

## PROBLEMS OF STATISTICAL THEORY OF INTERACTION OF ATOMS WITH RADIATION

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RECENTLY, interest has increased in the statistical theory of electromagnetic processes in different media. This is due primarily to the development of quantum radiophysics, and particularly to the theory of fluctuations of laser radiation<sup>[1-11]</sup> and statistical processes in nonlinear optics<sup>[12,13]</sup>. The development of a general statistical theory is stimulated also by the development of certain spectroscopic problems, namely the theory of spectral-line broadening in gases and in a plasma, excitation of atoms in strongly non-equilibrium systems<sup>[14,15]</sup>, etc.

The purpose of the present article is to develop one of the possible methods of describing statistical processes in classical and quantum systems of atoms and an electromagnetic field. Naturally, within the framework of one paper it is possible to present a sufficiently complete description only for a concrete system. The general theory is developed here using as an example a gas whose atoms interact via a transverse field. However, the results are given in such a form that a generalization to the case of interaction via a longitudinal field is obvious.

By way of a concrete application, we consider the theory of natural fluctuations of the radiation of the gas laser, which determine the radiation line width. The fundamental problem here is that of determining the non-equilibrium polarization noise of the working medium of a laser. The degree of non-equilibrium is determined by the magnitude of the generated field.

The paper consists of two chapters. In the first we develop the classical theory and in the second the quantum theory. Each of the chapters in turn can be subdivided into two. This subdivision is governed by the following.

The first problem is to obtain, on the basis of the initial microscopic equations, a system of kinetic equations for particles and fields, describing dissipative processes in the medium under consideration.

The starting point can be the Liouville equation for the distribution function  $f(x, X, t)$  of the variables of the particles and the field (here  $x$  is the aggregate of coordinates and momenta of the atoms, and  $X$  is the aggregate of the field oscillators).

From the Liouville equations it is possible to obtain a system of coupled equations for the simpler distribution functions—the distribution functions of the variables of one atom, the distribution functions of the variables of one oscillator, the second distribution functions, third distribution functions, etc. Such a system of equations is analogous to the system of equations used by Bogolyubov<sup>[16]</sup>, Born and Green<sup>[17]</sup>, and others in the theory of gases and plasma.

In place of the second and higher distribution functions, it is more convenient to use equations for the corresponding correlation functions.

Two methods are used for an approximate solution of the chain of equations and for obtaining kinetic equations, namely a closed system of equations for the first distribution functions  $f_1(x_1, t)$  and  $f_1(X_1, t)$ .

The first is based on the use of different variants of perturbation theory with respect to the interaction. Such a method was first used by Bogolyubov in the derivation of the Landau kinetic equation, and also by Bogolyubov and Gurov (see, for example,<sup>[18]</sup>) in the derivation of the corresponding quantum equation. A considerable number of papers are devoted to the application of perturbation theory with respect to the interaction to systems of charged particles and atoms interacting with an electromagnetic field. A review of a number of such papers can be found in the books<sup>[1,18,19]</sup>.

Besides the perturbation-theory method, a general method, which makes it possible to take into account the influence of the polarization of the medium in the collision integral, has been widely used recently. It is based on the approximation of the second correlation functions, when the third and higher correlation functions in the chain of equations are set equal to zero. Such an approach was formulated by Bogolyubov for a system of charged particles<sup>[16]</sup>. The existing classical kinetic equations were first obtained in the papers of Balescu<sup>[21]</sup> and Lenard<sup>[20]</sup>. The corresponding quantum equations were derived in<sup>[21,22]</sup>.

In this paper the kinetic equations are derived by a method close to that previously used by the author in plasma theory<sup>[23-26]</sup>. This method makes it possible to take into account in a relatively simple manner the contribution made to the kinetic equation by the resonant and nonresonant electromagnetic radiation and also the polarization of the medium.

After deriving the kinetic equations, we proceed to the next stage, the calculation of the natural fluctuations in He-Ne gas lasers.

An unbounded medium is characterized by four temporal parameters: the frequency of the oscillation or the transition, the Doppler width, the coefficient of the radiation friction, and the attenuation time of the field. In describing the statistical processes in a laser, there appear additional parameters: the attenuation time of the field in the resonator, and the correlation times of the fluctuations of the amplitude and of the phase of the laser emission. In a gas laser, these characteristic times are much longer than the times characterizing the processes of dissipation in an unbounded medium. This makes it possible to obtain the statistical description of the processes in two stages: first it is possible to obtain kinetic equations describing the dissipative processes in an unbounded medium, and then these kinetic equations can be used to describe the natural fluctuations of laser radiation.

The line width of laser radiation is determined by two factors: thermal fluctuations of the field in the resonator, and non-equilibrium fluctuations of the polarization of the medium. The thermal fluctuations in the resonator are determined by the well known Callen-Welton formula, and therefore the main problem reduces to a calculation of the non-equilibrium fluctuations of the polarization in the laser-generation regime.

The fluctuations of laser radiation are calculated in the present paper for both a classical and a quantum system. However, the calculation presented here for the line width of the coherent radiation of the classical generator is only illustrative in character, since there is no direct correspondence between the chosen model and the real system. This material is introduced to facilitate the understanding of the corresponding quantum calculation for the He-Ne laser.

## I. CLASSICAL THEORY

### 1.1. Initial Equations

We consider a system of atoms interacting with the electromagnetic field. We regard the atom as an oscillator having a dipole electric moment. The total number of atoms is denoted by  $N$ . The state of the oscillators is determined by specifying the coordinates and the momenta of the centers of gravity  $\mathbf{R}_i$  and  $\mathbf{P}_i$ , and the coordinates and momenta of the internal motion,  $\mathbf{r}_i$  and  $\mathbf{p}_i$ . The index  $i$  takes on values  $1, 2, \dots, N$ . The time variation of the oscillator state is determined by a system of corresponding equations of motion.

Instead of using the system of equations of motion, the evolution of the states of the system can be described by using an equation for the phase density  $N(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}, t)$  in the space of the variables  $\mathbf{R}, \mathbf{P}, \mathbf{r}$ , and  $\mathbf{p}$ . It is defined as follows:

$$N(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}, t) = \sum_{1 \leq i \leq N} \delta(\mathbf{R} - \mathbf{R}_i(t)) \delta(\mathbf{P} - \mathbf{P}_i(t)) \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{p} - \mathbf{p}_i(t)). \quad (1.1)$$

It follows therefore that the function

$$N(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}, t) d\mathbf{R} d\mathbf{P} d\mathbf{r} d\mathbf{p} \quad (1.2)$$

determines the number of atoms that have at the instant of time  $t$  variables  $\mathbf{R}_i, \mathbf{P}_i, \mathbf{r}_i$ , and  $\mathbf{p}_i$  in intervals  $d\mathbf{R}, d\mathbf{P}, d\mathbf{r}, d\mathbf{p}$  near the values  $\mathbf{R}, \mathbf{p}, \mathbf{r}$ , and  $\mathbf{p}$ .

The integral of the function  $N(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}, t)$  with respect to all the values of the variables  $\mathbf{R}, \mathbf{P}, \mathbf{r}$ , and  $\mathbf{p}$  is equal to the total number of atoms, i.e.,

$$\int N(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}, t) d\mathbf{R} d\mathbf{P} d\mathbf{r} d\mathbf{p} = N. \quad (1.3)$$

The equation for the phase density follows from the condition for the conservation of the total number of atoms. In the dipole approximation it can be written in the form

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + \mathbf{v} \frac{\partial}{\partial \mathbf{r}} - m\omega_0^2 \mathbf{r} \frac{\partial}{\partial \mathbf{p}} + e\mathbf{E}^M(\mathbf{R}, t) \frac{\partial}{\partial \mathbf{p}} \right) N = 0, \quad \mathbf{V} = \frac{\mathbf{P}}{M}. \quad (1.4)$$

In this equation  $m$  and  $M$  are the masses of the electron and the atom,  $\omega_0$  is the natural frequency of the oscillator, and  $\mathbf{E}^M$  is the microscopic intensity of the electric field. In the dipole approximation, the function  $\mathbf{E}^M$  does not depend on  $\mathbf{r}$ .

The equation for the function  $\mathbf{E}^M$  follows from the system of microscopic Lorentz equations. We write it in the form

$$\frac{\partial^2 \mathbf{E}^M}{\partial t^2} - c^2 \Delta \mathbf{E}^M = -4\pi \frac{\partial^2 \mathbf{P}^M}{\partial t^2}, \quad \text{div } \mathbf{E}^M = 0. \quad (1.5)$$

The microscopic polarization vector  $\mathbf{P}^M$  is determined by the phase density

$$\mathbf{P}^M(\mathbf{R}, t) = e \int \mathbf{r} N(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}, t) d\mathbf{P} d\mathbf{r} d\mathbf{p}. \quad (1.6)$$

When account is taken of the interaction with the transverse electromagnetic field in expression (1.6), it is necessary to take only the solenoidal part into account, and by the same token to use the condition  $\text{div } \mathbf{P}^M = 0$ .

We denote by  $\bar{N}$ ,  $\mathbf{E}$ , and  $\mathbf{P}$  the phase density, magnetic field, and polarization averaged over the ensemble. Using the identity  $\overline{\mathbf{N}\mathbf{E}^M} = \bar{N}\mathbf{E} + \overline{\delta N \delta \mathbf{E}}$ , where  $\delta N$  and  $\delta \mathbf{E}$  are the deviations from the mean values, we obtain from (1.4) the following equation for the function  $\bar{N}$ :

$$\hat{L}\bar{N} = -e \frac{\partial \overline{\delta N \delta \mathbf{E}}}{\partial \mathbf{p}} \equiv nJ, \quad \bar{N} = nf \quad (1.7)$$

( $f$  is the distribution function normalized to unity); here

$$\hat{L} = \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + \mathbf{v} \frac{\partial}{\partial \mathbf{r}} - m\omega_0^2 \mathbf{r} \frac{\partial}{\partial \mathbf{p}} + e\mathbf{E} \frac{\partial}{\partial \mathbf{p}}, \quad (1.8)$$

$n$  is the average number of atoms per unit volume, and  $J$  is the collision integral.

From (1.5) follows an equation for the average field

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2 \Delta \mathbf{E} = -4\pi \frac{\partial^2 \mathbf{P}}{\partial t^2}, \quad \text{div } \mathbf{E} = 0; \quad (1.9)$$

here  $\mathbf{P}$  is the average value of the polarization vector. From (1.6) we have

$$\mathbf{P}(\mathbf{R}, t) = e \int \mathbf{r} \bar{N}(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}, t) d\mathbf{P} d\mathbf{r} d\mathbf{p}. \quad (1.10)$$

The system (1.7)–(1.10) for the mean values of  $\bar{N}$  and  $\mathbf{E}$  (the first moments) is not closed, since Eq. (1.7) contains the function  $\overline{\delta N \delta \mathbf{E}}$ —the second central moment.

To obtain the equations for the second moments, we use the equations for the deviations  $\delta N$  and  $\delta \mathbf{E}$ .

From (1.4), (1.7), (1.5), and (1.9) we get

$$\hat{L}\delta N + e\delta \mathbf{E} \frac{\partial \bar{N}}{\partial \mathbf{p}} = -e \frac{\partial}{\partial \mathbf{p}} (\delta N \delta \mathbf{E} - \overline{\delta N \delta \mathbf{E}}) \equiv n\delta J, \quad (1.11)$$

$$\frac{\partial^2 \delta \mathbf{E}}{\partial t^2} - c^2 \Delta \delta \mathbf{E} = -4\pi \frac{\partial^2 \delta \mathbf{P}}{\partial t^2}, \quad \text{div } \delta \mathbf{E} = 0, \quad (1.12)$$

$$\delta \mathbf{P}(\mathbf{R}, t) = e \int \mathbf{r} \delta N d\mathbf{P} d\mathbf{r} d\mathbf{p}. \quad (1.13)$$

Equation (1.11) contains the product of the deviations  $\delta N$  and  $\delta \mathbf{E}$ . Consequently, the equation for the second moments contains the third moment, etc. Thus, we arrive at a chain of coupled equations, the exact solution of which, of course, is impossible.

The collision integral in (1.7) is determined by the second correlation function of the coordinates and momenta of the atomic and field oscillators. If we neglect this correlation, i.e., if we put  $J = 0$ , then we obtain a system of self-consistent equations for the first moments of  $\bar{N}$  and  $\mathbf{E}$ . The dissipative effects are discarded thereby. To take into account the dissipative processes it is necessary to take into account

at least the contribution from the second correlation function and by the same token, to include the second moments in addition to the first.

To explain more clearly how the chain of equations is to be terminated, let us consider the relations between the different temporal parameters characterizing the system in question.

### 1.2. Characteristic Temporal Parameters

We introduce the following notation:  $\omega_0$ —frequency of the axial oscillations of the atomic oscillator,  $\gamma$ —coefficient of radiative damping,  $\gamma_\omega$ —coefficient of field damping at the frequency  $\omega$  in an unbounded medium,  $\Delta\omega_D$ —Doppler width.

To simplify the calculation we disregard the influence of collisions on the Doppler broadening and the collision broadening. Allowance for this phenomena does not raise any fundamental difficulties. For the equilibrium state, a detailed calculation of the Doppler and collision broadenings are given in the review of Rautian and Sobel'man<sup>[14]</sup>.

In the coherent-radiation regime, the emission spectrum is characterized by the parameters  $\Delta\omega_{ph}$  and  $\Delta\omega_a$ —the spectral widths governed by the phase and amplitude fluctuations;  $\Delta\omega_r$  is the resonator bandwidth.

The seven temporal parameters  $\omega_0$ ,  $\gamma_\omega$ ,  $\Delta\omega_D$ ,  $\gamma$ ,  $\Delta\omega_r$ ,  $\Delta\omega_a$ , and  $\Delta\omega_{ph}$  can be broken up into two groups. One includes the "fast" parameters, and the other the "slow" ones.

We consider the case of inhomogeneous broadening, when  $\Delta\omega_D \gg \gamma$ . The field damping coefficient is proportional to the atom concentration  $n$  and depends essentially on the frequency  $\omega$ . It is maximal at the resonant frequency  $\omega_0$ . If  $\gamma \sim 10^8$  and  $\Delta\omega_D \sim 10^{10}$ , then already at  $n > 10^9 \text{ cm}^{-3}$  we have  $\gamma_{\omega_0} > \gamma$ . We assume that  $\gamma_{\omega_0} \gg \gamma$ .

