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STATUS OF THE THEORY OF PROPAGATION OF WAVES IN A RANDOMLY INHOMOGENEOUS MEDIUM*

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1. INTRODUCTION

CONTINUALLY more attention has been paid over the last 10–15 years to studying propagation of waves in randomly-inhomogeneous media. The heightened interest in this problem has been primarily due to the large number of pressing applied problems that arise in radio physics, acoustics, optics, plasma studies and certain other branches of physics. New phenomena pertaining to the same problem have gradually accrued to the classical objects of the theory, which are light scattering in the atmosphere and passage of radiation through the atmospheres of stars and planets. Diffuse reflection of radio waves from the ionosphere; scattering of sound and ultrasound in sea water; the so-called ultra-long-range propagation of ultrashort waves; incoherent scattering of radiowaves in the ionosphere; twinkling of extraterrestrial radio emission sources due to the ionosphere and the interplanetary plasma; and propagation of laser beams in air and in water: this is a characteristic (and of course not exhaustive) list of the problems that have arisen, not to mention the purely applied problems involving accuracy of measurement by radio methods of the coordinates of objects moving in the ionosphere or in outer space. Naturally, the abundance and variety of such problems has stimulated development and refinement of statistical methods for calculating wave fields propagating in a randomly-inhomogeneous medium or passing through a layer of such a medium.

This review is an attempt to outline the existing methods of the theory and their limits of applicability, as well as the role of the new methods for treating multiple scattering of waves that have recently developed at a heightened pace. These include the Markov approximation in the parabolic-equation method, or the application of methods that were originally developed in quantum electrodynamics for summing the series of the perturbation theory. The theory is being developed so intensively that a number of new results have not yet been reflected in the existing monographs and reviews.* Furthermore, the original papers are very numerous and scattered throughout journals concerned with the most varied topics. Therefore, it would probably be of some use to systematize them even partially from the standpoint of the theoretical methods used.

Propagation of waves in randomly-inhomogeneous media is such a vast field that we considered it expedient to limit our treatment only to problems of volume scattering in continuous media during free propagation. Thus, we do not touch upon reflection at randomly-uneven surfaces, nor upon scattering by dis-

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*Cf. the monographs by L. A. Chernov [1] and V. I. Tatarskiĭ, [2] individual sections in the books by E. L. Feinberg [3] and S. M. Rytov, [4] and the reviews. [5-15]

crete inclusions, such as artifical scatterers, aerosols, raindrops, and snow in the atmosphere, or by bubbles or fish in the water, nor propagation of waves in randomly-inhomogeneous feeders.*

However, even with these restrictions, the problems of both the theory itself and of its applications remain extremely varied. The object of applications of the theory might be in essence the propagation of waves in any real medium, ionized or neutral. This is because any such medium generally has random variations of the parameters, in addition to any possible regular inhomogeneity (which is usually very smooth on the scale of the wavelength λ). In a plasma, or in particular, in the interstellar and interplanetary media, as well as in the ionospheres of the planets, it is a question of random inhomogeneities of the electron concentration, temperature, and magnetic field, provided that the conditions for phenomenological description of a plasma have been met. In the lower atmosphere and in sea water it involves pulsations of density and temperature, and also fluctuations in the passive parameters, such as moisture in the atmosphere, salinity in sea water, etc. All of these lead ultimately to fluctuations in the velocity of wave propagation. Among the reasons for such random variations, we must mention above all turbulence and thermal fluctuations. We note that in essence we know the real statistical characteristics only for these fluctuations of the refractive index of the medium (see, e.g.[2,40-43]). Something is also known about non-thermal fluctuations in a plasma.[44-47]

In the propagation of waves, either electromagnetic or mechanical, fluctuations in the medium give rise to an entire series of phenomena that are superficially highly varied. Of course, in essence everything amounts to fluctuations in the amplitude, phase, and perhaps polarization of the wave. However, in observation, we have to deal with both twinkling of radiation sources and with variations in the angle of incidence, i.e., random refraction (as manifested, in particular, in blurring and trembling of the images of sources), and with fluctuations in the polarization (for transverse waves), and with scattering of waves from a limited volume of a medium, and with space-time correlation of fluctuations in the wave field and in its spectral composition and coherence, and with the behavior of directional waves, e.g., spreading of laser beams or directional diagrams of radio antennas, etc.

As we have said, the picture is no longer so variegated from the theoretical standpoint, but it is still far from being uniform.

To speak of posing the problem, in principle it involves a stochastic wave equation. Of course, in a number of cases the equations can be more complex than the Helmholtz equation. They can contain not only the second, but also the first derivatives of the wave field. They can be vector equations and form a system of simultaneous equations, as with electromagnetic waves or elastic waves in a solid. However, the main thing, which is characteristic even in the simplest case of a scalar wave equation, is that it involves a parametric equation, even if only a linear one: the random functions of position and time that describe the fluctuations in the properties of the medium do not enter additively, as "external forces," but as coefficients in the equation itself. If, for example, the time variations in the medium are so slow that we can take account of them in a quasi-steady-state manner, then with a monochromatic source (primary wave), the wave equation will be

$$\Delta u+k_0^2\varepsilon(\mathbf{r},\ t)\,u=0. \qquad (1)$$

The random "dielectric constant" $\epsilon(\mathbf{r}, t)$ enters here as a coefficient of the sought wave function u. This is the root of all the mathematical difficulties of the theory, since we don't know how to find an exact solution of such a wave equation.

This makes it necessary to apply certain approximate methods that make use of every "smallness" permitted by the conditions of the real problem. Most often this is a smallness of the fluctuations of ϵ, or its deviations from the mean value. It can also be the smallness of the wavelength in comparison with the dimensions of the inhomogeneities, etc. It is precisely these additional restrictions that the various approximate methods depend on. Quite evidently, one of the fundamental problems here is the region of applicability of the results provided by any of these methods. Often this problem proves to be far from simple, since the direct route of comparing the approximate with the exact solution is blocked. We should note that, even if it were open, i.e., we knew the exact solution, this wouldn't at all imply that the problem itself had been liquidated. It is by no means always expedient to find the exact solution and then simplify it according to the distinctive features of the problem, since the approximate method can give the same thing, but more quickly and perspicuously.

In practice, two types of problems arise: the direct problem, in which one has to find the statistics of waves propagating in this medium from the known statistics of the medium, and the inverse problem, which consists in drawing conclusions on the properties of random inhomogeneities from the measured moments of the field (correlation functions, spectra, etc.). However, from the standpoint of theory these problems are equivalent: we need the relation between the two statistics, which has been established as yet for a few first moments (the mean field, and the mean bilinear quantities, including the intensity and the fluctuation in the intensity itself).

In proceeding to describe the existing approximate methods, we shall assume for simplicity that the medium is on the average homogeneous and stationary, $\langle\epsilon\rangle$ = const., while the fluctuations are quasistatistical, i.e., $\epsilon = \langle\epsilon\rangle[1 + \tilde{\epsilon}(\mathbf{r})]$. Removing the two restrictions does not give rise to any difficulties in principle. We can rather easily extend the theory also to media that are on the average inhomogeneous and non-stationary. However, of course, this can be done under the condition that both processes occur smoothly and slowly enough. It is also not difficult to take into account a statistically steady-state time-dependence of the fluctuations.

*There is an extensive literature on scattering of waves at rough surfaces. See, e.g., the monographs [4,16] and the reviews. [17-20] A large number of studies have been devoted to scattering by discrete inclusions. [21-34] For waves in randomly-inhomogeneous waveguides, see [35-39].

2. METHOD OF SMALL PERTURBATIONS. FIRST APPROXIMATION

If $\tilde{\varepsilon}$ is small enough, then we can naturally resort to the method of perturbations, and expand u in a power series in $\tilde{\varepsilon}$, or more exactly, in $\sqrt{\langle\tilde{\varepsilon}^2\rangle}$. If we write (1) in the form

$$\Delta u + k^2 u = -k^2 \tilde{\varepsilon} u \qquad (k^2 = k_0^2 \langle\varepsilon\rangle) \tag{2}$$

and use the Green's function for a homogeneous medium $g(\mathbf{r}, \mathbf{r}')$, we can represent (2) as an integral equation

$$u(\mathbf{r}) = u_0(\mathbf{r}) - k^2 \int g(\mathbf{r}, \mathbf{r}')\, \tilde{\varepsilon}(\mathbf{r}')\, u(\mathbf{r}')\, d^3\mathbf{r}'. \tag{3}$$

Here $u_0(\mathbf{r})$ is the primary field that would have propagated in the medium in the absence of fluctuations. By solving (3) by iterations, we get the perturbation-theory series

$$u(\mathbf{r}) = u_0(\mathbf{r}) - k^2 \int g(\mathbf{r}, \mathbf{r}_1)\, \tilde{\varepsilon}(\mathbf{r}_1)\, u_0(\mathbf{r}_1)\, d^3\mathbf{r}_1 + k^4 \int\int g(\mathbf{r}, \mathbf{r}_1)\, g(\mathbf{r}_1, \mathbf{r}_2)\, \tilde{\varepsilon}(\mathbf{r}_1)\, \tilde{\varepsilon}(\mathbf{r}_2)\, u_0(\mathbf{r}_2)\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2 + \ldots \tag{4}$$

The nth term of the series describes n-fold scattering, and contains the n-fold product $\tilde{\varepsilon}(\mathbf{r}_1)\ldots\tilde{\varepsilon}(\mathbf{r}_n)$ inside an n-fold integral. Thus, even in calculating $\langle u\rangle$, we have to know the moments $\langle\tilde{\varepsilon}(\mathbf{r}_1)\ldots\tilde{\varepsilon}(\mathbf{r}_n)\rangle$ of $\tilde{\varepsilon}$ of all orders.

However, under certain conditions that we shall discuss below, we can restrict ourselves to the first (after u_0) term of the series. This is the single-scattering approximation, which even Rayleigh had used in optical problems, and which is often called the Born approximation in quantum mechanics. In this approximation, the amplitude u_0 of the primary field does not decline as it penetrates deeper into the medium (extinction is ignored), while the scattered field depends linearly on $\tilde{\varepsilon}$. Hence, in order to calculate the correlation function u_s of the scattered field, and particularly, its intensity $I_s \sim \langle|u_s|^2\rangle$, it suffices to know the correlation function of $\tilde{\varepsilon}$, i.e., $B_\varepsilon(\rho) = \langle\tilde{\varepsilon}(\mathbf{r}_1)\tilde{\varepsilon}(\mathbf{r}_2)\rangle$, where $\rho = \mathbf{r}_1 - \mathbf{r}_2$. All of this is obtained very easily, and one can easily make an estimate when this approximation is permissible.

Evidently, the energy scattered by a certain volume L^3 of the medium must be much smaller than the energy of the primary wave incident on it[2,4,48] For example, if the correlation function is Gaussian with a correlation radius l (this is the mean dimension of the inhomogeneities), i.e., if $B_\varepsilon(\rho) = \langle\tilde{\varepsilon}^2\rangle e^{-\rho^2/l^2}$, we get the condition

$$\langle\tilde{\varepsilon}^2\rangle kL \ll \frac{1}{kl\,(1 - e^{-2k^2l^2})} \equiv F(kl). \tag{5}$$

Thus, there are upper bounds both to the intensity of fluctuations $\tilde{\varepsilon}$ and to the path L of the wave in the medium: the latter must be small in comparison with the so-called extinction length $d = F(kl)/k\langle\varepsilon^2\rangle$. The trend of $F(kl)$ is shown in Fig. 1.

The restriction (5) becomes more rigid as $kl = 2\pi l/\lambda$ becomes larger, i.e., as the inhomogeneities become larger in comparison with λ. This is understandable, since with increasing kl the directional diagram (indicatrix) of the scattering by each inhomogeneity is extended more and more in the forward direction, in the direction of the wave incident on the inhomogeneities. Hence, re-emission by many inhomogeneities, or multiple scattering, acquires ever greater importance.

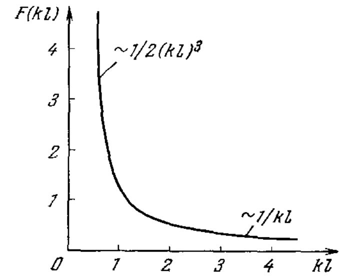

FIG. 1

The Born approximation proves quite sufficient in very many problems, not only for a scalar, but also for an electromagnetic field. This especially pertains to fluctuations $\tilde{\varepsilon}$ of thermal origin, which are usually rather weak. The Rayleigh theory of light scattering was thoroughly based on the Born approximation.[49-51] It also describes well the so-called "incoherent" scattering of radio waves in the ionosphere and in a laboratory plasma.[42,52-82] Along both the cited lines, the detailed theory of single scattering helps in solving the inverse problem, i.e., finding the parameters of the medium from the observed characteristics of the scattered field. When there are independent methods of measuring the parameters of the medium, we can test the conclusions of the scattering theory itself.

