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PROPERTIES OF REGGE POLE TRAJECTORIES

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INTRODUCTION

FOR some time the fruitful concept of Regge poles in the relativistic theory of strong interactions was not undisputed. After the "tempestuous blooming" of 1962—1963, this approach passed through a period of relative quiescence, due mainly to the difficulties in describing the available experimental data by means of a small number of Regge poles.

In recent years, owing to the accumulation of a large amount of experimental data and the elucidation of a number of theoretical points, the concept of Regge poles has, step by step, occupied stronger and stronger positions.

At present the main idea of the method—on the one hand the interrelations of resonances differing from each other in angular momentum but having their other quantum numbers equal (resonances in the t channel, described by a Regge trajectory $\alpha(t)$ for $t > 0$), and on the other hand the structure of the diffraction cone in the high-energy scattering in the cross or s channel (large s, negative t)—is generally accepted. We are also getting a good deal of light on the reasons for the temporary failures of the method, owing to the existence, besides the poles, of Regge cuts, and also on the complicated nature of the scattering amplitudes T(s, t) for $s \to \infty$, which in the t channel have vacuum quantum numbers. The structure of the vacuum Regge singularities has not been cleared up as yet.

In the first part of this review (Chapters 1 and 2) we shall give a summary of the experimental data on Regge pole trajectories, obtained both in the particle region [$\alpha(t)$ for $t > 0$] and in the scattering region [$\alpha(t)$ for $t < 0$]. Here we shall place the emphasis on those trajectories about which the available experimental material fits into a simple general picture. Primarily these are the ρ-meson trajectory and the pion-nucleon trajectories Δ_δ, N_γ, N_α.

Thereafter (Chapters 3 and 4) we consider questions of the correspondence between the experimental data and certain theoretical concepts—the spectral representations and the so-called parity doubling of the baryon trajectories, and also the problem of the asymptotic behavior of trajectories. In conclusion (Chapter 5) we discuss some general properties of Regge trajectories, which are evidently especially simple in the linear approximation which corresponds to negligibly small imaginary parts (the approximation of infinitely narrow resonances). Very recently linear Regge trajectories have been very effectively used in the so-called Veneziano model.

*Extended version of a survey report read at a scientific session of the Theoretical Physics Division of the U.S.S.R. Academy of Sciences (Leningrad, May, 1969).

Table I. The ρ-meson trajectory

Particle	I^P	M, MeV	Γ^{tot}, MeV	Reference
ρ	1^-	765±10	125±20	1
g	3^-	1660±20	110±30	1, 2
T	?	2190	~ 85	1

Table II

Reaction	Range of (−t)	ρ (0)	ρ′ (0)	Reference
πN, KN, $\bar{K}N$	0—1	0.53—0.54	0.65—0.78	3
πN–Ce	0—0.28	0.58±0.01	1.00±0.11	4
The same	0—1.1	0.57±0.01	0.91±0.06	4
πp, pp, $\bar{p}p$	0—1	0.57—0.58	0.94—1.01	5
KN	0—1	0.56	1.05	6
πN–Ce	FESR	0.55±0.58	—	7
The same	»	0.55±0.07	—	8
πN	IDR	0.56	—	

1. THE EXPERIMENTAL DATA ON MESON TRAJECTORIES

1.1. The ρ-meson trajectory is the meson trajectory that has been most completely studied. There are two reliably established points on the Chew-Frautschi diagram, and also a third candidate (Table I).

The Chew-Frautschi plot drawn for the ρ and g mesons is[2]

$$\rho_{CF}(t) = 0{,}46 \pm 0{,}03 + (0{,}92 \pm 0{,}03)\, t \tag{1.1}$$

and gives $M \approx 2220$ for $l = 5$. Accordingly the T meson is a good candidate for $J^P = 5^-$.

In the scattering region information about the ρ-meson trajectory has been obtained from a number of reactions (Table II). In the first five lines of the table we give data from analyses of the scattering through small angles with incident beam momenta $6\ \text{GeV/c} < P_L < \sim 20\ \text{GeV/c}$ (CE means charge exchange). The next two lines contain the results of analyses of experimental data by means of a new method for the theoretical analysis of scattering amplitudes, the "finite-energy sum rules" (FESR).[10] The last result was found by means of Hilbert-transform inverse dispersion relations (IDR).

Except for the data of the earliest paper, all of the results in Table II can be represented in the form

$$\rho_{scat}(t) = 0{,}57 \pm 0{,}02 + (0{,}95 \pm 0{,}1)\, t. \tag{1.2}$$

*Here and in what follows we depart from the usual practice, and denote a trajectory by its index symbol, writing $\rho(t)$ instead of $\alpha_\rho(t)$, and so on. Furthermore the index CF means the approximation from the data of the Chew-Frautschi plot, and the index scat means the approximation from scattering data.

Comparing (1.1) and (1.2), we see that the slopes do not disagree, but the "height" $\rho(0)$ differs by about 0.1. We note that this difference of the heights is clearly much larger than the possible errors in determining $\rho(0)$. We shall return to this point later.

1.2. The trajectory ω ($I^{GP} = 0^{--}$) in the scattering region has also been studied in a number of papers by analyzing the scattering through small angles in the range 6 GeV/c $< p_L < \sim 25$ GeV/c (Table III). We see that there is a much larger scatter in the data for this case than for $\rho(t)$. The average values can be represented in the form

$$\omega_{\text{scat}}(t) = 0.4 \pm 0.1 + (0.7 \pm 0.3)\,t. \qquad (1.3)$$

In the particle region a linear interpolation between the ω meson and $\omega(0) = 0.4 \pm 0.1$ gives

$$\omega(t) = 0.4 \pm 0.1 + (1.0 \mp 0.15)\,t. \qquad (1.4)$$

Thus there is apparently no contradiction between the data for $t > 0$ and for $t < 0$.

1.3. The trajectory A_2 ($I^{GP} = 1^{-+}$) (also called the R trajectory) is determined from scattering experiments with large errors (Table IV).

