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OSCILLATORY APPROACH TO THE SINGULAR POINT IN RELATIVISTIC COSMOLOGY

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1. INTRODUCTION

THE question of the existence of time singularities in the general cosmological solution of the gravitation equations has been the subject of an earlier paper in this journal[1]. Before we return to this question, let us recall first the gist of the problem.

As is well known, modern cosmology is based on the solutions first obtained by A. A. Friedman for the Einstein equations. These solutions describe completely a homogeneous and isotropic world ("closed" or "open" model, depending on whether the space is closed or infinite).

The main property of these solutions is that they are not stationary. The resultant concept of the expanding universe is fully confirmed by astronomical data, and by now it can be regarded that the isotropic model gives, in general outline, an adequate description of the modern state of the universe.

Another important property of the isotropic model is the presence in it of a time singularity of the space-time metric. The presence of such a singular point denotes, in other words, that the time is finite.

But the adequacy of the isotropic model for the description of the modern state of the universe does in itself not give grounds for expecting it to be just as suitable for the description of the earlier stages of the evolution of the world. Moreover, there is a question of the degree to which the existence of a singular point in time is in general an obligatory property of relativistic cosmological models, and whether it is connected with the specific simplifying assumptions on which they are based.

Independence of these assumptions would denote that the presence of the singularity is inherent not only to particular solutions but also to the general solution of Einstein's equations. A criterion of the generality of the solution is the number of arbitrary functions of the spatial coordinates contained in it. We have in mind only "physically arbitrary" functions, the number of which cannot be decreased by any suitable choice of the reference frame. In the general solution, the number of such functions should be sufficient for an arbitrary specification of the initial conditions (the distribution and motion of matter, the distribution of the gravitational field) at some instant of time which is chosen to be as the starting point. This number is equal to four for empty space and to eight for a space filled with matter (see[1], Sec. 1 or[2], Sec. 95).

To avoid misunderstandings, we emphasize immediately that for a system of nonlinear differential equations, such as Einstein's equations, the concept of a general solution is not unique. In principle there can exist several general integrals, each of which includes not the entire manifold of conceivable initial conditions, but only a finite part of it. Each such integral contains the entire required aggregate of arbitrary functions which, however, can be subjected to definite conditions (such as inequalities). The existence of a general solution with a singularity therefore does not exclude the additional existence of other general solutions having no singularity*.

It is of course impossible to determine the general integral in exact form for all of space and for all of time, but this is not needed for the solution of the problem, for it suffices to investigate the form of the solution near the singularity. This would clarify also another aspect of the problem, namely the character of the evolution of the space-time metric in the general solution on approaching the singular point. We emphasize that when we speak of the singular point we have in mind a physical singularity, wherein the density of

*For example, there are no grounds for doubting the existence of a general singularity-free solution describing an isolated body with not too large a mass.

matter and the invariance of the four-dimensional curvature tensor become infinite. We are interested here in the question of the singularity in the cosmological aspect. This means that we are dealing with a singular point reached by all of space (and not by a limited part of space, as in gravitational collapse of a finite body).

We shall see that the question of the existence of a general solution with a physical singularity in time can be answered in the affirmative. In this connection, let us say a few words about the relation between these results and the earlier investigations (reported in[1]), in which it was concluded that the general solution has no singularity.

Since there is no method for investigating the singularities of the solutions of Einstein's equations, searches for broader classes of solutions with a singularity must be carried out, in essence, by trial and error. Obviously, a negative result obtained in this manner cannot be fully convincing in itself; it is rescinded by constructing a solution with the required degree of generality, and at the same time all the obtained positive results pertaining to the concrete solutions remain in force.

It is natural to think, however, that if a singularity can exist in the general solution of Einstein's equations, then clues for its existence should be present in the general properties of these equations themselves (although these clues may by themselves be insufficient to establish the character of the singularity). The only presently known clue is connected with the form of the equations (in the synchronous reference frame, i.e., in the frame in which the element of the interval is

$$ds^2 = dt^2 - dl^2, \quad dl^2 = \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1.1}$$

(the spatial length element dl is separated from the temporal interval dt, and $x^0 = t$ is the proper time synchronized over all of space)*. The equation $R_0^0 = T_0^0 - T/2$, written in this reference frame, should cause the metric determinant g to vanish within a finite time, regardless of any assumptions concerning the distribution of matter (see[1], Sec. 2 or[2], Sec. 99).

But this clue has dropped out as soon as its connection became known with the purely geometrical properties peculiar to the synchronous reference frame, namely the intersection of the coordinate time lines with one another. Intersection occurs, generally speaking, on certain enveloping hypersurfaces—the four-dimensional analogs of the caustic surfaces of geometrical optics; it is precisely here that g vanishes[3]. Thus, although the singularity has a general character, it turns out to be fictitious and not physical. It vanishes when the reference frame is changed. By the same token, there might seem to be no need for further searches for a true singularity in the general solution.

The situation changed, however, after Penrose[4] discovered a theorem that connects the existence of a singularity (of unknown character) with certin quite general assumptions having no bearing on the choice of the reference frame. Other theorems of similar type were discovered subsequently by Hawking[5,6] and by Geroch[7]. It became clear that searches of a general solution with a singularity should be continued.

*We use the notation of the book [2]. Latin indices run through values 0, 1, 2, and 3 and Greek indices through the three spatial values 1, 2, 3. The metric g_{ik} has a signature (+ − − −); $\gamma_{\alpha\beta} = -g_{\alpha\beta}$ is a spatial three-dimensional matrix tensor. In addition, we use a system of units in which the velocity of light and the Einstein gravitational constants are equal to unity.

2. KASNER'S GENERALIZED SOLUTION

Let us recall certain properties of the previously obtain singularity-containing solutions that serve as the starting point for further generalization.

The Friedmann solution itself is a particular case of a class of solutions containing three physical arbitrary functions of the coordinates (see[1], Sec. 4). Although the space in it is inhomogeneous, but its contraction on approaching the singular point occurs in a "quasi-isotropic" manner—the linear distances in all directions decrease with the same power of the time. Just as in the fully homogeneous and isotropic case, this class of solutions exists only for a space filled with matter.

A much more general character is possessed by a class of solutions obtained as a generalization of the exact particular solution (belonging to Kasner[8]) for a field in vacuum, in which the space is homogeneous, and its metric in Euclidean, but depends on the time in accordance with

$$dl^2 = t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 \tag{2.1}$$

(see[2], Sec. 103). Here p_1, p_2, and p_3 are three arbitrary numbers connected with one another by the relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \tag{2.2}$$

By virtue of these relations, only one of the three numbers is independent. The numbers p_1, p_2, and p_3 never have equal values, and equality of two of them takes place only in the triads $(-1/3, 2/3, 2/3)$ and $(0, 0, 1)$*. In all other cases these numbers are different, and one of them is negative and the other two positive. If we arrange them in the order

$$p_1 < p_2 < p_3, \tag{2.3}$$

then the intervals of their variation are

$$-1/3 \leqslant p_1 \leqslant 0, \quad 0 \leqslant p_2 \leqslant 2/3, \quad 2/3 \leqslant p_3 \leqslant 1. \tag{2.4}$$

The numbers p_1, p_2, and p_3 can be represented in parametric form:

$$p_1(u) = \frac{-u}{1+u+u^2}, \quad p_2(u) = \frac{1+u}{1+u+u^2}, \quad p_3(u) = \frac{u(1+u)}{1+u+u^2} \tag{2.5}$$

All the different values of p_1, p_2, and p_3, subject to the order given in (2.3), are obtained if the parameter u runs through values in the region $u \geq 1$. On the other hand, the values $u < 1$ can be reduced to the same region by using

$$p_1(1/u) = p_1(u), \quad p_2(1/u) = p_3(u), \quad p_3(1/u) = p_2(u). \tag{2.6}$$

*If $(p_1, p_2, p_3) = (0, 0, 1)$ the space-time metric (1.1) with dl^2 from (2.1) can be reduced to Galilean form by the transformation $t \sinh z = \zeta$ and $t \cosh z = \tau$, i.e., the singularity is fictitious and we are dealing actually with flat space-time.

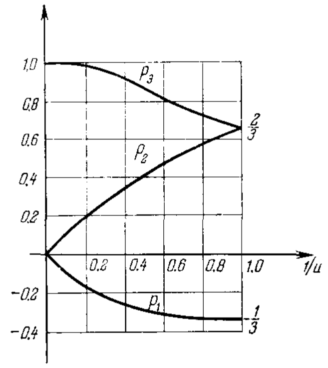

FIG. 1

Figure 1 shows plots of p_1, p_2, and p_3 as functions of 1/u. We note that $p_1(u)$ and $p_3(u)$ are monotonically increasing functions of the parameter u, and $p_2(u)$ decreases monotonically.

In the general solution, a form analogous to (2.1) pertains only to the limiting form of the metric (near the singular point t = 0), i.e., to the principal terms of its expansion in powers of t. In the synchronous reference frame, it is written in the form (1.1) with the spatial element of length

$$dl^2 = (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta)\, dx^\alpha\, dx^\beta, \tag{2.7}$$

where

$$a = t^{p_l}, \quad b = t^{p_m}, \quad c = t^{p_n}. \tag{2.8}$$

The three-dimensional vectors l, m, and n determine the directions along which the spatial distances change in time in accordance with the power laws (2.8). These vectors, and also the numbers p_l, p_m, and p_n (which as before are connected by relations (2.2)) are functions of the spatial coordinates. We shall denote here the exponents by p_l, p_m, and p_n, without stipulating their sequence in increasing order; on the other hand, we shall retain the symbols p_1, p_2, p_3 for the numbers in (2.5) satisfying the inequalities (2.3). The determinant of the metric (2.7) is equal to

$$-g = a^2 b^2 c^2 v^2 = t^2 v^2, \tag{2.9}$$

where $v = \mathbf{l}\cdot\mathbf{m}\times\mathbf{n}$. We also introduce a notation which will be found useful later on*

$$\lambda = \frac{\mathbf{l}\,\mathrm{rot}\,\mathbf{l}}{v}, \quad \mu = \frac{\mathbf{m}\,\mathrm{rot}\,\mathbf{m}}{v}, \quad \nu = \frac{\mathbf{n}\,\mathrm{rot}\,\mathbf{n}}{v}. \tag{2.10}$$

Since the exponents in (2.8) cannot be equal to one another, the spatial metric in (2.7) is anisotropic in principle. On approaching the singular point t = 0, the linear distances in each element of space decrease in two directions and increase in the third. The volume of each element, on the other hand, decreases in proportion to t.

Let us trace again (see[1], Sec. 3) how agreement between the metric (2.7) and the gravitational equations is obtained and what determines the number of physically arbitrary coordinate functions in it. Einstein's equations in vacuum in the synchronous system are

*Here and below, all the symbols for vector operations (vector products, the curl and gradient operations, etc.) should be understood in purely formal fashion, as operations on components (covariant) of the vectors l, m, and n–as if the coordinates x^1, x^2, and x^3 were Cartesian.

$$R^0_0 = -\tfrac{1}{2}\,\partial\varkappa^\alpha_\alpha/\partial t - \tfrac{1}{4}\varkappa^\beta_\alpha\varkappa^\alpha_\beta = 0, \tag{2.11}$$

$$R^\beta_\alpha = -(1/2\sqrt{-g})\,\frac{\partial}{\partial t}(\sqrt{-g}\,\varkappa^\beta_\alpha) - P^\beta_\alpha = 0, \tag{2.12}$$

$$R^0_\alpha = \tfrac{1}{2}(\varkappa^\beta_{\alpha;\beta} - \varkappa^\beta_{\beta;\alpha}) = 0, \tag{2.13}$$

where $\kappa_{\alpha\beta}$ denotes the three-dimensional tensor

$$\varkappa_{\alpha\beta} = \frac{\partial\gamma_{\alpha\beta}}{\partial t},$$

and $P_{\alpha\beta}$ is the three-dimensional Ricci tensor, expressed in terms of the three-dimensional metric tensor $\gamma_{\alpha\beta}$ in the same manner as R_{ik} is expressed in terms of g_{ik}; it contains only spatial (but not temporal) derivatives of $\gamma_{\alpha\beta}$.

Without defining the dependence of a, b, and c in (2.7) on t, we have (in place of (3.12) in[1])

$$\varkappa^\beta_\alpha = (2\dot a/a)\, l_\alpha l^\beta + (2\dot b/b)\, m_\alpha m^\beta + (2\dot c/c)\, n_\alpha n^\beta,$$

where the dot denotes differentiation with respect to t. Equation (2.11) takes the form

$$-R^0_0 = \ddot a/a + \ddot b/b + \ddot c/c = 0. \tag{2.14}$$

All the terms in it are of second order in the large (as $t \to 0$) quantity 1/t. Terms of the same order arise in Eqs. (2.12) only from terms with the time derivatives. If the components $P_{\alpha\beta}$ do not contain terms of still higher order, then

$$-R^l_l = (\dot a bc)^{\cdot}/abc = 0, \quad -R^m_m = (a\dot b c)^{\cdot}/abc = 0, \quad -R^n_n = (ab\dot c)^{\cdot}/abc = 0 \tag{2.15}$$

(the indices l, m, and n denote the components of the tensor in the directions of l, m, and n–see[1], Sec. 3). These equations, together with (2.14), lead to expressions (2.8) with exponents satisfying the conditions (2.2).

But the fact that one of the exponents p_l, p_m, or p_n is negative causes terms of higher order than t^{-2} to appear in the tensor $P_{\alpha\beta}$. If the negative exponent is p_l ($p_l = p_1 < 0$), then these terms contain the coordinate function λ and when they are taken into account the equations take the form

$$\begin{aligned} -R^l_l &= (\dot a bc)^{\cdot}/abc + \lambda^2 a^2/2b^2c^2 = 0,\\ -R^m_m &= (a\dot b c)^{\cdot}/abc - \lambda^2 a^2/2b^2c^2 = 0,\\ -R^n_n &= (ab\dot c)^{\cdot}/abc - \lambda^2 a^2/2b^2c^2 = 0. \end{aligned} \tag{2.16}$$

The second terms here are $\sim t^{-2(p_m+p_n-p_l)}$, with $p_m + p_n - p_l = 1 + 2|p_l| > 1$*. To eliminate these terms (and by the same token, in order for the solution (2.7) to be valid), it is necessary to impose on the coordinate functions the condition

$$\lambda = 0. \tag{2.17}$$

As to the three equations (2.13), containing only the first derivatives of the metric tensor with respect to time, they lead to three relations that do not contain the time; these should be imposed as the necessary condition on the coordinate functions in (2.7) (Eqs. (3.24) of[1]). We are left thus with only four conditions together with (2.17). These conditions relate ten different functions of the coordinates: three components each of the three vectors l, m, and n, and one function

*We exclude from consideration the case $(p_1, p_2, p_3) = (0, 0, 1)$, in which the singularity of the metric is fictitious.

in the exponents of t (one of the three functions p_l, p_m, and p_n, which are connected by relation (2.2)). In determining the number of physically arbitrary functions, it must be recognized also that the employed synchronous reference frame admits also of arbitrary transformations of the three spatial coordinates, without affecting the time. Therefore the considered solution contains only 10 − 4 − 3 = 3 physically arbitrary functions, or one less than necessary for the general solution in empty space.