Thus, for example, in[83], measurements of the electron concentration in the ionosphere by the incoherent-scattering method were compared with the results of pulse sounding of the ionosphere and with data obtained with a dispersion interferometer. Here they got rather good quantitative agreement.

The Born approximation also describes well ultrashort-wave scattering by fine-scale turbulent inhomogeneities in the troposphere and the lower ionosphere (one of the possible causes of tropospheric scattering and the so-called ultra-long-range propagation of ultrashort waves found in 1952).[85-105,2,5] In this problem we also have to take into account the regular refraction of radio waves. In the absence of reflection from the ionosphere, this can be taken into account in the geometrical-optics approximation. Refraction increases the intensity of the primary field in certain regions of space, and correspondingly, these regions contribute more to the scattered field.[98,106-111] This phenomenon is most clearly marked in scattering from the region of a caustic (a region of reflection of radio waves).[112-115] Incidentally, here we can no longer describe the primary field in the ray approximation. Calculations show that a layer situated near the reflection region, and amounting to 10–15% of the total thickness of the inhomogeneous layer, contributes about as much to the scattered field as the rest of the ionosphere does.

The literature has treated single scattering of electromagnetic waves in isotropic[2,3] and in anisotropic (gyrotropic) media,[116-118] scattering of sound waves,[119-130,2] scattering in an inhomogeneous medium in the presence of a phase boundary,[131-135], scattering of pulsed and modulated signals,[136-139] the effect of the

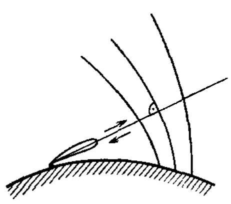

FIG. 2

directional diagrams of antennas,[140-143] scattering by anisometric inhomogeneities,[144,2] time and frequency correlation functions,[2,116,136,147,148] standing-wave scattering,[149] etc. We can point out another problem which contains no difficulties in principle in Born-approximation calculations, but is of great practical interest at present. We have in mind the scattering of radio waves by inhomogeneities of electron concentration in the polar ionosphere—the so-called "auroral radio reflections."

A quantitative theory of this phenomenon (in the single-scattering approximation) has been developed in[150-153]. In[152], they took into account practically all complicating factors: anisotropy of the medium, regular refraction, elongation of the inhomogeneities toward the geomagnetic pole, etc. Elongation of the inhomogeneities (anisometry) has the result that the backward scattering fundamentally arises from those inhomogeneities that are orthogonal to the wave vector of the incident wave (the condition of "aspect sensitivity," Fig. 2). This is precisely why the "auroral radio reflections" are preferentially observed in a northward direction in the Northern Hemisphere.

To return to theoretical problems, we should note that the correlation function B_ϵ proves to be unsuitable for describing turbulent fluctuations in ϵ. Turbulent fluctuations are characterized by a continuous range of scales l, and as the inhomogeneities become coarser, their specific weight increases very rapidly. However, over a broad range of l, from l_0 (the inner scale of the turbulence) to L_0 (the outer scale), we can consider the turbulent field to be locally homogeneous and isotropic, and describe it with the so-called structure function

$$D_\varepsilon(\rho) = \langle[\tilde\varepsilon(\mathbf{r}_1) - \tilde\varepsilon(\mathbf{r}_2)]^2\rangle, \quad \rho = |\mathbf{r}_1 - \mathbf{r}_2|,$$

which automatically eliminates the effect of the excessively large inhomogeneities.

If the fluctuations in ϵ are due to turbulent temperature fluctuations, then $\tilde\epsilon$ will obey, as the temperature itself does, the Kolmogorov "$2/3$ law":

$$D_\varepsilon(\rho) = C_\varepsilon^2 \rho^{2/3} \qquad (l_0 \ll \rho \ll L_0).$$

This corresponds to a power-law spatial spectrum having a density

$$\Phi_\varepsilon(\varkappa) = A C_\varepsilon^2 \varkappa^{-11/3} \qquad \left(\frac{2\pi}{L_0} \ll \varkappa \ll \frac{2\pi}{l_0}\right).$$

Changing from B_ϵ to D_ϵ does not affect the procedure for solving the wave equation, but, of course, it gives rise to its own peculiarities.[2] In a number of problems in which the scales of the most substantial inhomogeneities lie in the interval (l_0, L_0), it proves to be more convenient to describe the statistical structure of the ϵ field by means of the structure function than by B_ϵ, since then one doesn't introduce any "superfluous" parameters. However, such a description does not suffice for other problems in which the most essential scales lie outside the interval (l_0, L_0). For example, one can no longer find the extinction within the confines of the "$2/3$ law," since the effective scattering cross-section diverges as $\kappa \to 0$. This means that the extinction is determined by inhomogeneities that are so large $(l > L_0)$ that they no longer show isotropic behavior.

3. SCATTERING BY LARGE-SCALE INHOMOGENEITIES

As kl increases, we must either take into account the later terms in the perturbation-theory series (as will be discussed in Sec. 4), or go over to other approximate methods that deal with multiple scattering to some extent, and which use the smallness of the parameter $1/kl \sim \lambda/l \ll 1$, i.e., that approach geometrical optics. V. A. Krasil'nikov[154-157] was the first to use the geometrical-optics approximation in problems on propagation of waves in randomly-inhomogeneous media. This was later applied to various problems (some of which will be mentioned below) by Bergman,[158] Ellison,[159] V. Ya. Kharanen,[160] Muchmore and Wheelon,[5] L. A. Chernov,[1] Bremmer,[161] V. I. Tatarskii,[2] and many other authors.

Among the methods adapted to the case of large-scale inhomogeneities is the method of smooth perturbations proposed by S. M. Rytov[162] for the non-statistical problem of diffraction of light by ultrasound, and the parabolic-equation method, which was first used by by M. A. Leontovich[163] to solve the likewise non-statistical problem of propagation of radiowaves above the Earth's surface. A. M. Obukhovich[164] applied the former method (MSP) to propagation of waves in randomly-inhomogeneous media in 1953. As for the parabolic-equation method (PEM), it was first applied only in 1964 to volume statistical problems on wave propagation by L. A. Chernov[165] and L. S. Dolin.[166]

If we assume in Eq. (1) that $u = U(r)e^{-ikx}$, where the x coordinate is defined by the direction of propagation of the primary wave, we get the following equation for the complex amplitude U:

$$-2ik\frac{\partial U}{\partial x} + \Delta U = -k^2\tilde\varepsilon U.$$

If $x \gg l \gg \lambda$, then $|\partial^2 U/\partial x^2| \ll 2k|\partial U/\partial x|$ (of the order of l/x). Then we can replace the total Laplacian Δ by the transverse Laplacian $\Delta_\perp = (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$. This leads to the parabolic equation

$$-2ik\frac{\partial U}{\partial x} + \Delta_\perp U = -k^2\tilde\varepsilon U. \tag{6}$$

In using Eq. (6), people often speak of the differential approximation. The source of this terminology is evident from the form of the left-hand side (although the diffusion coefficient is imaginary, as it is in the Schrödinger equation), and the physical meaning consists in a slow transverse diffusion of the energy of the wave field with increasing x.

The method of smooth perturbations (MSP) is distinguished by the fact that one introduces the complex phase $\psi = S + i\chi$ in place of U. Here S is the phase proper, while χ is the logarithm of the amplitude, or the so-called level. Substituting $U = e^{-i\psi}$ into (6) gives the fundamental MSP equation:

$$2k\frac{\partial\psi}{\partial x}+i\Delta_{\perp}\psi+(\nabla_{\perp}\psi)^2=k^2\tilde{\varepsilon}, \quad (7)$$

In contrast to (6), it is no longer parametric, but instead, is non-linear.

The Green's function of the left-hand sides of the parabolic equation (6) and the linearized Eq. (7) (i.e., Eq. (6) with $\tilde{\epsilon}=0$ and Eq. (7) with $(\nabla_\perp\psi)^2 - k^2\tilde{\epsilon}=0$) correspond to the so-called parabolic or Fresnel approximation: $(1/4\pi R)e^{-ik(y^2+z^2)/2x}$ replaces the exact function $(1/4\pi R)e^{-ik(l-x)}$. This "abbreviated" Green's function is suitable for path lengths x of the wave in the medium such that the ratio of the area of the Fresnel zone ($x\lambda$) to the area of the inhomogeneities (l^2) is small in comparison with l^2/λ^2:

$$\frac{\lambda x}{l^2} \ll \frac{l^2}{\lambda^2}.$$

Thus, for large-scale inhomogeneities ($l/\lambda \gg 1$), the two equations make it possible to cover the regions of both Fraunhofer diffraction ($1 \ll (\lambda x/l^2) \ll l^2/\lambda^2$) and Fresnel diffraction ($\lambda x/l^2 \sim 1$), and of geometrical optics ($\lambda x/l^2 \ll 1$).* Here, even in the geometrical approximation, it is quite permissible that $x/l \gg 1$, since $\lambda/l \ll 1$. That is, a very great number of individual inhomogeneities can occur in the path of the beam. However, one must solve the complete equations (6) and (7). Then, in spite of the fact that they are first-order in x, one still can't achieve an exact solution, since (6) is parametric and (7) is non-linear. Consequently, one must also resort here to the procedure of perturbations and rely on the smallness of the fluctuations.

Before proceeding to describe this procedure, we note that, in spite of the equivalence in principle of the PEM equation (6) and the MSP equation (7), the results of solving them by the perturbation method do not by any means always agree, and they have different regions of applicability. In particular, the "sphere of action" of the first approximation of the MSP is restricted by the condition of smallness of the fluctuations in the level ($\langle\chi^2\rangle \lesssim 1$). On the other hand, the PEM is suitable also in the region of strong amplitude fluctuations (for more details on this, see parts (a)—(c) of this section). Hence, the problem of which of these two methods one prefers involves to a considerable extent how the problem is posed. For example, if one is interested in the behavior of the phase S, rather than the field strength, then the MSP is convenient: however, if one must calculate the moments of the field itself, then it is better to use the PEM.

We shall first take up the MSP.[1,2,4,164]

*In a turbulent medium, i.e., when there are inhomogeneities of many different scales, we must, of course, adopt the extreme values of l in these inequalities. For example, the Fraunhofer zone lies at distances x such that $L_0^2 \ll \lambda x \ll l_0^4/\lambda^2$. This is possible only when $\lambda L_0 \ll l_0^2$. The geometrical-optics region corresponds to $\lambda x \ll l_0^2$. [167,1,2] However, as Taylor [168,169] has shown, the region of applicability of the geometrical-optics approximation is broader for the phase shift, namely being $\lambda x \ll 21\, l_0^{7/6} L_0^{5/6}$.

a. Method of Smooth Perturbations

Expansion of the complex phase in the series $\psi = \psi_1 + \psi_2 + \ldots$, where ψ_n is of the order of $\langle\tilde{\epsilon}^2\rangle^{n/2}$, gives rise to a system of linear equations of successive approximations

$$\left.\begin{aligned} &2k\frac{\partial\psi_1}{\partial x}+i\Delta_\perp\psi_1=k^2\tilde{\varepsilon},\\ &2k\frac{\partial\psi_2}{\partial x}+i\Delta_\perp\psi_2=-(\nabla_\perp\psi_1)^2,\\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (8)$$

The right-hand sides of these equations rapidly become more complicated with increasing n. Just as in the Born approximation, one naturally considers first the problem of the conditions under which one can limit the treatment to the first approximation ψ_1. This problem is not simple, and a rather large number of people have dealt with it, both in our country in the early sixties,[2,3,170-173] and abroad, where a rather lively discussion set in between 1964 and 1969.[174-185,13,528]

The point is that we can expect to find an exhaustive solution of this problem only from a developed theory of multiple scattering. There is only one thing remaining within the confines of the perturbation theory itself: to compare the second approximation with the first, without any great reliance in the requirements set forth.

