SOVIET PHYSICS USPEKHI VOLUME 13, NUMBER 4 JANUARY-FEBRUARY 1971
539.121.7

THE IMPORTANCE OF INELASTIC PROCESSES AT HIGH ENERGIES AND THE THEORY OF FIREBALLS

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Usp. Fiz. Nauk 101, 385–428 (July, 1970)

I. INTRODUCTION

INELASTIC processes are of decisive importance in the physics of the strong interactions of high-energy particles. Nevertheless until very recently experimenters working with accelerators as well as theorists have preferred to deal with elastic scattering processes, or with processes involving production of at most one or two particles.

The experimental treatment of such events is less laborious and more reliable. The theorist finds advantages in the relative simplicity of the mathematical apparatus, in the somewhat distinguished role of the elastic scattering amplitude, which is connected with the total cross section by the optical theorem, and, finally, in the present availability of more complete and accurate experimental information on elastic collisions as compared with inelastic processes. Recently, owing to the development of new methods for treating data and the wide-spread use of computers there has been much more interest in high-multiplicity inelastic processes at energies accessible with accelerators. As for cosmic rays, their main use is with this sort of inelastic processes, since the study of elastic and quasielastic interactions encounters great experimental difficulties.

The interconnections of elastic and inelastic processes are of theoretical interest. As already mentioned, the inelastic processes are of predominant importance.

Firstly, most (about 80 percent) collisions of strongly interacting high-energy particles are inelastic.

Secondly, the elastic scattering amplitude itself is completely determined by the character of the inelastic processes at high energies.

Thirdly, at present accelerator energies ($E_{lab} \sim 25$–30 GeV at CERN and Brookhaven, and $E_{lab} \sim 70$ GeV at Serpukhov) the main part of the inelastic interactions consists of processes of multiple production, which do not reduce to binary processes and consequently cannot be treated by analogies with elastic processes. Therefore the problem of the theoretical description of inelastic processes is of primary importance.

In the present review we consider two main questions.

The first of these is, in what way high-multiplicity inelastic processes affect the characteristics of elastic scattering. Chapters II and III of this article are devoted to this. It is shown that the inelastic processes determine the elastic scattering both at small angles and also at intermediate and "large" angles.

The second question: What are the properties of the inelastic processes themselves at high energies, and how are they related with the elastic interactions in the "asymptotic" energy range? The later chapters of the review are devoted to this. Especial attention is given to inelastic processes in which "fireballs" are formed.

The review is theoretical in nature. Owing to this we have not undertaken a detailed and complete discussion of the experimental data. We even think that such a discussion would now be premature. The point is that, on one hand, not one of the theoretical schemes for the inelastic processes has been carried to the level of accurate quantitative predictions over a wide energy range (say from 10^{10} to 10^{15} eV), and on the other hand, the experimental information obtained with accelerators ($E_{lab} \lesssim 20$–30 GeV) and with cosmic rays ($E_{lab} \sim 10^{11}$ to 10^{13} eV) is different in character. In the former case most of the data relate to events of small multiplicity, whereas in the latter case essentially only many-prong events are examined. It may be expected that this difference will be decreased as the result of experiments with the Serpukhov accelerator. At any rate many vexed questions will be cleared up and a unified point of view will be developed, as is necessary for a detailed analysis of the experimental results. We shall discuss only some important qualitative experimental results which have played a large part in the formation of theoretical ideas.

First, we note the following data, originally obtained from cosmic rays, and then—much more accurately, but so far in a much smaller energy range—also with accelerators.

1. The total interaction cross sections are constant at energies 10^{10} eV to 10^{15} eV with accuracy ~ 50 percent. (Accelerator data show that the energy dependence of these cross sections is weak over the range from 10^{10} to 7×10^{10} eV.)

2. At high energies, in the c.m.s. the secondary particles are strongly collimated along the line of the collision. The angular collimation increases with increasing energy, so that the mean transverse momentum $\overline{p}_\perp$ remains constant ($\overline{p}_\perp \sim 2.5\,\mu$, where μ is the pion mass; here and in what follows we shall set $\hbar = c = 1$) over practically the entire investigated energy range.