The attained degree of generality does not decrease when matter is introduced: the matter is "written into" the metric (2.7) with all its ensuing four new coordinate functions needed for specifying the initial distribution of its density and the three components of its velocity. Namely, the evolution of the matter on approaching the singular point can be determined simply from the equations of its motion in a specified gravitational field. These equations are the hydrodynamic equations

$$(1/\sqrt{-g})\,\partial\,(\sqrt{-g}\,\sigma u^i)/\partial x^i = 0, \qquad (2.18)$$

$$(p+\varepsilon)\,u^k\left\{\frac{\partial u_i}{\partial x^k} - \frac{1}{2}u^l\,\frac{\partial g_{kl}}{\partial x^i}\right\} = -\frac{\partial p}{\partial x^i} - u_i u^k\,\frac{\partial p}{\partial x^k}, \qquad (2.19)$$

where u^i is the 4-velocity, and ϵ and σ are the energy and entropy densities of the matter (see, e.g.,[9], Sec. 125). For the ultrarelativistic equation of state $p = \epsilon/3$, the entropy is $\sigma \sim \epsilon^{3/4}$. The principal terms in (2.18) and (2.19) are those with the time derivatives. From (2.18) and the spatial components of (2.19) we get

$$\partial\,(\sqrt{-g}\,u_0\varepsilon^{3/4})/\partial t = 0, \quad 4\varepsilon\cdot\partial u_\alpha/\partial t + u_\alpha\cdot\partial\varepsilon/\partial t = 0,$$

whence

$$abcu_0\varepsilon^{3/4} = \text{const}, \quad u_\alpha\varepsilon^{1/4} = \text{const}, \qquad (2.20)$$

where the constants stand for quantities independent of the time. In addition, from the identity $u_i u^i = 1$ we have (recognizing that all the covariant components u_α are of the same order of magnitude)

$$u_0^2 \approx u_n u^n = u_n^2/c^2,$$

where u_n is the velocity component along the direction **n** connected with the largest (positive) degree of t (we assume that $p_n = p_3$). From the foregoing relations we get

$$\varepsilon \sim 1/a^2b^2, \quad u_\alpha \sim \sqrt{ab} \qquad (2.21)$$

or

$$\varepsilon \sim t^{-2(p_1+p_2)} = t^{-2(1-p_3)}, \quad u_\alpha \sim t^{(1-p_3)/2}. \qquad (2.22)$$

We can then easily verify that the components of the energy-momentum tensor of matter, in the right side of the equations

$$R_0^0 = T_0^0 - {}^1/_2 T, \quad R_\alpha^\beta = T_\alpha^\beta - {}^1/_2\delta_\alpha^\beta T,$$

are indeed of lower order in 1/t than the principal terms in their left sides. On the other hand, in the equations $R_\alpha^0 = T_\alpha^0$, the presence of matter leads only to a change of the relations imposed on the coordinate functions that enter in the solution (see[1], Sec. 3).

The fact that ϵ becomes infinite in accordance with (2.22) confirms that we are dealing in the solution (2.7) with a physical singularity at all values of the exponents (p_1, p_2, p_3), with the only exception of (0, 0, 1). For these last values, the singularity is not physical and can be eliminated by transforming the reference frame.

The fictitious singularity corresponding to the exponents (0, 0, 1) is the result of the intersection of the coordinate time lines on a certain two-dimensional "focal surface." As indicated in[1], Sec. 2, the synchronous reference frame can always be chosen such that the inevitable intersection of the time lines occurs on such a surface (in place of a three-dimensional caustic hypersurface). Therefore solutions with a such fictitious singularity that is simultaneous for all of space should exist with the complete set of arbitrary functions needed for the general solution. Near the point t = 0, it admits of a regular expansion in integer powers of t; it was analytically constructed in[10].

3. OSCILLATORY APPROACH TO THE SINGULAR POINT

Out of the four conditions that had to be imposed on the coordinate functions in the solution (2.7), three conditions, arising from the equations $R_\alpha^0 = 0$, are "natural"; they are the consequence of the very structure of the gravitational equations. The "loss" of one more derivative function is due to the imposition of the additional condition (2.17).

The general solution, by definition, is perfectly stable. The application of an arbitrary perturbation is equivalent to a change of the initial conditions at a certain instant of time, and since the general solution admits of arbitrary initial conditions, the perturbation cannot change its character. For the solution (2.7), on the other hand, the presence of the limiting condition $\lambda = 0$ denotes, in other words, instability with respect to perturbations that violate this condition. The application of such a perturbation should cause the model to go over to another regime, which by the same token will be already completely general. The perturbation, of course, need not be considered as being small—the transition to the new regime lies outside the region of arbitrarily small perturbations.

An investigation based on such an approach can indeed be carried out. It leads to a picture of a complex oscillatory approach to the singular point[11-13]. We still do not know all the details of this approach within the widest limits of the general case (see Secs. 7 and 8). Its main properties and its character, however, can be determined already from particular models that permit far-reaching analytic investigations.

We have in mind models with a homogeneous spatial metric of a definite type. As is well known, the assumption that space is homogeneous, without any additional symmetry, still leaves a considerable freedom in the metric. All the possible homogeneous (but anisotropic) spaces are customarily classified, following Bianchi, in nine types (see Appendix C). We are interested here in spaces of type VIII and IX.

If the spatial metric is represented in the form (2.7), then each of the types of the homogeneous spaces corresponds to a definite functional dependence of the reference vectors **l**, **m**, and **n** on the spatial coordinates. The concrete form of this dependence is immaterial here. All that matters is that for a space of type

VIII or IX the quantities λ, μ, and ν (2.10) reduce to constants, and all the "mixed" products of the type l curl m, l curl n, m curl l, etc. vanish. For a space of type IX, the quantities λ, μ, and ν have the same sign and we can put $\lambda = \mu = \nu = 1$ (simultaneous reversal of sign of all three constants changes nothing). For a space of type VIII, two constants have signs opposite to that of the third; we can put, for example, $\lambda = -1$ and $\mu = \nu = 1$*.

Our purpose is to determine the influence exerted on the "Kasner regime" by the perturbation represented in Einstein's equations by the terms containing λ. It was in this respect that models with spaces of type VIII and IX are suitable objects. Since all three quantities λ, μ, and ν differ from zero, the conditions (2.17) is certainly not satisfied, regardless of the direction l, m, or n to which the negative degree of the time pertains.

Einstein's equations for the models in question can be easily written down with the aid of the formulas given in[1] (Appendix C). They are of the form

$$\left.\begin{aligned} -R_l^l &= (\dot{a}bc)^{\cdot}/abc + (1/2a^2b^2c^2)\,[\lambda^2a^4 - (\mu b^2 - \nu c^2)^2] = 0,\\ -R_m^m &= (a\dot{b}c)^{\cdot}/abc + (1/2a^2b^2c^2)\,[\mu^2b^4 - (\lambda a^2 - \nu c^2)^2] = 0,\\ -R_n^n &= (ab\dot{c})^{\cdot}/abc + (1/2a^2b^2c^2)\,[\nu^2c^4 - (\lambda a^2 - \mu b^2)^2] = 0, \end{aligned}\right\} \quad (3.1)$$

$$-R_0^0 = \ddot{a}/a + \ddot{b}/b + \ddot{c}/c = 0 \quad (3.2)$$

(the remaining components R_l^0, R_m^0, R_n^0, R_l^m, R_l^n, and R_m^n vanish identically). We note that the equations contain only functions of the time; this is a manifestation of the homogeneity of space. We emphasize also that in this case Eq. (3.1) and (3.2) are exact equations, the validity of which is not connected with the closeness to the singular point $t = 0$†.

The time derivatives in (3.1) and (3.2) become simpler if we introduce in lieu of a, b, and c their logarithms α, β, and γ:

$$a = e^{\alpha}, \quad b = e^{\beta}, \quad c = e^{\gamma}, \quad (3.3)$$

and the variable t is replaced by τ in accordance with

$$dt = abc\, d\tau. \quad (3.4)$$

Then

$$\left.\begin{aligned} 2\alpha_{\tau\tau} &= (\mu b^2 - \nu c^2)^2 - \lambda^2 a^4,\\ 2\beta_{\tau\tau} &= (\lambda a^2 - \nu c^2)^2 - \mu^2 b^4,\\ 2\gamma_{\tau\tau} &= (\lambda a^2 - \mu b^2)^2 - \nu^2 c^4, \end{aligned}\right\} \quad (3.5)$$

$$\tfrac{1}{2}(\alpha + \beta + \gamma)_{\tau\tau} = \alpha_\tau\beta_\tau + \alpha_\tau\gamma_\tau + \beta_\tau\gamma_\tau. \quad (3.6)$$

Adding Eqs. (3.5) term by term and replacing in the left side the sum $(\alpha + \beta + \gamma)_{\tau\tau}$ in accordance with (3.6), we obtain an equation containing only first derivatives and constituting the first integral of the system (3.5):

$$\alpha_\tau\beta_\tau + \alpha_\tau\gamma_\tau + \beta_\tau\gamma_\tau = \tfrac{1}{4}(\lambda^2a^4 + \mu^2b^4 + \nu^2c^4 - 2\lambda\mu a^2b^2 - 2\lambda\nu a^2c^2 - 2\mu\nu b^2c^2). \quad (3.7)$$

This equation plays the roles of a constraint imposed on the initial conditions of (3.5).

The Kasner regime (2.8) is a solution of Eqs. (3.5) when it is possible to neglect in them all terms in the right-hand sides. But such a situation cannot continue (as $t \to 0$) without limit, since some of these terms always increase. Thus, if the negative exponent pertains to the function a(t) ($p_l = p_1$), then the perturbation of the Kasner regime is due to the terms λ^2a^4; The remaining terms will decrease with decreasing t.

Retaining in the right sides of (3.5) only these terms, we obtain the system

$$\alpha_{\tau\tau} = -\tfrac{1}{2}\lambda^2 e^{4\alpha}, \quad \beta_{\tau\tau} = \gamma_{\tau\tau} = \tfrac{1}{2}\lambda^2 e^{4\alpha} \quad (3.8)$$

(see (2.16); we put henceforth $\lambda^2 = 1$). The solution of these equations should describe the evolution of the metric from the initial state, in which it is described by formulas (2.8) with a definite set of exponents (with $p_l < 0$). Let $p_l = p_1$, $p_m = p_2$, and $p_n = p_3$, so that

$$a \sim t^{p_1}, \quad b \sim t^{p_2}, \quad c \sim t^{p_3}. \quad (3.9)$$

Here

$$abc = \Lambda t, \quad \tau = \Lambda^{-1}\ln t + \text{const}, \quad (3.10)$$

where Λ is a constant. Therefore the initial conditions for Eqs. (3.8) are formulated in the form*

$$\alpha_\tau = \Lambda p_1, \quad \beta_\tau = \Lambda p_2, \quad \gamma_\tau = \Lambda p_3 \quad \text{as} \quad \tau \to \infty. \quad (3.11)$$

Equations (3.8) can be readily integrated; its solution satisfying the condition (3.11) is

$$\left.\begin{aligned} a^2 = \frac{2|p_1|\Lambda}{\operatorname{ch}(2|p_1|\Lambda\tau)}, \quad b^2 = b_0^2 e^{2\Lambda(p_2 - |p_1|)\tau}\operatorname{ch}(2|p_1|\Lambda\tau),\\ c^2 = c_0^2 e^{2\Lambda(p_3 - |p_1|)\tau}\operatorname{ch}(2|p_1|\Lambda\tau), \end{aligned}\right\} \quad (3.12)$$

where b_0 and c_0 are two additional constants.

It is easy to verify that the asymptotic form of the function (3.12) as $\tau \to \infty$ indeed coincides with (3.9). The asymptotic expressions for these functions and the function $t(\tau)$ as $\tau \to -\infty$ is†

$$a \sim e^{-\Lambda p_1\tau}, \quad b \sim e^{\Lambda(p_2 + 2p_1)\tau}, \quad c \sim e^{\Lambda(p_3 + 2p_1)\tau}, t \sim e^{\Lambda(1 + 2p_1)\tau}.$$

Expressing a, b, and c as functions of t, we obtain

$$a \sim t^{p'_l}, \quad b \sim t^{p'_m}, \quad c \sim t^{p'_n}, \quad (3.13)$$

where

$$p'_l = \frac{|p_1|}{1 - 2|p_1|}, \quad p'_m = -\frac{2|p_1| - p_2}{1 - 2|p_1|}, \quad p'_n = \frac{p_3 - 2|p_1|}{1 - 2|p_1|}. \quad (3.14)$$

Here

$$abc = \Lambda' t, \quad \Lambda' = (1 - 2|p_1|)\Lambda. \quad (3.15)$$

---

*The constants λ, μ, and ν are the so-called structural constants of the group of motions of space (see (C.15)).

†Einstein's equations for homogeneous space in their exact form contain, generally speaking, six different functions of the time–the functions $\gamma_{ab}(t)$ in the metric (C.2). The fact that we have obtained in this case a non-contradictory system of exact equations for a metric defined only by three functions of the time ($\gamma_{11} = a^2$, $\gamma_{22} = b^2$, and $\gamma_{33} = c^2$) is connected with the symmetry that leads to the aforementioned identical vanishing of six components of the Ricci tensor.

*We recall once more that we are considering the evolution of a model in the direction $t \to 0$; therefore the "initial" conditions correspond to a later and not to an earlier time.

†We note that the asymptotic values of α_τ, β_τ, and γ_τ and $\tau \to -\infty$ can be obtained also without completely solving the equations (3.8). It suffices to note that the first of these equations has the form of uniform motion of a "particle" in the field of an exponential potential well, and α plays the role of the coordinate. In this analogy, the initial Kasner regime corresponds to free motion with constant velocity $\alpha_\tau = \Lambda p_1$. After being reflected from the wall, the particle moves freely with velocity $\alpha_\tau = -\Lambda p_1$. Noting also that $\alpha_\tau + \beta_\tau = \text{const}$ and $\alpha_\tau + \gamma_\tau = \text{const}$ by virtue of (3.8), we find that β_τ and γ_τ take on the values $\beta_\tau = \Lambda(p_2 + 2p_1)$ and $\gamma_\tau = \Lambda(p_3 + 2p_1)$.

Thus, the action of the perturbation leads to replacement of one Kasner regime by another, and the negative exponent of t is transferred from the direction l to the direction m: if previously $p_l < 0$, we have now $p'_m < 0$. During the course of the change, the function a(t) goes through a maximum, while b(t) goes through a minimum; the previously decreasing quantity b begins to increase, and the increasing quantity a begins to decrease, while the function c(t) continues to decrease. The perturbation itself ($\lambda^2 a^{4\alpha}$ in (3.8)), which was previously increasing, begins to decrease and attenuates. Further evolution leads in analogous fashion to an increase of the perturbation expressed by the terms with μ^2 (in place of λ^2) in Eqs. (3.5), to another alternation of the Kasner regime, etc.

The rule for the interchange of the exponents (3.14) is conveniently represented with the aid of the parametrization (2.5):

$$\left.\begin{array}{llll}\text{if} & p_l = p_1(u), & p_m = p_2(u), & p_n = p_3(u), \\ \text{then} & p'_l = p_2(u-1), & p'_m = p_1(u-1), & p'_n = p_3(u-1).\end{array}\right\} \quad (3.16)$$

The larger of the two positive exponents remains positive.

This alternation of the "Kasner epochs" with the switching ("bouncing") of the exponents p_l, p_m, and p_n in accordance with the rule (3.16) contains the key to the understanding of the character of the evolution of the metric as the singular point is approached.

The successive alternations (3.16) and the switching of the negative exponent (p_1) between the directions l and m continues until the integer part of the initial value of u is exhausted and u becomes less than one. The value $u > 1$ is transformed into $u < 1$ in accordance with (2.6). At this instant, the exponent p_l or p_m is negative, and p_n becomes the smaller of the two positive numbers ($p_n = p_2$). The next series of alternations will already switch the negative exponents between the directions n and l or between n and m. For an arbitrary (irrational) initial value of u, the alternation continues without limit*.

In an exact solution of the equations, the exponents p_l, p_m, and p_n lose, of course, their literal meaning. We note that a certain "fuzziness" in the definition of these numbers (and consequently also of the parameter u), introduced by this circumstance, while small, makes the consideration of any preferred (say, rational) values of u meaningless. This is precisely why a real meaning is possessed only by the regularities inherent in the general case of arbitrary irrational values of u.

Thus, the evolution of the model towards the singular point consists of successive periods (we shall call them eras), during which the spatial scales oscillate along two axes and decrease monotonically along the third; the volumes decrease approximately in proportion to t. On going from one era to the next, the direction along which the monotonic decrease of the distances occurs switches over from one axis to the other. The sequence of this switching acquires asymptotically the character of a random process. The same character is acquired also by the sequence of alternations of the lengths of the successive eras (by length of an era, to distinguish it from the time duration, we mean the number of Kasner epochs that alternate in it).

The successive eras condense on approaching t = 0. But the natural variable for the description of the time variation of this evolution is not the world time t itself, but its logarithm ln t, with respect to which the entire process of approaching the singular point stretches out to $-\infty$.

