THE PROBLEM OF SYMMETRY BREAKING AND INVARIANCE OF THE VACUUM IN QUANTUM FIELD THEORY

A. A. GRIB, E. V. DAMASKINSKIĬ, and V. M. MAKSIMOV

Leningrad State University

Usp. Fiz. Nauk 102, 587—620 (December, 1970)

INTRODUCTION

THIS review is devoted to the problem of symmetry breaking in elementary-particle theory and many-body theory. A special examination of the problem of symmetry in quantum field theory is necessary, since during the last decade essential differences have been found between the symmetry properties in quantum field theory and in quantum mechanics. These differences are due to the fact that the systems considered in field theory have an infinite number of degrees of freedom. In quantum mechanics a symmetry transformation always carries any vector of the Hilbert space of states into a vector of this same space. Because of the conservation of expectation values this transformation is unitary. An invariance of the Hamiltonian then always leads to the existence of corresponding selection rules and conservation laws.

In quantum field theory there can be two types of noninvariance of states.

1. A noninvariance of the same type as in quantum mechanics, when a state of a given Hilbert space goes over into a state of the same space. An example is the noninvariance of the one-particle states of the free field with respect to transformations of the inhomogeneous Lorentz group.

2. A noninvariance of the entire Hilbert space of the states of a system. The symmetry transformation is nonunitary and carries any state vector of the given space into a vector of a different Hilbert space of states.

The difference between the two types of noninvariance can be seen easily from the example of the quantum theory of many bodies. Suppose there are N atoms with spin unity in an external magnetic field directed along the z axis. In the ground state the spins of all the atoms are directed along the z axis, and the wave function of the ground state can be put in the form $\Psi^0 = \prod_{j=1}^{N} \psi_j^0$, where ψ_j^0 is the eigenfunction of the spin operator S_z with eigenvalue unity, which transforms according to the vector representation of the rotation group. The excited states are described by wave functions that differ from Ψ^0 by having the spins of some number ($\ll N$) of the atoms not directed along the z axis. Ψ^0 and the excited states above it form a Hilbert space H_0 of states. The state Ψ^0 is invariant under rotations around the z axis. The excited states are noninvariant under these rotations, but H_0 as a whole is invariant, since these states go over into each other. Thus we see that if the ground state is invariant, then the noninvariance of the excited states is of the first type.

Let us now consider rotations around the x axis. The ground state Ψ^0 itself is noninvariant under these rotations and goes over into $\Psi^\theta = \prod_{j=1}^{N} \psi_j^\theta$, where ψ_j^θ describes an atom with spin turned through the angle θ around the x axis. Since the wave functions of atoms with spin unity transform according to the vector representation of the rotation group, $(\psi_j^0, \psi_j^\theta) = \cos\theta$. Consequently,

$$(\Psi^0, \Psi^\theta) = \prod_{j=1}^{N} (\psi_j^0, \psi_j^\theta) = (\cos\theta)^N$$

and for $N \to \infty$,* $(\Psi^0, \Psi^\theta) \to 0$. Accordingly, the states Ψ^0 and Ψ^θ are orthogonal for all angles $0 < \theta < 2\pi$. Similarly, any excited state above Ψ^0 is orthogonal to any state excited above Ψ^θ. Thus Ψ^θ is orthogonal to any state of the Hilbert space H_0 and therefore does not belong to this space. The same is true for any state excited above Ψ^θ. Consequently, we arrive at the necessity of considering Hilbert spaces H_θ, in which the states Ψ^0 and the states excited above them are contained. In the spaces H_0, H_θ $(0 < \theta < 2\pi)$ there are realized various unitarily nonequivalent representations of observables[2-6] (in the present case the spin operators).

This noninvariance of Ψ^0 and the states excited above it is indeed a noninvariance of the second type. In many-body theory a noninvariance of the second type is a macroscopic noninvariance of a state of a system. Furthermore there always exists a noninvariant macroscopic observable, which behaves classically (see Chapter 1). In our example the magnetization $m = \lim_{N\to\infty} \frac{1}{N} \sum_j S_j$, where S_j is the spin operator of the j-th particle, is such a quantity. The value of such a macroscopic quantity is the same for all vectors of a fixed Hilbert space. The different values of these quantities correspond to different spaces.

In many-body theory a noninvariance of the ground state is always of the second type. In elementary-particle theory the vacuum plays the role of the ground state; it is defined as a state invariant with respect to space-time translations. A noninvariance of the vacuum is always a noninvariance of the second type.

The concept of a noninvariance of the vacuum was introduced by Heisenberg at the end of the 1950's.[7,8] Heisenberg's basic assertion was that the vacuum cannot be a state invariant under the breaking of internal symmetries: the properties of the vacuum reflect those of the world of physical phenomena, and therefore everything that is broken in this world is broken for the vac-

*$N \to \infty$ is a usually postulated condition in many particle theory and corresponds to the so-called thermodynamic limit (cf., e.g., [1], Chapters 1 and 7).

uum. After the appearance of Coleman's theorem,[9] which asserts that "invariance of the vacuum is invariance of the world", this proposition of Heisenberg's can be regarded as proved (not only for continuous internal symmetries, but also for the discrete P, CP, and T symmetries[10,11]). Accordingly it follows from Coleman's theorem that the vacuum in elementary-particle theory is noninvariant with respect to all the broken symmetries: isotopic symmetry, which is broken by the electromagnetic and weak interactions, the symmetry of the hypercharge group, which is broken by the weak interactions, the group SU(3), and also the C and CP reflections. However, the vacuum can also be invariant even for a completely invariant interaction. In many-particle theory the interaction is invariant under space rotations and translations, and nevertheless the ground state (vacuum) of a crystal is noninvariant with respect to these transformations, the ground state of a ferromagnetic is noninvariant with respect to rotations, and so on. Furthermore the noninvariance of the vacuum leads to the absence of the corresponding selection rules and conservation laws. For example, for the phonons of a crystal the law of conservation of momentum is not satisfied, and holds only up to a vector of the reciprocal lattice. Such a symmetry breaking, when the vacuum is noninvariant with an invariant Hamiltonian, is called spontaneous. In elementary-particle theory symmetry breaking is usually regarded as nonspontaneous, but there exist models which demonstrate the possibility of a spontaneous symmetry breaking in this theory also (see Chapter 7). A theory of spontaneous symmetry breaking is natural for cases in which the existence of an interaction breaking the symmetry is problematical, for example in the case of the group SU(3),[12,14] in the explanation of the mass difference of muon and electron,[12,13] and also in the theory of CP violation,[15,16] in K^0 decays which have not shown up in any other experiments, and so on.

At the present stage it is hard to decide which of the symmetry breakings in elementary-particle physics are spontaneous and which are not. This is because in the relativistic theory the symmetry properties of the Lagrangian are unknown; one knows only the conservation laws and selection rules in processes involving actual particles. But spontaneous symmetry breaking can lead to the same violations of conservation laws and selection rules as the existence of symmetry-breaking interactions. We also note that in a number of cases it is possible to regard a symmetry breaking as a combination of a spontaneous breaking with a subsequent nonspontaneous breaking.[17-19]

A general feature of any symmetry breaking is noninvariance of the vacuum. We recall that the invariance of the vacuum $|0\rangle$ with respect to the transformations of a group $U(\tau) = \exp(iQ\tau)$ means that the vacuum state is carried into itself by the transformations of the group: $U(\tau)|0\rangle = |0\rangle$. If the group is a continuous Lie group, the infinitesimal operator Q of the group then has the property $Q|0\rangle = 0$. If the vacuum is noninvariant, then $Q|0\rangle \neq 0$. Physically, noninvariance of the vacuum manifests itself in the fact that there exist noninvariant vacuum averages for certain local observables $A_\tau(x)$, i.e., observables for which the commutator with a field operator is zero for spacelike intervals. This is equivalent to the condition

$$\frac{1}{i}\frac{d}{d\tau}\alpha\tau(\tau)\Big|_{\tau=0} = \langle 0\,|\,[Q,\ A(x)]\,|\,0\rangle \equiv \beta \neq 0,$$

where $|0\rangle$ is the vacuum, $\alpha(\tau) = \langle 0|A_\tau(x)|0\rangle \equiv \langle 0|U(\tau)A(x)U(\tau)^{-1}|0\rangle$, with $U(\tau) = \exp(i\tau Q)$, where Q is the generator of the symmetry transformation. For the invariant vacuum $\beta = 0$. Conversely, if for arbitrary local A(x) we have $\beta = 0$, then the vacuum is invariant. This is proved in the framework of the axiomatic approach (cf., for example,[11,29]). For the noninvariant vacuum the generator Q does not exist as a Hermitian operator in Hilbert space (see Chapters 1 and 4), and the states do not transform according to a unitary representation of the symmetry group and cannot have corresponding quantum numbers assigned to them.

We shall call noninvariant vacuum averages anomalous averages. In many-body theory the presence of anomalous averages is associated with the existence of noninvariant macroscopic classical parameters such as magnetization, particle number density, and so on (see Chapter 1). In elementary-particle theory the physical meaning of such anomalous averages is not clear, but there exist models in which they can be given the meaning of mass, coupling constant, or particle number density in a cosmological model (see Chapters 5, 7, and 8).

There is a change of an anomalous average when there is a change of the vacuum. As can be seen from our example from many-particle theory, the transition from one vacuum to another already requires a nonunitary transformation. This means that the transition cannot be described by the laws of quantum mechanics[20] and requires a different logical scheme, distinct from that of quantum mechanics. In many-body theory this scheme turned out to be classical physics. In elementary-particle theory the question as to the possibility of many vacuums and the variation of the corresponding anomalous averages remains open (in a number of papers this variation is associated with development in time,[21] with the role of the gravitational field,[22,23] with a new role of measurements,[24,26] etc.).

In the theory of spontaneous symmetry breaking the concept of the noninvariant vacuum is closely connected with the degeneracy of the vacuum. The latter concept means that the same system, with a fixed value of the energy, can have different vacuums (ground states). The term degeneracy of the vacuum is used in two different senses: 1) the existence of different Hilbert spaces of the system, in each of which the vacuum is unique, and which differ by different values of some classical parameter; 2) the existence of different vacuums in a single Hilbert space, in which case the value of the classical parameter is not fixed—in nonrelativistic theory this corresponds to the absence of complete information about the system (the presence of a statistics).

If the value of the classical parameter in question is fixed and cannot vary, then we have the case of a single noninvariant vacuum. Such a situation is possible in elementary-particle theory. For example, the parity noninvariance in the two-component theory of the neutrino is due to the fact that the space-inversion transformation carries the neutrino over into a nonexistent state, since the neutrino is characterized by a definite helicity. Similarly, for continuous groups also, in the case of a unique but noninvariant vacuum a symmetry transformation takes the system over into a physically nonexistent state.

With spontaneous symmetry breaking the transition from one vacuum to another can be described as the excitation of an infinite number of particles (infinite, since with a change of the vacuum there is a change of a macroscopic parameter, for example, the magnetization of a ferromagnetic body), while there is no accompanying change of the total energy of the system, the state remains translationally invariant, and the energy of an individual particle must go to zero when its momentum goes to zero. This statement that when there is spontaneous symmetry breaking particles of zero mass must exist has been given the name of the Goldstone theorem in the literature. Examples of Goldstone particles are phonons and magnons in many-body theory. In elementary-particle theory photons could be such particles.

The present review is devoted to the problem of vacuum invariance and spontaneous symmetry breaking in quantum field theory. Owing to the fact that this problem arose first in many-body theory, we shall first expound the general properties of spontaneous symmetry breaking in this theory (Chapter 1), and illustrate these properties with examples from the theories of ferromagnetism (Chapter 2) and of superfluidity (Chapter 3). We then give a formulation and proof of Coleman's theorem (Chapter 4), which shows why the vacuum must be noninvariant in elementary-particle theory, and then expound the relativistic version of Goldstone's theorem (Chapter 5). The physical meaning of this theorem, and also its altered form when there is spontaneous symmetry breaking, are further discussed in Chapter 6. In Chapter 7 we give some typical examples of spontaneous symmetry breaking in elementary-particle theory (the models of Nambu and Jona-Lasinio, of Goldstone, and others). Finally, in connection with the problem of the classical parameter in elementary-particle theory, we examine in Chapter 8 the question of the quantization of fields in a space-time with a nonstationary metric. In this theory the vacuum changes with time, and its change corresponds to a change of a quantity of the type of a matter density; some authors[22,23] call this sort of process the process of creation of matter with the expansion of the universe.

1. GENERAL PROPERTIES OF SYMMETRY BREAKING IN MANY-PARTICLE THEORY

Spontaneous symmetry breaking is a widespread phenomenon in many-body physics. In any nonrelativistic many-body system there is breaking of the symmetry of the ground state under transformations of the Galilei group. Spontaneous symmetry breaking is often encountered in phase transitions; for example, in solidification the translational symmetry of the liquid is lost and there remains only the discrete group of translations by vectors of the crystal lattice. On going through the Curie point in the theory of ferromagnetism one must take into account a spontaneous breaking of the symmetry with respect to the rotation group, and in the theories of superconductivity and superfluidity the transition from the normal to the superconducting or the superfluid phase is accompanied by loss of gauge invariance.

In the nonrelativistic domain spontaneous symmetry breaking manifests itself in the fact that certain macroscopic parameters characterizing the physical system in question are noninvariant with respect to the symmetry transformations. The existence of macroscopic parameters, invariant as well as noninvariant, is a general property of many-body physics. By macroscopic parameters we mean those characteristics of a physical system whose variations are described by classical physics, in distinction from the quantum "microscopic" observables, which give us a view of the microscopic nature of the system, described by quantum theory. Observables of these two types differ in the methods by which they are measured. In fact, the methods of measurement of micro-observables, such as the energy and momentum of quasiparticle excitations, and of macro-observables, such as density, temperature, and magnetization, are entirely different.

When there is complete information about a system the macroscopic observables have exactly defined values. To each of the values of a macroquantity there corresponds a set of states of the physical system, differing from each other only by the values of the micro-observables. The manifold of these states forms a Hilbert space characterized by the given value of the macroscopic quantity. There are different Hilbert spaces for different values of the macroscopic variable, and this fact manifests itself physically in the absence of interference between states with different values of macroscopic quantities. In each of these spaces the micro-observables are described by operators which realize unitarily nonequivalent representations of the observables.

The value of a macro-observable in a given Hilbert space can be obtained as an average over the volume Ω, of the form

$$b=\lim_{\Omega\to\infty}\frac{1}{\Omega}\int_{\Omega}B(\mathbf{x})\,d^3x, \qquad (1.1)$$

where $B(\mathbf{x})$ is the corresponding local microobservable. That it is local means that the commutator of $B(\mathbf{x})$ with the creation (and annihilation) operators of the field, $[B(\mathbf{x}),\, a^{+}(\mathbf{y})]$ go to zero sufficiently rapidly for $|\mathbf{x}-\mathbf{y}|\to\infty$.

