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CLESCH-GORDAN COEFFICIENTS, VIEWED FROM DIFFERENT SIDES

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A generalized theory of angular momenta has been developed over the past few years. The new results account for a substantial change in the role played by Clebsch-Gordan coefficients both in physical and in mathematical problems. This review considers two aspects of the theory of Clebsch-Gordan coefficients, which forms a part of applied group theory. First, the close relation of these coefficients with combinatorics, finite differences, special functions, complex angular momenta, projective and multidimensional geometry, topology and several other branches of mathematics are investigated. In these branches the Clebsch-Gordan coefficients manifest themselves as some new universal calculus, exceeding substantially the original framework of angular momentum theory. Second, new possibilities of applications of the Clebsch-Gordan coefficients in physics are considered. Relations between physical symmetries are studied by means of the generalized angular momentum theory which is an adequate formalism for the investigation of complicated physical systems (atoms, nuclei, molecules, hadrons, radiation); thus, e.g., it is shown how this theory can be applied to elementary particle symmetries. A brief summary of results on Clebsch-Gordan coefficients for compact groups is given in the Appendix.

1. INTRODUCTION

CLEBSCH-Gordan coefficients (C-G coefficients) have long formed part and parcel of the mathematical apparatus of theoretical physics. These coefficients have been used in the computations of spectra—both atomic and nuclear—and have been used in various parts of scattering theory. Formulas and tables of various types can be found in many textbooks and monographs (and even in the "Pocket Diary for Physicists"), However in the majority of cases the concept of G-G coefficient is associated to formulas for the addition of angular momenta, and to many physicists their theory appears as a closed chapter.

In fact, in a certain sense, one may consider closed only what we call the classical theory of C-G coefficients, related to the expansion of products of representations of the three-dimensional rotation group (of real three-dimensional space) into irreducible components. Investigations over the last few years have taken the theory of C-G coefficients outside the narrow circle of its classical problems. The new developments are more and more intertwined with various sections of algebra, multidimensional geometry, topology, projective geometry, analytic function theory, the theory of special functions, differential equations, combinatorial analysis and the calculus of finite differences. One could say that the theory of C-G coefficients takes on the character of a new kind of calculus, going far beyond the scope of the classical theory.

However, many of the newer aspects of the theory of C-G coefficients can be found only in journal articles; the majority of these aspects are not fully developed and are not well known. At the same time this branch of mathematical physics has good chances to develop, and it seems useful to call attention to it. This is also important from the viewpoint of physical applications of the C-G coefficients, which have expanded recently in connection with the discovery of new symmetries of elementary particles, and also with the necessity of analyzing the interrelation between symmetries and the discovery of hidden symmetries in complicated physical systems: atoms, nuclei, hadrons, molecules. The theory of C-G coefficients is adequate for the study of such systems, which reflect the complicated character of the interactions of many particles. It should also be stressed that the theory of C-G coefficients, in distinction from the theory of characters of representations, makes it possible to use all the information stemming from the presence of symmetry in a physical system.

Thus, the purpose of the present review is to attract attention to the new aspects of the theory of C-G coefficients, to tell about the wealth of interrelations in this theory and to indicate a series of new possibilities of applications.

We shall restrict our attention below to the theory of C-G coefficients of the compact group SU(2) (which is also called the theory of angular momenta) and some of its generalizations, but we also list the results for C-G coefficients for other compact groups. The properties of C-G coefficients for noncompact groups, in particular for the group O(2, 1), necessitate a separate review article. The bases of the theory of angular momentum were laid in the fundamental work of Wigner[1] and Racah[2-4] with the purpose of carrying out practical calculations in atomic and nuclear spectroscopy. In the 1940-s and 50-s the C-G and Racah coefficients were studied in detail and tabulated; various combinations of these coefficients were discussed: generalized C-G coefficients, transformation matrices, j-symbols; graphical methods were developed. Toward the end of the fifties all concepts of angular momentum theory were unified into a consistent formalism and the theory seemed completed. The angular momentum theory as of that time, which we will call "classical," is exposed in a series of monographs[1,5-11].

Therefore, the discovery by Regge[12] of new symmetry properties of the C-G coefficients, overlooked in all earlier investigations, came completely unexpected. This discovery was the starting point of the new development of the theory[1)]. Over the past decade the theory of angular momenta was subject to qualitative changes. Among the new problems which have appeared in the past few years, one can indicate the generalization of C-G coefficients to arbitrary complex arguments, extension which was essential in connection with Regge trajectories[15]; generalizations of the C-G coefficients related to the analysis of complex physical systems; the close intertwining of the theory of angular momenta and the theory of C-G coefficients for compact groups. This situation of the theory makes a review of its state useful, in spite of the fact that owing to their incompleteness many of the questions touched upon in this review are exposed only schematically.

We have considered it very important to indicate directions which are almost undeveloped and would like to stress that our enumeration of such directions is probably incomplete, and that not all of those mentioned will turn out in the future to be equally fruitful. Undoubtedly, new relations will be discovered in future work.

The review consists of three parts. In the first part we list briefly the results of the classical theory of angular momenta and discuss the interrelations with discrete and continuous mathematics. In the second part we construct the generalized theory of angular momenta on the basis of the new symmetries (the Regge symmetry and higher ones) and discuss the physical aspects. In the third part we analyze the relation of the theory of C-G coefficients with geometric and topological concepts. The Appendix gives a brief listing of results in the theory of C-G coefficients for compact groups.

[1)]It is characteristic that these symmetries are already contained in the papers on hypergeometric functions [13,14]. However, nobody ever attempted to extract them from there.

I. C-G COEFFICIENTS AND THEIR RELATION WITH DISCRETE AND CONTINUOUS MATHEMATICS

2. The Classical Theory of Angular Momenta

The first part of this review treats problems which are directly related to the classical theory of angular momenta, such as: the relation between the theory of C-G coefficients and combinatorial analysis, the calculus of finite differences, the theory of special functions. integral representations and also the construction of C-G coefficients for complex angular momenta. However, first of all we must briefly discuss the classical theory of angular momenta[1-11]. At the basis of this theory lies the concept of C-G coefficients $(j_1j_2m_1m_2 \mid jm)$ which implement a transformation from the basis $\psi^{j_1}_{m_1} \times \psi^{j_2}_{m_2}$ to the basis ψ^{j}_{m}, where j_1, j_2, j are angular momenta and m_1, m_2, m are their projections. The C-G transformations are unitary, and thus verify the equations

$$\sum_{m_1m_2} (j_1j_2m_1m_2 \mid jm)\,(j_1j_2m_1m_2 \mid j'm') = \delta_{jj'}\delta_{mm'}, \tag{2.1}$$

$$\sum_{jm} (j_1j_2m_1m_2 \mid jm)\,(j_1j_2m_1'm_2' \mid jm) = \delta_{m_1m_1'}\delta_{m_2m_2'}. \tag{2.2}$$

The unitary relations (2.1) and (2.2) and the recurrence relations which are obtained by means of the infinitesimal operators of the group SU(2) allow one to compute all the C-G coefficients. The arbitrariness in the choice of phase is removed by imposing the supplementary condition[16]

$$(j_1j_2j_1m_2 \mid jm) \geqslant 0. \tag{2.3}$$

With this choice the C-G coefficients are always real. The symmetry properties of the Wigner coefficient (3j-symbol), which projects the product of three irreducible representations on an invariant subspace and is related to the C-G coefficient by the relation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2+m_3}(2j_3+1)^{-1/2}(j_1j_2m_1m_2 \mid j_3 - m_3), \tag{2.4}$$

are expressed by the equalities

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \varepsilon \begin{pmatrix} j_i & j_k & j_l \\ m_i & m_k & m_l \end{pmatrix}, \quad m_1+m_2+m_3=0, \tag{2.5}$$

where $\epsilon = 1$ if the permutation of the columns is even and equals $(-1)^{j_1+j_2+j_3}$ if the permutation of the columns is odd or there is a change of the sign of the projections; altogether there are 12 such symmetry operations. (The additional symmetries which were obtained by Regge[12] will be discussed in Chap. II.) Usually the index j is assumed to be positive, integer of half-integer (m takes on 2j + 1 values from −j to +j). However, the matrix elements of the square of the angular-momentum operator as well as the eigenvalue equations do not change under the substitution

$$j \to J = -j-1 \tag{2.6}$$

and hence one may consider negative values of the angular momenta in all formulas for the C-G coefficients[9,17]. The state vectors (wave functions) corresponding to j and J describe the same state and differ only by a phase factor

$$\mid jm\rangle = (-1)^{j-m} \mid Jm\rangle. \tag{2.7}$$

Thus, the substitution (2.6) denotes a definite symmetry of the C-G coefficients.

The matrix elements of irreducible representations of the group SU(2), i.e., the matrix elements $D(\omega)$ of finite rotations, are closely related to the C-G coefficients. If the rotation is parametrized by means of the Euler angles α, β, γ one can write the matrix elements $D^j_{mm'}(\alpha, \beta, \gamma)$ in the form

$$D^j_{mm'}(\alpha, \beta, \gamma) = e^{-im\alpha} D^j_{mm'}(\beta)\, e^{-im'\gamma}. \tag{2.8}$$

The quantity

$$D^j_{mm'}(\beta) = \frac{(-1)^{j-m}\, i^{m'-m}}{2^j} \sqrt{\frac{(j+m')!}{(j-m)!\,(j+m)!\,(j-m')!}} \times (1-\mu)^{-\frac{m'-m}{2}} (1+\mu)^{-\frac{m'+m}{2}} \times \frac{d^{j-m'}}{d\mu^{j-m'}} [(1-\mu)^{j-m}(1+\mu)^{j+m}] \quad (\mu = \cos\beta) \tag{2.9}$$

is, up to a numerical factor, a Jacobi polynomial[18]. The phase here has been chosen in agreement with[5], such that

$$D^j_{mm'} = (-1)^{m-m'} D^j_{mm'}. \tag{2.10}$$

A different choice of phase has been made in[19].

In special cases the finite rotation matrix can be expressed in terms of spherical harmonics and Legendre polynomials:

$$D^l_{m0}(\alpha, \beta, \gamma) = \left(\frac{4\pi}{2l+1}\right)^{1/2} Y^*_{lm}(\beta, \alpha), \qquad D^l_{00}(\alpha, \beta, \gamma) = P_l(\cos\beta). \tag{2.11}$$

The relation between the $D^j_{mm'}$ and the C-G coefficients is given by the relations[5]

$$D^{j_1}_{m_1m'_1}(\omega)\, D^{j_2}_{m_2m'_2}(\omega) = \sum_{jmm'} (j_1j_2m_1m_2 \mid jm)\, D^j_{mm'}(\omega)\, (j_1j_2m'_1m'_2 \mid jm'), \tag{2.12}$$

$$\frac{1}{8\pi^2} \int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi} D^{j_1}_{m_1m'_1}(\alpha, \beta, \gamma)\, D^{j_2}_{m_2m'_2}(\alpha, \beta, \gamma)\, D^{j_3}_{m_3m'_3}(\alpha, \beta, \gamma) \sin\beta\, d\beta\, d\alpha\, d\gamma = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}. \tag{2.13}$$

Important role in theory and applications are played by various combinations of the C-G coefficients (or Wigner coefficients) which are covariant sums of products of such quantities, e.g., the expression which appears when a scalar is formed from four or more angular momenta (addition of more than two angular momenta):

$$(j_1j_2m_1m_2 \mid j_{12}m_{12})\, (j_{12}j_3m_{12}m_3 \mid j_{123}m_{123})\, (j_{123}j_4m_{123}m_4 \mid jm) \tag{2.14}$$

(expressions of the type (2.14) are called generalized C-G coefficients). A special place among such combinations is occupied by the transformation matrices (and their symmetric forms, the so-called j-symbols) which implement the transition from one coupling of the angular momenta (e.g., in (2.14): $((j_1j_2)j_{12}j_3)j_{123}j_4j)$) to another coupling, differing from the first by the order in which the angular momenta are added. Arbitrary transformation matrices can be expressed in terms of a sum of products of the simplest matrices—the Racah coefficients. The relation between the transformation matrix and the corresponding Racah coefficient and 6j-symbol is given by the formula

$$((j_1j_2)j_{12}j_3j \mid j_1(j_2j_3)j_{23}j) = \sqrt{(2j_{12}+1)(2j_{23}+1)}\, W(j_1j_2jj_3;\, j_{12}j_{23}) = \sqrt{(2j_{12}+1)(2j_{23}+1)}\, (-1)^{j_1+j_2+j_3+1} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}. \tag{2.15}$$

According to the classical theory the 6j-symbol is invariant with respect to a permutation of columns with a simultaneous permutation of any two pairs of angular momenta which are situated in the same column (24 symmetry rules).

Transformations between different coupling schemes of four angular momenta lead to 9j-symbols related to the corresponding transformation matrix by

$$((j_1j_2)j_{12}(j_3j_4)j_{34}j \mid (j_1j_3)j_{13}(j_2j_4)j_{24}j) = \sqrt{(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)} \begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{pmatrix} \tag{2.16}$$

and satisfying 72 symmetry rules.

The theory becomes, of course, more complicated for the transformation matrices occurring in the addition of a larger number of angular momenta. Little has been done in this direction: there is not even an established notation for the j-symbols, the problem of their enumeration has not been solved, etc. The general structure of the transformation matrices can be seen from the following example. According to the definition

$$\sum_{m_{12}m_{123}} (j_1j_2m_1m_2 \mid j_{12}m_{12})\, (j_{12}j_3m_{12}m_3 \mid j_{123}m_{123})\, (j_{123}j_4m_{123}m_4 \mid jm) = \sum_{j_{14}j_{23}} (((j_1j_2)j_{12}j_3)j_{123}j_4j \mid (j_1j_4)j_{14}(j_2j_3)j_{23}j) \times \sum_{m_{14}m_{23}} (j_1j_4m_1m_4 \mid j_{14}m_{14})\, (j_2j_3m_2m_3 \mid j_{23}m_{23})\, (j_{14}j_{23}m_{14}m_{23} \mid jm). \tag{2.17}$$

In (2.17) the transformation matrix defines a recoupling between two coupling schemes with total angular momentum j and values of the intermediate momenta j_{12}, j_{123} and j_{14}, j_{23}. Multiplying both sides of the equation (2.17) by the appropriate C-G coefficients and carrying out the required summation, we obtain

$$(((j_1j_2)j_{12}j_3)j_{123}j_4j \mid (j_1j_4)j_{14}(j_2j_3)j_{23}j) = \frac{1}{2j+1} \sum (j_1j_2m_1m_2 \mid j_{12}m_{12})\, (j_{12}j_3m_{12}m_3 \mid j_{123}m_{123})\, (j_{123}j_4m_{123}m_4 \mid jm) \times (j_1j_4m_1m_4 \mid j_{14}m_{14})\, (j_2j_3m_2m_3 \mid j_{23}m_{23})\, (j_{14}j_{23}m_{14}m_{23} \mid jm). \tag{2.18}$$

The summation here goes over all projections of the angular momenta. Equations of the form (2.18) give general expressions for the transformation matrices. For concrete computations there exist many tables of C-G coefficients and of their combinations. The general expression of the C-G coefficient can be written in the form of a finite sum

$$(j_1j_2m_1m_2 \mid jm) = \Big[\frac{2j+1}{j_1+j_2+j+1} (j_2+j-j_1)!\, (j+j_1-j_2)! \times (j_1+j_2-j)!\, (j_1+m_1)!\, (j_1-m_1)!\, (j_2+m_2)!\, (j_2-m_2)!\, (j+m)!\, (j-m)!\Big]^{1/2} \times \sum_z \frac{(-1)^z}{z!\,(j_1+j_2-j-z)!\,(z+j-j_1-m_2)!\,(j_2+m_2-z)!\,(z+j-j_2+m_1)!\,(j_1-m_1-z)!}. \tag{2.19}$$

There are other equivalent representations, which will be analyzed in detail in Sec. 5.

On the other hand, the C-G coefficients and their combinations can be expressed in terms of special v values of generalized hypergeometric functions[20] (cf. Sec. 5):

$$(j_1j_2m_1m_2 \mid jm) = (-1)^{j_2+m_2} \Big[\frac{(j+j_1-j_2)!\,(j_1+j_2-j)!\,(j-m)!\,(j_1-m_1)!\,(2j+1)}{(j-j_1+j_2)!\,(j_1+j_2+j+1)!\,(j+m)!\,(j_1+m_1)!\,(j_2-m_2)!\,(j_2+m_2)!}\Big]^{1/2} \times \frac{(j+j_2+m_1)!}{(j_1-j_2-m)!}\, {}_3F_2(-j+j_1-j_2,\ j_1-m_1+1,\ -j-m;\ j_1-j_2-m+1,\ -j-j_2-m_1;\ 1). \tag{2.20}$$

From the expressions (2.19) which express the C-G coefficients and their combinations in terms of complicated sums over factorials one can glean the relation of the C-G coefficients to discrete mathematics, whereas the expression (2.20) indicates a connection with the theory of special functions and of differential equations.