3. The average multiplicity increases rather weakly with increasing energy.

4. When high-energy nucleons interact they retain a major fraction of the primary energy. The ratio of the energy E_1 that goes into the production of new particles to the energy E_0 of the primary particle in the c.m.s. is on the average $E_1/E_0 \sim 0.5$ (though the scatter in the values of this ratio is very large). This quantity is called the inelasticity coefficient

$$K = E_1/E_0.$$

5. From our point of view it is particularly important that we note the data on the formation of fireballs at high energies. These data are so far a matter for discussion. The disagreements are mainly due to the difficulties of obtaining and interpreting results with cosmic rays. This question has been treated in detail in a review by Miesowicz.[1] We shall not go into the details of the discussion, but only briefly describe the main conclusions. In the study of the interaction of nucleons with energies $\sim 10^{12}$ eV the idea has been proposed that two centers of emission of secondary pions are formed. These centers move with relativistic speed in the c.m.s. The number of charged particles emitted from each center is of the order of six or seven. The mass of an emission center is of the order $\mathfrak{M} \sim 3$ GeV.

These data were mainly obtained by Miesowicz and Gierula and their group,[2] and independently by Niu.[3] On the suggestion of Cocconi[4] these emission centers have been named fireballs.

Subsequently in papers by Dobrotin and Slavatinskiĭ and their colleagues[5] data were obtained which indicated that in collisions of nucleons with somewhat lower energies ($E_{lab} \sim 3 \times 10^{11}$ eV) a single emission center is formed. The main characteristics of the center (its mass, the number of secondary particles, and so on) are the same as already stated. The conclusion was that a single fireball is formed.

Then in papers by Hasegawa[6] and Rybicki[7] indications were found that at higher energies $\sim 10^{13}$ eV larger numbers of fireballs are formed. These data, however, cannot so far be regarded as well established even to the same extent as the conclusions about processes with formation of one or two fireballs.

A comparison of the data suggests that with increasing energy there is also an increase of the number of fireballs, but that their characteristics (primarily the mass) remain practically constant, that is, do not change with increase of the energy in the range from 3×10^{11} to $\sim 10^{13}$ eV. It is certainly a matter of interest to look for processes with the formation of one fireball at accelerator energies. The first preliminary results were obtained by Walker[8] in πp interactions at energy 25 GeV and by Zhdanov and others[9] in pp interactions at energies 21 and 24 GeV. At these energies, however, the cross sections for such processes are still very small, and their detailed investigation with accelerators will evidently be possible only at the energies of the Serpukhov accelerator.

We note that very recently extremely detailed information has been obtained on inelastic processes at accelerator energies up to 16 GeV.[10] The contribution of fireball processes at these energies is now being studied.

Simultaneously with the experiments (and under their influence) the theoretical ideas about inelastic processes have been developed. For a long time the only serious theoretical schemes for describing such processes were the hydrodynamical theory[11] used at ultrahigh energies $E_{lab} \gtrsim 10^{12}$ eV, and the statistical theory, which was applied in the energy range $10^9 \lesssim E_{lab} \lesssim 10^{10}$ eV. The Weizsäcker-Williams method was also used, but it contained many unknown parameters. Ideas were put forward about peripheral interactions of hadrons.[13-15] The use of Feynman diagrams with exchange of one meson, in papers by Goebel[16] and by Chew and Low,[17] made it possible to move further in the study of inelastic processes. Papers by Dremin and Chernavskiĭ[18] and by F. and G. Salzman[19] formulated the one-meson approximation; thereafter this was on one hand improved by the inclusion of effects of interaction in the initial and final states of such processes,[20] and on the other hand led to the more advanced idea of peripheral interaction, and to the treatment of multiperipheral diagrams, which were first considered in a paper by Berestetskiĭ and Pomeranchuk.[21] This development was most fully represented in papers by Amati, Fubini, Stanghellini, and Tonin,[22] who proposed a multiperipheral model (the AFST model) which, unlike previous models, made use of a closed equation describing a complete set of diagrams. The model has been studied very intensively and subjected to comparison with experiment. This revealed its practical shortcomings: it could not give asymptotically constant cross sections and did not describe the formation of fireballs. It was soon shown,[23] however, that these shortcomings were due only to an imprecise interpretation of the kernel of the equation used in the model of[22]. A subsequent treatment in the framework of the Bethe-Salpeter equation, carried out in a paper by Dremin, Roĭzen, White, and Chernavskii,[23] gave an interpretation of this kernel as the set of Feynman diagrams of a definite class and derived the AFST model as a special case of the Bethe-Salpeter equation. Moreover, it turned out that the Bethe-Salpeter equation together with the unitarity relation in the s channel also makes possible a direct examination of the properties of the elastic scattering associated with inelastic processes of various types. The restrictions imposed on the elastic scattering are then simultaneously restrictions on the inelastic processes. It turns out that as a result of the fulfilling of such seemingly "abstract" requirements as the condition of the existence of a solution and the presence of a vacuum singularity near $l = 1$ the theory predicts in a natural way an effect like fireball formation. Moreover, these "abstract" conditions make possible the determination (in order of magnitude) of parameters which were regarded as arbitrary in former models of peripheral processes.