According to formulas (3.12), the particular function (a, b, or c) that passes through a maximum during the alternation of the Kasner epochs, has a value at the maximum

$$a_{\max} = \sqrt{2\Lambda |p_1(u)|} \quad (3.17)$$

(it is assumed here that this value is large compared with b_0 and c_0); in (3.17), u is the value of the parameter corresponding to the epoch preceding the alternation. From this it is easy to conclude that the heights of the successive maxima decrease gradually during the course of each era. Indeed, in the next Kasner epoch the parameter has a value $u' = u - 1$, and the constant Λ is replaced, according to (3.15) by $\Lambda' = \Lambda(1 - 2|p_1(u)|)$. Therefore the ratio of the heights of successive maxima is

$$\frac{a'_{\max}}{a_{\max}} = \left[\frac{p_1(u-1)}{p_1(u)}(1 - 2|p_1(u)|)\right]^{1/2};$$

and finally

$$\frac{a'_{\max}}{a_{\max}} = \sqrt{\frac{u-1}{u}} = \sqrt{\frac{u'}{u}}. \quad (3.18)$$

So far we have considered the solution of Einstein's equations in empty space. Just as for a pure Kasner regime, matter does not change the qualitative properties of this solution and can be "written into" it if its reaction in the field is neglected.

However, if this were done for the model in question, taken to mean the exact solution of Einstein's equations, then the resultant picture of the evolution of the matter would have no general character at all, and would be specific precisely for the high symmetry possessed by this model. Mathematically this specific feature is connected with the fact that the components R^0_α of the Ricci tensor vanish identically for the considered homogeneous spatial geometry, and therefore the gravitational equations would not admit of motion of matter (which leads to non-zero components T^0_α of the energy-momentum tensor)*.

This difficulty disappears if we consider the model as consisting only of the principal term of the limiting form (as $t \to 0$) of the metric and "write in" the matter with an arbitrary initial distribution of the density and velocity. Then the evolution of the matter is determined by its general equations of motion (2.18)

*Note added in proof. The introduction of the nondiagonal components $\gamma_{ab}(t)$ into the metric (see the footnote on p. 749) leads to certain new properties of the model, namely rotation of the axes to which the exponents of the Kasner epochs belong; this question is investigated in an article by the authors in Zh. Eksp. Teor. Fiz. 60, No. 3 (1971) [Sov. Phys.-JETP 33, No. 3 (1971)].

*In others words, the synchronous system should also be co-moving with respect to the matter. By putting in (2.20) $u^\alpha = 0$ and $u^0 = 1$, we would obtain $\epsilon \sim (abc)^{4/3} \sim t^{-4/3}$.

and (2.19), which lead to formulas (2.22). During each Kasner epoch, the density increases like

$$\varepsilon \sim t^{-2(1-p_3)}, \tag{3.19}$$

where p_3, by agreement, is the largest of the numbers p_1, p_2, and p_3. The density of matter increases monotonically during the entire evolution towards the singular point.

The foregoing analysis must be supplemented because of the following circumstance:

Each (s-th) era corresponds to a series of values of the parameter u, starting with a certain maximum value $u^{(S)}_{max}$, and reaching a minimum $u^{(S)}_{min} < 1$ via the values $u^{(S)}_{max} - 1$, $u^{(S)}_{max} - 2, \ldots$ We put

$$u^{(s)}_{min} = x^{(s)}, \quad u^{(s)}_{max} = k^{(s)} + x^{(s)}, \tag{3.20}$$

i.e., $k^{(S)} = [u^{(S)}_{max}]$ (the square brackets denote the integer part of the number). The number $k^{(S)}$ determines the length of the era, measured in terms of the number of the Kasner epochs it contains. For the next era we have

$$u^{(s+1)}_{max} = 1/x^{(s)}, \quad k^{(s+1)} = [1/x^{(s)}]. \tag{3.21}$$

In an unlimited sequence of series of numbers u, made up in accordance with these rules, there will be observed arbitrarily small (but never vanishing) values of $x^{(S)}$, and accordingly arbitrarily large lengths $k^{(S+1)}$.

Large values of the parameter u correspond to Kasner exponents

$$p_1 \approx -1/u, \quad p_2 \approx 1/u, \quad p_3 = 1 - 1/u^2, \tag{3.22}$$

close to the values (0, 0 1). Two values close to zero are by the same token close to each other, and consequently the laws of variation of two of the three types of "perturbations"—the terms on the right sides of (3.5) (the terms with λ, μ, and ν)—are close. If at the start of such a long era the absolute magnitudes of these terms are also close to each other at the instant of alternation of two Kasner epochs (or if this is so stipulated in the initial conditions), then they will continue to be close to each other during the greater part of the entire duration of the era. In such a case (which we shall call the case of small oscillations), the investigation based on a consideration of the action of a perturbation of only one type is no longer valid. An analysis of the evolution of the metric then requires simultaneous account of two "perturbations"; this is done in Sec. 4.

4. EVOLUTION OF MODEL UNDER THE INFLUENCE OF TWO PERTURBATIONS

Thus, we consider a long era, during which two out of the three functions a, b, c (assume that these are a and b) experience small oscillations, and the third (c) decreases monotonically. The latter becomes small rapidly; let us consider the solution of the equations precisely in this region, where c can already be neglected in comparison with a and b. We first perform this calculations for a model of type IX, and accordingly we put $\lambda = \mu = \nu = 1$.[12].

After neglecting the function c, the first two equations of (3.4) yield

$$\alpha_{\tau\tau} + \beta_{\tau\tau} = 0, \tag{4.1}$$

$$\alpha_{\tau\tau} - \beta_{\tau\tau} = e^{4\beta} - e^{4\alpha}, \tag{4.2}$$

and as a third equation we use (3.7), which takes the form

$$\gamma_\tau (\alpha_\tau + \beta_\tau) = -\alpha_\tau \beta_\tau + {}^1/_4 (e^{2\alpha} - e^{2\beta})^2. \tag{4.3}$$

We write the solution of (4.1) in the form

$$\alpha + \beta = (2a_0^2/\xi_0)(\tau - \tau_0) + 2 \ln a_0,$$

where a_0 and ξ_0 are positive constants, and τ_0 stands for the upper limit of the era with respect to the variable τ. We shall find it convenient to introduce a new variable (in place of τ)

$$\xi = \xi_0 \exp \{(2a_0^2/\xi_0)(\tau - \tau_0)\}. \tag{4.4}$$

Then

$$\alpha + \beta = \ln (\xi/\xi_0) + 2 \ln a_0. \tag{4.5}$$

We also transform (4.2) and (4.3), introducing the symbol $\chi = \alpha - \beta$:

$$\chi_{\xi\xi} + \chi_\xi/\xi + {}^1/_2 \operatorname{sh} 2\chi = 0, \tag{4.6}$$

$$\gamma_\xi = -1/4'\xi + {}^1/_8 \xi (2\chi_\xi^2 + \operatorname{ch} 2\chi - 1). \tag{4.7}$$

A decrease of τ from τ_0 to $-\infty$ corresponds to a decrease of ξ from ξ_0 to zero. The long era of interest to us, with close values of a and b (i.e., with small χ) is obtained if ξ_0 is a very large quantity. Indeed, at large ξ the solution of (4.6) in the first approximation (in $1/\xi$) is

$$\chi = \alpha - \beta = (2A/\sqrt{\xi}) \sin (\xi - \xi_0), \tag{4.8}$$

where A is a constant; the factor $1/\sqrt{\xi}$ makes χ a small quantity (and consequently we can make in (4.6) the substitution $\sinh 2\chi \approx 2\chi$)*.

From (4.7) we now obtain

$$\gamma_\xi = {}^1/_4 \xi (\chi_\xi^2 + \chi^2) = A^2, \quad \gamma = A^2 (\xi - \xi_0) + \text{const}.$$

After determining α and β from (4.5) and (4.8) and expanding e^α and e^β in accordance with the assumed approximation, we obtain finally**:

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = a_0 \sqrt{\frac{\xi}{\xi_0}} \left[1 \pm \frac{A}{\sqrt{\xi}} \sin (\xi - \xi_0)\right], \tag{4.9}$$

$$c = c_0 e^{-A^2(\xi_0 - \xi)}. \tag{4.10}$$

On the other hand, the connection between the variable ξ and the time t is obtained by integrating the definition dt = abc dτ, and is given by the formula

$$t/t_0 = e^{-A^2(\xi_0 - \xi)}. \tag{4.11}$$

The value of the constant c at $\xi = \xi_0$ should already be $c_0 \ll a_0$.

We turn to the region $\xi \ll 1$. The principal terms in the solution of (4.6) are

$$\chi = \alpha - \beta = k \ln \xi + \text{const},$$

where k is a constant lying in the interval $-1 < k < 1$; this condition ensures smallness of the last term in

*The constant in the argument of the sine function need not necessarily coincide with the constant ξ_0 in (4.4)–(4.5); by setting them equal, however, we do not change the character of the solution at all.

*In a more exact calculation, a slowly varying logarithmic term appears in the argument of the sine function, and a pre-exponential factor appears in the expression for $c(\xi)$ (see Appendix B).

(4.6) (sinh 2χ contains ξ^{2k} and ξ^{-2k}). After determining α, β, and t, we obtain

$$a \sim \xi^{(1+k)/2}, \quad b \sim \xi^{(1-k)/2}, \quad c \sim \xi^{-(1-k^2)/4}, \quad t \sim \xi^{(3+k^2)/4}. \tag{4.12}$$

This is again the Kasner regime, with the negative power of t pertaining to the function c(t)***.

The results again lead to qualitatively the same picture of evolution of the model as described in Sec. 3.

We see that during a long time (corresponding to a large decrease in value of ξ), two of the functions (a and b) oscillate and remain close to each other in magnitude $((a - b)/a \sim 1/\sqrt{\xi})$; at the same time, the mean values of the functions a and b decrease slowly $(\sim \sqrt{\xi})$. The oscillations occur with a constant period with respect to the variable ξ: $\Delta\xi = 2\pi$ (or, equivalently with a constant period in the logarithmic time: $\Delta \ln t = 2\pi A^2$). The third function, on the other hand, decreases monotonically approximately like $c = c_0 t/t_0$.

Such an evolution continues until we get $\xi \sim 1$ and formulas (4.9) and (4.10) are no longer valid. Its time duration corresponds to a change of t from t_0 to a value t_1 connected with ξ_0 by

$$A^2\xi_0 = \ln (t_0/t_1). \tag{4.13}$$

On the other hand, the connection between ξ and t during this entire time can be represented in the form

$$\xi/\xi_0 = \ln (t/t_1)/\ln (t_0/t_1). \tag{4.14}$$

Then, as can be seen from (4.12), the decreasing function begins to increase, and the functions a and b begin to decrease. This Kasner epoch will continue until the terms c^2/a^2b^2 in (3.1) become $\sim t^{-2}$ and the next series of oscillations begins.

The laws governing the variation of the density of matter during the considered long era is obtained by substituting (4.9) in (2.21):

$$\varepsilon \sim (\xi_0/\xi)^2. \tag{4.15}$$

The density increases by a factor ξ_0^2 during the time that ξ changes from ξ_0 to $\xi \sim 1$.

We emphasize that although the function c(t) varies approximately in proportion to t, the metric (4.9) does not coincide in any way with the Kasner metric with exponents (0, 0, 1). The latter corresponds in this case to the exact solution obtained by Taub[15]; in this solution, which is consistent with Eqs. (3.5)–(3.6),

$$a^2 = b^2 = \frac{p}{2}\frac{\operatorname{ch}(2p\tau+\delta_1)}{\operatorname{ch}^2(p\tau+\delta_2)}, \quad c^2 = \frac{2p}{\operatorname{ch}(2p\tau+\delta_1)}, \tag{4.16}$$

where p, δ_1, and δ_2 are constant. In the limit as $\tau \to \infty$, the substitution $e^{p\tau} = t$ yields a = b = const and c = const·t. In this metric, the singularity at t = 0 is not physical.

Let us turn to the analogous investigation for the model of type VIII, putting now $\lambda = -1$ and $\mu = \nu = 1$ in (3.5)–(3.7)[13].

If the function a decreases monotonically during the long era, then nothing is changed in the investigation presented above: after neglecting a^2 in the right side of (3.5) and (3.7), we return to the same equations (4.6)–(4.7) (with suitable change of notation). Some changes occur, however, if the function b or c is the monotonically decreasing one; assume that this is c.

Using the same notation, we get as before Eq. (4.6), and accordingly the previous expressions (4.9) for the functions $a(\xi)$ and $b(\xi)$. On the other hand, (4.7) is replaced by

$$\gamma_\xi = -1/4\xi + {}^1/_8\xi\,(2\chi_\xi^2 + \operatorname{ch} 2\chi + 1). \tag{4.17}$$

The principal term at large ξ is now

$$\gamma_\xi \approx {}^1/_8\xi \cdot 2, \quad \gamma \approx {}^1/_8\,(\xi^2 - \xi_0^2),$$

so that

$$c/c_0 = t/t_0 = \exp\{-{}^1/_8\,(\xi_0^2 - \xi^2)\}. \tag{4.18}$$

As before, the dependence of c on t is given by $c = c_0 t/t_0$, but the connection between the variable ξ and the time is altered. The duration of the long era will be connected with ξ_0 by

$$\xi_0 = \sqrt{8 \ln (t_0/t_1)}. \tag{4.19}$$

On the other hand, ξ_0 determines the number (equal to $\xi_0/2\pi$) of oscillations of the functions a and b during the era. At a given duration of the era in terms of the logarithmic time (i.e., at a given ratio t_0/t_1), the number of oscillations for the model of type VIII will, generally speaking, be smaller than for the model of type IX. We now obtain for the period of the oscillations $\Delta \ln t = \pi\xi/2$; in contrast to the case of the model of type IX, the period does not remain constant now during the long era, but decreases slowly together with ξ.

5. EVOLUTION OF MODEL IN THE ASYMPTOTIC REGION OF ARBITRARILY SHORT TIMES

The long eras of the type investigated in Sec. 4 violate the "regular" course of the evolution, determined by the rules established in Sec. 3; this makes it difficult to investigate the evolution in time intervals spanning several eras. It can be shown, however, that such "anomalous" cases cease to appear in the as the model evolves spontaneously towards the singular point in the asymptotic region of arbitrarily short times t, sufficiently far from the initial instant at which the arbitrary initial conditions are specified. Even in long eras, both oscillating functions remain so different during the instants of alternation of the Kasner epochs, that the alternations themselves occur under the influence of only one perturbation. The present and succeeding sections are devoted to an analytic and statistical analysis of the evolution of homogeneous models in such an asymptotic region[14]. All the results pertain equally well to models of type VIII and IX.

During each Kasner epoch we have abc = Λt, i.e., $\alpha + \beta + \gamma = \ln \Lambda + \ln t$. On going from one epoch to another, the constant $\ln \Lambda$ changes by an amount of the order of unity (see (3.15)). In the asymptotic region of arbitrarily large values of $|\ln t|$, however, we can neglect not only these changes, but the entire constant $\ln \Lambda$. In other words, the employed approximation cor-

*By replacing sinh 2χ by 2χ in (4.6) and by solving this equation for all ξ, we obtain $\chi = c_1 J_0(\xi) + c_2 N_0(\xi)$, where J_0 and N_0 are Bessel functions of the first and second kind. This solution effects the interpolation between the limiting cases and makes it possible to connect, in order of magnitude, the constant parameters in (4.9) and (4.12).

responds to neglecting all the quantities the ratio of which to $|\ln t|$ tends to zero as $t \to 0$. We then have

$$\alpha+\beta+\gamma=-\Omega, \tag{5.1}$$

where Ω denotes the "logarithmic time"

$$\Omega=-\ln t. \tag{5.2}$$

In the same approximation, we can regard the alternations of the epochs as instantaneous. We can also neglect the constant in the right side of condition (3.17), $\alpha_{\max} = (1/2)\ln(2|p_1|\Lambda)$, which determines the instants of alternation, i.e., we choose this condition in the form $\alpha = 0$ (or similar equations for β and γ, if the initial negative exponent pertains to the functions b or c)*. We thus put

$$\alpha_{\max}=0, \quad \beta_{\max}=0, \quad \gamma_{\max}=0, \tag{5.3}$$

so that the quantities α, β, and γ run through only negative values connected with one another at each instant of time by the relation (5.2).