3. The Relation of C-G Coefficients with Combinatorics and the Calculus of Finite Differences

By their very nature the C-G coefficients belong to discrete mathematics: their arguments range over a discrete set of values, and their numerical value is expressed in terms of a sum of products of factorials. Therefore it is quite natural that in the analysis of the C-G coefficients and their combinations one can make use of combinatorial analysis. Such formulas have been used, e.g., in[9,21], to establish transformations between expressions of the C-G coefficients derived by various authors. However, the relation between combinatorics and the theory of angular momenta is more profound than might appear at a first glance. Combinatorial analysis[22,24] studies various complicated sums of factorials and binomial coefficients. Sums which are covariant with respect to some group or sums related to C-G coefficients are of interest in applications. Therefore a series of formulas from combinatorial analysis can be translated into the language of C-G coefficients. Consider, for example, the known relations

$$\sum_s \frac{1}{s!\,(b-s)!\,(c-s)!\,(a-b-c+s)!}=\frac{a!}{b!\,c!\,(a-b)!\,(a-c)!}, \tag{3.1}$$

$$\sum_s \frac{(a-s)!\,(b+s)!}{s!\,(c-s)!}=\frac{(a-c)!\,b!\,(a+b+1)!}{c!\,(a+b-c+1)!}. \tag{3.2}$$

The Vandermonde formula (3.1) can also be expressed in terms of binomial coefficients in the form

$$\binom{x+y}{n}=\sum_\alpha \binom{x}{n-\alpha}\binom{y}{\alpha}. \tag{3.3}$$

Rewritten in terms of the C-G coefficients these relations have the form

$$\sum_{m_1+m_2=m} (j_1 j_2 m_1 m_2 \,|\, j_1+j_2 m)=1, \tag{3.4}$$

$$\sum_{m_2=m-m_1} (j_1 j_2 m_1 m_2 \,|\, j_1+j_2 m)=\frac{2j+1}{2j_2+1}. \tag{3.5}$$

The transition from Eqs. (3.4) to (3.5) is simply the symmetry of the C-G coefficients.

The second Vandermonde formula, which occurs for the substitution $j \to J$

$$(-1)^n\binom{y-x+n-1}{n}=\sum_\alpha (-1)^\alpha \binom{y+\alpha-1}{\alpha}\binom{x}{n-\alpha} \tag{3.6}$$

corresponds to a C-G coefficient analogous to (3.4), but with negative angular momentum.

Another example is the product of two Fibonacci numbers[25], defined by

$$u_{2n}=\sum_{i=1}^{n}\binom{2n-i}{i}, \qquad u_{2n+1}=\sum_{i=1}^{n}\binom{2n+1-i}{i},$$

and given by the formula

$$u_{2j_1}u_{2j_2}=\sum_{m_1 m_2}\left(\tfrac{1}{3}(j_1+m_1)\ \tfrac{1}{3}(j_2+m_2)\ m_1 m_2 \,\middle|\, \tfrac{1}{3}(j_1+j_2+m_1+m_2)\ m_1+m_2\right) \times \binom{j_1+j_2}{j_1+j_2-m_1-m_2}. \tag{3.7}$$

The product of three Fibonacci numbers can be expressed in terms of a combination of two such C-G, etc. We note that here the argument of the angular momentum contains the projection m.

The equations (3.4), (3.5), (3.7) are illustrations of the interrelation between combinatorics and angular-momentum theory. It is likely that many other formulas from combinatorial analysis can be expressed in terms of the C-G coefficients.

As regards the combinatorial properties of the arguments in the C-G coefficients, their relation with the theory of magic squares, block-schemes, and finite geometries seems to be taking shape at the present time. However, it is more convenient to discuss these questions within the framework of the generalized theory of angular momenta, to which Chapter II is dedicated.

Another branch of discrete mathematics directly related to the C-G coefficients is the calculus of finite differences. We recall some basic definitions[26,27]:

1. Generalized power (the analog of ordinary power):

$$x^{(n)}=\begin{cases}\dfrac{x!}{(x-n)!}, & x\geqslant n;\\ 0, & x<n.\end{cases} \tag{3.8}$$

Its properties are:

$$\left.\begin{aligned} x^{(n)}&=x^{(m)}(x-m)^{(n-m)},\\ (x+1)^{(n)}&=(x+1)\,x^{(n-1)}.\end{aligned}\right\} \tag{3.9}$$

2. Finite differences (the analogs of derivatives):

$$\left.\begin{aligned} \Delta_h f(x)&=f(x+h)-f(x),\\ \Delta_h^{(2)} f(x)&=\Delta_h f(x+h)-\Delta_h f(x),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ \Delta_h^{(n)} f(x)&=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+kh).\end{aligned}\right\} \tag{3.10}$$

3. Operations with generalized powers (analogs of differentiation and integration):

$$\begin{aligned} \Delta x^{(k)}&=kx^{(k-1)},\\ \sum_{x=0}^{n} x^{(k)}&=\frac{(n+1)^{(k+1)}}{k+1}.\end{aligned} \tag{3.11}$$

The calculus of finite differences preserves the basic properties of the analogues indicated above.

In a cycle of papers[21,28,29], Ansary has shown that the numerical value of the C-G coefficients is determined by the expansion of the quasi-binomial

$$((ax-by))^{(n)}=\sum_\alpha (-1)^\alpha\binom{n}{\alpha}a^{(n-\alpha)}x^{(n-\alpha)}b^{(\alpha)}y^{(\alpha)} \tag{3.12}$$

according to the equation

$$\boxed{\begin{matrix}\alpha & \beta & \gamma\\ \alpha' & \beta' & \gamma'\\ \alpha'' & \beta'' & \gamma''\end{matrix}}=N'\,\frac{(\alpha'!)^{1/2}}{\alpha!}\left[\frac{\alpha''^{(\alpha''-\gamma')}}{\beta'^{(\alpha''-\gamma')}}\beta^{(\alpha''-\beta')}\gamma^{(\alpha''-\gamma')}\right]^{1/2}((\beta'\gamma''-\beta''\gamma'))^{(\alpha)}, \tag{3.13}$$

where the square matrix has the meaning of a Wigner coefficient. (Its arguments and symmetries are the same, up to a phase factor[12].) N′ is a normalization factor.

However, in order to apply finite-difference methods to the theory of C-G coefficients it is more convenient to use in place of (3.13) the relations derived in[10]. The quantity

$$\widetilde{T}_s^{\alpha\beta}(\mu,k)=(-1)^s\frac{\Delta^s}{(\Delta\mu)^s}\left[(k-\mu)_\mu^{(s+\alpha)}\,(k+\mu)_\mu^{(s+\beta)}\right] \tag{3.14}$$

is the difference analogue of a Jacobi polynomial. The C-G coefficients can be expressed in terms of these quantities by means of the formula

$$(j_1 j_2 m_1 m_2 \,|\, jm) = \delta_{m,\, m_1+m_2} \sqrt{\frac{(j_1+j-j_2)!\,(j_2+j-j_1)!\,(j_1+j_2-j)!\,(2j+1)!\,(j+m)!\,(j-m)!}{(j_1+j_2+1)!\,(j_1-m_1)!\,(j_1+m)!\,(j_2-m_2)!\,(j_2+m_2)!}} \times \widetilde{T}_s^{\alpha\beta}(\mu, k), \tag{3.15}$$

where

$$k = \frac{j_1+j_2+j}{2}, \quad \mu = \frac{j_1+j_2-j}{2} + m_1,$$
$$s = j - j_1 + j_2, \ \alpha = (j_1 - m_1) - (j+m), \ \beta = (j_1 + m_1) - (j+m).$$

Equations analogous to (3.14) and (3.15) can be constructed for some combinations of C-G coefficients. On the other hand, since one can express different relations in the calculus of finite differences in terms of C-G coefficients, it is not excluded that some of the problems of the theory of finite differences, namely finite difference equations, approximate calculation, approximations, could also be expressed in terms of C-G coefficients. This whole range of problems requires careful further examination.

4. Clebsch-Gordan Coefficients and Continuous Transformations

The C-G coefficients are closely related to continuous transformations. Here one can distinguish three different basic lines of development. The first is the obvious relation to Lie groups, infinitesimal transformations, the corresponding differential equations and their solutions, i.e., the special functions; the second line of approach is that of integral representations of the C-G coefficients; the third line consists in generalizing the C-G coefficients to continuous, and in general complex values. The relation with special functions (in particular, hypergeometric functions) are discussed below, in Chap. II (Sec. 6) on the basis of the symmetry of the C-G coefficients derived by Regge; in this section we briefly discuss the second and third lines.

An integral representation of the C-G coefficients is based on the relation (2.13). Since, according to (2.13) the C-G coefficients are defined as an integral of a product of three D-functions, using the differential representation of one of the D-functions and substituting for the first degenerate C-G[2] and for the other two D-functions the appropriate expressions, we obtain the integral representation

$$(j_1 j_2 m_1 m_2 \,|\, jm) = \frac{(-1)^{-j+j_1+m_2}}{2^{j_1+j_2+j+1}} \left[\frac{(2j+1)\,(j+m)!\,(j_1+j_2-j)!\,(j_1+j_2+j+1)!}{(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!(j-m)!(j+j_1-j_2)!(j-j_1+j_2)!}\right]^{1/2}$$
$$\times \int_{-1}^{+1} (1-x)^{j_1-m_1}(1+x)^{j_2-m_2} \frac{d^{j-m}}{dx^{j-m}} [(1-x)^{j-j_1+j_2}(1+x)^{j+j_1-j_2}]\, dx. \tag{4.1}$$

These representations yield the usual series for the C-G coefficients. Making use of the symmetry among different C-G coefficients one can also obtain other integral representations.

The third point of the interrelations between C-G coefficients and continuous transformations refers to a complexification of the variables. The generalization carried out below of the C-G coefficients of the group SU(2) is based on the properties of the representations of the group O(4) and of the Lorentz group SL(2, C). As is well known, the six generators of the group O(4) satisfy the commutation relations

$$\left.\begin{aligned} [\mathbf{M}, \mathbf{M}] &= i\varepsilon\mathbf{M}, \\ [\mathbf{M}, \mathbf{N}] &= i\varepsilon\mathbf{N}, \\ [\mathbf{N}, \mathbf{N}] &= i\varepsilon\mathbf{M}. \end{aligned}\right\} \tag{4.2}$$

Let us denote the appropriate quantum numbers by M, N, m, n (integers). Introducing the linear combinations

$$\left.\begin{aligned} \mathbf{A} &= \tfrac{1}{2}(\mathbf{M}+\mathbf{N}), \\ \mathbf{B} &= \tfrac{1}{2}(\mathbf{M}-\mathbf{N}), \end{aligned}\right\} \tag{4.3}$$

each three new generators satisfy the commutation relations of the group SU(2)

$$\left.\begin{aligned} [\mathbf{A}, \mathbf{A}] &= i\varepsilon\mathbf{A}, \\ [\mathbf{B}, \mathbf{B}] &= i\varepsilon\mathbf{B}, \\ [\mathbf{AB}] &= 0. \end{aligned}\right\} \tag{4.4}$$

Let us denote the quantum numbers belonging to these generators by l_A, m_A, l_B, m_B; this allows one to express the matrix elements of the operator N in terms of the known matrix elements of the operators A and B. Since

$$\mathbf{M} = \mathbf{A} + \mathbf{B}, \quad \mathbf{N} = \mathbf{A} - \mathbf{B},$$

the matrix element will depend on two C-G coefficients of the form

$$\left(\frac{n+M}{2}\,\frac{n-M}{2}\,\frac{m+\mu}{2}\,\frac{m-\mu}{2}\,\Big|\, Jm\right), \tag{4.5}$$

where Jm are the quantum numbers of the operators M and N, and $\frac{1}{2}(n \pm M)$ and $\frac{1}{2}(m \pm \mu)$ are the quantum numbers of the operators A, A_3 and B, B_3 respectively. This matrix element can be written in the form

$$\delta^{nM}_{JmJ'}(\delta) = \left[\frac{(n-J')!\,(n+J'+1)!}{(n-J)!\,(n+J+1)!}\right]^{1/2} \sum_{\mu} \left(\frac{n+M}{2}\,\frac{n-M}{2}\,\frac{m+\mu}{2}\,\frac{m-\mu}{2}\,\Big|\, Jm\right)$$
$$\times e^{i\delta\mu}\left(\frac{n+M}{2}\,\frac{n-M}{2}\,\frac{m+\mu}{2}\,\frac{m-\mu}{2}\,\Big|\, Jm\right) \quad (m \leqslant M). \tag{4.6}$$

This formula can be generalized to the Lorentz group. The six generators of the Lorentz group satisfy the commutation relations

$$\left.\begin{aligned} [\mathbf{M}, \mathbf{M}] &= i\varepsilon\mathbf{M}, \\ [\mathbf{M}, \mathbf{N}] &= i\varepsilon\mathbf{N}, \\ [\mathbf{N}, \mathbf{N}] &= -i\varepsilon\mathbf{M}. \end{aligned}\right\} \tag{4.7}$$

Their eigenvalues have the form $\sigma = -1 + ip$, ν, J, m (p is real, the other numbers are integers or half-integers). Introducing the (nonhermitian) generators

$$\mathbf{F} = \mathbf{M} + i\mathbf{N}, \quad K = \mathbf{M} - i\mathbf{N}, \tag{4.8}$$

we are led to complex eigenvalues and to the commutation relations of the complex rotation group SU(2, C)

$$\left.\begin{aligned} [\mathbf{F}, \mathbf{F}] &= i\varepsilon\mathbf{F}, \\ [\mathbf{K}, \mathbf{K}] &= i\varepsilon\mathbf{K}, \\ [\mathbf{F}, \mathbf{K}] &= 0. \end{aligned}\right\} \tag{4.9}$$

As a result of this we obtain for the matrix element of N_3 (a boost[3] by the "hyperbolic angle" θ) a formula[30-33] analogous to the one for O(4):

$$d^{\nu\sigma}_{JmmJ'}(\theta) = N \int_{-k-i\infty}^{-k+i\infty} dt\, \frac{e^{\pm i\pi t}}{\sin \pi t} \left(\frac{\sigma+\nu}{2},\ \frac{\sigma-\nu}{2},\ t - \frac{\sigma-\nu}{2} + m,\ -t + \frac{\sigma-\nu}{2}\,\Big|\, Jm\right)$$
$$\times \left(\frac{\sigma+\nu}{2},\ \frac{\sigma-\nu}{2},\ t - \frac{\sigma-\nu}{2} + m,\ -t + \frac{\sigma-\nu}{2}\,\Big|\, J'm\right) \tag{4.10}$$
$$\times \exp[-\theta(2t - \sigma + \nu + m)] \qquad (\pm \text{ for } \operatorname{Im} t \gtrless 0),$$

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*) $\begin{pmatrix} j_1 & j_2 & j \\ j_1 & -j_2 & -j_1+j_2 \end{pmatrix} = \left[\frac{(2j+1)\,(2j_1)!\,(2j_2)!}{(j_1+j_2-j)!\,(j_1+j_2+j+1)!}\right]^{1/2}.$

3) A boost is a pure, rotation-free, Lorentz transformation.

or simpler

$$d^{\nu\sigma}_{JmmJ'}(\theta)=N\int\limits_{-k-i\infty}^{-k+i\infty}dt\,\frac{e^{\pm i\pi t}}{\sin\pi t}\left(\frac{\sigma+\nu}{2},\ \frac{\sigma-\nu}{2},\ t+\frac{m}{2},\ -t+\frac{m}{2}\Big|Jm\right)\times\left(\frac{\sigma+\nu}{2},\ \frac{\sigma-\nu}{2},\ t+\frac{m}{2},\ -t+\frac{m}{2}\Big|J'm\right)e^{-2\theta t},\tag{4.11}$$

where the normalization factor is

$$N=\frac{i}{2}\,\frac{\Gamma(\sigma-J'+1)\,\Gamma(\sigma+J'+2)}{\Gamma(\sigma-J+1)\,\Gamma(\sigma+J+2)}.$$

These formulas contain quantities which are related to the C-G coefficients, but for two mutually complex conjugate angular momenta

$$j_1=\tfrac{1}{2}(-1+ip+\nu),\quad j_2=\tfrac{1}{2}(-1+ip-\nu)\qquad(j=J).\tag{4.12}$$

The "projections" of these angular momenta become continuous.

More general C-G coefficients occur for nonunitary representations. However, the theory of these objects is not yet developed.

If one replaces the generators $\mathbf{F}$ and $\mathbf{K}$ by the generators

$$\begin{aligned}f_{x,y}&=-iM_{x,y}+N_{x,y}, & k_{x,y}&=iM_{x,y}+N_{x,y},\\ f_z&=M_z+iN_z, & k_z&=M_z-iN_z,\end{aligned}\tag{4.13}$$

these generators form two algebras of the three-dimensional Lorentz group O(2, 1)

$$\left.\begin{aligned}&[f_x,f_y]=if_z, && [k_x,k_y]=ik_z,\\ &[f_y,f_z]=-if_x, && [k_y,k_z]=-ik_x, && [\mathbf{f},\mathbf{k}]=0.\\ &[f_z,f_x]=-if_y, && [k_z,k_x]=-ik_y,\end{aligned}\right\}\tag{4.14}$$

In this case, the computation of the matrix elements leads to a theory of C-G coefficients for the group O(2, 1)[30,34]; the properties of these coefficients go beyond the framework of the present review article.

In conclusion we write out the expressions of the C-G coefficients with the correct phase and normalization factors

$$(j_1j_2m_1m_2\,|\,jm)=e^{\mp i\pi(j_2+m_2)}\left[\Gamma(j_1-j_2-m_3+1,\ 2j_1+2)\right]^{-1}\times\left[\Gamma\begin{pmatrix}j_1+m_1+1,\,j_1-m_1+1,\,j_3-m_3+1,\,j_1+j_2-j_3+1,\,j_1+j_2+j_3+2\\ j_2+m_2+1,\,j_2-m_2+1,\,j_3+m_3+1,\,-j_1+j_2+j_3+1\end{pmatrix}\right]^{1/2}\times{}_3F_2(j_1-j_2-j_3,\,j_1-j_2+j_3+1,\,j_1-m_1+1;\,j_1-j_2-m_3+1,\,2j_1+2;\ 1);\tag{4.15}$$

here

$$\Gamma\begin{pmatrix}a,\,b,\,\ldots\\ l,\,f,\,\ldots\end{pmatrix}=\frac{\Gamma(a)\,\Gamma(b)\ldots}{\Gamma(l)\,\Gamma(f)\ldots},\quad j_1=\frac{\sigma+\nu}{2},\quad m_1=t'+m,$$

$$j_2=\frac{\sigma-\nu}{2},\quad m_2=-t',\quad j_3=J\ \text{or}\ J',\quad \sigma=-1+ip;$$

p and m are arbitrary real numbers.