The quantities  $\gamma_\omega$  and  $\gamma_{\omega_0}$  are described by the relation

$$\gamma_\omega = \gamma_{\omega_0} \exp[-(\omega - \omega_0)^2 / \Delta\omega_D^2].$$

For nonresonant frequencies, when  $|\omega - \omega_0| > \Delta\omega_D$ , we have  $\gamma_\omega \ll \gamma_{\omega_0}$ . We shall henceforth denote by  $\gamma_\omega^n$  the damping coefficient at nonresonant frequencies.

The introduced parameters satisfy the condition

$$\omega_0, \gamma_{\omega_0}, \Delta\omega_D \gg \gamma, \gamma_\omega^n, \Delta\omega_r, \Delta\omega_a, \Delta\omega_{ph}. \quad (1.14)$$

The slow parameters will be further subdivided into "fast" and "slow."

The ratio of the parameters  $\gamma_{\omega_0}$  and  $\Delta\omega_D$  depends on the concentration of the atoms  $\gamma_{\omega_0} > \Delta\omega_D$  when  $n > 10^{11} \text{ cm}^{-3}$ .

### 1.3. Approximation of Second Correlation Functions

The first problem is to obtain equations describing the processes in the system in times on the order of  $1/\gamma$ . The dissipative processes are determined in these equations by the fast fluctuations. The main assumption is that the triple and higher-order correlations of the fast fluctuations make a small contribution to the dissipative processes in times on the order of  $1/\gamma$ .

The second-correlation approximation was proposed first by Bogolyubov<sup>[16]</sup> in the analysis of a system of

charged particles. Further investigations have shown<sup>[20-26]</sup> that, in contrast to perturbation theory with respect to a small interaction, this approximation makes it possible to take the polarization of the medium immediately into account in the kinetic equations.

Obviously, the approximation with respect to the moments of the functions  $N$  and  $\mathbf{E}^M$  does not coincide with the approximation with respect to the correlation function. For example, in the second-moment approximation, just as in the self-consistent-field approximation, the medium is regarded as continuous.

In the second-correlation-function approximation, account is taken of the atomic structure of the medium. We shall show that this approximation is equivalent to equations for the second moments with a source that can be expressed in terms of the first moments.

In the second-moment approximation, the right side of (1.11) should be set equal to zero. As a result we obtain

$$\hat{L}\delta N + e\delta\mathbf{E} \frac{\partial \bar{N}}{\partial \mathbf{p}} = 0. \quad (1.15)$$

We multiply this equation by  $\delta N(\mathbf{x}', t)$  ( $\mathbf{x} = (\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p})$ ) and average:

$$\hat{L}\overline{\delta N(x, t)\delta N(x', t')} + e\overline{\delta\mathbf{E}(\mathbf{R}, t)\delta N(x', t')} \frac{\partial \bar{N}}{\partial \mathbf{p}} = 0, \quad t > t'. \quad (1.16)$$

We write down the corresponding equation in the second-correlation-function approximation. We denote by  $g_2(\mathbf{x}, \mathbf{x}', t, t')$  the correlation function of the variables of two atomic oscillators at the instants  $t$  and  $t'$ . When  $t = t'$  the function  $g_2$  is determined by the expression

$$\overline{\delta N(x, t)\delta N(x', t)} = n^2 g_2(x, x', t) + n\delta(x - x')f(x, t). \quad (1.17)$$

We can write an analogous relation also for  $t \neq t'$ :

$$\overline{\delta N(x, t)\delta N(x', t')} = n^2 g_2(x, x', t, t') + n f_2(x, t, x', t'). \quad (1.18)$$

The function  $f_2(\mathbf{x}, t, \mathbf{x}', t')$  determines the statistical connection between the states of one frequency at different instants of time. At  $t = t'$

$$f_2(x, t, x', t) = \delta(x - x')f(x, t). \quad (1.19)$$

In the second-correlation-function approximation, we obtain in lieu of (1.16)

$$\hat{L}n^2 g_2(x, x', t, t') + e\overline{\delta\mathbf{E}(\mathbf{R}, t)\delta N(x', t')} \frac{\partial \bar{N}}{\partial \mathbf{p}} = 0, \quad t > t'. \quad (1.20)$$

Using (1.18), we write this expression in the form

$$\hat{L}\overline{\delta N(x, t)\delta N(x', t')} + e\overline{\delta\mathbf{E}(\mathbf{R}, t)\delta N(x', t')} \frac{\partial \bar{N}}{\partial \mathbf{p}} = \hat{L}\overline{\delta N(x, t)\delta N(x', t')}^{(\text{source})} \quad (1.21)$$

We have introduced here the symbol

$$\overline{\delta N(x, t)\delta N(x', t')}^{(\text{source})} = n f_2(x, t, x', t'). \quad (1.22)$$

Thus, in the second-correlation-function approximation, Eq. (1.21) differs from (1.6) in that it contains the "source" (1.22).

The quantities  $x'$  and  $t'$  in the function  $f_2$  are parameters, so that the equation for this function coincides with the kinetic equation for the distribution function  $f(\mathbf{x}, t)$ .

The equation for the function  $f_2$  is solved subject to the initial condition (1.19).

It follows from (1.21) that the function  $\overline{\delta N(x, t)\delta N(x', t')}$  is determined completely by the

expression (1.22) if the field fluctuations make a negligibly small contribution, i.e., effects due to the polarization of the medium are of no importance.

It is convenient to introduce the source directly into the equation for the function  $\delta N$ . We then obtain in place of (1.15) the equation

$$\hat{L}\delta N + e\delta E \frac{\partial \bar{N}}{\partial p} = \hat{L}\delta N^{(source)} \quad (1.23)$$

The correlation of the functions  $\delta N^{(source)}(x, t)$  and  $\delta N^{(source)}(x', t')$  is determined by Eq. (1.22).

Since there is still no explicit expression for this collision integral  $J$ , we cannot write the equation for  $f_2$  in explicit form, and consequently we cannot determine the source. We shall show, however, that under the condition (1.14) the collision integral is determined by the fast fluctuations. To determine the spectral functions it suffices to know the function  $f_2$  in the time interval  $t - t' \ll 1/\gamma$ . The equation for this function is determined by the kinetic equation (1.7) with  $J = 0$ .

We have introduced here the source only for atomic oscillators. It can be introduced in similar fashion also for the oscillators of the transverse electromagnetic field.

Thus, just as in the statistical theory of non-equilibrium processes in a plasma (see the book<sup>[23]</sup>, Sec. 14–16, and the article<sup>[25]</sup>), we can use either the equation (1.20) for the correlation functions, or Eq. (1.23) for  $\delta N$ . The former method is more complicated, since to determine the two-time correlation function from (1.20) it is necessary to know the single-time correlation function. It can be obtained by approximately solving the chain of equations for the distribution functions.

The use of Eq. (1.23) makes it possible to find the spectral functions in a simpler manner, without first determining the single-time correlation function.

It must be borne in mind, however, that the proof of the equivalence of Eqs. (1.21) and (1.23) can be obtained without knowing the single-time correlation function. Therefore the use of (1.23) with the source (1.22) is not based on any definite assumption. The proof of the equivalence of (1.21) and (1.23) for a plasma is given in<sup>[23,25]</sup>.

#### 1.4. The Collision Integral

Let us consider the function

$$\overline{\delta N(x, t) \delta E(R', t')} = \overline{(\delta N \delta E)_{x, t, R-R', t-t'}}$$

We expand it in a Fourier integral in terms of  $t - t'$  and  $R - R'$ , and then put  $t = t'$  and  $R = R'$  and substitute the resultant expression in the collision integral. As a result we get

$$J(x, t) = -\frac{e}{n} \frac{1}{(2\pi)^4} \frac{\partial}{\partial p} \int \text{Re}(\delta N \delta E)_{x, t, \omega, k} d\omega dk \quad (1.24)$$

It follows from (1.24) that the relaxation time of the function  $f$  is of the order of  $1/\gamma$ . The collision integral is determined by the fast fluctuations. The slow fluctuations will be determined by the kinetic equation itself.

In order to find the spectral function  $(\delta N \delta E)_{\omega k}$  for the fast fluctuations, let us consider the solution of Eq. (1.23) in a time interval  $t - t'$  such that

$$1/\gamma \omega_0, \quad 1/\Delta \omega_D \ll t - t' \ll 1/\gamma. \quad (1.25)$$

The average field is limited by the condition

$$eE(t-t')/mV_\tau \ll 1, \quad \text{but } eE/mV_\tau \gamma \sim 1. \quad (1.26)$$

We then have in (1.23)

$$\dot{L} \rightarrow \hat{L}_0 = \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + \mathbf{v} \frac{\partial}{\partial \mathbf{r}} - m\omega_0^2 \mathbf{r} \frac{\partial}{\partial \mathbf{p}}, \quad (1.27)$$

and (1.23) can be rewritten in the form

$$\hat{L}_0 \delta N(x, t) = -e\delta E \frac{\partial \bar{N}}{\partial p} + \hat{L}_0 \delta N^{(source)} \quad (1.28)$$

In the right side of this equation there are two terms. Accordingly, we represent the spectral function  $\text{Re}(\delta N \delta E)_{\omega k}$  in the form

$$\text{Re}(\delta N \delta E)_{\omega k} = \text{Re}(\delta N \delta E)_{\omega k}^{(source)} + \text{Re}(\delta N \delta E)_{\omega k}^{(ind)}. \quad (1.29)$$

Let us find first the first part, due to the source. Under the conditions (1.25) and (1.26), the equation for  $f_2$  is determined by Eq. (1.7) with  $J = 0$  and  $\mathbf{E} = 0$ , i.e.,

$$\hat{L}_0 f_2(x, t, x', t') = 0. \quad (1.30)$$

We solve this equation subject to the initial condition (1.19). The solution can be written in the form

$$f_2(x, t, x', t') = \delta(x'(x, t-t') - x') f(x, t). \quad (1.31)$$

We have taken here into account the fact that in the interval  $t - t' \ll 1/\gamma$  we have  $f(x', t') = f(x, t)$ . The function  $x'(x, t - t')$  is determined by the solution of the system of equations of the characteristics.

From (1.31) we find an expression for the spectral function of the source

$$(\delta N \delta N)_{x t r' p' \omega k}^{(source)} = n \int_{-\infty}^{\infty} \delta[x'(x, \tau) - x'] e^{i\omega\tau - ik\rho} d\tau d\rho f(x, t), \quad (1.32)$$

where  $\tau = t - t'$  and  $\rho = \mathbf{R} - \mathbf{R}'$ .

In order to find with the aid of this expression the spectral function  $\text{Re}(\delta N \delta E)_{\omega k}^{(source)}$ , let us consider the solution of the equations for  $\delta \mathbf{P}$  and  $\delta \mathbf{E}$  in the interval (1.25).

From (1.28) we get an equation for  $\delta \mathbf{P}$

$$\left[ \left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} \right)^2 + \omega_0^2 \right] (\delta \mathbf{P} - \delta \mathbf{P}^{(source)}) = \frac{e^2 n}{m} f(\mathbf{P}) \delta \mathbf{E}; \quad (1.33)$$

here  $f(\mathbf{P}) = \int f(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}) d\mathbf{r} d\mathbf{p}$  is the distribution over the momenta of the centers of gravity.

In the approximation considered by us, it is natural to assume that the distribution of the atoms is spatially homogeneous, and  $f(\mathbf{P})$  is a Maxwellian distribution.

It follows from (1.33) that  $\delta \mathbf{P}$  consists of two parts:

$$\delta \mathbf{P} = \delta \mathbf{P}^{(ind)} + \delta \mathbf{P}^{(source)} \quad (1.34)$$

The induced part of  $\delta \mathbf{P}$  is determined by the field  $\delta \mathbf{E}$ . In the interval (1.25) we obtain for the induced part of the field  $\delta \mathbf{P}$  from (1.33)

$$\delta \mathbf{P}^{(ind)}(\omega, \mathbf{k}, \mathbf{P}) = [\epsilon_v(\omega, \mathbf{k}) - 1] f(\mathbf{P}) \delta \mathbf{E}(\omega, \mathbf{k})/4\pi; \quad (1.35)$$

here

$$\epsilon_v(\omega, \mathbf{k}) = 1 + (4\pi e^2 n/m) [\omega_0^2 - (\omega - \mathbf{k}\mathbf{v} + i\Delta)^2]^{-1} \quad (1.36)$$

is the dielectric constant for fast fluctuations, due to the atoms with velocity  $\mathbf{v}$ .

When solving (1.33), we let

$$\int_0^{t-t'} \dots dt \rightarrow \int_0^\infty e^{-\Delta t} \dots dt,$$

where

$$\gamma_{\omega_0}, \quad \Delta\omega_D \gg \Delta \gg \gamma. \quad (1.37)$$

From (1.35) and (1.36) we obtained, after integrating with respect to  $\mathbf{P}$ ,

$$\delta P^{(\text{ind})}(\omega, \mathbf{k}) = [\varepsilon(\omega, \mathbf{k}) - 1] \delta E(\omega, \mathbf{k}) / 4\pi, \quad (1.38)$$

$$\varepsilon(\omega, \mathbf{k}) = \int \varepsilon_0(\omega, \mathbf{k}) f(\mathbf{P}) d\mathbf{P} \equiv \varepsilon' + i\varepsilon''. \quad (1.39)$$

The real and imaginary parts of the dielectric constant are given under condition (1.37) by

$$\varepsilon' = 1 + \frac{4\pi e^2 n}{m\omega_0 \Delta\omega_D} \left( e^{-Z_+^2} \int_0^{Z_+} e^{t^2} dt - e^{-Z_-^2} \int_0^{Z_-} e^{t^2} dt \right),$$

$$\varepsilon'' = (\pi^{1/2}/2) (4\pi e^2 n / m\omega_0 \Delta\omega_D) (e^{-Z_+^2} - e^{-Z_-^2}); \quad (1.40)$$

Here  $Z_{\pm} = (\omega \pm \omega_0) / \Delta\omega_D$ ,  $\Delta\omega_D = k(2\kappa T/M)^{1/2}$ . When  $|Z_{\pm}| \gg 1$  we have

$$\text{здесь } Z_{\pm} = (\omega \pm \omega_0) / \Delta\omega_D, \quad \Delta\omega_D = k(2\kappa T/M)^{1/2}. \quad \text{При } |Z_{\pm}| \gg 1$$

$$\varepsilon' = 1 + [4\pi e^2 n / m(\omega_0^2 - \omega^2)]. \quad (1.41)$$

When  $|Z_{\pm}| \ll 1$  we have

$$\varepsilon' = 1 - \frac{4\pi e^2 n}{m(\Delta\omega_D)^2} \frac{\omega - \omega_0}{\omega_0}, \quad \varepsilon'' = \frac{\pi^{1/2}}{2} \frac{4\pi e^2 n}{m\omega_0 \Delta\omega_D}. \quad (1.42)$$

Under condition (1.37), formulas (1.40)–(1.42) do not depend on  $\Delta$ .