Such a comparison gives very rigid restrictions on $\langle\chi_1^2\rangle$ and $\langle S_1^2\rangle$. Indeed, the level χ must satisfy the conditions $|\langle\chi_2\rangle| \ll 1$ and $\langle\chi_2^2\rangle - \langle\chi_2\rangle^2 \ll \langle\chi_1^2\rangle$. However, we can show that $\langle\chi_2\rangle = -\langle\chi_1^2\rangle$, and $\langle\chi_2^2\rangle \sim a\langle\chi_1^2\rangle^2$ for a plane or spherical primary wave (a is a coefficient of the order of unity). Hence, for a medium having fluctuations on a single scale, the given inequalities for χ_2 (and analogous inequalities for the phase S_2) lead to the conditions $\langle\chi_1^2\rangle \ll 1$ and $\langle S_1^2\rangle \ll 1$. As we can easily see, these are equivalent to the conditions for applicability of the Born approximation. As we have said, the necessity of these extremely rigid conditions is far from evident. Therefore, we cannot regard as well-grounded the conclusion that the first MSP approximation is equivalent to the Born approximation, i.e., the theory of single scattering (see, e.g.,[186]).

In fact, as easly as 1962, V. I. Tatarskiĭ[173] took up the point that the conditions for smallness of the corrections of the second approximation must be imposed in a different way for the phase fluctuations than for the fluctuations in level, provided that, as usual, we are not interested in the overall shift of the phase S, but only in the phase difference, either ∇S (random refraction) or $S(\mathbf{r}_1) - S(\mathbf{r}_2)$ when $\rho = |\mathbf{r}_1 - \mathbf{r}_2|$ is not too great (interference, imaging of a source in the focal plane of a lens). In these cases, it suffices to require that the basis should obey the inequality $D_{S_2}(\rho) \ll D_{S_1}(\rho)$. For a locally homogeneous and isotropic turbulence, it turns out that $\langle\chi_2^2\rangle = a\langle\chi_1^2\rangle^2$, and $D_{S_2}(\rho) = b[D_{S_1}(\rho)]^2$. Here a and b are numbers of the order of unity that depend on the exponent in the spatial spectrum. Consequently, the imposed conditions take on the form

$$\langle\chi_1^2\rangle \ll 1, \quad D_{S1}(\rho) \ll 1.$$

In a turbulent medium obeying a "$^2/_3$ law," the first of these conditions gives

$$\langle \chi_1^2 \rangle \sim C_\varepsilon^2 k^{7/6} x^{11/6} \ll 1.$$

The condition that the structure function of the phase is small ($D_{S_1} \ll 1$) gives rise (in the "inertial" interval $l_0 \ll \rho \ll L_0$) to a quite different inequality, namely

$$C_\varepsilon^2 k^2 x \rho^{5/3} \ll 1,$$

That is, when the basis is small enough ($k\rho \ll \sqrt{kx}$), the restriction on the fluctuations in phase differences $S(r_1) - S(r_2)$ proves to be considerably weaker than the restriction on the dispersion.*

A number of attempts have been undertaken to estimate $\langle \chi^2 \rangle$ more accurately, based on the non-linear equation (7). This requires summation of a perturbation theory series expanded in the quantity $(\nabla_\perp \psi)^2$. Here the fundamental difficulty lies in the fact that turbulent fluctuations prove to follow the relation $\langle \chi_n^2 \rangle = c_n \langle \chi_1^2 \rangle^n$. Here the numerical constants c_n are multiple integrals. Hence, in the series for $\langle \chi^2 \rangle$, one can't isolate the principal subsequence that gives the fundamental term in the expansion in the new small parameter. The only thing that this approach provides is the conclusion that under a "$^2/_3$ law" (or any other self-modeling law)

$$\langle \chi^2 \rangle = f(\langle \chi_1^2 \rangle).$$

The attempts to establish the explicit form of the function f are too crude,[187-190,2] and we shall spend no time on them.

Since the theory cannot yet reliably indicate the limits of applicability of the first MSP approximation, it is natural to see what conclusions we can draw from the experimental data.

An experimental test of the course of $\langle \chi^2 \rangle$ as a function of the parameter $\langle \chi_1^2 \rangle$, i.e., the intensity of fluctuations of the level, as calculated in the first approximation, has been made in the Institute of Atmospheric Physics of the Academy of Sciences of the USSR. This was done initially by M. E. Gracheva and A. S. Gurvich,[191] M. E. Gracheva,[192] and A. S. Gurvich and M. A. Kallistratova[193] using an incoherent light source. It has since been done with a laser by the associates at the Institute of Atmospheric Physics[194,195] and by some American investigators.[196,197] Measurements of the twinkling of the light source made on an aboveground course gave the result[195] shown in Fig. 3. The first approximation holds up to $\langle \chi_1^2 \rangle \sim 1$, i.e., farther than indicated by the condition outlined above. Then $\langle \chi^2 \rangle$ leaves the besectrix, passes through a maximum, and begins gradually to decline.

Another consequence of the first MSP approximation is also confirmed experimentally: the logarithmic normal distribution law of the field amplitude, at least when $\langle \chi_1^2 \rangle \lesssim 1$.[191-195,198-201] (see also Part d of this section).

Thus, experiment says that the first MSP approximation for the level χ fails only when $\langle \chi_1^2 \rangle \gtrsim 1$. For this region (the so-called region of strong amplitude fluctuations), the theory of fluctuations of the level based on the MSP cannot yet be considered to be completed.

The situation differs completely with respect to phase fluctuations. As experiment shows,[193] calculation of the phase fluctuations (random refraction) by the first MSP approximation proves to be applicable even in the region of very strong fluctuations in the level. This conclusion confirms the theoretical estimates of V. I. Klyatskin.[202]

However, the results of calculating the phase fluctuations and angles of approach of the wave by the MSP are practically the same as those given by the method of geometrical optics (differing by no more than a factor of two[1,2]). Hence, what we have said implies that we can expect the corresponding geometrical-optics phase calculations to be also applicable in the region of strong fluctuations in the level. As was shown in[203], in the language of geometrical optics, the region of strong amplitude fluctuations is the region of strongly developed caustics (see Fig. 4, which shows schematically a picture of the caustics in a medium containing inhomogeneities as a plane wave is propagated through it). Naturally, in this region neither the MSP nor, a fortiori, geometrical optics can give correct results for the amplitude (in the first approximation of the perturbation theory). However, the caustics have practically no effect on the size of the fluctuations of the phase (the phase jumps of $\pi/2$ at each contact with a caustic are evidently insubstantial when $\langle S_1^2 \rangle \gg 1$). We can suppose that also the diffraction effects have a weaker influence on the behavior of the phase than of the amplitude. This is just why, in

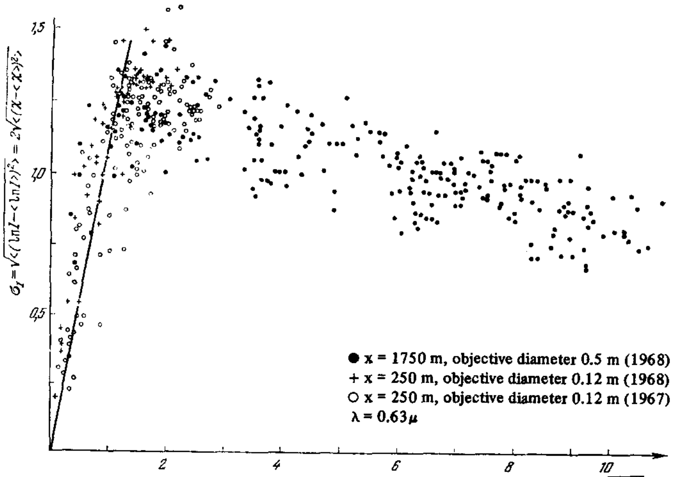

FIG. 3

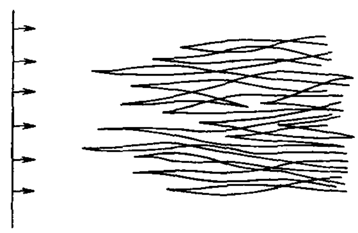

FIG. 4

*The restriction on the phase dispersion $\langle S_1^2 \rangle$ depends now on the size of the basis ρ. When $\rho \gtrsim L_0$ (where L_0 is the outer scale of the turbulence), the relation $D_{S_1} = 2\langle S_1^2 \rangle$ holds. Hence, the inequality $D_{S_1} \ll 1$ implies that $\langle S_1^2 \rangle \ll 1$. However, if $\rho \ll 1$, and simultaneously $\sqrt{\lambda x} \ll L_0$, then the condition $D_{S_1} \ll 1$ allows values of $\langle S_1^2 \rangle \gtrsim 1$ when the inequality $\langle \chi_1^2 \rangle \ll 1$ is obeyed. That is, we get a restriction on $\langle S_1^2 \rangle$ that is less rigid than in the Born approximation.

spite of the restriction $\langle \chi_1^2 \rangle \lesssim 1$, that the first MSP approximation in many cases proves adequate, and a great many studies have been based on it.

First of all, we note that the first of the equations (8) allows an exact solution for ψ_1 if we reinsert into it the total Laplacian:

$$\psi_1(\mathbf{r}) = \frac{k^2}{4\pi} \int \tilde{\varepsilon}(\mathbf{r}') \frac{U_0(\mathbf{r}')}{U_0(\mathbf{r})} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}',$$

Here $U_0(r)$ is the complex amplitude of the primary field in the medium without the fluctuations in ϵ. Thus, one can find ψ_1 for the most varied forms of the primary wave—not only for a plane wave,[164,1,2,4,204] but also for a spherical wave,[2,205-213] and also for spatially-limited beams.[214-229,529-532] If we know $\psi_1(r, \omega)$, we can calculate the various correlation functions of the phase and the level, both in space and in time and frequency,[1,2,4,6,13,230-243] and use them to solve various problems that arise, e.g., in interpreting observations of star twinkling,[2,244-251] in studying propagation of radio signals in interplanetary space,[252] propagation of pulsed or modulated signals,[253-257] etc.

The mentioned correlation functions are also used for finding the effective parameters of radiotechnical and radioastronomical antenna systems (the mean directivity pattern, the coefficient of directivity, the limiting resolving power in a turbulent medium, etc.),[258-266,533] the parameters of optical receiving systems (limiting resolution, image transmission quality through a medium containing inhomogeneities, signal/noise ratio in systems with heterodyne action, etc.)*,[174,208,267-284,511,534-537] in studying fluctuations in the focus of a lens,[1,2,285-292] in accounting for the effect of the finite dimensions of the receiver and source on fluctuations of the received signal,[2,212,247,294-298] in studying scattering by conductors placed in a medium containing random inhomogeneities,[299] etc.

There are no difficulties in principle in extending the MSP to the case of anisotropic fluctuations in a medium that is isotropic on the average,[206,257,300-303] and also to anisotropic randomly-inhomogeneous media,[161,304] to absorptive media,[305-309] to media showing gradual spatial variation in the mean value $\langle \epsilon \rangle$, the variance $\langle \tilde{\epsilon}^2 \rangle$, or the structure constant C_ϵ^2.[2,6,204,208,209,212,278,284,310-313] The MSP has also been used to treat problems of depolarization of a light wave,[314-316] and of scattering of a light wave during reflection from the ionosphere.[317] We note that an attempt had already been undertaken[189] to perform a partial summation of the perturbation-theory series for the complex phase $\psi = S + i\chi$ by integrating over random trajectories. In other words, the MSP, just like the Born approximation, permits one to take into account an entire series of all the possible complicating factors.

*The studies cited here on antennas and optical systems include studies in which the wave parameters were calculated not only by the MSP, but also by geometrical optics and the PEM, and also studies that used existing statistical results from other publications. Our collecting the references to all these articles together in one place is intended to emphasize here the antenna and optical specifics of the problems, rather than the features of the method of calculating the statistical parameters. Besides, within the framework of the assumptions made in the cited studies, the PEM and geometrical optics give practically the same results as the MSP.