By regarding the alternations of the epochs as instantaneous, we neglect the widths of the transition regions compared with the lengths of the epochs themselves; this condition is indeed satisfied (see the footnote on p. 000 below). On the other hand, the replacement of (3.17) by (5.3) requires that $\ln(|p_1|\Lambda)$ be small compared with the amplitudes of the oscillations of the corresponding functions α, β, and γ. But on going from one era to the next there can appear, as noted in Sec. 3, very small values of $|p_1|$, and neither these values nor the probability of their occurrence are connected in any way with the oscillation amplitude attained by that instant of time. One cannot exclude therefore, in principle, the appearance of small values of $|p_1|$ such that the required condition is violated. Such a strong decrease of $\alpha_{\max}$ can lead to different specific situations, in which the joining together of the Kasner epochs by means of the rule (3.16) becomes incorrect (including also the situation investigated in Sec. 4). This question was investigated also in[22]. These "dangerous" situations would violate the rules employed below for the statistical analysis in Sec. 6. However, as already mentioned, the probability of such violations tends asymptotically to zero; we shall return to this question at the end of Sec. 6.

Let us consider an era that contains k Kasner epochs, corresponding to a parameter u running through the values

$$u_n=k+x-1-n, \quad n=0,\ 1,\ \ldots,\ k-1, \tag{5.4}$$

and let the functions oscillating during that era be α and β (Fig. 2)†.

We denote the instants of the start of the Kasner epochs with parameters u_n by Ω_n. At each of these instances, one of the quantities α or β, is equal to zero, and the other has a minimum. The values of α or β at the successive minima, i.e., at the instants of Ω_n, will be denoted by

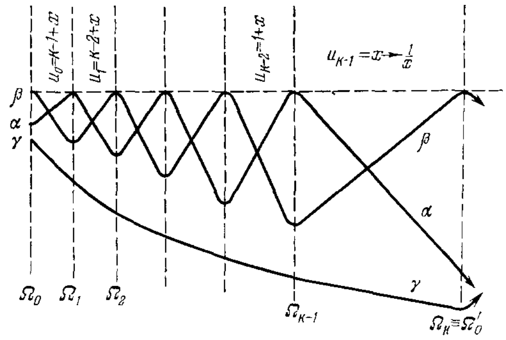

FIG. 2

$$\alpha_n=-\delta_n\Omega_n \tag{5.5}$$

(without distinguishing between the minima of α and β). The quantities δ_n, which measure these minima in units of the corresponding Ω_n, can have values between 0 and 1. The function γ, on the other hand, decreases monotonically during the given era; according to (5.1), its value at the instant Ω_n is

$$\gamma_n=-\Omega_n(1-\delta_n). \tag{5.6}$$

During the epoch that begins at the instant Ω_n and ends at the instant Ω_{n+1}, one of the functions, α or β, increases from $-\delta_n\Omega_n$ to zero, and the other decreases from zero to $-\delta_{n+1}\Omega_{n+1}$, in accordance with the respective linear laws

$$\text{const}+|p_1(u_n)|\Omega \text{ and } \text{const}-p_2(u_n)\Omega.$$

From this we obtain the recurrence relation

$$\delta_{n+1}\Omega_{n+1}=[(1+u_n)/u_n]\,\delta_n\Omega_n=[(1+u_0)/u_n]\,\delta_0\Omega_0 \tag{5.7}$$

and for the logarithmic duration of the epoch we get

$$\Delta_{n+1}\equiv\Omega_{n+1}-\Omega_n=\frac{f(u_n)}{u_n}\,\delta_n\Omega_n=\frac{f(u_n)\,(1+u_{n-1})}{f(u_{n-1})\,u_n}\,\Delta_n, \tag{5.8}$$

where we put for brevity $f(u) = 1 + u + u^2$. For the total duration n of the epochs we can obtain the formula

$$\Omega_n-\Omega_0=\left[n(n-1)+\frac{nf(u_{n-1})}{u_{n-1}}\right]\delta_0\Omega_0. \tag{5.9}$$

We see from (5.7) that $|\alpha_{n+1}| > |\alpha_n|$, i.e., the swing of the oscillations of the functions α and β increases during the entire era, whereas the coefficient δ_n can also be small. If the depth of the minimum at the start of the era was large, then it will no longer become small in the succeeding minima; in other words, the difference $|\alpha - \beta|$ remains large at the instants of alternation of the Kasner epochs. We emphasize that this statement does not depend on the length k of the era, since the alternations of the epochs will be determined by the usual rule (3.16) also for long eras.

The amplitude of the last oscillation of the function α or β in a given era is connected with the amplitude of the first oscillation by the relation $|\alpha_{k-1}| = |\alpha_0|(k + x)/(1 + x)$. Already at lengths k amounting to only several units we can neglect x compared with k, so that the increase of the amplitude of the oscillations of the function α or β is proportional to the

*By the same token, we neglect the effect of gradual lowering of the maxima of the oscillating functions during the era, described by formula (3.18).

†The definition of the limits of the era in accordance with (5.4) is natural in the sense that it unifies all the epochs during which the third function, $\gamma(t)$, decreases monotonically. If the era is defined in accordance with the sequence of values of u from k + x to 1 + x, then the monotonic decrease of $\gamma(t)$ would continue also during the first epoch of the next era.

length of the era. For the functions $a = e^{\alpha}$ and $b = e^{\beta}$ this means that if the amplitude of their oscillations at the beginning of the era was A_0, then at the end of the era it will be $A_0^{k/(1+x)}$.

During the course of the era, an increase takes place also in the duration (in logarithmic time) of the successive Kasner epochs; from (5.8) it is easy to conclude that $\Delta_{n+1} > \Delta_n$.* The total duration of the era is

$$\Omega_0' - \Omega_0 \equiv \Omega_k - \Omega_0 = k\left(k + x + \frac{1}{x}\right)\delta_0\Omega_0 \qquad (5.10)$$

(the term with $1/x$ is due to the last, k-th epoch, the duration of which is large at small x; see Fig. 2). The instant Ω_k of termination of the k-th epoch of the given era is at the same time the instant Ω_0' of the start of the next era.

In the first Kasner epoch of the new era, the first to increase is the function γ, from the minimum value $\gamma_k = -\Omega_k(1-\delta_k)$ reached by it in the preceding era; this value will play the role of the initial amplitude $\delta_0'\Omega_0'$ of the new series of oscillations. For it we readily obtain

$$\delta_0'\Omega_0' = (\delta_0^{-1} + k^2 + kx - 1)\,\delta_0\Omega_0. \qquad (5.11)$$

Obviously, $\delta_0'\Omega_0' > \delta_0\Omega_0$. Even at not very large lengths k, the increase of the amplitude is quite appreciable: the function $c = e^{\gamma}$ begins to oscillate from the amplitude $A_0' \sim A_0^{k^2}$. (We disregard the aforementioned "dangerous" cases of very strong lowering of the upper limit of the oscillations.)

According to (3.19), the increase of the density of matter during each of the first (k − 1)-st epochs is given by the formula

$$\ln(\varepsilon_{n+1}/\varepsilon_n) = 2\,[1 - p_3(u_n)]\,\Delta_{n+1}.$$

For the last, k-th epoch of the given era it must be recognized that when $u = x < 1$ the largest exponent is $p_2(x)$ (and not $p_3(x)$). As a result we obtain for the increase in density during the entire era

$$\ln(\varepsilon_k/\varepsilon_0) \equiv \ln(\varepsilon_0'/\varepsilon_0) = 2\,(k - 1 + x)\,\delta_0\Omega_0. \qquad (5.12)$$

Consequently we have $\epsilon_0'/\epsilon_0 \sim A_0^{2k}$ already at not very large values of k. During the next era (with length k′) the increase of the density will be even faster because of the increase in the initial amplitude A_0', namely $\epsilon_0''/\epsilon_0' \sim A_0'^{2k'} \sim A_0^{2k^2k'}$ etc. These formulas illustrate the vigorous character of the increase in the density of matter.

6. STATISTICAL ANALYSIS OF THE EVOLUTION OF THE MODEL ON APPROACHING THE SINGULAR POINT

The sequence of the length $k^{(s)}$ of the successive eras (expressed in terms of the number of Kasner epochs contained in them) acquires asymptotically the character of a random process. The source of this statistical behavior is the rule (3.20)–(3.21), according to which the transition from one era to the next is determined in an infinite numerical sequence of values of the parameter u.

We can change over to a statistical description of such a sequence by considering in lieu of the initial value $u_{max} = k^{(0)} + x^{(0)}$, values in which $x^{(0)}$ are distributed in the interval from 0 to 1 in accordance with a certain probability law. Then the values of $x^{(s)}$, which terminate each (s-th) series of numbers, will also be distributed in accordance with certain laws. It can be shown (see Appendix A) that with increasing s these distributions tend to a definite stationary (independent of s) probability distribution w(x), in which the initial conditions are already completely "forgotten":

$$w(x) = \frac{1}{(1+x)\ln 2}. \qquad (6.1)$$

From this we can find the probability distribution of the lengths of the series k:

$$W(k) = (\ln 2)^{-1}\ln[(k+1)^2/k(k+2)]. \qquad (6.2)$$

These formulas give grounds for investigating the statistical properties of the evolution of the model[14].

A complicating factor in such an investigation is the slow rate of decrease of the distribution function (6.2) at large values of k:

$$W(k) \approx 1/k^2 \ln 2. \qquad (6.3)$$

The mean value $\bar{k}$, calculated in accordance with this distribution, diverges logarithmically. For a sequence cut off at a very large but finite number N we would obtain $\bar{k} \sim \ln N$. However, in this case the meaning of the mean value is very limited because of its instability, namely, a slow decrease of W(k) causes the fluctuations of the number k to diverge even more rapidly than its mean value. A more adequate characteristic of the properties of the sequence in question is the probability that a number randomly chosen from it belongs to a series of length K, where K is large. This probability is equal to ln K/ln N. It is small if $1 \ll K \ll N$. In this sense one can say that a number chosen randomly from the sequence has a large probability of belonging to a long series.

Let us write out again the recurrence formulas which determine the rules for the transition from one error to the next one. The index s numbers the successive eras (and not the Kasner epochs in one era!), starting with a certain chosen initial era (s = 0). $\Omega^{(s)}$ and $\epsilon^{(s)}$ denote respectively the initial instant of time and the initial density of matter in the s-th era; $\delta_s\Omega_s$ is the initial amplitude of the oscillations of that pair from among the functions α, β, and γ, which experiences oscillations in the given era: $k^{(s)}$ is the length of the s-th era and $x^{(s)}$ determines the length of the next era in accordance with the formula $k^{(s+1)} = [1/x^{(s)}]$. According to (5.10)–(5.12) we have

$$\Omega^{(s+1)}/\Omega^{(s)} = 1 + \delta^{(s)}k^{(s)}(k^{(s)} + x^{(s)} + 1/x^{(s)}) \equiv e^{\xi_s}, \qquad (6.4)$$

$$\delta^{(s+1)} = 1 - \frac{(k^{(s)}/x^{(s)} + 1)\,\delta^{(s)}}{1 + \delta^{(s)}k^{(s)}(1 + x^{(s)} + 1/x^{(s)})}, \qquad (6.5)$$

$$\ln(\varepsilon^{(s+1)}/\varepsilon^{(s)}) = 2\,(k^{(s)} + x^{(s)} - 1)\,\delta^{(s)}\Omega^{(s)} \qquad (6.6)$$

(in (6.4) we introduce the symbol ξ_s for future use).

The quantities $\delta^{(s)}$ (which run through values from 0 to 1) also have their own stationary statistical dis-

*We note also that these durations are large compared with the widths of the transition regions between epochs; according to (3.12), these widths are large at small $|p_1|$ (i.e., large u) and amount to $\sim 1/|p_1| \sim u$. But even in this case $\Delta_n \sim u_n|\alpha_n| \gg u_n$.

tribution. It satisfies an integral equation expressing the fact that the quantities $\delta^{(S)}$ and $\delta^{(S+1)}$, which are connected by (6.5), have the same distribution; this equation can be solved numerically (see[14]). In view of the absence of any singularities in (6.5), the distribution has a perfectly stable character; the mean values of δ or its powers calculated from this distribution are definite finite numbers. In particular, the mean value of δ is $\overline{\delta} = 0.52$.

Let us examine the statistical connection between the large time intervals Ω and the number s of the errors alternating during that time.

Repeated application of (6.4) yields

$$\Omega^{(s)}/\Omega^{(0)} = \exp\left(\sum_{p=0}^{s-1} \xi_p\right). \tag{6.7}$$

A direct averaging of this equation, however, would be meaningless, since the slow decrease of the function W(k) makes the mean values of $e^{\xi s}$ unstable in the sense indicated above. This instability is eliminated by taking the logarithm: the "doubly logarithmic" time interval

$$\tau_s \equiv \ln(\Omega^{(s)}/\Omega^{(0)}) = \sum_{p=0}^{s-1} \xi_p \tag{6.8}$$

is expressed by the sum of the quantities ξ_p, which have a stable statistical distribution. The mean values of ξ_S, and also of their powers (calculated from the distributions of the quantities x, k, and δ) are finite; a numerical calculation yields $\overline{\xi} = 2.1$ and $\overline{\xi^2} = 6.8$.

Averaging (6.8) at a given s, we obtain

$$\overline{\tau}_s = 2{,}1s, \tag{6.9}$$

which yields the mean doubly-logarithmic time interval needed for s successive eras to occur.

On the other hand, to calculate the mean-square fluctuation of this quantity we write

$$\overline{(\tau_s - \overline{\tau}_s)^2} = \sum_{p,\,q=0}^{s-1} (\overline{\xi_p\xi_q} - \overline{\xi}_p\overline{\xi}_q) = s\sum_{p=0}^{s-1} (\overline{\xi_0\xi_p} - \overline{\xi}^2).$$

In the last equation we took into account the fact that in the stationary limit the statistical correlation between $\xi^{(S)}$ and $\xi^{(S')}$ depends only on the difference $|s - s'|$. In view of the presence of a recurrence relation between $x^{(S)}$, $k^{(S)}$, $\delta^{(S)}$ and $x^{(S+1)}$, $k^{(S+1)}$, and $\delta^{(S+1)}$ this correlation, strictly speaking, differs from zero. It decreases, however, very rapidly with increasing $|s - s'|$, and a numerical calculation shows that already at $|s - s'| = 1$ we have $\overline{\xi_{S+1}\xi_S} - \overline{\xi}^2 = -0.4$. Retaining the first two terms in the sum over p, we obtain

$$[\overline{(\tau_s - \overline{\tau}_s)^2}]^{1/2} = 1{,}4\sqrt{s}. \tag{6.10}$$

as $s \to \infty$, the relative fluctuation (i.e., the ratio of the mean square fluctuation (6.10) to the mean value (6.9)) tends consequently to zero like $s^{-1/2}$. In other words, the statistical relation (6.9) becomes almost certain at large s. Of course, this certainty is the consequence of the fact that, according to (6.8), τ_S can be represented by a sum of a large number of quasi-independent terms (i.e., it is of the same origin as the certainty of the values of the additive thermodynamic quantities of a macroscopic body). It follows hence that the probabilities of the different values of τ_S (at a given s) have a Gaussian distribution:

$$\rho(\tau_s) \propto \exp\{-(\tau_s - 2{,}1s)^2/4s\}. \tag{6.11}$$

The certain character of the relation (6.9) makes it possible also to invert it, i.e., to represent it as the dependence of the average number of eras s_τ, which alternate in a given interval of doubly-logarithmic time τ:

$$\overline{s}_\tau = 0{,}47\tau. \tag{6.12}$$

The corresponding statistical distribution is given by the same Gaussian distribution, in which the random quantity is now s_τ at a given τ:

$$\rho(s_\tau) \propto \exp\{-(s_\tau - 0{,}47\tau)^2/0{,}43\tau\}. \tag{6.13}$$

Turning to the density of matter, we rewrite (6.6), with allowance for (6.7), in the form

$$\ln\ln\frac{\varepsilon^{(s+1)}}{\varepsilon^{(s)}} = \eta_s + \sum_{p=0}^{s-1}\xi_p, \quad \eta_s = \ln[2\delta^{(s)}(k^{(s)} + x^{(s)} - 1)\Omega^{(0)}]$$

and then, for the total change of energy during s eras,

$$\ln\ln\frac{\varepsilon^{(s)}}{\varepsilon^{(0)}} = \ln\sum_{p=0}^{s-1}\exp\left\{\sum_{q=0}^{p}\xi_q + \eta_p\right\}. \tag{6.14}$$

The main contribution to this equation is made by the last term of the sum over p, which contains the exponential with the largest argument. Retaining only this term and averaging (6.14), we obtain in its right side an expression for $\overline{s\xi}$ coinciding with (6.9); all the remaining terms in the sum (and also the terms η_p in the exponents) lead only to corrections of relative order 1/s.