There has also been intensive development of another approach in the theory of inelastic processes—the many-reggeon model, which has been treated in most detail in papers by Ter-Martirosyan and his collaborators.[24] The diagrams discussed in this model are topologically equivalent to those of the multiperipheral model. The only difference is that it is a reggeon, not an elementary particle, that is exchanged. A phenomenological version of this model, supplemented with some specific assumptions about the behavior in the low-energy region, has been compared with the experimental data at 8 and 16 GeV in[25]. Chapter IX is devoted to a comparison of various models of inelastic processes, in particular the many-reggeon and multiperipheral models.

It must be noted that many models of inelastic processes have been proposed, for example the model of uncorrelated jets,[26,27] the bremsstrahlung model, and the quark model. They have not been much developed, however, although they have helped in the explanation of such important questions as the role of the phases of matrix elements of inelastic processes (cf. Chapter III), the isotopic relations between various channels, and so on. Recently there has been renewed interest in the statistical theory[12] in connection with accelerator experiments on particle spectra,[30] on production of particle-antiparticle pairs and the related problem of quarks,[31] and so on. It is clear, however, that the statistical model cannot pretend to describe the whole set of inelastic processes. Its interconnections with other models are also discussed in Chapter IX.

We would like to emphasize that in always using the word "model" we are indicating that the fundamental assumptions of the theory are insufficient for a complete quantitative description of inelastic processes. Owing to this one must at some stage make definite assumptions about certain quantities, and a number of parameters remain arbitrary prior to comparison with experiment. Nevertheless the fact that one can theoretically describe the qualitative, and in some cases also the quantitative, characteristics of inelastic processes, and connect them with the properties of the amplitude of the background elastic scattering represents an undoubted achievement of the work of recent years.

II. INTERRELATIONS OF ELASTIC AND INELASTIC PROCESSES AT HIGH ENERGIES

Our Chapters II and III are devoted to the question of the relations of elastic and inelastic processes. Therefore the reader who is acquainted with this problem or is interested only in questions of the description of inelastic processes may go directly to Chapter IV.

In the present chapter we shall show that the amplitude for elastic scattering through an arbitrary angle at high energies is practically completely determined by the character of the inelastic processes. At first glance this may seem a trivial assertion. On more detailed examination the following questions arise.

1. It is known from experiment that immediately beyond the diffraction cone[32] in the angular distribution of elastic scattering there is a region of weaker dependence on the angles.[33]

As will be shown in Chapters II, IV, and VI, the scattering in the diffraction cone is practically completely determined by precisely the peripheral inelastic processes with sufficiently high multiplicity. The fixing of the absolute values of the matrix elements of these processes* allows us to find the slope of the main vacuum singularity of the partial wave in the cross channel [see Chapter VI, Eq. (47)]. But this slope is too small to reproduce the experimentally observed width of the diffraction peak, which is mainly determined by the residue at the leading pole. At the same time, as shown in Chapter III, there are additive positive contributions from the phases and absolute values of the amplitudes for inelastic processes to the reciprocal width of the diffraction cone. Furthermore it follows from independent estimates that the contribution of the phases must be of just the same order of magnitude as must be ascribed to the residues in the phenomenological treatment of the data according to the Regge formulas. Therefore the question arises as to a possible connection of the Regge trajectories with the absolute values of these matrix elements. Accordingly, the study of the elastic scattering can give some information (though indirect and very incomplete) about the phases of the amplitudes for inelastic processes.

2. In the scattering outside the diffraction cone there is a region in which the cross section decreases exponentially with increase of the angle. Here the direct contribution of the inelastic processes may be small. But even in this case the characteristics of the elastic scattering, in particular the index of the exponential, are uniquely determined by the parameters of the diffraction peak. Accordingly, the elastic scattering is, in the final analysis, once again fixed by the character of the peripheral inelastic processes, even though the direct contribution may be small.