Let us consider the contribution of the resonant field. Owing to the condition (1.37), the field can be regarded as stationary when  $\omega \sim \omega_0$ .

From (1.12) with allowance for (1.34) and (1.38) we get

$$\delta E(\omega, \mathbf{k}) = -4\pi\omega^2 \delta P^{\perp(\text{source})} / [\omega^2 \varepsilon(\omega, \mathbf{k}) - c^2 k^2]; \quad (1.43)$$

here and below  $\mathbf{a}^{\perp} = \mathbf{k} \times [\mathbf{a} \times \mathbf{k}] / k^2$  is the transverse component of the vector  $\mathbf{a}$ .

Using this expression, we obtain

$$\text{Re}(\delta N \delta E)_{\omega \mathbf{k}}^{(\text{source})} = \text{Re} \frac{4\pi\omega^2 (\delta N \delta P^{\perp})_{\omega \mathbf{k}}^{(\text{source})}}{\omega^2 \varepsilon - c^2 k^2}. \quad (1.44)$$

Let us return to (1.32). We multiply this expression by  $e^{r'}$  and integrate with respect to  $\mathbf{r}'$  and  $\mathbf{p}'$ . This yields

$$e(\delta N \delta P^{\perp})_{\omega \mathbf{k}}^{(\text{source})} = \frac{2\pi e^2 n}{\omega} \frac{\delta(\omega - \mathbf{k}\mathbf{v} - \omega_0) + \delta(\omega - \mathbf{k}\mathbf{v} + \omega_0)}{2} \left( \omega \mathbf{r}^{\perp} - i \frac{\mathbf{p}^{\perp}}{m} \right) f(x, t). \quad (1.45)$$

Expressions (1.44) and (1.45) determine the sought spectral function.

The second term in (1.45), as we shall show, determines the damping, and the first determines the frequency shift of the atomic vibrator.

In order not to complicate the calculations, we retain only terms that determine the damping. The expression for the frequency shift will be written only in the final equation.

We then obtain from (1.44) and (1.45)

$$e \text{Re}(\delta N \delta E)_{\omega \mathbf{k}}^{(\text{source})} = \frac{8\pi e^2 n \omega^2 \varepsilon''}{m |\omega^2 \varepsilon - c^2 k^2|^2} \frac{\delta(\omega - \mathbf{k}\mathbf{v} - \omega_0) + \delta(\omega - \mathbf{k}\mathbf{v} + \omega_0)}{2} \mathbf{p}^{\perp} f(x, t). \quad (1.46)$$

This yields

$$\frac{e}{n(2\pi)^4} \int \text{Re}(\delta N \delta E)_{\omega \mathbf{k}}^{(\text{source})} d\omega d\mathbf{k} = -\gamma \mathbf{p} f(x, t); \quad (1.47)$$

we have introduced here the notation

$$\gamma = \frac{e^2}{2\pi^2 m} \int \frac{\omega^2 \varepsilon''}{|\omega^2 \varepsilon - c^2 k^2|^2} \frac{\delta(\omega - \mathbf{k}\mathbf{v} - \omega_0) + \delta(\omega - \mathbf{k}\mathbf{v} + \omega_0)}{2} \frac{\mathbf{p}\mathbf{p}^{\perp}}{\rho^2} d\omega d\mathbf{k}. \quad (1.48)$$

From this we get in the zeroth approximation with respect to  $\Delta\omega_D/\omega_0$ , after integrating with respect to  $\omega$  and  $\mathbf{k}$ ,

$$\gamma = (2e^2 \omega_0^2 / 3mc^3) [\varepsilon'(\omega_0)]^{1/2}. \quad (1.49)$$

We shall need later an expression for the spectral function  $\text{Re}(\delta \mathbf{j} \delta \mathbf{E})_{\omega \mathbf{k}}$ . From (1.46) we get

$$\text{Re}(\delta \mathbf{j} \delta \mathbf{E})_{\omega \mathbf{k}}^{(\text{source})} = -\frac{\omega}{4\pi} \varepsilon''(\omega, \mathbf{k}) \frac{8\pi\omega^2 \varepsilon''}{|\omega^2 \varepsilon - c^2 k^2|^2} \frac{\mathbf{p}^{\perp \perp}}{m}; \quad (1.50)$$

We have used here expression (1.40) for  $\varepsilon''$  and a Maxwellian distribution with respect to the velocities of the centers of the atoms.

In (1.46) and (1.50) we took into account only the resonant contribution, i.e., the contribution from the field  $\delta E$  at the frequencies  $\omega \sim \omega_0$ . When the nonresonant contribution is taken into account, for example, there appears in (1.50) an additional term of the order of  $\gamma/\Delta\omega_D$  of (1.50).

Let us find now an expression for the induced part of the spectral density  $\text{Re}(\delta N \delta E)_{\omega \mathbf{k}}$ .

Under the same assumptions we have

$$\text{Re}(\delta \mathbf{j} \delta \mathbf{E})_{\omega \mathbf{k}}^{(\text{ind})} = \frac{1}{4\pi} \left( \omega \varepsilon'' + \frac{1}{2} \frac{\partial \omega (\varepsilon' - 1)}{\partial \omega} \frac{\partial}{\partial t} \right) (\delta E \delta E)_{\omega \mathbf{k}}. \quad (1.51)$$

The second term takes into account the contribution due to the dispersion in the nonresonant region. In the resonant region the field is established within a time much shorter than  $1/\gamma$ , so that the contribution from the second term is of the order of  $\gamma/\Delta\omega_D$  of the first.

In order to find the expression for  $\text{Re}(\delta N \delta E)_{\omega \mathbf{k}}^{(\text{ind})}$ , from which formula (1.51) follows, we use the solution of (1.28) for the function

$$\delta N^{(\text{ind})} = \delta N - \delta N^{(\text{source})}$$

in the time interval (1.25). This yields

$$e \text{Re}(\delta N \delta E)_{\omega \mathbf{k}}^{(\text{ind})} = -\frac{m}{4\pi} \left( \varepsilon'' \omega + \frac{1}{2} \frac{\partial \omega (\varepsilon' - 1)}{\partial \omega} \frac{\partial}{\partial t} \right) (\delta E_i \delta E_j)_{\omega \mathbf{k}} \frac{\partial f}{\partial p_j}. \quad (1.52)$$

From (1.52), after integrating with respect to  $\omega$  and  $\mathbf{k}$ , we get

$$= -\frac{e^2}{6} \left[ (\delta E \delta E)_{\omega_0} + \frac{m}{4\pi e^2 n} \frac{1}{2\pi} \int \frac{\partial \omega (\varepsilon' - 1)}{\partial \omega} \frac{\partial (\delta E \delta E)_{\omega}}{\partial t} d\omega \right] \frac{\partial f}{\partial \mathbf{p}}. \quad (1.53)$$

The second term in this expression takes into account the contribution from the nonresonant field.

From (1.52) there follows expression (1.51). It is necessary to take into account here the fact that  $\mathbf{k} \cdot \delta \mathbf{E} = 0$ .

We now can write the sought kinetic equation. From (1.7) and (1.8) we get

$$\bar{L}f = -\frac{e}{n} \frac{\partial \delta N \delta E}{\partial \mathbf{p}} \equiv J. \quad (1.54)$$

On the basis of (1.52) and (1.47), we write the collision integral in the form

$$J = D \frac{\partial^2 f}{\partial \mathbf{p}^2} + \frac{\partial}{\partial \mathbf{p}} (A \mathbf{p} f). \quad (1.55)$$

The diffusion and friction coefficients are given by

$$D = \frac{e^2}{6} \left[ (\delta E \delta E)_{\omega_0} + \frac{m}{4\pi e^2 n} \frac{1}{2\pi} \int \frac{\partial \omega (\varepsilon' - 1)}{\partial \omega} \frac{\partial (\delta E \delta E)_{\omega}}{\partial t} d\omega \right], \quad (1.56)$$

$$A = \gamma = (2e^2 \omega_0^2 / 3mc^3) [\varepsilon'(\omega_0)]^{1/2}.$$

Thus, the diffusion coefficient is determined by the spectral function of the field.

### 1.5. Equation for the Spectral Function of the Field

From the field equation, for a spatially homogeneous distribution of the atoms, there follows the energy-balance equation

$$\frac{\partial}{\partial t} \left[ \frac{(\delta E \delta E)_{\omega k} + (\delta B \delta B)_{\omega k}}{8\pi} \right] = -\operatorname{Re} (\delta j \delta E)_{\omega k}; \quad (1.57)$$

$\delta B$  is the deviation of the magnetic-field intensity from the mean value.

Using (1.50) and (1.51), we can rewrite this equation in the form

$$\frac{1}{8\pi} \frac{\partial}{\partial t} \left[ \frac{\partial \omega \epsilon'}{\partial \omega} (\delta E \delta E)_{\omega k} + (\delta B \delta B)_{\omega k} \right] - \frac{\omega}{4\pi} \epsilon'' \left[ (\delta E \delta E)_{\omega k} - \frac{8\pi \omega^3 \epsilon''}{|\omega^2 \epsilon - c^2 k^2|^2} \frac{p^2}{m} \right]. \quad (1.58)$$

In the equilibrium case  $\overline{p^2}/m = 2kT$ , and from (1.58) we get an expression known from the theory of equilibrium electromagnetic fluctuations<sup>[27,28]</sup>

$$(\delta E \delta E)_{\omega k} = 16\pi \omega^3 \epsilon'' \kappa T / (|\omega^2 \epsilon - c^2 k^2|^2). \quad (1.59)$$

From (1.58) we obtain an equation for the spectral function  $(\delta E \delta E)_{\omega}$ :

$$\frac{1}{8\pi} \frac{\partial}{\partial t} \left[ \frac{\partial \omega \epsilon'}{\partial \omega} (\delta E \delta E)_{\omega} + (\delta B \delta B)_{\omega} \right] = -\frac{\omega}{4\pi} \epsilon'' \left( \omega, \frac{\omega}{c} \sqrt{\epsilon'} \right) \left[ (\delta E \delta E)_{\omega} - \frac{4}{3} \frac{\sqrt{\epsilon'}}{c^2} \omega^2 \frac{\overline{p^2}}{m} \right]. \quad (1.60)$$

In the equilibrium case we have  $\overline{p^2}/m = 3kT$ , and from (1.60) we obtain the well known expression with allowance for the polarization of the medium<sup>[27]</sup>:

$$(\delta E \delta E)_{\omega} = 4\pi^2 \rho_{\omega} = 4\omega^2 \sqrt{\epsilon'} \kappa T / c^3. \quad (1.61)$$

For the spectral function of the field at the resonant frequency  $\omega_0$ , by virtue of the condition  $\gamma \omega_0 \gg \gamma$ , we use the steady-state solution of (1.60)

$$(\delta E \delta E)_{\omega_0} = 4\omega^2 \overline{p^2} / 3c^3 m, \quad \epsilon'(\omega_0) = 1. \quad (1.62)$$

The kinetic equation (1.54), (1.55), even without allowance for the nonresonant field, differs from the usual Fokker-Planck equation for the system of atomic oscillators in the field in that in place of the expression for the diffusion coefficient

$$D = \gamma m \kappa T \quad (1.63)$$

it follows from (1.56) and (1.62) that

$$D = \frac{m \gamma \overline{p^2}}{3m} = \frac{1}{3} m \gamma \int \frac{p^2}{m} f dr dp, \quad (1.64)$$

i.e., the diffusion coefficient itself depends on the form of the distribution function. The kinetic equation is therefore nonlinear.

Let us write down the equation for the average energy of the atomic vibrators. From (1.54) and (1.55) we obtain

$$\frac{\partial}{\partial t} \left[ n \left( \frac{\overline{p^2}}{2m} + \frac{m \omega_0^2 \overline{r^2}}{2} \right) \right] = \frac{3nD}{m} - \gamma n \frac{\overline{p^2}}{m}. \quad (1.65)$$

From (1.60) we get an equation for the density of the electro-magnetic energy. It can be written in the form

$$\frac{1}{8\pi} \frac{\partial}{\partial t} \int [(\delta E \delta E)_{\omega} + (\delta B \delta B)_{\omega}] d\omega = - \left( \frac{3nD}{m} - \gamma n \frac{\overline{p^2}}{m} \right). \quad (1.66)$$

From (1.65) and (1.66) follows the law of conserva-

tion of the total energy of the atomic oscillators and of the field.

In the presence of a thermostat, the role of which can be played by the radiation field of another system of atoms, corresponding additional terms appear in the kinetic equation.

### 1.6. Slow Fluctuations at a Zero Average Field

In order to obtain an expression for the spectral functions of the slow fluctuations—fluctuations with a characteristic time on the order of  $1/\gamma$  it is necessary to repeat the foregoing procedure, the only difference being that the initial equations are not (1.4) and (1.5) but the kinetic equations.

We obtain in this manner, for example expressions for the spectral function of the source of slow polarization fluctuations.