Averaging these data, we get

$$R_{\text{scat}}(t) = 0.4 \pm 0.1 + (0.6 \pm 0.2)\,t. \qquad (1.5)$$

In the particle region the situation is not very simple. On one hand, the A_2 resonance has recently "split" into two resonances

$$A_2(1300) \begin{cases} \longrightarrow A_{2L}(1270,\ I^G = 1^-,\ J^P = N), \\ \longrightarrow A_{2H}(1315,\ IJ^{GP} = 12^{-+}). \end{cases}$$

On the other hand, the candidates S(1930) and U(2380) are not reliably enough established. The Chew-Frautschi plot constructed for $A_{2L,H}$, S, and U is of the form

$$R_{\text{CF}}(t) = (0.37 \pm 0.05) + (0.98 \pm 0.02)\,t. \qquad (1.6)$$

We see that the slopes R′ in (1.5) and (1.6) differ by more than their uncertainties (which, by the way, are rather large).

Table III. The trajectory ω

Reaction	Range of (−t)	ω (0)	ω' (0)	Reference
(πN), KN, $\bar{K}N$	0—1	0.50—0.52	0.5—0.6	3 *)
πp, pp, $\bar{p}p$	0—1	0.21—0.47	0.32—1.66	5
KN	0—1	0.32—0.36	0.8—1.0	6

*In [3] a combined analysis was made of the reactions of πN, KN, and $\bar{K}$N scattering. The symbol (πN) means (both here and in Table IV) that the ω trajectory does not make any direct contribution to πN scattering.

Table IV. The R trajectory

Reaction	Range of (−t)	R (0)	R' (0)	Reference
(πN), KN, $\bar{K}N$	0—1	0.31±0.01	0.55—0.80	3
KN	0—1	0.34	0.35	6
KN	0—1	0.48	0.60	6
$\pi^- p \to \eta^0 n$	0—0.85	0.3—0.40	0.6—0.8	11

Table V. Parameters of vacuum trajectories

Reaction	Range of (−t)	P' (0)	P_1 (0)	P_1' (0)	Reference
πN, KN, $\bar{K}N$	0—1	0.34	0.50	0.34	3
πN, $N\bar{N}$	0—1.5	0.33—0.49	0.62—0.66	1.3—1.6	12
πp, pp, $\bar{p}p$	0—1	0—0.3	0.57—0.75	1.5—2.2	5
πN	0—1	0.7±0.15	—	—	13
πN	—	Behavior of σ_{tot}	0.31±0.2	—	14
KN	—	Dispersion relations	0.89±0.1	—	15
πN	—	FESR	0.41±0.1	—	16
πN	—	FESR	0.49±0.02	—	17
KN	—	FESR	0.65±0.05	—	18

1.4. The vacuum trajectories are in the "most unfortunate" situation. To explain the behavior of the total and differential cross sections corresponding to the vacuum quantum numbers in the t channel, along with the Pomeranchuk vacuum trajectory $\alpha_P(t) \equiv P(t)$ that assures asymptotic constancy of the total cross sections [$P(0) = 1$] one introduces a second vacuum trajectory $\alpha_{P'} = P_1(t)$ (and sometimes also a third, $\alpha_{P''} = P_2$).

Table V gives a by no means complete list of the results on the slope $P'(0)$ of the main vacuum trajectory, and also of the parameters $P_1(0)$ and $P_1'(0)$ of the second vacuum trajectory. Even a cursory glance at Table V reveals the absence of any clear picture. The slope of the Pomeranchuk trajectory varies from 0 to 0.7. The slope $P_1'(0)$ of the second vacuum trajectory varies from 0.3 to 2.2. Relatively, the spread in the position $P_1(0)$ of the second trajectory is smaller; it can be seen that we can take

$$P_1(0) = 0.5 \pm 0.2. \qquad (1.7)$$

At the present time there are only two resonances belonging to the "vacuum family" that are reliably established: f(1260, $IJ^{GP} = 02^{++}$) and f′(1515, $IJ^{GP} = 02^{++}$). If we draw the Pomeranchuk trajectory through the f meson and P_1 through f_1 (sic), then when we use (2.7) we get in the linear approximation

$$P(t) = 1 + 0.64t, \qquad (1.8)$$

$$P_1(t) = (0.5 \pm 0.2) + (0.65 \mp 0.1)\,t. \qquad (1.9)$$

There is another point of view, according to which the main vacuum (Pomeranchuk) trajectory is either a trajectory with a very small slope not containing any resonances, or a stationary singularity. In this case, drawing P_1 through the f meson, we get

$$P_1(t) = 0.5 \pm 0.2 + (0.95 \mp 0.12)\,t. \qquad (1.10)$$

1.5. Exchange-degeneracy hypothesis and strange trajectories. It is a striking fact that the trajectories $\rho(I^P = 1^-)$ and $R(I^P = 1^+)$ are so close together. The A_2 meson lies practically on the linear ρ trajectory (1.1). This can be interpreted as evidence of the so-called "exchange degeneracy" (see [19,20]), owing to which trajectories differing in G parity and signature should coincide. If we accept this hypothesis, it is possible to construct two strange meson trajectories in the particle region [21]: a K trajectory passing through K(495, 0^+), K* (1230, 1^+?),

$$K^L_{\text{CF}}(s) = -0.19 + 0.78s, \qquad (1.11)$$

or through K (495, 0^-), K* (1320, 1^+?) (sic),

$$K^{H}_{\text{CF}}(s) = -0.16 + 0.66s, \qquad (1.12)$$

and a K* trajectory passing through K* (890, 1^-), K** (1420, 2^+),

$$K^{*}_{\text{CF}}(s) = 0.33 + 0.82s. \qquad (1.13)$$

2. THE EXPERIMENTAL DATA ON BARYON TRAJECTORIES

2.1. The trajectory Δ_δ. This trajectory $I^P = 3/2^+$, which passes through the Fermi resonance Δ_{33}, is the one "richest" in particles on the Chew-Frautschi plot. The following resonances (Table VI) fall on it.

In the particle region the trajectory Δ_δ is very well described by the linear approximation[22]

$$\Delta_\delta^{\text{CF}}(u) = 0.15 + 0.90u, \qquad (2.1)$$

which gives the masses to an error not exceeding 20 MeV (on the scale of $u = M^2$ the maximum error does not exceed 0.08 GeV^2).