At still larger angles the angular dependence is weak. Here the properties of the elastic scattering are again determined by the direct contribution of inelastic processes of the peripheral type. Accordingly, the angular dependence of the elastic scattering can give indications as to the existence of inelastic processes of various types. The present experimental data can be sufficiently well described as the consequence of processes of only two types: peripheral and nonperipheral (possibly statistical).

In discussing these questions it is best to use the unitarity condition, which directly connects the elastic scattering amplitude with the matrix elements for the inelastic processes. It can be written in the form

$$\operatorname{Im} A(p,\theta) = \frac{1}{32\pi^2}\int d\theta_1 \int d\theta_2 \frac{\sin\theta_1 \sin\theta_2 A(p,\theta_1) A^*(p,\theta_2)}{\{[\cos\theta - \cos(\theta_1+\theta_2)][\cos(\theta_1-\theta_2)-\cos\theta]\}^{1/2}} + F(p,\theta); \tag{1}$$

Here $A(p, \theta)$ is the elastic scattering amplitude, which depends on the momentum $p \equiv |\mathbf{p}|$ and the scattering angle θ (in the c.m.s.); $F(p, \theta)$ is the contribution to the imaginary part of the amplitude $\operatorname{Im} A(p, \theta)$ from all the inelastic processes (the so-called overlap function), which can be symbollically written in the following way:

$$F(p,\theta) = \sum_n \int M_{a\to n} M^*_{a'\to n}\, \delta\left(\hat{Q} - \sum_{i=1}^{n} q_i\right) d\Phi_n; \tag{2}$$

The quantities M are the matrix elements for the inelastic processes $a \to n$ and $a' \to n$ (a and a' are the initial and final states of the elastic scattering process in question, and n is an intermediate n-particle state); the integration is taken over the entire phase volume Φ_n admitted by the conservation-law δ functions for the process with total initial four-momentum $\hat{Q}$ and four-momenta q_i of the n final particles. The first term in (1) corresponds to the contribution of elastic processes, and in it the integration is taken over the angular ranges

$$|\theta_1 - \theta_2| \leqslant \theta, \quad \theta \leqslant \theta_1 + \theta_2 \leqslant 2\pi - \theta. \tag{3}$$

At angle $\theta = 0°$ the condition (1) leads to the usual

*The experimental distributions of the particles in inelastic processes give information only about the absolute values of the matrix elements for these processes.

optical theorem, the first term becoming the total cross section σ_{el} for elastic scattering, and the second becoming the total cross section σ_{in} for inelastic processes:

$$\operatorname{Im} A(p, 0) = 4p^2(\sigma_{el} + \sigma_{in}) \equiv 4p^2\sigma_{tot}, \tag{4}$$

and experiment shows that at high energies the largest contribution to (1) for $\theta = 0°$ is the second term (about 80 percent of the total).

The overlap function also gives the main contribution to the elastic scattering amplitude in the region of the diffraction cone. In fact, if we note that (as is shown by experiment) the elastic scattering amplitude at small angles is almost purely imaginary, and the differential cross sections show a Gaussian type of decrease with increasing angle, we can write with good accuracy for this range of angles

$$A(p, \theta) \approx 4ip^2\sigma_{tot}e^{-ap^2\theta^2/2} \qquad (\theta \leqslant \theta_d). \tag{5}$$

The parameter a is called the reciprocal width of the diffraction cone,* whose boundary we have denoted by θ_d. It can be easily verified by direct substitution of (5) in (1) that the elastic contribution in this region will be of the form $\exp(-ap^2\theta^2/4)$, i.e., it cannot fit the original form (5) and leads only to a small broadening of the angular distribution given by the function $F(p, \theta)$. It is not hard to show from (1) that for an $F(p, \theta)$ approximately described by the formula

$$F(p, \theta) \approx 4p^2\sigma_{in}\exp(-\alpha p^2\theta^2/2) \qquad (\theta \leqslant \theta_d), \tag{6}$$

the effective result of this broadening is that the parameter a in (5) is connected with α in the following way (for $\sigma_{el}/\sigma_{in} \ll 1$):

$$a \approx \alpha\,[1 - (\sigma_{el}/2\sigma_{tot})]. \tag{7}$$

Accordingly, the inelastic processes determine the form of the diffraction cone in the elastic scattering, since here the overlap function is much larger than the contribution of two-particle intermediate states [the first term in the right member of Eq. (1)].

