)^{(0)}_0(M|\tau| - 1 + e^{-M|\tau|}).$$

Accordingly, the signal spectrum is not of the Lorentz type but is determined by a more complicated expression. Without allowance for the amplitude fluctuations, we obtain for the spectral density of the beat signal

$$(S^2)_\omega = E_0^2 e^{-(\delta\dot{\Phi}^2)^{(0)}_0/8M} \lim_{\alpha\to 0} \mathrm{Re}\left[\frac{1}{\alpha - i\omega}\, {}_1F_1\left(\frac{\alpha - i\omega}{M};\ \frac{\alpha - i\omega}{M} + 1;\ \frac{(\delta\dot{\Phi}^2)^{(0)}_0}{8M}\right)\right]. \quad (5.12)$$

The spectral density of the radiation of each of the opposing waves is

$$(E^2_{1,2})_\omega = E_0^2 \exp\left[\frac{M_1M_2(\delta\dot{\Phi}^2)^{(0)}_0}{2M^3}\right] \mathrm{Re}\left[\frac{1}{(D_\varphi/2) - i(\omega - \omega_0)}\right.$$
$$\left.\times\, {}_1F_1\left(\frac{(D_\varphi/2) - i(\omega - \omega_0)}{M};\ \frac{(D_\varphi/2) - i(\omega - \omega_0)}{M} + 1;\ -\frac{M_1M_2(\delta\dot{\Phi}^2)^{(0)}_0}{2M^3}\right)\right]. \quad (5.13)$$

Near the boundary of the synchronization band, $M \ll (\delta\dot{\Phi})^{(0)}_0$ and the obtained line shape is the same as in the absence of coupling.

Near the center of the synchronization band, $M \approx \Omega_0$ and usually $(\delta\dot{\Phi})^{(0)}_0/M \ll 1$. In this case it follows from (5.12) that

$$(S^2)_\omega = \frac{E_0^2}{2} e^{-(\delta\dot{\Phi}^2)^{(0)}_0/8M} \times \left[\delta(\omega) + \frac{1}{4}\frac{(\delta\dot{\Phi}^2)^{(0)}_0}{\omega^2 + M^2}\right]. \quad (5.14)$$

The beat signal is thus a superposition of a dc component and a noise background. The intensity $(\delta\dot{\Phi}^2)_0/M$ of the latter is much smaller than the intensity of the dc component.

From (5.13) we obtain an approximate expression for the form of the emission spectrum of each of the opposing waves:

$$(E^2_{1,2})_\omega = \frac{E_0^2}{2} e^{M_1M_2(\delta\dot{\Phi}^2)^{(0)}_0/2M^3}\left\{\frac{(\delta\dot{\varphi}^2)_0}{(\omega - \omega_0)^2 + [(\delta\dot{\varphi}^2)^2_0/4]} - \frac{M_1M_2(\delta\dot{\Phi}^2)^{(0)}_0[(\delta\dot{\varphi}^2)_0 + 2M]}{2M^3[(\omega - \omega_0)^2 + ([(\delta\dot{\varphi}^2)_0/2] + M)^2]}\right\}.$$

It follows therefore that the form of the spectrum of each of the opposing waves near the center of the synchronization band, without allowance for the amplitude fluctuations, is a sum of Lorentz lines, a narrow and intense one of width $\Delta\omega_{1,2} = (\delta\dot{\varphi}^2_{1,2})_0$ and a broad one of width $(\delta\dot{\varphi}^2_{1,2})_0 + 2M$.

It follows from the foregoing that to measure the line width near the boundary of the synchronization region it suffices to measure the spectral density of the frequency fluctuations at frequencies much higher than Ω_0 (on the order of 10^3 sec^{-1}), and consequently one goes beyond the limits of the region of technical fluctuations (of the order of $10^4 - 10^5$ sec^{-1}). To determine the line width near the center of the synchronization band it is necessary to measure the spectral density of the frequency fluctuations at frequencies much smaller than Ω_0. Such measurements cannot be carried out directly, owing to the technical fluctuations.

An exact calculation of the radiation in the case when the beat frequency exceeds the width of the synchronization band entails great mathematical difficulties and has not yet been carried out. It is to be expected, however, that with increasing distance from the synchronization region the role of the coupling of the opposing waves will weaken and the results will agree with those obtained without allowance for the coupling.