From the kinetic equation (1.54) we obtain at  $\mathbf{E} = 0$  the following equation for the polarization vector  $\mathbf{P}(\mathbf{R}, \mathbf{P}, t)$ :

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} \right)^2 \mathbf{P} + \gamma \frac{\partial \mathbf{P}}{\partial t} + \omega_0^2 \mathbf{P} = 0. \quad (1.67)$$

Regarding this equation as a kinetic equation, we obtain the equation for the source polarization fluctuation correlation

$$\left[ \left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} \right)^2 + \gamma \frac{\partial}{\partial t} + \omega_0^2 \right] (\delta \mathbf{P} \delta \mathbf{P})_{\mathbf{R} \mathbf{P} t', \mathbf{R}' \mathbf{P}' t'}^{(\text{source})} = 0, \quad t > t'. \quad (1.68)$$

This equation is solved subject to the initial condition that follows from (1.19):

$$(\delta \mathbf{P} \delta \mathbf{P})_{\mathbf{R} \mathbf{P} t', \mathbf{R}' \mathbf{P}' t'}^{(\text{source})} = \delta(\mathbf{R} - \mathbf{R}') \delta(\mathbf{P} - \mathbf{P}') e^{2n} \int r^2 f(\mathbf{R}, \mathbf{P}, \mathbf{r}, \mathbf{p}, t) dr dp. \quad (1.69)$$

From (1.68) with the initial condition (1.69) and with allowance for the fact that  $\operatorname{div} \delta \mathbf{P} = 0$ , we get

$$(\delta \mathbf{P} \delta \mathbf{P})_{\omega \mathbf{k} t, \omega' \mathbf{k}' t'}^{(\text{source})} = m \omega \epsilon''(\omega, \mathbf{k}) (\overline{r^2})_{\mathbf{P}, t'} / 2\pi; \quad (1.70)$$

here

$$\epsilon''_V = (4\pi e^2 n / m) \omega \gamma \{ [\omega_0^2 - (\omega - \mathbf{k} \mathbf{v})^2]^2 + \omega^2 \gamma^2 \}^{-1}. \quad (1.71)$$

When  $\Delta \omega_D \gg \gamma$ , after integrating over the velocities we get from (1.70) the expression

$$(\delta \mathbf{P} \delta \mathbf{P})_{\omega \mathbf{k} t, \omega' \mathbf{k}' t'}^{(\text{source})} = m \omega \epsilon''(\omega, \mathbf{k}) (\overline{r^2})_{t'} / 2\pi. \quad (1.72)$$

The function  $\epsilon''$  is given by (1.40).

In the equilibrium state  $m \omega^2 \overline{r^2} = 2kT$ , and expression (1.72) takes the form

$$(\delta \mathbf{P} \delta \mathbf{P})_{\omega \mathbf{k} t, \omega' \mathbf{k}' t'}^{(\text{source})} = \epsilon''(\omega, \mathbf{k}) \kappa T / \pi \omega. \quad (1.73)$$

Expression (1.72) can be used to find the spectral function of the field.

### 1.7. Emission Line Width of a Classical Coherent-radiation Generator

A system of classical oscillators can be used as the working medium in coherent-radiation generators. In the review of Gaponov, Petelin, and Yulpatov<sup>[29]</sup> (see also Gal'tsov's dissertation<sup>[30]</sup>) there are detailed descriptions of the possible mechanisms of coherent induced radiation in classical systems.

In this section we consider the model of such a generator. The pumping and the nonlinearity will be introduced phenomenologically via the effective field.

This makes it possible to trace the occurrence of the radiation with a line width smaller than  $\gamma$ .

We denote the effective electric field acting on the oscillator by  $E_{\text{eff}}$ . We specify its connection with the coherent field  $E$  in the form

$$E_{\text{eff}} = E + \alpha \frac{\partial E}{\partial t}, \quad (1.74)$$

where  $\alpha$  is a nonlinear function of the field. In a weak field

$$\alpha = \alpha_0 - \beta E^2. \quad (1.75)$$

We shall consider only the phase fluctuations, which determine in the main the line width of the coherent radiation. We denote this width by  $\Delta\omega_{\text{ph}}$ . The designation "coherent field" will be justified if it turns out that  $\Delta\omega_{\text{ph}} = \ll \gamma$ .

When  $t - t_0 \gg 1/\Delta\omega_{\text{ph}}$ , a stationary regime is established, in which the mean value of the coherent field vanishes.

We consider a single-mode generation regime and represent the field  $E$  in the form

$$E(\mathbf{R}, t) = e_0 E(R, t), \quad E(R, t) = E_0 \cos(\Omega_0 t - \mathbf{k}_0 \mathbf{R} + \varphi), \quad \mathbf{k}_0 \parallel x; \quad (1.76)$$

here  $\Omega_0 = ck_0$  is the natural frequency of the resonator,  $E_0$  is the constant amplitude (the amplitude fluctuations are disregarded),  $\varphi = \varphi(\mu t)$  is a slowly varying phase, and  $e_0$  is a unit vector.

Since our example is for illustration purposes, we assume for simplicity that the atoms are at rest and that the concentration of the atoms is constant. In this approximation, the equation for the polarization vector is

$$\left( \frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t} + \omega_0^2 \right) \mathbf{P} = \frac{e^2 n}{m} E_{\text{eff}} = \frac{e^2 n}{m} \left( E + \alpha \frac{\partial E}{\partial t} \right). \quad (1.77)$$

We represent the polarization vector in the form

$$\mathbf{P} = \mathbf{P}^{(\text{ind})} + \mathbf{P}^{(\text{source})}. \quad (1.78)$$

The equation for the induced part of the polarization coincides with (1.77), and for  $\mathbf{P}^{(\text{source})}$  we have

$$\left( \frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t} + \omega_0^2 \right) \mathbf{P}^{(\text{source})} = 0. \quad (1.79)$$

Taking (1.78) into account, the field equation can be written in the form

$$\left( \frac{\partial^2}{\partial t^2} + \frac{\Omega_0}{Q} \frac{\partial}{\partial t} - c^2 \Delta \right) \mathbf{E} = -4\pi \frac{\partial^2}{\partial t^2} (\mathbf{P}^{(\text{ind})} + \mathbf{P}^{(\text{source})}) + \Omega_0^2 \mathbf{E}^{(\text{tr})}; \quad (1.80)$$

we have introduced here two new terms. The second term on the left side takes into account the loss of energy of coherent radiation in the resonator,  $Q$  is the corresponding quality factor,  $\mathbf{E}^{(\text{tr})}$  is the source of thermal noise in the resonator, and  $\Delta\omega_{\text{r}} = \Omega_0/Q \ll \gamma$ .

Let us find first the expression for  $\mathbf{P}^{(\text{ind})}$ . To this end, we separate the first harmonic of  $E_{\text{eff}}$ . Taking (1.76) into account, we obtain

$$E_{\text{eff}} = E(\mathbf{R}, t) - \Omega_0 \left( \alpha - \frac{\beta}{4} E_0^2 \right) E_0 \sin(\Omega_0 t - \mathbf{k}_0 \mathbf{R} + \varphi). \quad (1.81)$$

We substitute this expression in (1.77). As a result we obtain for the active part of the vector  $\mathbf{P}^{(\text{ind})}$  the expression

$$P = e_a^* E_0 \sin(\Omega_0 t - \mathbf{k}_0 \mathbf{R} + \varphi) / 4\pi, \quad (1.82)$$

where

$$e_a^* = -\frac{4\pi e^2 n}{m} \frac{(\omega_0 - \Omega_0) [\alpha_0 - (\beta/4) E_0^2] - \gamma/2 \Omega_0}{2[(\omega_0 - \Omega_0)^2 + (\gamma^2/4)]} < 0$$

is the imaginary part of the dielectric constant of the

active medium. The active part makes a contribution on the order of  $\gamma\omega_0/\omega_0$ .

The amplitude  $E_0$  is determined from the condition

$$-e_a^* = 1/Q, \quad (1.83)$$

which specifies the balance of the incoming and outgoing energy.

The equation for the phase  $\varphi$  is

$$d\varphi/dt = \Omega_0 \xi / E_0, \quad (1.84)$$

where

$$\xi = -\frac{1}{V} \int e_0 (4\pi P^{(\text{source})} + E^{(\text{tr})}) \cos(\Omega_0 t - \mathbf{k}_0 \mathbf{R} + \varphi) dV$$

is the noise source.

Averaging over the volume results from the fact that the phase depends only on the time.

The spectral noise density is given by

$$(\xi^2)_\omega = [(4\pi)^2 (e_0 \delta P^{(\text{source})})_{\Omega_0 k_0}^2 + (e_0 \delta E^{(\text{tr})})_{\Omega_0 k_0}] / 2V. \quad (1.85)$$

Equation (1.84) coincides with the corresponding equation used to calculate the line width in a self-oscillator<sup>[31,32]</sup>.

The spectral function of the thermal noise is determined by the Nyquist formula (or by the more general Callen-Welton formula). In our case it takes the form

$$(1/2V) (e_0 \delta E^{(\text{tr})})_{\Omega_0 k_0}^2 = 4\pi \kappa T / V \Omega_0 Q. \quad (1.86)$$

It remains for us to find the spectral density of the polarization noise. To this end we use Eq. (1.79) for  $\mathbf{P}^{(\text{source})}$ .

Since Eq. (1.79) coincides with (1.67) at  $V = 0$ , we can use expression (1.72) for the spectral source function.

As a result we get

$$[(4\pi)^2 / 2V] (e_0 \delta P^{(\text{source})})_{\Omega_0 k_0}^2 = (4\pi m \Omega_0 e^2 / V) (\bar{r}^2 / 3). \quad (1.87)$$

From (1.85)–(1.87) we find an expression for the spectral noise function  $\xi$ :

$$(\xi^2)_\omega = (4\pi / V \Omega_0) (Q^{-1} \kappa T + \varepsilon'' \bar{E}_{\Omega_0}), \quad \bar{E}_{\Omega_0} = m \Omega_0^2 \bar{r}^2 / 3; \quad (1.88)$$

$\bar{E}_{\Omega_0}$  is the average energy of the atomic oscillator.

The total width of the emission line, due to the phase fluctuations, is determined by<sup>[31-33]</sup>

$$\Delta\omega_{\text{ph}} \equiv 2D = \Omega_0^2 (\xi^2)_\omega / E_0^2; \quad (1.89)$$

Here  $D$  is the coefficient of phase diffusion (sometimes an expression is given for the line half-width  $\Delta\omega_{\text{ph}} = D$ ).

We denote by  $W$  the energy of the field in the resonator:

$$W = VE_0^2 / 8\pi, \quad (1.90)$$

and write down the expression for  $\Delta\omega_{\text{ph}}$  in the form

$$\Delta\omega_{\text{ph}} = (\Omega_0 / 2W) (Q^{-1} \kappa T + \varepsilon'' \bar{E}_{\Omega_0}). \quad (1.91)$$

If  $\bar{E}_{\Omega_0} = \kappa T$ , then expression (1.90) can be rewritten in the form

$$\Delta\omega_{\text{ph}} = (\Omega_0 / 2) (Q^{-1} + \varepsilon'') \kappa T / W. \quad (1.92)$$

We note that the quantity  $(\varepsilon'' + Q^{-1})\Omega_0$  determines the total bandwidth of a resonator filled with the medium. Thus, the spectrum is narrowed down by a factor  $\kappa T / 2W$  during the course of generation.

The formula for the line width is so simple only in the case of a weak field, when the influence of the field

on the spectrum of the polarization fluctuations can be neglected, and at the same time the threshold is exceeded enough to be able to use the correlation approximation.

If the  $Q$  remains constant when the generation energy is varied, then the bandwidth is inversely proportional to the energy. On the other hand, if the energy is varied by changing the  $Q$ , then to determine the dependence of  $\Delta\omega_{ph}$  on  $W$  it is necessary to eliminate  $1/Q$  from (1.91) with the aid of (1.85). A similar situation takes place also in quantum theory of the line width of a gas laser<sup>[33]</sup>.

## 2. QUANTUM THEORY

We shall describe the motion of the centers of the atoms classically, and the internal state of the atoms by quantum theory. We consider the function

$$N_{nm}(\mathbf{R}, \mathbf{P}, t), \quad (2.1)$$

which has the following properties:

$$\sum_n N_{nn}(\mathbf{R}, \mathbf{P}, t) = N(\mathbf{R}, \mathbf{P}, t) \quad (2.2)$$

is the phase density in space of the coordinates and momenta  $\mathbf{R}$  and  $\mathbf{P}$ :

$$\int N_{nm}(\mathbf{R}, \mathbf{P}, t) \frac{d\mathbf{R} d\mathbf{P}}{(2\pi\hbar)^3} = \hat{\rho}_{nm} \quad (2.3)$$

is the operator density matrix in terms of the variables  $n$  and  $m$ , the quantum numbers determining the internal state of the atom.

Thus, the function (2.1) is the quantum analog of the classical phase density (1.1) in the space  $\mathbf{r}, \mathbf{p}, \mathbf{R}, \mathbf{P}$ .

The normalization of the function (2.1) is

$$\sum_n \int N_{nn}(\mathbf{R}, \mathbf{P}, t) \frac{d\mathbf{R} d\mathbf{P}}{(2\pi\hbar)^3} = N; \quad (2.4)$$

$N$  is the total number of atoms.

In the dipole approximation, the equation for the function (2.1) is

$$\dot{L}N_{nm} = \frac{ie}{\hbar} \sum_{n_1} (\mathbf{r}_{nn_1} N_{n_1m}(\mathbf{R}, \mathbf{P}, t) - N_{nn_1}(\mathbf{R}, \mathbf{P}, t) \mathbf{r}_{n_1m}) \mathbf{E}^M(\mathbf{R}, t). \quad (2.5)$$

Here and below we use the symbol

$$\dot{L} = \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + i\omega_{nm},$$

and  $\mathbf{E}^M(\mathbf{R}, t)$  is the microscopic field. The field equation is

$$\left( \frac{\partial}{\partial t^2} - c^2 \Delta \right) \mathbf{E}^M = -4\pi \frac{\partial^2 \mathbf{P}^M}{\partial t^2}, \quad \text{div } \mathbf{E}^M = 0, \quad (2.6)$$

$$\mathbf{P}^M(\mathbf{R}, t) = e \sum_{nm} \int \mathbf{r}_{nm} N_{nm}(\mathbf{R}, \mathbf{P}, t) \frac{d\mathbf{P}}{(2\pi\hbar)^3}. \quad (2.7)$$

The last expression determines the microscopic polarization of the medium.