The hypergeometric function (and accordingly, the C-G coefficients) remain terminating series (finite sums) even after complexification. In the form above the series terminates for $j_1-j_2-j_3+z-1=-J+z-1$ vanishing.

Nonterminating series are characteristic for the noncompact group O(2, 1).

This example indicates the possibility of a complete complexification of the C-G coefficients and their utility in the theory of complex angular momenta.

In conclusion of this section it is necessary to note that both the investigation of the relations for C-G coefficients with differential and integral representations and their generalization to complex values of the variables require, in general, the usê of symmetries higher than SU(2), i.e., appeal to the theory of Lie groups, both compact and noncompact.

II. SYMMETRIES OF THE C-G COEFFICIENTS AND OF THEIR COMBINATIONS

5. The Regge Symmetry. Relations of the C-G Coefficients with Special Functions

In this second part of our review we consider the generalized theory of angular momenta. This theory is constructed on the basis of new quantities, the $n\times n$-symbols[35], which are closely related to the symmetries of the C-G coefficients and their combinations. The starting point of this development was the discovery by Regge of new symmetry properties of the C-G coefficients, not contained in the classical theory of angular momenta. We consider below the direct consequences of the Regge symmetries, including a reformulation of the theory of angular momenta in the so-called R-representation, and an investigation of its relation to generalized hypergeometric functions. According to Regge[12] the Wigner 3j-symbol can be represented in the form

$$\begin{pmatrix}j_1&j_2&j_3\\ m_1&m_2&m_3\end{pmatrix}=\begin{Vmatrix}-j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3\\ j_1-m_1 & j_2-m_2 & j_3-m_3\\ j_1+m_1 & j_2+m_2 & j_3+m_3\end{Vmatrix}=\begin{Vmatrix}R_{11}&R_{12}&R_{13}\\ R_{21}&R_{22}&R_{23}\\ R_{31}&R_{32}&R_{33}\end{Vmatrix}=\|R_{ik}\|,\tag{5.1}$$

where the 3×3 square symbol $\|R_{ik}\|$ is the coefficient in the expansion of the J-th power of the determinant[4)]:

$$\begin{Vmatrix}u_1&v_1&w_1\\ u_2&v_2&w_2\\ u_3&v_3&w_3\end{Vmatrix}^J=\sqrt{(J!)^3(J+1)}\sum_{\sum\limits_k R_{ik}=J}\begin{Vmatrix}R_{11}&R_{12}&R_{13}\\ R_{21}&R_{22}&R_{23}\\ R_{31}&R_{32}&R_{33}\end{Vmatrix}\times\frac{u_1^{R_{11}}u_2^{R_{12}}u_3^{R_{13}}v_1^{R_{21}}v_2^{R_{22}}v_3^{R_{23}}w_1^{R_{31}}w_2^{R_{32}}w_3^{R_{33}}}{(R_{11}!\,R_{12}!\,R_{13}!\,R_{21}!\,R_{22}!\,R_{23}!\,R_{31}!\,R_{32}!\,R_{33}!)^{1/2}}.\tag{5.2}$$

The entries of the symbol $\|R_{ik}\|$ are nonnegative integers. The sum of the elements in all rows and columns is the same and equals $j_1+j_2+j_3=J$. The numerical value is invariant under permutations of rows and columns and with respect to transposition, and is multiplied by $(-1)^J$ under odd permutations.

In distinction from the 12 symmetries of the classical angular momentum theory, corresponding to permutations of only the last two rows in the table, the symbol (5.1) exhibits 72 symmetry properties. (Transposition with respect to the second diagonal does not yield a new symmetry transformation.) These properties are hard to understand if one remains within the framework of three-dimensional space. In computing the C-G coefficients we study in fact a method of separating a single SU(2) group in the direct product SU(2) × SU(2) (addition of two angular momenta). This can be achieved by two essentially different methods. The transition from one method to the other scrambles the angular momenta and their projections. One should also remark that within the group O(4) = SU(2) × SU(2), j and m appear more or less with equal rights and

4) The normalization in Eq. (5.2) differs from Regge's paper [12].

are scrambled in different reductions.

In[36,37] it was indicated that there are definite functional relations between various Racah coefficients. These relations correspond to a new symmetry of the Racah coefficients (6j-symbols). Substituting into the definition of the 6j-symbol

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \sum \begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} \begin{pmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & -m \end{pmatrix}$$
$$\times \begin{pmatrix} j_3 & j_2 & j_{23} \\ -m_3 & -m_2 & m_{23} \end{pmatrix} \begin{pmatrix} j_1 & j & j_{23} \\ -m_1 & m & -m_{23} \end{pmatrix} (-1)^{\alpha},$$
$$\alpha = j_1 + j_2 + j_{12} + j_3 + j_{23} + j + m + m_{23} + m_3, \tag{5.3}$$

the Regge symbols according to (5.1), we obtain new symmetries of the 6j-symbol, which is conveniently considered in the form[38]

$$\begin{Vmatrix} R_{11} & R_{21} & R_{31} & R_{41} \\ R_{12} & R_{22} & R_{32} & R_{42} \\ R_{13} & R_{23} & R_{33} & R_{43} \end{Vmatrix} = \begin{Vmatrix} j_1+j-j_{23} & j+j_3-j_{12} & j_2+j_3-j_{23} & j_1+j_2-j_{12} \\ j_1+j_{12}-j_2 & j_3+j_{23}-j_2 & j_3+j_{12}-j & j_1+j_{23}-j \\ j+j_{12}-j_3 & j+j_{23}-j_1 & j_2+j_{12}-j_1 & j_2-j_{23}-j_3 \end{Vmatrix}$$
$$= \begin{Vmatrix} x_1+y_1 & x_2+y_1 & x_3+y_1 & x_4+y_1 \\ x_1+y_2 & x_2+y_2 & x_3+y_2 & x_4+y_2 \\ x_1+y_3 & x_2+y_3 & x_3+y_3 & x_4+y_3 \end{Vmatrix}. \tag{5.4}$$

Here all 12 elements are nonnegative integers. The differences between corresponding elements of rows and columns turn out to be constants. All in all there are $3! \times 4! = 144$ symmetry rules which follow from (5.1) and (5.3). We shall designate the quantities (5.1) and (5.4) as R-symbols. Since many quantities in the theory of angular momenta can be expressed in terms of combinations of Clebsch-Gordan and Racah coefficients, they can also be expressed as combinations of R-symbols. An essentially new element in the R-notation is the fact that we no longer distinguish here angular momenta from their projections. Linear combinations of j-s can play the roles of projections m and vice-versa. The R-notation contains, obviously, more information than the jm-notation. It also yields a series of new relations between the C-G coefficients, the Racah coefficients and the transformation matrices[38]. Thus, the first and second recurrence relation between the R-symbols has the form

$$\begin{Vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{25}+1 \\ R_{31} & R_{32} & R_{33}-1 \end{Vmatrix} \sqrt{R_{23}(R_{33}+1)} + \begin{Vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21}+1 & R_{22} & R_{23} \\ R_{31}-1 & R_{32} & R_{33} \end{Vmatrix} \sqrt{R_{21}(R_{31}+1)}$$
$$+ \begin{Vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22}+1 & R_{23} \\ R_{31} & R_{32}-1 & R_{33} \end{Vmatrix} \sqrt{R_{22}(R_{32}+1)} = 0. \tag{5.5}$$

This leads to recurrence relations which were not contained in the usual theory of angular momenta. For instance,

$$(j_1 j_2 m_1 m_2 \mid j_3 m_3) = \sqrt{\frac{(2j_3+1)(j_1+m_1+1)(j_1-m_1)}{2(j_3+1)(j_3+m_3+1)(j_3-m_3)}}$$
$$\times (j_1 - {}^1/_2 j_2\ m_1 - {}^1/_2\ m_2 \mid j_3 + {}^1/_2\ m_3 - {}^1/_2) \tag{5.6}$$
$$+ \sqrt{\frac{(2j_3+1)(j_2+m_2+1)(j_2-m_2)}{2(j_3+1)(j_3+m_3+1)(j_3-m_3)}}\ (j_1\ j_2 - {}^1/_2\ m_1\ m_2 - {}^1/_2 \mid j_3 + {}^1/_2\ m_3 - {}^1/_2).$$

Here are two more examples of new equations derived in this manner in the jm-notation. The orthogonality relation:

$$\sum_r (j_1+r+\gamma \quad j_2+r-\gamma \quad m_1+r+\gamma \quad m_2+r-\gamma \mid j_3 \quad m_3+2r)$$
$$\times (j_1+r+\gamma' \quad j_2+r-\gamma' \quad m_1+r+\gamma' \quad m_2+r-\gamma' \mid j_3 \quad m_3+2r)$$
$$\times \frac{j_1+j_2-m_1-m_2+4r+1}{2j_3+1} = \delta_{\gamma\gamma'}. \tag{5.7}$$

Relations between the 6j- and 3j-symbols:

$$(-1)^{j_{12}+j_2-j_1} \begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{pmatrix} \begin{pmatrix} j_{12} & j_3 & j \\ -m'_{12} & m_3 & m \end{pmatrix}$$
$$= (-1)^{2j} \sum_c (-1)^{c+\beta} (2c+1) \begin{Bmatrix} {}^1/_2(j_1+j_2-m_{12}) & {}^1/_2(j_1+j_2+m_{12}) & j_{12} \\ j_3 & j & c \end{Bmatrix}$$
$$\times \begin{pmatrix} j_3 & {}^1/_2(j_1+j_2+m_{12}) & c \\ m_3 & {}^1/_2(j_1-j_2-m_1+m_2) & \beta \end{pmatrix} \begin{pmatrix} {}^1/_2(j_1+j_2-m_{12}) & j & c \\ {}^1/_2(j_1-j_2+m_1-m_2) & m & -\beta \end{pmatrix}, \tag{5.8}$$

where $m'_{12} = j_2 - j_1$. The Racah coefficient here depends not only on the angular momenta j but also on their projections m. In this sense the distinction between the j- and jm-symbols disappears. From relations for third-order determinants and their expansions according to (5.1) and (5.2) follow a series of new relations between combinations of C-G coefficients and their combinations. We list several examples[38].

If the determinant in the left-hand side of (5.1) has two identical rows $v_i = u_i$, since the equation must be valid for arbitrary values of u_i and w_i, we obtain the relation

$$\sum_{R_{1k}+R_{2k}=J-R_{3k}} \begin{Vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{Vmatrix} (R_{11}!\, R_{12}!\, R_{13}!\, R_{21}!\, R_{22}!\, R_{23}!)^{-1/2} = 0. \tag{5.9}$$

In the jm-notation, this yields for even J

$$\sum_{m_i} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \Big[\prod_i (j_i - m_i)!\,(j_i + m_i)! \Big]^{-1/2} = 0. \tag{5.10}$$

Splitting the J-th power of the determinant into a product of determinants raised to powers $J_1, J_2, \ldots, J_n$ ($J_1 + J_2 + \ldots + J_n = J$), expanding the determinants according to Eq. (2.2), and equating the coefficients of equal powers, we obtain

$$\|R_{ik}\| = \Big[\frac{(J_1!)^3 \ldots (J_n!)^3 (J_1+1) \ldots (J_n+1)}{(J!)^3 (J+1)} \Big] \sum_{A'_{ik}+\ldots+A^{(n)}_{ik}=R_{ik}} \|A'_{ik}\| \ldots$$
$$\ldots \|A^{(n)}_{ik}\| \left[\frac{\prod_{ik} R_{ik}!}{\prod_i A'_{ik}! \ldots A^{(n)}_{ik}!} \right]^{1/2}, \tag{5.11}$$

where $\|R_{ik}\|$ and $\|A^{(r)}_{ik}\|$ ($r = 1, \ldots, n$) are the Regge symbols corresponding to the indicated determinants.

For $J_1 = J_2 = \ldots = J_n = 1$ the symbols $\|A^{(r)}_{ik}\|$ reduce to first-order symbols, equal to $\pm(2)^{-1/2}$, the sign being determined according to the distribution of units in the symbol. Substituting them into (5.11) we obtain a numerical expression for the Regge symbol

$$\|R_{ik}\| = \left[\frac{\prod_{ik} R_{ik}}{(J+1)!} \right] \frac{1}{J!} \Phi, \quad \Phi = \sum (-1)^{q_1+q_2+q_3} \frac{J!}{p_1!\, p_2!\, p_3!\, q_1!\, q_2!\, q_3!}. \tag{5.12}$$

The summation here is over all admissible values according to the scheme

$$\begin{Vmatrix} p_1+q_1 & p_3+q_3 & p_2+q_2 \\ p_2+q_3 & p_1+q_2 & p_3+q_1 \\ p_3+q_2 & p_2+q_1 & p_1+q_3 \end{Vmatrix} = \begin{Vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{Vmatrix}. \tag{5.13}$$

The numerical value of the 6j-symbol, represented in the form (5.4) is given by the expression

$$\left[\frac{\prod_{ik} R_{ik}!}{\prod_i \big(\sum_{k=1}^{3} R_{ik} + 1 \big)!} \right]^{1/2} \sum \frac{(-1)^z (z+1)}{\prod_{ik} x_i!\, y_k!}, \tag{5.14}$$

where $z = \sum_i x_i + \sum_k y_k$. The summation goes over all admissible values of x_i, y_k ($x_i + y_k = R_{ik}$), in analogy with Eq. (5.13).

The symmetries discovered by Regge not only allow one to formulate the R-representation of the theory of angular momenta, which contains a large quantity of new equations, but are also essential for the analysis of the relation between C-G coefficients and hypergeometric functions. For this purpose one must extend the Regge definition, removing the nonnegativity requirements for all entries of the table. We shall assume then that the table denotes the same C-G coefficient if one carries out in it one of the two independent substitutions (cf. (2.6))

$$j_1 \to -j_1 - 1 \tag{5.15}$$

or

$$j_2 \to -j_2 - 1, \quad j_3 \to -j_3 - 1. \tag{5.16}$$

A third substitution (three negative momenta)

$$j_1 \to -j_1 - 1, \; j_2 \to -j_2 - 1, \; j_3 \to -j_3 - 1, \tag{5.17}$$

is obviously the product of the first two. One can, of course, obtain other substitutions by combining these with permutations. After this completion one can consider that there exist $72 \times 4 = 288$ identical C-G coefficients with positive or negative values of j. If one considers the permutations of the angular momenta (123) and the substitutions $j \to -j - 1$ as trivial operations, then the permutation P of the first row of the Regge symbol with the second or third row and the transposition T will be nontrivial. This gives three nontrivial operations (e.g., P_{12}, T, $P_{12}T$). Together with the initial one, they yield four nontrivial forms of the C-G coefficients. As such one can select the formulas of Wigner (cf.[34], van der Waerden[40], Racah (cf.[1]) and Majumdar[41].

Looking at any of the formulas that express the C-G coefficients in the form of a finite sum (e.g., (2.19)) it is clear that in all of them the summation variable z occurs in five factorials. This immediately raises the suspicion of a connection with the generalized hypergeometric function of the type $_pF_q$ with $p + q = 5$.

We recall[42] that the generalized hypergeometric function is defined by the series (in general, an infinite series):

$$_pF_q(p_1p_2 \ldots p_p; q_1q_2 \ldots q_q; x) = \sum_z \frac{(p_1)_z (p_2)_z \ldots (p_p)_z}{(q_1)_z (q_2)_z \ldots (q_q)_z} \frac{x^z}{z!}, \tag{5.18}$$

where, e.g.,

$$(p_1)_z = \frac{\Gamma(p_1 + z)}{\Gamma(p_1)} = \frac{(p_1 + z - 1)!}{(p_1 - 1)!}.$$

Using the identity

$$\frac{(a-z)!}{a!} = \frac{(-a-1)!}{(-a+z-1)!}(-1)^z \tag{5.19}$$

one can change the sign of the summation variable, transfering the appropriate factorial from the numerator to the denominator, or vice versa.