6. MAXIMUM SENSITIVITY OF LASER GYROSCOPE

The question of the maximum sensitivity of an ideal laser gyroscope (without allowance for the coupling between the opposing waves through scattering) limited by the natural fluctuations of the radiation, was apparently first considered by Brunnet[54]. He stated that the minimum measureable frequency difference between the opposing waves is determined, without allowance for synchronization, by the natural line width. According to Brunnet's estimates, this minimal frequency distance corresponds to a laser rotary speed on the order of 0.1 deg/hr. However, as correctly noted by Rozanov[58], the maximum sensitivity of a laser gyroscope is determined not only by the natural line width but also by the measurement time.

It is shown in[59] that the average spread of the beat frequency far from the region of synchronization is determined by the formula

$$\delta\Omega = \frac{\overline{\langle(\dot{\Phi} - \overline{\langle\dot{\Phi}\rangle})^2\rangle}^{1/2}}{2} = \sqrt{\frac{\Delta\omega_{\mathrm{ph}}}{T}}.$$

The bar denotes averaging over the observation time T, and $\Delta\omega_{\mathrm{ph}}$ is the width of the beat signal line. Far from threshold it is determined by formulas (5.4) and (5.6). At $\Delta\omega_{\mathrm{ph}}/2\pi = 10^{-2}$ Hz (which corresponds to a power

$P = 0.25$ mW at $\Delta\omega_r/2\pi = 10^6$ Hz, $1 + (R^0/D^0) = 5$ and $\mu = \gamma_{ab}$) and $T = 10^2$ sec we have

$$\delta\Omega/2\pi = 10^{-2}/\sqrt{2\pi}\ \text{Hz}.$$

The corresponding minimum laser rotation speed is determined by the expression

$$\delta\dot{\theta} = \delta\Omega cL/2\omega_0 S,$$

where L is the laser perimeter, S the area of the contour, ω_0 is the oscillation frequency. Assuming $L = 40$ cm, $S = 100$ cm^2, and $\omega_0 = 3 \times 10^{15}$ rad/sec, we obtain $\delta\dot{\theta} = 10^{-2}$ deg/hr.

In[58,59] they investigated also the limiting sensitivity of a laser gyroscope with the speed of rotation measured within the synchronization region by a phase method. The following expression was obtained for the phase-method error, which is characterized by the variance of the phase difference:

$$M\langle\delta\Phi^2\rangle^{1/2} = \sqrt{\frac{\Delta\omega_{ph}}{2T}\left[1 - \frac{1}{MT}(1 - e^{-MT})\right]}.$$

At $T \gg 1/M$, the limiting sensitivity of the phase method coincides in order of magnitude with the limiting sensitivity of the frequency method.

On the other hand, if $T \ll 1/M$, then

$$M\langle\delta\Phi^2\rangle^{1/2} = \sqrt{D_{ph}M}$$

and consequently at small measurement times the limiting sensitivity of the phase method turns out to be higher than that of the frequency method. This result is physically obvious, for at short measurement times the averaging in the phase method is performed by the system itself within a time on the order of $1/M$, whereas there is no such averaging in the measurement by the frequency method.

7. AMPLITUDE AND PHASE FLUCTUATIONS IN A SOLID-STATE LASER

It was already noted in Chap. 2 that in a solid-state laser the relation between the dissipative parameters γ_a, γ_b, γ_{ab}, and $\Delta\omega_r$ is different than in a gas laser. In this case γ_a, $\gamma_b \ll \Delta\omega_r$, so that one cannot assume that the level populations "follow" the field as in a gas laser. Consequently, in the calculation of the population fluctuations, meaning also the polarization fluctuations, the field cannot be regarded as determined and the term $\delta P^{(ind)}$ can not be left out from formula (2.8). Thus, the induced part of the polarization is given by

$$P^{(ind)} = \langle P\rangle + \delta P^{(ind)}. \tag{7.1}$$

We shall present results for two regimes, standing wave and traveling wave. When (7.1) is taken into account, Eqs. (2.10) and (2.11) take the form

$$\frac{dE_{1,2}}{dt} = -\frac{\Delta\omega_r}{2}E_{1,2} - 2\pi\omega_0\varkappa''(\langle E_1\rangle, \langle E_2\rangle)\langle E_{1,2}\rangle + \omega_0\zeta_{a1,2}, \tag{7.2}$$

$$\frac{d\varphi_{1,2}}{dt} = -\frac{2\pi\omega_0}{E_{1,2}}\varkappa'(\langle E_1\rangle, \langle E_2\rangle)\langle E_{1,2}\rangle + \frac{\omega_0}{E_{1,2}}\zeta_{\Phi 1,2}. \tag{7.3}$$