As above, we shall denote by a bar averaging over the ensemble. We introduce the symbols for the mean values

$$\begin{aligned} \overline{N_{nm}}(\mathbf{R}, \mathbf{P}, t) &\equiv N f_{nm}(\mathbf{R}, \mathbf{P}, t), \\ \mathbf{E} &= \overline{\mathbf{E}^M}, \quad \mathbf{P} = \overline{\mathbf{P}^M}. \end{aligned} \quad (2.8)$$

The equation for the function  $f_{nm}$  is written in the form

$$\dot{L}f_{nm} - \frac{ie}{\hbar} \sum_{n_1} (\mathbf{r}_{nn_1} f_{n_1m} - f_{nn_1} \mathbf{r}_{n_1m}) \mathbf{E} = J_{nm}(\mathbf{R}, \mathbf{P}, t). \quad (2.9)$$

Equation (2.9) is similar to the kinetic equation (1.7), and therefore  $J_{nm}$  will also be called a collision integral.

It follows from (2.5) and (2.9) that the collision integral is determined by the expression

$$J_{nm} = \frac{ie}{\hbar N} \sum_{n_1} (\mathbf{r}_{nn_1} \overline{\delta N_{n_1m}} \delta \mathbf{E} - \overline{\delta N_{nn_1}} \delta \mathbf{E} \mathbf{r}_{n_1m}). \quad (2.10)$$

The equation for the deviation  $\delta N_{nm}$ , by analogy with Eq. (1.11), is written in the form

$$L\delta N_{nm} = \frac{ieN}{\hbar} \sum_{n_1} (\mathbf{r}_{nn_1} f_{n_1m} - f_{nn_1} \mathbf{r}_{n_1m}) \delta \mathbf{E} + N \delta J_{nm}; \quad (2.11)$$

we have left out immediately the term with the average field  $\mathbf{E}$ , since we shall need a solution of (2.11) in the interval  $t - t' \ll 1/\gamma$ , when condition (1.26) is satisfied.

From (2.6) we obtain the equations for the functions  $\mathbf{E}$  and  $\delta \mathbf{E}$ , for example,

$$\left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right) \delta \mathbf{E} = -4\pi \frac{\partial^2 \delta \mathbf{P}}{\partial t^2}, \quad \delta \mathbf{P} = e \sum_{nm} \int \mathbf{r}_{nm} \delta N_{nm} \frac{d\mathbf{P}}{(2\pi\hbar)^3}. \quad (2.12)$$

It is convenient in what follows to represent the function  $f_{nm}$  by

$$f_{nm} = \delta_{nm} f_n + f_{nm}^{(1)}; \quad (2.13)$$

we have separated there the diagonal part of the function  $f_{nm}$ .

We subject the matrix elements to the condition

$$f_{nn} = 0. \quad (2.14)$$

This means that we are considering atoms in which the polarization occurs only in transitions between different states.

Under the condition (2.14), the polarization vector is completely determined by the function  $f_{nm}^{(1)}$ , and consequently  $f_{nm}^{(1)} = f_{nm}^{(1)}(\mu t, t)$ , i.e., it depends on the fast time ( $\sim 1/\omega_{nm}$ ) and on the slow time ( $1/\gamma_{nm}$ ). On the other hand  $f_n = f_n(\mu t)$ , i.e., it depends only on the slow time.

We represent the deviation  $\delta N_{nm}$  likewise in the form of a rapidly and slowly varying functions

$$\delta N_{nm} = \delta_{nm} \delta N_{nn} + \delta N_{nm}^{(1)}. \quad (2.15)$$

We write down the equations for the functions  $f_n$  and  $f_{nm}^{(1)}$ .

It follows from (2.9) and (2.10) that

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} \right) f_n - \frac{ie}{\hbar} \sum_{n_1} (\mathbf{r}_{nn_1} f_{n_1n}^{(1)} - f_{nn_1}^{(1)} \mathbf{r}_{n_1n}) \mathbf{E} = J_n, \quad (2.16)$$

$$J_n = \frac{ie}{\hbar N} \sum_{n_1} (\mathbf{r}_{nn_1} \overline{\delta N_{n_1n}^{(1)}} \delta \mathbf{E} - \overline{\delta N_{nn_1}} \delta \mathbf{E} \mathbf{r}_{n_1n}), \quad (2.17)$$

$$\dot{L}f_{nm}^{(1)} - (ie/\hbar) \mathbf{r}_{nm} (f_m - f_n) \mathbf{E} = J_{nm}^{(1)}, \quad (2.18)$$

$$J_{nm}^{(1)} = (ie/\hbar N) \mathbf{r}_{nm} (\overline{\delta N_{nm}^{(1)}} \delta \mathbf{E} - \overline{\delta N_{nn}} \delta \mathbf{E}). \quad (2.19)$$

The equations for the deviations  $\delta N_{nn}$  and  $\delta N_{nm}^{(1)}$  in the approximation of the second correlation functions are similar to (1.23):

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} \right) (\delta N_{nn} - \delta N_{nn}^{(\text{source})}) - \frac{ie}{\hbar} \sum_m (\mathbf{r}_{nm} f_{mn}^{(1)} - f_{nm}^{(1)} \mathbf{r}_{mn}) \delta \mathbf{E} = 0, \quad (2.20)$$

$$L(\delta N_{nm}^{(1)} - \delta N_{nm}^{(1)(\text{source})}) = \frac{ieN}{\hbar} \mathbf{r}_{nm} (f_m - f_n) \delta \mathbf{E}. \quad (2.21)$$

The correlation of the fluctuations  $\delta N_{nm}^{(\text{source})}$  can be represented in the form

$$(\delta N_{nm} \delta N_{n'm'}^{(\text{source})})_{RR'PP'tt'} = N f_{nm} f_{n'm'} (\mathbf{R} \mathbf{R}' \mathbf{P} \mathbf{P}' t t'). \quad (2.22)$$



The equation for the function  $f_{nmn'm'}$  is determined by the kinetic equation (2.9) and is solved subject to the initial condition

$$(\delta N_{nm} \delta N_{n'm'}^{(source)})_{RR'PP't} = \frac{(2\pi\hbar)^3}{2} N \delta(\mathbf{R}-\mathbf{R}') \delta(\mathbf{P}-\mathbf{P}') (\delta_{nn'} f_{m'm}(\mathbf{R}, \mathbf{P}, t) + f_{nn'} \delta_{m'm}). \quad (2.23)$$

A derivation of this formula is given in the appendix of [34].

Expressions (2.22) and (2.23) correspond to the classical formulas (1.22) and (1.19).

Let us consider the particular case of the general formula (2.23).

When  $f_{nn'} = \delta_{nn'} f_n$  (for the slow part of the function  $f_{nn'}$ ), we obtain from (2.23)

$$(\delta N_{nm} \delta N_{n'm'}^{(source)})_{RR'PP't} = (2\pi\hbar)^3 \delta(\mathbf{R}-\mathbf{R}') \delta(\mathbf{P}-\mathbf{P}') N \delta_{nn'} \delta_{m'm'} (f_m + f_n)/2. \quad (2.24)$$

This expression for the spectral function of the source will be used in the derivation of the equation for the collision integral.

## 2.2. The Collision Integral $J_n$

Just as in the first part (see (1.24)), we express the collision integral (2.17) in terms of the spectral function

$$J_n = \frac{2e}{\hbar N (2\pi)^4} \sum_m \int \text{Im} (\delta N_{nm}^{(1)} \delta \mathbf{E})_{\omega \mathbf{k}} r_{mn} d\omega d\mathbf{k}. \quad (2.25)$$

The spectral function  $(\delta N_{nm} \delta \mathbf{E})_{\omega \mathbf{k}}$  will be represented here, too, in the form of a sum of two parts—the induced part and the contribution from the source (see (1.29)).

For the induced part, under the same assumptions as in the first part, we obtain the expression

$$\begin{aligned} & \frac{2e}{\hbar N} \text{Im} ((\delta N_{nm}^{(1)} \delta \mathbf{E})_{\omega \mathbf{k}} r_{mn})^{(ind)} \\ &= -\frac{2e^2 |r_{nm}|^2}{3\hbar^2} (f_m - f_n) \text{Im} \left\{ \frac{1}{\omega - \omega_{nm} - \mathbf{k}\mathbf{v} + i\Delta} (\delta \mathbf{E} \delta \mathbf{E})_{\omega \mathbf{k}} \right. \\ & \quad \left. + i \frac{\partial}{\partial \omega} \frac{1}{\omega - \omega_{nm} - \mathbf{k}\mathbf{v} + i\Delta} \frac{\partial}{\partial t} (\delta \mathbf{E} \delta \mathbf{E})_{\omega \mathbf{k}} \right\}. \quad (2.26) \end{aligned}$$

From this we get the corresponding expression for the function  $\text{Re}(\delta j \delta \mathbf{E})_{\omega \mathbf{k}}$ . This coincides in form with expression (1.51). The dielectric constant is now determined by

$$\epsilon(\omega, \mathbf{k}) = 1 + \sum_{nm} \frac{4\pi e^2 N |r_{nm}|^2}{3\hbar (2\pi\hbar)^3} \int \frac{f_n - f_m}{\omega - \omega_{nm} - \mathbf{k}\mathbf{v} + i\Delta} d\mathbf{P}. \quad (2.27)$$

If the velocity distribution is Maxwellian, we obtain after integration

$$\begin{aligned} \epsilon &= 1 + \frac{4\pi e^2 n}{3\sqrt{\pi} \hbar \Delta \omega_D} \sum_{nm} |r_{nm}|^2 (\rho_m - \rho_n) I_{nm}, \\ I_{nm} &= i\pi e^{-2\lambda_{nm}} - 2\pi^{1/2} e^{-2\lambda_{nm}} \int_0^{Z_{nm}} e^{t^2} dt, \quad (2.28) \\ Z_{nm} &= \frac{\omega - \omega_{nm}}{\Delta \omega_D}, \quad \rho_n = \frac{V}{(2\pi\hbar)^3} \int f_n d\mathbf{P}, \quad \Delta \omega_D = k(2\kappa T/M)^{1/2}. \end{aligned}$$

For frequencies close to resonance ( $|\omega - \omega_{nm}| \ll \Delta \omega_D$ ), it follows from (2.28) that

$$\begin{aligned} \epsilon' &= 1 - \frac{8\pi e^2 n}{3\hbar \Delta \omega_D} \sum_{n>m} |r_{nm}|^2 (\rho_m - \rho_n) \frac{\omega - \omega_{nm}}{\Delta \omega_D}, \\ \epsilon'' &= \pi^{1/2} \frac{4\pi e^2 n}{3\hbar \Delta \omega_D} \sum_{n>m} |r_{nm}|^2 (\rho_m - \rho_n). \quad (2.29) \end{aligned}$$

For the nonresonant region we get from (2.27)

$$\epsilon' = 1 + \frac{8\pi e^2 n}{3\hbar} \sum_{nm} \frac{|r_{nm}|^2 \omega_{nm}}{\omega_{nm}^2 - \omega^2} \rho_m. \quad (2.30)$$

For an oscillator, formulas (2.27)–(2.30) coincide with the corresponding classical formulas (1.40)–(1.42).

The contribution made by the source to the spectral function is given by

$$\begin{aligned} & \frac{2e}{\hbar N} \text{Im} ((\delta N_{nm}^{(1)} \delta \mathbf{E})_{\omega \mathbf{k}} r_{nm})^{(source)} \\ &= -\frac{16\pi^2 e^2 \omega^4 |r_{nm}|^2 e''}{3\hbar |\omega^2 \epsilon - c^2 k^2|^2} \delta(\omega - \omega_{nm} - \mathbf{k}\mathbf{v}) (f_n + f_m). \quad (2.31) \end{aligned}$$

Hence, recognizing that

$$\delta j = i \frac{e}{(2\pi\hbar)^3} \sum_{nm} \int \omega_{mn} r_{mn} \delta N_{nm} d\mathbf{P},$$

we obtain

$$\begin{aligned} \text{Re}(\delta j \delta \mathbf{E})_{\omega \mathbf{k}}^{(source)} &= \\ &= -\frac{\omega}{4\pi} \frac{32\pi^2 e^2 N \omega^4 e''}{|\omega^2 \epsilon - c^2 k^2|^2} \sum_{nm} \int \frac{|r_{nm}|^2}{3} \delta(\omega - \omega_{nm} - \mathbf{k}\mathbf{v}) (f_n + f_m) \frac{d\mathbf{P}}{(2\pi\hbar)^3}. \quad (2.32) \end{aligned}$$

Formulas (2.26) and (2.31) determine the collision integral  $J_n(\mathbf{R}, \mathbf{P}, t)$  (2.25).

The equation for the spectral function of the electromagnetic field differs from (1.58) only in that it is necessary to take for  $\epsilon(\omega, \mathbf{k})$  the quantum expression (2.27), and for the second term of the right side, which determines the contribution from the source, it is necessary to use (2.32).

From formulas (1.51) and (2.32) there follows in the equilibrium case, for the spectral field function, the well known expression (the Callen-Welton formula)

$$(\delta \mathbf{E} \delta \mathbf{E})_{\omega \mathbf{k}} = \frac{16\pi\hbar\omega^4 e''}{|\omega^2 \epsilon - c^2 k^2|^2} \left( \frac{1}{2} + \frac{1}{e^{\hbar\omega/\kappa T} - 1} \right). \quad (2.33)$$

At  $\hbar = 0$  it coincides with (1.59).

We integrate with respect to  $\mathbf{k}$  in the expression for  $J_n$ , and additionally with respect to  $\omega$  in the resonant terms. As a result, in the zeroth approximation in  $\Delta\omega_D/\omega_{nm}$ , we obtain the following expression (a more general result is given in [26]):

$$\begin{aligned} J_n &= \frac{e^2}{3\hbar^2} \sum_m |r_{nm}|^2 \left\{ [(\delta \mathbf{E} \delta \mathbf{E})_{\omega_{nm}} \right. \\ & \quad \left. + \frac{1}{2\pi} \int \frac{\partial}{\partial \omega} \frac{1}{\omega - \omega_{nm}} \frac{\partial}{\partial t} (\delta \mathbf{E} \delta \mathbf{E})_{\omega} d\omega \right] (f_m - f_n) - \frac{2\hbar\omega_{nm}^3}{c^3} \sqrt{\epsilon'} (f_m + f_n) \right\}. \quad (2.34) \end{aligned}$$

This expression corresponds to the classical equations (1.55) and (1.56).