Of course, one can also take into account the same complicating factors in the geometrical-optics approximation. Studies have been made in this approximation of fluctuations in the amplitude and phase of plane and spherical waves,[1,2,318-328] waves in a gyrotropic medium,[329-331] waves emerging from the ionosphere,[332] or passing through interplanetary space,[333] fluctuations in laser beams,[334-337] and changes in polarization of a wave.[335,338] A number of studies have been concerned with ray trajectories under various conditions: in isotropic media,[1,2,158,160,161,203,325,339-343,538] in gyrotropic media,[344,331] in the presence of refraction,[325,345,346] etc. With an appropriate generalization, the method of geometrical optics makes it possible to describe fluctuations in the parameters of waves even in the vicinity of caustics.[331,346-349]

We may note the double role of the method of geometrical optics in the theory of propagation of waves in media containing large-scale inhomogeneities. On the one hand, it serves as a heuristic basis for asymptotic methods that take into account diffraction effects (the MSP, PEM, and their modifications), as these methods often directly use ray representations.[2,187-189,202,342,350,351] On the other hand, the geometrical-optics approximation gives a satisfactory quantitative description of certain statistical characteristics of a wave, and first of all, its phase and angle of incidence, as has been discussed above. This is especially essential for media that are inhomogeneous on the average, for which calculations by the MSP are difficult.

b. The Parabolic-equation Method

We recall that in a homogeneous medium ($\tilde{\epsilon} = 0$), Eq. (6) admits an exact solution for a given value of the field at the boundary $x = 0$:

$$U(x, y, z) = \frac{k}{2\pi i x} \int\limits_S U_0(\eta, \zeta) \, e^{-\frac{ik}{2x}[(y-\eta)^2 + (z-\zeta)^2]} \, d\eta \, d\zeta, \qquad (9)$$

where $U_0(y, z) = U(x, y, z)|_{x=0}$. This parabolic-equation solution is also often used in statistical problems in which the field at the boundary is random, while the scales of the inhomogeneities of this field are large in comparison with the wavelength. Such an application of Eq. (9) is illustrated by the results from diffraction of a random (partially coherent) wave field by apertures whose dimensions are much larger than λ,[352-360] studies of the characteristics of large antennas with random variations of the currents in the aperture,[361-368] as well as the studies mentioned in the preceding section on the diffraction pattern at the focus of a lens,[2,285-292] (in the latter case, one introduces under the integral in (9) the focusing factor $\exp(ik(\eta^2 + \zeta^2)/2F)$, where F is the focal length).

The same type of problem with random boundary conditions includes also the frequently-applied model of a thin phase (or amplitude-phase) shield that is taken to replace a layer of a medium containing volume inhomogeneities. The widespread use of this replacement is mainly due to the substantial simplification of calculating the statistical characteristics of the field beyond the shield. This is just why this model is indispensable, although problems with random boundary conditions lie outside the framework of this review, which is concerned with problems of the theory of volume scattering.

The theory of a pure phase shield has been developed in greatest detail.[369-397] This model has been used to treat varied problems on propagation of waves through the ionosphere,[96,398-405] through the interplanetary plasma,[406-416] and very recently (since the discovery of pulsars) through the interstellar plasma as well.[417-419] The phase-shield model makes it possible to study propagation of pulsed signals,[420] the averaging action of a receiving aperture,[421,422,539] the effect of horizontal gradients,[423] and the possible appearance of caustics.[424]

However, the vulnerable side of this model is the problem of the equivalence of the shield.

If we understand the shield to be the exit surface of waves from a layer of the medium containing volume inhomogeneities and the phase fluctuations on this surface are calculated by solving the problem of bulk scattering in the layer (e.g., by the MSP or geometrical optics), then Eq. (9) will give the field beyond the shield quite correctly. However, the main advantage of the model will be lost: the simplification of calculation that it gives. However, if the properties of the shield are assigned, then evidently the results cannot nearly always pretend to give a good quantitative description of the fluctuations in a wave that has actually passed through a volume randomly-inhomogeneous medium. In particular, this pertains to the interpretation of observations of twinkling of radio sources (quasars) in the interplanetary plasma. When the source is at a large angular separation from the sun, not only the statistical characteristics of the equivalent shield are called into question, but even the distance at which one should put it. Thus, the necessity of applying the PEM to volume scattering with all the difficulties that arise here is in no way avoided by the simplifications provided by the thin-shield model.

In applying the PEM to thick layers of a randomly-inhomogeneous medium, the idea naturally intrudes of using a set of thin phase shields. By dividing the layer into a series of thin "plates" (the thickness of which, however, is much greater than the correlation radius of the inhomogeneities), one can try to apply to each such "plate" the thin-shield formula, while calculating the phase shift by the geometrical-optics method and using the "Fresnel" Green's function. This is precisely the path taken in several studies,[190,372,425-431] (a machine calculation for a multilayer model was undertaken in[432]).

It is more systematic to apply Eq. (6) directly, but with division of the medium into physically infinite thin "plates". L. A. Chernov has used this method, with the assumption that one can restrict the treatment within each "plate" to the first approximation of the method of small perturbations, and further, with account taken of the correlation of $U(r)$ with only those inhomogeneities that the wave has already passed. Thus he derived from Eq. (6) some approximate closed equations for the mean value $\langle U(r)\rangle$ and for the moments $\langle U(r_1)U(r_2)\rangle$ and $\langle U(r_1)U^*(r_2)\rangle$.[165,433] L. S. Dolin[166,434,435] has derived the same equations, but in a different way (in essence, by summing the perturbation-theory series to an accuracy of terms of the order of $\langle\tilde{\epsilon}^2\rangle$).* Thus, an approach has been developed in these studies to establish equations for the averaged bilinear quantities on the basis of the stochastic equation. Equations were subsequently derived as well for the moments of $U(r)$ of higher order.†[437-439] An analogous formulation of the problem that uses a division of the medium into "plates" is contained also in[441,442], with the assumption that the fluctuating component of the field is distributed according to a Gaussian law.

V. I. Tatarskiĭ[443] has recently developed a methodologically more refined approach as applied to the same parabolic equation (in particular, without division of the medium into auxiliary thin "plates"). We shall treat this study and the publications that have followed it[444-448] in somewhat more detail.

c. The Markov Approximation

In essence, the same physical assumptions are made in[443] as L. A. Chernov[165,433,438] makes. However, their mathematical formulation is more lucid, and hence, it permits a clearer formulation of the problem. The assumptions in question are based on the idea that the fluctuations of any parameter of the wave are usually related to inhomogeneities on a certain scale. For example, intensity fluctuations are mainly due to inhomogeneities whose transverse dimensions are of the order of the radius $\sqrt{\lambda x}$ of the Fresnel zone. Phase-difference fluctuations over the basis ρ are due to inhomogeneities of transverse scale $\sim\rho$, etc. However, if the medium is statistically isotropic, this means that also the longitudinal correlation radius of the most substantial inhomogeneities is of the same order of magnitude.

If the path length x is much greater than the dimensions of the substantial inhomogeneities, then the problem involves small parameters like ρx or $\sqrt{\lambda x}/x$, and we can seek a solution as an expansion in these small parameters. We can get the first term of such expansions by formally replacing the longitudinal correlation radius $\tilde{\epsilon}$ by zero, i.e., by replacing the true correlation function $B_\epsilon(x - x', \rho - \rho')$ by the effective function:

$$B_\epsilon^{\text{eff}}(x-x',\ \rho-\rho')=\delta(x-x')\,A(\rho-\rho'), \tag{10}$$

Here A is determined by the condition

$$\int_{-\infty}^{\infty} B_\epsilon\, dx=\int_{-\infty}^{\infty} B_\epsilon^{\text{eff}}\, dx=A(\rho-\rho').$$

As V. I. Tatarskiĭ[443] has shown if given a correlation function like (10), one can treat wave propagation in the parabolic-equation approximation as a Markov random process, and derive closed equations for the moments of U. Following the work of V. I. Klyatskin,[444] we shall demonstrate this for the example of the equation for $\langle U\rangle$.

*Among the foreign publications, the closest to the PEM is that of Hufnagel and Stanley, [174] although the parabolic equation (6) doesn't figure in it explicitly. This equation was derived in [436], but it was solved essentially in the Born approximation.

†Results equivalent to solving the derived equations by the perturbation method are to be found in [540].

One can easily transform (6) into the integro-differential equation

$$U(x, \boldsymbol{\rho}) = U(0, \boldsymbol{\rho})\, e^{-\frac{ik}{2}\int_0^x \tilde{\varepsilon}(\xi, \boldsymbol{\rho})\, d\xi} - \frac{i}{2k}\int_0^x e^{-\frac{ik}{2}\int_\eta^x \tilde{\varepsilon}(\xi, \boldsymbol{\rho})\, d\xi}\, \Delta_\perp U(\eta, \boldsymbol{\rho})\, d\eta. \tag{11}$$

Now, in averaging this equation, we must take into account the fact that the values of $\tilde{\epsilon}(\xi, \rho)$ are taken for $\xi > \eta$ in the exponent within the integral over η. Since the boundary condition for U in Eq. (6) is imposed at $x = 0$, $U(\eta, \rho)$ depends functionally only on the values of $\tilde{\epsilon}(\xi, \rho)$ for $\xi < \eta$, and does not contain the subsequent $\tilde{\epsilon}(\xi, \rho)$ for $\xi > \eta$. Hence, since $\tilde{\epsilon}(\xi, \rho)$ has a delta correlation function the values of $\Delta_\perp U(\eta, \rho)$ and the exponential within the integral over η in (11) are statistically independent, and the averaging of (11) gives the equation

$$\langle U(x, \boldsymbol{\rho})\rangle = U(0, \boldsymbol{\rho})\, \langle e^{-\frac{ik}{2}\int_0^x \tilde{\varepsilon}(\xi, \boldsymbol{\rho})\, d\xi}\rangle - \frac{i}{2k}\int_0^x \langle e^{-\frac{ik}{2}\int_\eta^x \tilde{\varepsilon}(\xi, \boldsymbol{\rho})\, d\xi}\rangle \cdot \Delta_\perp \langle U(\eta, \boldsymbol{\rho})\rangle\, d\eta.$$

If the distribution of $\tilde{\epsilon}$ is Gaussian, then this equation is reduced to the differential equation:

$$-2ik\frac{\partial\langle U\rangle}{\partial x} + \Delta_\perp\langle U\rangle - \frac{ik^3 A(0)}{4}\langle U\rangle = 0, \tag{12}$$

which has a solution of the form

$$\langle U(x, \boldsymbol{\rho})\rangle = U_0(x, \boldsymbol{\rho})\, e^{-x/2d_B}, \tag{13}$$

Here $U_0(x, \rho)$ is the solution of the same equation with the same boundary condition in a medium having no fluctuations (see Eq. (9)), and d_B is the extinction distance calculated in the Born approximation. One obtains the result (13) also in the second MSP approximation[2,385] (or by normalizing the energy of the field to the energy of the primary wave, as was done, e.g., in[1]), and also from the Bourret approximation, which will be discussed below. Of course, this doesn't imply that the cited approaches are fully equivalent. For example, the results of the Markov approximation for $\langle U \rangle$ and B_U (see Eq. (14)) coincide with those given by the second MSP approximation, but they already disagree for the fourth moment (Eq. (16)).

One can also derive in an analogous way equations for any of the moments of $\langle U(x, \rho_1) \ldots U^*(x, \rho_k)\rangle$, and also equations of the Einstein-Fokker type for the probability distribution of the field U. In particular, one gets the following equation for the second moment (the second-order coherence function) $B_U(x, \rho_1, \rho_2) = \langle U(x, \rho_1) U^*(x, \rho_2)\rangle$:

$$-2ik\frac{\partial B_U}{\partial x} + (\Delta_1 - \Delta_2) B_U - \frac{ik^3}{2}[A(0) - A(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)]\, B_U = 0, \tag{14}$$

It had previously been derived in a different way by L. S. Dolin and L. A. Chernov,[165,166,433-435] and it is equivalent to the transport equation in the small-angle approximation (see below).