One can thus reduce all sums to a standard form (with a plus sign in front of z)[5], and we see that (up to a factor) all four forms of the C-G coefficient represent values of the function $_3F_2$ for $x = 1$. Without calculating the numerical factor (one example is contained in Eq. (2.20), we only list the values of the arguments of the function $_3F_2$ for all four fundamental forms of the Wigner coefficient $\begin{pmatrix} j_1 j_2 j_3 \\ m_1 m_2 m_3 \end{pmatrix}$ (for greater clarity we write the arguments of the hypergeometric function in columns, omitting, as usual, the variable $x = 1$):

Wigner's form

$$_3F_2\begin{pmatrix} j_1 - m_1 + 1 & -j - j_2 - m_1 \\ -j - m & j_1 - j_2 - m + 1 \\ -j + j_1 - j_2 & \end{pmatrix}, \tag{5.20}$$

van der Waerden's form

$$_3F_2\begin{pmatrix} -j_1 - j_2 + j & j - j_2 + m_1 + 1 \\ -j_1 + m_1 & j - j_1 - m_2 + 1 \\ -j_2 - m_2 & \end{pmatrix}. \tag{5.21}$$

Racah's form

$$_3F_2\begin{pmatrix} j_1 + m_1 + 1 & -j - j_2 + m_1 \\ -j + m & j_2 - j + m_1 + 1 \\ -j_1 + m_1 & \end{pmatrix}, \tag{5.22}$$

Majumdar's form

$$_3F_2\begin{pmatrix} j_1 + j_2 - j + 1 & -2j \\ -j - m & j_1 - j - m_2 + 1 \\ j_1 - j_2 - j & \end{pmatrix}. \tag{5.23}$$

The functions $_3F_2$ appearing in connection with the classical C-G coefficients are not arbitrary functions (even if one forgets about the fact that we have set $x = 1$). These functions are distinguished by the fact that they degenerate into finite sums. It is proved in monographs on generalized hypergeometric functions (cf.[14]) that there exist altogether 18 such functions. In proving this one assumes, however, that the series for $_3F_2$ terminates for a highest coefficient equaling a negative integer, with all other numbers remaining arbitrary. If two coefficients are negative integers one can show that the number of terminating series is larger and equals 24. Recognizing that permutations of the arguments of the same type (those which are in the same columns in the above formulas) lead to equations of a different form for the C-G coefficients, one obtains altogether $24 \times 3! \times 2! = 288$ different forms for the C-G coefficients, a result obtained above from an analysis of the symmetries.

Thus, all functions $_3F_2$ which degenerate into finite sums are C-G coefficients. This result allows us to suspect that the other functions $_3F_2$ (for $x = 1$) are somehow related to generalized C-G coefficients. We have already encountered one example of such a generalization to complex arguments in Sec. 4. The relation between C-G coefficients and generalized hypergeometric functions is not an isolated fact. Various combinations of C-G coefficients can also be expressed in terms of generalized hypergeometric functions. Thus, Minton[42] has shown that the Racah coefficient satisfies the formula

$$\begin{aligned} W(abcd; ef) = {} & \Delta(abc)\,\Delta(cde)\,\Delta(acf)\,\Delta(bdf)\,[\Gamma(e + f + 1 - f - c)]^{-1} \\ & \times \Gamma\begin{bmatrix} a + b + c + d + 2 \\ a + b + 1 - e,\; c + d + 1 - e,\; a + c + 1 - f,\; b + d + 1 - f,\; e + f + 1 - a - d \end{bmatrix} \\ & \times {}_4F_3(e - a - b,\; e - c - d,\; f - c - a,\; f - b - d;\; -a - b - c - d - 1, \\ & \qquad e + f + 1 - a - d,\; e + f + 1 - b - c;\; 1), \end{aligned} \tag{5.24}$$

where we have used the notation

[5] A change of sign of all z is nothing other but a reversal of the finite sum: the last term becomes first, etc.

$$\Delta(xyz)=\left\{\Gamma\begin{bmatrix} x+y+1-z, \ x+z+1-y, \ y+z+1-x \\ x+y+z+2 \end{bmatrix}\right\}^{1/2},$$

$$\Gamma\begin{pmatrix} a, \ b \ \ldots \\ p, \ q \ \ldots \end{pmatrix}=\frac{\Gamma(a)\,\Gamma(b)\ldots}{\Gamma(p)\,\Gamma(q)\ldots}.$$

All properties of the C-G and Racah coefficients follow from those of the functions ${}_3F_2$ and ${}_4F_3$. The results for higher-order symbols are similar, and one can formulate a theory of angular momenta in the language of generalized hypergeometric functions ${}_mF_n$ with the argument $x = 1$.

Such a formulation opens up several directions for further investigations. Thus, since the C-G coefficient is a solution of the differential equation for ${}_3F_2$ at $x = 1$, and on the other hand, also a solution of a difference equation, there arises the problem of the relation between these two equations. It is also interesting to study the role of hypergeometric functions for values of $x \neq 1$ in the general theory of C-G coefficients. Thus, generalized hypergeometric functions and special degenerate cases of these (Bessel functions, Legendre functions, Jacobi, Chebyshev, and Hermite polynomials, etc.) are closely tied to the theory of C-G coefficients. An investigation of these aspects is essential both for the theory of special functions and for the physical applications.

6. Higher Symmetries

The Regge symmetry considered in the preceding section means, essentially, that an SU(3) symmetry is present in the theory of C-G coefficients of the SU(2) group. The J-th power of the determinant occurring in the left-hand side of Eq. (5.2) is an invariant of the group SU(3) and the symbol $\|R_{ik}\|$ is a special form of Wigner coefficient for the group SU(3) (at the same time it is a general Wigner coefficient of the group SU(2)). Together with the SU(3) symmetry an essential role may also be played in the theory of angular momenta by other symmetries, higher than SU(3) ("higher symmetries"). Thus, by analogy to the square 3×3 symbol $\|R_{ik}\|$ one can construct $n \times n$ symbols, corresponding to an arbitrary SU(n) group, and occurring in the expansion of the J-th power of a determinant of rank n[33]. For the group SU(4) the expansion of the type (5.1) takes on the form of rank

$$\begin{Vmatrix} u_1 & v_1 & w_1 & t_1 \\ u_2 & v_2 & w_2 & t_2 \\ u_3 & v_3 & w_3 & t_3 \\ u_4 & v_4 & w_4 & t_4 \end{Vmatrix}^J = \sqrt{(J!)^3(J+1)}\sum_{R_{ik}}\begin{Vmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{Vmatrix}\frac{\prod_i u_i^{R_{1i}} v_i^{R_{2i}} w_i^{R_{3i}} t_i^{R_{4i}}}{(\prod_{ik} R_{ik}!)^{1/2}}. \tag{6.1}$$

Here the 4×4 symbol $\|R_{ik}\|$ already exhibits $4! \times 4! \times 2 = 1152$ symmetry rules (permutations of rows and columns and transposition). For the group SU(n) the $n \times n$ symbol $\|R_{ik}\|$ corresponding to the expansion

$$|u_{ik}|^J = \sqrt{(J!)^3(J+1)}\sum \|R_{ik}\|\frac{\prod_{ik} u_{ik}^{R_{ik}}}{(\prod_{ik} R_{ik}!)^{1/2}}. \tag{6.2}$$

will exhibit $n! \times n! \times 2$ symmetries.

In the discussion of the $n \times n$-symbols there appear essentially new combinations of the C-G coefficients. As an example, we consider the group SU(4). Expanding the determinant with respect to a column, we obtain

$$\begin{vmatrix} u_1 & v_1 & w_1 & t_1 \\ u_2 & v_2 & w_2 & t_2 \\ u_3 & v_3 & w_3 & t_3 \\ u_4 & v_4 & w_4 & t_4 \end{vmatrix}^J = \sum_{\sum_k R_{ik}=J}\frac{J!}{R_{11}!\,R_{12}!\,R_{13}!\,R_{14}!}t_1^{R_{11}}t_2^{R_{12}}t_3^{R_{13}}t_4^{R_{14}}$$
$$\times\begin{vmatrix} w_2 & w_3 & w_4 \\ v_2 & v_3 & v_4 \\ u_2 & u_3 & u_4 \end{vmatrix}^{R_{11}}\begin{vmatrix} w_1 & w_3 & w_4 \\ v_1 & v_3 & v_4 \\ u_1 & u_3 & u_4 \end{vmatrix}^{R_{12}}\begin{vmatrix} w_1 & w_2 & w_4 \\ v_1 & v_2 & v_4 \\ u_1 & u_2 & u_4 \end{vmatrix}^{R_{13}}\begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}^{R_{14}}. \tag{6.3}$$

Making use of the relations (5.1) and (6.3) we obtain[35]

$$\sqrt{(J+1)!}\begin{vmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{vmatrix} =$$
$$=\sqrt{\prod_i (R_{1i}+1)!}\sum_{\sum_{\beta\neq\alpha} B_{i\alpha}^{\beta}=R_{i\alpha}}\frac{(\prod_{ik} R_{ik}!)^{1/2}}{\left(\prod_{\substack{\beta=1,2,3,4 \\ \alpha\neq\beta}} B_{2\alpha}^{\beta}!\,B_{3\alpha}^{\beta}!\,B_{4\alpha}^{\beta}!\right)^{1/2}} \tag{6.4}$$
$$\times\begin{Vmatrix} B'_{22} & B'_{23} & B'_{24} \\ B'_{32} & B'_{33} & B'_{34} \\ B'_{42} & B'_{43} & B'_{44} \end{Vmatrix}\begin{Vmatrix} B_{21}^2 & B_{23}^2 & B_{21}^2 \\ B_{31}^2 & B_{33}^2 & B_{34}^2 \\ B_{41}^2 & B_{43}^2 & B_{41}^2 \end{Vmatrix}\begin{Vmatrix} B_{21}^3 & B_{22}^3 & B_{24}^3 \\ B_{31}^3 & B_{32}^3 & B_{34}^3 \\ B_{11}^3 & B_{42}^3 & B_{41}^3 \end{Vmatrix}\begin{Vmatrix} B_{21}^4 & B_{22}^4 & B_{23}^4 \\ B_{31}^4 & B_{32}^4 & B_{33}^4 \\ B_{41}^4 & B_{42}^4 & B_{13}^4 \end{Vmatrix}.$$

In distinction from the convention of summing over combinations of two symbols, which is usual in the theory of C-G coefficients, here the summation goes over combinations of three symbols. A similar expansion can also be written for the 5×5-symbol; in this case four symbols participate in one summation, etc. We stress that each symbol (which is a factor in (6.4) or a similar equation) is a Wigner coefficient. The method of combining these is quite distinct from the usual method.

The direct physical interest in introducing the $n \times n$-symbols consists in a generalization of the concept of recoupling. In the classical theory the generalized C-G coefficients which appear in the addition of several angular momenta, are defined by specifying the intermediate momenta in the coupling scheme. Thus, the invariant formed from four spinors $x_i y_i$ $(i = 1, 2, 3, 4)$ has the form

$$\sum_{m_i}\begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{pmatrix}\begin{pmatrix} j_{12} & j_3 & j_4 \\ -m_{12} & m_3 & m_4 \end{pmatrix}\left[\frac{\prod_i (2j_i)!}{\prod_i (j_i-m_i)!\,(j_i+m_i)!}\right]^{1/2}\prod_i x_i^{j_i-m_i} y_i^{j_i-m_i}. \tag{6.5}$$

This is a consistent coupling scheme. In the case (6.5) the coupling is defined by specifying one intermediate angular momentum. One can generalize the concept of coupling scheme by means of $n \times n$-symbols; the new coupling schemes which appear are superpositions of couplings like (6.5), forming a complete system:

$$\sum_{m_i}\begin{Vmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ j_1-m_1 & j_2-m_2 & j_3-m_3 & j_4-m_4 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 & j_4+m_4 \end{Vmatrix}\left[\frac{\prod_i (2j_i)!}{\prod_i (j_i-m_i)!\,(j_i+m_i)!}\right]^{1/2} \tag{6.6}$$
$$\times\prod_i x_i^{j_i-m_i} y_i^{j_i+m_i}.$$

Since all angular momenta occurring in (6.6) are on an equal footing, one may call this coupling scheme symmetric. The transition between different coupling schemes (recoupling) is defined by the transformation matrix

$$\sum_{m_i}\begin{Vmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ j_1-m_1 & j_2-m_2 & j_3-m_3 & j_4-m_4 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 & j_4+m_4 \end{Vmatrix}\begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{pmatrix}\begin{pmatrix} j_{12} & j_3 & j_4 \\ -m_{12} & m_3 & m_4 \end{pmatrix}, \tag{6.7}$$

which extends the usual theory of transformation matrices.

The investigation of the $n \times n$-symbols is in many respects analogous to Sec. 5. We list the explicit expression for the $n \times n$-symbols, obtained by the same method as Eq. (5.12)[44,45]:

$$\left\| \begin{matrix} R_{11} \ldots R_{1n} \\ \cdots\cdots \\ R_{n1} \ldots R_{nn} \end{matrix} \right\| = \left[\frac{\prod\limits_{i,k=1}^{n} R_{ik}!}{(J+1)!} \right]^{1/2} \frac{\sum (-1)^{\sum\limits_{[l]} P_{l_1 \ldots l_n}}}{\prod\limits_{l_1 \ldots l_n} P_{l_1 \ldots l_n}!}. \tag{6.8}$$

Here $[l]$ denotes the set $(l_1 \ldots l_n)$ determined by the odd permutations, and $(l_1 \ldots l_n)$ denotes all the permutations of the indices. The summation is carried out over all nonnegative integers satisfying a system of D^2 equations of the form

$$R_{ik} = \sum_{l_1 \ldots l_n} R_{l_1 \ldots l_{i-1} k l_{i+1} \ldots l_n}, \tag{6.9}$$

analogous to the Eq. (5.13) for the 3×3-symbol.

The problems of the algebraic structure of the generalized theory are of great interest. Giovannini and Smith[46] have considered the $n \times n$-symbols as magic squares (magic squares have the property that the sum of the entries in all rows and columns is the same, where the entries are nonnegative integers). They have pointed out the special significance of the $n \times n$-symbols, which are the coefficients of the expansion of the determinant to the power $J = 1$, from which one can construct a symbol of arbitrary rank. These symbols form a group of $n \times n$-matrices isomorphic to the symmetric group (permutation group) S_n. We note that the Racah coefficient (6j-symbol) can also be represented in the form of a magic square[35]. In the notation (5.4) this coefficient was listed as a 3×4 table. The latter is part of a more general 4×4 table, corresponding to the magic square (the symbol to the right is given for illustration of a different order of arrangement of the arguments)

$$\left\| \begin{matrix} x_1+y_1 & x_4+y_2 & x_2+y_3 & x_3+y_4 \\ x_2+y_2 & x_3+y_1 & x_1+y_4 & x_4+y_3 \\ x_3+y_3 & x_2+y_4 & x_4+y_1 & x_1+y_2 \\ x_4+y_4 & x_1+y_3 & x_3+y_2 & x_2+y_1 \end{matrix} \right\| = \left\| \begin{matrix} x_1+y_1 & x_2+y_1 & x_3+y_1 & x_4+y_1 \\ x_1+y_2 & x_2+y_2 & x_3+y_2 & x_4+y_2 \\ x_1+y_3 & x_2+y_3 & x_3+y_3 & x_4+y_3 \\ x_1+y_4 & x_2+y_4 & x_3+y_4 & x_4+y_4 \end{matrix} \right\|. \tag{6.10}$$

In the j-notation the symbol $\begin{Bmatrix} j_1 j_2 j_4 \\ j_3 j_0 j_5 \end{Bmatrix}$ can be written in the form (the form given here differs from that in[35])

$$\left\| \begin{matrix} j_1+j_5-j_0 & j_0+j_5-j_1 & j_1+j_0-j_5 & j_2+j_3+j_4 \\ j_2+j_4-j_1 & j_1+j_4-j_2 & j_0+j_3+j_5 & j_1+j_2-j_4 \\ j_0+j_3-j_4 & j_1+j_2+j_5 & j_3+j_4-j_0 & j_0+j_4-j_3 \\ j_1+j_4+j_0 & j_2+j_3-j_5 & j_2+j_5-j_3 & j_3+j_5-j_2 \end{matrix} \right\|. \tag{6.11}$$

A discussion of 3j-, 6j-, and $n \times n$-symbols as magic squares is important not only from the viewpoint of symmetry, but has a deeper combinatorial meaning. Thus, a special case of magic squares are the so-called latin squares, in which the integers $a_1 \ldots a_n$ are arranged in such an order that each integer appears once and only once in each row and each column. Their theory is closely related to the general theory of block-schemes[22,24,27], and thus with problems of control theory, experiment planning, and coding. In this connection it is interesting to note a definite distinction between the 3×3-symbols and higher-rank symbols. The 3×3-symbols (considered as magic squares) are defined by specifying all their 9 elements. The condition of "magicity" for a fixed sum J yields five independent relations. Writing the elements in the form of a Regge table we can define four of them arbitrarily, e.g., $j_1 j_2 m_1 m_2$. Since $m_3 = -m_1 - m_2$ and the sum $j_1 + j_2 + j_3 = J$ has been fixed, the magic 3×3-square completely determines the C-G coefficient. In other words, a C-G coefficient is determined by 9 positive integers and the condition that the square be magic.

For 4×4-squares this situation no longer prevails. The listing of the entries does not determine the magic square uniquely (i.e., up to permutations). In this connection we note that the Racah coefficients are also not completely determined by simply listing the elements of the table—one must also indicate the order in which they occur in the table.