In the scattering region the trajectory Δ_δ has been studied by Barger and Cline[23] on the basis of the reactions $\pi^\pm p \to p\pi^\pm$ (sic) at energies 6–14 GeV in the momentum-transfer range $-1\ \text{GeV}^2 < u < 0$. Their result is

$$\Delta_\delta^{\text{scat}}(u) = 0.19 + 0.87u. \qquad (2.2)$$

These authors estimate the accuracy of the determination of the coefficients in (2.2) as ± 20 percent.

Finally, from an analysis of $\pi^- p \to p\pi^-$ Shih[24] has recently obtained

$$\Delta_\delta^{\text{scat}}(u) = 0.05 + 0.76u. \qquad (2.3)$$

Although (2.2) agrees with (2.1) within the limits of error, comparing it with (2.3) makes us feel cautious about the accuracy of the determination of $\Delta_\delta(0)$. We shall see later (Sec. 4.2) that this point may be of importance in principle.

2.2. The trajectory N_γ. On this trajectory with isospin-parity $I^P = 1/2^-$ there are two established resonances and two good candidates (Table VII).

The Chew-Frautschi plot is approximated by the linear expression[22]

$$N_\gamma^{\text{CF}}(u) = -0.94 + 0.92u, \qquad (2.4)$$

which gives a good description of the three leading resonances (accuracy $\Delta M^2 \approx 0.02\ \text{GeV}^2$). But N (1518) falls rather far from the linear approximation ($\Delta M^2 = 0.33\ \text{GeV}^2$).

In the scattering region the study of this resonance is difficult because of its low position.

2.3. The trajectory N_α. This trajectory ($I^P = 1/2^+$), passing through the nucleon N (938), contains one other resonance N (1688):

$$J^P = 5/2^+,\ M = 1690,\ \Gamma = 125,\ M^2 = 2.86.$$

The linear approximation in the particle region is

$$N_\alpha^{\text{CF}}(u) = -0.39 + 1.01u. \qquad (2.5)$$

The parameters of this trajectory in the scattering region are known from the work of Barger and Cline[23]:

$$N_\alpha^{\text{scat}}(t) = -0.38 + 0.88t. \qquad (2.6)$$

The authors estimate that the accuracy to which they are determined is $\lesssim 10$ percent. We note that there is a considerable difference between the slopes of (2.5) and (2.6).

2.4. Strange trajectories. In some cases one can use the Chew-Frautschi diagram to get linear approximations for baryon trajectories with strangeness (Table VIII).

A notable point here is that the slopes of the trajectories Σ_δ and Σ_γ are close to 0.90, i.e., to the slopes of Δ_δ and N_γ, and the slopes of Λ_α and Ξ_α are close to 1.0, i.e., to the slope of N_α.

In the scattering region the first attempt to analyze recently obtained experimental data on the backward "scattering" for the reaction $\pi^- p \to K^0\Lambda$ at energies 4.0 and 6.2 GeV was made by Barger, Cline, and Matos.[25]

Assuming that this process is due to exchange of Σ_α and Σ_γ, and treating these trajectories as degenerate, these authors obtained

$$\Sigma_{\alpha\gamma}^{\text{scat}}(u) = -0.84 + 1.0u. \qquad (2.7)$$

A comparison with the data of Table VIII shows that there is apparently no contradiction between the linear approximations for the trajectories in the particle region and in the scattering region.

2.5. Effects of inelasticity. The data on the change of the elastic properties of resonances along a trajectory are of interest. Figure 1 shows a plot of ln x ($x = \Gamma_{\text{el}}/\Gamma_{\text{tot}}$) as function of the angular momentum for the trajectories Δ_δ and N_γ. It can be seen that all of the resonances of Δ_δ and N_γ [and also N_α (1688)] lie practically on a single curve, which for $1 > 4$ can be

Table VI. The trajectory Δ_δ

Resonance	I^P	M, GeV	Γ, MeV	M^2
Δ (1236)	3/2+	1.236	120	1.53
Δ (1950)	7/2+	1.95	220	3.80
Δ (2420)	11/2+	2.42	310	5.86
Δ (2850)	?+	2.85	400	8.12
Δ (3230)	?	3.23	440	10.4

Table VII. The trajectory N_γ

Resonance	I^P	M, GeV	Γ, MeV	M^2
N (1518)	3/2−	1.515	115	2.30
N (2190)	7/2−	2.19	300	4.80
N (2650)	?−	2.65	360	7.02
N (3030)	?	3.03	400	9.18

Table VIII. Strange baryon trajectories

Trajectory	Calculated from the resonances	
$\Sigma_\delta = -0.25 + 0.91$	Σ (3/2+, 1385),	Σ (7/2+, 2030)
$\Lambda_\alpha = -0.70 + 0.97$	Λ (1/2+, 1115),	Λ (5/2+, 1815)
$\Sigma_\alpha = -0.78 + 0.90$	Σ (1/2+, 1190),	Σ (5/2+, 1915)
$\Sigma_\gamma = -0.90 + 0.87$	Σ (3/2−, 1660),	Σ (?, 2250)
$\Xi_\alpha = -1.26 + 1.01$	Ξ (1/2+, 1318),	Ξ (?, 1930)

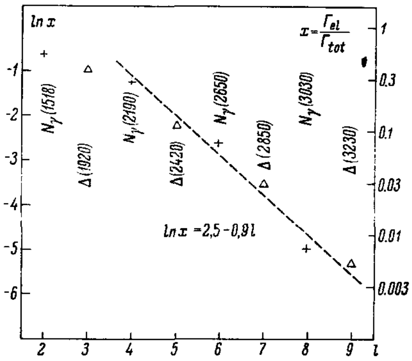

FIG. 1. Dependence of the inelasticity coefficient x on the orbital angular momentum l for the baryon trajectories Δ_δ and N_γ.

approximated by the expression

$$\ln x = 2.5 - 0.9l, \quad x = 12.2e^{-0.9l}. \tag{2.8}$$

This relation is in agreement with theoretical calculations by Jones and Teplitz[26] (cf. also[27]), who started from the requirement that the amplitude be bounded by a polynomial for indefinitely increasing Regge trajectories.

