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OSCILLATIONS OF INHOMOGENEOUS FLOWS OF PLASMA AND LIQUIDS

A. V. TIMOFEEV

I. V. Kurchatov Institute of Atomic Energy

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INTRODUCTION

IN this review we present results of the latest investigations of the theory of oscillations and stability of inhomogeneous flows of plasma and liquids. The theory of oscillations and stability of flow of an ordinary liquid has been under development for more than half a century.

Recently, however, a number of papers have been published, in which these problems are considered on the basis of new physical concepts developed in the theory of oscillations of another continuous medium, namely a plasma. Among such concepts are the concepts of resonant interaction of oscillations with the motion of the particles of the continuous medium. It was introduced by Landau in an analysis of the problem of the electronic Langmuir oscillations of a plasma that is in thermodynamic equilibrium and at rest[1]. In such a plasma, owing to the thermal velocity scatter, there are always particles whose velocity is equal to the phase velocity of the wave. For these particles, the field of the wave is constant in time, and they are therefore at resonance with the wave. Landau has shown that resonant particles absorb the wave energy and this leads to damping of the wave (Landau damping). It was also shown by means of numerous examples that in a nonequilibrium plasma the energy can be transferred via resonant interaction also in the opposite direction, from the plasma to the oscillations. As a rule, in these problems the resonant particles constitute a small fraction of the total number of particles at a given point. Therefore, to reveal resonant effects it was necessary to employ a kinetic analysis, and such resonances can be arbitrarily called kinetic.

However, the employed concepts of resonance turn out to be useful also in problems of another type, namely in the analysis of oscillations of a solid moving with a velocity that is variable in space, when at a certain (resonant) point its velocity coincides with the phase velocity of the wave. In this case the resonant particles are concentrated in the vicinity of the resonant point. Resonances of this type can be naturally called hydrodynamic. It turns out, for example, that the well-known Rayleigh theorem[2], according to which flow of an incompressible liquid with a velocity profile that has no inflection point is stable, is due to the fact that the oscillations are absorbed at the resonance point. Other important results are the theory of oscillations of flow of an incompressible liquid are also connected to one degree or another with resonant phenomena.

Resonant phenomena also play a decisive role in oscillations of a moving plasma, if its velocity changes sufficiently rapidly in a direction perpendicular to the motion. A study of the oscillations of a moving plasma is of considerable interest, since under real conditions the plasma is rarely at rest. Its motion may be due, for example, to electric fields which are easily produced in a plasma. At first glance it seems that the motion of a plasma can cause only additional instabilities, which are quite abundant in a plasma even without this. It is useful, however to turn here to examples from the theory of oscillation of flows of an ordinary liquid. As already mentioned, an ordinary liquid satisfies the Rayleigh theorem, according to which a definite type of flows is stable. It turns out that a similar situation takes place also for a moving plasma, thus, for example, in the "Ogra" and "Alice" thermonuclear installations the plasma stability was appreciably improved under conditions when the plasma rotated with a velocity that changed radially sharply and monotonically (these systems are axially symmetrical). A theoretical analysis shows that in such regimes, just as in the flow of an ordinary liquid, the stabilization is connected with the absorption of the oscillations at the resonant points where the angular velocity of rotation of the plasma coincides with the phase velocity of the wave. The same effect should become manifest also in oscillations of the electron cloud in a magnetron, and also in oscillations of an electron beam moving along a magnetic field with a velocity that is variable with the cross section. These two examples, because of their simplicity, have by now been analyzed in considerable detail.

The purpose of the present review is to demonstrate, by means of a very simple example, the most general laws characterizing the oscillations of moving continuous media (plasma and ordinary liquids). As already noted, these laws are connected mainly with processes that develop at the resonant points, at which the velocity of the continuous medium coincides with the phase velocity of the oscillations.

1. OSCILLATIONS OF PLANE-PARALLEL FLOWS OF INCOMPRESSIBLE LIQUIDS

1.1. Rayleigh's Theorem

An investigation of the resonant effects in oscillations of moving continuous media is best started with a consideration of the simplest and best-known problem of oscillations of plane-parallel flows of an incompressible liquid.

We introduce a rectangular coordinate system, the Oy axis of which is directed along the flow and the Ox axis along the direction in which its velocity changes (Fig. 1). The figure shows also the velocity profiles of the simplest flows. We assume that the velocity $V_0(x)$ varies monotonically, and its second derivative d^2V_0/dx^2 does not vanish anywhere (the velocity pro-

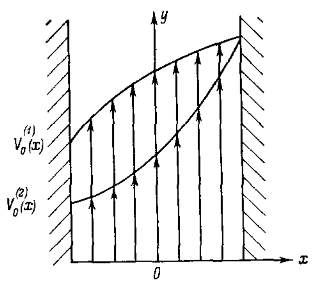

FIG. 1. Flow velocity profiles without inflection points. $d^2V_0^{(1)}/dx^2 < 0$, $d^2V_0^{(2)}/dx^2 > 0$.

file has no inflection points). We shall assume that the viscosity of the liquid is small (the Reynolds number is large), and we shall disregard it initially.

The motion of an ideal (nonviscous) incompressible liquid is described by the continuity equation

$$\operatorname{div} \mathbf{V} = 0 \tag{1.1}$$

and the equation of motion

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p; \tag{1.2}$$

here p is the pressure of the liquid and ρ its density.

For two-dimensional flows, Eq. (1.1) makes it possible to introduce the stream function φ ($V_x = \partial\varphi/\partial y$, $V_y = -\partial\varphi/\partial x$). Applying to (1.2) the operation of taking the z component of the curl, and expressing the velocity in terms of the stream function, we obtain

$$d\Delta\varphi/dt = 0. \tag{1.3}$$

This equation expresses the law of conservation of the curl of the velocity in an ideal incompressible liquid:

$$\operatorname{rot}_z \mathbf{V} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}.$$

In the absence of viscosity, the flow velocity profile can be arbitrary and accordingly Eq. (1.3) is satisfied by the arbitrary function x:

$$\varphi_0(x) = -\int^x V_{0y}(x)\,dx.$$

We now consider small perturbations of the flow $V_1 \ll V_0$ and $\varphi_1 \ll \varphi_0$. An equation describing such perturbations is obtained by linearizing (1.3):

$$\left(\frac{\partial}{\partial t} + V_0(x)\frac{\partial}{\partial y}\right)\Delta\varphi_1 - \frac{d^2V_0}{dx^2}\frac{\partial\varphi_1}{\partial y} = 0. \tag{1.4}$$

Since the flow is stationary and is homogeneous with respect to Oy, the perturbation $\varphi_1(x, y, t)$ can be chosen in the form $\varphi_1(x, y, t) = \varphi_1(x)e^{-i\omega t + iky}$. We then obtain from (1.4)

$$\frac{d^2\varphi_1}{dx^2} - k^2\varphi_1 - \frac{1}{V_0(x) - (\omega/k)}\frac{d^2V_0}{dx^2}\varphi_1 = 0. \tag{1.5}$$

On the surfaces bounding the flow ($x = x_{1,2}$), the normal velocity component vanishes. Therefore Eq. (1.5) should be supplemented with the boundary conditions $V_{1x}(x_{1,2}) = ik\varphi_1(x_{1,2}) = 0$. Thus, the problem reduces to finding the eigenfunctions of Eq. (1.5) and determining the eigenfrequency spectrum. If some of the frequencies have Im $\omega > 0$ at a specified velocity profile $V_0(x)$, then the corresponding flow turns out to be unstable, since the amplitudes of the natural (free) oscillations will grow in time.

In considering the flow stability, we shall follow Rayleigh[2]. We assume that the unstable oscillations exist, and accordingly (1.5) has at least one solution $\tilde{\varphi}_1(x)$ with Im $\omega > 0$. We multiply (1.5) by $\tilde{\varphi}_1^*(x)$ and integrate the result by parts:

$$\int_{x_1}^{x_2} dx\left\{-\left|\frac{d\tilde{\varphi}_1}{dx}\right|^2 - k^2|\tilde{\varphi}_1|^2 + \frac{i(\operatorname{Im}\omega/k) + [(\operatorname{Re}\omega/k) - V_0(x)]}{|(\omega/k) - V_0(x)|^2}\frac{d^2V_0}{dx^2}|\tilde{\varphi}_1|^2\right\} = 0. \tag{1.6}$$

If the flow velocity profile has no inflection points (d^2V_0/dx^2 does not vanish), then relation (1.6) cannot be satisfied, since the imaginary part of the integrand does not reverse sign in the integration interval (x_1, x_2). Therefore in flows with a velocity profile without inflection points, growing natural oscillations cannot exist, and consequently such flows are stable. This statement constitutes Rayleigh's theorem.

In this proof we used the inequality Im $\omega \neq 0$; on the other hand the sign of Im ω was immaterial. Therefore, on the basis of relation (1.6) only, it would be necessary to conclude that in flows with a velocity profile without inflection points there are likewise no damped oscillations (Im $\omega < 0$). Moreover, if the neutral oscillations (Im $\omega = 0$) are understood as the limiting case of oscillations with Im $\omega \neq 0$ as Im $\to 0$, then it can be shown that such oscillations are impossible in the flows under consideration[3] (see also Appendix I.1 at the end of the review).

1.2. Landau's Circuiting Rule and Absorption of Oscillations at Resonant Points

We have found that in flows having a velocity profile without inflection points no natural oscillations whatever are possible. This unusual situation calls for a more detailed discussion. The problem is facilitated by the fact that the propagation of the perturbations in inhomogeneous flows of an ordinary liquid have certain common features with the propagation of electromagnetic perturbations in a plasma situated in an inhomogeneous magnetic field (concerning the latter question, see, for example,[4,6] and also Sec. 5.1 of the present review). This analogy was noted and used in[7], which we shall mainly follow.