Examples of macroquantities are: the particle number density

$$\rho=\lim_{\Omega\to\infty}\frac{1}{\Omega}\int_{\Omega}a^{+}(\mathbf{x})\,a(\mathbf{x})\,d^3x,$$

the kinetic energy density

$$\varepsilon=\lim_{\Omega\to\infty}\frac{1}{\Omega}\int_{\Omega}a^{+}(\mathbf{x})\frac{1}{2m}\nabla^2 a(\mathbf{x})\,d^3x \text{ and so on.}$$

Quantities of the type (1.1) commute with all the creation and annihilation operators. For example, we indeed have

$$[b,\, a^{+}(\mathbf{y})]=\lim_{\Omega\to\infty}\frac{1}{\Omega}\int_{\Omega}d^3x\,[B(\mathbf{x}),\, a^{+}(\mathbf{y})]=0$$

because of the local nature of $B(\mathbf{x})$. It can be shown that all quantities of the form (1.1) also commute with each other, so that they can be regarded as c numbers. In theories with a translationally invariant ground state (vacuum) the values of these observables are equal to the vacuum expectations of the corresponding local quantities:

$$\lim_{\Omega\to\infty}\left\langle 0\left|\frac{1}{\Omega}\int_\Omega B(\mathbf{x})\,d^3x\right|0\right\rangle = \lim_{\Omega\to\infty}\frac{1}{\Omega}\int_\Omega \langle 0|B(\mathbf{x})|0\rangle\,d^3x = \langle 0|B(\mathbf{x})|0\rangle = b\,^*). \quad (1.2)$$

We shall show that to different values of macroquantities there correspond unitarily nonequivalent representations of the observables, which means that a change of state of the system in which a macroquantity changes cannot be described by a unitary operator. Let $a_i^+(\mathbf{x})$ and $a_i(\mathbf{x})$ be creation and annihilation operators in Hilbert space H_i (i = 1, 2), and let b_i be the value in H_i of the macroscopic quantity defined by (1.1). Let us suppose that the representations in question are connected by a unitary operator, i.e.,

$$a_1(\mathbf{x}) = Ua_2(\mathbf{x})U^{-1}, \quad a_1^+(\mathbf{x}) = Ua_2^+(\mathbf{x})U^{-1}.$$

Then

$$B_1(\mathbf{x}) = UB_2(\mathbf{x})U^{-1}, \quad (1.3)$$

where $B_i(\mathbf{x})$ is the operator representing the macro-observable in H_i. The limit in (1.1) is to be understood in the weak sense, i.e., as convergence of the matrix elements. Then we have

$$b_1 = \langle\psi_1|b_1|\psi_1\rangle = \lim_{\Omega\to\infty}\frac{1}{\Omega}\int_\Omega \langle\psi_1|B_1(\mathbf{x})|\psi_1\rangle\,d^3x \quad (\psi_1\in H_1). \quad (1.4)$$

Using (1.3), we get

$$\langle\psi_1|B_1(\mathbf{x})|\psi_1\rangle = \langle U^{-1}\psi_1|B_2(\mathbf{x})|U^{-1}\psi_1\rangle = \langle\psi_2|B_2(\mathbf{x})|\psi_2\rangle \ (\psi_2\in H_2). \quad (1.5)$$

Substituting (1.5) in (1.4) and using (1.2), we have

$$b_1 = \lim_{\Omega\to\infty}\frac{1}{\Omega}\int_\Omega \langle\psi_2|B_2(\mathbf{x})|\psi_2\rangle\,d^3x = \langle\psi_2|b_2|\psi_2\rangle = b_2. \quad (1.6)$$

Accordingly, if the representations are equivalent, the macroquantity has the same value in both of them. Consequently, representations corresponding to different values are unitarily nonequivalent.

We note in passing that the Fock representation ordinarily used for creation and annihilation operators describes systems with zero density, i.e., $\rho = \lim_{\Omega\to 0}(N/\Omega) = 0$ in any state of the system. Consequently, all macroscopic bodies must be described by representations unitarily nonequivalent to the Fock representation. With a change of the density, since it is a macroscopic parameter, there must be a change of the representation (of the Hilbert space).

This division of observables into macroscopic and microscopic allows us to define the concept of an exact symmetry in the following way. A symmetry is exact if the transformations corresponding to it do not change the values of the macroparameters. In this case a symmetry operation takes the vectors of a given Hilbert space into vectors of the same space, and can be described by a unitary operator. The ground state (vacuum) is not changed by the symmetry operations.

If, on the other hand, a symmetry transformation changes the value of a macroscopic observable, then it takes a vector of a given Hilbert space into a vector of a different Hilbert space. In this case the symmetry cannot be described by a unitary operator, and if the Hamiltonian is invariant under the symmetry transformations this means that there is spontaneous symmetry breaking. The ground state is then noninvariant.

For example, the noninvariance of the ground state of any nonrelativistic many-body system with respect to the Galilei group is due to the fact that in any actual system the mean particle number density ρ is different from zero. This density is a macroscopic observable. A macroscopic quantity is also a function of the momentum distribution function $\rho(\mathbf{p})$ of the particles, which cannot be a constant. If $\rho(\mathbf{p})$ were a constant, the system would be characterized by an infinite energy density per unit volume. Under Galilei transformations $\rho(\mathbf{p})$ changes:

$$\rho(\mathbf{p}) \to \rho(\mathbf{p}') = \rho(\mathbf{p}+m\mathbf{v}_0),$$

where $\mathbf{v}_0$ is the velocity of the motion of the reference system. A change of $\rho(\mathbf{p})$ requires a transition to a different Hilbert space, which means that the ground state (vacuum) is noninvariant.

Conversely, noninvariance of the ground state of a many-particle system leads to the appearance of a noninvariant macroscopic parameter which is not identically equal to zero. As was noted in the Introduction, noninvariance of the vacuum or ground state manifests itself through the existence of a nonvanishing vacuum average $\alpha(\tau) \equiv \langle 0|A_\tau(\mathbf{x})|0\rangle \neq 0$ of a local operator $A_\tau(\mathbf{x})$. Owing to the translational invariance of the ground state α_τ can be represented as the vacuum average of the operator $C_\tau = \lim_{\Omega\to\infty}\int_\Omega d^3x A_\tau(\mathbf{x})$, which is the corresponding noninvariant macro-observable. The average of the operator C_τ over the vacuum $|0\rangle$ coincides with the average over any normalized state $\psi(\mathbf{x})$ of the Hilbert space containing $|0\rangle$:

$$\alpha_\tau = \langle 0|A_\tau(\mathbf{x})|0\rangle = \langle 0|C_\tau|0\rangle = \langle\psi(\mathbf{x})|C_\tau|\psi(\mathbf{x})\rangle.$$

We shall show that the generator Q of a transformation

$$A_{\tau_1}(\mathbf{x}) \to A_{\tau_2}(\mathbf{x}) = \exp\{iQ(\tau_2-\tau_1)\}A_{\tau_1}(\mathbf{x})\exp\{-iQ(\tau_2-\tau_1)\},$$

which is accompanied by a change of C_τ (so that $C_{\tau_1} \to C_{\tau_2}$) is not an observable (measurable) quantity. In fact, according to the quantum mechanical theory of measurement,[20] in a measurement of Q the system is taken over into a state described by an eigenvector of the observable Q. Because Q and C_τ do not commute, the average value of C_τ in such a state is undefined, which clearly contradicts the statement proved above, that this average has the same value for any state $\psi(\mathbf{x})$. Accordingly, in the case of a noninvariant vacuum Q cannot be measured and is not an operator in the Hilbert space.* The physical meaning of this fact is that C_τ is

*Owing to the translational invariance of the vacuum $\langle 0|B(\mathbf{x})|0\rangle = \langle 0|\exp(i\mathbf{Px})B(0)\exp(-i\mathbf{Px})|0\rangle = \langle 0|B(0)|0\rangle = b$ = const, where **P** is the momentum operator.

*In the literature this sort of situation has been given the name "superselection". [27,29] In the case of a continuous group one speaks of continuous superselection, [28] and in that of a discrete group, of discrete superselection.[30] For example, the discrete superselection rule with respect to the baryon or electric charge in quantum field theory manifests itself as the impossiblity of finding a system in an eigenstate of an operator (C or CP) which does not commute with the operators for these charges (if they are nonzero).

a classical characteristic of the system, so that the value of this quantity does not change in a measurement (absence of Bohr complementarity).

On the other hand, the impossibility of representing Q and $U(\tau) = \exp(iQ\tau)$ by operators in the Hilbert space means, as we already indicated in the Introduction, that there is breaking of the symmetry. In the case of an invariant ground state (vacuum) $C_\tau \equiv 0$, so that a change of the ground state from invariant to noninvariant must be manifested physically as the appearance of a new classical characteristic of the system, in accordance with our earlier statements.

This sort of situation occurs in some phase transitions: the new phase is characterized by a new parameter which is absent in the normal phase. For example, in the transition of liquid helium from the normal to the superfluid state this new parameter is the so-called classical wave function of the condensate particles, $C_\tau = \lim_{\Omega\to\infty} (1/\Omega) \int_\Omega a^+(x)e^{i\tau}d^3x$, which is noninvariant under the gauge transformation $a^+(\mathbf{x}) \to a^+(\mathbf{x})e^{i\tau}$. In the transition of a metal from the normal to the superconducting state the gap function

$$\Delta(\mathbf{x}) = \lim_{\Omega\to\infty} \frac{1}{\Omega} \iint_\Omega b(\mathbf{x}, \mathbf{y}-\mathbf{z})\, a^+(\mathbf{z})\, a^+(\mathbf{y})\, d^3z\, d^3y,$$

appears, where $b(\mathbf{x}, \mathbf{y}-\mathbf{z}) \to 0$ for fixed $\mathbf{x}$ and $|\mathbf{y}-\mathbf{z}| \to \infty$. $\Delta(\mathbf{x})$ is noninvariant under gauge transformations. In the transition through the Curie point in a ferromagnetic material a nonzero magnetization appears, which is noninvariant under rotations.

For spontaneous symmetry breaking the noninvariant macroscopic observables can be described by means of the concept of quasiaverages introduced by Bogolyubov.[31,32] It is well known that in statistical mechanics the average value $\langle A\rangle$ of any dynamical quantity A is defined by the equation

$$\langle A\rangle = \frac{\mathrm{Sp}\, Ae^{-\beta\mathscr{H}}}{\mathrm{Sp}\, e^{-\beta\mathscr{H}}}; \qquad (1.7)$$

here $e^{-\beta\mathscr{H}}$ is the density matrix that describes the ground state of the system, where $\mathscr{H}$ is the Hamiltonian and $\beta = 1/kT$.*

If, however, there is a macroscopic quantity whose change is not accompanied by a change of the energy (the vacuum is degenerate), then the averaging (1.7) contains an extra averaging over the different values of this quantity. Accordingly, to calculate averages with a fixed value of the macroscopic parameter it is necessary to modify the definition (1.7) of the averages. The modified definition of the averages for spontaneous symmetry breaking has been worked out by Bogolyubov, who introduced into the treatment an infinitely weak external field which breaks the symmetry of the system. For this purpose a new term $\nu\mathscr{H}_{\rm int}$ is introduced into the Hamiltonian H of the system, where ν is a small parameter. The new Hamiltonian $\mathscr{H}_\nu = \mathscr{H} + \nu\mathscr{H}_{\rm int}$ no longer commutes with the symmetry operation owing to the unsymmetrical added term. In this case the average of a dynamical variable A, defined by (1.7) and equal to zero for $\nu \equiv 0$, will be different from zero for $\nu \neq 0$.

*We note that there always exists a Hilbert space such that in it the average (1.7) for any fixed T can be written as an expectation with respect to a translationally invariant vector (vacuum). [33]

Moreover, it can remain different from zero even in the limit $\nu \to 0$.

Following Bogolyubov, we give the name of the quasiaverage $\langle A\rangle$ to this double limit:

$$<A> = \lim_{\nu\to 0}\lim_{\Omega\to\infty} \frac{\mathrm{Sp}\, Ae^{-\beta\mathscr{H}_\nu}}{\mathrm{Sp}\, e^{-\beta\mathscr{H}_\nu}} = \lim_{\nu\to 0}\langle A\rangle_\nu. \qquad (1.8)$$

It is important to emphasize that the limit $\nu \to 0$ must be taken after the thermodynamic limit. A difference between $\langle A\rangle$ and $\langle A\rangle$ for some $\mathscr{H}_{\rm int}$ indicates a spontaneous symmetry breaking. It is also easy to see that the usual average is the quasiaverage further averaged over the group of the broken symmetry.

Accordingly, the procedure of taking the limit $\nu \to 0$ is equivalent to the choice of a ground state, or, what is the same thing, a concrete value of the macroscopic parameter characterizing the representation.

The point of view has been advanced[37] that real systems have spontaneous symmetry breaking because they are in an external field which does not have the complete symmetry of the system. Owing to a further ordering which exists in some systems, the effect of singling out a particular one of the various vacuums (of fixing a definite value of a macroparameter) can be secured by means of a very weak external action.

We note that in nonrelativistic many-body theory every local change of state can be described as a production of quasiparticles. Let us consider a certain system. We apply to it a local transformation of a spontaneously broken symmetry. For example, in the case of a ferromagnetic, this corresponds to the situation in which all the spins of the atoms in a certain subvolume of the system are turned through the same angle relative to the spins of the main set of atoms. If the interaction in the system is of finite range, the change of energy caused by this local symmetry transformation is proportional to the area of the boundary of the selected subvolume Ω (i.e., $E \sim \Omega^{2/3}$). The number of excited quasiparticles is itself proportional to Ω. Consequently, the energy per quasiparticle is $\sim \Omega^{2/3}/\Omega$ and goes to zero for $\Omega \to \infty$, i.e., there must exist in the system excitations with energies that go to zero in the dipole limit. If there are long-range (for example, Coulomb) forces the change of the energy will no longer be proportional to $\Omega^{2/3}$, since there is a contribution to the energy not only from the particles that are near the boundary of the subvolume Ω, but also from all the particles in this subvolume. In this case an energy gap in the spectrum of the quasiparticles can exist in the long-wavelength limit.[1,39]

The existence of gap-free excitations in the case of a system with only short-range forces is the content of the nonrelativistic Goldstone theorem, which is also known as the Bogolyubov "$1/q^2$ theorem"[31,32] and as the Hugenholtz-Pines theorem[38] in the theory of superfluidity. We can give a general formulation of this theorem as follows:

Theorem. Let there exist a continuous symmetry group G whose generators can be written in the form of integrals of local densities over all space, and let the Hamiltonian of the system (or the equations of motion) be invariant with respect to this group. If the ground state is noninvariant with respect to this symmetry group and the interaction falls off sufficiently rapidly with the distance, then in the spectrum of the Hamiltonian there necessarily exists a branch of elementary excitations whose energy goes to zero as the momentum goes to zero (i.e., in the long-wavelength limit).

The proof of this theorem is given in a paper by Lang[40] (or see the book by Hugenholtz[1]), and also in[39,41,42]. To prove the nonrelativistic Goldstone theorem we must consider the commutator of a generator Q of a spontaneously broken symmetry group with some local operator A(x), $[Q, A(\mathbf{x})] = \eta(\mathbf{x})$, which is such that

$$\eta = \langle 0 | \eta(\mathbf{x}) | 0 \rangle \neq 0. \tag{1.9}$$

The condition (1.9) means that the macroscopic quantity A_T is noninvariant under the symmetry transformations. The essence of the Goldstone theorem consists of the assertion that the average $\langle 0 | [Q, A(\mathbf{x})] | 0 \rangle = \eta$ is different from zero only if there exist states $|\psi_j\rangle$ whose energy goes to zero in the long-wavelength limit, and which are intermediate states in the matrix element

$$\sum_j \langle 0 | Q | \psi_j \rangle \langle \psi_j | A(x) | 0 \rangle. \tag{1.10}$$

It can be seen from this that the properties of the Goldstone excitations are closely connected with the properties of the local density A(x) of the noninvariant macroscopic quantity. When Galilei invariance of the ground state is broken the momentum distribution function $\rho(\mathbf{p})$ is such a macroscopic quantity and the Goldstone excitations are phonons.