Let us now examine the part played by $F(p, \theta)$ at large scattering angles ($\theta > \theta_d$). In this case the main contribution to the integral is from regions where one of the angles θ_i is small (smaller than θ_d) and the other is large (of the order of θ). Substituting in (1) the amplitude in the form (5) for small angles, we get the following integral equation for $\theta > \theta_d$:

$$\operatorname{Im} A(p, \theta) = \frac{p\sigma_{tot}}{4\pi(2\pi a)^{1/2}}\int_{-\infty}^{\infty} dv \exp[-ap^2(\theta - v)^2/2]\operatorname{Im} A(p, v) + F(p, \theta), \tag{8}$$

whose solution is of the form

$$\operatorname{Im} A(p, \theta) = F(p, \theta) - \frac{i\sigma_{tot}p}{8\pi^2 a}\int_{-\infty}^{\infty} dv F(p, v)\int_{-i\infty}^{i\infty} dr \frac{\exp[-rp(\theta - v)]}{\exp(-r^2/2a) - (\sigma_{tot}/4\pi a)} + \sum_{k=-\infty}^{\infty} C_k(p)\exp[-b_k(p)p\theta], \tag{9}$$

where in general C_k and b_k are complex [but still the whole sum is real, see (10)]. If in some subregion of angles overlap is unimportant, then $\operatorname{Im} A(p, \theta)$ must be described by the solution of the homogeneous equation [i.e., the last term in (9)]:

$$\operatorname{Im} A(p, \theta) = C_0(p)\exp[-b_0(p)p\theta] + \sum_{k=1}^{\infty} 2|C_k(p)|e^{-(\operatorname{Re} b_k)p\theta}\cos(|\operatorname{Im} b_k|p\theta - \varphi_k), \tag{10}$$

where the $b_k(p)$ are given by the formulas

$$b_0 = [2a\ln(4\pi a/\sigma_{tot})]^{1/2}, \qquad b_k \approx (2\pi a|k|)^{1/2}(1 + i\,\operatorname{sign} k) \quad (|k| \geqslant 1); \tag{11}$$

the coefficients $C_k(p)$ are undetermined functions of the energy, and the phases φ_k can be taken equal to $\pi/4$ (cf.[34]).* For elastic scattering all processes studied at present $b_0 < \operatorname{Re} b_k$ ($k \geq 1$). Therefore for large values of $p\theta$ only the first term in (10) is important. As we go toward smaller values of $p\theta$ other terms in (10) can become important; that is, there will be oscillations superposed on the exponential decrease, and their amplitude increases with decreasing $p\theta$. Furthermore the parameters b_0 and b_k are determined by the total cross section and the reciprocal width of the diffraction peak, whose values are primarily due to the inelastic processes, as we have indicated above.

Accordingly, even in the range of angles where the overlap function may be negligibly small, the inelastic processes determine the functional dependence of the elastic scattering amplitude on the angle, owing to the fact that they play the decisive role in the region of the diffraction cone.† This is a factual proof of our assertion that the *inelastic processes determine the* elastic scattering at any angle.

The question as to whether or not there actually exists a range of angles where $F(p, \theta) \ll \operatorname{Im} A(p, \theta)$ can be settled only by comparison with experiment. It has been found[35] that our formula (10) gives a good description of the very exact data on proton-proton scattering at l.s. momenta from 81 to 21.1 GeV/c[33] in the range $1 \lesssim p\theta \lesssim 2.4$ GeV/c (i.e., $1 \lesssim |t| \lesssim 6$ $(\text{GeV}/c)^2$. Furthermore the index of the main exponential is exactly given, and also the damping of the amplitude of the oscillations with increase of $p\theta$, their period, and the sign and the position of the zeroes. This justifies the assumption that at these energies and in this range of values of $p\theta$ the contribution of $F(p, \theta)$ to $\operatorname{Im} A(p, \theta)$ can be neglected.

Nevertheless it is possible that with increase of the energy there is some sort of change in this range. For example, in the reggeon-exchange model,[36,37] or indeed in the Chou-Yang model,[38] both of which claim to give an asymptotic description of the elastic scattering, it is easily verified that the overlap function is always of the same order of magnitude as the imaginary part of the amplitude.