The physical processes that lead to elimination of the natural oscillations are particularly easy to analyze using as an example oscillations whose characteristic spatial scale in the Ox direction is small compared with the characteristic scale of variation of the initial velocity $V_0(x)$. When we analyze these oscillations we can use the quasiclassical approximation, choosing the perturbations in the form of waves traveling along Ox:

$$\varphi_1(x, y, t) \approx (k_x)^{-1/2}\exp\left(-i\omega t + iky + i\int^x k_x\,dx\right).$$

In fact, the spatial scales of the initial and perturbed velocities turn out to be the same in order of magnitude. Therefore, strictly speaking, the use of the quasiclassical approximation would be incorrect. Our purpose, however, is a physical interpretation of the results obtained in the preceding section. This is done most simply in the "quasiclassical language" of waves and wave packets made up of such waves. We shall therefore consider in place of (1.5) a simplified model equation, which, while retaining the characteristic features of (1.5), admits at the same time of the possibility of using the quasiclassical approximation. To

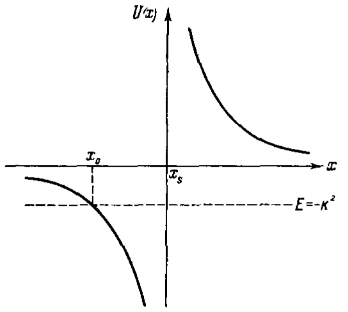

FIG. 2. Effective potential U(x) for Eqs. (1.5) and (1.7). x_S and x_0 are the singular and ordinary turning points.

this end we replace the last term in the left side of (1.5) by $A\varphi_1/(x - x_S)$:

$$\frac{d^2\varphi_1}{dx^2} - k^2\varphi_1 - \frac{A}{x - x_s}\varphi_1 = 0. \quad (1.7)$$

If we define x_S by the equation $\omega/k = V_0(x_S)$ and put $A = V_0''(x_S)/V_0'(x_S)$, then in the vicinity of the point x_S Eqs. (1.5) and (1.7) will coincide. We shall show later that it is precisely this region which is of greatest interest, since the physical phenomena leading to the elimination of the natural oscillations develop in it. We shall assume A to be a sufficiently large quantity, $A \gg \max\{a^{-1}, k^2 a\}$, where a is the dimension of the flow along Ox. This makes it possible to use the quasiclassical approximation in the vicinity of the point x_S. In the region where $|A/(x - x_S)| \gg k^2$, the quasiclassical wave vector is equal to $k_x \approx \pm[-A/(x - x_S)]^{1/2}$ and accordingly the solutions (see (1.10) below take the form

$$\varphi_1(x) \approx (x_s - x)^{1/4} \exp[\pm 2iA^{1/2}(x_s - x)^{1/2}]. \quad (1.8)$$

For concreteness we have put $A > 0$.

Equations (1.5) and (1.7) can be regarded as Schrödinger equations describing the motion of a particle with energy $E = -k^2$ in a potential $U(x)$ equal to $[V_0(x) - (\omega/k)]^{-1} d^2V_0/dx^2$ or $A/(x - x_S)$ respectively. At the resonance point x_S, where the phase velocity of the oscillations along the Oy axis coincides with the flow velocity $\omega/k = V_0(x_S)$, $U(x)$ becomes infinite, and this point is a singular turning point (Fig. 2)*. If $\text{Im}\,\omega \neq 0$, then the resonant point shifts into the complex plane. It then becomes necessary to continue analytically the solutions of (1.5) and (1.7) to the plane of complex variables x. We shall consider the case $\text{Im}\,\omega = 0$, for at real values of the variable the physical meaning of the solution is much simpler.

Let us find the formulas for "joining together" the quasiclassical solutions on different sides of the singular turning point x_S. From (1.8) it follows that the point x_S is simultaneously a branch point. Let us consider a solution that decreases when $x > x_S$, i.e., in the nontransparency region: $\varphi_1 \approx (x_S - x)^{1/4} \exp[-2A^{1/2}(x - x_S)^{1/2}]$. If in the analytic continuation of this solution into the region of free propagation of the oscillations ($x < x_S$) (see Fig. 2) the singular point x_S is circuited in the complex plane from the top, then $\arg(x - x_S)$ receives an increment $+\pi$ and the solution under consideration goes over into a wave that travels to the left: $\varphi_1 \approx (x_S - x)^{1/4}\exp[-2iA^{1/2}(x_S - x)^{1/2}]$. If x_S is circuited from below, $\arg(x - x_S)$ receives an increment $-\pi$ and the same solution goes over into a wave traveling to the right: $\varphi_1 \approx (x_S - x)^{1/4}\exp[2iA^{1/2}(x_S - x)^{1/2}]$.

The question of the choice of the contour around singular points is encountered in many problems involving the oscillations of a continuous medium, where the oscillations can resonate with the motion of the medium. Thus, in the problem of Langmuir oscillations of a plasma at rest, allowance for the resonant particles, i.e., particles with thermal velocity v coincides with the phase velocity of the oscillations ω/k, leads to the appearance of a singularity in the distribution functions of the particles with respect to the velocities $f(v) \sim [(\omega/k) - v]^{-1}$. For this case, Landau has indicated that neutral oscillations with $\text{Im}\,\omega = 0$ should be regarded as the limits of growing oscillations ($\text{Im}\,\omega > 0$) as $\text{Im}\,\omega \to 0$ and therefore the resonant point should be circuited from below if $k > 0$ and from above if $k < 0$*. This rule of circuiting is obtained automatically if the problem of the natural oscillations is considered as a part of the more general problem of the temporal evolution of the initial perturbations, and if this problem is solved by the Laplace-transform method. On the basis of this consideration, one could attempt to use the Landau circuiting rule also in the analysis of oscillations of the flows of an incompressible liquid. In all physical problems, however, the use of singular expressions, generally speaking, should be justified by the results of a more complete investigation with allowance for the additional factors that eliminate the singularity. In the present case, to eliminate the singularity it is necessary to take into account in the vicinity of the resonant point x_S the finite value of the viscosity of the medium, i.e., to consider in lieu of (1.5) the equation (see, for example,[3]):

$$-i\nu\left(\frac{d^2}{dx^2} - k^2\right)^2\varphi_1 + (\omega - kV_0(x))\left(\frac{d^2}{dx^2} - k^2\right)\varphi_1 + k\frac{d^2V_0}{dx^2}\varphi_1 = 0; \quad (1.9)$$

here ν is the kinematic viscosity coefficient.

In fact, the general solution of (1.5) and (1.7) contains a singularity of the type $(x - x_S)\ln(x - x_S)$, and therefore on approaching x_S the higher derivatives of φ_1 increase rapidly, $d^n\varphi_1/dx^n \sim (x - x_S)^{1-n}$ and consequently at sufficiently small distances from the resonant point x_S the small-scale effects such as viscosity become significant. A mathematically correct analysis of this question was presented by Wasow[8] (see Appendix II.1). From this result it follows, in particular, that if we are not interested in the details

*A similar singularity is possessed at the origin by a Coulomb potential. However, in the problem of the motion of an electron in the field of the nucleus one of the boundary conditions for the ψ function of the electron is set at the singular point, and therefore this condition is satisfied by that of the two linearly independent conditions of the wave equation which is regular. Therefore the singular solution is discarded. In the present case the singularity falls inside the interval in question. Therefore, in order to satisfy the boundary conditions on both ends of the interval, it is necessary to retain both the regular and the singular solution.

*The quasiclassic approach gives an asymptotic representation of the exact solution. This representation remains unchanged (there is no Stokes phenomenon) if we do not cross the Stokes line (the imaginary-phase line) on going around the point x_S. It is easy to see that in this case this condition is satisfied.

of the behavior of the solution in the vicinity of the resonant point, then in the investigation of oscillations of a real liquid it is possible to use the singular equation describing the oscillations of an ideal liquid. Then the resonant point must be circuited in accordance with Landau's instructions. Such an approach is apparently valid in all problems involving oscillations of moving continuous media, if the energy of the oscillations becomes dissipated in the vicinity of the resonant point. However, if there is no energy dissipation, as in the case of flute oscillations of a dense plasma (see Ch. V), then the use of the singular equation for finding the eigenfunctions is incorrect.

Using in this case the Landau circuiting rule, i.e., regarding the neutral oscillations as the limiting case of growing oscillations as $\mathrm{Im}\,\omega \to 0$ ($V_0(x_s) = \omega/k$, $\mathrm{Im}\,x_s = \mathrm{Im}\,\omega/k V_0'(x_s)$), we find that the point x_s should be circuited from below when $kV_0'(x_s) > 0$ and from above when $kV_0'(x_s) < 0$. As a result, a solution that decreases to the right from x_s (the region of nontransparency) goes over when $kV_0'(x_s) > 0$ in the region to the left of x_s into a wave traveling to the left, and when $kV_0'(x_s) < 0$ into a wave traveling to the right:

$$(x_s-x)^{1/4}\exp\{-\mathrm{sgn}\,[kV_0'(x_s)]\cdot 2iA^{1/2}(x_s-x)^{1/2}\} \rightleftarrows (x_s-x)^{1/4}\exp[-2A^{1/2}(x-x_s)^{1/2}]. \quad (1.10)$$

We recall that the time dependence of the perturbations is taken in the form $e^{-i\omega t}$.

Let us now consider the physical results of the use of (1.10). To this end, we obtain an expression for the propagation velocity of a wave packet made up of short-wave quasiclassical perturbations. This quantity, as is well known, determines the energy-transfer rate. In the quasiclassical approximation we obtain from (1.5) the following expression for the oscillation frequency as a function of the coordinate x and of the wave vector k_x:

$$\omega = kV_0(x) + [kV_0''(x)/(k_x^2+k^2)],$$

whence we get for the group velocity

$$d\omega/dk_x = -2k_x kV_0''/(k_x^2+k^2)^2.$$

In the vicinity of the resonant point x_s, where $k_x \gg k$, we get

$$k_x \approx \pm[kV_0''/(\omega - kV_0)]^{1/2} \approx \pm(A/|x-x_s|)^{1/2}.$$

This expression was used to obtain (1.8), since the phase of the exponential in (1.8) is

$$\varphi(x) = \int_{x_s}^{x} k_x(x)\,dx.$$

Relation (1.10) makes it possible to determine the sign of $d\omega/dk_x$ for perturbations localized in a liquid (decreasing in the nontransparency region). For such perturbations we get

$$d\omega/dk_x = 2\,\mathrm{sgn}\,(kV_0'(x_s))\,kV_0''(x)\,|x-x_s|^{3/2}A^{-3/2}. \quad (1.11)$$

It must be recalled now that (1.10) was derived for the case $A = V_0''(x_s)/V_0'(x_s) > 0$. Therefore the group velocity is positive at both signs of $kV_0'(x_0)$, and consequently the energy can be transferred only in the direction towards the resonant point. It follows from (1.11) that when the wave packet approaches the resonant point, its velocity decreases. Since there is no reflection from the resonant point, the energy of the perturbation accumulates in the vicinity of the resonant point, and no matter how low the viscosity of the liquid, this energy will be dissipated after a sufficiently large time interval.*

With the aid of (1.10) it is easy to show that (1.5) and (1.7) cannot have quasiclassical eigenfunctions. Indeed, such solutions should be localized in a potential well between the turning points x_s and x_0 (see Fig. 2). It is known that a solution decreasing beyond the usual turning point, at which $k_x(x_0) = 0$, goes over in the transparency region into a standing wave. This rule for joining together the solutions follows from the conservation of the energy upon reflection from an ordinary turning point. Indeed, the standing wave can be represented as a sum of an incident and a reflected wave with equal amplitudes. At the same time, according to (10), a solution that decreases beyong the singular turning point goes over into a traveling wave. Therefore the joining of solutions that decrease beyond the points x_0 and x_s is obviously impossible, and consequently Eqs. (1.5) and (1.7) have no quasiclassical eigenfunctions at $\mathrm{Im}\,\omega = 0$. Oscillations with $\mathrm{Im}\,\omega > 0$ must be regarded on the basis of the general rules for the construction of asymptotic solutions (see, for example,[9-11]). The proof of the absence of eigenfunctions with $\mathrm{Im}\,\omega > 0$ does not differ in practice in this case from the proof given above. At the same time, it turns out that damped oscillations cannot be considered within the framework of the approximation of an ideal liquid. Therefore the conclusion that there are no damped oscillations in flows with a velocity profile having no inflection points, which could be based on relation (1.6), is generally speaking incorrect. Incidentally, damped oscillations are usually of little interest.