Phonons are also the Goldstone particles associated with the breaking of gauge invariance in superfluid helium and with the breaking of the translational invariance in crystals. In a ferromagnetic the Goldstone particles are spin waves (magnons) associated with a local deviation of the spins of the atoms from the direction of the total magnetization (see Chapter 2).

In a superconductor there are no quasiparticles with energy approaching zero for zero momentum. This is due to the presence of the long-range Coulomb interaction. Therefore in the spectrum of the excitations of the phonon type associated with local changes of the electron density there is a gap caused by the Coulomb interaction. These excitations are called plasmons and lead to a screening of the Coulomb interaction. This situation has been studied in detail in papers by Anderson.[43-46]

We have expounded the characteristic properties of spontaneous symmetry breaking in the nonrelativistic region. The next two chapters are devoted to illustrations of these properties with the concrete examples of the Heisenberg ferromagnetic (Chapter 2) and the theory of superfluidity (Chapter 3).

2. SYMMETRY BREAKING IN THE HEISENBERG MODEL OF A FERROMAGNETIC

We shall examine the main properties of spontaneous symmetry breaking with the example of the Heisenberg model of a ferromagnetic material,[34-36] in which it is described as a set of stationary atoms with spin S located at the sites of a lattice. We assume that the spin is $\frac{1}{2}$ (in a system of units in which $\hbar = 1$).

In the simple case of a cubic lattice the sites can be numbered with three integers (n_1, n_2, n_3). We denote this set of three integers by Z. Then the coordinates of any site in the lattice can be written as (n_1a, n_2a, n_3a), where a is the lattice constant.

The Hamiltonian of the model is of the form

$$\mathscr{H} = -\frac{1}{2} \sum_{n,m} v_{nm} (\mathbf{S}_n, \mathbf{S}_m) \quad (n, m \in Z), \tag{2.1}$$

where the spin operators $S^{(x)}$, $S^{(y)}$, $S^{(z)}$ satisfy the commutation relations

$$[S_n^{(x)}, S_m^{(y)}] = iS_n^{(z)}\delta_{nm} \tag{2.2}$$

(the other relations are obtained by cyclic permutation of x, y, z). It is convenient to introduce the operators

$$S_n^{(\pm)} = S_n^{(x)} \pm iS_n^{(y)}. \tag{2.3}$$

The commutation relations for these operators are

$$[S_n^{(+)}, S_m^{(-)}] = 2\delta_{nm}S_n^{(z)}, \quad [S_n^{(\pm)}, S_m^{(z)}] = \mp \delta_{nm}S_n^{(\pm)}. \tag{2.4}$$

It is seen that the operator $S_n^{(+)}$ increases the projection of the spin of atom "n" along the z axis by unity, and the operator $S_n^{(-)}$ decreases it by unity. With the operators (2.3) the Hamiltonian (2.1) takes the form

$$\mathscr{H} = -\frac{1}{2} \sum_{n,m} v_{nm} (S_n^{(z)}S_m^{(z)} + S_n^{(-)}S_m^{(+)}). \tag{2.5}$$

Since the system is invariant under translations by lattice vectors, v_{nm} depends only on the difference (n – m): $v_{nm} = v(n - m)$. Since the Hamiltonian (1.1) involves the operators S_n only through the scalar products (S_n, S_m), the Hamiltonian is invariant with respect to the group of spin rotations SU(2). Under these rotations *only the spins are transformed, not the lattice.* Accordingly, the Hamiltonian of the model is invariant with respect to the group of translations Z and the group of spin rotations SU(2).

Let us consider the state of a ferromagnetic described by this model. We confine ourselves to the simplest states, i.e., those in which there are no correlations between the states of different atoms. In the ground state of the ferromagnetic the spins of all the atoms are in the same direction. Let l be the unit vector in this direction. The state of the n-th atom is then described by the vector ψ_{nl} (in two-dimensional Hilbert space), which satisfies the equation

$$(S_n, l)\,\psi_{nl} = \psi_{nl}/2. \tag{2.6}$$

This equation defines ψ_{nl} up to a phase factor. The scalar product of the vectors ψ_{nl_1} and ψ_{nl_2} is

$$(\psi_{nl_1}, \psi_{nl_2}) = e^{i\varphi} \sqrt{\frac{1 + (l_1, l_2)}{2}} = e^{i\varphi} \cos\frac{\theta}{2}, \tag{2.7}$$

where θ is the angle between l_1 and l_2. The state of the entire ferromagnetic, in which all spins are directed along l, can then be described by giving the states of all the atoms: $\Psi_l = \prod_n \psi_{nl}$. States in which the spins of all the atoms except a finite number are directed along l, and also linear combinations of such states, differ from Ψ_l only microscopically. We are here considering the case of an infinite number of atoms. If the total number of atoms N were finite, the number $\tilde{N}$ of "incorrectly" directed spins would have to be much smaller than N. If their number were of the same order as N, we would have to speak of macroscopically different states.

The states that differ microscopically from Ψ_l form the Hilbert space H^l. The spin observables of individual atoms are described by operators in H^l and realize a representation of the commutation relations (2.2). States that differ in the direction of the total magnetization (i.e., macroscopically different states) are orthogonal to each other. In fact,

$$(\Psi_{l_1}, \Psi_{l_2}) = \prod_{n=1}^{N} (\psi_{nl_1}, \psi_{nl_2}) = \left[\cos\frac{\theta}{2}\right]^N e^{iN\varphi} \xrightarrow[N\to\infty]{} 0.$$

It is easy to see that also any state of H^{l_1} is orthogonal to any state of H^{l_2}. States belonging to the same H^l are characterized by the same value of the macroscopic parameter (the magnetization) $\mathbf{m} = \lim_{N\to\infty} (1/N) \sum_{n=1}^{N} \mathbf{S}_n$. We note that when we speak of the total magnetization $\mathbf{S} = \sum_{n=1}^{N} \mathbf{S}_n$ we actually mean simply $\mathbf{m}N$, where N is the total number of atoms, and are neglecting the fluctuations of $\mathbf{S}$, which are of the order of $\mathbf{S}/N^{1/2}$. The magnetization $\mathbf{m}$ commutes with all the individual spin operators $\mathbf{S}_n$. For example,

$$[m_z, S_n^{(x)}] = \lim_{N\to\infty} \frac{1}{N} [S_n^{(z)}, S_n^{(x)}] = \lim_{N\to\infty} \frac{1}{N} iS_n^{(y)} = 0.$$

It is easily verified that m_x, m_y, and m_z commute with each other. Consequently, $\mathbf{m}$ behaves like a classical quantity; in particular, all the components of $\mathbf{m}$ are simultaneously measurable and are c numbers in the Hilbert space H^l. It is not hard to verify that $\mathbf{m} = (\Psi_l, \mathbf{S}_n\Psi_l) = 1/2$. All other averages of products of the $\mathbf{S}_n$ can be expressed in terms of $\mathbf{m}$. For T = 0° K, $\mathbf{m}$ is the only macroscopic parameter.

Under spin rotations the magnetization transforms as a vector, and the ground state Ψ_{l_1} goes over into a ground state Ψ_{l_2}. Consequently, under a rotation state of the Hilbert space H^{l_1} go over into states of a different space H^{l_2}, where l_2 is obtained from l_1 by the rotation in question. If the ground state were invariant under rotations, then $((\Psi_l, \mathbf{S}_n\Psi_l)$ would have to be equal to zero. The fact that $\mathbf{m}$ is not zero means that there is symmetry breaking. Since the Hamiltonian of the model is invariant, it is a spontaneous symmetry breaking.

When there is symmetry breaking in an infinite system the symmerty transformations cannot be accomplished with unitary operators; that is, there does not exist any unitary operator U that satisfies the relations $S_n^{(k)'} = US_n^{(k)}U^{-1}$ (k = x, y, z), where $S_n^{(k)'}$ is obtained from $S_n^{(k)}$ by the transformation in question (for example, for a rotation around the z axis through angle φ

$$S_n^{(x)} \to S_n^{(x)'} = S_n^{(x)} \cos\varphi + S_n^{(y)} \sin\varphi,$$
$$S_n^{(y)} \to S_n^{(y)'} = -S_n^{(x)} \sin\varphi + S_n^{(y)} \cos\varphi,$$
$$S_n^{(z)} \to S_n^{(z)'} = S_n^{(z)}).$$

We shall now show that in the Heisenberg model of a ferromagnetic material there are gapless excitations. The assertion that such excitations exist when there is spontaneous symmetry breaking is the content of the Goldstone theorem (see Chapter 1).

In the Heisenberg model the equation of motion is of the form

$$\frac{1}{i}\frac{d}{dt} S_n^{(-)} = [H, S_n^{(-)}] = \sum_m v_{nm} [S_m^{(z)} S_n^{(-)} - S_m^{(-)} S_n^{(z)}].$$

Let us make a Fourier transformation

$$S_q^{(z)} = \frac{1}{\sqrt{N}} \sum_n S_n^{(z)} e^{i\mathbf{q}\mathbf{R}_n}, \quad S_q^{(\pm)} = \frac{1}{\sqrt{N}} \sum_n S_n^{(\pm)} e^{\mp i\mathbf{q}\mathbf{R}_n}.$$

Then in the momentum representation the equations of motion are

$$-i\dot{S}_q^{(-)} = \frac{1}{\sqrt{N}} \sum_{q'} [v(\mathbf{q}') - v(\mathbf{q}-\mathbf{q}')] S_{q'}^{(z)} S_{q-q'}^{(-)}. \tag{2.8}$$

We shall consider the case in which the z axis is along the direction of the magnetization. It is easily seen that

$$\frac{1}{\sqrt{N}} S_{q'}^{(z)} \xrightarrow[N\to\infty]{} m\delta_{0q'} = \frac{1}{2}\delta_{0q'}. \tag{2.9}$$

Substituting (2.9) in (2.8), we get the equations of motion in the form

$$-i\dot{S}_q^{(-)} = \omega_0(\mathbf{q}) S_q^{(-)}, \quad \omega_0(\mathbf{q}) = [v(0) - v(\mathbf{q})], \tag{2.10}$$

where $v(\mathbf{q})$ is the Fourier transform of the interaction potential v_{nm}:

$$v_{nm} = v(\mathbf{R}_n - \mathbf{R}_m) = \sum_q v(\mathbf{q}) \exp\{i\mathbf{q}(\mathbf{R}_n - \mathbf{R}_m)\}.$$

The frequency $\omega_0(\mathbf{q})$ goes to zero for $\mathbf{q} \to 0$. Accordingly, the operator $S_q^{(-)}$ is the creation operator of a Goldstone particle with momentum $\mathbf{q}$, which in the present case is a spin wave (magnon). For large N, up to terms of order $N^{-1/2}$, the operators $S_q^{(z)}$, $S_q^{(\pm)}$ satisfy the commutation relations (the z axis is in the direction of the total magnetization):

$$[S_q^{(+)}, S_{q'}^{(-)}] = \delta_{qq'}, \quad [S_q^{(z)}, S_{q'}^{(\pm)}] = 0 \qquad (\mathbf{q} \neq 0).$$

The operators $S_0^{(z)}$, $S_0^{(\pm)}$ are connected with the total spin operator: $S_0^{(z)} = N^{-1/2}S^{(z)} = m^{(z)}N^{1/2}$, $S_0^{(\pm)} = N^{-1/2}S^{(\pm)}$. For $N \to \infty$, the $S_0^{(z)}$ diverge, but the $S_0^{(\pm)}$ have meaning.

We have seen that spontaneous symmetry breaking in a ferromagnetic is accompanied by the appearance of a macroscopic quantity, the magnetization. Since in the absence of an external field all directions are equivalent, we have an infinite set of ground states corresponding to different directions of the magnetization. The energies of all these states are equal. In field theory every change of state can be described as the excitation of quasiparticles. Macroscopic changes, of the type of a rotation of the entire ferromagnetic, involve the appearance of an infinite number of particles. These quasiparticles excited in macroscopic changes must be condensed in a state with zero energy and zero momentum, since the energy does not change in these changes, and the state remains translationally invariant.

We shall show that a change of the direction of the magnetization occurs with the condensation of magnons with zero momentum, i.e., that for a change of the direction of the magnetization it is necessary that a number ~ N of magnons appear. The operator for the number of magnons is $M = \sum_q S_q^{(-)} S_q^{(+)} = \sum_n S_n^{(-)} S_n^{(+)}$ (the latter equality follows from the unitarity of the Fourier transformation). It is well known that $S_n^{(-)} S_n^{(+)} = 1/2 - S_n^{(z)}$. Consequently, $M = 1/2 N - \sum_n S_n^{(z)}$. Then the magnetization can be expressed in terms of the number of spin waves by the relation

$$m^{(z)}=\frac{1}{N}\sum_n S_n^{(z)}=\frac{1}{2}-\frac{M}{N}. \quad (2.11)$$

It can be seen from (2.11) that for a change of **m** it is necessary that a number ~ N of spin waves be produced.

In concluding this chapter we note that in (2.10) we have assumed that v(n − m) falls off sufficiently rapidly for |n − m| → ∞ so that v(q) is nonsingular for q → 0. If, on the other hand, there is long range interaction in the system, then in general this condition is not satisfied and a gap can appear in the spectrum of the magnons. Obviously this is due to the fact that with long-range interactions the gas of quasiparticles excited when there is a macroscopic change in the system cannot be regarded as a free-particle gas for arbitrarily small densities.

3. SYMMETRY BREAKING IN THE THEORY OF SUPERFLUIDITY

A typical example of spontaneous symmetry breaking is given by the theory of the superfluidity of a nonideal Bose gas. The Hamiltonian of this system in the coordinate representation is of the form

$$\mathscr{H}=-\frac{1}{2m}\int \nabla\psi^+(\mathbf{x})\nabla\psi(\mathbf{x})\,d^3x + \frac{1}{2}\int\int d^3x d^3y \psi^+(\mathbf{x})\psi^+(\mathbf{y})V(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x}). \quad (3.1)$$

It is easy to verify that this Hamiltonian is invariant with respect to the gauge transformation

$$\psi(\mathbf{x})\to e^{i\alpha}\psi(\mathbf{x}),\quad \psi^+(\mathbf{x})\to e^{-i\alpha}\psi^+(\mathbf{x}), \quad (3.2)$$

and this leads to conservation of current,

$$\rho(\mathbf{x},t)=\psi^+(\mathbf{x},t)\psi(\mathbf{x},t),\quad \mathbf{J}(\mathbf{x},t) = \frac{1}{i}[(\nabla\psi^+(\mathbf{x},t))\psi(\mathbf{x},t)-\psi^+(\mathbf{x},t)\nabla\psi^+(\mathbf{x},t)], \quad (3.3)$$

$$\partial_t\rho+\nabla\mathbf{J}=0. \quad (3.4)$$

Superfluidity of a Bose gas is due to the fact that at low temperatures the number of quasiparticles with zero momentum is of the order of the total number of quasiparticles, i.e., there is a condensate in the system. The presence of the condensate manifests itself in the existence of an anomalous average over the ground state (vacuum) $|0\rangle$ (see the Introduction):

$$\langle 0|\psi|0\rangle=\lim_{\Omega\to\infty}\frac{1}{\Omega}\int\psi(x)\,d^3x=\sqrt{\rho_0}\,e^{i\alpha}, \quad (3.5)$$

where, owing to the translational invariance, $\rho_0^{1/2}$ is a constant. The appearance of $\rho_0^{1/2}\neq 0$ means that there is a new macroscopic characteristic of the system—the classical wave function of the condensate, which is noninvariant under the gauge transformation (3.2). The fact that the Hamiltonian $\mathscr{H}$ in (3.1) is invariant under transformations (3.2) means that we have spontaneous breaking of gauge invariance.