At larger angles the experimentally found behavior of the cross sections differs from that predicted by Eq. (10).

In the region $p\theta \gtrsim 2.4$ GeV/c [i.e., $|t| \gtrsim 6$ $(\text{GeV}/c)^2$] the differential cross sections for pp scat-

*It is different for different processes and can depend on the energy.

*Inclusion of the real part of the elastic scattering amplitude would require that in the formulas (11) the quantity σ_{tot} be replaced by the expression $\sigma_{tot}(1 + \bar{\delta}_d\bar{\delta}_1)$, where $\delta = \operatorname{Re}A/\operatorname{Im}A$, $\bar{\delta}_1$ means the average value of δ at large angles, and $\bar{\delta}_d$ is the average value of δ in the region of the diffraction cone.

†Consequently, neglect of the function $F(p, \theta)$ by no means means a complete denial of the importance of inelastic processes.

tering in the energy range indicated above drop off more weakly than exponentially with increase of the angle [see Eq. (10)]. This can be understood if we suppose that $F(p, \theta)$ again becomes important. The weak angular dependence of the differential cross sections allows us to assume that here there is also little change in $\operatorname{Im} A(p, \theta)$. Then it is easily shown from (8) and (9) that $\operatorname{Im} A(p, \theta)$ and $F(p, \theta)$ are connected by the relation

$$\operatorname{Im} A(p, \theta) = F(p, \theta)/(1-\sigma_{tot}/4\pi a). \tag{12}$$

Consequently, at large angles the inelastic processes again determine the elastic scattering directly [through $F(p, \theta)$].

We can conclude that in the description of elastic scattering the primary problem is to explain the behavior of the overlap function $F(p, \theta)$ at small angles, since this will make clear the structure of the diffraction peak, i.e., the main part of the elastic scattering processes. At the same time at large angles the behavior of this function can be connected with such theoretical questions of principle as the treatment of microparticles as statistical objects.

Here we only note briefly that attempts to describe the diffraction peak as the consequence of a definite class of inelastic processes are contained in the uncorrelated-jet model,[26,27] and in models of the multiperipheral type,[22,24] while the region of scattering at very large angles is usually associated with the presence of inelastic processes of the statistical type.[31]

III. THE ROLE OF THE PHASES OF THE MATRIX ELEMENTS FOR INELASTIC PROCESSES

Let us examine in more detail what sort of properties of the matrix elements for inelastic processes can determine the form of the overlap function $F(p, \theta)$ at small scattering angles. We have found that the unitarity condition along with the experimental data on elastic scattering allow us to draw some conclusions about the behavior of the function $F(p, \theta)$. Thus in the region of small angles $F(p, \theta)$ is well approximated by a function with a Gaussian decrease with increasing angle [see Eq. (6)]. The question naturally arises as to what determines the rate of decrease of this function. The point is that the matrix elements of the inelastic processes, which determine $F(p, \theta)$ according to (2), are complex functions, having both absolute values and phases. Fukuda and Iso[39] were the first to point out the importance of the phases. It will be shown here that the parameter α in (6) can be represented as the sum of two terms, one of which is determined by the absolute value of the matrix element, and the other by the phase.[49] At the same time all the possible distributions of particles in inelastic processes are determined only by the absolute value of the matrix element (since only squares of absolute values of matrix elements appear in the expressions for the differential cross sections*). Therefore an examination of the diffraction peak of the elastic scattering can give some additional information about the inelastic processes.

Let us write the overlap function $F(p, \theta)$ in (2) as the sum of the contributions from all inelastic channels:

$$F(p, \theta) = \sum_n F_n(p, \theta), \tag{13}$$

where

$$F_n(p, \theta) = \langle p;\, n \,|\, \exp(-iJ_y\theta)\,\delta^4\,(Q - \sum_j^n q_j) \,|\, p;\, n\rangle \tag{14}$$

and we have introduced the notation $M_{a\to n} \equiv \langle q_1, \ldots, q_n | p; n\rangle$.