Thus, in the particular example of shortwave quasiclassical oscillations, which is somewhat of a model, we have shown that in flows having a velocity profile without an inflection point there are no undamped natural oscillations (the Rayleigh theorem), because the oscillations are absorbed at the resonant points. It turns out (see, for example,[3] and also Appendix I) that the stability of such flows as the Poiseuille flow, flow of the boundary-layer type, and flow with a velocity profile having an inflection point, is also determined by phenomena that occur in the vicinity of the resonant points.

1.3. Evolution of Initial Perturbations

In the preceding section it was shown that during the first stage of the evolution perturbations with frequency ω and having a wave vector equal to k along Oy accumulate in the vicinity of the resonant point $x_s(\omega = kV_0(x_s))$. A more complete picture of the evolution can be obtained with the aid of Eq. (1.4). In[12-14]

*The propagation of electromagnetic oscillations in a plasma situated in an inhomogeneous magnetic field is described by an equation similar to (1.7) (see [4-6]) and also Sec. 5.2). In this problem, the use of the Landau circuiting rule, usually justified by the presence of small collisions (see, however, [45]), also leads to the conclusion that the oscillations incident on the resonant point are completely absorbed.

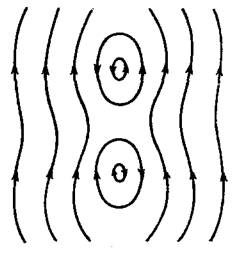

FIG. 3. Elementary flow perturbation corresponding to a single Van Kampen-Case wave.

they used for this purpose the Laplace-transform method:

$$\frac{d^2\varphi_{p,k}}{dx^2} - k^2\varphi_{p,k} - \frac{V_0''(x)}{V_0(x)-(ip/k)}\varphi_{p,k} = \frac{\Delta\varphi_k(x,0)}{p+ikV_0(x)}; \qquad (1.12)$$

here

$$\varphi_{p,k}(x) = \int_0^\infty dt e^{-pt}\varphi_k(x,t),$$

and $\Delta\varphi_k(x, 0)$ is the initial perturbation of the velocity curl.

The solution of (1.12) was determined with the aid of the Green's function $G_{p,k}(x, x_0)$:

$$\varphi_{p,k}(x) = \int_{x_1}^{x_2} dx_0 G_{p,k}(x,x_0)\frac{\Delta\varphi_k(x_0,0)}{p+ikV_0(x_0)}, \qquad (1.13)$$

$$G_{p,k}(x,x_0) = W_{p,k}^{-1}\begin{cases} g_{p,k}^{+}(x)\, g_{p,k}^{-}(x_0) & (x>x_0), \\ g_{p,k}^{-}(x)\, g_{p,k}^{+}(x_0) & (x<x_0); \end{cases} \qquad (1.14)$$

here $g_{p,k}^{\pm}(x)$ are the solutions of the homogeneous equation corresponding to (1.12) and satisfy the boundary conditions on the right (left) end of the interval (x_1, x_2); $W_{p,k} = W(g_{p,k}^{+}; g_{p,k}^{-})$ is the functional determinant*.

In order to determine the dependence of the perturbations on the time, we take the inverse Laplace transform

$$\varphi_k(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp e^{pt}\varphi_{p,k}(x). \qquad (1.15)$$

As is well known, the asymptotic form of expressions of the type (1.15) should be determined by the singularities of the integrand. If the homogeneous equations had eigenfunctions, i.e., functions satisfying the boundary conditions on both ends of the interval (x_1, x_2), then for the corresponding eigenvalues of the frequency ($\omega = ip$) the function $g_{p,k}^{\pm}$ would coincide, apart from a factor. The determinant $W_{p,k}$ would then vanish, and the Green's function would have a pole. In the preceding section it was shown that the homogeneous equation (1.5) has no eigenfunctions corresponding to undamped oscillations ($\mathrm{Re}\, p \geq 0$). In this case the main contribution to the asymptotic form is made by the zeroes of the resonant denominator $p + ikV_0(x)$. Substituting (1.3) in (1.15) and taking the residue at the point $p = -ikV_0(x_0)$, we obtain

$$\varphi_k(x,t) \approx \int_{x_1}^{x_2} dx_0 \Delta\varphi_k(x_0,0)\, e^{-ikV_0(x_0)t} G_{k,-ikV_0(x_0)}(x,x_0). \qquad (1.16)$$

This expression has a simple physical meaning, namely, it means that a perturbation modulated in the direction of the flow with a wave vector equal to k breaks up into individual jets that are carried along the flow with its local velocity $V_0(x_0)$ (Fig. 3). In the laboratory frame, the frequency of the perturbation connected with such a jet turns out to be $kV_0(x_0)$. To each individual jet of liquid there corresponds a local perturbation of the velocity curl $\Delta\varphi_k(x_0, 0) = \delta(x - x_0)$, and its influence on the flow is described by the Green's function $G_{k,-kV_0(x_0)}(x, x_0)^*$.

Thus, the Laplace-transform method gives the same result as the analysis of wave packets, namely, perturbations with frequency ω and wave vector along Oy equal to k are localized in the vicinity of the point $x_s(\omega = kV_0(x_s))$. The asymptotic expression (1.6) becomes valid when $t \gtrsim |kV_0(x_1) - kV_0(x_2)|^{-1}$. It is interesting to note that the same condition can be obtained from the quasiclassical formulas by using them at the limit of applicability and estimating the time of passage of the wave packet through the characteristic spatial scale.

In order to determine the further evolution of the perturbations, we integrate (1.6) twice by parts:

$$\varphi_k(x,t) \approx t^{-2} e^{-ikV_0(x)t} C_k(x,t); \qquad (1.17)$$

here

$$C_k(x,t) = -\int_{x_1}^{x_2} dx_0 e^{ikt[V_0(x)-V_0(x_0)]} \times \frac{d}{dx_0}\left\{\frac{1}{kV_0'(x_0)}\frac{d}{dx_0}\left[\frac{1}{kV_0'(x_0)}\Delta\varphi_k(x_0,0)\, G_{k,-ikV_0(x_0)}(x,x_0)\right]\right\}.$$

The asymptotic form of $C_k(x, t)$ as $t \to \infty$ is determined by the singularities of the integrand. Since $G_{k,-ikV_0(x)}(x, x_0)$ is a Green's function of a second-order equation, its second derivative should have a singularity of the type $G''(x, x_0) \approx \delta(x - x_0)$ as $x \to x_0$. Therefore we have approximately $C_k(x, t) \approx -(kV_0'(x))^{-2}\Delta\varphi_0(x)$, and consequently $\varphi_k(x, t)$ should attenuate with time like t^{-2}.

The damping $\varphi_k(x, t)$ is brought about by the circumstance that the dimensions of the region that contributes to $\varphi_k(x, t)$ contract with increasing time. Indeed, if the elementary perturbations (jets) lie at a characteristic distance Δx from the point x, then at $t \gg (kV_0'\Delta x)^{-1}$ their phases shift by an amount much larger than π, and therefore the joint contribution of the perturbations as a result of the interference will tend to zero.

A more accurate calculation of the asymptotic form[14] shows that allowance for the singularity in the left-hand side of (1.12) can lead to a change of $C_k(x, t)$ in (1.17), so that the latter acquires, in addition to the constant term, also a term that depends on the time logarithmically: $C_k(x, t) \approx C_{1k}(x) + C_{2k}(x) \ln t$. The latter term takes into account the resonant response of the flow at the point x to the sudden occurrence of an elementary perturbation at the point $x_0 \neq x$. We recall that when the Laplace-transform method is used it is assumed that the perturbations occur instan-

*When determining the Green's functions, the singularity in the left-hand side of (1.12) turns out to be negligible (see below); therefore, to simplify the derivation, we disregard it.

*These perturbations were first considered by Case [12] and are analogous to the waves introduced by Van Kampen in the investigation of oscillations of a hot plasma at rest [15] (see also the end of the present section); they are therefore sometimes called Van Kampen-Case waves.

taneously at $t = 0$. However, if a periodic perturbation is suddenly turned on at the point x_0, there is excited besides the frequency $\omega = kV_0(x_0)$, also the entire spectrum of frequencies from zero to ∞, including the frequency $\omega = kV_0(x)$, although the perturbation with $\omega = kV_0(x_0)$ enters in the spectrum with the largest weight.

We now take into account the influence of the viscosity on the asymptotic form of $\varphi_k(x, t)$. The equation of motion of a viscous incompressible liquid can be reduced to the form (compare with (1.3))

$$\left(\frac{d}{dt} - \nu\Delta_{x,y}\right)(\operatorname{rot} \mathbf{V})_z = 0; \tag{1.18}$$

here ν is the kinematic viscosity coefficient.

It follows from (1.18) that under the influence of the viscosity the initial perturbation of the velocity curl spreads within a time Δt over a distance $\Delta x \approx (\nu\Delta t)^{1/2}$. In view of the fact that the perturbations located at different points along x move with different velocities along Oy, their phases are shifted relative to one another. Thus, for example, after a time Δt, the phase shift of the perturbations located at a distance Δx turns out to be $\Delta\varphi \approx kV_0' \cdot \Delta x \cdot \Delta t$. Substituting here $\Delta x \approx (\nu\Delta t)^{1/2}$, we find that within a time $\Delta t \gtrsim (kV_0')^{-2/3}\nu^{-1/3}$ the action of the viscosity leads to a *smearing of the most elementary perturbations*, since perturbations with a phase that changes by more than π appear at the same point.

It is interesting to compare the problem considered here with the problem of the evolution of a perturbation in a hot plasma at rest, considered by Van Kampen[15]. He indicated that in order to trace the fate of an arbitrary initial perturbation, it was necessary, besides the ordinary plasma oscillations, to introduce a new class of elementary excitations. These perturbations are freely moving beams of charged particles with a density modulated in the direction of motion. If the beam velocity is equal to v and the modulation wavelength is $\lambda = 2\pi/k$, then such a beam in the laboratory frame will give rise to a wave with frequency $\omega = kv$. The influence of the Coulomb collisions on the Van Kampen waves was considered in[16], where the following damping time was obtained:

$$t \approx \nu^{-1/3}(kV_0)^{-2/3}.$$

This expression is similar to that obtained by us for the damping of perturbations in a viscous liquid. If we assume, as is done in[16], that Coulomb collisions lead to diffusion in velocity space, then we can visualize for the damping a qualitative picture that is perfectly analogous to that given by us above.