In accordance with the nonrelativistic Goldstone theorem (Chapter 1), with certain additional restrictions on the interaction potential V(x − y) we must have a gapless spectrum of elementary excitations.*

*For a simple proof for the model considered see [48,49,41].

Following Bogolyubov,[51,52] we shall find the explicit form of the spectrum of elementary excitations and show that there is no gap.

In the momentum representation the Hamiltonian (3.1) can be written in the form

$$\mathscr{H}=\sum_{\mathbf{k}}\frac{|\mathbf{k}|^2}{2m}a_{\mathbf{k}}^+a_{\mathbf{k}}+\frac{1}{2\Omega}\sum_{\substack{\mathbf{k}_1,\mathbf{k}_2\\ \mathbf{k}_1',\mathbf{k}_2'}}v(\mathbf{k}_1-\mathbf{k}_1')a_{\mathbf{k}_1'}^+a_{\mathbf{k}_2'}^+a_{\mathbf{k}_2}a_{\mathbf{k}_1}\delta(\mathbf{k}_1'+\mathbf{k}_2'-\mathbf{k}_1-\mathbf{k}_2), \quad (3.6)$$

where v(**k**) is the Fourier transform of the interaction potential, $a_{\mathbf{k}}^+$, $a_{\mathbf{k}}$ are the creation and annihilation operators, and Ω is the quantization volume.

To take the condensate into account explicitly we make the canonical transformation $\psi(x)\to\varphi(x)=\psi(x)+\rho_0^{1/2}e^{i\alpha}$. In the momentum representation it takes the form

$$\left.\begin{aligned} a_{\mathbf{k}}^+\to b_{\mathbf{k}}^+ &= a_{\mathbf{k}}^+ + \sqrt{\rho_0\Omega}\,e^{-i\alpha}\delta_{\mathbf{k}0},\\ a_{\mathbf{k}}\to b_{\mathbf{k}} &= a_{\mathbf{k}}+\sqrt{\rho_0\Omega}\,e^{i\alpha}\delta_{\mathbf{k}0}\end{aligned}\right\} \quad (3.7)$$

These operators have the same commutation relations as the a operators. In terms of the b operators, up to constant terms and terms of orders higher than the second, the Hamiltonian (3.6) takes the form[1]

$$\mathscr{H}'=\mathscr{H}-\mu N = \sum_{\substack{\mathbf{k}\\(\mathbf{k}\neq 0)}}\left(\frac{\mathbf{k}^2}{2m}+\rho_0 v(\mathbf{k})b_{\mathbf{k}}^+b_{\mathbf{k}}\right)+\frac{1}{2}\rho_0\sum_{\substack{\mathbf{k}\\(\mathbf{k}\neq 0)}}v(\mathbf{k})(b_{\mathbf{k}}^+b_{-\mathbf{k}}^+ + b_{-\mathbf{k}}b_{\mathbf{k}}); \quad (3.8)$$

here N is the operator of the number of particles, and μ is the chemical potential, equal in first approximation to $\rho_0 v(0)$. The higher order terms can be taken into account with perturbation theory. $\mathscr{H}'$ can be diagonalized by means of the Bogolyubov canonical transformation

$$\left.\begin{aligned} b_{\mathbf{k}} &= u_{\mathbf{k}}\beta_{\mathbf{k}}+v_{\mathbf{k}}\beta_{-\mathbf{k}}^+, & u_{\mathbf{k}}&=u_{-\mathbf{k}},\\ b_{\mathbf{k}}^+ &= u_{\mathbf{k}}\beta_{\mathbf{k}}^+ + v_{\mathbf{k}}\beta_{-\mathbf{k}}, & v_{\mathbf{k}}&=v_{-\mathbf{k}}.\end{aligned}\right\} \quad (3.9)$$

The requirement that the new creation and annihilation operators (the β operators) satisfy the Bose commutation relations imposes on the real functions $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ the restriction

$$u_{\mathbf{k}}^2-v_{\mathbf{k}}^2=1. \quad (3.10)$$

The values of the coefficients $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ are fixed by the condition that there be no terms of the type $\beta_{\mathbf{k}}^+\beta_{-\mathbf{k}}^+$ in the transformed Hamiltonian. The result is that up to a constant

$$\mathscr{H}'=\sum_{\mathbf{k}>0}\varepsilon_{\mathbf{k}}(\beta_{\mathbf{k}}^+\beta_{\mathbf{k}}+\beta_{-\mathbf{k}}^+\beta_{-\mathbf{k}}), \quad (3.11)$$

where $\epsilon_{\mathbf{k}}$ is the energy of an excitation quasiparticle. For small values of the momentum $\epsilon_{\mathbf{k}}$ is of the form

$$\varepsilon_{\mathbf{k}}=c|\mathbf{k}|=|\mathbf{k}|\sqrt{\frac{\rho_0 v(0)}{m}}. \quad (3.12)$$

Accordingly, we see that $\epsilon_{\mathbf{k}}\to 0$ for $\mathbf{k}\to 0$, as was to be expected from the Goldstone theorem. The elementary excitations are phonons propagated with the speed of sound c.

4. NONINVARIANCE OF THE VACUUM AND COLEMAN'S THEOREM

In a Lagrangian field theory the existence of a symmetry is expressed by invariance of the Lagrangian with respect to a group of transformations of the fields.

If this group is continuous, it follows from Noether's theorem that there exists a current-density four-vector $j_\mu(x)$ which satisfies the local conservation law

$$\partial_\mu j^\mu(x) = 0 \tag{4.1}$$

and which generates the group transformation in the sense that the change of any local operator $A(\mathbf{y}, t)$ is given by

$$\delta A(\mathbf{y}, t) = i\delta\lambda \int d^3x [A(\mathbf{y}, t), j^0(x)]|_{x^0=t}. \tag{4.2}$$

The operator $A(\mathbf{y}, t)$ is a function of the fields at the time t. It is also assumed that the current $j^\mu(x)$ is local, i.e., $[A(y), j^\mu(x)] = 0$ for spacelike $|y - x|$, so that the integral in (4.2) converges. If the symmetry is exact, there exists a time-independent charge operator

$$Q = \int d^3x j^0(x). \tag{4.3}$$

The charge Q is the generator of the symmetry group, i.e., (4.2) can be written in the form

$$\delta A(y) = i\delta\lambda [A(y), Q]. \tag{4.4}$$

It is important to note that (4.2) remains valid even if Q does not exist as an operator [it is sufficient that $j^0(x)$ exist].

We shall show that if the vacuum is noninvariant (i.e., if $Q|0\rangle \neq 0$) Q cannot be an operator in Hilbert space, and consequently the symmetry transformation takes the vacuum out of H. Let us assume that Q is an operator in H and $Q|0\rangle = |\psi\rangle$. Then because Q commutes with the momentum operator $\mathbf{p}$, $|\psi\rangle$ is a translationally invariant state. Its norm is infinite,

$$\|\psi\|^2 = \langle 0|QQ|0\rangle = \left\langle 0 \left| \int j^0(\mathbf{x}, t) d^3x \right| \psi \right\rangle = \int \langle 0|j^0(\mathbf{x}, t)|\psi\rangle d^3x = \infty,$$

since, $|0\rangle$ and $|\psi\rangle$ being translationally invariant, the matrix element $\langle 0|j^0(\mathbf{x}, t)|\psi\rangle$ is independent of $\mathbf{x}$. Consequently, our assumption is untrue and Q does not exist as an operator in Hilbert space.

Coleman[9] has put forward a theorem asserting that invariance of the vacuum with respect to a group of transformations means the invariance of the corresponding Hamiltonian. It follows from this theorem that the vacuum cannot be invariant with respect to a broken symmetry* (if the vacuum were invariant, the Hamiltonian would be invariant under the same transformations) and a transformation of the broken symmetry (see Introduction) cannot be represented by a unitary operator. The Coleman theorem was further discussed in papers by Fabri and Picasso,[53] Swieca,[54] and others.[10,55,56] For the case of a discrete symmetry group (the P, CP, C, and T transformations) the proof of the analogous theorem can be found in[11] (page 386), where the theorem is given the statement that it is impossible to describe any broken symmetry with a unitary operator.

The following statements are direct physical consequences of Coleman's theorem in elementary particle physics:

1. In the theory of electromagnetic interactions the physical vacuum cannot be an isotopically invariant state.

*We emphasize that this applies to both Abelian and nonabelian groups, such as SU(2) or SU(3), for example.

2. In the theory of weak interactions the physical vacuum cannot be a state with zero strangeness and zero isotopic spin. Moreover, the vacuum cannot be a P invariant state (nor CP, in the theory of K^0 mesons).

3. The physical vacuum cannot be invariant with respect to the group SU(3), which is broken in all interactions.

We shall formulate and prove Coleman's theorem in the framework of relativistic local field theory.

Theorem. Let there exist a local four-current $j^\mu(x)$ such that the charge $Q = \int j^0(\mathbf{x}, t)d^3x$ is a self-adjoint operator in the Hilbert space of states. Then the current j^μ satisfies the local conservation law $\partial_\mu j^\mu(\mathbf{x}, t) = 0$, and the charge Q does not depend on the time.

Proof. We have already shown that Q exists as an operator only if

$$Q(t)|0\rangle = 0. \tag{4.5}$$

Let us define the operators

$$\varphi(x) = \partial_\mu j^\mu(x), \quad \pi(x) = \partial_t \varphi(x).$$

It then follows from (4.5) that

$$\langle 0|[Q, \pi(0)]|0\rangle = \left\langle 0 \left| \left[\int j^0(\mathbf{x}, t) d^3x, \pi(0) \right] \right| 0 \right\rangle = 0,$$

Differentiating with respect to the time and adding terms $\partial_i j^i(\mathbf{x}, t)$, which give zero when integrated over d^3x, since owing to local commutativity the terms on an infinitely remote surface do not contribute to the commutator, we have $\langle 0|[\int \varphi(\mathbf{x}, 0)d^3x, \pi(0)]|0\rangle = 0$. Applying the Källen-Lehmann expansion $[\rho(\kappa^2) \geq 0]$

$$\langle 0|[\varphi(x), \varphi(y)]|0\rangle = \int_0^\infty \rho(\kappa^2)\Delta(x-y, \kappa^2) d\kappa^2 \qquad (\rho(\kappa^2) \geqslant 0)$$

and using the equation $(\partial/\partial y^0)\Delta(x-y, \kappa^2)|_{x^0=y^0} = -\delta(\mathbf{x}-\mathbf{y})$, we get $\int_0^\infty \rho(\kappa^2)d\kappa^2 = 0$, i.e., $\rho(\kappa^2) = 0$. But then we also have

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int_0^\infty \rho(\kappa^2)\Delta^+(x-y, \kappa^2) d\kappa^2 = 0,$$

from which, since the metric in the Hilbert space is positive, we get $\varphi(x)|0\rangle = 0$. The Federbush-Johnson theorem,[57] which states that if a local operator acting on the vacuum gives zero the operator itself is equal to zero, then gives $\varphi(x) = \partial_\mu j^\mu = 0$. It follows that the charge is conserved: $Q(t) = Q(0)$.

In particular it follows from Coleman's theorem that if the Hamiltonian is noninvariant with respect to some group, then the vacuum is also noninvariant.

Let us make clear why such a statement does not hold in nonrigorous quantum field theory, which assumes the possibility of a unitary connection between the physical and mathematical vacuums. These vacuums are connected with each other by the relation $|0\rangle_{ph} = S(0, -\infty)|0\rangle_m$, where $S(0, -\infty)$ is the half-way S matrix. The symmetry breaking caused by the noninvariance of $S(0, -\infty)$ with respect to the transformation in question has the result that under the transformation $|0\rangle_{ph}$ goes over into $|0'\rangle_{ph} = U|0\rangle_{ph}$, where U is the operator of the symmetry transformation. If an operator $S(0, -\infty)$ actually existed (as is true in the case of nonrelativistic quantum mechanics), then $|0\rangle_{ph}$ would exist in the same Hilbert space as $|0\rangle_m$. In this same space $|0'\rangle_{ph}$ would also exist, so that transformations of the broken symmetry would be represented by unitary operators in this space. This, however, is the essential difference between relativistic quantum field theory and quantum mechanics—that, by Haag's theorem,[11,58] $S(0, -\infty)$ does not exist and $|0\rangle_{ph}$ is not in the same Hilbert space as $|0\rangle_m$ (this fact is sometimes expressed in the words that "the mathematical vacuum is orthogonal to the physical vacuum"). Therefore, if a transformation of the group is represented by a unitary operator on $|0\rangle_m$, in the relativistic theory it does not follow from this that it is also represented unitarily in action on the physical vacuum; that is, in general (and by Coleman's theorem, always), $|0'\rangle_{ph}$ is not in the same Hilbert space as $|0\rangle_{ph}$. This means that there is no representation of the broken symmetry by unitary operators. Thus we see that there is a deep connection between Coleman's theorem and Haag's theorem.

We shall now make some comments on the consequences of Coleman's theorem in elementary-particle theory. That the coupling constant of the interaction breaking the symmetry is small is of no importance in principle for Coleman's theorem; an arbitrarily small symmetry breaking is enough to prevent the representation by unitary operators. Symmetry breaking in the Hamiltonian means noninvariance of the vacuum, which manifests itself in the existence of the anomalous average $\langle 0|[Q(t), A(\mathbf{x}, t)]|0\rangle \neq 0$.

In the axiomatic approach there is a proof[10] that in a case of symmetry breaking there always exists a local observable whose anomalous vacuum average does not depend on the coupling constant at all. However, the specific form of this observable is still unknown.

In a Lagrangian theory using the concept of a single physical vacuum the noninvariant terms not taken into account in the Lagrangian because of the smallness of the coupling constant can nevertheless act on the invariant part "through the noninvariant vacuum."