We thus have

$$\overline{\ln\ln(\varepsilon^{(s)}/\varepsilon^{(0)})} = \overline{\ln(\Omega^{(s)}/\Omega^{(0)})}. \tag{6.15}$$

By virtue of the almost certain character of the connection between τ_S and s, which we have established above, relation (6.15) can be written in the form

$$\overline{\ln\ln(\varepsilon_\tau/\varepsilon^{(0)})} = \tau \quad \text{or} \quad \overline{\ln\ln(\varepsilon^{(s)}/\varepsilon^{(0)})} = 2{,}1s,$$

where it determines the value of the double-logarithm of the density increase, averaged over a specified doubly-logarithmic time interval τ or over a specified number of eras s.

We emphasize once more that stable statistical relations exist just for doubly-logarithmic time intervals and density increments. On the other hand, for quantities such as $\ln(\epsilon^{(S)}/\epsilon^{(0)})$ the relative fluctuation increases exponentially with the increasing averaging region, thereby depriving the mean-value concept of a stable meaning.

It remains to show that the dangerous cases mentioned in Sec. 5, which violate the regular course of the evolution as expressed by the recurrence relations (6.4)--(6.6), do not arise in the asymptotic limiting regime.

The dangerous cases are those in which excessively small values of the parameter u = x (and hence also $|p_1| \approx x$) appear at the end of the era. We choose as the criterion for the selection of such cases the inequalities

$$x^{(s)}\exp|\alpha^{(s)}| \leqslant 1, \tag{6.16}$$

where $|\alpha^{(S)}|$ is the initial depth of the minima of the oscillating functions in the s-th era (it would be preferable to take the final amplitude, but this would only strengthen the selection criterion).

The value of $x^{(0)}$ in the initial era is specified by the initial conditions. The dangerous values lie in the interval $\delta x^{(0)} \sim \exp(-|\alpha^{(0)}|)$, and also in those intervals that lead to the dangerous case in succeeding eras. In order for $x^{(S)}$ to fall in the dangerous interval $\delta x^{(S)} \sim \exp(-|\alpha^{(S)}|)$, the initial value of $x^{(0)}$ should lie, according to (A.7), in an interval of width $\delta x^{(0)} \sim \delta x^{(S)}/k^{(1)2}\ldots k^{(S)2}$. Altogether, consequently, in an initial unit interval of all possible values of $x^{(0)}$, the values leading to the appearance of the dangerous case lie in a fraction λ of this interval, with

$$\lambda = \exp(-|\alpha^{(0)}|) + \sum_{s=1}^{\infty}\sum_{k} \frac{\exp(-|\alpha^{(s)}|)}{k^{(1)2}k^{(2)2}\ldots k^{(s)2}} \qquad (6.17)$$

(the interval sum is taken over all $k^{(1)}, k^{(2)}, \ldots, k^{(S)}$ from 1 to ∞). It is easy to see that this series converges to a value $\lambda \ll 1$, the order of magnitude of which is determined already by the first term in (6.17).

It suffices to prove this by strongly majoring the series, and to this end we put $|\alpha^{(S)}| = (s+1)|\alpha^{(0)}|$, independently of the lengths of the eras $k^{(1)}, k^{(2)}, \ldots$ (Actually the $|\alpha^{(S)}|$ increase much more rapidly; even in the most inconvenient case $k^{(1)} = k^{(2)} = \ldots = 1$, the values of $|\alpha^{(S)}|$ increase more readily like $q^S|\alpha^{(0)}|$ with $q > 1$.) Noting that

$$\sum_k 1/k^{(1)2}k^{(2)2}\ldots k^{(s)2} = (\pi^2/6)^s,$$

we then obtain

$$\lambda = \exp(-|\alpha^{(0)}|)\sum_{s=0}^{\infty}[(\pi^2/6)\exp(-|\alpha^{(0)}|)]^s \approx \exp(-|\alpha^{(0)}|),$$

q. e. d.

If the initial value $x^{(0)}$ lies outside the dangerous section λ, then the dangerous cases do not arise at all. On the other hand, if it lies in this section, then a dangerous case occurs, but after leaving this section the model beings a "regular" evolution with a new initial value, which can only accidentally (with a probability λ) fall in the dangerous interval. Repetitions of such cases can lead to a dangerous situation with probabilities only $\lambda^2, \lambda^3, \ldots$, which tend asymptotically to zero. This reasoning proves the foregoing statement.

7. CONSTRUCTION OF A GENERAL SOLUTION FOR A LONG ERA WITH SMALL OSCILLATIONS

In Secs. 3–6 we investigated the evolution of the metric near the singular point using spatially-homogeneous models as examples. From the character of this evolution it is clear that the analytic construction of a general solution with a singularity of this type should be carried out separately for each of the main elements of the evolution: for the Kasner epochs, for the process of alternation of the epochs under the influence of the "perturbation," for a long era with simultaneously acting perturbations of two types. The answer to the first question is obvious: during the Kasner epoch (i.e., so long as the perturbations are small), the metric is given by expression (2.7) without the additional condition (2.17). In the present section we answer the third of the posed questions, namely we construct a solution for a long era with small oscillations of the oscillating functions, considered in Sec. 4 for the particular case of homogeneous models. We shall show that the time dependence of such a solution exhibits a far-reaching analogy with the solution that holds in the particular cases of homogeneous models, and the latter are obtained from the general solutions by a special choice of the arbitrary functions contained in it*.

The construction of the general solution is best carried out, however, in a reference frame differing somewhat from the synchronous one. Namely, we assume as before that $g_{0\alpha} = 0$, but in place of the condition $g_{00} = 1$ we choose $g_{00} = -g_{33}$. Introducing again the spatial metric tensor $\gamma_{\alpha\beta} = -g_{\alpha\beta}$, we thus have

$$g_{00} = \gamma_{33}, \quad g_{0\alpha} = 0. \qquad (7.1)$$

We denote the preferred spatial coordinate by $x^3 = z$, and denote the temporal variable by $x^0 = \xi$ (to distinguish it from the proper time t); we shall see that ξ corresponds precisely to the variable introduced in Sec. 4 for the homogeneous models. Differentiation with respect to ξ and z will be denoted by a dot and by a prime, respectively. The Latin indices a, b, and c in this section assume values 1 and 2, corresponding to the spatial coordinates x^1 and x^2, which will also be denoted by x and y. Thus, the metric takes the form

$$ds^2 = \gamma_{33}(d\xi^2 - dz^2) - \gamma_{ab}\,dx^a\,dx^b - 2\gamma_{a3}\,dx^a\,dz. \qquad (7.2)$$

As we shall see, the solution corresponding to our problem is obtained assuming the inequalities

$$\gamma_{33} \ll \gamma_{ab}, \qquad (7.3)$$

$$\gamma_{a3}^2 \ll \gamma_{aa}\gamma_{33} \qquad (7.4)$$

(these conditions generalize the condition for the smallness of one of the functions a^2, b^2, or c^2 compared with the other two, as was postulated in the homogeneous models).

The inequality (7.4) denotes that the components γ_{a3} are small in the sense that for any ratio of the displacements dx^a and dz one can omit from the square of the spatial length element dl^2 the terms with the products $dx^a dz$. Thus, the first approximation to the solution will be the metric (7.2) with $\gamma_{a3} = 0$†.

$$ds^2 = \gamma_{33}(d\xi^2 - dz^2) - \gamma_{ab}\,dx^a\,dx^b. \qquad (7.5)$$

By calculating the components of the Ricci tensor R_0^0, R_3^0, R_3^3, and R_a^b from the metric (7.5), we can easily verify that by virtue of the condition (7.3) all the terms in these components, containing differentiation with respect to the coordinates x^a, are small compared with terms containing derivatives with respect to ξ and z (the ratio of the former to the latter is $\sim\gamma_{33}/\gamma_{ab}$). In other words, to obtain the equations of the main approximation it is necessary to differentiate γ_{33} and γ_{ab} in (7.5) as if they were independent of x^a. Putting

$$\gamma_{33} = e^{\psi}, \quad \dot{\gamma}_{ab} = \varkappa_{ab}, \quad \gamma'_{ab} = \lambda_{ab}, \quad |\gamma_{ab}| = G^2, \qquad (7.6)$$

*The material in this section is based on [15,16].

†We call attention to the fact that this metric admits of additional arbitrary transformations of the type.

we obtain the following equations**:

$$2e^{\psi}R_a^b = G^{-1}(G\lambda_a^b)' - G^{-1}(G\varkappa_a^b)^{\cdot} = 0, \quad (7.7)$$

$$2e^{\psi}R_3^0 = \frac{1}{2}\varkappa\psi' + \frac{1}{2}\lambda\dot{\psi} - \varkappa' - \frac{1}{2}\varkappa_a^b\lambda_b^a = 0, \quad (7.8)$$

$$2e^{\psi}(R_0^0 - R_3^3) = \lambda\psi' + \varkappa\dot{\psi} - \dot{\varkappa} - \lambda' - \frac{1}{2}\varkappa_b^a\varkappa_b^a - \frac{1}{2}\lambda_a^b\lambda_b^a = 0. \quad (7.9)$$

The raising and the lowering of the two-dimensional indices is carried out here with the aid of γ_{ab}. κ and λ are the contractions of κ_a^a and λ_a^a, with

$$\varkappa = 2\dot{G}/G, \quad \lambda = 2G'/G. \quad (7.10)$$

As to the Ricci-tensor components R_a^0 and R_a^3, they vanish identically in such a calculation. In the next approximation, however (i.e., when account is taken of small γ_{a3} and of the derivatives with respect to x and y), they determine the values of γ_{a3} from the already known γ_{33} and γ_{ab}.

The contraction of Eqs. (7.7) yields $G'' - \ddot{G} = 0$, whence

$$G = f_1(x, y, \xi + z) + f_2(x, y, \xi - z). \quad (7.11)$$

We can have here different cases, depending on the values of $N = g^{ik}G, {}_iG, {}_k$, i.e., depending on the character of the variable G. In the approximation in question, $g^{00} = \gamma^{33} \gg \gamma^{ab}$, and therefore $N \approx g^{00}(\dot{G})^2 - \gamma^{33}(G')^2 = 4\gamma^{33}\hat{f}_1\hat{f}_2$. The time singularities of interest to us result from the case $N > 0$ (G is time-like). Putting in (7.11) $f_1 = (1/2)(\xi + z)\sin y$ and $f_2 = (1/2)(\xi - z)\sin y$, we represent G in the form

$$G = \xi \sin y. \quad (7.12)$$

Such a choice does not make the analysis less general; it can be shown that it is made possible (in the main approximation considered here!) simply by the remaining still-permissible variable transformations†. The factor sin y is introduced in (7.12) for convenience in subsequent comparison with the homogeneous models. When (7.12) is taken into account, Eqs. (7.7)–(7.9) take the form

$$\dot{\varkappa}_a^b + \xi^{-1}\varkappa_a^b - \lambda_a^{b\prime} = 0, \quad (7.13)$$

$$\dot{\psi} = -\xi^{-1} + \frac{1}{4}\xi(\varkappa_a^b\varkappa_b^a + \lambda_a^b\lambda_b^a), \quad (7.14)$$

$$\psi' = \frac{1}{2}\xi\varkappa_a^b\lambda_b^a. \quad (7.15)$$

The fundamental equation here is (7.13), which determine the components γ_{ab}; the function ψ is then determined by simply integrating Eqs. (7.14)–(7.15).

The variable ξ runs through values from 0 to ∞. Let us consider the solution of (7.13) in two limiting regions, $\xi \gg 1$ and $\xi \ll 1$.

In the region of large ξ we can (as is confirmed by the result) seek a solution in the form of an expansion in $1/\sqrt{\xi}$:

$$\gamma_{ab} = \xi[a_{ab}(x, y, z) + O(1/\sqrt{\xi})], \quad (7.16)$$

with

$$|a_{ab}| = \sin^2 y \quad (7.17)$$

(Eq. (7.17) is needed to satisfy the condition (7.12)). Substituting (7.16) in (7.13), we obtain in the principal order of magnitude

$$(a^{ac\prime}a_{bc})' = 0, \quad (7.18)$$

where the quantities a^{ac} form a matrix inverse to a_{ac}. The solution of (7.18) is written in the form

$$a_{ab} = l_a l_b e^{-2\rho z} + m_a m_b e^{2\rho z}, \quad (7.19)$$

$$l_1 m_2 - l_2 m_1 = \sin y, \quad (7.20)$$

where l_a, m_a, and ρ are arbitrary functions of the coordinates x and y and are connected by the condition (7.20) which is derived from (7.17).

To find the succeeding terms of the expansion, it is convenient to represent the matrix of the unknown quantities γ_{ab} in the form

$$\gamma_{ab} = \xi(\tilde{L}e^H L)_{ab}, \quad (7.21)$$

where

$$L = \begin{pmatrix} l_1 e^{-\rho z} & l_2 e^{-\rho z} \\ m_1 e^{\rho z} & m_2 e^{\rho z} \end{pmatrix}, \quad (7.22)$$

and the symbol ~ denotes the transpose. The matrix H is symmetrical, and its trace is equal to zero. The representation (7.21) ensures symmetry of γ_{ab} and satisfaction of the condition (7.12). If we replace exp H by unity, then we obtain from (7.21) $\gamma_{ab} = \xi a_{ab}$ with a_{ab} from (7.19). In other words, the principal term of the expansion γ_{ab} corresponds to H = 0; the further terms are obtained by power expansion of the matrix H, the components of which are regarded as small quantities.

We denote the independent components of the matrix H by σ and φ, writing

$$H = \begin{pmatrix} \sigma & \varphi \\ \varphi & -\sigma \end{pmatrix}. \quad (7.23)$$

Substituting (7.21) in (7.13) and retaining only the terms linear in H, we obtain for σ and φ the equations

$$\ddot{\sigma} + \xi^{-1}\dot{\sigma} - \sigma'' = 0,$$
$$\ddot{\varphi} + \xi^{-1}\dot{\varphi} - \varphi'' + 4\rho^2\varphi = 0. \quad (7.24)$$

If we seek the solution of these equations in the form of Fourier series in the coordinate z, then we obtain Bessel equations for the coefficients of the series as functions of ξ. The principal asymptotic terms of the solution at large values of ξ are*

$$\sigma = \frac{1}{\sqrt{\xi}}\sum_{n=-\infty}^{\infty}(A_{1n}e^{in\omega\xi} + B_{1n}e^{-in\omega\xi})e^{in\omega z},$$

$$\varphi = \frac{1}{\sqrt{\xi}}\sum_{n=-\infty}^{\infty}(A_{2n}e^{i\omega_n\xi} + B_{2n}e^{-i\omega_n\xi})e^{in\omega z}, \quad (7.25)$$

$$\omega_n^2 = n^2\omega^2 + 4\rho^2.$$

The coefficients A and B are arbitrary complex functions of the coordinates x and y, and satisfy the necessary conditions for σ and φ to be real; The

*On the other hand, the equation $R_0^0 + R_3^3 = 0$ is a direct consequence of the system (7.7)–(7.9), if $G \neq 0$ or $G' \neq 0$. The case $G = G' = 0$ does not require special consideration, for it can be shown that the space-time metric reduces in this case (in first approximation) to the Galilean one.

†When $N < 0$ (G is space-like) we can put G = z, and this leads to a generalization of the well known Einstein-Rosen metric [17]. At N = 0 we arrive at the Robinson-Bondi wave metric, which depends only on $\xi + z$ or only on $\xi - z$ (see [2], Sec. 103).

*It is possible that a solution can be sought also in the form of Fourier integrals; this question has not been thoroughly investigated. We therefore do not state here that the expandability in Fourier series is a necessary requirement imposed on the coordinate dependence of the functions σ and φ.

fundamental frequency ω is an arbitrary real function of x and y. From (7.14)–(7.15) it is now easy to obtain the first term of the expansion of the function ψ:

$$\psi = \rho^2\xi^2 \tag{7.26}$$

(this term vanishes if $\rho \equiv 0$; in this case the principal term is the expansion term linear in ξ: $\psi = \xi q(x, y)$, where q is a positive function; see[15]).

Thus, in the region of large ξ, the components of the metric tensor γ_{ab} oscillate with decreasing ξ against a background of a slow decrease due to the factor ξ in (7.21). On the other hand, the component $\gamma_{33} = e^{\psi}$ decreases rapidly approximately like $\exp(\rho^2\xi^2)$; this ensures the possibility of satisfying the condition (7.3)*.