2. FLUTE OSCILLATIONS OF A RAREFIED UNCOMPENSATED PLASMA IN A MAGNETIC FIELD

2.1. Oscillations of a Gas of Particles with Charges of the Same Sign

It was shown in the preceding chapter that oscillations of flows of an incompressible liquid are determined to a considerable degree by the physical processes that act in the vicinity of the resonant points, at which the phase velocity of the oscillations coincides with the flow velocity. The role of resonant processes may also be no less important in the case of oscillations of a moving plasma To this end it is necessary that the plasma velocity change sufficiently steeply in a direction transverse to the motion. In this case oscillations with phase velocity changes in a wide range will resonate with the motion of the plasma.

An examination of oscillations in a moving plasma is best started with flute oscillations of a rarefied uncompensated plasma in a magnetic field. In such a plasma, the fact that the space charge is not compensated gives rise to electric fields that cause the plasma to drift with a velocity $\mathbf{V} = c[\mathbf{H}\nabla\varphi]/H^2$ (here $\mathbf{H}$ is the magnetic field and φ the electric potential). Our problem is to investigate the influence exerted by such a motion on the plasma oscillations.

It is useful first to simplify the problem and to consider the lack of compensation to be maximal, assuming the density of one of the plasma components (electrons or ions) to be equal to zero. We assume also that the magnetic field is homogeneous and the density of the charged particles along the magnetic field is constant. In such a system, greatest interest attaches to oscillations in which the charged particles shift across the magnetic field and accordingly only transverse electric fields are excited. These oscillations are called flute oscillations, since during the course of their development the equal-density surfaces forms grooves elongated along the magnetic field. An investigation of flute oscillations of a gas of charged particles in a magnetic field is of interest principally in connection with the problem of the stability of the cloud of electrons (ions) in a magnetron and similar systems.

Let us assume that the density of the charged particles n_j ($j = e, i$) is small enough so that the plasma frequency $\omega_{pj} = (4\pi e_j^2 n_j/m_j)^{1/2}$ is much smaller than the cyclotron frequency $\omega_j = |e_j| H/m_j c$. In this case the frequency of the flute oscillations turns out to be small compared with the cyclotron frequency, and therefore we can disregard the inertia of the particles in the analysis of the latter. The low-frequency flute oscillations are described by Eq. (1.3) (see above) in which the stream function is replaced by $c\varphi/H$[7,17,18]. This can be readily verified by expressing in the continuity equation

$$\frac{\partial n_j}{\partial t} + \operatorname{div} n_j \mathbf{V}_j = 0 \tag{2.1}$$

the density of the charged particles in terms of the electric potential from the Poisson equation

$$\Delta\varphi = -4\pi e_j n_j$$

and by taking into account the equation $\operatorname{div} \mathbf{V}_j = \operatorname{div}(cH^{-2}\mathbf{H} \times \nabla\varphi) = 0$. It is interesting to note that the curl of the velocity determines the space-charge density: $(\operatorname{curl} \mathbf{V}_j)_H = c\Delta\varphi/H = -4\pi c e_j n_j/H$ (here $(\operatorname{curl} \mathbf{V}_j)_H$ is the projection of the velocity curl on the direction of the magnetic field). We recall that in flute oscillations the charged particles move in a plane perpendicular to the magnetic field, so that the condition $\operatorname{curl} \mathbf{V}_j \parallel \mathbf{H}$ is satisfied.

Let us assume that in the initial state the density of the charged particles changes in one direction. We introduce a rectangular coordinate system, directing

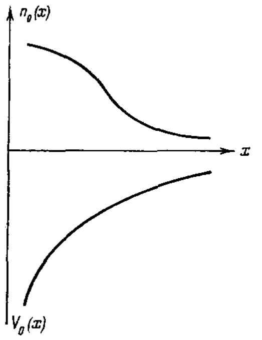

FIG. 4. Density profile (and velocity profile) of a gas of charged particles for which the Rayleigh theorem is satisfied.

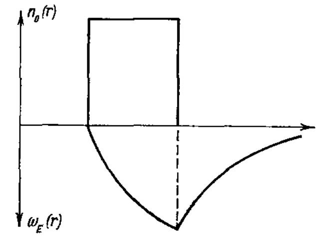

FIG. 5. Density (and velocity) profile of rotation of a gas of charged particles, in which "diokotron" instability builds up.

the Ox axis along the density gradient and the Oz axis along the magnetic field. The initial electric field is parallel to the density gradient of the charged particles, and consequently the particles drifting in the crossed fields will move along Oy. Linearizing (1.3) with respect to small perturbations of the electric potential, we arrive at the equations (1.4) and (1.5).

The system whose oscillations are described by (1.5) is stable: oscillations with $\operatorname{Im}\omega \geq 0$ are impossible in it if the velocity profile $V_{0y}(x)$ has no inflection points (Rayleigh theorem). From the unperturbed Poisson equation, written in the form

$$(c/H)\,d^2\varphi_0/dx^2 = dV_{0j}/dx = -4\pi\frac{c}{H}e_j n_j,$$

it follows that the inflection points in the velocity profile are simultaneously extremal points for the density of the charged particles. Therefore for flute oscillations the Rayleigh theorem can be reformulated as follows: the gas of charged particles in a magnetic field is stable against flute oscillations if the density of the charged particles changes monotonically[7,18] (see, for example, Fig. 4).

The analogy between flute oscillations and oscillations in the flow of an incompressible liquid makes it possible to conclude that the perturbations of the electric potential will evolve in accordance with (1.17): $\varphi_k(x, t) \approx t^{-2}e^{-ikV_0(x)t}C(x)$*). In an incompressible liquid, the action of the viscosity after a sufficiently large time interval would lead to a more rapid damping. A similar influence is exerted on the flute oscillations by effects connected with the finite size of the Larmor radius of the charged particles[19]. These effects must also be taken into account when establishing the rules of circuiting around the resonant points in (1.5) (see[18]). It turns out that the mechanism of collisionless Landau absorption acts in a rarefied plasma in the vicinity of the resonant points. In this region, the charged particles move with a velocity close to the phase velocity of the wave. The field of the wave for such particles is constant in time, and therefore the wave can perform work on the field and give up its energy. Just as in the case of an incompressible liquid, this absorption can be taken into account within the framework of Eq. (1.5), if the resonant point is circuited during the construction of the solution in the complex plane after Landau[18].

*In this case the role of the elementary perturbations (Van Kampen—Case waves) is played by plane layers of charged particles parallel to yOz, with a density modulated along Oy.

In concluding this section, let us make a few remarks. Real systems such as a magnetron have, as a rule, axial symmetry. The magnetic field in them is parallel to the symmetry axis and the electric field is radial, while the charged particles drift in azimuth. It is shown in[18] that the formulation of the Rayleigh theorem, as given above, remains in force also for axially-symmetrical systems, namely, such systems are stable against flute oscillations if the density of the charged particles varies monotonically along the radius.

Usually axially-symmetrical systems are enclosed by metallic shells with specified potentials, on which the perturbation of the potential vanishes. It turns out (see [20]) that even in the case when the density has a monotonic profile there are no undamped natural oscillations ($\operatorname{Im}\omega = 0$) with an azimuthal wave number $m = 1$, for which the resonance condition $\omega = \omega_E(r)$ is satisfied on the surface of the shell (here $\omega_E = (c/Hr)\,d\varphi_0/dr$ is the angular velocity of the drift in crossed fields). These oscillations are similar to the oscillations of the Poiseuille flow (see Appendix 1.3) which also resonate with motion at the boundary itself and build up under the influence of viscosity. It is possible that in the case of flute oscillations a similar result is produced by the influence of effects of the finite Larmor radius.

If the plasma is separated from the shell by a vacuum region, then neutral oscillations ($\operatorname{Im}\omega = 0$) with $m > 1$ become possible. It is interesting to note that the resonance condition, which in the case of axially-symmetrical systems takes the form $\omega/m = \omega_E(r)$, must be satisfied for such oscillations in a region free of plasma[18].

If the density of the charged particles does not have a monotonic variation, then such a system is unstable, and this instability is perfectly analogous to the instability of flows having an inflection point in the velocity profile[3]. An example of such an instability for systems with planar symmetry was considered in[21]. In the case of axially-symmetrical systems, most investigations were devoted to the instability of the distributions of the density of the charged particles in the form of a step: $n(r) = n_0$ $(r_1 < r < r_2)$, $n(r) = 0$ $(0 < r < r_1,\ r > r_2)$ (Fig. 5). This was called "diokotron" instability (see, for example,[17,22]). Flute oscillations at relatively large charged-particle density ($\omega_{pj} \approx \omega_j$) were considered in[23,24]. The stability of the state in which all the particles move with the same value of the generalized momentum was investigated (in axially-symmetrical systems such a state is called the Brillouin state). In both investigations, it was concluded that the system is unstable. We note that in[23] they investigated, apparently for the first time, the

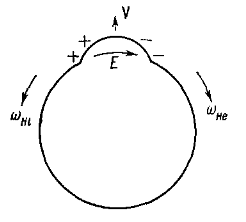

FIG. 6

stability of plasma flow with a velocity that varies in the transverse direction, i.e., flow in which the plasma layers slip relative to one another. The instability observed in[23] is therefore called slipping instability.

2.2. Oscillations of a Rarefied Plasma

The investigation of flute oscillations is also of interest in connection with the problem of the stability of a hot (thermonuclear) plasma in magnetic traps. Magnetic traps are axially symmetrical systems with a magnetic field parallel to the axis. The intensity of the magnetic field increases towards the end of the system, so that the so-called magnetic mirrors are produced at the ends and prevent the charged particles from leaving the trap. Simultaneously, the inhomogeneity of the magnetic field leads to a drift of the particles in azimuth, the electrons and ions drifting in opposite directions with angular velocity

$$\omega_{Hj} = \frac{c}{H}\frac{T_j}{e_j}\frac{1}{r}\frac{dH}{dr},$$

where e_j is the charge of the particle of type j (j = e, i) and T_j is the temperature.