The noise sources differ from ξ_a and ξ_{ph} in (2.10) and (2.11) by the additional terms

$$\zeta_{a1,2} = \xi_{a1,2} + \xi_{a1,2}^{(ind)}, \tag{7.4}$$

$$\zeta_{\Phi 1,2} = \xi_{\Phi 1,2} + \xi_{\Phi 1,2}^{(ind)}. \tag{7.5}$$

The noise-source intensities $\xi_{ph1,2}$ and $\xi_{a1,2}$ for the traveling-wave and standing-wave regimes were calculated by us earlier (see (2.28), (2.30), (2.15)).

We therefore need to consider here only the induced fluctuations. For the induced parts of the fluctuations of the density matrix elements with the field given in the form (2.9) we obtain a linearized system of equations in which we can no longer neglect the field fluctuations[29].

Recognizing that the relation $\gamma_{ab} \gg \Delta\omega_r$, γ_a, γ_b is satisfied in a solid-state laser and assuming for simplicity $\gamma_a = \gamma_b$, we obtain in the two cases the following expressions for the Fourier components of the induced-fluctuation sources:

1) Traveling-wave regime:

$$\omega_0(\xi_a^{(ind)})_\omega = \frac{\Delta\omega_r}{2}\,\frac{i\omega + \gamma_a(1 - gaE_0^2)}{i\omega + \gamma_a(1 + gaE_0^2)}(\delta E)_\omega, \tag{7.6}$$

$$(\xi_{ph}^{(ind)})_\omega = \frac{\mu}{\gamma_{ab}}(\xi_a^{(ind)})_\omega. \tag{7.7}$$

2) Standing-wave regime:

$$\omega_0(\xi_a^{(ind)})_\omega = \frac{\Delta\omega_r}{2}\left\{1 - \frac{1}{1 - (1/\sqrt{1 + 4gaE_0^2})} \times\left[1 - \frac{1}{i\omega}\,\frac{i\omega + \gamma_a}{\sqrt{1 + [4\gamma_a gaE_0^2/(i\omega + \gamma_a)]}} - \frac{\gamma_a}{\sqrt{1 + 4gaE_0^2}}\right]\right\}(\delta E)_\omega, \tag{7.8}$$

$$(\xi_{ph}^{(ind)})_\omega = \frac{\mu}{\gamma_{ab}}(\xi_a^{(ind)})_\omega. \tag{7.9}$$

The amplitude and phase fluctuations of a solid-state laser will also be considered for two regimes.

7.1. Traveling-wave regime. Substituting the noise sources in (7.2) and taking (2.28) into account, we obtain an expression for the spectral density of the field amplitude fluctuations[29]:

$$(\delta E^2)_\omega = \frac{\omega^2 + \gamma_a^2(1 + gaE_0^2)^2}{(\omega^2 - \Delta\omega_r\gamma_a gaE_0^2)^2 + \omega^2\gamma_a^2(1 + gaE_0^2)^2}\,\omega_0^2(\xi_a^2)_0. \tag{7.10}$$

The curve of the spectral density of the field amplitude fluctuations can be approximately regarded as a sum of two lines: a broad line

$$(\delta E^2)_\omega = \frac{\omega_0^2(\xi_a^2)_0}{\omega^2 + [\Delta\omega_r gaE_0^2/(1 + gaE_0^2)]^2} \tag{7.11}$$

and a narrow peak at the frequency

$$\omega_{max} = \sqrt{\Delta\omega_r\gamma_a gaE_0^2 - \gamma_a^2(1 + gaE_0^2)^2} \tag{7.12}$$

(when $\Delta\omega_r \gg \gamma_a$ we have $\omega_{max} \approx \sqrt{\Delta\omega_r\gamma_a gaE_0^2}$). The width of the peak at half-height is $\Delta\omega = \gamma_a(1 + gaE_0^2)$, and the spectral density of the field fluctuations at the maximum is

$$(\delta E^2)_{\omega=\omega_{max}} = \omega_0(\xi_a^2)_0/\gamma_a^2(1 + gaE^2). \tag{7.13}$$