In other words, since the unique physical vacuum must be noninvariant with respect to all broken symmetries, already in the theory of the strong interactions the noninvariance of the vacuum must lead to anomalous averages whose values cannot in general be expressed analytically in terms of the coupling constants of the interactions breaking the symmetry. In such a case ordinary perturbation theory means that one assumes that these averages are analytic, so that one can neglect them if one neglects the corresponding interactions. An alternative way is to construct the theory of broken symmetries in two stages: one first considers a theory with a symmetric Lagrangian but an unsymmetric vacuum (spontaneous symmetry breaking), and then uses perturbation theory to take the noninvariant correction to the Lagrangian into account. This approach, as we shall see below (Chapter 7), already in the first stage presupposes that we renounce perturbation theory and the assumption that the anomalous averages are analytic in the coupling constant (and then these averages are also nonanalytic in the coupling constant of the interaction which conserves the symmetry—see Chapter 7). This approach differs physically from the perturbation-theory method by taking into account the appearance of new particles—Goldstone zero-mass bosons (Chapters 5 and 6). Inclusion of the noninvariant correction in the Lagrangian leads to the appearance of a mass for these particles (see Chapter 6).

5. SPONTANEOUS SYMMETRY BREAKING AND GOLDSTONE'S THEOREM

In the preceding chapter we have seen that the vacuum is noninvariant if the Lagrangian is noninvariant. But the vacuum can be noninvariant even with an invariant Lagrangian. Such a symmetry breaking is called spontaneous. An important consequence of this is the presence in the theory of zero-mass particles. This statement is known as Goldstone's theorem[59,60] (see also[61-64]). We shall here give the formulation and proof of this theorem in the framework of relativistic quantum field theory, assuming, as usual, that the symmetry of the theory can be expressed as the conservation of a local current $j^\mu(x)$.

Theorem. In a local* translationally invariant field theory with a conserved local four-current $j^\mu(x)$ and vacuum noninvariant with respect to the continuous symmetry group whose generator is the charge $Q = \int j^0(\mathbf{x}, t)d^3x$ there are necessarily particles of mass zero.

In other words, the theorem states that the conditions

$$\partial_\mu j^\mu(\mathbf{x}, t) = 0 \quad \text{(conservation of current)} \tag{5.1}$$

$$\langle 0|[Q, \Phi(x)]|0\rangle \neq 0 \quad \text{(noninvariance of the vacuum)} \tag{5.2}$$

[here $\Phi(x)$ is some local operator] are compatible only if there are zero-mass particles in the theory. We shall see from the proof of the theorem that these particles give a contribution to the matrix element $\langle 0|[j^\mu(x), \Phi(y)]|0\rangle$.

Proof. Let us consider the spectral expansion of the matrix element $\langle 0|[j^\mu(x), \Phi(y)]|0\rangle$:

$$\langle 0|[j^\mu(x), \Phi(y)]|0\rangle = \int \rho_\mu(p)e^{ip(x-y)}d^4p, \tag{5.3}$$

$$\rho_\mu(p) = \sum_g [\delta^4(p-p_g)\langle 0|j_\mu(0)|g\rangle\langle g|\Phi(0)|0\rangle - \delta^4(p-p_g)\langle 0|\Phi(0)|g\rangle\langle g|j_\mu(0)|0\rangle]\,\theta(p_g^0)\,\theta(p_g^2)\dagger; \tag{5.3'}$$

here $|g\rangle$ are intermediate states and form a complete system. For $\rho = 0$ the intermediate states are not states $(p_g^\mu) = 0$ (states with $p_0 = 0$ and $p_j = 0$ are sometimes called "vacuumlike states" or spurions). We shall show that the $|g\rangle$ are states with zero mass, $p_g^2 = 0$.

From considerations of relativistic invariance we have

$$\rho_\mu(p) = p_\mu g(p^2) + p_\mu \varepsilon(p_0)\,\theta(p^2)\,h(p^2),$$

where $\epsilon(p_0) = 1$ if $p_0 > 0$, and $\epsilon(p_0) = -1$ if $p_0 < 0$. On the other hand, by current conservation, Eq. (5.1), $p_\mu \rho_\mu(p) = 0$, from which we have $\rho_\mu(p) = p_\mu \delta(p^2) \times [C\epsilon(p_0) + D]$, where C and D are constants. The pres-

*The requirement that the theory be local is very essential; it replaces the restrictions placed on the potential in the nonrelativistic formulation (see Chapter 1).

†Here $\theta(p) = \begin{cases} 1, & \text{if } p > 0. \\ 0, & \text{if } p < 0. \end{cases}$

ence of $\delta(p^2)$, in the case of nonzero C, D, is what was to be proved. In fact, the expression (5.3) is nonzero only owing to intermediate states with zero mass. But it is easily verified that the constant C is nonzero in the case of noninvariant vacuum, since

$$\langle 0 | [Q, \Phi(y)] | 0 \rangle = (2\pi)^3 \int \rho_0(p^2)|_{\mathbf{p}=0}\, dp_0 = C$$

and is not zero owing to (5.2). D = 0 owing to locality.

The group of transformations with respect to which the symmetry is broken according to (5.1) and (5.2) does not necessarily have to be Abelian (commutative). The possibility of considering nonabelian groups is extremely important, since while in the case of an Abelian group the theorem applies only to violation of the law of conservation of hypercharge (strangeness) in elementary-particle theory, in the case of nonabelian groups the theorem also applies to the cases of the isotopic group SU(2) and the group SU(3),[60,65,66] and also to the case of spontaneous breaking of the symmetry with respect to the Lorentz group and the rotation group.[67-69] In the case of a nonabelian group one needs not one but several vector (or axial-vector) currents. For example, for the group SU(2) one needs three, and for the group SU(3), eight such currents j_μ^α (α is an internal index). For the case of a nonabelian group the constant C must be supplied with additional indices:

$$C_a^\alpha = \langle 0 | [Q^\alpha \Phi_a(x)] | 0 \rangle;$$
$$Q^\alpha = \int j_0^\alpha(\mathbf{x}, t)\, d^3x \quad (\alpha = 1, 2, \ldots, m;\ a = 1, 2, \ldots, n). \tag{5.4}$$

For example, if $\Phi_a(x)$ are the operators of spinless fields, then under the action of the transformation group they undergo changes

$$\Phi_a \to \Phi_a + \delta_\alpha \Phi_a, \text{ where } \delta_\alpha \Phi_a = \varepsilon T_{ab}^\alpha \Phi_b \quad (a, b = 1, 2, \ldots, n),$$

and T_{ab} are matrix elements of the generators T^α (α = 1, ..., m). In this case

$$C_a^\alpha = T_{ab}^\alpha \langle 0 | \Phi_b | 0 \rangle. \tag{5.5}$$

It must be emphasized that in the case of a nonabelian group the particles of mass zero are always connected not with the fields with quantum numbers Φ_b for which $\langle 0|\Phi_b|0\rangle \neq 0$, but with fields regarding which we in general cannot assume anomalous properties of the type $\langle 0|\Phi_a|0\rangle \neq 0$ [in (5.4) the commutator contains the field Φ_a, and $C_a \neq 0$ owing to $\langle 0|\Phi_b|0\rangle \neq 0$ in (5.5)].

Let us make clear what the property $\langle 0|\Phi_b|0\rangle \neq 0$ means and in what way it is anomalous. In the Fock representation for the field operators $\langle 0|\Phi(x)|0\rangle = \langle 0|(\Phi^+(x) + \Phi^-(x))|0\rangle = 0$, where $\Phi^+(x)$ and $\Phi^-(x)$ are the positive and negative frequency parts of the field operator. Therefore for the particular case in which $\Phi(x)$ is a field operator, noninvariance of the vacuum means that we are considering a nonfock representation of the field $\Phi(x)$. The transition from a field $\Phi'(x)$, for which $\langle 0|\Phi'(x)|0\rangle = 0$ to a field $\Phi(x)$ with $\langle 0|\Phi(x)|0\rangle \neq 0$ can be accomplished easily by adding constants to the field [necessarily constants, since owing to the translational invariance of the vacuum $\langle 0|\Phi(x)|0\rangle$ does not depend on the coordinates]. In physical applications $\langle 0|\Phi(x)|0\rangle \neq 0$, where Φ is a field operator, is encountered, for example, in the relativistic theory of K^0 mesons of Salam and Ward,[70] in the transition from strong to weak interactions, when it is assumed that $\langle 0|K^0(x)|0\rangle \neq 0$. Here there is spontaneous breaking of the symmetry with respect to the hypercharge group and the isotopic group SU(2). The anomalous vacuum average that appears is identified with the weak interaction constant. Owing to the nonabelian nature of the isotopic group this situation has the consequence that on the hypothesis as to the isotopic properties of K mesons (K mesons transform as isospinors) the condition $\langle 0|K^0(x)|0\rangle \neq 0$ means that there are particles with mass zero and the quantum numbers of K^+ and K^- mesons. Then, if one identifies these with the actual K mesons, the appearance of their mass must be explained on the basis of a different, nonspontaneous, symmetry breaking (see the altered formulation of the Goldstone theorem for nonspontaneous symmetry breaking in the following chapter).

In the general case the operator $\Phi(x)$ [or $\Phi_a(x)$ in (5.2)] is not the operator of the fundamental field, as in the example we have given. It can be some combination (bilinear or more complicated) of other fields considered at a point (in a finite region) of space-time. For example, in the model of Nambu and Jona-Lasinio[71,72] to be analyzed later (Chapter 7), $\Phi(x) = \bar{\Psi}(x)\Psi(x)$, where Ψ is a spinor field of mass zero. Then $\langle 0|\Phi|0\rangle = \langle 0|\bar{\Psi}\Psi|0\rangle \neq 0$ means spontaneous breaking of CP invariance and strangeness in K^0 decay[16] one assumes not $\langle 0|K^0(x)|0\rangle \neq 0$, but $\langle 0|K^0(x)K^0(x)|0\rangle \neq 0$, i.e., $\Phi = K^0(x)K^0(x)$, and so on. In these examples the appearance of anomalous averages is due to the fact that the fields are not described by the Fock representation and the transition from the bare vacuum to the physical vacuum can no longer be made by adding constants to the field, but is obtained by a Bogolyubov transformation (Chapter 8). The Goldstone theorem we have proved applies to these cases also.

Finally we note that in the proof of the theorem we have used concepts such as conserved current and a charge which is an integral over space of the zeroth component of the current density. In the case of a noninvariant vacuum, however, (see Introduction and Chapter 4) the charge must not be given the meaning of an operator in Hilbert space, and this makes our argument nonrigorous. But the point is that in the proof of Goldstone's theorem the existence of the charge as an operator in Hilbert space is indeed not required; it is enough to know its commutation properties with the field, [Q, Φ]—in other words, it suffices that it exist as an "operator on the operators Φ(x)" (a so-called automorphism). A rigorous derivation of Goldstone's theorem, with the definitions of all the mathematical concepts used, is given in axiomatic quantum field theory.[66,73,74]

6. THE PHYSICAL MEANING OF GOLDSTONE'S THEOREM IN ELEMENTARY-PARTICLE THEORY

Goldstone's theorem is essential for the determination of the type of a symmetry breaking, i.e., whether or not it is spontaneous. Only two mass-zero particles are known to us, the photon and the neutrino. But these particles are obviously insufficient if we regard the breaking of such symmetries as SU(2) or SU(3), for example, as spontaneous.[65,75] Moreover, there is not a single physical example of spontaneous symmetry breaking with the neutrino (or, more generally, any fermion) as the Goldstone particle. An objection against a Goldstone neutrino is the fact that already the simplest condition for symmetry breaking, $\langle 0|\Psi|0\rangle \neq 0$, where Ψ is a spinor, contradicts not only Lorentz invariance but also the Pauli principle. Among the various efforts in the direction of a Goldstone interpretation of neutrinos we may indicate two papers[76,77] in which the attempt is made to avoid the difficulties that arise in the theory by introducing an indefinite metric in the space of states. The interpretation of the photon as the Goldstone particle in a theory with $\langle 0|A_\mu(x)|0\rangle = b_\mu \neq 0$ also leads to a contradiction with relativistic invariance (the vacuum cannot be a relativistically invariant state if $b_\mu \neq 0$). Nevertheless, we shall see below that there exists a model[67,68] with the photon (and also with the graviton) as a Goldstone particle, in which the indicated violation of relativistic invariance does not manifest itself physically.

The difficulties associated with the attempt to interpret the Goldstone particles directly as physical particles with mass zero have led to a tendency to regard Goldstone's theorem as an argument against the consideration of the very possibility of spontaneous symmetry breaking in elementary-particle physics. From this point of view the Goldstone theorem reduces to a trivial statement about the properties of fields with mass zero. In fact, suppose there exists a free scalar field with mass zero[81] and with the equation of motion $\Box\varphi(x) = 0$. This equation can be understood as the conservation law of a "local current" $\partial_\mu j_\mu(x) = 0$, with $j_\mu(x) = \partial_\mu\varphi(x)$. The corresponding Abelian group (invariance of the Lagrangian with respect to which group leads to conservation of the current) has as its generator $Q = \int \partial_0\varphi(x)d^3x$ and leads to the transformation

$$U(\eta)\varphi(x)U^{-1}(\eta) = \varphi(x) + \eta, \quad U = \exp(i\eta Q), \tag{6.1}$$

which follows from the commutation relations

$$[\partial_0\varphi(x), \varphi(x')]\,\delta(x^0 - x^{01}) = -i\delta(x - x').$$

The equation $\Box\varphi = 0$ is obviously invariant with respect to (6.1) for η = const. But (6.1) means a transition to fields $\varphi'(x) = \varphi(x) + \eta$, for which $\langle 0|\varphi'(x)|0\rangle = \eta \neq 0$; that is, a nonzero anomalous average appears. In this case Goldstone's theorem reduces to the trivial remark that the field φ is a zero-mass field, which is already obvious from the field equation $\Box\varphi = 0$. An analogous treatment illustrating Goldstone's theorem for a free field of mass zero can be carried out for the case of fields with nonzero spin.[80]

There are, however, several arguments indicating that the situation with Goldstone's theorem in elementary-particle physics is not so simple as this. These are as follows.