The (n + m)-th order moment

$$M_{n,m}(x;\, \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n;\, \boldsymbol{\rho}_1', \ldots, \boldsymbol{\rho}_m') \equiv \langle U(x, \boldsymbol{\rho}_1) \ldots U(x, \boldsymbol{\rho}_n)\, U^*(x, \boldsymbol{\rho}_1') \ldots U^*(x, \boldsymbol{\rho}_m')\rangle$$

satisfies the equation[445]

$$\frac{\partial M_{n,m}}{\partial x} = \frac{1}{2k}(\Delta_1 + \ldots + \Delta_n - \Delta_1' - \ldots - \Delta_m')\, M_{n,m} - \frac{k^2}{8} Q_{n,m}(\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n;\, \boldsymbol{\rho}_1', \ldots, \boldsymbol{\rho}_m')\, M_{n,m}, \tag{15}$$

where

$$Q_{n,m} = \sum_{i=1}^{n}\sum_{j=1}^{n} A(\boldsymbol{\rho}_i - \boldsymbol{\rho}_j) - \sum_{k=1}^{m}\sum_{j=1}^{n} A(\boldsymbol{\rho}_k' - \boldsymbol{\rho}_j) - \sum_{i=1}^{n}\sum_{l=1}^{m} A(\boldsymbol{\rho}_i - \boldsymbol{\rho}_l') + \sum_{k=1}^{m}\sum_{l=1}^{m} A(\boldsymbol{\rho}_k' - \boldsymbol{\rho}_l').$$

In particular, as regards the fourth-order moment $\Gamma_4 = M_{2,2}$ that determines the intensity fluctuation:

$$\Gamma_4(x;\, \mathbf{R}, \mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\rho}) = \Big\langle U\Big(x, \mathbf{R} + \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} + \frac{\boldsymbol{\rho}}{4}\Big)\, U\Big(x, \mathbf{R} - \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} + \frac{\boldsymbol{\rho}}{4}\Big) \times U^*\Big(x, \mathbf{R} + \frac{\mathbf{r}_1 - \mathbf{r}_2}{2} - \frac{\boldsymbol{\rho}}{4}\Big)\, U^*\Big(x, \mathbf{R} - \frac{\mathbf{r}_1 - \mathbf{r}_2}{2} - \frac{\boldsymbol{\rho}}{4}\Big)\Big\rangle,$$

one can derive from (15) the equation[443]

$$\frac{\partial\Gamma_4}{\partial x} = \frac{i}{k}(\nabla_R\nabla_\rho + \nabla_{r_1}\nabla_{r_2})\,\Gamma_4 - \frac{\pi k^2}{4} F(\mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\rho})\,\Gamma_4, \tag{16}$$

where

$$F(\mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\rho}) = H\Big(\mathbf{r}_1 + \frac{\boldsymbol{\rho}}{2}\Big) + H\Big(\mathbf{r}_1 - \frac{\boldsymbol{\rho}}{2}\Big) + H\Big(\mathbf{r}_2 + \frac{\boldsymbol{\rho}}{2}\Big) + H\Big(\mathbf{r}_2 - \frac{\boldsymbol{\rho}}{2}\Big) - H(\mathbf{r}_1 + \mathbf{r}_2) - H(\mathbf{r}_1 - \mathbf{r}_2)$$

and

$$H(\boldsymbol{\rho}) = \frac{1}{\pi}[A(0) - A(\boldsymbol{\rho})] = 2\iint_{-\infty}^{\infty}[1 - \cos\boldsymbol{\varkappa}\boldsymbol{\rho}]\,\Phi_\varepsilon(0, \boldsymbol{\varkappa})\, d^2\boldsymbol{\varkappa}.$$

A special case of Eq. (16) was derived by V. I. Shishov[437] by selective summation of the perturbation-theory series, and in[439] by dividing the medium into layers.* L. S. Dolin[449] (see also[450] and[435]) has derived a general solution of Eq. (14) by a Fourier transformation over the variable $R = \frac{1}{2}(\rho_1 + \rho_2)$. It has the following form:

$$B_U(x, \mathbf{R}, \boldsymbol{\rho}) \equiv \Big\langle U\Big(x, \mathbf{R} + \frac{1}{2}\boldsymbol{\rho}\Big)\, U^*\Big(x, \mathbf{R} - \frac{1}{2}\boldsymbol{\rho}\Big)\Big\rangle = \frac{1}{4\pi^2}\iint_{-\infty}^{+\infty} d^2\mathbf{R}' \iint_{-\infty}^{+\infty} d^2\boldsymbol{\varkappa}\, \exp\Big\{i\boldsymbol{\varkappa}(\mathbf{R} - \mathbf{R}') - \frac{\pi k^2}{4}\int_0^x H\Big(\boldsymbol{\rho} - \frac{\boldsymbol{\varkappa}(x - \xi)}{k}\Big)\, d\xi\Big\}\, U_0\Big(\mathbf{R}' + \frac{\rho}{2} - \frac{\varkappa x}{2k}\Big)\, U_0^*\Big(\mathbf{R} - \frac{\rho}{2} + \frac{\varkappa x}{2k}\Big). \tag{17}$$

Here $U_0(\rho) = U(0, \rho)$ is the assigned initial distribution of the complex amplitude of the field.† If, for example,

$$U_0(\rho) = U_0 \exp\Big\{-\frac{\rho^2}{2a^2} + \frac{ik\rho^2}{2F}\Big\}$$

(the initial beam has a Gaussian amplitude distribution and a quadratic phase distribution) and the fluctuations of the refractive index are described by a "$\frac{2}{3}$ law",

*DeWolf [190] has also tried to sum the perturbation-theory series for Γ_4. However, in this study he unjustifiably omitted some strongly connected diagrams. According to [189], this leads to a normal probability distribution for the field, and a Rayleigh distribution of the amplitude. A numerical solution of the equations that were derived in [437] is given in [540].

†If the initial field distribution is random, then in the integral in (17) one must replace the product $U_0U_0^*$ by the initial value of the second moment

$$B_{U_0} \equiv B_U(0; \mathbf{R}', \rho - x\kappa/k),$$

Then formula (17) will describe the propagation of a partially coherent wave in a medium containing inhomogeneities. We note that the state of the theory of partially coherent fields in a vacuum, i.e., in the absence of random inhomogeneities, has been covered in [451-453]. The elements of this theory were initially introduced to describe waves in randomly-inhomogeneous media in [454-455], where the Born approximation was used, and in [174], where an approach resembling the PEM was applied.

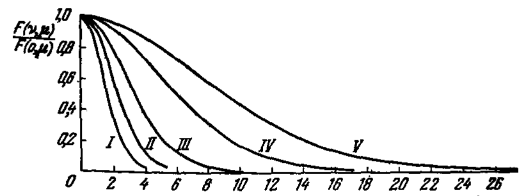

FIG. 5

then we can derive from (17) the following expression for the mean intensity in the beam $\langle I(x, R)\rangle = \langle |U(x, R)|^2\rangle = B_U(x, R, 0)$:[446]

$$\langle I(x, \mathbf{R})\rangle = I_0(x) F(\nu, \mu),$$

where

$$I_0(x) = \frac{k^2 a^4 U_0^2}{x^2 g^2(x)}, \quad g(x) = \sqrt{1 + k^2 a^4\left(\frac{1}{x} + \frac{1}{F}\right)^2},$$

$$\nu = \frac{2kaR}{x g(x)}, \quad \mu = 0{,}27 C_\varepsilon^2 k^2 x \left(\frac{2a}{g(x)}\right)^{5/3},$$

$$F(\nu, \mu) = 2\int_0^\infty J_0(\nu t)\exp\left\{-t^2 - \frac{1}{2}\mu t^{5/3}\right\} t\, dt.$$

Figure 5 shows graphs of the function $F(\nu, \mu)/F(0, \mu)$, which is the profile of the mean intensity in the beam normalized to the intensity $\langle I(x, 0)\rangle$ at the center of the beam. It is plotted against the dimensionless argument $\nu = R/R_0$, where $R_0 = xg(x)/2ka$, for several values of the parameter μ, which characterizes the intensity of fluctuations $\tilde{\varepsilon}$. Figure 6 shows the intensity at the axis of the beam normalized to the value of $I_0(x)$, which is the intensity at the axis of the beam in the absence of fluctuations: $F(0, \mu) = \langle I(x, 0)\rangle / I_0(x)$.

As for the equation (16) for Γ_4, one cannot solve it in the general case. We note that Eq. (16) does not fit an expression of Γ_4 in the form of a sum of products of B_U, i.e., in the form that Γ_4 would have with a Gaussian field distribution. This directly indicates that the distribution of U is generally not Gaussian. However, one can easily obtain one of the integrals of this equation. For an infinite incident wave, we have the relation

$$\iint_{-\infty}^{\infty} [\Gamma_4(x; \mathbf{R}, \mathbf{r}, 0, \rho) - |B_U(x, \mathbf{R}, 0)|^2]\, d^2\mathbf{r} = 0. \qquad (18a)$$

Restricted beams obey the following formula

$$\iint_{-\infty}^{\infty} d^2\mathbf{R} \iint_{-\infty}^{\infty} d^2\mathbf{r}\, [\Gamma_4(x; \mathbf{R}, \mathbf{r}, 0, \rho) - |B_U(x, \mathbf{R}, 0)|^2] = 0. \qquad (18b)$$

Relations (18a) and (18b) have a simple physical meaning: they stem from the law of conservation of energy. Hence, the fluctuations of intensity involve only a redistribution of energy within the beam.

Just like the equation for B_U, the equation for the fourth moment can be written in the form of a radiation-transport equation in the small-angle approximation.[446] If we seek Γ_4 in the form

$$\Gamma_4(x; \mathbf{R}, \mathbf{r}_1, \mathbf{r}_2, \rho) = \iint_{-\infty}^{\infty} e^{i\mathbf{p}\mathbf{R}}\, d^2\mathbf{p} \iint_{-\infty}^{\infty} e^{i\varkappa\mathbf{r}_2} d^2\varkappa\, \varphi\left(x; \mathbf{p}, \mathbf{r}_1, \varkappa, \rho - \frac{\mathbf{p}x}{k}\right),$$

then we can derive from (16) an equation for the function $\varphi(x; \mathbf{p}, \mathbf{r}_1, \kappa, \rho)$:

$$\frac{\partial\varphi}{\partial x} + \frac{\varkappa}{k}\nabla_{r_1}\varphi + \frac{\pi k^2}{4}\left[H\left(\mathbf{r}_1 + \frac{\rho}{2}\right) + H\left(\mathbf{r}_1 - \frac{\rho}{2}\right)\right]\varphi \qquad (16a)$$
$$= \pi k^2 \iint_{-\infty}^{\infty} \Phi_\varepsilon(\varkappa')\left[\cos\frac{\varkappa'\rho}{2} - \cos\varkappa'\mathbf{r}_1\right]\varphi(x; \mathbf{p}, \mathbf{r}_1, \varkappa - \varkappa', \rho)\, d^2\varkappa'.$$

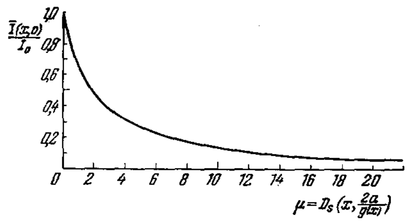

FIG. 6

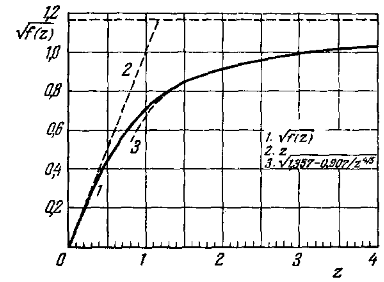

FIG. 7

The quantity φ is a formal analog of the energy flux in the radiation-transport equation. As we see from (16a), the "scattering indicatrix" and the "extinction coefficient" here are functions of the coordinates.

We can solve Eq. (16a) approximately by replacing the value of φ within the integral by the known initial value $\varphi_0 = \varphi(0; \mathbf{p}, \mathbf{r}, \kappa, -\kappa', \rho)$ (the "single-scattering approximation" in the sense in which this term is used in the theory of radiation transport).

If the fluctuations in the dielectric constant of the medium obey a "$^2/_3$ law," then in this approximation, the mean square of the relative fluctuations of intensity of a plane wave $\beta^2(x) = [\langle (I - \langle I\rangle)^2\rangle]/\langle I\rangle^2$ is a function of the quantity $\beta_0^2 = 0.3\, C_\varepsilon^2 k^{7/6} x^{11/6}$, as found in the first MSP approximation. Figure 7[445] shows the relation $\beta = f(\beta_0)$. In the region of strong fluctuations ($\beta_0 \gg 1$), the correlation function of the intensity fluctuations calculated in the single-scattering approximation is characterized by two scales: $l_1 = \sqrt{\lambda x}\, \beta_0^{6/5}$, and $l_2 = (C_\varepsilon^2 k^3)^{-3/11} = \sqrt{\lambda x}/\beta_0^{6/11}$, with $l_1 \gg l_2$. The scale l_2 is related to the dimensions of the inhomogeneities of intensity, while the scale l_1 can be interpreted as the mean distance (in the plane x = const.) between positive and negative fluctuations in I.