The self-consistent flute oscillations considered in the preceding section were connected with displacements of the space charge in the field of the wave. This displacement excited, in turn, alternating electric fields. If the magnetic field is inhomogeneous, then the flute oscillations can be maintained even in a neutral plasma. Indeed, let us assume that a flute elongated along the magnetic field was produced on a plasma cylinder (Fig. 6 shows the projection on a plane perpendicular to the magnetic field). The electric charges, drifting in the inhomogeneous magnetic field in opposite directions, separate. This results in electric fields that lead to further motion of the plasma with velocity $\mathbf{V} = c\mathbf{H} \times \nabla\varphi / H^2$.

The flute oscillations of a neutral plasma become unstable at a density larger than critical, which is determined from the condition $r_{di}^2 \approx a_n a_H$, where $r_{di} = (T_i/m_i)^{1/2}\omega_{pi}^{-1}$ is the average Debye radius of the ions, a_p and a_H are the characteristic scales of variation of the density of the plasma and of the magnetic field, respectively[25]. In the experiment, this instability leads to appreciable losses of particles from the magnetic traps. At the same time it was noted that if the density of the space charge is sufficiently large ($[(n_{0i} - n_{0e})/n_0]^2 \gtrsim r_{di}^2/a_n a_H$), then the stability of the plasma improves[26] (see also [27]). A theoretical analysis shows that when this condition is satisfied, the effect of displacement of the initial charge in the field of the oscillations prevails over the effect of charge separation.* On the other hand, in the limiting case when $[(n_{0i} - n_{0e})/n_0]^2 \gg r_{di}^2/a_n a_H$, the flute oscillations of a rarefied plasma apparently do not differ in any way from the oscillations of a gas of charged particles, considered in the preceding section. Therefore the Rayleigh theorem should hold also for the flute oscillations of a rarefied plasma, according to which the plasma is stable if the space-charge density varies montonically[7,18]. The influence of decompensation on the flute oscillations of a rarefied plasma was considered also in[27-33]. In most investigations, the authors reach the conclusion that the space charge exerts a stabilizing action. They used, however, methods developed for the investigation of oscillations of a quiescent plasma, which do not make it possible to consider the characteristic (resonant) phenomena resulting from the plasma drift. Therefore, although the stabilization conditions obtained in some of them[27,28] are of the correct order of magnitude, the region of applicability of these results is limited to small values of the space-charge density, when the plasma drift velocity is much smaller than the phase velocity of the oscillations.

In conclusion, we emphasize that in this section we have confined ourselves to the case of a rarefied plasma with $\omega_{pi} \ll \omega_i$. A correct examination of the influence of electric fields on flute oscillations of a denser plasma encounters considerable mathematical difficulties and has therefore not yet been performed.

3. OSCILLATIONS OF AN ELECTRON BEAM IN A LONGITUDINAL MAGNETIC FIELD

The example of the flute oscillations shows that the motion of a plasma can be a stabilizing factor, and that the stabilization, just as in the case of an ordinary liquid, is due to the absorption of the oscillations at the resonant points. Consequently, the statement that plasma and liquid flows with very simple velocity profiles (such as profiles without inflection points) can be called the generalized Rayleigh theorem.† The proof of this statement for flute oscillations (see the preceding section) was made easy by the similarity between the behavior of an ideal liquid and the behavior of a rarefied gas of charged particles in the magnetic field. We shall show in the present chapter that the generalized Rayleigh theorem is also valid for oscillations of an electron beam in a strong longitudinal magnetic field. The oscillations of such a system constitute the simplest but also most characteristic example of oscillations in a moving plasma.

We introduce a rectangular coordinate system whose Oz axis is parallel to the direction of motion of the electrons, and whose Ox axis is parallel to the direction in which the electron velocity changes (Fig. 7). We assume that a strong magnetic field is directed along Oz and prevents the transverse displacements

*On the other hand, this condition means that the drift velocity in crossed fields exceeds the phase velocity of the flute oscillations.

†For stabilization, as a rule, it is necessary to have sufficiently large velocity gradients, in order that the change of the velocity within the limits of the system greatly exceed the velocity of the unstable oscillations in the plasma at rest.

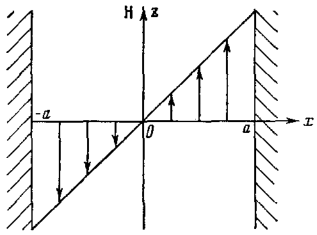

FIG. 7

of the electrons without at the same time influencing the longitudinal displacements. The presence of the magnetic field greatly simplifies the problem. The electron density will be assumed constant. We assume that the electron velocity varies linearly, $V_{0z}(x) = V_0 x/a$ $(-a \le x \le a)$, and that at $x = \pm a$ the system is bounded by metallic surfaces on which the perturbation of the electron potential vanishes.

The oscillations of such a system should be described with the aid of the Poisson equation, the continuity equation (see (2.1)), and the z-component of the equation of motion:

$$\frac{dV_{ze}}{dt} = \frac{e}{m}\frac{\partial\varphi}{\partial z}. \qquad (3.1)$$

From these equations, linearized with respect to small perturbations, we find that the self-consistent perturbations of the potential φ_1 should satisfy the equation

$$\frac{d^2\varphi_1}{dx^2} + \left[\frac{\omega_p^2}{(\omega - k_z V_0(x))^2} k_z^2 - k_z^2 - k_y^2\right]\varphi_1 = 0. \qquad (3.2)$$

By virtue of the stationary character of the system and its homogeneity with respect to y and z, the potential perturbations were chosen in the form $\varphi_1(\mathbf{r}, t) = \varphi_1(x) e^{-i\omega t + ik_y y + ik_z z}$. The main difference between the equations investigated in the preceding sections and (3.2) is that the latter has a second-order pole at the resonant point.

If the electrons are at rest ($V_0 = 0$), then the solutions of (3.2) are of the form $\varphi_1(x) = C\sin(k_x x)$, where $k_x^2 = k_z^2[(\omega_p^2/\omega^2) - 1] - k_y^2$. In order to satisfy the boundary conditions $\varphi_1(\pm a) = 0$, we put $k_x = n\pi/a$. This condition determines the spectrum of the frequencies of the natural oscillations $\omega = \omega_{pe} k_z/k$, or otherwise $\omega = \omega_{pe}\cos\theta$, where $\cos\theta = k_z/k$, $k^2 = k_z^2 + k_y^2 + (n\pi/a)^2$. We recall that in a homogeneous and isotropic plasma the frequency of the Langmuir oscillations is ω_{pe}. Thus, the influence of the magnetic field reduces to a decrease of the frequency of the oscillations propagating at an angle to the magnetic field.

If $V_0 \ne 0$ ($V_0(x) = V_0 x/a$), then the solutions of (3.1) are $\varphi_1(x) = C_1 x_1^{1/2} J_\nu(x_1) + C_2 x_1^{1/2} J_{-\nu}(x_1)$, where $x_1 = ik_1[x - (\omega a/k_z V_0)]$, $k_1^2 = k_z^2 + k_y^2$, $\nu^2 = (1/4) - (\omega_{pe}^2 a^2/V_0^2)$, and $C_{1,2}$ are arbitrary constants.

The frequencies of the natural oscillations are also determined by the boundary conditions $\varphi_1(\pm a) = 0$. A study of these conditions shows that at any value of the velocity gradient $dV_0/dx = V_0/a$ the oscillations are stable, whereas if $V_0/a > 2\omega_{pe}$ there are no undamped oscillations (with $\mathrm{Im}\,\omega \ge 0$)[34]*. That is to say, the situation realized in this case is the same as in the flow of an incompressible liquid with a velocity profile without inflection points (see Ch. 1).

The oscillations of the flow in question were considered also in[35], where it was concluded that this flow was unstable if the condition $V_0/a > 2\omega_{pe}$ is satisfied. In[34], however, it is shown that this result is due to mathematical errors made in[35].

Equation (3.2) has a singularity at the point x_s, where the phase velocity of the wave with respect to Oz coincides with the stream velocity $V_0(x_s) = \omega/k$. In the present case, just as in the case of flute oscillations, the Landau mechanism of collisionless absorption is effective at the resonant points, and leads to the absorption of the oscillation energy. A correct examination of this process calls for allowance for the thermal scatter of the electron velocities. However, if, as before, we are not interested in details of the behavior of the solution in the vicinity of the resonant point, we can use the simplified Eq. (3.2), supplemented by the Landau rule for circuiting around the resonant point (see [34]).

When the condition $V_0/a > 2\omega_{pe}$ is satisfied, the natural oscillations do not form a complete system, since it is impossible to expand the potential perturbation, which depends arbitrarily on x, in terms of these oscillations. For this case, the evolution of the initial perturbations in time was considered directly in[34] by the Laplace-transform method. It turned out that the perturbations attenuate like t^α, with the exponent α depending on whether it is the plasma density or its velocity that is perturbed at t = 0. In the former case $\alpha = \nu - (3/2)$, and in the latter $\alpha = 2\nu - 1$ (here $\nu = [(1/4) - (\omega_p^2 a^2/V_0^2)]^{1/2}$). The contribution made to the temporal asymptotic expression is due in part to beams of electrons with a density that is modulated in the direction of motion, and in part to the resonant response of the flow to the occurrence of the perturbations (see Sec. 1.3). After a sufficiently large time ($t \gtrsim (k_z v_T)^{-1}$), when the effects of thermal velocity scatter become appreciable, the perturbations attenuate more rapidly (compare with attenuation of perturbations in flow of an incompressible liquid under the influence of viscosity).

Such an analysis is valid for a sufficiently strong magnetic field, when $k_y V_0/k_z a\omega_e \ll 1$, where ω_e is the electron cyclotron frequency. If this condition is not satisfied, as frequently occurs in the experiment, it is necessary to take into account the transverse displacements of the electrons in the oscillations. This case was investigated in[36,38,39] and also by V. M. Kostin (thesis). The general result of these investigations is as follows: the flow is stable if the condition $\omega_{pe}^2 a k_y/V_0\omega_e k_z \gtrsim 1$. is satisfied. It follows from this criterion that at a sufficiently large velocity gradient the flow still remains stable.

*More accurately speaking, even when the condition $V_0/a > 2\omega_{pe}$ is satisfied, there remain for each value of k_z two natural oscillations that resonate with the stream at the metallic walls ($\omega/k_z = \pm V_0$, $\mathrm{Im}\,\omega = 0$). These oscillations are analogous to the first mode of the flute oscillations in axially-symmetrical systems (see Sec. 2.1), and also to oscillations of Poiseuille flow (see Appendix I.3).

Oscillations of the electron beam in the absence of a magnetic field were considered in[37,38]. It was shown that at sufficiently small velocity gradient ($k_z V_0(x) \ll \omega_{pe}$), when the effects of resonant interaction of the oscillations with the flow are negligible, the oscillations are unstable. The case of a large velocity gradient has not yet been considered to date.