It is seen from (7.12) that for a peak to exist on the amplitude-fluctuation spectral-density curve it is necessary to satisfy the condition

$$\gamma_a/\Delta\omega_r < gaE_0^2 < \Delta\omega_r/\gamma_a. \tag{7.14}$$

This condition is practically always satisfied in a solid-state laser. The appearance of a peak in the spectrum of the amplitude fluctuations of solid-state lasers is due to the large inertia of the working-level populations. In gas laser $\gamma_a \sim \gamma_{ab} \gg \Delta\omega_r$ and the amplitude of the radiation field, at small deviations from the stationary state, approaches the stationary state aperiodically. In solid-state lasers, since $\gamma_a \ll \Delta\omega_r \ll \gamma_{ab}$, the approach to the stationary state is oscillatory with frequency ω_{max}. Analogous phenomena are observed also in vacuum-tube oscillators with inertial non-linearity[61].

The existence of a peak of this type in the emission of lasers was indicated in[20,29,51,62]. It was observed experimentally in[52].

Integrating $(\delta E^2)_\omega$ with respect to the frequencies, we obtain an expression for the field-amplitude variance

$$\langle\delta E^2\rangle = \frac{2\pi\hbar\omega_0}{V}\left(\frac{1+gaE_0^2}{gaE_0^2}+\frac{\Delta\omega_r}{\gamma_a}\frac{1}{1+gaE_0^2}\right)\left(\bar{n}+\frac{1}{2}+\frac{1}{2}\frac{R^0}{D^0}\right). \quad (7.15)$$

For the variance of the number of photons we have accordingly

$$\langle\delta n^2\rangle = \langle n\rangle\left(\frac{1+\alpha\langle n\rangle}{\alpha\langle n\rangle}+\frac{\Delta\omega_p}{\gamma_a}\frac{1}{1+\alpha\langle n\rangle}\right)\left(\bar{n}+\frac{1}{2}+\frac{1}{2}\frac{R^0}{D^0}\right), \quad (7.16)$$

where

$$\alpha = \frac{8\pi\hbar\omega_0}{V}a.$$

This result coincides with that given by Lax[20].

We consider now the phase fluctuations in a solid-state laser. From (7.3) we obtain for the phase fluctuations the equation

$$\frac{d\delta\varphi}{dt} = -\frac{\omega_0}{E_0}\left[\xi_{ph}^{(ind)}+\xi_{ph}^{(sp)}-\varkappa'\delta E\right]. \quad (7.17)$$

Using (7.9), (2.28), and (7.12) we obtain for the spectral density of the frequency fluctuations the expression

$$(\delta\dot{\varphi}^2)_\omega = \frac{\omega_0^2}{E_0^2}\left[(\xi_\varphi^2)_0+\frac{\mu^2}{\gamma_{ab}^2}\frac{(\Delta\omega_r\gamma_a gaE_0^2)^2}{(\omega^2-\Delta\omega_r\gamma_a gaE_0^2)^2+\omega^2\gamma_a^2(1+gaE_0^2)^2}(\xi_a^2)_0\right]. \quad (7.18)$$

This yields for the emission line width

$$\Delta\omega = (\delta\dot{\varphi}^2)_0 = \frac{\omega_0^2}{E_0^2}\left[(\xi_\varphi^2)_0+\frac{\mu^2}{\gamma_{ab}^2}(\xi_a^2)_0\right].$$

This expression coincides with that obtained in[20,37]. The spectral density of the frequency (7.18) was obtained in[53]. It follows from this formula that when $\mu \neq 0$ the amplitude fluctuations on the frequency-fluctuation spectral density curve give rise to a peak at the frequency ω_{max}.

On the basis of (7.18) we can find the form of the emission spectral line, in analogy with the procedure used for the gas laser. The difference lies in the fact that the broad line due to the amplitude fluctuations has two symmetrically located small peaks of width γ_a.

7.2. Standing-wave regime. In this case the calculation of the fluctuations must be carried out with allowance for the spatial modulation of the populations. In the general case the result is quite complicated[47], and we present therefore only the result for the case of a weak field, from which we can assess the role of the population-difference modulation.