From (1.51) and (2.32) follows a corresponding expression for the function  $\text{Re}(\delta j \delta \mathbf{E})_{\omega}$ .

In the equilibrium state, the spectral function of the field is determined by the expression

$$(\delta \mathbf{E} \delta \mathbf{E})_{\omega} = \frac{4\pi\hbar\omega^3 \sqrt{\epsilon'}}{c^3} \left( \frac{1}{2} + \frac{1}{e^{\hbar\omega/\kappa T} - 1} \right) = 4\pi^2 \rho_{\omega}^{(r)} + \frac{2\hbar\omega^3 \sqrt{\epsilon'}}{c^3}. \quad (2.35)$$

It follows, for example, from the equation for  $\text{Re}(\delta j \delta \mathbf{E})_{\omega} = 0$ .

In the right side,  $\rho_{\omega}^{(T)}$  is the temperature part of the Planck formula. We introduce the corresponding function also for the non-equilibrium state

$$\rho_{\omega} = [(\delta \mathbf{E} \delta \mathbf{E})_{\omega}/4\pi^2] - (\hbar\omega^3 \sqrt{\epsilon'}/2\pi^2 c^3). \quad (2.36)$$

Using this function, we can write the collision integral (2.34) in the form

$$\begin{aligned} J_n &= \sum_{m<n} \left[ B_m^n \left( \rho_{\omega_{nm}} + \frac{1}{2\pi} \int \frac{\partial}{\partial \omega} \frac{1}{\omega - \omega_{nm}} \frac{\partial \rho_{\omega}}{\partial t} d\omega \right) (f_m - f_n) - A_m^n f_n \right] \\ & \quad + \sum_{m>n} \left[ B_n^m \left( \rho_{\omega_{mn}} + \frac{1}{2\pi} \int \frac{\partial}{\partial \omega} \frac{1}{\omega - \omega_{mn}} \frac{\partial \rho_{\omega}}{\partial t} d\omega \right) (f_m - f_n) + A_n^m f_m \right]; \quad (2.37) \end{aligned}$$

here

$$B_m^n = \frac{4\pi^2 e^2}{3\hbar^2} |r_{nm}|^2, \quad A_m^n = \frac{4e^2}{3\hbar c^3} |r_{nm}|^2 \omega_{nm}^3 \sqrt{\epsilon'} \quad (2.38)$$

are the Einstein coefficients.

In (2.37) we take into account both the decrease of the population of the level  $n$  as a result of spontaneous transitions to lower levels, and the increase of the population as a result of spontaneous transitions from the upper levels to the given level.

### 2.3. The Collision Integral $J_{nm}^{(1)}$ . The Equation for the Polarization Vector

In the collision integral  $J_{nm}^{(1)}$  it is also possible to separate two contributions: the induced contribution and the one from the source.

It follows from (2.19) and (2.12) that to determine the contribution from the source it is necessary to know the spectral function

$$(\delta N_{nm} - \delta N_{nn}, \quad \delta N_{n_1 m_1}^{(1)})^{(\text{source})}$$

at frequencies close to the transition frequencies.

From (2.23) we obtain the corresponding initial condition

$$N_{nm} - \delta N_{nn}, \quad \delta N_{n_1 m_1}^{(1)*} \quad (\text{source}) \\ - (2\pi\hbar)^3 \delta(\mathbf{R} - \mathbf{R}') \delta(\mathbf{P} - \mathbf{P}') (\delta_{m m_1} + \delta_{m n_1} - \delta_{n m_1} - \delta_{n n_1}) f_{m_1 n_1} / 2.$$

This expression vanishes if the resonance condition  $\omega_{nm} - \omega_{n_1 m_1}$  is satisfied only at  $n_1 = n$  and  $m_1 = m$ , i.e., there is no resonant interaction between the different transitions. As a result, the contribution from the source to the collision integral  $J_{nm}^{(1)}$  also vanishes.

We confine ourselves here to this case. Then it remains for us to determine only the induced part of the collision integral. Under the same assumptions as above, we obtain for it the following expression:

$$J_{nm}^{(1)} = -\gamma_{nm} f_{nm}^{(1)}, \quad \gamma_{nm} = e^2 |r_{nm}|^2 (\delta E \delta E)_{\omega_{nm}} / 3\hbar^2. \quad (2.39)$$

As a result, the kinetic equation for  $f_{nm}^{(1)}$  takes the form

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + \gamma_{nm} + i\omega_{nm} \right) f_{nm}^{(1)} = \frac{ie}{\hbar} \mathbf{r}_{nm} (f_m - f_n) \mathbf{E}. \quad (2.40)$$

Thus, the collision frequency  $\gamma_{nm}$  is determined by the spectral function of the field at the transition frequency. In the expression for  $\gamma_{nm}$  it is possible to take into account also the nonresonant contribution, as is done in the kinetic equation for the functions  $f_n$ .

For the spectral function at the resonant frequency, we can use the steady-state solution, which follows from the equation  $\text{Re}(\delta j \delta \mathbf{E})_{\omega_{nm}} = 0$ . It takes the form

$$(\delta E \delta E)_{\omega_{nm}} = (2\hbar \omega_{nm}^3 \sqrt{\epsilon'} / c^3) (\rho_m + \rho_n) / (\rho_m - \rho_n). \quad (2.41)$$

In the equilibrium state, this expression coincides with (2.35).

The collision integral  $J_n$  defined by (2.34) contains also the spectral function  $(\delta E \delta E)_{\omega}$  at the nonresonant frequencies. The equation for this function is analogous to the corresponding classical equation (1.60), and is given by

$$\frac{1}{8\pi} \frac{\partial}{\partial t} \left[ \frac{\partial \omega \epsilon'}{\partial \omega} (\delta E \delta E)_{\omega} + (\delta \mathbf{B} \delta \mathbf{B})_{\omega} \right] = \\ = - \frac{\omega \epsilon''(\omega, \omega \sqrt{\epsilon'} / c)}{4\pi} \left[ (\delta E \delta E)_{\omega} - \frac{8\pi^2 e^2 n}{3 \sqrt{\pi} \Delta \omega_D} \frac{\omega^3 \sqrt{\epsilon'}}{c^3 e^{\sigma}} \sum_{nm} |r_{nm}|^2 e^{-Z_{nm}^2} (\rho_m + \rho_n) \right]. \quad (2.41')$$

The functions  $\epsilon'$  and  $\epsilon''$  in this equation are determined by (2.28).

In the equilibrium state, the right side of the equation vanishes and we obtain for  $(\delta E \delta E)_{\omega}$  the expression (2.35).

From the kinetic equation (2.40) follows an equation for the polarization vector of the transition  $n \rightleftharpoons m$

$$\mathbf{P}_{nm}(\mathbf{R}, \mathbf{P}, t) = eN (r_{mn} f_{nm}^{(1)} + r_{nm} f_{mn}^{(1)}).$$

It takes the form

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} \right)^2 \mathbf{P}_{nm} + 2\gamma_{nm} \frac{\partial \mathbf{P}_{nm}}{\partial t} + \omega_{nm}^2 \mathbf{P}_{nm} = - \frac{2e^2 N}{3\hbar} \omega_{mn} |r_{nm}|^2 (f_m - f_n) \mathbf{E}. \quad (2.42)$$

Let us compare this equation with the corresponding classical equation (1.67).

When  $\mathbf{E} = 0$ , these equations coincide in form. The main difference is that in the classical equation the damping is due completely to the contribution from the source, and in the quantum equation the contribution to  $\gamma_{nm}$  from the source is equal to zero. Consequently the damping coefficient  $\gamma_{nm}$ , as follows from (2.39) and (2.41), depends on the functions  $f_m$  and  $f_n$ .

We introduce in place of the spectral function of the field the function  $\rho_{\omega}$  (2.36) and express  $\gamma_{nm}$  in terms of the Einstein coefficients:

$$2\gamma_{nm} = 2B_m^n \rho_{\omega_{nm}} + A_m^n, \quad \rho_{\omega_{nm}} = \frac{\hbar \omega_{nm}^3 \sqrt{\epsilon'}}{\pi^2 c^3} \left( \frac{1}{2} \frac{\rho_m - \rho_n}{\rho_m - \rho_n} - \frac{1}{2} \right). \quad (2.43)$$

In the case of equilibrium distribution of the field, we obtain from (2.43) the well known expression for  $\gamma_{nm}$ .

It is seen from (2.39) that the line width of the transition  $n \rightleftharpoons m$  is determined by the spectral function of the field at the frequency  $\omega_{nm}$ . The field fluctuations determine also the frequency shift of the transition. The value of the shift is given by

$$\delta \omega_{nm} = \frac{2}{3} \frac{e^2}{\hbar^2} |r_{nm}|^2 \frac{1}{2\pi} \int \frac{(\delta E \delta E)_{\omega}}{\omega_{nm} - \omega} d\omega;$$

here  $\int$  denotes an integral in the sense of the principal value, so that the frequency shift is determined by the spectral function at the nonresonant frequencies.

We note in conclusion that the kinetic equations for the functions  $f_n$  and  $f_{n,m}^{(1)}$  remain valid also when account is taken of interaction via a potential field. We then have

$$(\delta E \delta E)_{\omega} \rightarrow (\delta E \delta E)_{\omega}^{(n)} + (\delta E \delta E)_{\omega}^{(B)},$$

i.e., it is replaced by a sum of the spectral functions of the potential and solenoidal fields. Expressions for these functions can be found, for example, in [26].

### 2.4. Spectral Function of the Polarization Fluctuations

From (2.42) at  $\mathbf{E} = 0$  there follows an equation for the correlation of the polarization fluctuations

$$\left[ \left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} \right)^2 + 2\gamma_{nm} \frac{\partial}{\partial t} + \omega_{nm}^2 \right] \overline{\delta \mathbf{P}_{nm}(\mathbf{R}, \mathbf{P}, t) \delta \mathbf{P}_{n'm'}^*(\mathbf{R}', \mathbf{P}', t')} \stackrel{(\text{source})}{=} 0. \quad (2.44)$$

The corresponding initial condition follows from (2.24)

$$\overline{(\delta \mathbf{P}_{nm} \delta \mathbf{P}_{n'm'}^*)} \stackrel{(\text{source})}{=} (2\pi\hbar)^3 \delta(\mathbf{R} - \mathbf{R}') \delta(\mathbf{P} - \mathbf{P}') \delta_{nm} \delta_{m'n'} |r_{nm}|^2 (f_m + f_n). \quad (2.45)$$

Using the solution of (2.44) subject to initial condition (2.45), we obtain an expression for the spectral polarization function

$$(\delta P \delta P)_{\omega k}^{(source)} = \sum_{n>m} \frac{\hbar \omega \epsilon_{nm}''(\omega, k)}{\pi \omega_{nm}} \frac{1}{2} \frac{\rho_m + \rho_n}{\rho_m - \rho_n}; \quad (2.46)$$

$\epsilon_{nm}''$  is the imaginary part of the dielectric constant for the transitions  $n \rightleftharpoons m$ , and  $f(\mathbf{P})$  is the Maxwellian distribution.

Equation (2.46) contains a product of two functions

$$\epsilon_{nm}'' \approx \frac{1}{2} \frac{\rho_m + \rho_n}{\rho_m - \rho_n} = \frac{1}{2} + \frac{1}{e^{\hbar \omega_{nm}/\kappa T} - 1}.$$

The former has a characteristic width  $\Delta \omega_D$ , and the latter  $\kappa T/\hbar$ . If

$$\kappa T/\hbar \gg \Delta \omega_D \quad (2.47)$$

we can rewrite (2.46) in the form

$$(\delta P \delta P)_{\omega k}^{(source)} = \frac{\hbar \epsilon''(\omega k)}{\pi} \left( \frac{1}{2} + \frac{1}{e^{\hbar \omega/\kappa T} - 1} \right); \quad (2.48)$$

Here  $\epsilon'' = \sum_{n>m} \epsilon_{nm}''$  is the imaginary part of the dielectric constant. The expression for it at  $\Delta \omega_D \gg \gamma_{nm}$  coincides with (2.28). When  $\hbar = 0$ , (2.48) coincides with the classical expression (1.73).

The condition (2.47) was used already in fact when separating the resonant contribution in the derivation of the expressions for the collision integrals.

If the condition  $\hbar \omega_{nm} \gg \kappa T$  is satisfied, then the second term of expression (2.43) for  $\gamma_{nm}$  is much larger than the first, and  $\gamma_{nm}$  does not depend on the distribution functions  $f_n$  and  $f_m$ . Under this condition, expression (2.46) is valid also for the non-equilibrium state.

## 2.5. Laser Radiation Line Width

Assume now that the system of atoms considered by us is the active medium of a gas laser. This may be, for example, neon in an He-Ne laser. Let us consider the fluctuations of the phase of the laser emission. These fluctuations determine the emission line width. We designate by the indices a and b the levels with inverted population; a is the upper level and b the lower.

In the two-level approximation, the kinetic equations (2.16), (2.34), and (2.40) can be written in the form of a system of four equations for the functions  $f_a$ ,  $f_b$ ,  $f_{ab}$  and  $f_{ba}$ . If the condition  $\hbar \omega_{ab} \gg \kappa T$  is satisfied, then we can neglect in the collision integrals the terms containing the function  $\rho_\omega$ . The collision integrals (2.37) and (2.39) then become linear functions of  $f_n$  and  $f_{nm}$ . On this basis, the system of kinetic equations for the distribution functions  $f_a$ ,  $f_b$ ,  $f_{ab}$ , and  $f_{ba}$  can be written in the form<sup>[35-37]</sup>

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} \right) f_a = \frac{ie}{\hbar} (r_{ab} f_{ba} - f_{ab} r_{ba}) \mathbf{E} - \gamma_a (f_a - f_a^{(0)}), \quad (2.49)$$

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} \right) f_b = -\frac{ie}{\hbar} (r_{ab} f_{ba} - f_{ab} r_{ba}) \mathbf{E} - \gamma_b (f_b - f_b^{(0)}), \quad (2.50)$$

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + i\omega_{ab} + \gamma_{ab} \right) f_{ab} = \frac{ie}{\hbar} r_{ab} (f_b - f_a) \mathbf{E}, \quad (2.51)$$

$$f_{ba} = f_{ab}^*$$

where  $f_a^{(0)}$  and  $f_b^{(0)}$  are the populations of the levels a and b and are determined by the pumping.