We point out here that if Eq. (2.8) is sufficiently accurate in the region $l \sim 10$, then it follows that the experimental discovery of πN resonances with $l \gtrsim 10$ is an extremely complicated task, so that the pion-nucleon part of the Rosenfeld table is evidently already filled almost to its upper end.

3. SPECTRAL REPRESENTATIONS AND THE PROBLEM OF SUBTRACTION

3.1. Meson trajectories.

It is usually assumed that the meson trajectories are analytic functions in the complex plane of the invariant variable s with a cut along part of the positive real semiaxis, $s_{cut} < s < \infty$. The beginning s_{cut} of the cut is at the two-particle threshold of the continuous spectrum of the reaction.

The two-particle unitarity condition enables us to determine[28] the form of the behavior of Im α at the threshold

$$\operatorname{Im}\alpha(s)|_{s\sim s_{cut}} \approx c\,(s-s_{cut})^{\operatorname{Re}\alpha(s_{cut})+1/2}. \tag{3.1}$$

The problem of determining the asymptotic behavior of Regge trajectories has not yet yielded to any sort of simple theoretical treatment.

In the course of the last year the hypothesis that $\alpha(s)$ increases without limit for $s \to \infty$ has been gaining more and more popularity. On one hand this idea does not contradict the experimental material which we have already presented on the trajectories "rich in particles", Δ_δ and N_γ (and also ρ), and on the other hand it leads to some beautiful theoretical consequences (for example, to the Veneziano model[29]).

Assuming that

$$\alpha(s) \sim s^n, \quad 0 < n < 1,$$

we get

$$\alpha(s) = \alpha(0) + \frac{s}{\pi}\int_{s_{cut}}^{\infty}\frac{ds'}{s'}\frac{\operatorname{Im}\alpha(s')}{s'-s}. \tag{3.2}$$

In the case of a stronger increase a second subtraction is necessary:

$$\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{s^2}{\pi}\int_{s_{cut}}^{\infty}\frac{ds'}{s'^2}\frac{\operatorname{Im}\alpha(s')}{s'-s}. \tag{3.3}$$

According to Eq. (3.1) the index of the increase of Re $\rho(s)$ is unity, or close to it. Using the relation

$$\operatorname{Im}\alpha(m_i^2) = m_i\Gamma^{tot}\alpha'(m_i^2), \tag{3.4}$$

we calculate Im ρ at the points s_ρ, s_g, and s_T. We get

$$\operatorname{Im}\rho(m_\rho^2) = 0.09 \pm 0.02, \quad \operatorname{Im}\rho(m_g^2) = 0.15 \pm 0.05, \quad \operatorname{Im}\rho(m_T^2) = 0.17. \tag{3.5}$$

If we believe the third point $s_T = m_T^2$, then at any rate Im ρ does not increase more rapidly than $s^{1/2}$. In this case one could use the representation (3.2). But it is not hard to verify that in this case we do not get anything like a linear increase of Re $\rho(t)$ of the form (1.1). In other words, we get the impression that there is a contradiction between the behaviors of Im ρ and Re ρ.

There are two possibilities here: it may be that beginning with $t \gtrsim 4M_N^2$ there is a sharp change in the behavior of Im ρ, so that asymptotically Im ρ increases almost linearly, and consequently Re ρ is described by a formula with two subtractions, Eq. (3.3).

There are at least two arguments in favor of this possibility:

1) The analogy with the behavior of Im Δ_δ and Im N_γ (see Chapter 4).

2) Summation of the logarithmic diagrams in the theory with the Lagrangian $\lambda(\pi_\sigma\pi_\sigma)^2$ leads[30] to Regge poles (and also stationary singularities) in all three isotopic t amplitudes, I = 0, 1, 2. Furthermore the pole I = 1 lies below the pole I = 2. For the relative suppression of the pole I = 2 it is necessary to take into account the pion-nucleon interaction $g(\overline{N}\tau_\sigma N)\pi_\sigma$, i.e., states $N\overline{N}$ as intermediate states from the point of view of the meson trajectories.

The subtraction of the linear term is not due to a divergence of the integral with a single subtraction in the right member of (3.2), and is, so to speak, a "superfluous luxury."

In this case the representation (3.3) can be rewritten in the form

$$\alpha(s) = \alpha(0) + s\alpha'(\infty) + \frac{s}{\pi}\int_{s_{cut}}^{\infty}\frac{ds'}{s'}\frac{\operatorname{Im}\alpha(s')}{s'-s}, \tag{3.6}$$

with

$$\alpha'(\infty) = \alpha'(0) - \frac{1}{\pi}\int_{s_{cut}}^{\infty}\frac{ds}{s^2}\operatorname{Im}\alpha(s). \tag{3.7}$$

The question of the nature of the "extra" subtraction and of the numerical value of $\alpha'(\infty)$ remains completely open.

3.2. The ρ Trajectory

According to all appearances, the experimental data on scattering are a reliable enough way of determining $\rho(0)$:

$$\rho(0) = 0.57 \pm 0.01.$$

Extrapolating to the threshold with Eq. (1.2), we find

$$\rho(4\mu^2) = 0.64 \pm 0.02. \tag{3.8}$$

Therefore, according to (3.1),

$$\operatorname{Im}\rho(s)|_{s-4\mu^2}\approx c(s-4\mu^2)^{1,14}. \quad (3.9)$$

The threshold behavior (3.9) leads to a singularity in the behavior of Re $\rho(s)$ near the point $s = 4\mu^2$ described by the formulas

$$\Delta\operatorname{Re}\rho(s)=\frac{c}{\sin(0,14\pi)}(4\mu^2-s)^{1,14},\quad s\leqslant 4\mu^2;$$
$$\Delta\operatorname{Re}\rho(s)=-c\operatorname{ctg}(0,14\pi)(s-4\mu^2)^{1,14},\quad s\geqslant 4\mu^2. \quad (3.10)$$

It follows from these formulas that the integral of Im ρ has a tendency to shorten the vertical gap of 0.1 between Eqs. (1.1) and (1.2).