For combinatorial applications the transformed $n \times n$-symbols $\langle | R_{ik} | \rangle$ might present some interest. These symbols are the coefficients in the expansion of an arbitrary determinant

$$|u_{ik}|^J = \sum_{R_{ik}} \langle | R_{ik} | \rangle \prod_{ik} u_{ik}^{R_{ik}}. \tag{6.12}$$

They are a direct generalization of the binomial coefficients

$$\left\langle \begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{vmatrix} \right\rangle,$$

which occur in the expansion for $n = 2$:

$$\begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix}^J = \sum_{R_{ik}} \left\langle \begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{vmatrix} \right\rangle u_{11}^{R_{11}} u_{12}^{R_{12}} u_{21}^{R_{21}} u_{22}^{R_{22}}. \tag{6.13}$$

The properties of the symbols $\langle | R_{ik} | \rangle$ are obtained from the properties of the determinants according to the method of Sec. 5. The Vandermonde formula (3.1) can be rewritten in these notations:

$$\sum_{m_1+m_2=m} \left\langle \begin{vmatrix} j_2+m_2 & j_2-m_2 \\ j_2-m_2 & j_2+m_2 \end{vmatrix} \right\rangle \left\langle \begin{vmatrix} j_1+m_1 & j_1-m_1 \\ j_1-m_1 & j_1+m_1 \end{vmatrix} \right\rangle = \left\langle \begin{vmatrix} j_1+j_2+m & j_1+j_2-m \\ j_1+j_2-m & j_1+j_2+m \end{vmatrix} \right\rangle. \tag{6.14}$$

In spite of the importance of the new symmetries of the C-G coefficients and their combinations, the total number of papers on this subject is relatively small. In addition to the already quoted papers, one should particularly point out the papers[9,48], where it is shown that the use of the Regge symmetries has definite advantages for computer calculations, the paper[49], where the Regge symmetries were discussed on the basis of the theory of entire analytic functions, and the papers[21,26,27], where the Regge symmetry was discussed on the basis of the quasibinomial representations of the C-G coefficients and the relation between the symmetries of the squares in (3.14) and (3.15) and the Regge symbols was pointed out. On the basis of Schwinger's boson operator technique[50], Bincer[51] has proposed the interpretation of the Regge and higher symmetries as reduction symmetries. The Regge symmetry is related to different ways of reducing the Kronecker product, expressed in terms of boson operators. It was shown that in the classical limit the Regge symmetry is equivalent to the $m \longleftrightarrow \mu$ symmetry in the finite rotation matrix $D^j_{m\mu}$. A number of other aspects related to the new symmetries are discussed in in[52,53,54].

7. Generalized Angular Momentum Theory

The new symmetries, combinations, and methods of coupling of the C-G lead to a generalization of the theory of angular momenta. The object of the generalized theory are the $n \times n$-symbols of various ranks and their possible combinations, as well as the appropriate generating invariants[44,45,55]. The classical theory of angular momenta is a special case, containing the restricted class of combinations of 3×3-symbols. The generalized angular momentum theory generates a wide variety of new forms, presenting interest both from the viewpoint of mathematics and its applications. Even consideration of combinations of 3×3-symbols corresponding to Wigner coefficients yields many new facts. In the usual construction of combinations of C-G coefficients (jm- and j-symbols) there occur two types of invariant summation, dyadic and triadic[57]. The dyadic method corresponds to summing over the projections of the angular momentum (with respect to two elements of the Regge symbol); the triadic method corresponds to a combination of transformation matrices and involves a sum over a triad of angular momenta (the upper row (R_{11}, R_{12}, R_{13}) of the Regge symbol (5.1). However, from the viewpoint of the generalized theory, according to (5.2) all R_{ik} entering in the 3×3-symbols are to be treated on an equal footing. Therefore for invariant summation of products of Regge symbols (the construction of combinations) one may select any elements R_{ik} (three elements for triadic summation, two for dyadic summation, from any row or column). Thus, in addition to the usual summation of Regge symbols, carried out over $R_{21}R_{31}$, $R_{22}R_{32}$, $R_{23}R_{33}$, $R_{11}R_{12}R_{13}$, one can consider invariant expressions obtained by summing over $R_{11}R_{12}R_{13}$, $R_{21}R_{22}R_{23}$ or over $R_{21}R_{31}$, $R_{22}R_{23}$, $R_{32}R_{33}$, etc. This yields a rich class of "nonstandard" combinations. However, up to the present, only a small number of such combinations has been investigated or used. Combinations of the form

$$(j_1 j_2 \nu_1 \nu_2 \mid j\nu)\,(T_1 T_2 t_1 t_2 \mid Tt), \tag{7.1}$$

where $j_1 + \nu_1 = -T_1 + T_2 + T$, $j_2 + \nu_2 = T_1 - T_2 + T$, and

$$(j_1' j_2' \nu_1' \nu_2' \mid j'\nu')\,(j_1 j_2 \nu_1 \nu_2 \mid j\nu)\,(T_1 T_2 t_1 t_2 \mid Tt), \tag{7.2}$$

where $j_1' + \nu_1' = -j_1 + j_2 + j$, $j_2' + \nu_2' = j_1 - j_2 + j$, $j_1 + \nu_1 = -T_1 + T_2 + T$, $j_2 + \nu_2 = T_1 - T_2 + T$, both derived from the usual generalized C-G coefficients[8] by means of the Regge symmetries, are of importance in the theory of C-G coefficients for the SU(n)-groups. Another special form of "nonstandard" combinations, containing simultaneously dyadic and triadic summations, are the formulas obtained by Vanagas and Batarunas[58] with the aid of the characters of the symmetric group. Of interest is also the formula obtained with the aid of dyadic summations over elements situated in different rows and columns:

$$\sum_{\varkappa} (-1)^{R_{12}} \left\langle \begin{vmatrix} R_{11}+\varkappa & R_{12}-\varkappa & R_{13} \\ R_{21} & R_{22}+\varkappa & R_{23}-\varkappa \\ R_{31}-\varkappa & R_{32} & R_{33}+\varkappa \end{vmatrix} \right\rangle = (-1)^{R_{21}} \frac{J!}{R_{13}!\,R_{32}!\,(J-R_{13})!\,(J-R_{23})!} \cdot \tag{7.3}$$

Although it is premature to talk about physical applications of the "nonstandard" combinations one may expect a change in this situation in the future. The completely new coupling types (6.4) seem to be very promising, as well as the use of various combinations of $n \times n$-symbols of higher rank than the Regge symbols. In this connection it is interesting to note that according to[44,45,55] the C-G coefficients of the group SU(n) can be constructed as combinations of $n \times n$-symbols, i.e., the theory of C-G coefficients of higher groups is an integral part of the generalized angular momentum theory. However, the investigation of this group of problems is still in its infancy.

In addition to the consideration of $n \times n$-symbols and of their combinations, an important part of the generalized theory are the generating invariants[44,45,55,56]. Since higher symmetries manifest themselves in the generalized theory, particularly symmetries corresponding to the groups SU(n) (and other semisimple Lie groups), it becomes necessary to use fully the theory of invariants of the classical groups. This theory discussed in detail in Weyl's book[60].

In analogy with the way in which a Wigner coefficient combines three irreducible representations into an invariant (and the generalized Wigner coefficient combines several such representations), one can associate combinations of $n \times n$-symbols of different ranks with definite generating invariants. According to (6.1) the role of generating invariant for the $n \times n$-symbol is played by the determinant of rank n, raised to the J-th power. In particular, for the Wigner symbol (the 3×3-symbol), it is $(\epsilon_{ikl} u_{1i} u_{2k} u_{3l})^J$, and for the metric tensor (the 2×2-symbol $\begin{matrix} R_{11} R_{12} \\ R_{21} R_{22} \end{matrix}$) it is $(\epsilon_{\lambda\mu} u_{1\lambda} u_{2\mu})^{J'}$. The generating invariants of any standard combinations of C-G coefficients of the group SU(2) can be constructed by means of the tensors ϵ_{ikl} and $\epsilon_{\lambda\mu}$. Thus, e.g., the generating invariant for the product of two C-G coefficients, summed over the projections m_1 and m_2, is of the form

$$(\varepsilon_{lmn} u_{1\underline{l}} u_{2\underline{m}} u_{3n})^J (\varepsilon_{l'm'n'} u_{1\underline{l}'} u_{2\underline{m}'} u_{3n'})^{J'} (\varepsilon_{\lambda\lambda'})^{B_1} (\varepsilon_{\mu\mu'})^{B_2}. \tag{7.4}$$

Here the Latin indices l, m, n take on the whole set of possible values 1, 2, 3; the underlined latin indices $\underline{l}$, $\underline{m}$, $\underline{n}$ take on only the value 1; the Greek indices λ, μ, ν corresponding to l, m, n take on the remaining values 2 and 3. In this notation the generating invariant for the Racah coefficient (6j-symbol) can be written in the form

$$\begin{aligned}(\varepsilon_{l_1 m_1 n_1} u'_{1\underline{l}_1} u'_{2\underline{m}_1} u'_{3\underline{n}_1})^{J_1} (\varepsilon_{l_2 m_2 n_2} u^2_{1\underline{l}_2} u^2_{2\underline{m}_2} u^2_{3\underline{n}_2})^{J_2} \\ \times (\varepsilon_{l_3 m_3 n_3} u^3_{1\underline{l}_3} u^3_{2\underline{m}_3} u^3_{3\underline{n}_3})^{J_3} (\varepsilon_{l_4 m_4 n_4} u^4_{1\underline{l}_4} u^4_{2\underline{m}_4} u^4_{3\underline{n}_4})^{J_4} \\ \times (\varepsilon_{\lambda_1\lambda_2})^{B_{12}} (\varepsilon_{\mu_1\mu_5})^{B_{13}} (\varepsilon_{\nu_1\nu_4})^{B_{14}} (\varepsilon_{\lambda_3\lambda_4})^{B_{34}} (\varepsilon_{\mu_2\mu_4})^{B_{24}} (\varepsilon_{\nu_2\nu_3})^{B_{23}}.\end{aligned} \tag{7.5}$$

Similarly one can write out the generating invariants for an arbitrary combination of Wigner coefficients and for any transformation matrix. The expansion coefficients of the generating invariants for expansions in powers of u_{ik} are combinations of Wigner coefficients. Thus, the product of Wigner coefficients corresponding to the generating invariant (7.4) is

$$\sum \begin{Vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{Vmatrix} \begin{Vmatrix} R'_{11} & R'_{12} & R'_{13} \\ R'_{21} & R'_{22} & R'_{23} \\ R'_{31} & R'_{32} & R'_{33} \end{Vmatrix} \begin{Vmatrix} R_{21} & R'_{21} \\ R_{31} & R'_{31} \end{Vmatrix} \cdot \begin{Vmatrix} R_{22} & R'_{22} \\ R_{32} & R'_{32} \end{Vmatrix}. \tag{7.6}$$

The summation is carried out over repeated R_{ik}; the 6j-symbol corresponding to the generating invariant (7.5) is

$$\sum \left\| \begin{matrix} R'_{11} & R'_{12} & R'_{13} \\ R'_{21} & R'_{22} & R'_{23} \\ R'_{31} & R'_{32} & R'_{33} \end{matrix} \right\| \left\| \begin{matrix} R^2_{11} & R^2_{12} & R^2_{13} \\ R^2_{21} & R^2_{22} & R^2_{23} \\ R^2_{31} & R^2_{32} & R^2_{33} \end{matrix} \right\| \left\| \begin{matrix} R^3_{11} & R^3_{12} & R^3_{13} \\ R^3_{21} & R^3_{22} & R^3_{23} \\ R^3_{31} & R^3_{32} & R^3_{33} \end{matrix} \right\| \left\| \begin{matrix} R^4_{11} & R^4_{12} & R^4_{13} \\ R^4_{21} & R^4_{22} & R^4_{23} \\ R^4_{31} & R^4_{32} & R^4_{33} \end{matrix} \right\|$$
$$\times \left\| \begin{matrix} R'_{21} & R^2_{21} \\ R'_{31} & R^2_{31} \end{matrix} \right\| \left\| \begin{matrix} R'_{22} & R^3_{22} \\ R'_{32} & R^3_{33} \end{matrix} \right\| \left\| \begin{matrix} R'_{23} & R^4_{23} \\ R'_{33} & R^4_{33} \end{matrix} \right\| \left\| \begin{matrix} R^3_{21} & R^4_{21} \\ R^3_{31} & R^4_{31} \end{matrix} \right\| \left\| \begin{matrix} R^2_{22} & R^4_{22} \\ R^2_{32} & R^4_{32} \end{matrix} \right\| \left\| \begin{matrix} R^2_{23} & R^3_{23} \\ R^2_{33} & R^3_{33} \end{matrix} \right\|. \tag{7.7}$$

All combinations of the classical angular momentum theory can be expressed in terms of a sum of products of 2×2-symbols and 3×3-symbols. An example of a generating invariant for combinations of $n\times n$-symbols of higher rank is given by the expression

$$\sum_k \begin{vmatrix} u_0 & v_0 & w_0 & t_0 \\ u_1 & v_1 & w_1 & \delta_{1k_1} \\ u_2 & v_2 & w_2 & \delta_{2k_2} \\ u_3 & v_3 & w_3 & \delta_{3k_3} \end{vmatrix}^{J_1} \begin{vmatrix} \delta_{1k'_1} & x_1 & y_1 \\ \delta_{2k'_2} & x_2 & y_2 \\ \delta_{3k'_3} & x_3 & y_3 \end{vmatrix}^{J_2} \Big(\prod_i \delta_{k_i k'_i}\Big)^I. \tag{7.8}$$

The coefficients in the expansion of (7.8) are given by the combinations

$$\sum_{R_{24}R_{34}R_{44}} \left\| \begin{matrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{matrix} \right\| \left\| \begin{matrix} R_{24} & R_{25} & R_{26} \\ R_{34} & R_{35} & R_{36} \\ R_{44} & R_{45} & R_{46} \end{matrix} \right\|. \tag{7.9}$$

The possibility of formulating the generalized theory of angular momenta in terms of generating invariants makes the role of the latter quite prominent, allowing one to solve applied problems directly in terms of the invariants. The theory of invariants[60], which has been actively developed at the end of the last century, thus acquires new applications. Let us consider as an example the methods of coupling angular momenta. Out of k spinors u_i ($i = 1, \ldots, k$) of the group SU(2) one can construct generating invariants for the Wigner coefficients of rank k. (Here the rank corresponds to the number of component angular momenta; the notation $[u_i u_k]$ corresponds to the determinant.) The invariants

$$[u_1u_2]^{R_{13}}[u_1u_3]^{R_{12}}[u_2u_3]^{R_{11}} \tag{7.10}$$

correspond to the usual Wigner coefficient of the group SU(2). The invariants

$$[u_1u_2]^{A_{12}}[u_1u_3]^{A_{13}}[u_1u_4]^{A_{14}}[u_2u_3]^{A_{23}}[u_2u_4]^{A_{24}}[u_3u_4]^{A_{34}} \tag{7.11}$$

corresponds Wigner coefficients of rank 4. In operations with generating invariants one should keep in mind that not all of them are linearly independent. Thus, for instance, the identity

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = 0 \tag{7.12}$$

implies the following relation among the determinants of order 2:

$$[u_1u_2][u_3u_4] + [u_3u_1][u_2u_4] + [u_1u_4][u_2u_3] = 0, \quad u_i = \{x_i y_i\}. \tag{7.13}$$

Therefore, in distinction from the Wigner coefficients of rank 3, those of rank 4 are not unique and depend on the choice of independent invariants (in other words, on the method of coupling the angular momenta). Owing to the relations of type (7.13) (syzygies) there appears a host of forms in the theory of invariants, and many possibilities arise for the choice of independent invariants[6]. The group-theoretic meaning of such relations is that the Kronecker product of two irreducible representations may contain the same irreducible representation D_j several times. The operator which distinguishes the basis functions of multiply occuring representations is not contained in the group and must be additionally specified, e.g., by specifying the coupling scheme of the angular momenta. Equations (6.5) and (6.6) illustrate two methods of constructing the generating invariants. The completeness property of the symmetric coupling method (6.6) implies the possibility of representing the 6j-symbols in terms of linear combinations of the quantities τ_n:

$$\tau_n = \sum_{m_i} \left\| \begin{matrix} R_{11} & R_{12} & \ldots & R_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ R_{n-21} & R_{n-22} & \ldots & R_{n-2n} \\ j_1 - m_1 & j_2 - m_2 & \ldots & j_n - m_n \\ j_1 + m_2 & j_2 + m_2 & \ldots & j_n + m_n \end{matrix} \right\| \left\| \begin{matrix} R'_{11} & R'_{12} & \ldots & R'_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ R'_{n-21} & R_{n-22} & \ldots & R_{n-2n} \\ j_1 + m_1 & j_2 + m_2 & \ldots & j_n + m_n \\ j_1 - m_1 & j_2 - m_2 & \ldots & j_n - m_n \end{matrix} \right\|. \tag{7.14}$$

On the other hand, the generating invariant for any coupling method can always be written in the form of a linear combination of quantities of the type (7.11). Thus, the Wigner coefficient of rank 5 for the $((j_1 j_2) j_{12} j_3) j_{123} j_4 j)$-coupling

$$\begin{pmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{pmatrix} \begin{pmatrix} j_{12} & j_3 & j_{123} \\ m'_{12} & m_3 & m_{123} \end{pmatrix} \begin{pmatrix} j_{123} & j_4 & j \\ m'_{123} & m_4 & m \end{pmatrix} \begin{pmatrix} & j_{12} & \\ m_{12} & & m'_{12} \end{pmatrix} \begin{pmatrix} \varepsilon & j_{12} & \\ m_{123} & & m'_{123} \end{pmatrix} \tag{7.15}$$

corresponds (apart from the normalization) to the generating invariant

$$\sum [uv]^{A_1}[u\varepsilon_{kl}]^{A_2}[v\varepsilon_{kl}]^{A_3}[w\varepsilon_{kl}]^{A'_1}[w\varepsilon_{kl'}]^{A'_2}[\varepsilon_{kl}\varepsilon_{kl'}]^{A'_3}[ts]^{A''_1}[t\varepsilon_{kl'}]^{A''_1}[s\varepsilon_{kl'}]^{A''_3}$$
$$= \sum \frac{[uv]^{\beta_{12}}[uw]^{\beta_{13}}[vw]^{\beta_{23}}[ut]^{\beta_{14}}[vt]^{\beta_{24}}[us]^{\beta_{15}}[wt]^{\beta_{34}}[ws]^{\beta_{35}}[st]^{\beta_{45}}[vs]^{\beta_{25}}}{\prod_{ik}\beta_{ik}!}, \tag{7.16}$$

where

$$[uv] = (\varepsilon^{ik}u_i v_k), \quad [u\varepsilon_{kl}] = (\varepsilon^{ik}u_i\varepsilon_{kl}) = u_l \text{ и т. д.},$$
$$\beta_{12} = A_1 = j_1 + j_2 - j_{12}, \quad \beta_{13} + \beta_{23} = A'_1, \quad \beta_{15} = A''_1,$$
$$\beta_{13} + \beta_{14} + \beta_{15} = A_2, \quad \beta_{34} + \beta_{35} = A'_2, \quad \beta_{11} + \beta_{24} + \beta_{34} = A''_2,$$
$$\beta_{23} + \beta_{24} + \beta_{25} = A_3, \quad \beta_{14} + \beta_{24} + \beta_{15} + \beta_{25} = A'_3, \quad \beta_{15} + \beta_{25} + \beta_{35} = A''_3.$$

Expanding the invariant I_A corresponding to the coupling scheme A in terms of the invariants I_B, corresponding to the coupling scheme B, we have

$$I_A = \sum_B c_{A,B} I_B. \tag{7.17}$$

Making use of the expansion of invariants in powers of the components of the spinors which make them up we obtain relations between quantities and transformation matrices. The generating invariants also throw light on a series of other problems, of which the interrelation of symmetries merits special attention.