In a weak field we have for the spectral density of the amplitude fluctuations

$$(\delta E^2)_\omega = \frac{(\omega^2+\gamma_a^2)\,\omega_0^2\,(\xi_a^2)_0}{(\omega^2-3\Delta\omega_r\gamma_a aE_0^2)+\omega^2\gamma_a^2}.$$

If no account is taken of the modulation, the number 3 in the denominator is replaced by 2. This leads to a corresponding change in the value of the spectral density at $\omega = 0$ and in the value and position of the maximum. The reason is that when allowance is made for spatial modulation of the populations the laser radiation intensity depends somewhat differently on the excess of the pump level over threshold than when allowance is made for this modulation[47]. The dependence of the amplitude-fluctuation spectrum on the excess of pump over threshold is the same for both cases.

8. CONCLUSION

As already indicated in the Introduction, natural fluctuations of laser radiation can be separated by spectral methods against the background of slower technical fluctuations. We have presented in the review the results of a calculation of the natural fluctuations for different emission regimes of gas and solid-state lasers at practically arbitrary ratios of pump to threshold. The main results consist in the following. We have obtained the spectral characteristics of the laser emission. In particular, we have determined the form of the spectrum and the emission line width. In the simplest case, just as in a vacuum tube oscillator, the emission spectrum is a superposition of two lines. One, narrow and intense, is due to the fluctuations of the emission frequency. Its width, depending on the radiation power, is of the order of 10^{-1}–10^{-3} Hz. The second line, broad and weak, is due to fluctuations of the emission amplitude. Its width is of the order of 10^5–10^7 Hz. The width of the narrow line (at a specified observation time) determines the limiting sensitivity of the laser gyroscopes. The amplitude and frequency fluctuations of opposing waves in a ring laser have been considered in detail. The fluctuations in gas and solid-state lasers were calculated by different methods.

For an He-Ne gas laser the polarization has time to follow the field (the inequalities $\gamma_a \sim \gamma_b \sim \gamma_{ab} \gg \Delta\omega_r$) are satisfied, so that the problem of calculating the natural fluctuations of the laser radiation can be reduced to a solution of a system of equations for the field amplitudes and phases with random sources. The main task is to determine the statistical characteristics of the random forces. The spectral densities of the fluctuation sources can be represented in the form of a sum of two parts. One of them is due to the equilibrium fluctuations of the field in the resonator, and the other to the medium-polarization fluctuations connected with the spontaneous emission. The equilibrium field fluctuations are determined on the basis of the Kallen-Welton formula. The polarization fluctuations are non-equilibrium. Their spectral density is indeed one of the main problems in the calculation of natural fluctuations in a laser. Since the number of photons in the resonator in the generated mode is large ($\sim 10^3$) even at the very threshold of the generation, we can use the classical field equations. The quantum character of the radiation of the atoms of the working medium is accounted for through the spectral densities of the fluctuation sources and through the polarizability of the medium. Thus, this theory is semi-phenomenological.

However, calculations based on the semi-classical equations are simpler and therefore make it possible to consider more complicated cases. We note also that in quantum theory, as a rule, one calculates only the moments of the numbers of photons. The calculation of the spectral characteristics of the fluctuations within the framework of the quantum theory is a complicated problem, but its solution on the basis of the semi-classical theory entails no difficulty. The use of the semi-classical theory makes it possible to calculate fluctuations in lasers by the statistical methods used in the theory of radio oscillators.

The spectral densities of the amplitude and phase fluctuations can be obtained at not too high an excess above threshold by using correlation theory, and near the generation threshold by using the Fokker-Planck equation. In the case of a ring gas laser, there are two regions where it is necessary to employ the method of the Fokker-Planck equation in the investigation of the amplitude and intensity fluctuations of the opposing waves. These are the region near the generation threshold and the region of small deviations of the generation frequency from the center of the Doppler line, where the two-wave regime becomes unstable.

When considering the frequency and phase fluctuations in a ring laser, it is necessary to take into account the coupling between the opposing waves. This coupling determines the synchronization region in which the frequencies of the opposing waves become locked.

In the case of a solid-state laser, in view of the fact that the polarization of the medium assumes its steady-state more slowly than the field amplitude, it is necessary to take into account also induced fluctuations, namely polarization fluctuations due to the field fluctuations. The statistical characteristics of the radiation are then significantly altered.

The method developed above for calculating natural fluctuations in lasers is sufficiently general and can be used successfully to calculate non-equilibrium fluctuations in other systems.