A numerical calculation shows, however, that the gap is not reduced by more than 0.03. The point is that the threshold approximation (3.9) breaks down rather rapidly already at $s \sim 1\ \text{GeV}^2$.

In fact, relying on the numerical values (3.5), we can easily verify that in the region $4\mu^2 < s < 3\ \text{GeV}^2$ we have Im $\rho \sim 0.1\ (s - 4\mu^2)^{1/3}$. It can be seen from this that not only does the threshold behavior break down very rapidly, but also the absolute values of Im ρ are small. This also has the consequence that the absolute values of the integral term in the right member of (3.6) are also small ($\lesssim 0.05$) in the range $0 < s < 3\ \text{GeV}^2$, and this in turn makes it impossible to close the "gap" by more than 0.03.

Therefore we arrive at the conclusion that the experimental data on the masses and widths of the ρ and g mesons, on one hand, and the data for determining the trajectory $\rho(t)$ in the scattering region, on the other hand, are in contradiction with each other in the framework of the spectral representation (3.2), (3.3).

This contradiction is perhaps an indication of the presence of an additional close-in singularity (a stationary[30] or moving[31] cut).

4. THE SPECTRAL REPRESENTATION OF BARYON TRAJECTORIES

4.1. The Spectral Representation and the Parity Doubling

According to the usual recipe for reggeizing the amplitudes for meson-baryon scattering,[32,33] which takes into account the square-root kinematic singularity $u^{1/2}$ associated with the presence of a spin, baryon trajectories that differ from each other in parity are closely connected. Such a pair of trajectories $\alpha_+(u)$ and $\alpha_-(u)$ must coincide at $u = 0$, and for $u < 0$ they are complex conjugates:

$$\alpha_+(u)=(\alpha_-(u))^*, \quad (4.1)$$

i.e.,

$$\alpha_\pm(u)=\alpha_1(u)\pm i\alpha_2(u)\quad \text{for}\quad u<0, \quad (4.2)$$

where α_1 and α_2 are real for $u < 0$. The asymptotic form of the amplitude for backward meson-baryon scattering is then determined simultaneously by the trajectories α_+ and α_-, in such a way that

$$d\sigma/d\Omega\sim s^{2\alpha_1(u)}. \quad (4.3)$$

In the region $u > 0$ the trajectories are in general not equal to each other. However, they can be represented as functions of the total energy $W = u^{1/2}$, and are then connected by the so-called MacDowell symmetry conditions

$$\alpha_+(W)=\alpha_-(-W).$$

Accordingly, the two trajectories can be reduced to a single function $\alpha(u^{1/2})$ satisfying the spectral representation[32]

$$\alpha(W)=a+bW+\frac{W^2}{\pi}\int_{W_0}^{\infty}\frac{\operatorname{Im}\alpha_+(W')\,dW'}{W'^2(W'-W)}+\frac{W^2}{\pi}\int_{W_0}^{\infty}\frac{\operatorname{Im}\alpha_-(W')\,dW'}{W'^2(W'+W)}. \quad (4.4)$$

We then have

$$\alpha_+(u)=\begin{cases}\alpha(\sqrt{u}),\\ \alpha(i\sqrt{-u}),\end{cases}\quad \alpha_-(u)=\begin{cases}\alpha(-\sqrt{u}) & \text{for}\quad u>0,\\ \alpha(-i\sqrt{-u}) & \text{for}\quad u<0.\end{cases} \quad (4.5)$$

It is convenient to transform (4.4) by explicitly separating the parts odd and even with respect to W,

$$\alpha(W)=\alpha_1(u)+Wa(u), \quad (4.6)$$

$$\alpha_1(u)=a+\frac{u}{\pi}\int_{W_0^2}^{\infty}\frac{du'}{u'(u'-u)}\{\operatorname{Im}\alpha_-(\sqrt{u'})+\operatorname{Im}\alpha_-(-\sqrt{u'})\}, \quad (4.7)$$

$$a(u)=b+\frac{2u}{\pi}\int_{W_0^2}^{\infty}\frac{d\sqrt{u'}}{u'(u'-u)}\{\operatorname{Im}\alpha_+(\sqrt{u'})-\operatorname{Im}\alpha_-(-\sqrt{u'})\}. \quad (4.8)$$

Experimental information about a given baryon trajectory can consequently be "gathered from three sources":

a) The positive-parity resonances determine the function

$$\alpha_+(u)=\alpha_1(u)+\sqrt{u}\,a(u)\quad \text{for}\quad u>0. \quad (4.9)$$

b) The negative-parity resonances determine the function

$$\alpha_-(u)=\alpha_1(u)-\sqrt{u}\,a(u)\quad \text{for}\quad u>0. \quad (4.10)$$

c) The differential cross section for meson-nucleon scattering in the neighborhood of $\theta = 180^\circ$ gives information about the quantity

$$\operatorname{Re}\alpha(W)=\alpha_1(u)\quad \text{for}\quad u<0. \quad (4.11)$$

The problem of reconciling these various experimental data is not a trivial one.

4.2. The Pion-nucleon Trajectory with I = 3/2

Unlike the positive-parity trajectory Δ_δ, the negative-parity trajectory with I = 3/2, Δ_γ, is practically unobserved in the particle region. Recently two resonances $I^P = 3/2$ have been found: Δ (1630, $J^P = 1/2^-$) and Δ (1690, $J^P = 3/2^-$). When we take the signature into account only Δ (1690) can be ascribed to the trajectory Δ_γ. The linear Barger-Cline formula given in Chapter 4

$$\Delta_\delta(u)=0.15+0.90u$$

gives an error in J of the order of ± 0.05. Through the points of the Chew-Frautschi diagram one can draw a series of parabolas $a + bw + cw^2$ which will give errors of the same order of magnitude. One can also choose the coefficients a, b, c so that for $W = -1.69$ the parabola will pass through Δ (1690):

$$\Delta(W)=-0.25+0.35W+0.83W^2. \quad (4.12)$$

For all the Δ_δ resonances and Δ (1690) Eq. (4.12) gives an error $|\Delta J| \lesssim 0.05$ in the angular momentum. It predicts a resonance $J^P = 7/2^+$ at a mass ≈ 2340. In the scattering region, however, it gives

$$\mathrm{Re}\,\Delta(u) = -0.25 + 0.83u,$$

which disagrees seriously with the experimental data (2.2) and (2.3).