4. FLUTE OSCILLATIONS OF A QUASINEUTRAL PLASMA

In the problems considered in the preceding chapters, the resonance of the oscillations with the motion of the continuous medium (plasma or liquid) led to the absorption of oscillation energy. In an inhomogeneous plasma in a magnetic field, a special type of motion appears, the so-called Larmor or gradient drift, resonance with which should lead to a different result. The Larmor drift is possible even in a homogeneous magnetic field, when each charged particle rotates on a Larmor circle whose center is at rest. However, if the density of the Larmor centers or the average energy of the Larmor rotation vary across the magnetic field, a macroscopic motion sets in with a velocity $V_j = (c/n_j e_j H^2) H \times \nabla p_j]$, where p_j is the pressure of the particles of type j(e, i) (see, for example,[40,41]). This expression for the velocity can be obtained from the hydrodynamic equation of motion

$$-\nabla p_j + (en_j/c)\,[\mathbf{V}_j\mathbf{H}] = 0. \qquad (4.1)^*$$

If the wave resonates with such a motion, then its phase for each individual particle varies with time, and therefore the mechanism of collisionless Landau absorption is no longer effective. Special interest therefore attaches to the influence of the resonances with the Larmor drift on the oscillations of not too dense a plasma, when there is likewise no collision dissipation.

It turns out that if the condition $\omega_{pi} \gg \omega_i$ is satisfied, then the effects connected with the Larmor drift of the ions must be taken into account in the analysis of the flute oscillations. In this case the equation for the flute oscillations has the following form (see, for example,[42]):

$$\frac{d}{dx} n_0 [\omega - kV_{\text{Л}i}(x)] \frac{d\varphi_1}{dx} - k^2 n_0 [\omega - kV_{\text{Л}i}(x)]\, \varphi_1 = 0. \qquad (4.2)$$

We use the same system of coordinates as in Ch. 2 (the Oz axis is parallel to the magnetic field, and the Ox axis is directed along the density gradient); the perturbations of the electric potential are chosen in the form $\varphi_1(x) \exp[-i\omega t + iky]$. We recall that in Ch. 2 we considered a low-density plasma ($\omega_{pj} \ll \omega_j$) with an uncompensated space charge ($n_{0e} \neq n_{0i}$). Now we assume, on the other hand, that in the initial state the plasma is neutral and that its density is sufficiently high ($\omega_{pj} \gg \omega_j$). In this case the perturbations of the charged-particle density also cancel each other in the zeroth approximation in $(\omega_i/\omega_{pi})^2$. Such a plasma is called quasineutral. We shall assume also that the magnetic field is homogeneous. The latter assumption is valid if the condition

$$(kV_{\text{Л}i})^2 \gg \frac{1}{H}\frac{dH}{dx}\frac{1}{nm_i}\frac{dp_i}{dx}.$$

*$[\mathbf{V}_j\mathbf{H}] \equiv \mathbf{V}_j \times \mathbf{H}$.

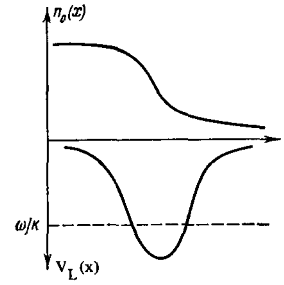

FIG. 8

is satisfied. The influence of the Coulomb collisions can be disregarded if $kV_{Li} \gg e^4 n_0/m_j^{1/2} T_j^{3/2}$. A plasma-density distribution typical of experimental conditions is shown in Fig. 8. The temperature (average energy) is assumed for simplicity to be independent of the coordinates.

In the region where n_0 = const, Eq. (4.2) takes the form $d^2\varphi_1/dx^2) - k^2\varphi_1 = 0$. Its solutions are the functions $\varphi_1(x) = \exp(\pm kx)$. Therefore the eigenfunctions of (4.2), if they do exist, should be localized in the region where $dp_i/dx \neq 0$, and should decrease exponentially outside this region, $\varphi_1 \sim \exp(-k|x|)$. Let us assume that such a solution $\tilde{\varphi}_1(x)$ exists. We multiply (4.2) by $\tilde{\varphi}_1^*(x)$ and integrate by parts:

$$\int_{-\infty}^{\infty} dx\, n_0(x)\,(\omega - kV_{\text{Л}i}(x)) \left(\left|\frac{d\tilde{\varphi}_1}{dx}\right|^2 + k^2 |\tilde{\varphi}_1|\right) dx = 0. \qquad (4.3)$$

It follows from (4.3) that the imaginary part of the frequency of the natural oscillations should vanish and that at some point x_s there should be satisfied the resonant condition $\omega/k = V_{Li}(x)$.

Eq. (4.2) was obtained by expansion with respect to a dimensionless parameter ($r_{Li}\nabla$), and takes into account the first two terms of the expansion (the wavelength of the oscillations is assumed large compared with the Larmor radius). However, in the vicinity of the resonant point the higher derivatives of $\varphi_1(x)$ increase very rapidly: $\varphi_1 \sim \ln(x - x_s)$, $d^n\varphi_1/dx^n \sim (x - x_s)^{-n}$. In this region it is necessary to take into account the next terms of the expansion. If we take into account only the first correction, then, for an ion velocity distribution (perpendicular to the magnetic field) close to a δ function $f_0(v_\perp, x) \approx n_0(x) 1/2\pi\nu_\perp\, \delta(\nu_\perp - \nu_{\perp 0})$), then the equation for the flute oscillations takes the form[42]

$$\frac{p_{0i}}{m_i\omega_i^2}(\omega - k\tilde{V}_{\text{Л}i}(x))\frac{d^4\varphi_1}{dx^4} + \frac{d}{dx} n_0 (\omega - kV_{\text{Л}i}(x))\frac{d\varphi_1}{dx} - k^2 n_0 (\omega - kV_{\text{Л}i}(x))\, \varphi_1 = 0, \qquad (4.4)$$

where

$$\tilde{V}_{\text{Л}i}(x) = -\frac{c}{Hep_{0i}}\frac{d\pi_{0i}}{dx}, \quad \pi_{0i} = 2\pi m_i \int_0^{\infty} dv_\perp v_\perp^5 f_0(v_\perp, x).$$

Equation (4.4) recalls the equation for the oscillations of an incompressible liquid at large Reynolds numbers, where the small-scale effects due to the viscosity also turned out to be significant in the vicinity of the resonant point. The viscosity has led to the absorption of the oscillation energy, a factor accounted for within the framework of the approximation of the ideal hydrodynamics by using the Landau rule for circuiting around the resonant points. In the case of resonance with the Larmor drift, as already noted above, the energy of the oscillations cannot be absorbed. Therefore the Landau rule cannot be used and allowance for the small-scale effects of a finite Larmor radius leads to a change in the character of the solution at the resonant point x_s, namely, a solution that varies slowly in space on one side of x_s, which is well described within the framework of Eq. (4.2), becomes small-scale on the other side of the point x_s, and must be analyzed with the aid of Eq. (4.4).

It was shown in[42] that in the simplest cases Eq. (4.4) has no eigenfunctions. Therefore, just as in the analogous situation in Ch. 1, the problem of the evolution of the initial perturbations was considered directly in[42] with the aid of the Laplace-transform method. The ion velocity distribution was assumed Maxwellian. The effects of the finite Larmor radius were taken into account completely. It was found that when $k(dV_{Li}/dx)at \gg 1$ (here a is the characteristic scale of the variation of the Larmor drift velocity), the asymptotic form of the perturbation periodic in $y(\varphi_1 \sim e^{iky})$ takes the form

$$\varphi_k(x, t) \approx [1 - I_0(\zeta) e^{-\zeta}]^{-1} e^{-ikV_{Li}(x)t}, \tag{4.5}$$

where I_0 is the Bessel function of imaginary argument, $\zeta = k(dV_{Li}/dx)r_{Li}t$. If the time is not too large, $\zeta \ll 1$ ($I_0 e^{-\zeta} \approx 1 - \zeta$), then $\varphi_k \approx t^{-2}\exp[-ikV_{Li}t]$. We recall that in an incompressible liquid the perturbations attenuated in accordance with the same law (see Sec. 1.2). This similarity is not accidental but is connected with the fact that when $\zeta \ll 1$, when the effects of the finite Larmor radius are negligible, it is possible to describe the flute oscillations by means of the simplified Eq. (4.2), which has the same structure as Eq. (1.5), which describes the oscillations of an incompressible liquid (both are second-order equations and their solutions have a logarithmic singularity).

When $\zeta \gg 1$ ($I_0 e^{-\zeta} \approx (2\pi\zeta)^{1/2} \ll 1$), the amplitude $\varphi_k(x, t)$ ceases to depend on the time. In this case the perturbation breaks up into an aggregate of modulated beams that do not interact with one another and are transported along the Oy axis with the local velocity of the Larmor drift. This means that the elementary excitation, which depends on the time like $\exp[-ikV_{Li}(x_1)t]$, should include a singular part of the type $\varphi_k(x) \sim \delta(x - x_1)$. This is confirmed by an analysis of the spatial dependence of the perturbations.

The problem of flute oscillations was considered, from a different point of view, in a number of other papers (see, for example,[43,44]), where principal attention was paid to establishment of the conditions of stabilization of the oscillations on going from a rarefied plasma (see Sec. 2.2) to a quasineutral one. It was assumed that the space charge is compensated. The influence of the resonance effects on the flute oscillations was investigated in[45]. The principal means of eliminating the singularity in the corresponding differential equation was proposed to be a transition to complex values of the frequency with $\operatorname{Im} \omega > 0$, which is essentially equivalent to the use of the Landau circuiting rule. We see that in our case this would be incorrect.

5. RESONANT PHENOMENA IN A PLASMA AT REST

5.1. Cyclotron Resonance in an Inhomogeneous Magnetic Field

A plasma, and all the more a plasma in a magnetic field, may exhibit qualitatively new properties compared with an ordinary liquid. In particular, in a plasma the oscillations may resonate with the motion of the charged particles even in the case when the plasma is at rest. By way of the simplest example, let us consider the propagation of electromagnetic oscillations with circular polarization, and with frequency close to the electron cyclotron frequency, along an inhomogeneous magnetic field. If the motion of the electrons in the field of the wave is described with the aid of the hydrodynamic equations, then it is easy to obtain for the amplitude $E(z)$ of the electric field the following equation (see, for example[4-6]):

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2} E - \frac{\omega_{pe}^2 \omega}{c^2 [\omega - \omega_e(z)]} E = 0; \tag{5.1}$$

where the Oz axis is directed along the magnetic field. We consider oscillations whose electric vector rotates in the same direction as electrons in the magnetic field: $\mathbf{E} = (1/\sqrt{2})\{E(z); iE(z); 0\}e^{-i\omega t}$.