APPENDIX 1

We denote by q the aggregate of variables of the atom (quantum numbers n and coordinates r and p) and by $\rho(q, q', t) = \psi^{+}(q, t)\psi(q', t)$ the operator density matrix. Here $\psi^{+}(q', t)$ and $\psi(q, t)$ are quantum wave functions satisfying known commutation relations. Using these commutation relations, we can write[56]

$$\begin{aligned} N^2\langle\rho(q_1, q_1', t)\,\rho(q_2, q_2', t)\rangle &= N^2\langle\psi^{+}(q_1', t)\,\psi(q_1, t)\,\psi^{+}(q_2', t)\,\psi(q_2, t)\rangle \\ &= N(N-1)\,F_2(q_1, q_1', q_2, q_2', t) + N\delta(q_1-q_2')\langle\rho(q_2, q_1', t)\rangle; \end{aligned} \tag{1}$$

we have introduced here the symbol for the second distribution function

$$F_2 = \langle\psi^{+}(q_1', t)\,\psi^{+}(q_2', t)\,\psi(q_1, t)\,\psi(q_2, t)\rangle. \tag{2}$$

The function F_2 is connected with the correlation function G by the relation

$$F_2 = \langle\rho(q_1, q_1', t)\rangle\langle\rho(q_2, q_2', t)\rangle + G(q_1, q_1', q_2, q_2', t). \tag{3}$$

The spontaneous fluctuations are calculated in an approximation in which the correlation function is neglected.

From (1) and (2) at G = 0, after symmetrizing with respect to the particles, we obtain for the deviations $\delta\rho = \rho - \langle\rho\rangle$ the expression

$$\begin{aligned} \langle\delta\rho(q_1, q_1', t)\,\delta\rho(q_2, q_2', t)\rangle &= \frac{1}{2N}[\delta(q_1-q_2')\langle\rho(q_2, q_1', t)\rangle \\ &+ \delta(q_2-q_1')\langle\rho(q_1, q_2', t)\rangle - 2\langle\rho(q_1, q_1', t)\rangle\langle\rho(q_2, q_2', t)\rangle]. \end{aligned} \tag{4}$$

If the motion of the mass centers of the atoms is described classically, then we can change over from the functions $\langle\rho(q', q, t)\rangle$ to the functions $\rho_{nm}(r, p, t)$, where r and p are the coordinates and momentum of the mass center. As a result, expression (4) becomes

$$\begin{aligned} &\langle\delta\rho_{nm}(r, p, t)\,\delta\rho_{n'm'}(r', p', t)\rangle \\ &= \frac{1}{2N}\{\delta(r-r')\,\delta(p-p')\,[\delta_{mn'}\rho_{nm'}(r, p, t) + \delta_{nm'}\rho_{n'm}(r, p, t)] \\ &\qquad - 2\rho_{nm}(r, p, t)\,\rho_{n'm'}(r', p', t)\}. \end{aligned} \tag{5}$$

Thus, expression (5), which is used in the text to calculate the spectral densities of the amplitude and phase fluctuation sources, holds for G = 0. By the same token, the fluctuation sources are determined by the motions of the individual atoms. The correlations themselves are expressed in terms of the fields produced by these sources. This question is discussed in greater detail in[28]. The last term of the right-hand side of (5) is important only when $\rho_a + \rho_b \approx 1$[30]. Allowance for this term in a two-level laser scheme leads to the correction obtained by Kazantsev and Surdutovich to the formula for the variance of the number of photons[37].

APPENDIX 2

We present an expression for the spectral densities of the polarization noise at zero frequency for the case when the amplitudes of the opposing waves differ little, i.e., when $(E_1 - E_2)/(E_1 + E_2) \ll 1$. At $E_1 = E_2 = E$ the following expressions were obtained in[26b]:

$$(\xi_{a1,2}^{(\pi)2})_0 = \frac{2\pi\hbar\,\Delta\omega_p}{V\omega_0}\left\{\frac{R^0}{D^0}\left(1+\frac{gFaE_0^2}{1+F}\right)-\frac{\gamma_-}{\gamma_+}\frac{gFaE_0^2}{1+F}\right\}, \tag{6}$$

$$(\xi_{\phi 1,2}^{(\pi)2}) = \frac{2\pi\hbar\,\Delta\omega_p}{V\omega_0}\left\{\frac{R^0}{D^0}\left(1+aE_0^2-\frac{gFaE_0^2}{1+F}\right)-\frac{\gamma_-}{\gamma_+}\frac{gFaE_0^2}{1+F}\left(1+\frac{\mu^2}{\gamma_{ab}^2}f_1\right)\right\}.$$