Let us briefly consider the problem of the limits of applicability of the Markov approximation in the PEM. When one estimates the limits of applicability of an approximate theory by estimating the next term in an expansion, then one usually has no assurance that the subsequent terms (neglected in the expansion) will not change the result.

This is not the case with the Markov approximation, since an exact solution of the problem corresponding to $l = 0$ can be derived in an independent way, and it coincides with the principal term of the expansion in l.

Corrections to Markov-type solutions arising from the finite longitudinal correlation radius of the dielectric constant have been discussed in[444,445], while the conditions have been studied in[445,447] under which one can use the parabolic equation itself in a medium containing random inhomogeneities. It turned out that the restrictions involving both the use of the parabolic equation and the Markov approximation are practically identical. They have the form:

a) $\langle\tilde{\varepsilon}^2\rangle kl \ll 1$;   b) $\sigma_\alpha^2 \ll 1$;   c) $\langle\tilde{\varepsilon}^2\rangle kx \ll 1$.

Here $\sigma_\alpha^2 \sim \langle\tilde{\epsilon}^2\rangle x/l$ is the mean square of the fluctuations in the direction of propagation. Condition a) can also be written in the form $\lambda\alpha \ll 1$, where $\alpha \sim \langle\tilde{\epsilon}^2\rangle k^2 l$ is the extinction coefficient, and $\lambda\alpha$ is the attenuation per unit wavelength. Condition c) can be written in the form $\alpha x \ll l/\lambda$. Its meaning consists in the smallness of the back-reflected waves, and it can be satisfied even at large values of αx, since $l \gg \lambda$. In addition to these conditions, naturally, the purely "geometric" conditions for applicability of the parabolic equation, as indicated at the beginning of Sec. 3, must be satisfied ($\lambda \ll l$, $(\lambda x/l^2) \ll (l^2/\lambda^2)$). Furthermore, the Markov approximation becomes valid only when $x \gg l$, i.e., the "process of establishment" of a Markov system occurs in the region $x \sim l$. No further limitations are imposed on the fluctuations in the field amplitude within the framework of these conditions. We shall point out also that V. I. Klyatskin[448] has discussed longitudinal field correlations in the Markov approximation.

d. Distribution Laws of the Fluctuations in the Scattered Field

Now we shall take up the rather complicated problem of the distribution laws of the fluctuation probabilities. It isn't hard to write equations for the characteristic functional of the field u or the logarithm of the field ln u. However, these equations contain variational derivatives, and can't be solved. However, substitution into these equations of Gaussian-type characteristic functionals shows that a normal distribution for u or ln u is not a solution for them.

Whenever the first approximations of the perturbation theory or the MSP are suitable, i.e., when the fluctuations in u or ln u are small enough, the solution is a linear functional of $\tilde{\epsilon}$. In view of the central limit theorem, we can then state that the distribution law of u (in the region of applicability of the Born approximation) or of ln u (in the region of applicability of the first MSP approximation) must be normal. This is the point where the substantial difference between the two cited approximations is manifested. Formally, they are limited by the condition of smallness of the fluctuations of u or ln u, and at first glance, this seems to be equivalent. However, we can conclude from comparing the probability-distribution laws that a normal distribution for u can be considered to be a special case of the logarithmic normal law, in which not only the fluctuations in ln u, but in u itself become small. That is, the Born approximation can be treated as a special case of the first MSP approximation involving scattering at small angles (of course, conditions can occur in which the Born approximation is applicable, but the first MSP approximation is not, e.g., when $kl \ll 1$).

One could draw more definite conclusions on the distribution laws of the fluctuations by comparing the approximate with the exact solutions. For lack of the latter, we have to appeal to the experimental data.

Comparison with such data has been carried out in[191-195,198-201,541], in which light fluctuations were studied. It turned out that the distribution law for ln u in the region of relatively weak amplitude fluctuations ($\sigma_\chi = \sqrt{\langle x^2\rangle} < 1$) is very close to normal. Here we should note that the observed values of the fluctuations, which again confirmed well the log-normal distribution of the amplitude, cannot be explained within the framework of the Born approximation, in which the amplitude is distributed according to the law of Rice (a mixed Rayleigh distribution), and the mean square of its fluctuations has an upper limit.*

I. G. Kolchinskiĭ[251] has experimentally studied the distribution law for the phase difference by observations of the "trembling" of the images of stars, and again, it proved to be close to normal.

One can compare the probability distribution laws with results based on exact solution of the problem in the above-described Markov approximation. Here it turns out that the second moments of the field $\langle U_1U_2^*\rangle$, as obtained by solving Eq. (14), exactly coincide with the result based on the MSP (taking account of the second approximation for $\langle\chi\rangle$) and on the assumption that the field is log-normal. However, there is no longer such an agreement for the fourth moments of the field $\langle U_1U_2U_3^*U_4^*\rangle$, which describe the fluctuations of intensity. Hence we can draw the qualitative conclusion that the log-normal distribution law of u is obeyed well in the region of small field values, and fails in the region where its values are large (because the contribution to $\langle uu^*\rangle$ arises from a region on the distribution-density curve closer to $|u| = 0$ than the contribution to $\langle(uu^*)^2\rangle$ does). In other words, we can expect that a normal law for the fluctuations in the level χ will be well obeyed when $\chi < 0$, but violated in the region of large positive χ.

We can draw another qualitative conclusion by comparing the behavior of the second and fourth moments of the field with increasing distance x. For those models of the fluctuations of ϵ that have no finite outer scale (e.g., the "$^2/_3$ law"), the fourth moments of the field as $x \to \infty$ approach a limit that depends on the type of spectrum that ϵ has. This means that the distribution law of the field is not universal in this case. However, if the fluctuations of ϵ have a fixed correlation radius l, then as $x \to \infty$ ($\sqrt{\lambda x} \gg l$), the limiting distribution of the field will apparently be normal, as in the problem of the equivalent shield.

4. THEORY OF MULTIPLE SCATTERING

The general theory of multiple scattering has been developed in the last seven years. During this time, the Green's-function method, which had previously been developed in quantum electrodynamics, was applied to

*DeWolf [351] has recently discussed the problem of the distribution law of the amplitude. He considered an interesting result of Mitchell, [456] which consists in the fact that the sum of a moderate number of quantities that are distributed according to log-normal laws is distributed according to a log-normal, rather than a Gaussian law. This perhaps explains why the amplitude distribution law in the region of strong fluctuations, where a multiray situation occurs, is closer to log-normal than to the Rice law (or at greater optical depth, to the Rayleigh law).

the discussed macroscopic problems. To speak more concretely, people have used the Dyson equation[459] for the mean field $\langle u(r)\rangle$, and the Bethe-Salpeter equation[460] for the covariance $B_U = \langle u_1 u_2^*\rangle$ (or the correlation function $\Psi_u = B_u - \langle u_1\rangle\langle u_2^*\rangle$). These equations were derived by using the graph technique of Feynman.[457,458] Thus it is again a question of deriving equations for averaged quantities. However, one cannot derive closed equations of this type by averaging the original differential equations for the field u because of their parametric nature: the moments of different orders are coupled together. Hence, one must resort to solving (4), although it is written in the form of a perturbation-theory series. The graph technique makes it possible formally to sum this series, as well as the product of two such series, and this leads to the Dyson (D.) equation and the Bethe-Salpeter (B.-S.) equation.

Of course, problems of multiple scattering of various wave fields by random assemblies of scatterers arose in physics considerably earlier than the analogous problems of quantum electrodynamics, in which the topic is, e.g., propagation of electron waves with account taken of their interaction with the vacuum fluctuations of the electromagnetic field (emission and absorption of virtual photons), or, conversely, propagation of electromagnetic waves that interact with the electron-positron vacuum (creation and annihilation of virtual electron-positron pairs). People had also encountered multiple scattering long ago in the problem of passage of radiation through the atmospheres of stars and planets,[461-466] and in problems of scattering of thermal neutrons,[467,468] charged particles,[469] etc. Here they usually used the linearized integro-differential equation of Boltzmann. In view of the classical description of motion of particles or radiation along trajectories (rays) (i.e., description in the geometrical-optics approximation), this equation is a so-called transport equation (of particles of energy). Typically, of course, wave-interference effects are not taken into account here.

The first person to pose the problem of multiple scattering of waves and to solve it for a model of point, isotropic scatterers distributed in an uncorrelated way was Foldy.[470] This study preceded that of Dyson by four years, and that of Salpeter and Bethe by six years. However, owing to some simplifications (neglect of the distinction between conditional and unconditional averages), equations were derived in it of the D. type for the mean field $\langle u\rangle$, and of the B.-S. type for the mean intensity $\langle |u|^2\rangle$. Without treating the further development of the results of this study (anisotropic scatterers whose positions are correlated, cf.[471-473]), we note that Yu. N. Gnedin and A. Z. Dolginov[474] were the first to use the graph technique in the theory of scattering by discrete scatterers in a problem of scattering of a flux of particles by a target consisting of an assembly of uncorrelated scatterers having an arbitrary scattering amplitude. Then Frisch[475] applied it, but now with arbitrary correlation in the positions of the scatterers. The results of these studies as well were equations of the D. and B.-S. types.

Application of the graph technique to scattering of waves in a continuous fluctuating medium began with the studies of Bourret.* [476-477] He assumed that the parameters of the medium fluctuate according to the normal law, and are (locally) statistically independent of the sought field. This led to the D. and B.-S. integral equations with approximate expressions for the kernels that were proportional to the correlation function (B_ϵ) of the medium. The cited expressions are called the Bourret approximation for the kernel of the D. equation, and the ladder approximation for the kernel of the B.-S. equation (the latter term is taken from quantum electrodynamics). Furthermore, Bourret introduced the Green's function of free space ($\tilde{\epsilon} = 0$) into the B.-S. equation instead of the average Green's functions.

A more general derivation of the two equations assuming a normal law for a fluctuating medium is due to V. I. Tatarskiĭ and M. E. Gertsenshteĭn[482] and V. I. Tatarskiĭ[483,2] (see also[484,542]), and to Frisch[475,11], who admitted deviations from the normal law.

What do the D. and B.-S. equations look like?

Let $G(\mathbf{r}, \mathbf{r}_0)$ be the sought Green's function, i.e., a solution of the following equation that satisfies the condition of radiation to infinity:

$$\Delta G + k^2(1+\tilde{\varepsilon})G = \delta(\mathbf{r}-\mathbf{r}_0), \tag{19}$$

and, as before, let $g(\mathbf{r}, \mathbf{r}_0)$ be the Green's function in a homogeneous medium ($\tilde{\epsilon} = 0$). The D. equation for $\langle G\rangle$ is

$$\langle G(\mathbf{r}, \mathbf{r}_0)\rangle = g(\mathbf{r}, \mathbf{r}_0) + \int\int g(\mathbf{r}, \mathbf{r}_1) M(\mathbf{r}_1, \mathbf{r}_2)\langle G(\mathbf{r}_2, \mathbf{r}_0)\rangle\, d^3\mathbf{r}_1 d^3\mathbf{r}_2, \tag{20}$$

or in symbolic (operator) form,

$$\langle G\rangle = g + \hat{g}\hat{M}\langle G\rangle, \tag{21}$$

Here the "kernel" M, which is called the mass operator,† is an infinite series, being the sum of the so-called strongly connected graphs having no external propagation lines.

The B.-S. equation has an analogous form. For the mixed moment $B_G = \langle G(\mathbf{r}, \mathbf{r}_0)G^*(\mathbf{r}', \mathbf{r}_0')\rangle$, it can be written as follows:

$$\begin{aligned} B_G(\mathbf{r}, \mathbf{r}_0; \mathbf{r}', \mathbf{r}_0') &= \langle G(\mathbf{r}, \mathbf{r}_0)\rangle\langle G^*(\mathbf{r}', \mathbf{r}_0')\rangle \\ &+ \int\int\int\int \langle G(\mathbf{r}, \mathbf{r}_1)\rangle\langle G^*(\mathbf{r}', \mathbf{r}_3)\rangle K(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) \\ &\times B_G(\mathbf{r}_2, \mathbf{r}_0; \mathbf{r}_4, \mathbf{r}_0')\, d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{r}_3 d^3\mathbf{r}_4, \end{aligned} \tag{22}$$

or, in operator form:

$$B_G = \langle G\rangle\langle G^*\rangle + \langle\hat{G}\rangle\langle\hat{G}^*\rangle\hat{K}B_G. \tag{23}$$

It contains as a "kernel" the intensity operator K (a name introduced by Frisch[475]), which depends on four arguments, and is also a sum of strongly connected, but two-row graphs having no external propagation lines.