At the point z_s, where the electron cyclotron frequency coincides with the oscillation frequency, Eq. (5.1) has a singularity. For its elimination it is necessary to consider the resonant interaction of oscillations with motion of electrons rotating on Larmor circles[46-49]. It turns out that this interaction leads to the absorption of the energy of the oscillations and that its influence on the oscillations can be taken into account within the framework of the simplified Eq. (5.1), if the resonant point is circuited in the complex plane in accordance with the Landau rule[46]. Then (5.1) becomes close to Eq. (1.5) (see Ch. 1). If, by analogy with the analysis given in Ch. 1, we treat (5.1) as a Schrödinger equation, then it differs from (1.5) only in the sign of the "total energy" (Fig. 9). This difference makes possible the existence of oscillations that propagate freely as $z \to \pm\infty$.

Let us consider a wave radiated at $z = -\infty$ and incident on the resonant point from the left. At $\omega_{pe}^2 \gg c\,|d\omega_e/dz|$, when it is legitimate to use the quasiclassical approximation, an analysis analogous to that given in Sec. 1.2 leads to the conclusion that the oscillations are completely absorbed[4-6,46,48]. If the plasma density is small, $\omega_{pe}^2 \ll c\,|d\omega_e/dz|$, so that the plasma exerts a small action on the propagation of the oscillations, then the absorption coefficient

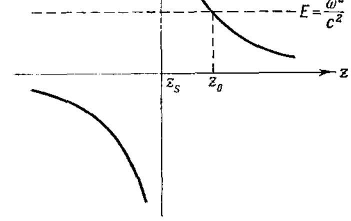

FIG. 9. Effective potential U(z) for Eq. (5.1). z_s and z_0 are the singular and ordinary turning points.

η is also small: $\eta \approx \pi\omega_{pe}^2/c\,|\,d\omega_e/dz\,|$ (see [48,49]). The general expression for the absorption coefficient, covering both limiting cases, is $\eta = 1 - \exp[-\pi\omega_{pe}^2/c\,|\,d\omega_e/dz\,|]$. It is interesting to note that for a wave propagating with $z = -\infty$, the absorption coefficient has a smaller value, $\eta' = \exp[-\pi\omega_{pe}^2/c\,|\,d\omega_e/dz\,|]\eta$ [4]. Indeed, such a wave first runs up to the usual turning point, from which it is partly reflected. Therefore only a certain fraction of the wave energy, on the order of $\exp[-\pi\omega_{pe}^2/c\,|\,d\omega_e/dz\,|]$, reaches the resonant point, where the oscillations are absorbed.

5.2. Cyclotron Instability in an Inhomogeneous Magnetic Field

In the preceding section we considered oscillations with a frequency close to the cyclotron frequency and radiated by an external source. At the same time it is known that the cyclotron oscillations can occur spontaneously in a plasma with a particle velocity distribution that is not in thermodynamic equilibrium. If the magnetic field is inhomogeneous, then the region of the resonant cyclotron absorption becomes smaller. In this case, in analogy with the problems considered in Chs. 1–3 of the review, it is natural to expect the appearance of a stabilizing effect. Such a result was indeed obtained in[50], where ion cyclotron oscillations in a thermodynamically nonequilibrium plasma were considered. In ion cyclotron oscillations, the electrons take part in addition to the ions, and the properties of the oscillations depend significantly on the electron temperature. It was shown in[50] that if the electrons are not $(\epsilon_{\parallel e}\omega_i^2/\epsilon_{\perp i}\,\omega_{pi}^2 \gg 1)$, then at sufficiently large inhomogeneity of the magnetic field $(a/L \gg \omega_{pi}^2/\omega_i^2)$ the cyclotron instability can be stabilized (here a is the dimension of the system, L the characteristic scale of variation of the magnetic field, and $\epsilon_{\parallel j}(\epsilon_{\perp j})$ is the average energy of the thermal motion of the charged particles along (across) the magnetic field). If the electrons are cold $(\epsilon_{\parallel e}\,\omega_i^2/\epsilon_{\perp i}\omega_{pi}^2 \ll 1)$, then, as shown in[50], the inhomogeneity of the magnetic field only decreases the instability increment.

5.3. Conversion of Oscillations in a Plasma

The plasma contains a large number of various types of oscillations. Frequently, oscillations with different wavelengths (large-scale and small-scale) have the same frequency. If the medium is homogeneous, then such oscillations are not coupled in any way with one another. However, in an inhomogeneous medium, oscillations with different space scales cease to be independent. A coupling between them is particularly effective at points at which the refractive index (the wave number) of the large-scale oscillations becomes infinite. Indeed, on approaching such points, the characteristic wavelength of the large-scale oscillations decreases, so that ultimately the distinction between large-scale and small-scale oscillations becomes meaningless.

There is a considerable number of investigations of the conversion of different types of oscillations (see, for example,[5,6,51–54]). As the simplest example, let us consider the conversion of longwave electron Langmuir oscillations into shortwave ones[53]. If the plasma is in the magnetic field, then such oscillations are described by the equation

$$\frac{3}{2}\,\frac{\omega_{pe}^2 v_{Te}^2}{\omega^4}\,\frac{d^4\varphi_1}{dz^4}+\frac{d}{dz}\left(1-\frac{\omega_{pe}^2}{\omega^2}\right)\frac{d\varphi_1}{dz}-k_\perp^2\varphi_1=0; \tag{5.2}$$

here v_{Te} is the thermal velocity of the electrons. It is assumed that the magnetic field is directed along Oz and that the plasma density varies in the same direction, $\omega_{pe}^2(z) = 4\pi e^2 n_0(z)/m_e$. The perturbations of the potential are chosen in the form $\varphi_1(\mathbf{r}, t) = \varphi_1(z)\exp[-i\omega t + i\mathbf{k}_\perp \mathbf{r}_\perp]$ ($\mathbf{k}_\perp$ is the component of the wave vector perpendicular to the magnetic field). Equation (5.2) is approximate, since the influence of the finite electron temperature has been taken into account in it only partially (the term of the smallest degree in v_{Te} was retained). This, however, is perfectly sufficient for our purposes. If the characteristic scale of the oscillations along Oz greatly exceeds the Debye radius $r_{de} = v_{Te}/\omega_{pe}$, then we can omit the first term from (5.2):

$$\frac{d}{dz}\left(1-\frac{\omega_{pe}^2(z)}{\omega^2}\right)\frac{d\varphi_1}{dz}-k_\perp^2\varphi_1=0. \tag{5.3}$$

If at the same time the space scale is small compared with the distance over which the plasma density varies, then we can describe the oscillations by using the quasiclassical approximation, putting

$$\varphi_1(z) \approx k_\parallel^{-1/2}\exp\left(i\int^z k_\parallel dz\right).$$

We then obtain from (5.3) $k_\parallel^2 = k_\perp^2/(\omega_{pe}^2/\omega^2) - 1]$, or in a different form, $\omega^2 = \omega_{pe}^2 k_\parallel^2/k^2$. Thus, neglecting thermal effects, we have obtained magnetized electron Langmuir oscillations, which were considered from a different point of view in Ch. 3.

Let us assume that the density first decreases from left to right. Then, in the region where the density is sufficiently low $(\omega_{pe} < \omega)$, short-wave oscillations, for which in the quasiclassical approximation we have $k_\parallel^4 = 2\omega^4 k^2/3\omega_{pe}^2 v_{Te}^2$, can propagate. This relation has been obtained from (5.2) neglecting the last term.

Let the oscillations be radiated at $z = -\infty$ and propagate in the plasma from left to right. In the region where the quasiclassical approximation is valid, Eq. (5.3) can be regarded as a Schrödinger equation describing the motion of a particle with zero total energy in a potential $U(z) = k_\perp^2/[1 - (\omega_{pe}^2(z)/\omega^2)]$ (Fig. 10). At the point z_s, where $\omega = \omega_{pe}(z_s)$, the effective potential becomes infinite. Similar equations were considered in Ch. 1 and in Sec. 5.1, where it was

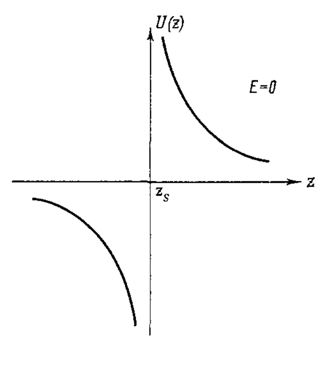

FIG. 10. The effective potential U(z) for Eq. (5.3). z_s—singular turning point.

shown that at the point at which the potential becomes infinite the oscillations are completely absorbed. An analysis of the complete equation (5.2) shows that in the present case the energy of the longwave oscillations is converted completely into energy of the shortwave oscillations in the present case at the point z_s. (We recall that the shortwave oscillations can indeed propagate in the region $z > z_s$. For them, the point z_s is an ordinary turning point.) The similarity between the present problem and that considered earlier increases because, as noted in[54], it is possible to use a certain modification of the Landau circuiting rule when determining the longwave solutions of (5.2). Namely, if we are not interested in a shortwave solution, then the point z_s can be circuited in the complex plane. The rule for circuiting is obtained by adding to the frequency a small positive imaginary part ν; then the point z_s shifts from the real axis by an amount $\delta z_s = \nu(d\omega_{pe}/dz)^{-1}$.

6. CONCLUSION

Thus, the stability and the vibrational properties of plasma flow in an ordinary liquid are determined to a considerable degree by the physical processes that act in a small vicinity of the resonant points. In particular, the stability of the flow having the simplest velocity profile is ensured by resonant absorption of the oscillations. However, it is natural to assume that these results are valid only for oscillations with sufficiently small amplitude. Indeed, if the amplitude of the particle displacement in the oscillations greatly exceeds the width of the absorption layer, then the physical processes acting on this layer cannot exert any appreciable influence on the evolution of the oscillations. In this case it becomes necessary to take into account the nonlinear effect in the vicinity of the resonant points. Thus, the next task is to develop a nonlinear theory of resonant phenomena in inhomogeneous flows of a plasma and of an ordinary liquid. It is possible, in particular, that this method will reveal the mechanism that leads to instability of Couette flow.