We of course should not overestimate the accuracy of these latter data, nor the reliability of the approximation (4.12). For comparison, we mention that the formula of James[34]

$$\Delta(W) = -0.19 + 0.37W + 0.81W^2,$$

which has its "intercept" $\Delta(0)$ higher by 0.06, has an accuracy only slightly inferior to our Eq. (4.12).

4.3. The Pion-nucleon Trajectory $N_{\alpha,\beta}$

It would seem that here parity doubling is on a relatively stronger footing. The point is that the resonances N_α (1688, $5/2^+$) and N_β (1680, $5/2^-$) are practically parity-degenerate. An obvious difficulty is that there is no partner N_β (~ 1000, $1/2^-$) with negative parity for the nucleon. Adherents of the degeneracy usually assume (see references in[34]) that for some reason or other the Regge residue $\beta(W)$ is equal to zero for N_β (1000, $1/2^-$).

A second possible view[35] is that the trajectory $N_{\alpha,\beta}$ is strongly asymmetric, the coincidence of the masses N_α (1688) and N_β (1680) is accidental, and the parity partner for N_α (940, $1/2^-$) (sic) is N (1550, $1/2^-$).

The first possibility (degeneracy) can of course not be excluded by simple arguments. However, the solution of the problem of the absence of N (1000, $1/2^-$) is here rather artificial. Besides this, a situation in which N_α has a degeneracy and N_γ and Δ_γ do not also fails to inspire much confidence from a theoretical point of view.

The second point of view does not seem to us to be acceptable. The point is that the masses of the particles N (1550) and N_β (1680) are "impermissibly close together"; the mean slope of the trajectory is here $N_\beta' \sim 5$. In view of the definite absence of a resonance N_β ($9/2^-$) with mass < 2 GeV the mean slope N_β' of the trajectory above N (1680) is at any rate not larger than 1.5. Therefore the trajectory must have a sharp local maximum in its derivative at $W \approx 1600$. It does not seem possible to reconcile such a maximum with the spectral representation. Accordingly we conclude that the problem of reconciling the experimental data on the pion-nucleon trajectories with the dispersion relations (4.5) may be decidedly difficult. The difficulties, as we have seen, show up clearly in the trajectory Δ_δ. Therefore the obtaining of more reliable experimental information about the behavior of the trajectory $\Delta(u)$ for $u < 0$ from πN scattering experiments is now a pressing matter.

If the results of[23] and[24] are confirmed, the accepted scheme for reggeization of the baryon amplitudes will come into contradiction with experiment.

On the other hand, if the formulas of[23] and[24] have to be amended ($\Delta_{\mathrm{scat}}(0) \lesssim -0.10$) and an approximation of the type of (4.12) is confirmed, we shall have a general picture of the pion-nucleon trajectories which is regrettably variegated. In this case it will turn out that

1) the trajectory N_α is almost ideally symmetric;

2) the trajectories Δ_δ and N_γ are decidedly asymmetric:

$$\Delta_\delta = -0.25 + 0.35W + 0.83W^2, \qquad (4.12)$$

$$N_\gamma = -0.57 + 0.25W + 0.73W^2 \qquad (4.13)$$

(a formula obtained by means of the resonances N_γ (1525, $3/2^-$), N_γ (2200, $7/2^-$) and N_δ (1860, $3/2^+$).

4.4. The Asymptotic Behavior

Let us turn to the question of the asymptotic behavior of the nucleon trajectories and the problem of subtractions. Here we shall ignore the odd part of the dispersion integral with two subtractions. We consider

$$\delta\alpha(u) = \frac{u}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{du'}{u'(u'-u)}\{\mathrm{Im}\,\alpha_+(u') + \mathrm{Im}\,\alpha_-(u')\}. \qquad (4.14)$$

The imaginary parts of the trajectories Δ_δ and N_γ at the points of the known resonances are plotted in Fig. 2.

One is struck by the fact that the imaginary parts of Δ_δ and N_γ lie practically on a single curve, very close to the linear function

$$\mathrm{Im}\,\alpha \sim 0.14\,(s - 1.17). \qquad (4.15)$$

Accordingly it is quite possible that the integral (4.14) does not exist for Δ_δ and N_γ and that the dispersion relations require a further subtraction.

If we take this point of view, the constant of the linear (in $u = W^2$) subtraction is an independent (of the behavior of Im α) new parameter. The fact that the slopes of the baryon and (with less accuracy) the meson trajectories are approximately equal can be ascribed in this case to a sort of universality of the linear-subtraction constant.

On the other hand, it is possible to assume that the imaginary parts of the trajectories increase slightly more slowly than linearly, so that the integral (4.14), though it does converge, still behaves "almost linearly" over a wide range of the variable. In this case the universality of the slope of Re α can be connected with the universality of Im α at large energies. This version can be illustrated with the following model approximate expressions for $\delta\alpha(u)$:

a) $$\delta\alpha(u) = c(u-u_0)\left[1 - a\frac{u-u_0}{\pi}\int_{u_0}^{\infty}\left(\frac{u'-u_0}{u'}\right)^{1/2}\frac{du'}{(u'-u_0)(u'-u)}\right]^{-1} + cu_0\left[1 + a\frac{u_0}{\pi}\int_{u_0}^{\infty}\frac{du}{u^{3/2}(u-u_0)^{1/2}}\right]^{-1},$$

b) $$\delta\alpha(u) = -c\,(u_0 - u)^{1-\varepsilon} + cu_0^{1-\varepsilon}.$$

Setting $c \approx 0.95$, $u_0 \approx 1.2$, $a = \pi\epsilon \approx 0.15$, we get a

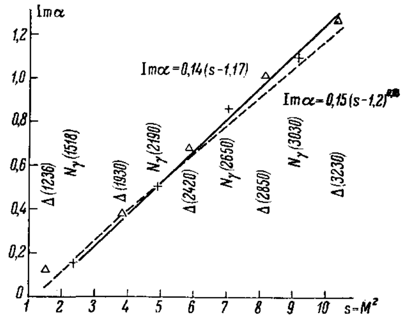

FIG. 2. The imaginary parts of the baryon trajectories Δ_δ and N_γ.