Superficially, the two equations look like linear integral equations, but actually this isn't so. The "kernels" M and K are infinite series whose summation cannot actually be carried out. Also, we know

*Independently of Bourret, Furutsu [478-481] has pointed out the applicability of the graph technique for normal fluctuations in ϵ.

†In the theory of multiple scattering of waves, this operator is more suitably called the polarization operator, especially if one is treating electromagnetic waves. [15]

nothing about their convergence in the presence of strong fluctuations in ϵ. Furthermore, even if we could sum these series, then we would get M and K in the form of functionals of the sought functions $\langle G \rangle$ in B_G. That is, the equations would prove to be nonlinear.

Then what is the worth of these equations?

First, they help in studying a number of the general problems of the theory of multiple scattering: in linking the problems of continuous randomly-inhomogeneous media and of sets of discrete scatterers (which we are not treating here), in revealing the various approximate approaches and elucidating the relation between them, and in particular, in providing a basis for the transport equation (see below).

Second, they become an actual means of solving concrete problems whenever one can replace the "kernels" M and K with approximate (abbreviated) expressions, i.e., known functions of the coordinates. Then, the D. and B.-S equations become linear integral equations that can be solved under certain supplementary assumptions. It is essential to emphasize that the approximate solutions thus obtained now sum a certain subsequence of the perturbation-theory series. Thus, they now describe multiple scattering to some extent. Let us first take up precisely this second aspect of the problem.

When the fluctuations of ϵ are normal, if we restrict ourselves to the first terms of the series for the kernels M and K (let us denote these terms, which are proportional to the correlation functions B_ϵ, as M_1 and K_1), we get the Bourret approximation for the D. equation, and the ladder approximation for the B.-S. equation. If in addition the random field $\tilde{\epsilon}$ is statistically homogeneous and isotropic (all quantities depending only on the moduli of the differences of the vectors r), then the D. equation can be completely solved. That is, one gets an expression for the mean Green's function $\langle G \rangle$ in the randomly-inhomogeneous medium.* In the region $R \gg l$ in the case of fine-scale inhomogeneities ($kl \ll 1$), this expression is:

$$\langle G\rangle_1 = -\frac{e^{-ik_{\text{eff}}R}}{4\pi R}, \tag{24}$$

where the effective wave number† k_{eff} is

$$k_{\text{eff}} = k\,(1 + 2k^2l^2\langle\tilde{\varepsilon}^2\rangle - i4k^3l^3\langle\tilde{\varepsilon}^2\rangle). \tag{25}$$

Thus, k_{eff} describes the decrease in phase velocity and extinction. Analysis of the following terms of the expansion of the mass operator sh~ws that the given result holds under the condition of relative smallness of both of the corrections to k, i.e., when $k^2l^2\langle\tilde{\epsilon}^2\rangle \ll 1$ (this implies, in particular, that the extinction per wavelength is small). Since $kl \ll 1$, this condition permits even large values of $\langle\tilde{\epsilon}^2\rangle$. Furthermore, in contrast to the condition (5) for validity of the Born approximation, here we don't have such a rigid limitation on the wave path L in the medium (a residual limitation on L is required in order that the product of L by the correction to k that is of second order in $\langle\tilde{\epsilon}^2\rangle$ should be small in comparison with unity).

The representation (24) proves to be also applicable to the case of large scale inhomogeneities ($kl \gg 1$), but then the extinction must be small, not per wavelength λ, but over the length l of a single inhomogeneity.[483,498-500]

If the field $\tilde{\epsilon}$ is not Gaussian, then it is natural to use for the kernels M and K the so-called one-group approximation introduced by V. M. Finkel'berg,[29,501] in which M and K are linear functionals of B_ϵ. When ϵ has a Gaussian distribution, the one-group approximation coincides with the Bourret approximation for M and the ladder approximation for K. Matters are more complicated with the B.-S. approximation. Even with a normal, homogeneous, and isotropic field $\tilde{\epsilon}$, one can derive a closed expression for B_{G_1} (again the subscript 1 indicates the first approximation, in this case, the ladder approximation) only at the expense of a number of supplementary assumptions.

One of the fundamental difficulties in operating with the D. and B.-S. equations for scattering media of infinite extent involves the fact that each term of the perturbation-theory series proves to diverge.† People usually eliminate this divergence by a well-known method, by introducing a small real absorption.[2] V. N. Alekseev and V. M. Komissarov[503-505] were able to derive an altered form in place of the usual integral form (3) of the initial equation (2) for the field

$$u = u_0 - k^2\hat{g}\tilde{\varepsilon}u \tag{3a}$$

Apparently, this corresponds better to the physical pattern of propagation in a randomly-inhomogeneous medium, and it automatically relieves the stated difficulty. The topic is the subsequent modification of (3a).

Of course, the D. equation holds not only for the Green's function (point source), but also for the field $u(r)$ arising from any primary field $u_0(r)$. Then we can write it in operator form as

$$\langle u\rangle = u_0 + \hat{g}\hat{M}\,\langle u\rangle. \tag{26}$$

By using this equation, the initial wave equations for u and G, and the reciprocity theorem, the cited authors derived the following equation for u:

$$u = \langle u\rangle + k^2\,\langle\hat{G}\rangle\,\hat{e}u, \tag{27}$$

where the operator $\hat{e}$ is expressed in terms of the mass operator and the unit operator:

$$\hat{e} = -\left(\frac{\hat{M}}{k^2} + \tilde{\varepsilon}\hat{I}\right).$$

In distinction from (3a), in (27) the primary field u_0 and the Green's function g corresponding to free

*A rather great number of studies have been devoted to deriving the dispersion relations for the mean field (both the exact relations and in the Bourret approximation, as well as by using the effective dielectric-constant tensor for the electromagnetic waves and the effective elasticity tensor for elastic waves in solids) (to supplement those cited, see, e.g. [486-497,484]). We shall spend no time on them, but refer the reader to the review article of Yu. A. Ryzhov and V. V. Tamoĭkin, [15] which also discusses the important problem of emission from antennas and moving charges in randomly-inhomogeneous media.

†One can find $\langle G\rangle$ in the geometrical-optics approximation for a statistically and regularly inhomogeneous medium having a smooth variation of the parameters. [485]

†Evidently, such difficulties do not arise for scattering in a limited volume. For example, Austin [502] has carried out an accounting for extinction for the mean field without introducing real absorption.

space ($\tilde{\varepsilon} = 0$) are respectively replaced by the mean field $\langle u \rangle$ and the mean Green's function $\langle G \rangle$ in the medium containing inhomogeneities. Hereby extinction is taken into account from the very outset. Thus the solution of Eq. (27) by the perturbation method (using the Bourret approximation for M) is free from the difficulties involving the divergence of the terms in the usual perturbation-theory series.*

Now we shall take up some general results concerning the D. and B.-S. equations. There are several relations involving their kernels that arise from the most general physical principles, and which can be used, both for constructing approximate expressions for the kernels, and for solving the equations themselves. They include the reciprocity relation, † [32] the dispersion relation, which arises from the causality principle,[478,15] the constancy of sign of the imaginary part of the Fourier image of the operator $\hat{M}$ and the optical theorem.[508] The latter stems from the conservation of energy, and in particular, it shows that if the intensity operator $\hat{K}$ is taken in the ladder approximation, then one should take the Bourret approximation for the mass operator $\hat{M}$.

As we have said, the kernels M and K are infinite series. The terms of these series are divided into two classes.

The first class includes the rapidly-declining terms, i.e., those that decline as rapidly upon displacing the arguments as the correlation functions of the field $\tilde{\varepsilon}$ do. Their nonlocality radius l determines the scale of the effective inhomogeneities of the medium, and it is of the order of the correlation radius of the fluctuations in $\tilde{\varepsilon}$. If the field $\tilde{\varepsilon}$ is Gaussian, then the Bourret approximation and the ladder approximation are rapidly-declining. The one-group approximation of V. M. Finkel'berg[29,501] is rapidly-declining for a non-Gaussian field $\tilde{\varepsilon}$. In this approximation, the kernels M and K have the meaning of the scattering operators of the volume in question (which is small in comparison with the scale of the extinction length). In other words, they determine the mean and the covariance of the field that is scattered by a small volume, and are interrelated by the ordinary optical theorem[509] that says that the imaginary part of the forward scattering amplitude is proportional to the total effective scattering cross-section.

The second class includes the slowly-declining terms of the series, i.e., those that decline as the arguments are displaced as some positive integral power of the Green's function g of free space. They include the two-group, three-group approximations, etc.

When one solves the D. and B.-S. equations, and also when one derives the transport equations from them, one usually restricts the expressions for the kernels M and K to the declining terms alone, although the slowly-declining terms may also prove to be substantial in some cases. This is indicated by the exact solutions of certain problems on wave propagation in a one-dimensional model of a scattering medium.[510,511] These solutions call into question the applicability of the transport equation (or, at least, the possibility of restricting the treatment to the Bourret and ladder approximations) in one-dimensional problems, in which the Green's function generally does not decline with distance. In this regard, the problem of the conditions under which one may omit the slowly-declining terms in the kernels M and K is especially substantial.

If the scale l is small enough, then we can adopt it as the small parameter, and correspondingly, construct a solution of the D. and B.-S. equations. This is the so-called weak-nonlocality approximation of the kernels M and K.* Here, various methods of solution are possible, using ether: 1) slowly varying ray amplitudes $U(r)$, which play a role in going over to the parabolic equation (6), or 2) the spatial spectral field density $u(r)$. In addition, the Fraunhofer approximation also is a variety of weak-nonlocality approximation. We shall briefly explain what these methods consist in.

Let the field $u_0(r) = U_0(r)e^{-ikx}$ be normally incident on the plane $x = 0$, which is the boundary of an inhomogeneous medium ($x > 0$). If we express the sought field in the same form, i.e., we assume that $u(r) = U(r)e^{-ikx}$, and assume that $\langle U \rangle$ and $B_U = \langle U_1 U_2^* \rangle$ vary little over the course of the nonlocality radius l, we can approximately reduce the D. and B.-S. equations to purely differential equations.[512] When $U_0 = \text{const.}$ (plane primary wave) and there are large-scale inhomogeneities ($kl \gg 1$), the conditions for applicability of these differential equations have the form

$$\frac{l}{d} \ll 1, \quad \frac{1}{kl} \cdot \frac{x}{d} \ll 1, \tag{28}$$

where d is the extinction length. Further, one can solve the simplified B.-S. equation also by the perturbation method for the exponent φ in the expression for $B_U = e^{\varphi}$, i.e., essentially, by using the MSP.† [512] The first MSP approximation is applicable here when

$$\frac{x}{d} \ll \sqrt{\frac{kl^2}{d} \cdot kl}, \tag{29}$$

Here we must consider the parameter kl^2/d to be small (kl^2 is of the order of magnitude of the longitudinal correlation radius of the field), since backward scattering is ignored. If

$$\frac{1}{kl} < \frac{kl^2}{d} \ll 1 \quad \left(\text{in other words, } \frac{1}{(kl)^4} < \langle \tilde{\varepsilon}^2 \rangle \ll \frac{1}{kl^3}\right),$$

then the condition (29) permits values for the distance x that the wave travels in the medium that exceed the extinction length d.

It is essential to emphasize that even the first approximation for the covariance of the field $B_U = \langle U(r_1) U^*(r_2) \rangle$ thus obtained satisfies the requirement of conservation of energy, i.e., $\langle |U|^2 \rangle = 1$. Also, it transforms into the result that one gets from the ordinary first MSP approximation for the stochastic equation (7), but with extinction neglected.

*Brown [506] has applied essentially the same method, but for the B.-S. equation in spectral form, and avoiding the introduction of a small real absorption.

†The reciprocity relation for partially-coherent fields has been derived in a different way in [507].

*We should note that this approximation imposes no restrictions on the size of the parameter kl.

†Brown [506] has also obtained results similar to [512].