1. It does not follow from the proof of the Goldstone theorem that the Goldstone states must exist as free asymptotic fields. The corresponding particles can be purely virtual. Electrodynamics gives a typical example of such a situation. In the Lorentz gauge $\partial_\mu A_\mu = 0$ the field equations are $\Box A = e_0 j_\mu(x)$, where $\partial_\mu j_\mu = 0$, and are invariant with respect to the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \eta_\mu, \tag{6.2}$$

where η_μ is a constant four-vector. According to the commutation relations the generator of the transformation (6.2) is

$$L(\eta) = \int d^3x\eta_\mu(\partial_0 A_\mu - e_0 x^\mu j_0),$$

so that the field equations can be understood as the "conservation of a current": $\partial_\nu J^{\nu\mu} = 0$, where $J^{\nu\mu} = \partial_\nu A_\mu - e_0 x^\mu j_\nu$. In this case Goldstone's theorem simply states that it is necessary that mass-zero particles exist in the theory under the condition $\langle 0|A'_\mu|0\rangle \neq 0$, where $|0\rangle$ is the vacuum. But the cases $\langle 0|A_\mu|0\rangle = 0$ and $\langle 0|A'_\mu|0\rangle \neq 0$ differ only by the transformation (6.2), which has nothing to do with physical photons, so that the Goldstone bosons here are not physical particles and reduce to the unphysical longitudinal and timelike photons, whose number changes under gauge transformations.[30]

2. One can construct a model of electrodynamics without photons.[67,68] This is a theory in which the original Lagrangian contains only the interacting electron-positron field. In this theory the photons arise as Goldstone particles owing to spontaneous symmetry breaking. An example of such a theory is Bjorken's model.[67] The original Lagrangian is of the form

$$\mathcal{L} = \overline{\Psi}(x)(i\gamma_\mu\partial_\mu - m)\Psi(x) - \tfrac{1}{8}G[\overline{\Psi}(x), \gamma_\mu\Psi(x)][\overline{\Psi}(x), \gamma_\mu\Psi(x)], \tag{6.3}$$

where $\Psi(x)$ is a spinor field of mass m and G is the coupling constant. In this theory it is assumed that the vacuum $|0\rangle$ is not a Lorentz invariant and CPT invariant state, so that

$$\langle 0|[\overline{\Psi}(x), \gamma_\mu\Psi(x)]|0\rangle = \langle 0|j_\mu(x)|0\rangle = F\eta_\mu \neq 0. \tag{6.4}$$

Accordingly we have a theory with a Lorentz invariant and CPT invariant Lagrangian, but with a noninvariant vacuum. The property (6.4) assumes the presence of a nonzero charge density of the vacuum, so that the vacuum is not a charge-invariant state. Owing to the translational invariance of the vacuum F is a constant and η_μ is a unit vector, which is assumed in[67] to be timelike. One then examines the Feynman diagrams in perturbation theory with the Lagrangian (6.3) and the anomalous averages (6.4) and shows that a certain class of the diagrams is equivalent to the diagrams of the usual quantum electrodynamics with intermediate photon lines. The photon Green's function that appears (there was no photon field in the Lagrangian) contains along with a part identical with the usual photon propagator a part which depends on the noninvariant anomalous averages. However, the contribution of this part to the corresponding diagrams (as well as the contribution of a number of new diagrams different from the diagrams of standard electrodynamics[67]) turns out to be inversely proportional to $G\Lambda^2$, where G is the coupling constant in (6.3) and Λ is an effective cutoff necessary for the derivation of the conditions of existence of the nonvanishing anomalous averages. Accordingly this theory is equivalent (as shown with the perturbation-theory diagrams[67]) to ordinary quantum electrodynamics in the limit $G \to \infty$ or $\Lambda \to \infty$. For a more detailed acquaintance with this theory we refer the reader to the original papers.[67,68]

This model is a theory in which a spontaneous breaking of Lorentz invariance is unobservable, and the

Goldstone bosons (their appearance in the models of[67] and[68] is essentially due to the asymmetry of the vacuum) are physical particles, photons.

3. Particles of zero mass may not appear in a theory with spontaneous symmetry breaking if the local current operator itself cannot be defined as an operator in Hilbert space, since it may be singular when the arguments of the operators in terms of which the current is expressed coincide. Some authors[82] suppose that in such a theory there is "local spontaneous symmetry breaking." This situation (unlike the "global" symmetry breaking for which Goldstone's theorem applies) is possible in theories with an interaction which is strongly singular on the light cone. An example of such a theory is the electrodynamics of particles of zero mass, in which the fermions acquire a mass through the spontaneous breaking of γ^5 symmetry.[82,83] In this theory one can redefine the axial-vector current operator (taking vacuum polarization into account), but the new current will no longer be conserved.[83]

4. Goldstone particles may not arise if there were certain mass-zero fields in the original Lagrangian. It is known from the nonrelativistic example of the theory of superconductivity that the Goldstone excitations in this theory acquire mass owing to the presence of the long-range Coulomb interaction[43,44,84] and manifest themselves as plasmon vibrations. Therefore in relativistic theory also there are attempts to construct analogous examples of spontaneous symmetry breaking without the appearance of mass-zero particles (here the original zero-mass vector fields acquire a finite mass, so that there are no zero-mass physical particles at all in the theory).[85,86] This effect of a long-range interaction in the original Lagrangian, which leads to failure of the Goldstone theorem in the nonrelativistic domain (see Chapter 1), is possible in a relativistic theory owing to violation of the condition of locality, whose significance for Goldstone's theorem we indicated in the preceding chapter. In fact, a peculiarity of the quantum theory of zero-mass vector fields is the possibility of using the Coulomb (radiation) gauge. An advantage of this gauge is that there is no indefinite metric in the space of states, but this is achieved at the price of renouncing explicit Lorentz invariance (nevertheless there is no physical manifestation of this noninvariance) and locality.[87] Because of the lack of locality, there is no Goldstone theorem, and spontaneous symmetry breaking is possible without zero-mass particles. Since the theory is gauge invariant, the final results must not depend on the gauge, and therefore we can state that in a Lorentz invariant gauge (with which an indefinite metric is necessary) these particles will appear but will be unphysical because of the indefinite metric.

One model of this sort has been proposed by Kibble.[87] This is a model of the Yang-Mills[88] type, and describes the interaction of complex scalar (pseudoscalar) fields with mass and vector fields with zero mass. The model possesses the property of local gauge invariance.[88] The author shows that the condition of noninvariance of the vacuum with respect to the SU(2) group of intrinsic transformations leads to the appearance of a mass of the vector fields, and Goldstone zero-mass particles do not appear.

In conclusion we point out that the Goldstone particles acquire a nonzero mass if on the spontaneous symmetry breaking there is superposed an additional nonspontaneous symmetry breaking. By this we mean that the complete Lagrangian can be divided into two parts, one invariant and the other noninvariant. The vacuum state of the invariant part of the Lagrangian is assumed noninvariant, i.e., there is a spontaneous symmetry breaking for this part of the Lagrangian. According to Goldstone's theorem there are then zero mass particles. The noninvariant part of the total Lagrangian can be taken into account with perturbation theory and leads to the appearance of a mass for the Goldstone particles. This effect of the appearance of a mass when the symmetry breaking is by stages can be formulated as a modified Goldstone theorem[17]:

Theorem. In a local field theory with partially conserved current and noninvariant vacuum there must exist Goldstone bosons with finite mass.

To prove the theorem we consider[17] a Lagrangian with the density $\mathscr{L} = \mathscr{L}_0 + \epsilon_i \Phi_i$, where $\mathscr{L}_0$ is invariant with respect to some, in general nonabelian, group G, the ϵ_i are constants, and the Φ_i are local fields which form a basis of a definite representation of the group G. The currents J^a_μ, which are conserved for $\epsilon_i = 0$, will for $\epsilon_i \neq 0$ satisfy a condition of "partial conservation" and commutation relations associated with the transformation of the fields Φ_i according to a representation of the group:

$$\partial_\mu J^a_\mu = \varepsilon_i T^a_{ij} \Phi_j, \tag{6.5}$$

$$\left[\int J^a_0(\mathbf{x}, t)\, d^3x,\ \Phi_i(\mathbf{y}, t) \right] = T^a_{ij} \Phi_j(\mathbf{y}, t), \tag{6.6}$$

where T^a_{ij} is the real antisymmetric matrix of the generator T^a. We define one-point and two-point functions

$$\lambda_i \equiv \langle 0 | \Phi_i | 0 \rangle, \tag{6.7}$$

$$\Delta_{ij}(p^2) \equiv i \int d^4x e^{-ipx} \langle 0 | T\{\Phi_i(x), \Phi_j(0)\} | 0 \rangle. \tag{6.8}$$

We assume that owing to the noninvariance of the vacuum there are certain $\lambda_i \neq 0$. From (6.5) and (6.6) it is easy to derive

$$\varepsilon_i T^a_{ij} \lambda_j = 0, \tag{6.9}$$

$$\Delta^{-1}_{ij}(0) \lambda^a_j = \varepsilon^a_i, \tag{6.10}$$

where $\lambda^a_j = T^a_{jk}\lambda_k$, $\epsilon^a_i \equiv T^a_{ik}\epsilon_k$. Equation (6.10) is the result we need. In fact, if we had $\epsilon_i \equiv 0$ and $\lambda_i \neq 0$, this would be the case of the ordinary Goldstone theorem, since $\Delta^{-1}_{ij}(0) = 0$ would mean that there was a singularity of $\Delta_{ij}(p^2)$ at $p^2 = 0$. If, on the other hand, $\epsilon_i \neq 0$ (i.e., if there is symmetry breaking in the Lagrangian) then no zero-mass particles appear. Moreover, certain arguments[17] can be given in favor of a linear dependence of $\Delta^{-1}_{ij}(p^2)$ on p^2, so that $\Delta(p^2) = Z^{1/2}_T (p^2 + \mu^2)^{-1} Z^{1/2}$, where $Z^{1/2}$ is a positive wave-function renormalization matrix and μ^2 is a mass matrix. It can then be seen from (6.10) that $\mu^2 \to 0$ for $\epsilon_i \to 0$. We note that in this proof, as in the Goldstone theorem given earlier, the Φ_i are not necessarily the fundamental field; the Φ_i can be complex structures made up of other fields. That the Φ_i are scalars is due to the Lorentz invariance of the Lagrangian $\mathcal{L}$. The modified Goldstone theorem allows us to state that a mass will appear for the particles even if there are only massless terms in the original Lagrangian. It is well known[18,19] that in the theory of strongly interacting particles (hadrons) there is a higher symmetry SU(3) ⊗ SU(3) (chiral symmetry) in the case of massless particles. Breaking of the chiral symmetry in the Lagrangian allows us to conclude from the modified Goldstone theorem that masses will appear for the Goldstone bosons and to derive a number of experimentally verifiable relations between these masses[17,89,90] (as we shall see in the next chapter, the original spontaneous symmetry breaking owing to noninvariance of the vacuum allows us to get a nonzero mass for the fields appearing in the axial currents). The modified Goldstone theorem is naturally connected with the Gell-Mann-Levy[91] hypothesis of partial conservation of the axial-vector current (PCAC), according to which the axial-vector current $j^i_{\mu A}$ satisfies the relation

$$\partial_\mu j^i_{\mu A} = \frac{m_\pi^2}{\sqrt{2}} b_\pi \Phi^i(x), \qquad \begin{cases} i=1,2,3 & \text{for } SU(2), \\ i=1,\ldots,8 & \text{for } SU(3), \end{cases} \tag{6.11}$$

where m_π is the mass of the π meson, Φ^i is the Heisenberg operator of the interacting pion field, and b_π is a form-factor $[b_\pi(m_\pi^2) = 0.96\, m_\pi)$. It is seen from (6.11) that $\partial_\mu j^i_{\mu A} = 0$ if $m_\pi^2 = 0$.

Accordingly, from the point of view of the modified Goldstone theorem the Goldstone particles can be the π mesons in the case of the group SU(2) and an octet of mesons in the case of SU(3).

7. RELATIVISTIC MODELS OF SPONTANEOUS SYMMETRY BREAKING

In the present chapter we shall present some typical models of spontaneous symmetry breaking in relativistic field theory. A feature of these models is that one renounces the use of ordinary perturbation theory, since the resulting expressions are nonanalytic in the coupling constant. Spontaneous symmetry breaking is closely connected with the transition to the physical vacuum—a transition which, according to Haag's theorem,[11,58] which we have already mentioned, cannot be regarded as a transformation in the Hilbert space of the bare particles. From this point of view the models of spontaneous symmetry breaking are examples of situations in which the symmetry properties of the physical particles are different from those of the bare particles. Therefore some authors[92-94] prefer to speak not so much of spontaneous symmetry breaking as of a change of symmetry in the transition from the mathematical vacuum to the physical vacuum. In particular, there exists a model of spontaneous (dynamic) production of symmetry,[95] in which the symmetry of the physical fields is higher than the symmetry of the bare particles.

Here we shall deal in detail with two typical models of spontaneous symmetry breaking, and shall confine ourselves in other cases to references to the original papers.

1. The first attempt to apply the idea of the noninvariant vacuum to obtain physical particles with finite mass from zero-mass particles, in analogy with the appearance of the gap in the theory of superconductivity, was made in papers by Nambu and Jona-Lasinio[71,72] and by Vaks and Larkin.[96,97] We shall first point out why models in which the mass of the physical particles is different from zero for zero-mass bare particles are so interesting.

First, in the case of fermion fields a zero bare mass leads to the presence of a special symmetry of the bare fields (the so-called γ_5 invariance), which is broken when a mass of the physical fields appears. However, a "trace" of this original symmetry remains: the original exact symmetry manifests itself physically as a broken symmetry, and, as we know from the theory of strong interactions, the existence of a broken symmetry is sufficient for the prediction of experimentally verifiable facts. Thus recently a number of papers have appeared[18,19] which propose to use as the broken symmetry of the strong interactions not simply SU(3), but the so-called chiral group SU(3) ⊗ SU(3), which is exact if the particle masses are zero.

Second, in these models the non-zero particle mass is a function of the other parameters of the theory, such as the coupling constant. This possibly indicates a way in which one may find the answer to the question as to why the particles have definite masses. Usually the mass is regarded as an external parameter of the theory and the question of its origin is not raised.

The original Lagrangian of[71,72] is a γ_5 invariant Lagrangian describing the four-fermion interaction of certain fermions with zero bare mass. The Lagrangian density is

$$\mathcal{L} = -\overline{\Psi}\gamma_\mu\partial_\mu\Psi + g_0[(\overline{\Psi}\Psi)^2 - (\overline{\Psi}\gamma_5\Psi)^2], \tag{7.1}$$

where g_0 is a coupling constant which is assumed to be positive. By γ_5 invariance we mean invariance under the transformations

$$\Psi \to e^{i\tau\gamma_5}\Psi, \quad \overline{\Psi} \to \overline{\Psi}e^{i\tau\gamma_5}, \tag{7.2}$$

which leads to conservation of the helicity $\chi = \int \overline{\Psi}\gamma_0\gamma_5\Psi d^3x$.