8. The Interrelation of Symmetries

The generalized theory of angular momenta, which in distinction from the classical theory, contains formulas referring to higher symmetries, allows one to analyze the interrelations of symmetries in a real system. An essential role in such a theory is played by the generating invariants. This is due to the fact

[6]One can select as independent invariants three- and higher-dimensional determinants [61]. Although the use of such spatial (3 × 3 × 3 and higher) determinants seems promising for the generalized theory, such problems have not yet been investigated.

that the same quantity can be the generating invariant for the C-G coefficients of different groups. The methods of deriving and studying the character of such interrelations among C-G coefficients was discussed in[56], using as an example the expression

$$\psi_{p_{ik}}(x_i y_i) = \left[\frac{P!}{\prod\limits_{i<k} p_{ik}!}\right]^{1/2} \prod_{i<k\leqslant 4} \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}^{p_{ik}}, \qquad (8.1)$$

which represents the normalized product of six second-order determinants raised to the powers p_{ik}. In analogy to the way in which the third order determinant raised to the J-th power plays the role of the generating invariant for the C-G coefficients of the group SU(2) and for C-G coefficients of a special type of the group SU(3), the expression (8.1) is the starting point for the derivation of the C-G coefficients of different groups. This expression can be treated as the basis of the representation D(0P0) of the group SU(4)[59], by attributing to the quantities x_i and y_i the transformation properties of the basis vectors of the fundamental representation D(100). On the other hand, attributing to the pairs x_i and y_i the transformation properties of spinors, one may consider them as the basis of the generating invariants for the Wigner coefficients of rank 4 of the group SU(2), of type (7.11). One may also attribute to (8.1) the meaning of a basis for the group SU(3). This yields the relation among the C-G coefficients of the groups SU(2), SU(3) and SU(4), based on the relation (8.1). Such an approach can be used in every concrete case. In[62] the symmetry of the C-G coefficients was used to obtain a unique classification of the invariants of the group SU(2) in terms of the representations of the group SU(3), and of the invariants of the group SL(2, C) in terms of the representations of SL(3, C) and its compact subgroup SU(3). The transformation properties of the Lagrangians obtained in this way were compared with symmetry properties of elementary particles. Attributing to the quantities u_i, v_i, w_i in (5.2) the meaning of basis vectors for the representation D(10) of SU(3), one can write down a relation which defines the contravariant representation of SU(3):

$$\frac{U_1^{R_{11}} U_2^{R_{12}} U_3^{R_{13}}}{(R_{11}!\,R_{12}!\,R_{13}!)^{1/2}} = \sqrt{(J+1)!} \sum_{\sum\limits_i R_{2i}=J} \begin{Vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{Vmatrix} \frac{\prod\limits_i v_i^{R_{2i}} w_i^{R_{3i}}}{\left(\prod\limits_i R_{2i}!\,R_{3i}!\right)^{1/2}}. \qquad (8.2)$$

On the other hand, if the quantities v_1, w_1, v_2, w_2, v_3, w_3 in the right-hand side of (8.2) are formally considered as spinors of the group SU(2), the right-hand side represents the normalized product of powers of the minors of the determinant (5.2), and according to[7], is an invariant of the group SU(2)

$$\frac{U_1^{R_{11}} U_2^{R_{12}} U_3^{R_{13}}}{(R_{11}!\,R_{12}!\,R_{13}!)^{1/2}} = \frac{1}{(R_{11}!\,R_{12}!\,R_{13}!)^{1/2}} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}^{R_{11}} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}^{R_{12}} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}^{R_{13}}$$

$$= \sum_{m_1+m_2+m_3=0} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \psi_{m_1}^{j_1} \psi_{m_2}^{j_2} \psi_{m_3}^{j_3};$$

here

$$\psi_{m_i}^{j_i} = \left[\frac{(2j_i)!}{(j_i+m_i)!\,(j_i-m_i)!}\right]^{1/2} v_i^{j_i+m_i} w_i^{j_i-m_i}. \qquad (8.3)$$

Thus, all invariants of the group SU(2) can be considered as basis vectors in the space of the representation SU(3). The converse is also true.

In analogy to SU(2) the invariants of the proper Lorentz group SL(2, C) form a basis for the representation of SL(3, C). The vectors e_{pq}^{PQ} of a canonical basis, corresponding to the representation $\tau(PQ)$ of SL(2, C), are expressed in terms of the vectors $e'^{P_1Q_1}_{p_1q_1}$ and $e''^{P_2Q_2}_{p_2q_2}$, corresponding to the representations $\tau(P_1Q_1)$ and $\tau(P_2Q_2)$, in the following manner

$$e_{pq}^{PQ} = \sum_{\substack{p_1p_2 \\ q_1q_2}} (P_1P_2p_1p_2 \mid Pp)\,(Q_1Q_2q_1q_2 \mid Qq)\, e'^{P_1Q_1}_{p_1q_1} e''^{P_2Q_2}_{p_2q_2}. \qquad (8.4)$$

In the right-hand side of (8.4) are the usual Clebsch-Gordan coefficients which can be written in the form of 3 × 3-symbols. Accordingly, any invariant of the Lorentz group can be written in the form

$$\sum_{\substack{p_1p_2p_3 \\ q_1q_2q_3}} \begin{Vmatrix} -P_1+P_2+P_3 & P_1-P_2+P_3 & P_1+P_2-P_3 \\ P_1-p_1 & P_2-p_2 & P_3-p_3 \\ P_1+p_1 & P_2+p_2 & P_3+p_3 \end{Vmatrix}$$

$$\times \begin{Vmatrix} -Q_1+Q_2+Q_3 & Q_1-Q_2+Q_3 & Q_1+Q_2-Q_3 \\ Q_1-q_1 & Q_2-q_2 & Q_3-q_3 \\ Q_1+q_1 & Q_2+q_2 & Q_3+q_3 \end{Vmatrix} \psi_{1\,p_1q_1}^{P_1Q_1} \psi_{2\,p_2q_2}^{P_2Q_2} \psi_{3\,p_3q_3}^{P_3Q_3}. \qquad (8.5)$$

The invariant (8.5) is determined by the six indices:

$$\pi_1 = -P_1+P_2+P_3,\quad \pi_2 = P_1-P_2+P_3,\quad \pi_3 = P_1+P_2-P_3,$$
$$\varkappa_1 = -Q_1+Q_2+Q_3,\quad \varkappa_2 = Q_1-Q_2+Q_3,\quad \varkappa_3 = Q_1+Q_2-Q_3;$$
$$\pi_1+\pi_2+\pi_3 = P_1+P_2+P_3 = \pi,\quad \varkappa_1+\varkappa_2+\varkappa_3 = Q_1+Q_2+Q_3 = \varkappa.$$

Since SU(3) is a subgroup of SL(3, C), each relativistic invariant is a basis vector of some reducible representation of SU(3). The invariants which differ in the order of ψ_1, ψ_2, ψ_3 can form the six components of a representation of SU(3). In order to construct a full basis it is necessary, in general, to use invariants with other ψ_1, ψ_2, ψ_3. In some simple cases the indices $[\pi_1\pi_2\pi_3][\kappa_1\kappa_2\kappa_3]$ characterize directly the basis vectors $M^{\pi_1\pi_2\pi_3}_{\kappa_1\kappa_2\kappa_3}$ of the representation $D(\pi\kappa)$ of SU(3) in a symmetric basis[44]. An example of the correspondence between representations of the Lorentz group, according to which transform the basis vectors ψ_1, ψ_2, ψ_3, and the indices is:

$$\left.\begin{array}{ll} \text{a)} & [\tau(\tfrac{1}{2}\,0)\,\tau(\tfrac{1}{2}\,0)\,\tau(00)] \to [100]\,[000], \\ \text{b)} & [\tau(\tfrac{1}{2}\,0)\,\tau(00)\,\tau(\tfrac{1}{2}\,0)] \to [010]\,[000], \\ \text{c)} & [\tau(\tfrac{1}{2}\,0)\,\tau(0\,\tfrac{1}{2})\,\tau(\tfrac{1}{2}\,\tfrac{1}{2})] \to [010]\,[100], \\ \text{d)} & [\tau(\tfrac{1}{2}\,\tfrac{1}{2})\,\tau(\tfrac{1}{2}\,\tfrac{1}{2})\,\tau(00)] \to [001]\,[001]. \end{array}\right\} \qquad (8.6)$$

The cases (a) and (b) differ by a permutation of the vectors ψ_i. The cases (c) and (d) correspond to the reducible representation D(11) + D(00) of SU(3), spanned by a totality of 9 vectors. In addition to the indices π_i, κ_i one can formally introduce other indices: π, κ, the hypercharge $Y = \kappa_1 - \pi_1 + \tfrac{1}{2}(\pi - \kappa)$, the isospin T and the isospin projection $t_3 = \tfrac{1}{2}(\pi_2 - \pi_3 + \kappa_3 - \kappa_2)$. Thus, the invariant $[\tau(\tfrac{1}{2}0)\tau(\tfrac{1}{2}0)\tau(00)]$ corresponds to three states (quarks), transforming according to the representation D(10):

$$\left.\begin{array}{llll} «\lambda»: & [100]\,[000], & Y = -\frac{2}{3}, & t_3 = 0, \\ «p»: & [010]\,[000], & Y = \frac{1}{3}, & t_3 = \frac{1}{2}, \\ «n»: & [001]\,[000], & Y = \frac{1}{3}, & t_3 = -\frac{1}{2}. \end{array}\right\} \qquad (8.7)$$

Such a classification also applies to the currents, since the decisive element is the presence of the C-G coefficient.

The theory of C-G coefficients gives a prescription for the classification of relativistic invariants according to the representations of SU(3). Each of the tensors which forms an invariant is, in distinction from the latter, not a basis vector of a representation of SU(3), i.e., there appears a new symmetry which characterizes the interaction and the corresponding Lagrangians. Let us classify the interactions of elementary particles according to the representations of the group SU(3). Consider the weak interaction Lagrangian

$$\bar{\psi}_a\gamma_\lambda(1+\gamma_5)\psi_b\bar{\psi}_c\gamma_\lambda(1+\gamma_5)\psi_d. \tag{8.8}$$

Rewriting this expression in terms of 3×3-symbols we find that by its transformation properties the weak current belongs to the octet representation of the group SU(3), and according to what was said before, it corresponds to the six basis vectors of D(11)

$$[010][001],\ [001][010],\ [100][001],\ [100][010],\ [010][100],\ [001][100]. \tag{8.9}$$

The first two vectors correspond to $Y = 0$, $t_3 = 1$, i.e., belong to the strangeness-conserving currents. The remaining vectors correspond to $Y = +1$, $t_3 = \frac{1}{2}$, i.e., belong to strangeness-changing currents. The weak interaction Lagrangian, as can be seen from (8, 9) is also a member of the SU(3)-octet. These are the same transformation properties as those ascribed to the Lagrangian and currents by Cabibbo[63], based on the experimental selection rules of the weak interactions, with the participation of hadrons. The electromagnetic interaction Lagrangian $(\bar{\psi}\gamma_\mu\psi)A_\mu$ is a sum of two invariants, constructed from three representations $[\tau(0\frac{1}{2})\tau(\frac{1}{2}\frac{1}{2})\tau(\frac{1}{2}0)]$ of the Lorentz group, and consequently form an incomplete set of basis vectors for the SU(3) octet. For the electromagnetic interaction there is also no contradiction between the transformation properties of the Lagrangian which follow from the theory of C-G coefficients, and the properties derived from the selection rules.

The fact that one obtains the right transformation properties is also an indication that if one considers the hadrons as a composite system, in distinction from the quark model, there is, in principle, no necessity to attribute to the subparticles the SU(3) quantum numbers, i.e., a fractional electric charge. These quantum numbers may refer only to the interaction and to the hadronic states in toto, in the same manner as the groups O(4) (O(5)) describe the hydrogen atom, but not separately the proton and electron, i.e., the system as a whole only. One may hope that further development of the problem of interrelations between C-G coefficients will help carry out a "target-oriented" search for elementary particle symmetries, to discoveries of hidden symmetries in the theory of nuclei and molecular spectroscopy. In the light of the SU(3)-example discussed above the idea of the concept of noninvariance group, introduced by Eddington[64], becomes clearer. According to this concept, physical systems can be characterized by symmetries which are not symmetry groups of the Lagrangian or the Hamiltonian. For example, in molecular spectroscopy one makes use of the groups R(5) and O(4, 1) for the classifications of the state of an electron in the field of many Coulomb force centers[65,67]. Among other results on noninvariance group one should point out the papers[68-72]. In a certain sense the three-body problem[73,74] belongs to this class of problems, and the generalized theory of angular momenta could be quite essential for it. In conclusion, we stress the fact that the technique of C-G coefficients is particularly important for complex physical systems, exhibiting a whole set of interrelated symmetries. Atoms, nuclei and hadrons are just such systems. One may expect in the future a widening of the sphere of applications of the theory of C-G coefficients, in particular to include problems related to the symmetries of leptons, the theory of coherent states, etc.

III. RELATIONS OF THE C-G COEFFICIENTS TO GEOMETRY AND TOPOLOGY

9. Geometric Interpretation

This third chapter is dedicated to the least developed part of the theory: the geometric and topologic interpretation of the C-G coefficients. From general principles the relation to geometric concepts seems to be quite natural. Already in 1872 in his famous "Erlangen program" Felix Klein (cf.[60,75]) has developed the group-theoretic approach to geometry. A typical example of a systematic exposition of geometry on the basis of the symmetry concept is Bachmann's monograph[76], which also contains an extensive bibliography of the subject. The combinatorial aspects of the theory of C-G coefficients, which were discussed above, immediately implies a relation to finite geometries[24].

One should think that an investigation of the relation to geometric characteristics should be useful also from the viewpoint of geometrization of physical concepts closely related to C-G coefficients.

Before discussing purely geometric problems we briefly consider some graphical methods.

In the classical theory of angular momenta graphical methods have been developed by Yutsis, Levinson and Vanagas[8] (cf. also[9]). The C-G coefficients were represented by a three-line vertex (Fig. 1a) and the summation over projections by joining lines together (Fig. 1b). The Regge symmetry implies a natural generalization of this approach, by assigning to the Wigner coefficient 9 free line-ends (Fig. 2,a) and all the other known graphical methods are obtained as special cases of this[77]. The reduction of the graphical methods is illustrated in Fig. 2. Figure 2b corresponds to the jm-formulation of the theory, when the summation is carried out over the upper row and over the columns of the two lower rows; Fig. 2c corresponds to summation over triads (j-symbols); Figs. 2d and e are simplified graphs; cf. also[8]); Fig. 2f represents the usual summation over projections[9], and Fig. 2g corresponds to the graph of the R-symbol, considered as the metric matrix in the representation space of the group SU(3). The indicated methods may turn out to be useful in the discussion of nonstandard combinations of C-G coefficients.

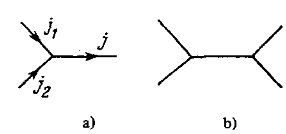

FIG. 1. Graphs of the classical theory.

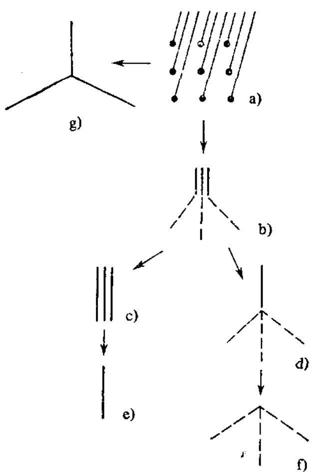

FIG. 2. Reduction of the graphical methods.

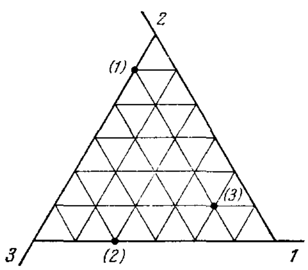

FIG. 3. Geometric interpretation of the Regge symbol.