Just as in the classical case, we consider a single-mode generation regime. We specify the field in the form of a travelling wave, as is the case, for example, in a ring laser when one of the waves is suppressed.

The spectral function of the noise that determines the line width will be again determined by formula (1.85). However, the expressions for the spectral functions of the thermal and polarization noise will now, naturally, be different.

The spectral function of the thermal noise is determined by

$$(1/2V) (e_0 \delta \mathbf{E}^{(T)})_{\omega k}^2 = (4\pi \hbar / VQ) [\bar{n} + (1/2)], \quad \bar{n} = (e^{\hbar \omega / \kappa T} - 1)^{-1}. \quad (2.52)$$

When  $\hbar = 0$ , this expression coincides with (1.86).

Let us find the expression for the spectral function of the polarization noise.

The use of the term 'polarization noise' is justified because the width of the polarization fluctuation spectrum (of the order of  $\Delta \omega_D$ ) is much larger than the width of the laser emission spectrum  $\Delta \omega_{ph}$ . The amplitude fluctuations will not be considered in this case.

To find the spectral function of the polarization noise, let us consider, as in Sec. 1.7, the solution of Eqs. (2.49)–(2.51) in an interval  $t - t'$  such that

$$1/\gamma_a, 1/\gamma_b, 1/\gamma_{ab} \ll t - t' \ll 1/\Delta \omega_{ph}, 1/\Delta \omega_p. \quad (2.53)$$

In this interval, the phase fluctuations can be neglected, and consequently the field in (2.49)–(2.51) can be regarded as determined. It is determined by Eq. (1.76).

Equations (2.49)–(2.51) at a given field are linear with respect to the functions  $f_a$ ,  $f_b$ , and  $f_{ab}$ , with sources  $f_a^{(0)}$  and  $f_b^{(0)}$  as the pump.

Since the working medium is discrete, consisting of excited neon atom, and since there is thermal noise in the resonator, (2.49)–(2.51) must be regarded as equations for random functions.

Averaging these equations, we obtain the corresponding kinetic equations for the functions  $\bar{f}_a$ ,  $\bar{f}_b$ , and  $\bar{f}_{ab}$ . In accordance with condition (2.53), the field is regarded as determined during the course of the averaging, and therefore the equations for the mean values coincide with the equations for the random functions.

Let us consider the steady-state solution of the kinetic equations. It is determined by the specified functions  $f_a^{(0)}$  and  $f_b^{(0)}$ , i.e., by the pump and by the field.

Since the steady-state solution for specified functions  $f_b^{(0)}$  and  $f_a^{(0)}$  is determined only by the field, it can be called induced.

We denote by  $\delta f_a$  and  $\delta f_{ab}$  the deviation of the functions  $f_a$  and  $f_{ab}$  from this solution:

$$f_a = \bar{f}_a + \delta f_a, \quad f_{ab} = \bar{f}_{ab} + \delta f_{ab}.$$

The equations for the deviations differ from Eqs. (2.49)–(2.51) only in the absence of terms with functions  $f_a^{(0)}$  and  $f_b^{(0)}$ .

In place of the equations for  $f_a$  and  $f_b$ , it is more convenient in what follows to use the equations for the functions

$$R = f_a + f_b, \quad D = f_a - f_b. \quad (2.54)$$

The system of equations for the deviations is

$$\left\{ \begin{aligned} \left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + \gamma_{ab} + i\omega_{ab} \right) \delta f_{ab} &= -\frac{ie}{\hbar} \mathbf{r}_{ab} \delta D \mathbf{E}, \quad \delta f_{ba} = \delta f_{ab}^*, \quad (2.55) \\ \left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + \frac{\gamma_a + \gamma_b}{2} \right) \delta D + \frac{\gamma_a - \gamma_b}{2} \delta R &= \frac{2ie}{\hbar} (\mathbf{r}_{ba} \delta f_{ab} - \delta f_{ba} \mathbf{r}_{ab}) \mathbf{E}, \quad (2.56) \\ \left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + \frac{\gamma_a + \gamma_b}{2} \right) \delta R + \frac{\gamma_a - \gamma_b}{2} \delta D &= 0. \quad (2.57) \end{aligned} \right.$$

We consider again the approximation with second correlation functions. In accordance with (1.23), (2.20), and (2.21), we make the following substitutions

$$\delta f_{ab} \rightarrow \delta f_{ab} - \delta f_{ab}^{(\text{source})}, \quad \delta D \rightarrow \delta D - \delta D^{(\text{source})}, \quad \delta R \rightarrow \delta R - \delta R^{(\text{source})} \quad (2.58)$$

in Eqs. (2.55)–(2.57), which correspond to the second-moment approximation.

Since the field does not fluctuate in the time interval (2.53), Eqs. (2.55)–(2.57), unlike (2.20) and (2.21), have no terms proportional to  $\delta \mathbf{E}$ , i.e., the induced parts of the random deviations  $\delta f_{ab}$ ,  $\delta D$ , and  $\delta R$  are equal to zero. Therefore

$$\delta f_{ab} = \delta f_{ab}^{(\text{source})}, \quad \delta D = \delta D^{(\text{source})}, \quad \delta R = \delta R^{(\text{source})} \quad (2.59)$$

Thus, the correlations of the deviations  $\delta f_{ab}$ ,  $\delta D$ , and  $\delta R$  are equal to the corresponding correlations of the deviations  $\delta f_{ab}^{(\text{source})}$ ,  $\delta D^{(\text{source})}$ , and  $\delta R^{(\text{source})}$ .

In accordance with the definition of the correlation of the source (1.22) and (2.22), it is necessary to find the solution of the kinetic equations subject to the initial conditions (1.19) and (2.23), respectively.

The kinetic equations (2.49)–(2.51) are inhomogeneous. Their solution for a given field  $\mathbf{E}$  can be represented in the form of a sum of two parts: the solutions of the homogeneous and the inhomogeneous equations. The first solution is determined by the initial conditions.

The homogeneous kinetic equations coincide in form with Eqs. (2.55)–(2.57); therefore, taking (2.49) into account, we use these equations to determine the correlation and spectral functions of the random deviations  $\delta F_{ab}$ ,  $\delta R$ , and  $\delta D$  of interest to us.

We write down the equation for the two-time correlation of the deviations  $\delta f_{ab}^{(\text{source})}$ . From (2.55) we get

$$\left( \frac{\partial}{\partial t} + \mathbf{V} \frac{\partial}{\partial \mathbf{R}} + i\omega_{ab} + \gamma_{ab} \right) (\delta f_{ab} \delta f_{ab}^{(\text{source})})_{\mathbf{R}\mathbf{R}'\mathbf{P}\mathbf{P}';t} = -\frac{ie}{\hbar} \mathbf{r}_{ab} \mathbf{E}(\mathbf{R}, t) (\delta D \delta f_{ab}^{(\text{source})})_{\mathbf{R}\mathbf{R}'\mathbf{P}\mathbf{P}';t}. \quad (2.60)$$

This equation contains a new function—the correlation of the deviation  $\delta D$  and  $\delta F_{ab}^*$ . The equation for it follows from (2.56) and contains two new functions,  $(\delta R \delta f_{ab}^*)$  and  $(\delta f_{ba} \delta f_{ab}^*)$ . The equation for the former follows from (2.57), and for the latter from the equation for the function  $\delta f_{ba}$ . These equations do not contain new functions. As a result we obtain a system of four equations. Let us write down the corresponding initial conditions.

From (2.23) and (2.24) we obtain for the case of two levels

$$(\delta f_{ab} \delta f_{ab}^{(\text{source})})_{\mathbf{R}\mathbf{R}'\mathbf{P}\mathbf{P}';t} = \frac{(2\pi\hbar)^3}{2N} \delta(\mathbf{R} - \mathbf{R}') \delta(\mathbf{P} - \mathbf{P}') (\bar{f}_a + \bar{f}_b), \quad (\delta f_{ba} \delta f_{ab}^{(\text{source})})_{\mathbf{R}\mathbf{R}'\mathbf{P}\mathbf{P}';t} = 0, \quad (2.61)$$

$$(\delta D \delta f_{ab}^*) = 0, \quad (\delta R \delta f_{ab}^*)_{\mathbf{R}\mathbf{R}'\mathbf{P}\mathbf{P}';t} = (2\pi\hbar)^3 N^{-1} \delta(\mathbf{R} - \mathbf{R}') \delta(\mathbf{P} - \mathbf{P}') \bar{f}_{ba}. \quad (2.62)$$

Let us return to formula (1.58) for the spectral function of the noise that determines the phase fluctuations.

The second term, which takes into account the contribution of the thermal noise of the resonator, is de-

termined in the quantum case by formula (2.52). The first term, which determines the contribution of the polarization fluctuations, is expressed completely in terms of the spectral function  $(e_0 \delta \mathbf{P})_{\Omega_0 \mathbf{k}_0}^2$  only in the case of a weak field, when the saturation effect can be neglected.\* In the case of a strong field it is more convenient to express the contribution made to  $(\xi^2)_{\omega}$  by the polarization noise directly in terms of the spectral functions of the deviations  $\delta f_{ab}$  and  $\delta f_{ba}$ .

We introduce in lieu of the functions  $\delta f_{ab}$  and  $\delta f_{ba}$  the slowly-varying functions  $\delta \tilde{f}_{ab}$  and  $\delta \tilde{f}_{ba}$ :

$$\delta f_{ab} = \delta \tilde{f}_{ab} e^{-i(\Omega_0 t - \mathbf{k}_0 \mathbf{R} + \varphi)}, \quad (2.63)$$

$$\delta f_{ba} = \delta \tilde{f}_{ba} e^{i(\Omega_0 t - \mathbf{k}_0 \mathbf{R} + \varphi)}. \quad (2.64)$$

Then the first term in Eq. (1.84) for  $\xi$  can be represented in the form

$$\xi(\mu t) = -\frac{4\pi e N}{2V} \int (e_0 \mathbf{r}_{ba} \delta \tilde{f}_{ab} + e_0 \mathbf{r}_{ab} \delta \tilde{f}_{ba}) \frac{d\mathbf{R} d\mathbf{P}}{(2\pi\hbar)^3}. \quad (2.65)$$

This leads to an expression for the spectral function

$$(\xi^2)_{\omega} = \frac{(4\pi)^2 e^2 N^2}{4V} \int \left\{ \frac{|r_{ab}|^2}{3} (\delta \tilde{f}_{ab} \delta \tilde{f}_{ab}^*)_{\omega, 0} + (e_0 \mathbf{r}_{ab})^2 (\delta \tilde{f}_{ba} \delta \tilde{f}_{ab})_{\omega, 0} + \text{c. c.} \right\} \frac{d\mathbf{P} d\mathbf{P}'}{(2\pi\hbar)^6}. \quad (2.66)$$

Thus, the spectral function  $(\xi^2)_{\omega}$  is determined by the spectral functions

$$(\delta \tilde{f}_{ab} \delta \tilde{f}_{ab}^*)_{\omega, 0}, \quad (\delta \tilde{f}_{ba} \delta \tilde{f}_{ab})_{\omega, 0}. \quad (2.67)$$

These functions are connected with the spectral functions of the deviations  $\delta f_{ab}$  and  $\delta f_{ba}$  by the equations

$$\left. \begin{aligned} (\delta f_{ab} \delta f_{ab}^*)_{\omega + \Omega_0, \mathbf{k} + \mathbf{k}_0} &= (\delta \tilde{f}_{ab} \delta \tilde{f}_{ab}^*)_{\omega, \mathbf{k}}, \\ (\delta f_{ba} \delta f_{ab}^*)_{\omega - \Omega_0, \mathbf{k} - \mathbf{k}_0} &= (\delta \tilde{f}_{ba} \delta \tilde{f}_{ab})_{\omega, \mathbf{k}} e^{i2(\Omega_0 t - \mathbf{k}_0 \mathbf{R} - \varphi)}. \end{aligned} \right\} \quad (2.68)$$

Expression (2.66) contains the spectral functions at  $\mathbf{k} = 0$ . Since  $\omega \ll \Omega_0$  we can put  $\omega = 0$  when determining the spectral functions (2.68).

As a result of calculations we obtain for the spectral functions (2.67) the following expressions:

$$(\delta \tilde{f}_{ab}(\mathbf{P}) \delta \tilde{f}_{ab}^*(\mathbf{P}'))_{\omega, 0} = \frac{(2\pi\hbar)^3}{2N\Gamma_E} \gamma_{ab} \left[ (2 + aE_0^2) (\bar{f}_a + \bar{f}_b) - \frac{\gamma_a - \gamma_b}{\gamma_a + \gamma_b} aE_0^2 \bar{D} \right] \delta(\mathbf{P} - \mathbf{P}'), \quad (2.69)$$

$$\begin{aligned} & (e_0 \mathbf{r}_{ab})^2 (\delta \tilde{f}_{ba}(\mathbf{P}) \delta \tilde{f}_{ab}(\mathbf{P}'))_{\omega, 0} = \\ & = \frac{(2\pi\hbar)^3}{2N\Gamma_E} \frac{|r_{ab}|^2}{3} \gamma_{ab} aE^2 \left[ (\bar{f}_a + \bar{f}_b) - \frac{\gamma_a - \gamma_b}{\gamma_a + \gamma_b} \bar{D} \frac{(\omega_{ab} - \Omega_0 + \mathbf{k}_0 \mathbf{V})^2 - \gamma_{ab}^2}{\Gamma_0} \right] \delta(\mathbf{P} - \mathbf{P}'); \end{aligned} \quad (2.70)$$

we have introduced here the notation

$$\Gamma_E = (\omega_{ab} - \Omega_0 + \mathbf{k}_0 \mathbf{V})^2 + \gamma_{ab}^2 (1 + aE_0^2), \quad \Gamma_0 = \Gamma_{E=0}, \quad (2.71)$$

where

$$a = \frac{e^2 |r_{ab}|^2 (\gamma_a + \gamma_b)}{6\hbar^2 \gamma_a \gamma_b \gamma_{ab}} \quad (2.72)$$

is the saturation parameter.