Equation (14) is a formal expression for F_n, whose physical meaning is that one takes the product of the matrix element for the transition of two particles with momenta p and $-p$ into an n-particle state times the conjugate of the matrix element for the transition of these n particles into two with the momenta p' and $-p'$, the angle between the vectors p and p' being θ. This last fact is expressed by writing $\langle p', n| = \langle p, n | \exp(-iJ_y\theta)$, where J_y is the component of the total angular momentum operator in the direction perpendicular to the plane of the scattering. Since we are considering spinless particles, J_y can be replaced by L_y, the component of the orbital angular momentum in that direction. If, guided by the general form (6) of the overlap function, we now assume that each n-particle contribution to it is also well approximated by a Gaussian function, i.e., that

$$F_n(p, \theta) \approx 4p^2\sigma_n \exp(-\alpha_n p^2\theta^2/2), \tag{15}$$

or for very small angles

$$F_n(p, \theta) \approx 4p^2\sigma_n\,[1-(\alpha_n p^2\theta^2/2)], \tag{16}$$

then

$$\alpha_n = \frac{\langle p, n \,|\, L_y\delta^4\,(Q - \sum\limits_{j=1}^n q_j)\, L_y \,|\, p, n\rangle}{p^2\langle p, n \,|\, \delta^4\,(Q - \sum\limits_{j=1}^n q_j) \,|\, p, n\rangle}. \tag{17}$$

Then also

$$\alpha \approx \sum_n \sigma_n\alpha_n/\sigma_{in}. \tag{18}$$

We rewrite (17) in the form

$$\alpha_n = \frac{\int \cdots \int |\mathscr{L}_y R_n e^{i\varphi_n}|^2\,\delta^4\,(Q - \sum\limits_{j=1}^n q_j) \prod\limits_{j=1}^n d^4q_j\delta_+(q_j^2+m_j^2)}{p^2\int \cdots \int |R_n e^{i\varphi_n}|^2\,\delta^4\,(Q - \sum\limits_{j=1}^n q_j) \prod\limits_{j=1}^n d^4q_j\delta_+(q_j^2+m_j^2)}, \tag{19}$$

where we have introduced the concrete expression for the operator L_y in the form

$$\mathscr{L}_y = \sum_{j=1}^n \mathscr{L}_y^{(j)} = i\sum_{j=1}^n \left(q_{jx}\frac{\partial}{\partial q_{jz}} - q_{jz}\frac{\partial}{\partial q_{jx}}\right), \tag{20}$$

Where δ_+ is the positive-frequency delta function, $M_{a\to n} = R_n e^{i\varphi_n}$, with real R_n, and m_j is the mass of the j-th particle. It is not hard to see that

$$\alpha_n = \frac{\int \cdots \int \{|\mathscr{L}_y \ln R_n|^2 + |\mathscr{L}_y\varphi_n|^2\}\, R_n^2\delta^4\,(Q - \sum\limits_{j=1}^n q_j) \prod\limits_{j=1}^n d^4q_j\delta_+(q_j^2+m_j^2)}{p^2\int \cdots \int R_n^2\delta^4\,(Q - \sum\limits_{j=1}^n q_j) \prod\limits_{j=1}^n d^4q_j\delta_+(q_j^2+m_j^2)}, \tag{21}$$

since $\mathscr{L}_y$ is an antihermitian operator (sic).

Consequently, the quantity α_n consists of two positive additive terms, one of which is determined by the absolute value of the matrix element, and the other by its phase. By (18) this is also true for the quantity α,

*In fact, as can be seen from (2), for $\theta \neq 0$ we have $a \neq a'$ and the phases of the quantities M are important, while the total cross section for the inelastic processes is determined by the quantity $F(p, 0)$ and the phases play no part, and the various distributions in inelastic processes are obtained if we omit the integrations over the variables in question in the expression for $F(p, 0)$.

i.e.,

$$\alpha = \alpha_R + \alpha_\varphi, \qquad (22)$$

where $\alpha_R \geq 0$, $\alpha_\varphi \geq 0$, and we have also for the quantity a [see (7)]:

$$a = a_R + a_\varphi \qquad (23)$$

($a_R \geq 0$, $a_\varphi \geq 0$). There is no doubt that these results hold only approximately in the region of small angles. Owing to the assumptions we have made. If we use the formal apparatus of work with wave packets, it can be shown that the phase φ_n is due to the shift of the center of the wave packet in the interaction process, while the absolute value R_n is determined by the deformation of the wave packet.