The geometric interpretation of the Wigner coefficient is also closely related to the Regge symmetry[12]. We shall use triangular (barycentric) coordinates in the plane. Consider an equilateral triangle, the sides of which serve as coordinate axes. The values of the coordinates are counted perpendicularly to the axes, with positive values lying in the interior of the triangle. For any point the sum of the three triangular coordinates is a constant equal to J. We shall consider the values of the three lines of the Regge symbol as coordinates of three points. Since the R_{ik} are nonnegative integers, these points lie at the vertices of a coordinate net (Fig. 3). If two points are given in this system, the third is automatically defined. The 72 symmetry rules allow us to permute the axes and the points and in a certain sense to exchange their places. Thus, the Regge symmetry consists of the permutation symmetry (where any two of the three points can be interchanged), the coordinate symmetry, and the replacement of axes by points and vice-versa. As an example, Fig. 3 illustrates the graphical representation of the symbol

$$\left\|\begin{matrix} 5 & 1 & 0 \\ 0 & 4 & 2 \\ 1 & 1 & 4 \end{matrix}\right\| = \begin{pmatrix} 1/2 & 5/2 & 3 \\ 1/2 & -3/2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & -2 \end{pmatrix}.$$

When considering the addition of fixed angular momenta j′ and j″ one may also use a coordinate net consisting of equilateral triangles[77] (Fig. 4a). The coordinates are defined as the distances from the axes OM and ON. The angular momentum j with projection m is represented by the point with coordinates $(j - m, j + m)$, the projection is the distance from the point to the bisector of the coordinate angle. When the angular momenta $\{j'm'\}$ and $\{j''m''\}$ are added, the points corresponding

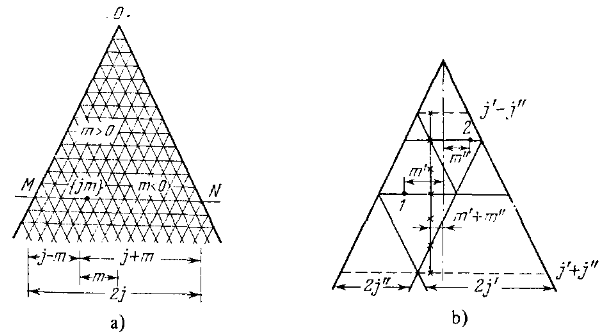

FIG. 4. Addition of angular momenta.

to the resulting angular momenta $\{jm\}$ are situated on a vertical at a distance $m' + m''$ from the bisector and between horizontal lines corresponding to $j' + j''$ and $|j' - j''|$ (Fig. 4,b).

For arbitrary $n \times n$-symbols the generalization of the described geometric construction is obvious. Whereas for the 3×3-symbol the values of R_{ik} are the barycentric coordinates of three points in a plane, for the 4×4-symbol the values of R_{ik} will be represented by barycentric coordinates of four points in space. The coordinate system is given by a regular tetrahedron. The coordinates of a point are its distances from the sides of the tetrahedron. The sum of the four coordinates of a point as well as the sum of the distances of all four points from a given coordinate plane are constant and equal to J. For an arbitrary $n \times n$-symbol the values of R_{ik} are the barycentric coordinates of n points in an $(n - 1)$-dimensional space. These points form a regular $(n - 1)$-dimensional simplex; the coordinate system is defined by this $(n - 1)$-simplex. The position of the points of an n-simplex within the regular coordinate simplex defines the numerical value of the $n \times n$-symbol. The determination of the numerical values of the $n \times n$-symbols by this geometric method was discussed with the Wigner symbol as an example in[77]. Different types of combinations of $n \times n$-symbols of various ranks, including their numerical values, as well as the symbols themselves, can be discussed in the language of higher-dimensional geometry, which thus can be used to express the generalized theory of angular momenta.

Together with n-dimensional geometry, the relation with projective geometry is of great interest[78,79]. Giovannini and Smith[46] who have considered the $n \times n$-symbols as magic squares, have also carried through this generalization. In distinction from the $n \times n$-symbols, which have nonnegative integers as their elements, the Q_n-symbols introduced in[46] have as elements R_{ik} arbitrary rational numbers. At the same time one requires, as before, that $\sum_i R_{ik} = \sum_k R_{ik} = J$. The algebra Q_n is a vector space over the rational number field. The totality of subspaces of this space forms a projective geometry satisfying the appropriate axioms[78,79]. The symmetry between rows and columns in the Q_n-symbols corresponds to the duality between points and straight lines in projective geometry. Based on the magic-square representation of the 6j-symbol (6.11), Geovannini and Smith[46] interpret the well-known relations[80]

$$\lim_{p\to\infty}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3+p & j+p & j_{23}+p \end{Bmatrix}=\begin{pmatrix} j_1 & j_2 & j_{12} \\ j-j_3 & j_{23}-j_3 & j_3-j \end{pmatrix} \qquad (9.1)$$

as a projection of the vector space Q_4 onto a certain subspace. As a whole, the relation between C-G coefficients and projective geometry is not sufficiently developed. It is conceivable that problems related to finite rotation matrices[51] and to vector parametrizations of the rotation and Lorentz groups[81,82] are also associated to these ideas. However, this direction requires further research. As regards multidimensional geometry, one has to stress particularly the aspects relating the C-G coefficients to algebraic topology.

10. Angular Momentum Theory and Topology

The theory of angular momenta can make effective use of the methods of algebraic topology[83,84]. Various combinations of C-G and Racah coefficients are characterized by topological invariants—Betti groups (homology groups).

At present the sphere of applications of topological methods to physics is perpetually growing: general relativity[85-87], solid state physics[88,89], quantum field theory[90,92].

Topology as a branch of mathematics originated toward the beginning of the twentieth century, mainly through the work of Henri Poincaré, who studied the structure of complicated geometric multidimensional formations. In distinction from analytic geometry, where the structure of a complex body is defined by a system of inequalities or the equations of the boundaries, the topological approach decomposes multidimensional geometric objects into their simplest elements, called simplexes.

A simplex is an elementary building-block, from which complicated geometric figures-polyhedra can be built according to definite rules. The scheme for decomposing a polyhedron into simplexes is called a complex. A line-segment, a triangle, a tetrahedron are, resepctively, simplexes of the one-, two- and three-dimensional space. In general an r-dimensional simplex $[a_0 a_1 \ldots a_r]$ is defined as the set of points

$$z=\sum_{i=0}^{r}\lambda^i a_i, \qquad (10.1)$$

where $a_0, a_1, \ldots, a_r$ is a system of independent points of the space $R_n (r \le n)$ and $\lambda^0, \lambda^1, \ldots, \lambda^r$ are real numbers satisfying the conditions

$$\sum_{i=0}^{r}\lambda^i=1 \qquad (\lambda^i \geqslant 0;\ i=0,\ 1,\ \ldots,\ r). \qquad (10.2)$$

The quantities $[a_0 a_1 \ldots a_{i-1} a_{i+1} \ldots a_r]$ are called the (r − 1)-edges of the simplex A^r. A finite set of simplexes form a complex K, if K contains together with each simplex its edges and any two simplexes in K either do not intersect, or their intersection is an edge of either simplex (correct incidence relations). This defines the structure of the polyhedron, which is the set of all points of the complex.

An important application of topology is the possibility (in the obvious absence of geometric intuition in higher dimensions) to characterize the structure of complicated polyhedra by their topological invariants: the homology groups (Betti groups). The homology groups are defined as follows[93-95]. One introduces the concept of chain

$$x=\sum_i \tau^i A_i^s, \qquad (10.3)$$

where τ^i are integers, and A_i^s the set of all s-dimensional simplexes of the complex K. The associated set of all (r − 1)-dimensional edges of the simplexes A is the boundary Δx of the chain x:

$$\Delta x=\sum_{i=0}^{n}\tau_i \Delta A_i^s. \qquad (10.4)$$

The boundary ΔA^r of a simplex is determined according to the symbolic expression

$$\Delta A^r=\sum_{i=0}^{n}(-1)^i\frac{\partial A^r}{\partial a^i}. \qquad (10.5)$$

An r-chain is called a cycle if its boundary vanishes. The collection of cycles forms an abelian group Z^r. A cycle is said to be homologous to zero if it is the boundary of an (r + 1)-chain in K. These boundary-cycles also form a group B^r, which is a subgroup of Z^r. The factor-groups $H^r = Z^r/B^r$ are the homology groups. They are abelian groups and are completely determined by their invariants: the Betti numbers p^r (the number of infinite cyclic summands in the canonical direct sum decomposition of H^r), and the torsion coefficients t_q (essentially the orders of the finite cyclic groups in the same decomposition).

The indicated concepts are easily visualized on using as an example two-dimensional complexes: a) A plane with n noles. Here each line is a chain, any closed line is a cycle; if there is no hole inside a closed line, the cycle is homologous to zero; the one-dimensional Betti number p^1 is equal to the number of holes n. b) For a sphere $p^2 = 1$, $p^1 = 0$, $p^0 = 1$. The zero-dimensional Betti number p^0 always equals the number of disconnected pieces of the polyhedron. The topological concepts, which are trivial in simple cases, are important characteristics of the structure of multidimensional manifolds.

We note that the classification of polyhedra is a narrow branch of combinatorial topology, which practically does not use the modern topologic techniques, based on the simultaneous use of both homology and cohomology, closely related to the classification of differential operators[60]. (In distinction from Eq. (10.3) a cochain y_s is defined on the simplexes A_i^s by the linear functional $(y_s A_i^s) = \tau_i$. One then defines cocycles, coboundaries, the appropriate abelian groups, and their quotient, the cohomology groups.) The modern topological machinery allows one to analyze the analytic structure of multidimensional integrals, and to study their singularities[90-92]; this approach reduces differential and integral relations to purely algebraic ones. In order to apply the topological methods to the theory of angular momenta it is necessary to associate to the concepts of the latter (angular momenta, C-G, Racah and other coefficients, etc.) geometric objects in multidimensional spaces, i.e., polyhedra. We shall consider the triangle with sides j_1, j_2, j_3 as the geometric image of the Wigner coefficient (cf. the three points in a triangular coordinate net, Fig. 3, as well as the geometric interpretation[1]). Then the 6j-symbol (with four faces which are C-G coefficients) is represented by a tetrahedron. To any combination of C-G coefficients, j-symbols, one can associate a definite polyhedron, constructed from the indicated simplexes. Due to the structure of the classical theory of angular momenta the correct incidence conditions for simplexes are automatically valid. (Of course, the usual graphs[8] (cf. Fig. 1) can also be characterized topologically. However, here the rules of formation of complexes do not correspond to the rules of combination of the C-G coefficients.) The homology (Betti) groups of the complex K are invariants of the polyhedron, and are thus characteristic for the combinations of C-G coefficients, to which the polyhedron is associated. It should be stressed here that not all properties of the geometric concepts are reflected by the topological invariants, and some essential geometric features are lost in this process. The determination of the Betti numbers for combinations of C-G coefficients reduces either to a direct computation of all the cycles of given dimension, of the cycles which are homologous to zero or to one another, or to a use of the Mayer-Vietoris formula[96] which relates the Betti numbers of the complexes K_1, K_A, K_B, K_0, where $K_A \subset K_1$, $K_0 \subset K_1$, such that $K_A \cup K_B = K_1$, $K_A \cap K_B = K_0$ (the Mayer-Vietoris formula is an analog, sui generis, of Clebsch-Gordan expansions in topology). For the description of concrete polyhedra it is convenient to use a collection of complexes K_s ($s = 0, 1, 2, \ldots$ are the dimensions of the complex). Then, in the three-dimensional case a polyhedron is described by the following collection of Betti numbers p_s^r:

$$\begin{matrix} p_0^0 & p_1^1 & p_2^2 & p_3^3 \\ p_0^1 & p_1^2 & p_2^3 & \\ p_0^2 & p_1^3 & & \\ p_0^3 & & & \end{matrix} \tag{10.6}$$

As examples we indicate the collections (10.6) for the Racah coefficient and the combination $\sum_{m_{12}} (j_1 j_2 m_1 m_2 | j_{12} m_{12})(j_{12} j_3 m_{12} m_3 | jm)$, respectively

$$\begin{matrix} 4310 & & 420 \\ 100 & & 10 \\ 10 & & 1. \\ 1, & & \end{matrix} \tag{10.7}$$

In[83], where concrete examples of topological characterizations of combinations of C-G coefficients are given, use has been made of the collection of Betti numbers p_3^3, p_2^2, p_1^1, $p_0^0 - 1$. The topological approach turned out to be effective for a series of problems of angular momentum theory, including an enumeration and classification of j-symbols, derivation of relations among j-symbols, and an analysis of the structure of various combinations. A simple topologically covariant treatment of the theory of angular momenta[84] already leads to interesting results. We denote the metric matrix $\left\{ {j \atop mm'} \right\}$ corresponding to the one-dimensional simplex $[a_1 a_2]$ by X_{12}; the Wigner coefficient $\left\{ {j_1 j_2 j_3 \atop m_1 m_2 m_3} \right\}$ corresponding to the two-dimensional simplex $[a_1 a_2 a_3]$ will be denoted by X_{123}; the 6j-symbol $\left\{ {j_1 j_2 j_{12} \atop j_3 j \; j_{23}} \right\}$, corresponding to three-dimensional simplex $[a_1 a_2 a_3 a_4]$ will be denoted by X_{1234}. In terms of the $X_{ik\ldots}$ one can write down any relation which does not depend on the concrete values of the angular momenta. As already remarked, the summation is carried out over general simplexes, and will be denoted by including the simplexes to be summed over in square brackets. Thus, the 6j-symbol and the contraction of two 6j-symbols can be written respectively in the form (to the right are the traditional notations)

$$X_{1234} = [X_{123} X_{234} X_{134} X_{124}] = \begin{Bmatrix} X_{12} & X_{23} & X_{13} \\ X_{34} & X_{14} & X_{24} \end{Bmatrix}, \tag{10.8}$$

$$[X_{1234} X_{2345}] = \begin{Bmatrix} X_{12} & X_{23} & X_{13} \\ X_{34} & X_{14} & X_{24} \end{Bmatrix} \begin{Bmatrix} X_{23} & X_{24} & X_{34} \\ X_{45} & X_{35} & X_{25} \end{Bmatrix}. \tag{10.9}$$

The j-symbols are closely related to transformation matrices, which determine the transition between various angular momentum coupling schemes. The usual notation (but in terms of the $X_{ik\ldots}$) of the transformation matrices has the form

$$((((X_{36} X_{46}) X_{34} X_{24}) X_{23} X_{13}) X_{12} X_{15} X_{25} | (((X_{36} X_{13}) X_{16} X_{15}) X_{56} X_{46}) X_{45} X_{21} X_{25}). \tag{10.10}$$

For applications, however, the following notation is more convenient

$$(X_{346} X_{234} X_{123} X_{125} | X_{136} X_{156} X_{456} X_{245}). \tag{10.11}$$

This expression can be rewritten in terms of a contraction of 6j-symbols

$$[X_{1346} X_{1234} X_{1245} X_{1456}]. \tag{10.12}$$

The operation rules with transformation matrices in the form (10.11) reduces to contracting repeated X_{ikl}, e.g.,

$$\begin{aligned} &(X_{123} X_{345} X_{134} X_{146} | X_{126} X_{235} X_{256} X_{456})(X_{123} X_{134} X_{347} X_{148} | X_{128} X_{237} X_{278} X_{478}) \\ &\quad = (X_{345} X_{146} X_{128} X_{237} X_{278} X_{478} | X_{126} X_{235} X_{256} X_{456} X_{347} X_{148}). \end{aligned} \tag{10.13}$$

Not every transformation matrix can be represented in a form of the type (4.11) or (4.12), e.g., the 9j-symbol. However a 9j-symbol can be considered as a 12j-symbol of the second kind with one of the Racah coefficients of a special form

$$\begin{Bmatrix} j' j'' j \\ j' j'' j \end{Bmatrix}.$$

Symbolically, the 9j-symbol can be represented in the form

$$[X_{1234} X_{2345} X_{1246} Y_{(24)(36)}]. \tag{10.14}$$

Here $Y_{(ik)(lm)}$ is a special value of the Racah coefficient, with $X_{ik} = X_{lm}$, $X_{il} = X_{km}$, $X_{kl} = X_{im}$. The notation in the other similar j-symbols is analogous. Thus, any j-symbol can be represented in topological form as a combination of Racah coefficients. The structure of such combinations is considerably more complex than the structure of combinations of C-G coefficients. However, different combinations of Racah coefficients can be reduced to the standard form with the help of the relations

$$[X_{1345} X_{2345} X_{1245}] = [X_{1234} X_{1235}], \tag{10.15}$$

$$[X_{1267} X_{1247} X_{2347} X_{2357} X_{2567}] = \begin{bmatrix} X_{1234} & X_{1235} & X_{1256} \\ X_{1347} & X_{1357} & X_{1567} \end{bmatrix}, \tag{10.16}$$

$$[X_{1234} X_{1235} X_{1256} X_{1267} X_{1278} X_{1248}] = \begin{bmatrix} X_{1345} & X_{1456} & X_{1467} & X_{1478} \\ X_{2345} & X_{2456} & X_{2467} & X_{2478} \end{bmatrix}. \tag{10.17}$$

$$\begin{bmatrix} X_{1234} & X_{2345} & X_{2357} \\ X_{1246} & X_{2456} & X_{2567} \\ X_{1468} & X_{4568} & X_{5678} \end{bmatrix} = \begin{bmatrix} X_{1267} & X_{1678} & \\ X_{1237} & X_{1378} & X_{3578} \\ & X_{1348} & X_{3458} \end{bmatrix} = \begin{bmatrix} X_{2349} & X_{1234} & X_{1347} \\ X_{2359} & X_{1235} & X_{1357} \\ & X_{1256} & X_{1567} \end{bmatrix}. \tag{10.18}$$

For combinations of Racah coefficients the graphical method is quite useful. Figure 5a represents the 6j-symbol. Here four lines originate in one vertex, the lines corresponding to the C-G coefficient X_{ikl}. Figure 5b represents the 12j-symbol. Figure 6 represents graphically the equations (10.15)—(10.18), which allows one to reduce various graphs to tree-like diagrams with a double line, of the type represented in Fig. 7.