Let us consider now the region $\xi \ll 1$. The main approximation to the solution of (7.13) is obtained from the assumption (confirmed by the result) that it is possible to leave out from these equations the terms with the derivatives with respect to the coordinates:

$$\dot{\varkappa}_a^b + \xi^{-1}\varkappa_a^b = 0. \tag{7.27}$$

This equation together with the condition (7.12) yields

$$\gamma_{ab} = \lambda_a\lambda_b\xi^{2s_1} + \mu_a\mu_b\xi^{2s_2}, \tag{7.28}$$

where λ_a, μ_a, s_1, and s_2 are arbitrary functions of all three coordinates x, y, and z, and are connected with one another by the conditions

$$\lambda_1\mu_2 - \lambda_2\mu_1 = \sin y, \quad s_1 + s_2 = 1. \tag{7.29}$$

Equations (7.14)–(7.15) now yield

$$\gamma_{33} = e^{\psi} \sim \xi^{-(1-s_1^2-s_2^2)}. \tag{7.30}$$

The derivatives $\lambda_a^{b'}$, calculated from (7.28), contain the terms $\sim \xi^{4S_1-2}$ and $\sim \xi^{4S_2-2}$, whereas the terms remaining in (7.27) are proportional to ξ^{-2}. Therefore in order for the transition from (7.13) to (7.27) to be valid it is necessary to have $s_1 > 0$, $s_2 > 0$, and $1 - s_1^2 - s_S^2 > 0$.

Thus, in the region of small ξ, the oscillations of the functions γ_{ab} cease, and the function γ_{33} begins to increase with decreasing ξ. This is the Kasner regime, and when γ_{33} becomes comparable with γ_{ab} the conditions for the validity of the approximation in question no longer hold.

To check whether the foregoing analysis is self-consistent, it is necessary also to consider the equations $R_a^0 = 0$ and $R_a^3 = 0$, to calculate from them the components γ_{a3}, and to check whether the proposed inequality (7.4) holds. Such an investigation (see[16]) shows that in both asymptotic regions the components γ_{a3} turn out to be $\sim \gamma_{33}$. Therefore satisfaction of the inequality (7.3) automatically ensures validity also of the inequality (7.4).

The obtained solution contains, as it should for the general case of a field in vacuum, four arbitrary functions of three spatial coordinates x, y, and z. In the region $\xi \ll 1$ these functions are, for example, λ_1, λ_2, μ_1, and s_1. In the region $\xi \gg 1$ the four functions are the four Fourier series with respect to the coordinate z, which enter in (7.25), with coefficients are are functions of x and y; although the Fourier expansion (or integral?) does single out a certain special class of functions, this class is sufficiently broad to include a finite fraction of the entire manifold of conceivable initial conditions.

The solution contains also a certain number of other arbitrary functions of the two coordinates x and y. Such "two-dimensional" arbitrary functions appear, generally speaking, because the connections between these three-dimensional functions, which appear in the solution of the Einstein's equations, are differential (and not algebraic); we leave aside the deeper question of the geometrical meaning of such functions. We shall likewise not calculate the number of independent two-dimensional functions, all the more since in this case it is difficult to impart an unambiguous meaning to this question, since the three-dimensional functions are also specified in terms of sets of two-dimensional ones*. (The number of two-dimensional functions in the solution considered here is discussed in greater detail in[16].)

Let us show, finally, that the obtained general solution contains the particular solutions considered in Sec. 4 for the homogeneous models.

Taking the reference vectors for the space of type IX from (D.2) and substituting in (2.7), we write the space-time metric of this model in the form

$$\begin{aligned} ds_{\mathrm{IX}}^2 = dt^2 &- [(a^2\sin^2 z + b^2\cos^2 z)\sin^2 y + c^2\cos^2 y]\,dx^2 \\ &- [a^2\cos^2 z + b^2\sin^2 z]\,dy^2 - c^2 dz^2 \\ &+ (b^2 - a^2)\sin 2z \sin y\,dx\,dy - 2c^2\cos y\,dx\,dz. \end{aligned} \tag{7.31}$$

In the case when $c^2 \ll a^2, b^2$, we can neglect c^2 everywhere except in the term c^2dz^2. To change over from the synchronous reference frame, in which (7.31) is written, to the system satisfying the conditions (7.1), we make the transformation $dt = c\,d\xi/2$ and the substitution $z \to z/2$. Assuming also that $\chi \equiv \ln(a/b) \ll 1$, we obtain from (7.31) in the main approximation:

$$\begin{aligned} ds_{\mathrm{IX}}^2 = &\frac{1}{4}c^2(d\xi^2 - dz^2) \\ &- ab\{\sin^2 y(1-\chi\cos z)\,dx^2 + (1+\chi\cos z)\,dy^2 + 2\chi\sin z\sin y\,dx\,dy\}. \end{aligned} \tag{7.32}$$

Analogously, for a model of type VIII with reference vectors from (D.11), we obtain

$$\begin{aligned} ds_{\mathrm{VIII}}^2 = &\frac{1}{4}c^2(d\xi^2 - dz^2) \\ &- ab\{\sin^2 y(\operatorname{ch} z - \chi)\,dx^2 + (\operatorname{ch} z + \chi)\,dy^2 - 2\operatorname{sh} z\sin y\,dx\,dy\}. \end{aligned} \tag{7.33}$$

According to Sec. 4, we have in this case in both cases $ab = \xi$ (we assume for simplicity $a_0^2 = \xi_0$) and formula (4.8) for χ; the function $c(\xi)$, on the other hand, is given by (4.10) or (4.18) for models of type IX and VIII, respectively.

We obtain a similar metric of type VIII from (7.22),

*The terms quadratic in H in (7.13) lead only to small corrections ($\sim 1/\xi$) to σ and φ. On the other hand, allowance for the cubic terms leads to a weak dependence of the amplitudes A and B on ξ; this dependence can be represented as the appearance of logarithmic phases in the oscillating factors of (7.25). The corresponding calculations for the case $\rho \equiv 0$ are given in [15] (see also the analogous situation for homogeneous models; Appendix B).

*The regular expansion of the general solution of Einstein's equations contains (besides the four three-dimensional functions) also three independent functions of two coordinates (see [18], Sec. 40, and also Appendix A of [1]).

(7.25), and (7.26) by choosing the two-dimensional vectors l_a and m_a in the form

$$l_1 = -m_1 = (1/\sqrt{2}) \sin y, \quad l_2 = m_2 = 1/\sqrt{2} \tag{7.34}$$

and putting

$$\rho = 1/2, \quad A^*_{20} = B_{20} = iAe^{i\xi_0}, \qquad A_{1n} = A_{2n} = B_{1n} = B_{2n} = 0 \quad (n \neq 0). \tag{7.35}$$

To obtain a metric of type IX it is necessary to put

$$\left.\begin{array}{l} \rho = 0, \quad \omega = 1, \\ A_{11} = -B^*_{11} = A^*_{1,-1} = -B_{1,-1} = -\frac{1}{2} Ae^{-i\xi_0}, \\ A_{21} = B^*_{21} = A^*_{2,-1} = B_{2,-1} = -\frac{1}{2} iAe^{-i\xi_0}, \\ A_{1n} = A_{2n} = B_{1n} = B_{2n} = 0 \qquad (n \neq \pm 1) \end{array}\right\} \tag{7.36}$$

(the approximation (7.26) is not sufficient in this case for the calculation of $c(\xi)$, and it is necessary to calculate the term of ψ that is linear in ξ; this is done in[15].

In the foregoing investigation, space was assumed empty. Inclusion of matter does not change the generality of the solution and does not change its qualitative properties (see[15,16]).

8. CONCLUDING REMARKS

Thus, we have described in the preceding sections singularities of a new type in the cosmological solutions of Einstein's equations; these singularities have a complicated oscillatory character. Although we studied these singularities mainly with special homogeneous models as examples, there are convincing reasons for assuming that the singularities in the general solution of the gravitational equations have a similar character; it is precisely this circumstance which makes it particularly significant for cosmology.

Such a statement is based, first, on the reasoning indicated at the start of Sec. 3, namely, that the oscillatory approach to the singular point is the result of just the unique type of perturbation with respect to which the generalized Kasner solution is unstable. A confirmation of the generality of the solution is also the analytic construction given in Sec. 7 for a long era with small oscillations. Although this case is not an obligatory element of the evolution of the metric (in light of the results of Secs. 5 and 6) in the asymptotic vicinity of the singular point, it contains all the qualitative features, namely, oscillation of the metric in two spatial dimensions with a monotonic decrease in the third dimension, with obligatory violation of such a regime at the end of a definite time interval. What still remains unclear, however, are the details of the single alternation of the Kasner epochs in the general case of a spacelike-inhomogeneous metric.

A special study is also needed to determine whether the existence of the singular point imposes any limitations on the properties of the spatial geometry. So far it can be stated only that there is no direct connection with the finite or infinite character of the space; this is evidenced by the existence of both a closed and an open homogeneous model with an oscillatory singular point.

The oscillatory approach to the singular point casts new light on the very concept of finite time. Between any finite instant of world time t and the instant t = 0 there is contained an infinite number of oscillations. In this sense, the process acquires an infinite character. It turns out that the more natural variable for its description is not the time t itself but its logarithm ln t, with respect to which the process is stretched out to $-\infty$.

We have spoken throughout of the direction of the approach to the singular point as being the direction of the decrease of time. But in view of the symmetry of the gravitational equations with respect to the reversal of the sign of time, we are equally justified of speaking of an approach to the singularity in the direction of increasing time. Actually, however, in view of the physical nonequivalence of the future and of the past, there is a substantial difference between the two cases with respect to the very formulation of the problem. A singularity in the future can have a physical meaning only if it is admissible under arbitrary initial conditions, specified at some preceeding instant of time. It is clear that there are no grounds whatever for having the distribution of matter and field, attained at any particular instant during the evolution of the universe, correspond to specific conditions required to realize some particular solution of the equations of gravitation.

An investigation based on the gravitational equations alone is hardly capable of determining the type of singularity in the past. It is natural to assume that the choice of the solution corresponding to the real world is connected with some deep physical requirements, the establishment of which on the basis of only the existing gravitational theory is impossible, and which can be clarified only as a result of further syntheses of physical theories. In this sense, it may turn out in principle that this choice corresponds to some particular (say, isotropic) type of singularity. Nonetheless, it is more natural to assume a priori that by virtue of the general character of the oscillatory regime, it is just this regime that should describe the initial stages of the evolution of the world.

In this connection, considerable interest may attach to a property of the model indicated by Misner[19]; this property pertains to the propagation of light signals. Let us recall first the situation that takes place in this respect in the Friedmann model.

In the isotropic model there exists a "light horizon" for the propagation of signals. This means, that for each given instant of time there exists a certain largest distance, beyond which exchange of light signals is impossible, and therefore a causal connection between signals is impossible, namely, a signal does not have time to propagate over such distances in the time elapsed from the singular point t = 0. Indeed, the propagation of a signal is determined by the equation ds = 0. In the isotropic model near the singular point t = 0 the element of the interval is of the form $ds^2 = dt^2 - 2t\, dl^2$, where dl^2 stands for a special differential form that does not contain the time (see[2], Secs. 107–109). By the substitution $t = \eta^2/2$ it reduces to the form

$$ds^2 = \eta^2 (d\eta^2 - d\bar{l}^2). \tag{8.1}$$

From this we obtain for the "distance" $\overline{\Delta l}$, traversed by the signal, the expression

$$\Delta\bar{l} = \Delta\eta. \qquad (8.2)$$

Since the variable η, together with the time t, runs only through values starting from zero, signal can traverse by the "instant" η only distances $\overline{\Delta l} \leq \eta$, and this established the distance to the horizon.

The existence of a light horizon in the isotropic model raises definite difficulties in the question of the origin of the presently observed isotropy of the relict black radio emission. Indeed, from the point of view of this model, the observed isotropy would denote identity of the properties of radiation arising to the observed also from such regions of space, the history of which could not be in any causal connection with one another. The situation in the model with the oscillatory evolution near the singular point, on the other hand, may turn out to be different. Let us illustrate this with the homogeneous model of type IX as an example.

Namely, we consider the propagation of a signal in that direction, in which the scales change during the course of a long era in accordance with a law close to $\sim$t. The square of the element of length in this direction is given by $dl^2 = t^2 d\bar{l}^2$, and the corresponding element of the four-dimensional interval is $ds^2 = dt^2 - t^2 d\bar{l}^2$. The substitution $t = e^{\eta}$ transforms it into

$$ds^2 = e^{2\eta}(d\eta^2 - d\bar{l}^2), \qquad (8.3)$$

from which we obtain again for signal propagation an equation of the type (8.2). The essential difference consists, however, in the fact that the variable η now runs through values from $-\infty$ (if the metric (8.3) holds for all t starting from t = 0). Therefore for each specified "instant" η there are preceeding intervals $\Delta\eta$ sufficient for the signal to cover any finite distance.

Thus, during a long era, the light horizon opens up in a definite direction in space. Although the duration of each of the long eras is still finite, during the course of the evolution of the world they alternate an infinite number of times in different directions in space. This circumstance allows us to hope that in the model considered here it is possible to obtain a causal connection between events in the entire volume of space*. There is still no exhaustive investigation of this question. There is likewise no investigation of the question for the analogous open model.

During the course of time, with increasing distance from the singular point, the influence of the matter on the evolution of the metric, which is immaterial at earlier stages of the evolution, gradually increases and ultimately becomes predominant. One can expect that this influence will lead to a gradual "isotropization" of space, as a result of which its properties approach the Friedmann model, which describes satisfactorily the present state of the universe. Of course, this question still requires further investigation. The question of establishing a connection between the parameters of the theory and the time scale of the real world still remains open.

One final remark concerning the general validity of considering the question of a "singular state" of a world with such arbitrarily large densities of matter on the basis of the existing theory of gravitation. Of course, the physical applicability of Einstein's equations in their present form can be verified in the indicated conditions only in the process of future syntheses of physical theories, and in this sense this question cannot be answered at present. It is important, however, that the gravitational theory itself does not cease to be logically consistent (i.e., it does not lead to internal contradictions) at any density of matter. In other words, this theory is not limited, as such, by certain conditions that follow from it and are capable to make its application logically invalid and contradictory at very large densities; limitations could arise in principle only as a result of factors that are "extraneous" with respect to the theory of gravitation itself. This circumstance makes it in any case formally valid and necessary to consider the question of singularities in cosmological models already within the framework of the existing theory.

*This property caused Misner to call the model the "mixmaster universe."

APPENDICES

A. SOME INFORMATION FROM THE THEORY OF CONTINUOUS FRACTIONS

Let us consider an infinite sequence of positive numbers u, consisting of series, each (s-th) of which begins with a certain (irrational) number $u_{max}^{(s)} = k^{(s)} + x^{(s)}$, and reaches via values $k^{(s)} + x^{(s)} - 1, k^{(s)} + x^{(s)} - 2, \ldots$ to a value $x^{(s)} < 1$; the transition to the next series is in accordance with the rule

$$u_{max}^{(s+1)} = k^{(s+1)} + x^{(s+1)} = 1/x^{(s)}. \qquad (A.1)$$

The integers $k^{(s)}$ determine the lengths of the series.

If the entire sequence begins with the number $k^{(0)} + x^{(0)}$, then the lengths $k^{(1)}, k^{(2)}, \ldots$ are the numbers which enter in the expansion of $x^{(0)}$ in an infinite continuous fraction:

$$x^{(0)} = \cfrac{1}{k^{(1)} + \cfrac{1}{k^{(2)} + \cfrac{1}{k^{(3)} + \ldots}}}. \qquad (A.2)$$

Therefore to study the properties of the sequences of interest to us we can use a number of known results of the theory of continuous fractions (see, e.g.,[20]).

As was noted in Sec. 3, we can be interested only in those properties of the sequence, which are inherent in the general case of an arbitrary irrational number $x^{(0)} < 1$. It is precisely for this reason that there is no need to consider the case of rational numbers $x^{(0)}$ (for which the expansion in a continuous fraction is finite). Nor is there any interest in specific properties inherent in period continuous fractions (such fractions result from the expansion of quadratic irrational numbers, i.e., numbers that are roots of quadratic equations with integer coefficients)*. We note that in both these cases all the elements of the expansion (the numbers $k^{(1)}, k^{(2)}, \ldots$) are finite in magnitude in an obvious manner. This property is also exclusive: the set of all numbers $x^{(0)} < 1$ whose expansion has these

*The simplest example of an expansion in a periodic continuous fraction is

$$\frac{\sqrt{5}-1}{2} = \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}.$$

properties has a measure zero compared with the set of all numbers in the segment (0, 1).