APPENDICES

I. OSCILLATIONS OF CERTAIN TYPES OF FLOWS OF AN INCOMPRESSIBLE LIQUID

1. Following[3], we shall show that in a flow having a velocity profile without an inflection point there are neither growing natural oscillations ($\operatorname{Im}\omega > 0$) nor neutral ones ($\operatorname{Im}\omega = 0$). We make in (1.5) the substitution $\varphi_1 = (\omega/k - V_0(x))\psi_1$:

$$\frac{d}{dx}\left[\left(\frac{\omega}{k}-V_0(x)\right)^2\frac{d\psi_1}{dx}\right]-k^2\left(\frac{\omega}{k}-V_0(x)\right)^2\psi_1=0. \qquad (I.1)$$

We assume that this equation has a solution $\Psi_1(x)$, satisfying the boundary conditions $\tilde{\varphi}_1(x) = (\omega/k - V_0(x))$ and $\tilde{\psi}_1(x) = 0$ on both ends of the interval (x_1, x_2). We multiply (I.1) by $\tilde{\psi}_1^*$ and integrate by parts:

$$\int_{x_1}^{x_2}\left(\frac{\omega}{k}-V_0(x)\right)^2\left[\left|\frac{d\tilde{\psi}_1}{dx}\right|^2+k^2|\tilde{\psi}_1|^2\right]dx=0. \qquad (1.2)$$

If $\operatorname{Im}\omega = 0$, $\omega/k \neq V_0(x)(x_1 < x < x_2)$, then (I.2) cannot be satisfied, and therefore there are no oscillations

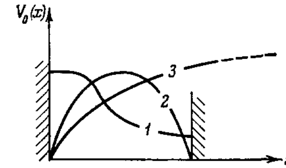

FIG. 11. Profiles of flow velocity 1—with an inflection point, 2—Poiseuille, 3—boundary-layer type.

with such a frequency. It is easily seen that (I.2) remains valid also in the case when the resonant point x_s, at which the condition $\omega/k = V_0(x_s)$ is satisfied, coincides with one of the ends of the interval (x_1, x_2). If the point x_s falls inside this interval, then relation (I.2) becomes meaningless, since the integral diverges. Returning in this case to (1.6) and recalling the definition $\lim_{\alpha\to 0}[\alpha/(x^2 + a^2)] = \delta(x)(\alpha = \operatorname{Im}\omega/[kV_0'(x_s)])$, we obtain for the imaginary part of the integral in (1.6)

$$\pi\,\mathrm{sgn}(\operatorname{Im}\omega)\,|\tilde{\varphi}_1(x_s)|^2 V_0''(x_s)\,|V_0'(x_s)|^{-1} \neq 0. \qquad (1.3)$$

Thus, relation (1.6), which should be satisfied for all the eigenfunctions of (1.5), is not satisfied if $\operatorname{Im}\omega \to 0$. If $\varphi_1(x_s) = 0$, then it is necessary to consider one of the intervals (x_1, x_s) or (x_s, x_2). In this case we are unable to satisfy a condition analogous to (I.2).

2. If the profile of the flow velocity has an inflection point $x_0(V_0''(x_0) = 0)$, then particular interest attaches to oscillations that resonate with the flow at the same point, for which $V_0(x_0) = \omega/k$ (see, for example,[3]). For such oscillations (Fig. 11), Eq. (1.5) is regular. We rewrite it in the form

$$\varphi_1''-\lambda\varphi_1+K(x)\varphi_1=0; \qquad (I.4)$$

Here $\lambda = k^2$, $k(x) = -V_0''(x)/(V_0(x) - V_0(x_0))$. If $(V_0'''V_0')_{x_0} < 0$ then $K(x) > 0$ at least in the vicinity of the point x_0, where $K(x) \approx -V_0'''(x_0)/V_0'(x_0)$. In this case, according to the Sturm-Liouville theorem, Eq. (1.4) should have eigenfunctions φ_n with corresponding eigenvalues $\lambda_n = k_n^2$. For such oscillations, $\omega_n = k_n V_0(x_0)$, $\operatorname{Im}\omega_n = 0$. It can be shown (see, for example,[3]), that oscillations with smaller values of k^2 are unstable. In general, apparently, with decreasing k the stability of the flow becomes worse. The most unstable oscillations, with $k = 0$, were considered in[55], where it is shown that the condition for the vanishing of V_0'' is necessary and sufficient for the instability of flows of an ideal liquid.

3. The analysis in Appendix I.1 is not quite complete. Indeed, let us assume that the velocity of the flow along O_x varies nonmonotonically and vanishes on the boundaries (x_1, x_2). It is easy to see that in this case Eqs. (I.1) and (I.2) are satisfied at $k = 0$, $\omega/k = 0$ and $\varphi_1 = CV_0(x)$. Such "oscillations" resonate with the flow at its boundaries; it is therefore impossible to use the Landau rule in their analysis, and the viscosity of the liquid must be taken into account, no matter how small it may be. Heisenberg has shown that under the influence of viscosity the oscillations become unstable. With decreasing viscosity, the increment of the oscillations and their frequency tend to zero, so that in the limiting case we actually obtain the solution described above. An analysis of the "viscous" equation (1.9) is cumbersome and calls for the use of numerical methods (see, for example,[3]), and is therefore not

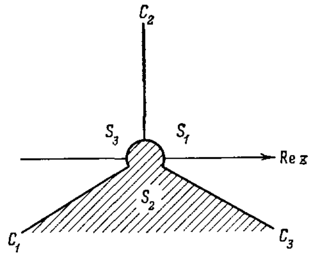

FIG. 12. Plane of complex variable for Eq. (1.9) in the vicinity of the point x_s. The region in which viscosity must be taken into acount is shown shaded.

given here. A similar instability was observed for a flow of the boundary-layer type, where $V_0(x) \underset{x\to\infty}{\rightarrow}$ const. In this case the boundary condition at infinity is $V_1(\infty) = 0$.

II. EQUATIONS WITH A SMALL PARAMETER IN FRONT OF THE HIGHEST-ORDER DERIVATIVE

1. An equation of this type is (1.9), if the viscosity of the liquid is sufficiently low. In view of the presence of a small parameter in front of the fourth derivative, its solutions can be classified as slowly-varying and rapidly-varying. In considering the slowly-varying solutions, we can disregard the viscosity, confining ourselves to the approximation of ideal hydrodynamics. For the rapidly-varying solutions, the viscosity is important. However, such a separation is incorrect in the vicinity of the resonant point, where solutions with different space scales become "tied up" with one another. To investigate the properties of the solutions in this region, Wasow [8] introduced seven functions: A_k, U_k, $V_k (k = 1, 2, 3)$ and connected them by the three relations

$$A_i = e_{ikl}U_l + \delta_{i2}V; \tag{II.1}$$

here e_{ikl} is an absolutely antisymmetrical tensor and δ_{i2} is the Kronecker symbol.

It turned out that the separation of the solutions into rapidly-varying (A_k) and slowly-varying (U_k, V) solutions occurs already in a small vicinity of the point x_s: $|z| \gtrsim \max(|\Lambda|^{-1}, |\Lambda|^{-3})$ (here $z = (x_s - x)V_0''(x_s)/V_0'(x_s)$, $\Lambda = ikV_0'^4(x_s)/\nu V_0''^3(x_s) \approx ika\cdot\mathrm{Re}$, where $\mathrm{Re} = V_0 a/\nu$ is the Reynolds number of the unperturbed flow and a is the characteristic scale of variation of V_0). Following[8], let us consider the analytic continuation of these solutions on the plane of complex variables z in the vicinity of $|z| \ll 1$, which we break up into sectors S_k (Fig. 12). For concreteness we assumed that $kV_0'(x_s) > 0$, $V_0''(x_s) > 0$.

For A_k everywhere with the exception of the line C_k arg $C_k = 11/6\pi - 2\pi k/3$), the following asymptotic representation is valid:

$$A_k = \alpha_k \pi^{1/2}\Lambda^{-3/4}z^{-5/4}\exp\{\beta_k[(2i/3)\Lambda^{1/2}z^{3/2} + (i\pi/4)]\}; \tag{II.2}$$

here $-\alpha_1 = -\alpha_2 = \alpha_3 = -\beta_1 = \beta_2 = -\beta_3 = 1$.

The U_k in the sectors $S - S_k$ are expressed in terms of Hankel functions

$$U_k = \pi i\sqrt{z}\,H_1(2\sqrt{z}) \underset{|z|\to 0}{\approx} 1 + z\ln z + \ldots \tag{II.3}$$

Finally, the function V in the entire z plane is expressed in terms of the Bessel function

$$V = 2\pi i\sqrt{z}\,J_1(2\sqrt{z}) \underset{|z|\to 0}{\approx} 2\pi i\left(z - \frac{z^2}{2} + \ldots\right). \tag{II.4}$$

In those regions where (II.2) and (II.3) are not valid, it is necessary to use relation (II.1) to determine A_k and U_k.

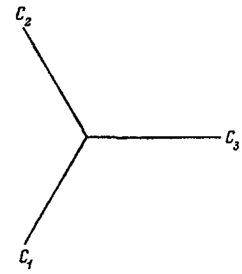

FIG. 13

Expressions (II.3) and (II.4) could be obtained by assuming the viscosity equal to zero, i.e., in the approximation of ideal hydrodynamics (see (1.5)). However, it follows from (II.1) that in the sectors S_k the spatial scale U_k changes—these solutions become "viscous." Thus, the use of the approximation of ideal hydrodynamics in the entire plane of the complex variable z becomes impossible. Nonetheless, if we choose V and U_2 as the linearly independent solutions of (1.5) and circuit the resonant point $x_s (z = 0)$ in the complex z plane from above, then everywhere on the real axis, with the exception of a small vicinity of the point $x_s(|z| \lesssim \min(|\Lambda|^{-1}, |\Lambda|^{-3}))$ the ideal-hydrodynamics approximation will be valid (see Fig. 12). It is easy to see that this circuiting rule coincides with the Landau circuiting rule.

2. The equation of flute oscillations of a quasineutral plasma (4.4) is close in its properties to Eq. (1.9). In particular, its solutions also separate into rapidly-varying A_k and slowly varying solutions U_k, V. It is important, however, that in this case the sectors S_k, in which the spatial scale of the solutions U_k varies, are rotated through an angle $\pi/6$ (Fig. 13). This figure shows the vicinity of the point x_{s1} (see Fig. 7). In order to obtain the corresponding picture for the vicinity of x_{s2} (see Fig. 13), it is necessary to perform a reflection at the resonance point. It follows from the figure that no matter what function we choose from U_k, the solution will be slowly-varying on the real axis only on one side of x_s. A solution that varies slowly on the left of x_s becomes rapidly-varying on the right of x_s, whereas a solution that is slowly varying on the right of x_s becomes rapidly varying and growing on the left of x_s. We recall that on the lines C_k the phase A_k is real (see (II.2)).

3. In the case of electron Langmuir oscillations described by Eq. (5.2), the location of the sectors S_k is given by Fig. 13. In the region $x < 0$ a wave travelling to the right is given by the solution $U_3 - V \sim e^{-2ix^{1/2}}$ ($|x| \gg 1$, $x = (z_s - z)k_\perp^2 \omega^2/\omega_{pe}^2$). According to (II.1), at $x > 0$ it goes over into a rapidly oscillating solution describing a wave traveling to the right, and into a slowly-varying solution that decreases to the right from $x = 0$. The latter could be obtained with the aid of the simplified Eq. (5.3) by using the circuiting rule given at the end of Sec. 5.3.

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