In addition to the above problems, the topological method allows one to consider higher-order simplexes, corresponding to some supersymbols, which define the transitions between various coupling schemes of Racah coefficients, similar to the way in which the Racah coefficient determines the transformation between various coupling schemes of C-G coefficients. Here, in distinction from (10.14), the 9j-symbol plays an independent role, forming part of the boundary of the four-dimensional simplex. The practical construction of the four-dimensional simplexes (and of the corresponding polyhedra) is based on using the form (6.11) for the 6j-symbol.

The relationship between C-G coefficients and topology implies the possibility of using homology groups for a characterization of physical states in processes taking place in complex atomic systems. The wave function of a complex system (e.g., an atom with a definite way of coupling several angular momenta) can be characterized by means of homology groups. The same is true of the matrix elements for such a system. The topologization of the theory which occurs in this way involves the use of the new quantum numbers p_s^r and of the corresponding selection rules.

The possibilities of the topological approach are not limited to the above. As already indicated, the generalized theory of angular momenta includes the higher symmetries. At the same time, the differential operators which characterize definite Lie groups, are described by cohomology groups, and can be specified in terms of the appropriate Betti numbers[70,97]. The simultaneous presence of homology and cohomology groups determines structures in the theory of C-G coefficients which are related to the use of modern topological methods which are analogous to those used in quantum field theory for analyzing the analytic properties of Feynman integrals[90-92]. These problems are almost completely undeveloped and one can hardly say anything at present about their practical value.

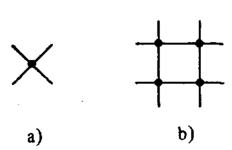

FIG. 5. The 6j-symbol and the 12j-symbol.

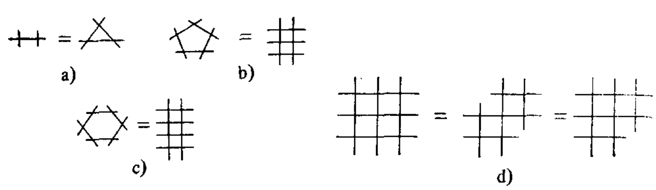

FIG. 6. Graphical interpretation of relations between the j-symbols.

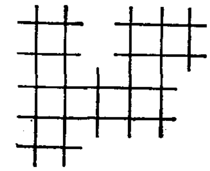

FIG. 7. A tree-like graph with a double line.

11. Relations Between the C-G Coefficients and Multidimensional Complex Integrals

Among the various geometric relations between C-G coefficients attention should be given to the Hilbert spaces F_n, the elements of which are entire analytic functions. The use of the Hilbert spaces F_n in the study of the rotation group is based on the fact that the irreducible representations of that group can be obtained by considering homogeneous polynomials of two complex variables (for the group SU(n)—of n complex variables). All these polynomials are elements of F_2 and can be discussed simultaneously. A systematic discussion of the relation between C-G coefficients and multidimensional complex integrals was carried out by Bargmann[49] on the basis of the Hilbert spaces F_n. According to Bargmann, the elements of F_n are entire analytic functions $f(z)$, where $z = (z_1, z_2, \ldots, z_n)$ is a point of a complex n-dimensional Euclidean space. Any such function $f(z)$ admits everywhere a power-series expansion

$$f(z) = \sum_{h_1 h_2 \ldots h_n} \alpha_{h_1 h_2 \ldots h_n} z_1^{h_1} z_2^{h_2} \ldots z_n^{h_n}. \tag{11.1}$$

The inner product of two elements f, f' F is defined by

$$(ff') = \int \overline{f(z)}\, f'(z)\, d\mu_n(z), \tag{11.2}$$

where the measure is defined by

$$d\mu_n(z) = \pi^{-n} \exp(-\bar{z}z) \prod_k dx_k\, dy_k \qquad (z = x + iy).$$

In discussing the Kronecker product $D_{j_1} x D_{j_2}$ of two representations of SU(2), Bargmann[49] makes use of the space F_6 and shows that the generating function of the Wigner symbol can be written in the form

$$\Phi(\tau\xi\eta) = \exp(D(\tau, \xi, \eta)), \tag{11.3}$$

where

$$D(\tau, \xi, \eta) = \begin{Vmatrix} \tau_1 & \tau_2 & \tau_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{Vmatrix}, \tag{11.4}$$

and $\tau = (\tau_1\tau_2\tau_3)$, $\xi = (\xi_1\xi_2\xi_3)$, $\eta = (\eta_1\eta_2\eta_3)$ are triples of complex numbers. The Wigner coefficient is defined by expanding the generating function in powers of τ_i^p, ξ_k^q, η_l^r. For fixed τ the generating function $\Phi_\tau = \Phi(\tau, \xi, \eta)$ is an element of F_6:

$$(\Phi_\tau, \Phi_{\tau'}) = \int \left[\int \overline{\exp(D(\tau, \xi, \eta))} \exp D(\tau, \xi, \eta)\, d\mu_3(\eta) \right] d\mu_3(\xi). \tag{11.5}$$

It can be seen from Eqs. (11.3) and (11.4) that the generating function satisfies the Regge symmetry. The generating function for the Racah coefficient can be expressed in the form of an integral over a product of four generating functions Φ and is an element of F_{12}. The substantial advantages of Bargmann's method consist in the fact that the formulas can be written simultaneously for all C-G coefficients, independently of the concrete values of the angular momenta. This method can be easily generalized to arbitrary combinations of the C-G coefficients. By analogy to (11.5), the generating functions for these combinations can be expressed as multidimensional complex integrals. Such formulas encompass all special values of the combinations obtained in the expansions of type (11.1). Thus, the topological investigation of combinations of C-G coefficients carried out in Sec. 11 is closely related to the topological characteristics of multidimensional complex integrals. On the other hand, Fotiadi, Froissart, Lascoux and Pham[90] have proposed to make use of homological algebra methods in the investigation of the analytic structure of multidimensional Feynman integrals occurring in quantum field theory. Thus there appear diverse (and hopefully, fruitful) relations between the theory of C-G coefficients with geometric and topological concepts on the one hand, and complex integrals, on the other.

12. Conclusion

In this necessarily condensed review we have attempted to show to what extent the formulas of the theory of C-G coefficients are related to other chapters of modern mathematics. The majority of these relations became apparent only during the past few years, and there is no doubt, that this circle will be enlarged. Among the closest-lying directions of development of the theory one should mention generalizations to other compact Lie groups, different from SU(2). The theory of C-G coefficients which exists at this moment also contains the C-G coefficients of the semisimple Lie groups, with the C-G coefficients of SU(2) being the basis of the whole theory (cf.[44,45,55,56,98-100]). A second promising direction is the study of C-G coefficients for values of their arguments different from integers and half-integers. This direction is closely related to the theory of special functions. Without doubt, investigations will continue into the group-theoretic, combinatoric and geometric aspects of the theory of C-G coefficients. To the authors of this review the present state of the theory of C-G coefficients appears as a collection of fragments, giving a quite hazy impression of the whole picture. The reason for writing this review was to call to the attention of the readers the great variety of unsolved (and even unformulated) problems.

Only the future will tell how important the role of C-G coefficients will be for physics. However the consistency and beauty of the theory in statu nascendi, on the one hand, and the continuous widening of the physical applications, on the other hand, force one to think that the physical side of the theory of C-G coefficients is quite important. The use of C-G coefficients in physics is still rather limited, in spite of their various applications. This is related to the insufficient development of the theory and to the underestimation of their role and effectiveness.

APPENDIX

THE CLEBSCH-GORDAN COEFFICIENTS OF COMPACT GROUPS

The theory of C-G coefficients of higher compact groups which contain the group SU(2) (O(3)) as a subgroup, has much in common with the classical theory of angular momenta and, as was mentioned in Section 12, should in the future become a new branch of the generalized theory. However, at present, the problem of creating a theory of C-G coefficients which is convenient for physical applications is far from being completed. From this point of view a detailed review of results for the higher groups seems to be premature. Below we briefly go over the peculiarities, the principles of computational methods, and the problem of tabulation for the C-G coefficients of the higher compact groups.

Compared to the classical theory of angular momenta, the general theory of C-G coefficients of compact groups exhibits a series of peculiarities. First, multiplicities may appear in the Clebsch-Gordan series (for the group SU(2) the same representation may appear with multiplicity higher than one only for the addition of several angular momenta). In order to distinguish the multiply occurring representations one makes use of an external factor, not contained in the group (in the group SU(2) this factor is the order in which the angular momenta are coupled). A second important peculiarity is the nonuniqueness in the determination of the canonical basis, its dependence on the choice of a chain of subgroups. Various reduction schemes in terms of subgroups may be essential for different concrete physical problems, where a hierarchy of physical symmetries is observed. We note that in the presence of a chain of subgroups the C-G coefficients can be factorized in terms of the subgroups. Thus, the SU(3) C-G coefficients in the chain SU(3) $\supset$ SU(2) consist of a C-G coefficient of SU(2) and an isoscalar factor. For more complicated chains there appears a system of factors. One has to keep in mind these peculiarities both in computing the C-G coefficients and in using data from the literature, in particular tables. In the presence of multiplicities the factor distinguishing the multiple representation is not contained in the group, depends on the concrete problem at hand and often has the character of a convention. Therefore one must first construct C-G coefficients corresponding to a Clebsch-Gordan series not containing multiple representations. These coefficients are uniquely determined and it makes sense to tabulate them. The tabulation of C-G coefficients for higher groups is just as unjustifiable as the tabulation of generalized C-G coefficients in the theory of angular momenta[55].

It is also necessary to indicate that the transition between C-G coefficients of the same group, but for different schemes of reduction (with respect to subgroups) is considerably more complicated than their direct computation. The results obtained by means of one reduction scheme are useless for another reduction scheme. Therefore it is imperative to understand the limited character of formulas and tables of C-G coefficients and the conditions in which they have been derived, before making use of them. For the C-G coefficients of higher groups some of the other concepts known in the case of SU(2) undergo certain changes. Thus, for the Wigner coefficients of the general case there is no longer any symmetry with respect to permutation of the representations[101].

The theory of Lie algebras, which is exposed in a series of reviews and monographs[60,102-106], is at the basis of the general theory of C-G coefficients for compact groups. All simple Lie algebras have been classified and studied; there are four infinite families of "classical" groups corresponding to these algebras: the unitary unimodular groups $SU(n)\,(n \geq 1)$, the orthogonal groups $SO(2n+1)\,(n \geq 2)$ and $SO(2n)\,(n \geq 3)$, the symplectic groups $Sp(2n)$ and the five "exceptional" groups: G_2, F_4, E_6, E_7, E_8. The Lie algebras allow one, in principle, to determine any quantities characterizing the appropriate groups. However, there is a large gap between the general formulation and realistic computational schemes for physical problems. In the discussion of concrete methods for the construction of C-G coefficients one may, conventionally, distinguish three approaches:

1. The starting point of the infinitesimal approach are the commutation relations between the operators representing the Lie algebra. It is convenient to write these commutation relations in the Cartan-Weyl basis, containing l mutually commuting operators H_i (with $[H_i, H_j] = 0$, $i, j = 1, 2, \ldots, l$ (l denotes the "rank" of the group) and $r - 1$ noncommuting operators E_α (r is the dimension of the Lie algebra). These operators define the root vectors $r_i(\alpha)$ ($[H_i, E_\alpha = r_i(\alpha) E_\alpha$), the weights $m = (m_1, \ldots, m_l)$ of the representation ($H_i\psi = m_i\psi$), and the matrix elements of the representation of the group[106,107]. In[107] there were also considered methods for the construction of C-G coefficients. The basis plays an important role in such a construction. The problem of constructing and labeling a canonical basis was discussed in[108-110]. The basis vectors of irreducible representations are labeled by means of the eigenvalues of the additive operators, $[H_i] = m_i$, and also by the eigenvalues of some nonadditive operators, such as the Casimir operators[108,109,111]. Thus, for the representations of the groups $SU(n)$, $D(P_1, \ldots, P_{n-1})$ the signature may be related to the Casimir operators of that group. As nonadditive parameters one may select the Casimir operators K^α_κ of the groups $SU(n-1), \ldots, SU(2)$ ($\kappa = 2, \ldots, n$, $= 2, \ldots, \kappa$) i.e., the labeling of the canonical basis consists of the three blocks of numbers $|\, P_1, \ldots, P_{n-1}; [K^\alpha_\kappa]; [H_i]\rangle$.

The calculation of matrix elements of the representations and of C-G coefficients in the infinitesimal method is a rather lengthy and complicated affair. Each group and reduction scheme requires a separate discussion. We indicate some papers containing concrete computations of matrix elements (SU(3)[112,113], SU(4)[114], SU(6)[115,116], Sp(6)[117,118], G_2[119], F_4[120], SO(5)[121,122]) and to the calculation of the Clebsch-Gordan coefficients by means of the infinitesimal method (SU(3)[122,123-126], SU(6)[116,127-130] SU(n)[131], SO(5)[132-134]). Further development of this approach presents great methodical interest.

2. A second approach, related to the use of algebraic methods, is closest in ideology to the classical theory of angular momenta. An essential element in this approach is the definition of a polynomial basis. The problems of construction of a polynomial basis were discussed in a series of papers (SU(3)[135-138], SU(4)[139], SO(5)[140-142], SO(n)[143-145]). An important special case of a polynomial basis is the symmetric basis introduced for the groups SU(n)[44,45,55]. Characteristic properties of this basis are:

a) Redundancy (not all components are linearly independent, although the expansions in terms of the basis are unique).

b) Symmetry (the contraction of an arbitrary basis vector of the representation $D(P_1, \ldots, P_{n-1})$ of $SU(n)$ with the conjugate vector yields a product of determinants of order $(n+1)$ raised to the powers P_1, $P_2, \ldots, P_{n-1}$, i.e., there is a close relation to $n \times n$-symbols).

c) Factorization (the symmetric basis consists of separate factors which are bases of the representations $D(P00\ldots)$, $D(0Q0\ldots)$, $D(00R\ldots)\ldots$) These properties are convenient for the construction of transformations among different reduction schemes.

The polynomial basis allows for a wide use of generating invariants, a method which permits one to study the structure and interrelation between C-G coefficients of different groups[55,56]. An important stage in the development of an algebraic computation scheme were the papers[137-138], where for the computation of individual C-G coefficients use was made of the rules for combining them in order to form more general coefficients. Among the other papers on the algebraic method it is worth mentioning[146-151]. In spite of known difficulties (finding the construction rules, computation of the normalization), the algebraic approach allows one to obtain results which are of general validity for a given group. It is, of course, incorrect to compare it directly with the infinitesimal method, since in algebraic calculations one makes widespread use of the knowledge of general properties of representations, which in turn are derived with the help of Lie algebras.

3. A third direction in the theory of C-G coefficients is based on the close interrelation between representations of the semisimple Lie groups and the representations of the symmetric group, π_f[60]. For a long time this approach played the main role in applications and was related to the calculation of fractional parentage ("genealogic") coefficients. The latter are the factors in the C-G coefficients of higher groups. They are quite convenient for the construction of wave functions for many-particle systems exhibiting definite permutational symmetries. The results of this approach are systematized in the monographs[152-156]. The computational machinery of the symmetric group is a useful tool in the representation theory of compact Lie groups. A characteristic example is the pletism method[156-159]. The relationship between the matrix elements and the C-G coefficients of the symmetric group π_f and the C-G coefficients of the compact Lie groups has been discussed in several papers[160,161]. The reduction coefficients (S-coefficients) have been defined in[100]; these coefficients realize the reduction of the space

$R^f = R \times R \times \ldots \times R$ into spaces irreducible with respect to $\pi_f \times G$ (R is the representation space of the group G, corresponding to the symmetry of the physical system under consideration). The S-coefficients define formally the relation between the computational machinery of the symmetric group and the compact groups.

Notwithstanding the accomplishments in all directions of investigation of the C-G coefficients of the higher groups, the fragmentary character of the results still needs to be overcome. A series of papers have computed tables of the C-G coefficients (and factors) for various groups, with applications in view. We point out the tables of isoscalar factors for the group SU(3)[112,162,163], the numerical tables, obtained with the help of computers, for the reduction $SU(3) \supset SU(2)$, the tables of isoscalar factors of the groups SU(4)[114], SU(6)[116], the tables of reduction coefficients for $D(11) \times D(11) \times D(11)$ of SU(3)[165], and the table of C-G coefficients of the group SO(5)[132,138]

Acknowledging a certain value of these tables (if one takes into account the limitations listed above), one should remark that the reduction of the factors of the C-G coefficients of higher groups to the C-G coefficients of the groups SU(2) (or their combinations) makes such tables practically unnecessary. Thus, the reduction of some isoscalar factors of the group SU(3) to the C-G coefficients of SU(2)[55,98], makes a large portion of the tables[112,162] useless. Quite important and interesting is an analogous result for SO(5)[99]. Further development of the problem of reduction of the C-G coefficients for the compact groups to C-G coefficients of the group SU(2) would allow one to use ready-made universal tables of the C-G coefficients of SU(2), in place of the new tables. The methodical importance of such a reduction was already mentioned in Sec. 12, from the point of view of a unified theory of C-G coefficients.

The latter is important also for the theory of C-G coefficients of noncompact groups. Insofar as the physical aspects of the problems of higher symmetries are concerned, this set of problems is undergoing a significant revision (cf. Section 8) and the set of physical objects to which one applies the theory of C-G coefficients of compact groups is also expanding (cf., e.g., the coherence problem[166-168]).

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