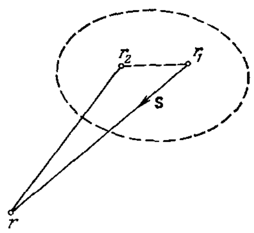

FIG. 8

In the case of a discrete medium, if we take the kernels M and K in the first approximation over the density of scatterers, then the first approximation for the covariance B_U coincides with the result of averaging the bilinear combination $u(\mathbf{r}_1)u^*(\mathbf{r}_2)$. Here the field $u(\mathbf{r})$ is obtained by applying to the stochastic wave equation (1) a special variant of the MSP that has been developed by N. P. Kalashnikov and M. I. Ryazanov.[513,514] The transverse spectrum of the field u satisfies the transport equation in the small-angle approximation.[166,434,435,449,450]

Neglecting spatial dispersion in the D. equation leads to the weak-nonlocaity approximation,[29] which is equivalent to transforming the iteration series for the D. equation in the Fraunhofer approximation.[32] The transformation consists in the following. If the observation point r (Fig. 8) lies within the Fraunhofer zone with respect to the effective inhomogeneity containing the points $\mathbf{r}_1$ and $\mathbf{r}_2$, then the Green's function for free space $g(\mathbf{r} - \mathbf{r}_2)$ can be approximately written in the form

$$g(\mathbf{r}-\mathbf{r}_2) \cong g(\mathbf{r}-\mathbf{r}_1)^{-ik\mathbf{s}(\mathbf{r}_2-\mathbf{r}_1)},$$

which can be integrated with respect to $\mathbf{r}_2$ (the unit vector $\mathbf{s}$ is directed from $\mathbf{r}_1$ to $\mathbf{r}$).

An analogous transformation can also be made in the terms of the iteration series for the B.-S. equation, but with the difference that not two, but four points are associated here with each effective inhomogeneity, and the problem doesn't involve the vacuum Green's functions g, but the mean values $\langle G \rangle$. Consequently, the B.-S. equation is reduced to the transport equation.[30,32]

The second of the methods cited above uses the spectral density of the field, i.e., the Fourier transform $B_U(\mathbf{R}, \varkappa)$ of the covariance $B_U(\mathbf{R}, \rho)$ over $\rho = \mathbf{r}_1 - \mathbf{r}_2$ (the dependence on $\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ arises from the possible presence of a smooth, regular, and random inhomogeneity in the medium, and the limited nature of the wave beam). The B.-S. equation written for $B_U(\mathbf{R}, \varkappa)$ is called the generalized transport equation.[515] Lax[471] has pointed out the possibility of formulating this equation and usefulness thereof.

The generalized transport equation has the following form:

$$B_u(\mathbf{R}, \varkappa) = B_u^0(\mathbf{R}, \varkappa) + \iiiint F(\mathbf{R}, \varkappa; \mathbf{R}'', \varkappa'')\, Q(\mathbf{R}'', \varkappa''; \mathbf{R}', \varkappa')\, B_u(\mathbf{R}', \varkappa')\, d^3\mathbf{R}''d^3\mathbf{R}'d^3\varkappa''d^3\varkappa', \qquad (30)$$

Here $B_u^0(\mathbf{R}, \varkappa)$ and $F(\mathbf{R}, \varkappa; \mathbf{R}', \varkappa')$ denote the spatial spectral densities of the mean field and the mean Green's function. That is, they are the Fourier transforms, respectively, of $\langle u(\mathbf{r}_1)u^*(\mathbf{r}_2)\rangle$ with respect to $\rho = \mathbf{r}_1 - \mathbf{r}_2$, and of $\langle G(\mathbf{r}_1, \mathbf{r}_1)G^*(\mathbf{r}_2, \mathbf{r}_2')\rangle$ with respect to $\rho = \mathbf{r}_1 - \mathbf{r}_2$ and $\rho' = \mathbf{r}_1' - \mathbf{r}_2'$. The kernel Q of Eq. (27) is the Fourier transform of the intensity operator $K(\mathbf{r}_1, \mathbf{r}_1', \mathbf{r}_2, \mathbf{r}_2')$ with respect to $\rho = \mathbf{r}_1 - \mathbf{r}_2$ and $\rho' = \mathbf{r}_1' - \mathbf{r}_2'$.

The ordinary transport equation is written as an equation for the ray intensity $I(\mathbf{R}, \mathbf{s})$, where $\mathbf{R}$ is the radius vector of the point in space, and $\mathbf{s}$ is a unit vector in the ray direction. In integro-differential form, this equation looks like

$$(\mathbf{s}, \nabla_R)\, I(\mathbf{R}, \mathbf{s}) = -\alpha I(\mathbf{R}, \mathbf{s}) + \oint f(\mathbf{s}, \mathbf{s}')\, I(\mathbf{R}, \mathbf{s}')\, d^2\mathbf{s}', \qquad (31)$$

where the integral over $\mathbf{s}'$ is taken over the entire solid angle 4π. The quantities α and $f(\mathbf{s}, \mathbf{s}')$ are called the extinction and scattering coefficients, respectively. For a non-absorbing medium, they are related by

$$\alpha = \oint f(\mathbf{s}, \mathbf{s}')\, d^2\mathbf{s}', \qquad (32)$$

which is an expression of the law of conservation of energy.

If radiation is being propagated along the x axis in a medium containing large-scale inhomogeneities ($kl \gg 1$), so that the scattering coefficient $f(\mathbf{s}, \mathbf{s}')$ is large in the forward direction, then we can simplify Eq. (31) by integrating to infinity along the component $\mathbf{s}_\perp'$ of the unit vector $\mathbf{s}'$ transverse to the x axis. Then the equation takes on the form

$$\left(\frac{\partial}{\partial x} + \mathbf{s}_\perp \nabla_{R_\perp}\right) I(\mathbf{R}, \mathbf{s}_\perp) = -\alpha I(\mathbf{R}, \mathbf{s}_\perp) + \int_{-\infty}^{\infty} f(|\mathbf{s}_\perp - \mathbf{s}_\perp'|)\, I(\mathbf{R}, \mathbf{s}_\perp')\, d^2\mathbf{s}_\perp' \qquad (33)$$

and is called the transport equation in the small-angle approximation.[449,450] Eq. (33) is considerably simpler than (31), and when subjected to Fourier transformation, it allows integration by quadrature, as has been discussed above.

The transport equation (31) was originally formulated by O. D. Khvol'son, and then by Schwartzschild. In modern form (see the monographs[463-465]), it has been independently derived by Chandrasekhar,[516] G. V. Rozenberg,[517] and somewhat earlier by V. V. Sobolev,[518] who treated the special case of Rayleigh scattering. The derivation of the transport equation by the cited authors is based on considerations of energy balance, with no explicit microscopic interpretation given to the extinction and scattering coefficients entering into the equation.

There is an extensive literature on methods of solving the transport equation, and the circle of phenomena that it is used to describe is continually expanding. Thus, along with the classical problem of incidence of a plane monochromatic (or quasi-monochromatic) wave on a half-space filled with a statistically and regularly homogeneous medium, problems have been treated in the literature on the behavior of a narrow beam of light,[519] on the directional diagram of a receiver in a scattering medium,[520] on the propagation of brief pulses,[521] on the features of radiation transport in a magnetoactive plasma,[522] etc.

A number of studies have appeared in the last 10–15 years concerned with deriving the transport equation within the framework of the theory of multiple scattering and with elucidating the limits of its applicability.[28,30,32,474,515,523-527] In treating these problems, it is convenient to distinguish the case of large-scale inhomogeneities, which involves the transport equation in the small-angle approximation (33). The derivation of (33) is much easier than that of (31), which holds for inhomogeneities of any arbitrary scale. In particular, as we have mentioned, the transport equation in the small angle approximation can be directly derived by solving the simplified B.-S. equation by using the MSP. Here the ray intensity $I(\mathbf{R}, \mathbf{s}_\perp)$ coincides with field spectrum $B_u(\mathbf{R}, \kappa_\perp)$ perpendicular to the direction of propagation of radiation.

Of course, a thorough derivation of the transport equation (31) should include an elucidation of the conditions under which (31) can be derived from the generalized (and correspondingly more complex) transport equation (30). In particular, the spectral density $B_u(\mathbf{R}, \kappa)$ depends both on the direction and on the modulus of the wave vector κ, while the ray intensity $I(\mathbf{R}, \mathbf{s})$ is actually a function only of the ray direction $\mathbf{s} = \kappa/\kappa$, whereas the modulus of κ is fixed, being equal to the k-wave number in a homogeneous medium. The generalized transport equation (30) describes a wider set of phenomena, including spatial dispersion of waves, spatial variation in spectral density over the course l of an effective inhomogeneity, and phenomena involving the mutual arrangement of inhomogeneities within a Fresnel diffraction zone, and also, if there is a phase boundary, it takes into account the fluctuations of this boundary and the existence of reflection and refraction of the mean field $\langle u \rangle$ at it.

The ordinary equation can be derived from the generalized equation only if one neglects the listed phenomena. Here one gets the two following important results.

First, the spectral density $B_u(\mathbf{R}, \kappa)$ proves to be related to the ray intensity $I(\mathbf{R}, \mathbf{s})$ by

$$B_u(\mathbf{R}, \varkappa) \cong k^{-2}\delta(\varkappa - \mathbf{k})\, I(\mathbf{R}, \mathbf{s}), \tag{34}$$

where $\mathbf{s} = \kappa/\kappa$. Thus, by inverse Fourier transformation we get the formula

$$\langle u(\mathbf{r}_1)\, u^*(\mathbf{r}_2)\rangle = \oint e^{ik\mathbf{s}(\mathbf{r}_1-\mathbf{r}_2)} I(\mathbf{R}, \mathbf{s})\, d^2\mathbf{s}, \tag{35}$$

which relates the ray intensity with the covariance of the field.

Second, the extinction and scattering coefficients α and $f(\mathbf{s}, \mathbf{s}')$ of the transport equation are expressed in terms of the Fourier transforms $\widetilde{M}$ and $\widetilde{K}$ of the kernels M and K:

$$\widetilde{M}(\varkappa, \varkappa') = (2\pi)^3\,\delta(\varkappa - \varkappa')\,\widetilde{M}_0(\varkappa),$$
$$\widetilde{K}(\varkappa_1, \varkappa_1'; \varkappa_2, \varkappa_2') = (2\pi)^3\,\delta(\varkappa_1 - \varkappa_1' - \varkappa_2 + \varkappa_2')\,\widetilde{K}_0(\varkappa_1, \varkappa_1'; \varkappa_2, \varkappa_2'),$$

while, namely,

$$\alpha = -\operatorname{Im}\frac{\widetilde{M}_0(k)}{k}, \qquad f(\mathbf{s}, \mathbf{s}') = \frac{1}{(4\pi)^2}\widetilde{K}_0(k\mathbf{s}, k\mathbf{s}'; k\mathbf{s}, k\mathbf{s}'). \tag{36}$$

These relations reveal the microscopic meaning of the extinction and scattering coefficients. The conditions of applicability of the transport equation are restricted by the parameters of both the medium and the wave field.

The restrictions on the parameters of the medium arise even when we neglect spatial dispersion. They consist in the requirement that the extinction distance d should be large in comparison, both with the wavelength $\lambda = 2\pi/k$ in a homogeneous medium, and with the scale l of the effective inhomogeneities:

$$d \gg \lambda, \quad d \gg l.$$

The restrictions on the wave field appear because we have neglected the spatial nonlocality of the kernel Q in the generalized transport equation, as well as the finite width of the spatial spectrum of the field. If L_R is the scale of the inhomogeneity of the spectral density $B_U(\mathbf{R}, \kappa)$ with respect to $\mathbf{R}$, then L_R must be large in comparison with: first, the dimension l of the effective scattering inhomogeneity; and second, the wavelength $\lambda = 2\pi/k$ in a homogeneous medium; and third, (if we are interested in the correlation function of the field), with the distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$ between observation points. Thus,

$$L_R \gg l, \quad L_R \gg \lambda, \quad L_R \gg r,$$

That is, the wave field must be weakly inhomogeneous on the scale of all three of the cited quantities.

* * *

In estimating the current status of the theory of volume scattering as a whole, we can state that, in spite of the substantial growth of the general theory of multiple scattering, the most productive methods from the standpoint of concrete results are still the original methods: the method of small perturbations, the MSP, and the PEM. Among the assets of the general theory of multiple scattering we can list that it provides a basis for the transport equation. However, no one has yet achieved an analogous basis of the MSP and the PEM, which are akin to the geometrical-optics approximation, and which take into account to some extent both multiple scattering and diffraction effects. Perhaps further development of the general theory of multiple scattering will make it possible to solve this problem, which owes its importance primarily to the fact that the cited asymptotic methods most probably will remain even in the future the working apparatus for solving concrete problems.

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