We substitute (2.69) and (2.70) in (2.66). As a result we obtain

$$\begin{aligned} (\xi^2)_{\omega} &= \frac{(4\pi)^2 e^2 N}{6V} \gamma_{ab} \int \frac{d\mathbf{P}}{(2\pi\hbar)^3} \times \\ & \times \frac{1}{\Gamma_E^2} \left[ (1 + aE_0^2) (\bar{f}_a(\mathbf{P}) + \bar{f}_b(\mathbf{P})) - \frac{\gamma_a - \gamma_b}{\gamma_a + \gamma_b} \frac{\bar{D}(\mathbf{P}) aE^2}{\Gamma_0} (\omega_{ab} - \Omega_0 + \mathbf{k}_0 \mathbf{V})^2 \right]. \end{aligned} \quad (2.73)$$

Let us express the functions  $\bar{f}_a + \bar{f}_b$  and  $\bar{f}_a - \bar{f}_b$  in terms of the functions  $f_a^{(0)}$  and  $f_b^{(0)}$ , which are determined by the pumping.

To this end it is necessary to use the equations for the induced parts of the functions  $\bar{f}_a$ ,  $\bar{f}_b$ , and  $\bar{f}_{ab}$ . We

\*This circumstance was not taken into account in [33] (see [34]).

recall that the equations of these functions coincide with Eqs. (2.49)–(2.51).

As a result of the solution we obtain the following expressions:

$$f_a(\mathbf{P}) + \bar{f}_b(\mathbf{P}) = f_a^{(0)} + f_b^{(0)} + \frac{\gamma_a - \gamma_b}{\gamma_a + \gamma_b} D^{(0)}(\mathbf{P}) \frac{\gamma_{ab}^2 a E_0^2}{\Gamma_E}, \quad (2.74)$$

$$\bar{D}(\mathbf{P}) = D^{(0)}(\mathbf{P}) \Gamma_0 / \Gamma_E. \quad (2.75)$$

We substitute these expressions in (2.73). We recognize that

$$V(2\pi\hbar)^{-3} f_a^{(0)}(\mathbf{P}) = \rho_a^{(0)} f(\mathbf{P});$$

here  $\rho_a$  is the level population due to the pumping and  $f$  is the Maxwellian distribution.

As a result we obtain

$$(\xi^2)_\omega = \frac{(4\pi)^2 e^2 n |r_{ab}|^2}{6V} \gamma_{ab} \int d\mathbf{P} f(\mathbf{P}) \left\{ \frac{(1 + aE_0^2)(\rho_a^{(0)} + \rho_b^{(0)})}{\Gamma_E} - \frac{1}{\Gamma_E^2} \frac{\gamma_a - \gamma_b}{\gamma_a + \gamma_b} aE_0^2 D^{(0)} [(\omega_{ab} - \Omega_0 + \mathbf{k}_0 \mathbf{V})^2 - \gamma_{ab}^2 (1 + aE_0^2)] \right\}. \quad (2.76)$$

Let us consider two limiting cases:

a) Immobile atoms ( $f(\mathbf{P}) = \delta(\mathbf{P})$ ):

$$(\xi^2)_\omega = \frac{(4\pi)^2 e^2 n |r_{ab}|^2}{6V} \gamma_{ab} \left[ \frac{(1 + aE_0^2)(\rho_a^{(0)} + \rho_b^{(0)})}{(\omega_{ab} - \Omega_0)^2 + \gamma_{ab}^2 (1 + aE_0^2)} - \frac{\gamma_a - \gamma_b}{\gamma_a + \gamma_b} aE_0^2 D^{(0)} \frac{(\omega_{ab} - \Omega_0)^2 - \gamma_{ab}^2 (1 + aE_0^2)}{[(\omega_{ab} - \Omega_0)^2 + \gamma_{ab}^2 (1 + aE_0^2)]^2} \right]. \quad (2.77)$$

b) Inhomogeneous broadening ( $\gamma_{ab} \ll \Delta\omega_D$ ). In the zeroth approximation in  $\gamma_{ab}/\Delta\omega_D$  we obtain from (2.76), after integrating with respect to  $\mathbf{P}$ ,

$$(\xi^2)_\omega = \sqrt{\pi} (4\pi)^2 e^2 n |r_{ab}|^2 (6V\Delta\omega_D)^{-1} (1 + aE_0^2)^{1/2} (\rho_a^{(0)} + \rho_b^{(0)}). \quad (2.78)$$

It is convenient to represent (2.77) and (2.78) in explicit form, using the expressions for the imaginary part of the dielectric constant

$$\varepsilon''(\Omega_0) = -4\pi e^2 |r_{ab}|^2 n \gamma_{ab} D^{(0)} / 3\hbar [(\omega_{ab} - \Omega_0)^2 + \gamma_{ab}^2 (1 + aE_0^2)] \quad (2.79)$$

for immobile atoms and

$$\varepsilon''(\Omega_0) = -\pi^{1/2} \cdot 4\pi e^2 n |r_{ab}|^2 D^{(0)} / 3\hbar \Delta\omega_D (1 + aE_0^2)^{1/2} \quad (2.80)$$

when  $\gamma_{ab} \ll \Delta\omega_D$ ,  $\omega_{ab} - \Omega_0 \ll \Delta\omega_D$ .

As a result, expressions (2.77) and (2.78) take the form

$$(\xi^2)_\omega = \frac{4\pi e^2 \hbar}{2V} \left[ (1 + aE_0^2) \frac{\rho_b^{(0)} + \rho_a^{(0)}}{\rho_b^{(0)} - \rho_a^{(0)}} + \frac{\gamma_a - \gamma_b}{\gamma_a + \gamma_b} aE_0^2 \frac{(\omega_{ab} - \Omega_0)^2 - \gamma_{ab}^2 (1 + aE_0^2)}{(\omega_{ab} - \Omega_0)^2 + \gamma_{ab}^2 (1 + aE_0^2)} \right] \quad (2.81)$$

for immobile atoms and

$$(\xi^2)_\omega = \frac{4\pi e^2 \hbar}{2V} (1 + aE_0^2) \frac{\rho_b^{(0)} + \rho_a^{(0)}}{\rho_b^{(0)} - \rho_a^{(0)}} \quad \text{if } \gamma_{ab} \ll \Delta\omega_D. \quad (2.82)$$

The approximation of immobile atoms is justified for lasers with homogeneous line broadening.

Let us consider the limiting cases of weak and strong fields:

a) Weak field ( $aE_0^2 \ll 1$ ). In the zeroth approximation in  $aE_0^2$ , formulas (2.81) and (2.82) have the same form:

$$(\xi^2)_\omega = (4\pi e^2 \hbar / 2V) [\rho_b^{(0)} + \rho_a^{(0)}] / [\rho_b^{(0)} - \rho_a^{(0)}]. \quad (2.83)$$

The only different expressions are those for the functions  $\varepsilon''_0$ . They follow from formulas (2.79) and (2.80) at  $aE_0^2 = 0$ .

Using formulas (1.89), (2.52), and (2.83), we obtain for this case the following expression for the emission line width:

$$\Delta\omega_{\text{ph}} \equiv 2D = \frac{\hbar\Omega_0^2}{2W} \left( \frac{1}{Q} \left( \bar{n} + \frac{1}{2} \right) + \varepsilon''_0 \frac{1}{2} \frac{\rho_b^{(0)} + \rho_a^{(0)}}{\rho_b^{(0)} - \rho_a^{(0)}} \right). \quad (2.84)$$

Here the first term is determined by the thermal noise of the resonator, and the second by the polarization noise. The second term contains the product of two negative quantities  $\varepsilon''_0 < 0$  and  $\rho_b^{(0)} - \rho_a^{(0)} < 0$ .

Expression (2.84) is usually written in explicit form.

We introduce a symbol for the power

$$P = \Delta\omega_p W = \Delta\omega_p E_0^2 V / 8\pi, \quad \Delta\omega_p = \Omega_0 / Q$$

and use the self-excitation condition

$$Q^{-1} + \varepsilon''_0 = 0. \quad (2.85)$$

Then, eliminating  $\varepsilon''_0$ , we can write expression (2.84) in the form

$$\Delta\omega_{\text{ph}} = \frac{\hbar\Omega_0}{2P} (\Delta\omega_p)^2 \left( \bar{n} + \frac{1}{2} + \frac{1}{2} \frac{\rho_a^{(0)} + \rho_b^{(0)}}{\rho_a^{(0)} - \rho_b^{(0)}} \right). \quad (2.86)$$

This expression coincides, for example, with that obtained by Haken<sup>[4]</sup>. The calculated widths of the gas laser were given in a number of other papers, for example those of Lamb<sup>[35]</sup>, Lax<sup>[3]</sup>, Willis<sup>[5]</sup>, and others.

In the first approximation in  $aE_0^2$ , we get from formulas (2.81) and (2.82)

$$(\xi^2)_\omega = \frac{4\pi e^2 \hbar}{2V} \left[ \frac{\rho_b^{(0)} + \rho_a^{(0)}}{\rho_b^{(0)} - \rho_a^{(0)}} + \frac{\gamma_a - \gamma_b}{\gamma_a + \gamma_b} aE_0^2 \frac{(\omega_{ab} - \Omega_0)^2 - \gamma_{ab}^2}{(\omega_{ab} - \Omega_0)^2 + \gamma_{ab}^2} \right],$$

$$(\xi^2)_\omega = \frac{4\pi e^2 \hbar}{2V} \left( 1 + \frac{aE_0^2}{2} \right) \frac{\rho_b^{(0)} + \rho_a^{(0)}}{\rho_b^{(0)} - \rho_a^{(0)}}.$$

b) Strong field ( $aE_0^2 \gg 1$ ). For immobile atoms it follows from (2.81) and (2.77) that

$$(\xi^2)_\omega = \frac{4\pi e^2 \hbar}{2V} aE_0^2 \left[ \frac{\rho_b^{(0)} + \rho_a^{(0)}}{\rho_b^{(0)} - \rho_a^{(0)}} + \frac{\gamma_b - \gamma_a}{\gamma_b + \gamma_a} \right] = \frac{(4\pi)^2 e^2 n |r_{ab}|^2}{6V\gamma_{ab}} D^{(0)} \left[ \frac{\rho_a^{(0)} + \rho_b^{(0)}}{\rho_a^{(0)} - \rho_b^{(0)}} + \frac{\gamma_a - \gamma_b}{\gamma_a + \gamma_b} \right]. \quad (2.87)$$

Thus, the polarization-noise spectral function at  $aE_0^2 \gg 1$  does not depend on the field.

If the change of power is due to the change of the  $Q$  at constant pumping, then, with allowance for the condition (2.85) expression (2.52) can be written in the form

$$(\xi^2)_\omega^T = 4\pi\hbar |e''| V^{-1} (\bar{n} + 1/2). \quad (2.88)$$

Inasmuch as in a strong field

$$\varepsilon'' = -4\pi e^2 |r_{ab}|^2 n D^{(0)} / 3\hbar \gamma_{ab} aE_0^2 \sim 1/aE_0^2,$$

it follows that the spectral function of the thermal noise decreases with increasing field. Consequently, the line width at  $aE_0^2 \gg 1$  is determined by the polarization noise.

As a result we obtain the expression

$$\Delta\omega_{\text{ph}} = \frac{\hbar\Omega_0^2}{2W} \varepsilon'' aE_0^2 \cdot \frac{1}{2} \left[ \frac{\rho_b^{(0)} + \rho_a^{(0)}}{\rho_b^{(0)} - \rho_a^{(0)}} + \frac{\gamma_b - \gamma_a}{\gamma_b + \gamma_a} \right]. \quad (2.89)$$

Let us compare formulas (2.84) and (2.89). It follows from them that both in a weak and in a strong field the line width is inversely proportional to the field energy, i.e.,  $\Delta\omega_{\text{ph}} \sim 1/W$ . However, the proportionality coefficient is different.

In the case of a gas laser at  $aE_0^2 \gg 1$ , but at  $\gamma_{ab}^2 aE_0^2 \ll \Delta\omega_D^2$ , it follows from (2.82) that

$$(\xi^2)_\omega = (4\pi\hbar\varepsilon''/2V) aE_0^2 (\rho_b^{(0)} + \rho_a^{(0)}) / [\rho_b^{(0)} - \rho_a^{(0)}]. \quad (2.89a)$$

Since now we have (see (2.80))

$$e'' = -\sqrt{\pi} 4\pi e^2 n |r_{ab}|^2 / 3\hbar \Delta\omega_D (aE_0^2)^{1/2} \sim 1/(aE_0^2)^{1/2}, \quad (2.90)$$

the polarization noise increases like  $(aE_0^2)^{1/2}$ .

The line width is given by

$$\Delta\omega_{ph} = (\hbar\Omega_0^2/2W) e'' aE_0^2 (\rho_b^{(0)} + \rho_a^{(0)}) / [\rho_b^{(0)} - \rho_a^{(0)}] \sim 1/W^{1/2} \quad (2.91)$$

and consequently the decrease of the line width with increasing power slows down when  $aE_0^2 \gg 1$ .

The results (2.89a) and (2.91) are in qualitative agreement with those given in<sup>[33]</sup>.

By now there is already a considerable number of experimental papers devoted to the determination of the emission line width and to the investigation of the spectrum of the amplitude fluctuations of gas lasers. In order of magnitude, the results of the calculation of the line width agree with the experimental data in those regions above threshold where the correlation theory is valid.

For a detailed comparison with the experimental data it is necessary to generalize the results presented above. The point is that the greatest interest attaches to an investigation of fluctuations in a linear laser, where the field form is close to a standing wave, and in a ring laser operating in the regime of two opposing waves. In these cases, in calculating the phase fluctuations, it is necessary to take into account the influence of the amplitude fluctuations and of the coupling of the opposing waves. A review of these results cannot be presented within the framework of the present article, so that we confine ourselves only to citing the literature<sup>[34,38-47]</sup>.

In conclusion we note that although the exposition was carried out here using as an example a gas whose atoms interact only via the transverse field, a similar method can be used to determine the nonequilibrium fluctuations also in other systems.

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