The functions f_1 and f_2 are determined by the expressions

$$f_1 = f^2\left\{2\left(\frac{\mu^2}{\gamma_{ab}^2}-1\right)(F^2-1)-\frac{3}{gF}(F^2-1)-2F^2aE_0^2\right\},$$

$$\begin{aligned} f_2 = \frac{1}{1+F}-(1-g)\frac{F}{1+F}+\frac{2f^2g^2F^3aE_0^2}{1+F}\Big[\left(\frac{\mu^2}{\gamma_{ab}^2}-1\right)(1+2aE_0^2) \\ -2\frac{\mu^4}{\gamma_{ab}^4}+\frac{(2\mu^2/\gamma_{ab}^2)-(1+2aE_0^2)}{gF}\Big]. \end{aligned}$$

In a strong field, the spectral densities of the amplitude and phase correlations of the opposing waves also differ from zero. They are determined by the expressions

$$(\xi_{a1}^{(\pi)}\xi_{a2}^{(\pi)})_0 = -\frac{2\pi\hbar\,\Delta\omega_p}{V\omega_0}\frac{gFaE_0^2}{1+F}\left(\frac{R^0}{D^0}-\frac{\gamma_-}{\gamma_+}\right),$$

$$(\xi_{\phi 1}^{(\pi)}\xi_{\phi 2}^{(\pi)})_0 = -\frac{2\pi\hbar\,\Delta\omega_p}{V\omega_0}aE_0^2\left\{\frac{R^0}{D^0}\frac{1}{1+F}[1+(g-1)F]+\frac{\gamma_-}{\gamma_+}f_2\right\}. \tag{7}$$

LIST OF SYMBOLS

ω_0—generation frequency
Q—figure of merit of resonator
$\Delta\omega_r = \omega_0/Q$—resonator band width
V—resonator volume
L—resonator length
$\Delta\omega = 2\pi\Delta$—laser emission line width
a, b—indices of working levels (a—upper level),
$\rho_a(v)$, $\rho_b(v)$, $\rho_{ab}(v)$, $\rho_{ba}(v)$—matrix-density elements for the working levels of an atom possessing a velocity v.
$\rho_a^{(0)}(v)$, $\rho_b^{(0)}(v)$—level populations in the absence of a field,
$\rho_a^0 = \int\rho_a^{(0)}(v)\,(e^{-mv^2/2kT}/\sqrt{2\pi kT/m})dv$, $\rho_b^0 = \int\rho_b^{(0)}(v)\times(e^{-mv^2/2kT}/\sqrt{2\pi kT/m})dv$—working-level populations averaged over the velocities.

$D(v) = \rho_a(v) - \rho_b(v)$—difference between the populations of the working levels ($D^0 = \rho_a^0 - \rho_b^0$),

$R(v) = \rho_a(v) + \rho_b(v)$—sum of populations of working levels

$\gamma_a, \gamma_b, \gamma_{ab}$—relaxation constants of the corresponding density-matrix elements,

$$\gamma^2 = \frac{\gamma_a\gamma_b\gamma_{ab}}{\gamma_a+\gamma_b}, \quad \gamma_+ = \frac{\gamma_a+\gamma_b}{2}, \quad \gamma_- = \frac{\gamma_b-\gamma_a}{2},$$

$\mu = \omega_0 - \omega_{ab}$—deviation of generation frequency from the center of the gain profile;

$g = (\gamma_{ab}^2/\mu^2 + \gamma_{ab}^2)$, $\kappa'_{1,2}$, $\kappa''_{1,2}$—real and imaginary parts of the polarizability;

$a = (e^2|r_{ab}|^2/6\hbar^2\gamma^2)$—saturation parameter;

$d = (4\pi^2 e^2 n|r_{ab}|^2 D^0/3\hbar\sqrt{2\pi}k_0 u)$—pump parameter;

$\Phi(x) = \sqrt{(2/\pi)}\int_0^x e^{-x^2/2}dx$—probability integral

$D_\nu(z)$—parabolic cylinder function.

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