We assume that the physical particles have a nonzero mass (which means breaking of the γ_5 invariance). In this case the standard perturbation theory, in which $\mathcal{L}_0 = -\overline{\Psi}\gamma_\mu\partial_\mu\Psi$ and $\mathcal{L}_{int} = g_0[(\overline{\Psi}\Psi)^2 - (\overline{\Psi}\gamma_5\Psi)^2]$ are γ_5 invariant, cannot be applied. The fundamental idea of[71,72] is that the free-field Lagrangian $\mathcal{L}_0'$ is chosen to have the symmetry of the physical fields, in the form

$$\mathcal{L} = (\mathcal{L}_0 + \mathcal{L}_S) + (\mathcal{L}_{int} - \mathcal{L}_S) = \mathcal{L}_0' + \mathcal{L}_{int}', \tag{7.3}$$

where $\mathcal{L}_S = -m\overline{\Psi}\Psi$, after which $\mathcal{L}_{int}'$ can be taken into account with perturbation theory. This choice of $\mathcal{L}_0'$ is connected with the assumption that the physical vacuum is γ_5 noninvariant. According to Coleman's theorem (Chapter 4) the vacuum $|\Phi(m)\rangle$ for $\mathcal{L}_0'$ is also γ_5 noninvariant. The change to a "different" vacuum, expressed in the change from $\mathcal{L}_0$ to $\mathcal{L}_0'$, means the consideration of a different and unitary-nonequivalent representation of the canonical anticommutation relations. In fact, let

$$\gamma_\mu\partial_\mu\Psi^{(0)}(x) = 0, \quad (\gamma_\mu\partial_\mu + m)\Psi^{(m)}(x) = 0.$$

We shall assume that the initial conditions are the same for the two fields:

$$\Psi^{(0)}(x)=\Psi^{(m)}(x) \quad (7.4)$$

at $x^0 = 0$. Here

$$\Psi_\alpha^{(\lambda)}(x)=\frac{1}{\sqrt{\Omega}}\sum_{\substack{\mathbf{p},\sigma\\ p_0^2=\mathbf{p}^2+\lambda^2}}\{u_\alpha^{(\lambda)}(\mathbf{p},\sigma)a_{(\lambda)}(\mathbf{p},\sigma)e^{ipx}+v_\alpha^{(\lambda)}(\mathbf{p},\sigma)b_{(\lambda)}^+(\mathbf{p},\sigma)e^{-ipx}\}, \quad (7.5)$$

where $\lambda = 0$ or m, and $u_\alpha^{(\lambda)}(\mathbf{p},\sigma)$ and $v_\alpha^{(\lambda)}(\mathbf{p},\sigma)$ are normalized spinor eigenfunctions for particles and antiparticles with *momentum* **p** and *spin projection* σ along the direction of the momentum. The operators $a_{(\lambda)}$, $a^+_{(\lambda)}$, $b_{(\lambda)}$, $b^+_{(\lambda)}$ satisfy the commutation relations

$$\{a_{(\lambda)}(\mathbf{p},\sigma),\ a^+_{(\lambda)}(\mathbf{p}',\sigma)\}_+=\delta_{\mathbf{pp}'}\delta_{\sigma\sigma'},$$
$$\{b_{(\lambda)}(\mathbf{p},\sigma),\ b^+_{(\lambda)}(\mathbf{p}',\sigma)\}_+=\delta_{\mathbf{pp}'}\delta_{\sigma\sigma'}.$$

It follows from (7.4) that the operators for the particles with mass are connected with those for the mass-zero particles by a Bogolyubov transformation (through a somewhat more complicated one than in the theory of superconductivity, owing to the presence of antiparticles):

$$\left.\begin{aligned}&a_{(m)}(\mathbf{p},\sigma)=\xi_{\mathbf{p}}a_{(0)}(\mathbf{p},\sigma)+\eta_{\mathbf{p}}b^+_{(0)}(-\mathbf{p},\sigma),\\&b_{(m)}(\mathbf{p},\sigma)=\xi_{\mathbf{p}}b_{(0)}(\mathbf{p},\sigma)-\eta_{\mathbf{p}}a^+_{(0)}(-\mathbf{p},\sigma),\\&|\xi_{\mathbf{p}}|^2=\frac{1}{2}\left(1+\frac{|\mathbf{p}|}{\sqrt{\mathbf{p}^2+m^2}}\right),\ |\eta_{\mathbf{p}}|^2=\frac{1}{2}\left(1-\frac{|\mathbf{p}|}{\sqrt{\mathbf{p}^2+m^2}}\right).\end{aligned}\right\} \quad (7.6)$$

The connection between the vacuums $|\Phi(0)\rangle$ and $|\Phi(m)\rangle$, defined by

$$a_{(\lambda)}(\mathbf{p},\sigma)|\Phi(\lambda)\rangle=b_{(\lambda)}(\mathbf{p},\sigma)|\Phi(\lambda)\rangle=0 \quad (\lambda=0 \text{ or } m), \quad (7.7)$$

can be derived from (7.6) and (7.7) and is

$$|\Phi(m)\rangle=\prod_{\mathbf{p},\sigma}\{\xi_{\mathbf{p}}-\eta_{\mathbf{p}}a^+_{(0)}(\mathbf{p},\sigma)b^+_{(0)}(-\mathbf{p},\sigma)\}|\Phi(0)\rangle. \quad (7.8)$$

The scalar product

$$\langle\Phi(0)|\Phi(m)\rangle=\exp\left(\sum_{\mathbf{p}}\ln\xi_{\mathbf{p}}\right)=\prod_{\mathbf{p}}\xi_{\mathbf{p}} \quad (7.9)$$

goes to zero when the normalization volume Ω goes to infinity ($|\xi_{\mathbf{p}}| < 1$), so that in a translationally invariant theory $|\Phi(0)\rangle$ and $|\Phi(m)\rangle$ are orthogonal to each other. The expression (7.8) for $|\Phi(m)\rangle$ of course has only a formal meaning [in the more rigorous axiomatic theory the concept of the noninvariant vacuum is formulated without the use of expansions of the form (7.8)], but it enables us to see the feature of γ_5 noninvariance of $|\Phi(m)\rangle$, which leads to an infinite degeneracy of the vacuum. In fact, under the transformation (7.2)

$$\begin{aligned}|\Phi(m)\rangle\rightarrow|\Phi(m)\rangle_\tau&=e^{-i\tau\chi}|\Phi(m)\rangle\\&=\prod_{\mathbf{p},\pm}\{\xi_{\mathbf{p}}-\eta_{\mathbf{p}}e^{\pm 2i\tau}a^+_{(0)}(\mathbf{p},\pm)b^+_{(0)}(-\mathbf{p},\pm)\}|\Phi(0)\rangle,\end{aligned} \quad (7.10)$$

and we have $_\tau\langle\Phi(m)|\Phi(m)\rangle_{\tau'} = 0$ [$\tau' \neq \tau \pmod{2\pi}$]. Accordingly, there is an infinite set of vacuums $|\Phi(m)\rangle_\tau$ ($0 \leq \tau \leq 2\pi$). But all of these vacuums reduce essentially to a physically equivalent description of the system, so that the only actual difference is that between $|\Phi(0)\rangle$ and $|\Phi(m)\rangle$, since the γ_5 noninvariance of $|\Phi(m)\rangle$, allowed for in the Bogolyubov transformation (7.6), manifests itself in the existence of the anomalous average $\langle\Phi(m)|\bar{\Psi}\Psi|\Phi(m)\rangle$, which is identified with the mass.

By putting L in the form (7.3), Nambu and Jona-Lasinio then calculate the proper-energy part $\Sigma(p, m, g_0, \Lambda)$ where Λ is a cutoff, so that the physical particle satisfies the equation

$$[i\gamma p+\Sigma(p,m,g_0,\Lambda)]\Psi=0,$$

if $(i\gamma p + m)\Psi = 0$, i.e.,

$$m=\Sigma(p,m,g_0,\Lambda)|_{(i\gamma p+m)\Psi=0}. \quad (7.11)$$

From (7.11), calculating Σ in first order by perturbation theory and using the expression S_F^m for the propagators,[30] we get a necessary condition for the existence of a nontrivial solution ($m \neq 0$)

$$1=-\frac{ig_0}{(2\pi)^4}\int\frac{d^4p}{p^2+m^2-i\varepsilon}F(p,\Lambda), \quad (7.12)$$

where $F(p, \Lambda)$ is the cutoff form-factor. From (7.12) we can find a connection between m, g_0, and Λ. For a Lorentz invariant cutoff at $p^2 = \Lambda^2$ we get the relation

$$\frac{2\pi^2}{g_0\Lambda^2}=1-\frac{m^2}{\Lambda^2}\ln\left(\frac{\Lambda^2}{m^2}+1\right), \quad (7.13)$$

from which it can be seen that a nontrivial solution exists only for

$$0<\frac{2\pi^2}{g_0\Lambda^2}<1. \quad (7.14)$$

We see from (7.13) and (7.14) that the resulting theory is nonanalytic at $g_0 = 0$; m cannot be expanded in a power series in g_0.

In[71,72] there is further discussion of the question of the Goldstone theorem (Chapters 5 and 6). The theory in question is a theory with a γ_5 invariant Lagrangian but a noninvariant vacuum. In it the axial current

$$\partial_\mu j_{\mu 5}=0,\quad j_{\mu 5}=i\bar{\Psi}\gamma_\mu\gamma_5\Psi. \quad (7.15)$$

is conserved. The Dirac equation for a particle with mass does not conserve this type of current, since

$$\partial_\mu(\bar{\Psi}^{(m)}\gamma_\mu\gamma_5\Psi^{(m)})=2im\bar{\Psi}^{(m)}\gamma_5\Psi^{(m)}. \quad (7.16)$$

The question arises, how are (7.15) and (7.16) to be reconciled? The answer proposed in[71,72] is that owing to polarization corrections the current $j_{\mu 5}$ must be redefined. In particular, between one-nucleon states it will have the form

$$\langle p'|j_{\mu 5}|p\rangle=\bar{u}(p')X_\mu(p',p)U(p),$$

where

$$X_\mu(p',p)=F(q^2)\left(i\gamma_\mu\gamma_5+\frac{2m\gamma_5 q_\mu}{q^2}\right),\quad q=p-p'',\ p^2=p^2=m^2.$$

Then Eqs. (7.15) and (7.16) can be compatible if

$$X_\mu(p',p)=F(q^2)\left(i\gamma_\mu\gamma_5+\frac{2m\gamma_5 q_\mu}{q^2}\right),$$

which corresponds to the appearance of a pole at $q^2 \rightarrow 0$; i.e., a Goldstone boson ($m = 0$) appears with the quantum numbers of a nucleon-antinucleon pair (a pseudoscalar meson with $m = 0$).

An obvious shortcoming of the model of Nambu and Jona-Lasinio is the explicit dependence on the cutoff parameter Λ, whose physical meaning is still unclear.

2. In the case of scalar (or pseudoscalar) fields Goldstone[59] has proposed a simple example of a nontrivial *symmetry breaking* with the appearance of bosons of mass zero.

Let the Lagrangian for interacting charged scalar fields

$$\varphi=\frac{1}{\sqrt{2}}(\varphi_1+i\varphi_2)\ \text{and}\ \varphi^*=\frac{1}{\sqrt{2}}(\varphi_1-i\varphi_2)$$

be of the form

$$\mathcal{L} = \partial_\mu \varphi^* \partial_\mu \varphi - \mu_0^2 \varphi^* \varphi - \frac{\lambda_0}{6} (\varphi^* \varphi)^2 = \partial_\mu \varphi^* \partial_\mu \varphi - V(\varphi^* \varphi), \qquad (7.17)$$

where $\mu_0^2 < 0$. The Lagrangian (7.17) is invariant under the transformation $\varphi \to \varphi e^{i\alpha}$, $\varphi^* \to \varphi^* e^{-i\alpha}$ (or $R\varphi_1 = \varphi_2$). The function $V(\varphi^*\varphi)$ has a maximum at φ^*, $\varphi = 0$, and therefore it is natural to suppose that the vacuum $|0\rangle$, for which $\langle 0|V|0\rangle = 0$, is unstable, so that the states of the system are to be looked for on a vacuum $|0'\rangle$ such that $\langle 0'|V|0'\rangle \neq 0$. This can be done by setting $\langle 0'|V|0'\rangle = \chi \neq 0$, where χ is determined from the condition that the total energy be a minimum

$$\left\langle 0' \left| \frac{\partial V}{\partial \varphi^*} \right| 0' \right\rangle = \langle 0' | \varphi V' (\varphi^* \varphi) | 0' \rangle = 0.$$

Carrying out the calculations, we get $|\chi|^2 = -3\mu_0^2/\lambda_0$. The difference of χ from zero can be taken into account explicitly by making a canonical transformation

$$\varphi = \varphi' + \chi. \qquad (7.18)$$

Choosing χ real and substituting (7.18) in (7.17), we get

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi_1' \partial_\mu \varphi_1' + 2\mu_0^2 \varphi_1'^2) + \frac{1}{2} \partial_\mu \varphi_2' \partial_\mu \varphi_2' - \frac{\lambda_0}{6} \chi \varphi_1' (\varphi_1'^2 + \varphi_2'^2) - \frac{\lambda_0}{24} (\varphi_1'^2 + \varphi_2'^2)^2, \qquad (7.19)$$

i.e., instead of two fields with imaginary mass φ_1 and φ_2 (or φ and φ^*) we get a field with the real mass $2^{1/2}\mu_0$ and a field with mass zero. The Lagrangian (7.19) is already noninvariant with respect to the transformation $R\varphi_1' = \varphi_2'$. This is obviously due to the transformation (7.18), since it is this transformation that has explicitly introduced an asymmetry between φ_1 and φ_2. In terms of φ_1 and φ_2 the transformation (7.18) with real χ means a change to φ_1' and φ_2' according to the formulas

$$\varphi_1 = \varphi_1' + \sqrt{2}\,\chi, \quad \varphi_2 = \varphi_2'.$$

In (7.19) there is also obvious breaking of the symmetry with respect to the gauge transformation indicated above, which is due to the noninvariance of the vacuum, which manifests itself in $\langle 0'|\varphi|0'\rangle = \chi \neq 0$.

The main features of the model are:

a) the initial fields φ and φ^* (or φ_1 and φ_2) are characterized by a negative value of the square of the mass (an imaginary mass), but the physical particles acquire a real mass and a zero mass, respectively;

b) the fields φ_1' and φ_2' interact with each other, so that the Goldstone bosons φ_1' act as physical particles;

c) the theory is nonanalytic in the coupling constant λ_0, as is seen from the expression given for χ.

The models of Nambu and Jona-Lasinio and of Goldstone are examples of nontrivial relativistic models of spontaneous symmetry breaking.

To conclude this chapter we shall briefly list some other models of spontaneous symmetry breaking in elementary-particle theory.

a) Breaking of SU(3). A model of spontaneous breaking of SU(3), of the type of the model of Nambu and Jona-Lasinio, which we have analyzed, has been constructed in[12,98]. Instead of a single spinor field Ψ with zero mass in Eq. (8.1), in the model of SU(3) breaking a triplet of spinor fields is taken. A nonlinear interaction of the self-action type [as in (7.1)] is invariant with respect to SU(3). The vacuum is assumed invariant only with respect to the isospin rotation group and the hypercharge group. As in the model of Namby and Jona-Lasinio, the noninvariance of the vacuum with respect to the full symmetry group of the Lagrangian leads to the appearance of nonzero masses for the particles, and relations determined by the Gell-Mann-Okubo formula hold between these masses.

A model of spontaneous breaking of SU(3) (and also of the group $RO_{(n)}$ of internal rotations[99]) can also be constructed in analogy with the Goldstone model; the basic fields in this model are not spinor fields, but scalars.

An open question in these models is that of the physical meaning of the Goldstone bosons.

A model of spontaneous breaking of SU(3) without massless particles can be constructed in analogy with the Kibble model (see Chapter 6), to which the Goldstone theorem does not apply owing to peculiarities of the quantization of the vector fields (cf. the corresponding model in[14]).

b) Explanation of the mass difference of the muon and the electron. This problem is treated in[12,13]. In these papers isotopic properties are ascribed to the muon and the electron: an isotopically invariant Lagrangian is taken for the bare fields with zero mass. The vacuum is assumed to be an isotopically noninvariant state, so that spontaneous symmetry breaking leads to the appearance of a mass difference between the particles. This is interpreted as the mass difference of the physical muon and electron. In these models difficulties arise associated with the physical interpretation of the Goldstone bosons.

c) Spontaneous breaking of CP invariance. According to the idea of spontaneous breaking of CP, the violation of CP invariance in the decays of K^0 mesons is not due to the existence of a new CP-noninvariant interaction, but to CP noninvariance of the vacuum with an invariant Lagrangian. In[15,99] CP noninvariance of the vacuum manifests itself in the existence of a CP-noninvariant vacuum average $\langle 0|\varphi|0\rangle$ for a certain field with which the K^0 mesons interact. The Lagrangian of this field is of the form of the Lagrangian in the Goldstone model. The breaking of CP is then a breaking of the symmetry under R-reflection (see discussion of the Goldstone model above).