To change over to a probabilistic description, we shall consider in place of the definite value $x^{(0)}$ the values $x^{(0)} = x$ distributed in an interval from 0 to 1 in accordance with a certain specified probability $w_0(x)$. Then the values $x^{(s)}$ that terminate each series will also have a certain probability. Let the $w_s(x)dx$ be the probability that the s-th series is terminated with the value $x^{(s)} = x$ lying in a specified interval dx.

In order for the s-th series to have a length k, the preceding series should terminate with a number in the interval between $1/(k+1)$ and $1/k$. Therefore the probability that the series will have a length k is

$$W_s(k) = \int\limits_{1/(k+1)}^{1/k} w_{s-1}(x)\,dx. \qquad (A.3)$$

The value $x^{(s+1)} = x$, which terminates the (s + 1)-st series, can result from the initial (for this series) values $u_{max}^{(s+1)} = x + k$, where $k = 1, 2, \ldots$; they correspond to the values $x^{(s)} = 1/(k + x)$ for the preceding series. Noting this, we can write the following recurrence relation for the probability distribution $w_{s+1}(x)$ in terms of the distribution $w_s(x)$:

$$w_{s+1}(x)\,dx = \sum_{k=1}^{\infty} w_s\left(\frac{1}{k+x}\right)\left| d\,\frac{1}{k+x}\right|,$$

or

$$w_{s+1}(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2}\, w_s\left(\frac{1}{k+x}\right). \qquad (A.4)$$

If the distributions $w_s(x)$ tend to a stationary (independent of s) limiting distribution $w(x)$ with increasing s, then the latter should satisfy the equation

$$w(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2}\, w\left(\frac{1}{k+x}\right). \qquad (A.5)$$

This equation actually has a solution*

$$w(x) = [(1+x)\ln 2]^{-1} \qquad (A.6)$$

(normalized to unity). This can be readily verified by noting that with this function the sum in the right side of (A.5) becomes equal to

$$\sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k+1)} = \sum_{k=1}^{\infty} \frac{1}{x+k} - \sum_{k=1}^{\infty} \frac{1}{x+k+1} = \frac{1}{x+1}.$$

The corresponding stationary probability distribution of the lengths of the series is obtained by substituting (A.6) in (A.3); it is written out in the text (formula (6.2)).

An idea of the rate at which the stationary distribution (A.6) is established can be gained from the following example. Let the initial values $x^{(0)}$ be distributed in a narrow interval of width $\delta x^{(0)}$ about a certain definite number. From the recurrence relation (A.4) (or directly from the expansion (A.2)) we can readily conclude that the widths of the distributions $w_s(x)$ (about other definite numbers) will then be equal to

$$\delta x^{(s)} = k^{(1)2} k^{(2)2} \ldots k^{(s)2}\,\delta x^{(0)} \qquad (A.7)$$

(this expression is valid only to the extent to which $\delta x^{(s)} \ll 1$).

*This result was already known to Gauss.

B. REFINEMENT OF THE CALCULATIONS IN SECTION 4

In Sec. 4, in obtaining a solution describing small oscillations in the region of large values of the variable ξ, we have confined ourselves to the first term of the expansion of $\sinh 2\chi$ in (4.11). Such an approximation is equivalent to retaining in the solution of only the term of lowest order in the small quantity $1/\sqrt{\xi}$. We present here more accurate calculations with allowance of the next terms of the expansion in powers of $1/\sqrt{\xi}$.

After making the substitution

$$\chi = \varphi/\sqrt{\xi} \qquad (B.1)$$

Eq. (4.6) takes the form

$$\ddot{\varphi} + \varphi = -{}^2/_3\,\xi^{-1}\varphi^3 - \xi^{-2}\,({}^1/_4\varphi + {}^2/_{15}\varphi^5). \qquad (B.2)$$

We have retained here the first three terms of the expansion of $\sinh 2\chi$; the omitted terms $\sim \varphi^2/\xi^3$ (a dot over a letter denotes in this section differentiation with respect to ξ).

The first approximation to the solution, which is sought in the form of an expansion in powers of $1/\xi$, is the solution of the equation without the right-hand side, in the form

$$\varphi_0 = 2A \sin(\xi - \xi_0). \qquad (B.3)$$

The next term of the expansion in φ should be $\sim 1/\xi$ and should be determined from (B.2) with allowance for the term $-2\varphi^3/3\xi$ in the right-hand side. Writing $\varphi = \varphi_0 + \varphi_1$ and assuming that $\varphi_1 \sim 1/\xi$, we obtain for φ_1 the equation

$$\ddot{\varphi}_1 + \varphi_1 = -\frac{2}{3\xi}\varphi_0^3 = \frac{4A^3}{3\xi}\sin(3\xi - 3\xi_0) - \frac{4A^3}{\xi}\sin(\xi - \xi_0). \qquad (B.4)$$

However, the second term in the right-hand side of this equation has a resonant frequency (which coincides with the frequency of the solution of the homogeneous equation $\ddot{\varphi}_1 + \varphi_1 = 0$), leading to the appearance in φ_1 of logarithmically-diverging terms and thereby violating the initial assumption of the order of smallness of φ_1. The appearance of resonance denotes in reality a weak change of the phase of the sine function in the first-approximation function φ_0. Accordingly, we write it in the form

$$\varphi_0 = 2A \sin(\xi - \xi_0 + \psi) \qquad (B.5)$$

in place of (B.3), where $\psi(\xi)$ is assumed to be a slowly varying function in the sense that $\ddot{\psi} \ll \dot{\psi} \ll 1$. The expression $\ddot{\varphi}_0 + \varphi_0$ now does not vanish rigorously, but will be a small quantity $\dot{\psi}$:

$$\ddot{\varphi}_0 + \varphi_0 = -4A\dot{\psi}\sin(\xi - \xi_0 + \psi) \qquad (B.6)$$

(we have omitted terms $\sim \ddot{\psi}$ and $\sim \dot{\psi}^2$). Accordingly we now obtain for the correction φ_1, in place of (B.4), the equation

$$\ddot{\varphi}_1 + \varphi_1 = \frac{4A^3}{3\xi}\sin(3\xi - 3\xi_0 + 3\psi) + 4A\left(\dot{\psi} - \frac{A^2}{\xi}\right)\sin(\xi - \xi_0 + \psi), \qquad (B.7)$$

and ψ is obtained precisely from the condition that the resonant terms drop out from the right-hand side. Hence (with the definite choice of the integration constant)

$$\psi = A^2 \ln(\xi/\xi_0). \qquad (B.8)$$

We can now represent the correction φ_1 in the form

$$\varphi_1=\frac{1}{\xi}(C_1\sin\Delta+C_2\cos\Delta)-\frac{A^3}{6\xi}\sin 3\Delta, \quad (B.9)$$

$$\Delta=\xi-\xi_0+A^2\ln(\xi/\xi_0). \quad (B.10)$$

Here C_1 and C_2 are constants that are arbitrary for the time being. It is easy to verify that the expression (B.9) satisfies the equation (B.7) in the order $1/\xi$. As to the terms $\sim\xi^{-2}$ etc., which appear from $\ddot{\varphi}_1$ as the result of the differentiation of the factor $1/\xi$ and the logarithmic phase, they affect only the determination of the corrections $\sim 1/\xi^2$ etc.

The constants C_1 and C_2 are found from the condition that there be no resonant terms in the right-hand side of the equation that determines the correction $\varphi_2\sim 1/\xi^2$. Writing $\varphi=\varphi_0+\varphi_1+\varphi_2$ with φ_0 and φ_1 from (B.5) and (B.9) and substituting in (B.2), we obtain, in order $1/\xi^2$, the equation for φ_2

$$\ddot{\varphi}_2+\varphi_2=\xi^{-2}[2(C_1+A^3)\cos\Delta-(2C_2+4C_1A^2+A^5+{}^1/_2A)\sin\Delta+ \\ +(2A^2C_1-A^5)\sin 3\Delta+(2A^2C_2-A^3)\cos 3\Delta-{}^3/_5A^5\sin 5\Delta]. \quad (B.11)$$

The terms proportional to $\cos\Delta$ and $\sin\Delta$ in the square brackets would make a contribution of the order $1/\xi$ to φ_2, thereby contradicting the initial assumption $\varphi_2\sim 1/\xi^2$, the consequence of which is (B.11) itself. The requirement that these terms be equal to zero yields

$$C_1=-A^3,\quad C_2={}^1/_4(6A^5-A).$$

Proceeding analogously, we can obtain a solution for φ accurate to any order in $1/\xi$.

Thus, accurate to $\xi^{-3/2}$ inclusive, we have ultimately

$$\chi=\frac{2A}{\sqrt{\xi}}\left[\sin\Delta-\frac{A^2}{12\xi}(6\sin\Delta+\sin 3\Delta)+\frac{6A^4-1}{8\xi}\cos\Delta+O\left(\frac{1}{\xi}\right)\right]. \quad (B.12)$$

The corresponding functions $a(\xi)$ and $b(\xi)$ are given by

$$a=a_0\sqrt{\xi/\xi_0}\,e^{\chi/2},\quad b=a_0\sqrt{\xi/\xi_0}\,e^{-\chi/2}. \quad (B.13)$$

These formulas pertain to the homogeneous models of both types, IX and VIII. Substituting (B.12) in (4.7) or (4.17), we get

$$\gamma=\frac{1}{16}1(\mp 1)(\xi^2-\xi_0^2)+A^2(\xi-\xi_0)+\left(\frac{1}{2}A^4-\frac{1}{4}\right)\ln(\xi/\xi_0)+ \\ +\ln c_0+\frac{1}{4}A^2\xi^{-1}\cos 2\Delta+O(\xi^{-2}), \quad (B.14)$$

where the upper and lower signs in the first term pertain respectively to the metric of type IX and VIII. From this we have for the function $c(\xi)$

$$\left.\begin{aligned} c(\xi)&=c_0(\xi/\xi_0)^{(2A^4-1)/4}e^{-A^2(\xi_0-\xi)}\ \text{(type IX)},\\ c(\xi)&=c_0(\xi/\xi_0)^{(2A^4-1)/4}e^{-\frac{1}{8}(\xi_0^2-\xi^2)-A^2(\xi_0-\xi)}\ \text{(type VIII)}\end{aligned}\right\} \quad (B.15)$$

(only the term with the highest power of ξ has been retained in the three-exponential factors). Finally, for the connection between the world time t and the variable ξ we obtain

$$\begin{aligned} t/t_0&=c(\xi)/c_0\ \text{(type IX)},\\ t/t_0&=c(\xi)\,\xi_0/c_0\xi\ \text{(type VIII)}\end{aligned} \quad (B.16)$$

with the same accuracy with which formulas (B.15) are valid.

C. HOMOGENEOUS SPACES

For the reader's convenience, we present here a brief exposition of the theory of homogeneous spaces.

Homogeneity denotes identity of the metric properties in all points of space. An exact definition of this concept is connected with a consideration of the aggregate of coordinate transformations that make the space congruent with itself, i.e., leave its metric unchanged: if prior to the transformation the length element is

$$dl^2=\gamma_{\alpha\beta}(x^1,x^2,x^3)\,dx^\alpha dx^\beta,$$

then after the transformation the same element has the form

$$dl^2=\gamma_{\alpha\beta}(x'^1,x'^2,x'^3)\,dx'^\alpha dx'^\beta$$

with the same functional dependence of $\gamma_{\alpha\beta}$ on the new coordinates. The space is homogeneous if it admits of the aggregate of transformations (or, as is said, the group of motions) that make it possible to align any specified point of the space with any other point. By virtue of the three-dimensional character of the space it is obvious that to this end different transformations of the group should be determined by the values of three parameters.

Thus, in Euclidean space homogeneity is expressed by invariance of the metric with respect to parallel translation of the Cartesian system of coordinates. Each translation is determined by three parameters—the components of the vector of the displacement of the origin. All these transformations leave invariant the three independent differentials (dz, dy, dz) of which the length element is constructed. In the general case of a non-Euclidean homogeneous space, the transformations of its group of motion also leave invariant three differential forms which, however, do not reduce to complete differentials of any coordinate functions. We write these forms in the form

$$e_\alpha^a dx^\alpha, \quad (C.1)$$

where the Latin index a numbers three independent vectors (coordinate functions); we shall call these the reference vectors.

With the aid of the forms (C.1), a spatial metric that is invariant against a given group of motions is constructed as follows:

$$dl^2=\gamma_{ab}(e_\alpha^a dx^\alpha)(e_\beta^b dx^\beta),$$

i.e., the metric tensor is

$$\gamma_{\alpha\beta}=\gamma_{ab}e_\alpha^a e_\beta^b, \quad (C.2)$$

where the coefficients of γ_{ab}, which are symmetrical in the indices a and b, are functions of the time*. The contravariant components of the metric tensors are written in the form

$$\gamma^{\alpha\beta}=\gamma^{ab}e_a^\alpha e_b^\beta, \quad (C.3)$$

where the coefficients γ^{ab} form a matrix that is the inverse of the matrix $\gamma_{ab}(\gamma_{ac}\gamma^{bc}=\delta_a^b)$, and the

*Throughout this section, we stipulate summation over repeated indices, both Greek and Latin (a, b, c, . . .)which number the reference vectors.

quantities e^{α}_{a} form three vectors "inverse" to the vectors e^{a}_{α}:

$$e^{\alpha}_{a}e^{b}_{\alpha}=\delta^{b}_{a}, \quad e^{\alpha}_{a}e^{a}_{\beta}=\delta^{\alpha}_{\beta} \tag{C.4}$$

(each of these equations follows automatically from the other). We note that the connection between e^{α}_{a} and e^{a}_{α} can be written in explicity form as follows:

$$\mathbf{e}_1=(1/v)[\mathbf{e}^2\mathbf{e}^3], \quad \mathbf{e}_2=(1/v)[\mathbf{e}^3\mathbf{e}^1], \quad \mathbf{e}_3=(1/v)[\mathbf{e}^1\mathbf{e}^2], \tag{C.5}$$

where $v = \mathbf{e}^1\cdot\mathbf{e}^2\times\mathbf{e}^3$, and $\mathbf{e}_a$ and $\mathbf{e}^a$ must be understood as Cartesian vectors with respective components e^{α}_{a} and e^{a}_{α}.* The determinant of the metric tensor (C.2) is

$$\gamma=|\gamma_{ab}|\,|e^{a}_{\alpha}|^2=|\gamma_{ab}|\,v^2, \tag{C.6}$$

where $|\gamma_{ab}|$ is the determinant of the matrix γ_{ab}.†

The invariance of the differential forms (C.1) denotes that

$$e^{a}_{\alpha}(x)\,dx^{\alpha}=e^{a}_{\alpha}(x')\,dx'^{\alpha}, \tag{C.7}$$

with e^{a}_{α} in both sides of the equation being the same functions of the old and new coordinates, respectively. Multiplying this equation by $e^{\beta}_{a}(x')$, making the substitution $dx'^{\beta}=(\partial_{\alpha}x'^{\beta})dx^{\alpha}$, and comparing coefficients of identical differentials dx^{α}, we obtain

$$\partial_{\alpha}x'^{\beta}=e^{\beta}_{a}(x')\,e^{a}_{\alpha}(x). \tag{C.8}$$

These equations are a system of differential equations determining the functions $x'^{\beta}(x)$ from the specified reference vectors‡. For them to be integrable, Eqs. (C.8) must identically satisfy the conditions

$$\partial_{\alpha}\partial_{\gamma}x'^{\beta}=\partial_{\gamma}\partial_{\alpha}x'^{\beta}.$$

Calculating the derivatives, we obtain

$$[\partial'_{\delta}e^{\beta}_{a}(x')\cdot e^{\delta}_{b}(x')-\partial'_{\delta}e^{\beta}_{b}(x')\cdot e^{\delta}_{a}(x')]\,e^{b}_{\gamma}(x)\,e^{a}_{\alpha}(x)=e^{\beta}_{a}(x')\,[\partial_{\alpha}e^{a}_{\gamma}(x)-\partial_{\gamma}e^{a}_{\alpha}(x)]$$