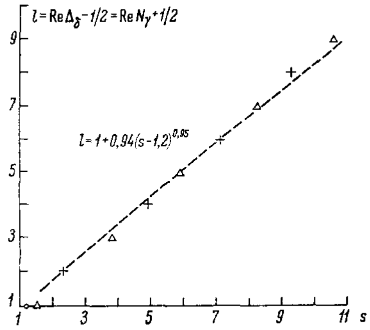

FIG. 3. The real parts of the baryon trajectories Δ_δ and N_γ..

good numerical description of both Im Δ_δ and Im N_γ, and also of the quasilinear terms in Re Δ_δ, Re N_γ (see Figs. 2 and 3).

5. THE GENERAL PICTURE

5.1. The Universality of the Linear Approximation

As was pointed out in Chapters 1 and 2, for the well studied trajectories ρ, Δ_δ, and N_γ the linear approximation is surprisingly "weakly broken." From the discussion in Chapters 3 and 4 it follows that after the second subtraction is performed the linear approximation corresponds to complete neglect of Im α, i.e., to the "infinitely narrow resonance" (INR) approximation. Since the slopes of these trajectories are nearly equal,

$$\rho' = 0.92, \quad \Delta'_\delta = 0.90, \quad N'_\gamma = 0.92, \tag{5.1}$$

and since as a rule the slopes of the other meson and baryon resonances [see (1.4), (1.5), (1.6), (2.5), (2.6) and Table IV] differ from (5.1) by not more than ±0.1 (so that they are equal to the accuracy with which they are determined), we get the impression that in the INR approximation all of the trajectories have the same universal slope

$$\alpha' \approx 0.90. \tag{5.2}$$

It was remarked in Sec. 4.4 that the equality of the slopes of the baryon resonances may be connected with the universality of the imaginary parts of the baryon trajectories in the asymptotic region. From this point of view the fact that the slopes of the meson trajectories are equal is extremely remarkable, since the behavior of the imaginary part Im ρ of the ρ trajectory does not correspond at all to the quasilinear increase of the nucleon trajectories Im Δ_δ, Im N_α, Eq. (4.15).

If we suppose that the fact that the slopes are equal is not accidental, the following interpretations are possible:

a) The imaginary parts of all trajectories (both baryon and meson) in the asymptotic region have a universal value and increase approximately linearly.

If this is so, then in the observed region $0 < s < 4\ (\mathrm{GeV/c})^2$ the quantity Im ρ has not yet reached its asymptotic behavior. It must be expected that, beginning with some value $s \sim 4M_N^2$, Im ρ will come to a behavior of the form (4.15). In this case, in the region $s > 4M_N^2$ we must expect broad meson resonances with $\Gamma_i^{tot} \sim 0.15\ M_i$.

b) The true asymptotic behavior of the imaginary parts is much weaker than linear (for example, Im $\alpha \sim s^{1/2}$), but for some unknown reason there is an "extra" linear term in the spectral representations, with universal properties.

It is therefore extremely important to solve the problem of determining the asymptotic form of the trajectories theoretically. There have recently been more and more papers on the asymptotic properties of trajectories. Among them we note a very curious paper by Tiktopoulos.[36]

Starting from the Logunov-Tavkhelidze quasipotential equation[37] for the partial waves of the scattering, Tiktopoulos studied the conditions for the occurrence of quasistable states with large orbital quantum numbers. In the case of a quasipotential $V(s, r)$ which has no singularities for small r and increases with the energy as

$$V(s, r) \sim v(s)\, u(r), \quad v(s) \to \infty \quad \text{for} \quad s \to \infty, \tag{5.3}$$

he found that the leading Regge trajectory increases as

$$l(s) \sim [v(s)]^{1/2}. \tag{5.4}$$

If we now assume that for $s \to \infty$ the quasipotential (5.3) is determined by the Pomeranchuk reggeon[38] and take $v(s) = s$, we get from (5.4) as the asymptotic form of the trajectory

$$l(s) \sim s^{1/2}.$$

The importance of this result lies in its universality. Since for any elastic scattering we have the Pomeranchuk reggeon as the $V(s, r)$, the asymptotic forms of all leading trajectories (both M and B) are the same, differing only by a numerical coefficient.

Unfortunately, one cannot get linear asymptotic forms in this way. However, this sort of argument throws definite light on the generality of the asymptotic *forms of different meson and baryon trajectories.*

5.2. The Veneziano Model and the Properties of Linear Trajectories

The linear approximation for trajectories with a universal slope has become very popular during the past year owing to the successes of the so-called Veneziano model[29] for the scattering amplitude.

This model is essentially based on the linear approximation for the trajectories and amounts to a construction of the invariant components of the scattering amplitude from linear combinations of "generalized" Euler beta functions, with the linear trajectories as their arguments:

$$B_N(\alpha_i(s), \alpha_j(t)) = \frac{\Gamma(1-\alpha_i(s))\,\Gamma(1-\alpha_j(t))}{\Gamma(N-\alpha_i(s)-\alpha_j(t))}, \tag{5.5}$$

where N is a positive integer.

The expression (5.5) has the following properties:

1) in the region $t < 0$, $s > 0$ it has poles at the points $\alpha_i(s) = n$, i.e., it describes resonances in the s channel;

2) in the asymptotic scattering region $t < 0$, $s \approx -u \to -\infty$ it has the Regge form, for example:

$$B_1(\alpha_i(s), \alpha_j(t)) \to \frac{\Gamma(1-\alpha_i(-u))\,\Gamma(1-\alpha_j(t))}{\Gamma(1-\alpha_i(-u)-\alpha_j(t))} \to (\alpha_i' u)^{\alpha_j(t)}\,\Gamma(1-\alpha_j(t)); \tag{5.6}$$

3) it possesses crossing symmetry $s \leftrightarrow t$.