In[16,100] spontaneous breaking of CP is associated not with the existence of a new field $\varphi(x)$, but with the properties of the K^0 mesons themselves; here $\langle 0|K^0K^0|0\rangle \neq 0$, i.e., $\varphi(x) = K^0(x)K^0(x)$.

Models of spontaneous breaking of CP reduce phenomenologically to models of the Wolfenstein superweak interaction.[101]

Other interesting models which use the concept of noninvariance of the vacuum are the models of breaking of SU(3) ⊗ SU(3) in strong-interaction theory, which we have already mentioned earlier, and also the calculation of the Cabibbo angle, which plays an essential part in the definition of the currents in the weak interaction.[102-106]

8. PROPERTIES OF THE VACUUM AND THE CURVATURE OF SPACE-TIME

We here consider an interesting property of field quantization in space-time with a nonstationary metric.

The theory of a quantized field in such a metric (with the gravitational field not quantized) is equivalent to the quantum theory of a field interacting with an external classical field. As is well known,[107] in the case of an external field depending on the time it is necessary to take into account the production of real particles by this field, which is interpreted by a number of authors[22,23,108,109] as the production of matter in the expansion of the universe. The vacuum state, defined as a state invariant with respect to the symmetry group of the Lagrangian, is noninvariant under translations in time, so that each instant of time has its own vacuum corresponding to it. The change of the vacuum with time is accompanied by the change of a certain classical quantity, defined as the number density of the particles in the vacuum and interpreted in[22,23] as the classical matter density, in analogy with what is done in the examples from nonrelativistic physics which we gave in Chapter 1, where a change of the vacuum is necessarily associated with a change of some classical quantity. However, calculations[23] for the scalar (pseudoscalar) π-meson field in the quasieuclidian Friedman model gave for the present stage of evolution of the universe a value for this density of the order $\approx (mH^2/16(2\pi)^3)\sin^2 m\eta \approx 10^{-46}\ \mathrm{cm}^{-3}$, where η plays the part of the time, H is Hubble's constant, and m is the mass of the particle. This small value indicates that the effect of matter production is unimportant at the present time, but it can have been important at early stages of the evolution of the universe. In particular it is possible that all of the matter in the universe originated in such a way.[109] To elucidate this question it is necessary to make a combined treatment of quantum field theory and the Einstein equations.

A nonvanishing matter density in the vacuum can also lead to consequences such as the appearance of a nonzero cosmological constant (see the review by Ya. B. Zel'dovich[110]). An interesting question in theories that connect properties of the vacuum with macroscopic characteristics in cosmology is that of the macroscopic consequences of noninvariance of the vacuum in elementary particle theory (Coleman's theorem) with respect to transformations such as spatial reflection, the CP reflection, and gauge transformations of the strangeness (hypercharge). In particular CP noninvariance of the vacuum can have the consequence that the corresponding macroscopic quantity is different from zero only for particles. If this quantity is identified with the matter density in the universe, its being equal to zero for antimatter will manifest itself physically as the absence of antigalaxies.

[1] N. M. Hugenholtz, Quantum Theory of Many-body Systems (Russ. transl.), Mir, 1967.

[2] I. E. Segal, Mathematical Problems of Relativistic Physics, Providence, R. I., Am. Math. Soc., 1963.

[3] O. I. Zav'yalov and V. N. Sushko, Teor. Matem. Fiz. 1, 153 (1955).

[4] A. S. Wightman and S. S. Schweber, Phys. Rev. 98, 812 (1955).

[5] R. Haag and D. Kastler, J. Math. Phys. 5, 848 (1964).

[6] H. Araki, Einführung in die axiomatische Quantenfeldtheorie, I, II. Lecture notes, ETH, Zürich, 1961–1961–1962.

[7] In collection: Nelineĭnaya kvantovaya teoriya polya (Nonlinear Quantum Field Theory), IL, 1959.

[8] W. Heisenberg, Introduction to the Unified Field Theory of Elementary Particles, New York, Interscience, 1966.

[9] S. Coleman, J. Math. Phys. 7, 787 (1966).

[10] E. Fabri, F. Picasso and F. Strocchi, Nuovo Cim. 48A, 376 (1967).

[11] N. N. Bogolyubov, A. A. Lggunov, and I. T. Logunov, Osnovy aksiomaticheskogo podkhoda v kvantovoĭ teorii polya (Fundamentals of the Axiomatic Approach in Quantum Field Theory), "Nauka", 1969.

[12] M. Baker and S. L. Glashow, Phys. Rev. 128, 2462 (1962).

[13] R. Arnowitt and S. Deser, Phys. Rev. 138, B712 (1965).

[14] Y. S. Kim and F. Landis Markley, Nuovo Cim. 63A, 60 (1969).

[15] G. Marx, Phys. Rev. Lett. 14, 334 (1965).

[16] A. A. Grib, ZhÉTF Pis. Red. 2, 14 (1965) [JETP Lett. 2, 8 (1965)].

[17] S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968).

[18] M. Gell-Mann, R. Oakes and B. Renner, Phys. Rev. 175, 2195 (1968).

[19] R. Dashen and W. Weinstein, Phys. Rev. 183, 1261 (1969).

[20] J. Von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton Univ. Press, 1955.

[21] P. A. M. Dirac, Phys. Rev. 139, B684 (1965).

[22] L. Parker, Phys. Rev. 183, 1057 (1969).

[23] A. A. Grib and S. G. Mamaev, Yad. Fiz. 10, 1276 (1969) [Sov. J. Nucl. Phys. 10, 722 (1970)].

[24] G. Wakita, Prog. Theor. Phys. 23, 32 (1960).

[25] A. Komar, Phys. Rev. 133, B542 (1964).

[26] A. A. Grib, Vestnik LGU (Leningrad State Univ.) No. 16, 16 (1968).

[27] G. C. Wick, A. S. Wightman and E. P. Wigner, Phys. Rev. 88, 101 (1952).

[28] G. Emch and M. Guenin, J. Math. Phys. 7, 915 (1966).

[29] R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and all that, New York, W. A. Benjamin, 1964.

[30] S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Evanston, Ill., Row, Peterson and Co., 1961.

[31] N. N. Bogolyubov, Kvasisrednie v zadachakh statisticheskoĭ mekhaniki (Quasiaverages in Problems of Statistical Mechanics), OIYaI (Joint Inst. Nucl. Res.), Dubna, D 781, 1961; 2nd Edition p 511, 1963.

[32] N. N. Bogolyubov, Physica 26, 1 (1960).

[33] a) R. Haag, N. M. Hugenholtz and M. Winnink, Commun. Math. Phys. 5, 215 (1967); b) N. M. Hugenholtz and J. D. Wieringa, ibidem 11, 183 (1969); c) M. Winnink, An Application of C-algebras to Quantum Statistical Mechanics of Systems in Equilibrium, Thesis, Univ. of Groningen, 1968; d) J. D. Wieringa, Thermodynamic Limit and K. M. S. States in Quantum Statistical Mechanics, a C-algebraic Approach, Thesis, Univ. of Groningen, 1970.

[34] W. Heisenberg, Zs. Phys. 49, 619 (1928).

[35] D. C. Mattis, The Theory of Magnetism, New York, Harper and Row, 1965.

[36] R. F. Streater, Commun. Math. Phys. 6, 233 (1967).
[37] R. H. Brout, Phase Transformations, New York, W. A. Benjamin, 1965.
[38] N. M. Hugenholtz and D. Pines, Phys. Rev. 116, 489 (1959).
[39] H. Wagner, Z. Phys. 195, 273 (1966).
[40] R. V. Lange, Phys. Rev. 146, 301 (1966).
[41] L. Leplae and H. Umezawa, Nuovo Cim. 44B, 410 (1966).
[42] a) A. Klein and B. W. Lee, Phys. Rev. Lett. 12, 266 (1964); b) W. Gilbert, Phys. Rev. Lett. 12, 713 (1966).
[43] P. W. Anderson, Phys. Rev. 110, 827 (1958).
[44] P. W. Anderson, Phys. Rev. 112, 1900 (1958).
[45] P. W. Anderson, Phys. Rev. 130, 439 (1963).
[46] P. W. Anderson, Lectures on the Many-Body Problem, Vol. 2, Ed. E. R. Caianiello, 1964.
[47] Collection "Teoriya sverkhprovodimosti" (Theory of Superconductivity"), IL, 1960.
[48] A. Coniglio and M. Marinaro, Nuovo Cim. 48B, 249 (1967).
[49] A. Coniglio and M. Marinaro, Nuovo Cim. 48B, 267 (1967).
[50] J. A. Swieca, Commun. Math. Phys. 4, 1 (1967).
[51] N. N. Bogolyubov, V. V. Tolmachev, and D. V. Shirkov, A New Method in the Theory of Superconductivity, New York, Consultants Bureau, 1959.
[52] I. M. Khalatnikov, An Introduction to the Theory of Superfluidity, New York, W. A. Benjamin, 1965.
[53] E. Fabri and L. E. Picasso, Phys. Rev. Lett. 16, 408 (1966).
[54] J. A. Swieca, Phys. Rev. Lett. 17, 974 (1966).
[55] G. F. Dell'Antonio, Nuovo Cim. 47, A1 (1967).
[56] B. Schroer and P. Stichel, Commun. Math. Phys. 3, 258 (1966).
[57] P. Federbush and K. Johnson, Phys. Rev. 120, 1926 (1960).
[58] R. Haag, Kgl. Dan. Vid. Selsk, Mat.-fys. Medd. 29, No. 12 (1955).
[59] J. Goldstone, Nuovo Cim. 19, 154 (1961).
[60] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 962 (1962).
[61] S. Bludman and A. Klein, Phys. Rev. 131, 2363 (1963).
[62] T. W. B. Kibble,[111] p. 277.
[63] G. S. Guralnik, C. R. Hagen and T. W. B. Kibble, Advances in Particle Physics, Ed. R. L. Cool and R. E. Marshak, 1968, page 567.
[64] J. T. Lopuszanski, Fort. der Phys. 15, 684 (1967).
[65] R. F. Streater, Phys. Rev. Lett. 15, 475 (1965).
[66] R. F. Streater, Proc. Roy. Soc. 287, A510 (1965).
[67] J. D. Bjorken, Ann. Phys. 24, 174 (1963).
[68] I. Bialynicki-Birula, Phys. Rev. 130, 465 (1963).
[69] a) G. S. Guralnik, Phys. Rev. 136, B1404, B1417 (1964); b) G. S. Guralnik and C. R. Hagen, Phys. Rev. 171, 1431 (1968).
[70] A. Salam and I. C. Ward, Phys. Rev. Lett. 5, 390 (1960).
[71] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).
[72] G. Jona-Lasinio and Y. Nambu, Phys. Rev. 124, 246 (1961).
[73] a) D. Kastler, D. W. Robinson and A. Swieca, Commun. Math. Phys. 2, 108 (1966); b) D. Kastler,[111], page 305.
[74] A. V. Bulinskiĭ, Vestnik MGU (Moscow State Univ.) No. 2, 75 (1969).
[75] R. F. Streater, Mathematical Theory of Elementary Particles, Ed. R. Goodman and I. E. Segal, MIT Press, 1966, p. 10.
[76] J. A. Swieca, Nuovo Cim. 57A, 573 (1968).
[77] H. Leutwyler, Helv. Phys. Acta 38, 431 (1965).
[78] H. J. Treder, Monatshcr. Deutsh. Acad. Wiss. 9, 427 (1967).
[79] a) P. R. Phillips, Phys. Rev. 146, 966 (1966); b) H. C. Ohanian, Phys. Rev. 184, 1305 (1969).
[80] G. S. Guralnik and C. R. Hagen, Nuovo Cim. 61A, 98 (1969).
[81] G. S. Guralnik and C. R. Hagen, Nuovo Cim. 43A, 1 (1966).
[82] Th. A. J. Maris, G. Jakob and B. Liberman, Nuovo Cim. 52A, 116 (1967).
[83] K. Johnson, Phys. Lett. 5, 253 (1963).
[84] J. A. Cronin, Amer. J. Phys. 33, 319 (1965).
[85] A. A. Migdal and A. M. Polyakov, Zh. Eksp. Teor. Fiz. 51, 135 (1966) [Sov. Phys.-JETP 24, 91 (1967)].
[86] P. W. Higgs, Phys. Rev. 145, 1156 (1966).
[87] T. W. B. Kibble, Phys. Rev. 155, 1554 (1967).
[88] Collection "Élementarniye chastitsy i kompensiruyushchie polya" ("Elementary Particles and Compensating Fields"), Moscow, "Mir", 1964.
[89] S. Okubo and V. S. Mathur, Phys. Rev. Lett. 23, 1412 (1969).
[90] R. F. Dashen, Phys. Rev. 183, 1245 (1969).
[91] M. Gell-Mann and M. Levy, Nuovo Cim. 16, 705 (1960).
[92] Proc. of the Hungar. Conf., Acta Phys. Hung. 19, f. 1–4 (1965).
[93] a) L. Leplae and H. Umezawa, Nuovo Cim. 44B, 410 (1966); b) K. Nakagawa, Nuovo Cim. 50A, 37 (1967).
[94] H. Umezawa, Nuovo Cim. 40A, 450 (1965).
[95] S. A. Bludman and N. G. Despande, Nuovo Cim. 45, A656 (1966).
[96] V. G. Vaks and A. I. Larkin, Zh. Eksp. Teor. Fiz. 40, 282 (1961) [Sov. Phys.-JETP 13, 192 (1961)].
[97] V. G. Vaks and A. I. Larkin, Proc. 1960 Rochester Conf. on High-energy Physics (1960), p. 871.
[98] H. Miyasawa, Recent Developments in Particle Physics, ser. Nuclear Physics, Vol. 2, 1966, p. 22.
[99] G. Marx, Phys. Rev. 140, 1068 (1966).
[100] A. A. Grib, Vestnik LGU (Leningrad State Univ.) No. 22, 50 (1967).
[101] L. B. Okun', Usp. Fiz. Nauk 89, 603 (1966) [Sov. Phys.-Uspekhi 9, 574 (1967)].
[102] R. Gatto, G. Sartori and M. Tonin, Phys. Lett. 28, B128 (1968).
[103] N. Cabibbo and L. Maiani, Phys. Lett. 28, B131 (1968).
[104] N. Cabibbo, "Hadrons and their Interactions", 1967 School of Physics; E. Majorana, Academic Press 1968, p. 381.
[105] A. Pais, Phys. Rev. 173, 1587 (1968).
[106] A. I. Solomon and Y. Neeman, Nuovo Cim. 60, A108 (1969).
[107] J. Schwinger, Theory of Quantized Fields (Russ. Transl.) IL, 1956.
[108] L. Parker, Phys. Rev. Lett. 21, 562 (1968).
[109] R. Sexl and H. Urbantke, Phys. Rev. 179, 1247 (1969).
[110] Ya. B. Zel'dovich, Usp. Fiz. Nauk 95, 209 (1968) [Sov. Phys.-Uspekhi 11, 381 (1968)].
[111] Proc. of the 1967 Conference on Particles and Fields, Ed. C. R. Hagen et al., 1968.

Translated by W. H. Furry