Multiplying both sides of the equation by $e^{\alpha}_{d}(x)e^{\gamma}_{c}(x)e^{f}_{\beta}(x')$ and transferring the differentiation from some factors to the others with allowance for (C.4), we obtain in the left side

$$e^{f}_{\beta}(x')\,[e^{\delta}_{c}(x')\cdot\partial'_{\delta}e^{\beta}_{d}(x')-e^{\delta}_{d}(x')\cdot\partial'_{\delta}e^{\beta}_{c}(x')]=e^{\beta}_{c}(x')\,e^{\delta}_{d}(x')\,[\partial'_{\delta}e^{f}_{\beta}(x')-\partial'_{\beta}e^{f}_{\delta}(x')],$$

and in the right side the same expression as a function of x. Since x and x′ are arbitrary, these expressions should reduce to constants:

$$e^{\alpha}_{a}e^{\beta}_{b}(\partial_{\beta}e^{c}_{\alpha}-\partial_{\alpha}e^{c}_{\beta})=C^{c}_{ab}. \tag{C.9}$$

The constants C^{c}_{ab} are called the structure constants of the group. Multiplying by e^{γ}_{c}, we can rewrite (C.9) in the form

$$e^{\alpha}_{a}\partial_{\alpha}e^{\gamma}_{b}-e^{\beta}_{b}\partial_{\beta}e^{\gamma}_{a}=C^{c}_{ab}e^{\gamma}_{c}. \tag{C.10}$$

As seen from the definition, the structure constants are antisymmetrical in the lower indices

$$C^{c}_{ab}=-C^{c}_{ba}. \tag{C.11}$$

One more condition for them can be obtained by noting that Eq. (C.10) can be written in the form of a commutation rule

$$[X_aX_b]\equiv X_aX_b-X_bX_a=C^{c}_{ab}X_c \tag{C.12}$$

for the linear differential operators*

$$X_a=e^{\alpha}_{a}\partial_{\alpha}. \tag{C.13}$$

Then the aforementioned condition results from the identity

$$[[X_aX_b]\,X_c]+[[X_bX_c]\,X_a]+[[X_cX_a]\,X_b]=0$$

(called the Jacobi identity) and takes the form

$$C^{f}_{ab}C^{d}_{fc}+C^{f}_{bc}C^{d}_{fa}+C^{f}_{ca}C^{d}_{fb}=0. \tag{C.14}$$

We note that Eqs. (C.9) can be written in vector form:

$$[\mathbf{e}_a\mathbf{e}_b]\,\mathrm{rot}\,\mathbf{e}^c=-C^{c}_{ab},$$

where again the vector operations are carried out as if the coordinates x^{α} were Cartesian. With the aid of (C.5) we obtain from this

$$\frac{1}{v}(\mathbf{e}^1\,\mathrm{rot}\,\mathbf{e}^1)=C^{1}_{32}, \quad \frac{1}{v}(\mathbf{e}^2\,\mathrm{rot}\,\mathbf{e}^1)=C^{1}_{13}, \quad \frac{1}{v}(\mathbf{e}^3\,\mathrm{rot}\,\mathbf{e}^1)=C^{1}_{21} \tag{C.15}$$

and six more equations obtained by cyclic permutation of the indices 1, 2, and 3.

The Einstein equations for a world with homogeneous space can be represented in the form of a system of ordinary differential equations containing only the functions of the time. To this end, all the three-dimensional vectors and tensors must be expanded in terms of the triad of reference vectors of the given space. Denoting the components of such expansions by the indices a, b, ..., we have, by definition

$$R_{ab}=R_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b}, \quad R_{0a}=R_{0\alpha}e^{\alpha}_{a}, \quad u_a=u_{\alpha}e^{\alpha}_{a},$$

all these quantities are already functions of t only (the scalar quantities, the density ϵ and the pressure of matter p, are also functions of the time). Further raising and lowering of the indices is carried out with the aid of the quantities γ^{ab}: $R^{b}_{a}=\gamma^{cb}R_{ac}$, $u_a=\gamma_{ab}u^{b}$, etc.

Einstein's equations in the synchronous reference frame are expressed in terms of the three-dimensional tensors $\kappa_{\alpha\beta}$ and $P_{\alpha\beta}$ (see (2.11)–(2.13)). For the former we have simply

$$\kappa_{ab}=\dot{\gamma}_{ab} \tag{C.16}$$

(the dot denotes differentiation with respect to t). The components of P_{ab} can be expressed in terms of the quantities γ_{ab} and the structure constants of the group:

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*One must not confuse e^{α}_{a} with the contravariant components of the vector e^{a}_{α}! The latter are equal to $e^{a\alpha}=\gamma^{\alpha\beta}e^{a}_{\beta}=\gamma^{ab}e^{\alpha}_{b}$.

†The representation of the spatial metric in the form $\gamma_{\alpha\beta}=a^2l_{\alpha}l_{\beta}+b^2m_{\alpha}m_{\beta}+c^2n_{\alpha}n_{\beta}$, used in Sec. 4, corresponds to a diagonal matrix γ_{ab} with components $\gamma_{11}=a^2$, $\gamma_{22}=b^2$, $\gamma_{33}=c^2$; the vectors **l**, **m**, and **n** correspond to the vectors $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$.

‡For transformations of the form $x'^{\beta}=x^{\beta}+\xi^{\beta}$, where ξ^{β} are small quantities, we obtain from (C.8) the equations

$$\partial_{\alpha}\xi^{\beta}=\xi^{\gamma}e^{a}_{\alpha}\partial_{\gamma}e^{\beta}_{a}.$$

Three linearly independent solutions of these equations, ξ^{β}_{b} (b = 1, 2, 3), determine infinitesimally small transformations of the group of motions of space. The vectors ξ^{β}_{b} are called Killing vectors.

*To avoid misunderstandings in comparison with other papers, we note that the systematic theory of continuous groups is usually constructed on the basis of the operators (generators of the group) defined in terms of the Killing vectors: $X_a=\xi^{\alpha}_{a}\partial_{\alpha}$.

$$\left.\begin{aligned} P_{ab} &= -a^c_{ad}a^d_{bc} - C^d_{dc}a^c_{ab}, \\ a^c_{ab} &= {}^1/_2\,(C^c_{ab} + C^f_{bd}\gamma_{fa}\gamma^{dc} - C^f_{da}\gamma_{fb}\gamma^{dc}). \end{aligned}\right\} \tag{C.17}$$

The same quantities can be used to express the covariant derivatives $\kappa^{\beta}_{\alpha;\gamma}$, and we obtain for R^0_a

$$R^0_a = -{}^1/_2\dot{\gamma}_{bc}\gamma^{bd}\,(C^c_{da} - \delta^c_a C^f_{fd}). \tag{C.18}$$

We emphasize that to set up the Einstein equations there is no need to use the explicit expressions for the reference vectors as functions of the coordinates*.

The choice of three reference vectors in differential from (C.1) (and with them also of the operators (C.13)) is of course not unique. They can be subjected to any linear transformation with constant (real) coefficients:

$$e'^{\alpha}_a = A^b_a e^{\alpha}_b. \tag{C.19}$$

With respect to such transformations, the quantities γ_{ab} behave like covariant tensors, and the constants C^c_{ab} like a tensor that is covariant in the indices a and b and contravariant in the index c.

The conditions (C.11) and (C.14) are the only ones that the structure constants must satisfy. However, among the sets of constants allowed by these conditions there are equivalent ones, in the sense that their difference is connected only with the transformations (C.19). The question of the classification of homogeneous spaces reduces to a determination of all the non-equivalent sets of structure constants.

The simplest method of doing it (following Beer) is to use the "tensor" properties of the constants C^c_{ab} and express these nine quantities in terms of six components of the symmetric "tensor" n^{ab} and the three components of the "vector" a_c in accordance with

$$C^c_{ab} = e_{abd}n^{dc} + \delta^c_b a_a - \delta^c_a a_b, \tag{C.20}$$

where e_{abd} is a unit antisymmetrical "tensor." The antisymmetry condition (C.11) is already taken into account here, and the Jacobi identity (C.14) leads to the condition

$$n^{ab}a_b = 0. \tag{C.21}$$

By means of the transformations (C.19) the symmetric "tensor" n^{ab} can be reduced to diagonal form; let $n^{(1)}$, $n^{(2)}$, and $n^{(3)}$ be its principal values. The equality (C.21) shows that the "vector" a_b (if it exists) lies on one of the principal directions of the "tensor" n^{ab}, namely the direction corresponding to a zero principal value. Without loss of generality, we can therefore put $a_b = (a, 0, 0)$. Then (C.21) reduces to $an^{(1)} = 0$, i.e., either a or $n^{(1)}$ must vanish. The commutation rules (C.12) become

$$[X_1X_2] = aX_2 + n^{(3)}X_3, \quad [X_2X_3] = n^{(1)}X_1, \quad [X_3X_1] = n^{(2)}X_2 - aX_3. \tag{C.22}$$

We are still free to reverse the signs of the operators X_a and to transform their scale arbitrarily (to multiply them by constants). This enables us to change simultaneously the sign of all the $n^{(a)}$, and also to make a positive (if it differs from zero). It is also possible to transform all the structure constants into ± 1, if at least one of the quantities a, $n^{(2)}$, or $n^{(3)}$ is equal to zero. If all three quantities differ from zero, then the scale transformations leave the ratio $a^2/n^{(2)}n^{(3)}$ invariant.

Thus, we arrive at the following list of all the possible types of homogeneous spaces; the Roman number in the first column of the table is the customary Bianchi classification:

Type of space	a	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$	Type of space	a	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$
I	0	0	0	0	V	1	0	0	0
II	0	1	0	0	IV	1	0	0	1
VII	0	1	1	0	VII	a	0	1	1
VI	0	1	−1	0	III ($a=1$) VI ($a\neq 1$)	a	0	1	−1
IX	0	1	1	1					
VIII	0	1	1	−1					

The parameter a runs through all positive values. The corresponding types constitute one-parameter families of different groups.

Type I is Euclidean space (all the components of the spatial curvature tensor vanish). Besides the trivial case of Galilean metric, this includes the metric (2.1).

If we put in the metric tensor (C.2) for the space of type IX $\gamma_{ab} = (1/2)a^2\delta_{ab}$, then we obtain with the aid of (C.17) for the Ricci tensor

$$P_{ab} = \frac{1}{2}\delta_{ab}, \quad P_{\alpha\beta} = P_{ab}e^a_{\alpha}e^b_{\beta} = \frac{2}{a^2}\gamma_{\alpha\beta},$$

corresponding to a space of constant positive curvature (with curvature radius a; see[2], Sec. 107); this space is contained thus in type IX as a particular case.

Analogously, a space of constant negative curvature is contained as a particular case in type V. This can be easily verified by transforming first the structure constants of this group by means of the substitutions $X_2 + X_3 = X'_2$, $X_2 - X_3 = X'_3$, and $X_1 = X'_1$. We then have $[X'_1X'_2] = X'_2$, $[X'_2X'_3] = 0$, and $[X'_3X'_1] = -X'_3$. Putting then $\gamma_{ab} = a^2\delta_{ab}$, we obtain the Ricci tensor $P_{\alpha\beta} = -2\delta_{\alpha\beta}/a^2$, corresponding to a space of constant negative curvature.

D. HOMOGENEOUS SPACES OF TYPE VIII AND IX

For the space of type IX, the commutation rules of the operators X_a are:

$$[X_1X_2] = -X_3, \quad [X_2X_3] = -X_1, \quad [X_3X_1] = -X_2,$$

i.e., the non-zero structure constants are*

$$C^1_{32} = C^2_{13} = C^3_{21} = 1. \tag{D.1}$$

According to (C.15), these constants coincide with λ, μ, and ν, [Eq. (2.10)], respectively.

The reference vectors corresponding to the constants (D.1) are

$$\left.\begin{aligned} \mathbf{l} &\equiv \mathbf{e}^1 = (\sin z \sin y,\ \cos z,\ 0), \\ \mathbf{m} &\equiv \mathbf{e}^2 = (-\cos z \sin y,\ \sin z,\ 0), \\ \mathbf{n} &\equiv \mathbf{e}^3 = (\cos y,\ 0,\ 1), \end{aligned}\right\} \tag{D.2}$$

where the coordinates are denoted $x^1 = x$, $x^2 = y$, and $x^3 = z$. According to (C.6), the volume element is

$$dV = \sqrt{\gamma}\,dx\,dy\,dz = \sqrt{|\gamma_{ab}|}\,\sin y\,dx\,dy\,dz. \tag{D.3}$$

*A derivation of formulas (C.17) and (C.18) can be found in Schucking's article in the book [21].

*The common sign of the structure constants is reversed here compared with the table given above.

The coordinates run through values in the intervals

$$0 \leqslant x \leqslant 4\pi, \quad 0 \leqslant y \leqslant \pi, \quad 0 \leqslant z \leqslant 2\pi \tag{D.4}$$

(see below). The space is closed and its volume is

$$V = 16\pi^2 \sqrt{|\gamma_{ab}|}. \tag{D.5}$$

As already indicated, the particular case $\gamma_{ab} = (1/4)a^2\delta_{ab}$ corresponds to a space of constant positive curvature. With these values of γ_{ab} and with the reference vector (D.2), the length element is

$$dl^2 = {}^1/_4 a^2 (dx^2 + dy^2 + dz^2 + 2\cos y \, dx \, dz). \tag{D.6}$$

Let us show how it can be transformed to the form customarily used for a space of constant positive curvature:

$$dl^2 = dX^2 + dY^2 + dZ^2 + \frac{(X\,dX + Y\,dY + Z\,dZ)^2}{a^2 - X^2 - Y^2 - Z^2}, \tag{D.7}$$

or

$$dl^2 = a^2 (d\chi^2 + \sin^2\chi \sin^2\theta \, d\varphi^2 + \sin^2\chi \, d\theta^2), \tag{D.8}$$

where χ, θ, and φ are angles of a four-dimensional spherical coordinate system connected with X, Y, and Z in (D.7) by

$$X = a \sin\chi \sin\theta \cos\varphi, \quad Y = a \sin\chi \sin\theta \sin\varphi, \quad Z = a \sin\chi \cos\theta$$

(see[2], Sec. 107). By means of the substitution

$$X = \rho \cos(\beta/2), \quad Y = \rho \sin(\beta/2), \quad Z = \sqrt{a^2 - \rho^2} \sin(\alpha/2)$$

and $\rho = a \sin(y/2)$ we transform the element (D.7) into

$$dl^2 = {}^1/_4 (a^2 - \rho^2)\, d\alpha^2 + {}^1/_4 \rho^2 \, d\beta^2 + (1 - \rho^2/a^2)^{-1} d\rho^2$$
$$= {}^1/_4 a^2 (\cos^2(y/2)\, d\alpha^2 + \sin^2(y/2)\, d\beta^2 + dy^2).$$

The same form is assumed by (D.6) if the substitution $z + x = \alpha$ and $z - x = \beta$ is made. Combining now all the successive substitutions, we find that the transformation form (D.6) to (D.8) is effected by the formulas

$$\left.\begin{aligned} \sin(y/2)\cos[(z-x)/2] &= \sin\chi \sin\theta \cos\varphi, \\ \sin(y/2)\sin[(z-x)/2] &= \sin\chi \sin\theta \sin\varphi, \\ \cos(y/2)\sin[(z+x)/2] &= \sin\chi \cos\theta. \end{aligned}\right\} \tag{D.9}$$

Variation of the coordinates χ, θ, and φ in the intervals $0 \leq \chi$, $\theta \leq \pi$, and $0 \leq \varphi \leq 2\pi$ corresponds to variation of the coordinates x, y, and z in the intervals (D.4).

For homogeneous space of type VIII, the commutation rules are

$$[X_1 X_2] = -X_3, \quad [X_2 X_3] = X_1, \quad [X_3 X_1] = -X_2 \tag{D.10}$$

i.e., the structure constants are $C^1_{32} = -1$ and $C^2_{13} = C^3_{21} = 1$.

The corresponding reference vectors are

$$\mathbf{l} = (-\operatorname{sh} z \sin y, \operatorname{ch} z, 0), \quad \mathbf{m} = (-\operatorname{ch} z \sin y, \operatorname{sh} z, 0), \quad \mathbf{n} = (\cos y, 0, 1). \tag{D.11}$$

The coordinate z now runs through values from 0 to ∞, and the volume of space is infinite.

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