A crossing-symmetric linear combination of expressions (5.4) has the Regge asymptotic form with the correct signature factor, and also satisfies[39] the sum rules at finite energy (FESR). It does, however, have one important shortcoming: the successive resonances in the direct channel, corresponding to a trajectory $\alpha_i(s)$, differ by one unit of angular momentum. To eliminate the superfluous poles in the amplitude one can impose a condition of the form

$$\alpha_i(s)+\alpha_j(t)+\alpha_j(u)=\text{const}. \tag{5.7}$$

For example, the condition

$$\alpha_i(s)+\alpha_j(t)+\alpha_j(u)=1$$

eliminates from the sum

$$B_1(\alpha_i(s),\ \alpha_j(t))+B_1(\alpha_i(s),\ \alpha_j(u))$$

all poles with odd $\alpha_i(s) = 2n + 1$.

In the model for the process $\pi + \pi \to \pi + \omega$, considered in Veneziano's original paper,[29] the condition

$$\rho(s)+\rho(u)+\rho(t)=2$$

eliminates the poles with even $\rho(s) = 2n$.

We see that to secure the Regge asymptotic form (5.6) it is necessary to have a linear behavior of the trajectory $\alpha_i(s)$ for $|s| \to \infty$. The condition for cancellation of the superfluous poles of (5.5) obviously requires not only linearity in the region of small values of s, t, u, but also equality of the slopes of two different trajectories, $\alpha_i' = \alpha_j'$. In the treatment of meson-baryon scattering the baryon trajectories of the s and u channels are connected by a relation of the form (5.7) with the meson trajectory of the t channel. Therefore the Veneziano model requires that the slopes of all meson and baryon trajectories be equal [i.e., that the condition (5.2) hold]. Accordingly, all of the trajectories are parallel.

By further arguments, using additional symmetry conditions, one can get connections between the various parallel trajectories.

5.3. Quantization of the Trajectories

Such connections in the framework of the Veneziano model have been obtained by Ademollo, Veneziano, and Weinberg,[40] starting from the Adler self-consistency condition for the amplitude for scattering with a soft pion. Introducing the concept of the "normality" of a trajectory, equal to $P(-1)^J$ for bosons and to $P(-1)^{J-1/2}$ for fermions (where P is the parity), they found that "if a particle lying on a trajectory α_1 can decay into a pion and a particle of the opposite normality lying on a trajectory α_2, then these trajectories must be parallel, and their intercepts $\alpha_i(0)$ differ by a half-integral number (equal to $\frac{1}{2}$)." Such pairs of trajectories are assigned to ρ and π, to K^* and K_L, and to Δ_δ and N_α. The respective differences of the intercepts, calculated from data in the particle region, are 0.48, 0.53, and 0.54.

In a recent paper[41] it was pointed out that the "quantization" of trajectories is of a more general nature than that found in[40].

For the analysis of the linear approximation of trajectories it is convenient to use the mass scale introduced in[42]:

$$x_{ik}=s-m_i^2-m_k^2; \tag{5.8}$$

here s is the square of the mass of the resonance, and m_i and m_k are the masses of the products of the main two-particle decay.

Let us write the Regge trajectory in the linear approximation in terms of the variable x:

$$\alpha_{ik}(s)=a_{ik}+\alpha'_{ik}(s-m_i^2-m_k^2). \tag{5.9}$$

The "absolute intercepts" a_{ik} of the trajectories as introduced in (5.9) have curious properties. To good accuracy they are the same for trajectories belonging to the same multiplet of the group SU(3), and thus are a simple characteristic of the properties of the trajectories passing through a multiplet J^P and its recurrences. It is found that[41]:

for baryons	for mesons	
$a_\delta\equiv a\,[10\,(3/2^+)]\approx 1,$	$a\,(1^-)\approx 1/2,$	(5.10)
$a_\alpha\equiv a\,[8\,(1/2^+)]\approx 1/2,$	$a\,(0^-)\approx 0.$	
$a_\gamma\equiv a[8\,(3/2^-)]\approx 0,$		

The deviations from the mean values are as a rule not larger than ±0.05. Accordingly, the "quantization rule" (5.10) is satisfied to the accuracy to which the coefficients of the Regge trajectories are determined in the linear approximation.

The conditions (5.10) are more general than the corresponding formulas of[40]. First, the "condition of opposite normality" is not essential for them. For example, the octet trajectories $8(\frac{1}{2}^+)$ and $8(\frac{3}{2}^-)$ have the same "normality," but the difference of their intercepts is equal to $\frac{1}{2}$. On the other hand, the trajectories $10(\frac{3}{2}^+)$ and $8(\frac{3}{2}^-)$ have opposite "normalities," owing to a transition with emission of a pion, but nevertheless the difference of their intercepts is 1.

Second, the quantization conditions (5.10) include trajectories with spinless decays (φ trajectory), and also with three-particle decays. We get the impression that the conditions for quantization of trajectories are not connected with the Veneziano model and the soft-pion approximation, but are of a deeper nature. In particular, they explicitly reflect the SU(3) symmetry.

6. CONCLUSION

In summary, it must be said that the phenomenological description of hadron interactions by means of Regge poles is a very fruitful idea, whose basic premises are now not subject to doubt.

We shall now list some problems whose solution is important for the further progress of the Regge-pole scheme:

1) The problem of vacuum poles, or more exactly, of the leading Pomeranchuk singularity.

2) The problem of the asymptotic forms of trajectories.

3) The problem of the universality of the nucleons in the linear approximation, associated with Problem 2).

4) The problem of the parity doubling of baryon trajectories.

5) The problem of the connection of the various symmetry breakings [of SU(3) symmetry, of chiral symmetry] with the properties of linear approximations, and of the nature of the quantization.

It is quite possible that these problems are more strongly interconnected than they may seem to be at first glance. Even a partial solution of these problems will aid in the conversion of the phenomenological scheme which the Regge-pole theory now provides into a full-fledged theory.

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