We remark that the phases φ_n can contribute to a only if they depend on the momenta q_j of the secondary particles. If the phases are constant or if they depend only on the total energy of the colliding particles, then $a_\varphi \equiv 0$ (cf.[21,28]) Accordingly, the assumption that the phases are constant or that they depend only on the total energy of the collision leads directly to the largest width of the diffraction cone, $a^{-1} = a_R^{-1}$. The introduction of a phase that depends on the momenta of the secondary particles can only narrow the cone.

This general treatment cannot answer the question as to which of the terms in (23) makes the main contribution to a, i.e., determines the width of the diffraction cone. For this it is necessary to make some sort of concrete assumptions about the form of the matrix elements of inelastic processes. At present the role of the phases has been studied in the uncorrelated-jet model,[40,42-45] in the many-reggeon model,[40,43,44] and in the hydrodynamic model.[39] The quantitative results obtained depend on the assumptions about the behavior of the phases, and are somewhat different in different papers. But from the whole set of papers we can draw the general qualitative conclusion that: 1) in the absence of correlations between the secondary particles the effects of the phases predominate (a_φ is about an order of magnitude larger than a_R); 2) when such correlations are introduced the importance of the phases decreases, but in the many-reggeon model, for example, they are still very important ($a_\varphi > a_R$) at energies now attainable with accelerators ($\ln s \sim 3$ to 4, with s in GeV^2), although it may become small asymptotically; 3) for very strong correlations (in the hydrodynamic model) we have $a_R \sim a_\varphi$.

Accordingly, the phases of the matrix elements of inelastic processes are very important in the formation of the diffraction peak of the elastic scattering. Unfortunately, at the present time one sees no further ways of studying these phases and their dependence on the momenta of the secondary particles experimentally.

Inelastic processes in which the number of secondary particles is very large ($n \gg 1$) were the ones discussed in the papers cited.[38,40,42-45] The question naturally arises as to whether reactions with production of a small number of particles, and in which one or both of the colliding particles are converted into resonances (so-called inelastic-binary or quasi-two-particle reactions) can determine the form of the diffraction cone. The experimental data on such reactions[10,46-48] at energies from 10 to 30 GeV indicate that such events are a small fraction of all the inelastic processes ($\lesssim 10$ percent). According to the optical theorem this means that at $\theta = 0$ the fraction of the overlap function caused by them is equally small ($\lesssim 10$ percent). At nonzero angles, according to (2), the contribution of the binary processes to the differential cross section for elastic scattering must have an upper limit[49] given by the inequality

$$d\sigma_{el}^{(b)}/dt \leqslant \sum_i (16\pi c_i^2)^{-1} [(d\sigma_B^{(i)}/dt_i)_{t_i=0}]^2 e^{c_i t/2}, \qquad (24)$$

if we take into account the experimental fact[10,46-48] that the cross section for binary reactions* falls off exponentially with increasing $|t|$: $d\sigma_b^{(i)}/dt_i = [(d\sigma_b^{(i)}/dt_i)_{t_i=0}] e^{c_i t}$. Substituting in (24) the values of c_i and $(d\sigma_b^{(i)}/dt_i)_{t_i=0}$ taken from experiment,[10,46-48] we can easily show that in the entire diffraction cone the contribution of the shadow scattering caused by the binary reactions is unimportant. Therefore we can conclude that the small-angle elastic scattering is the result of diffraction caused by the presence of large-multiplicity inelastic processes.

IV. THE BETHE-SALPETER EQUATION

We now pass directly to the description of large-multiplicity inelastic processes. As has been shown above, they not only predominate in the total cross section, but also essentially determine the form of the diffraction cone in the elastic scattering. Therefore without a detailed understanding of the character of such processes any description of elastic scattering must be regarded as only a formal phenomenological approximation.

At the same time, as we have pointed out, to calculate the differential cross sections for inelastic processes one needs to know only the absolute values of the matrix elements for these processes, while the *shadow elastic scattering* caused by them depends largely on their phases, about which we still have very little information. Therefore at present it is more realistic to solve the problem of the interconnection of elastic and inelastic processes by dealing only with the question of which singularities of the partial amplitude for elastic scattering in the angular-momentum plane of the cross channel correspond to given inelastic processes in the direct channel. This restricted statement of the problem will be the basis of all the following exposition.

The theoretical treatment of essentially inelastic collisions is undoubtedly an extremely complicated problem. In this field of physics at present there are a few general relations and a multitude of different models.

The theory we shall expound here, which was proposed in[23], is an attempt to describe and classify precisely such processes and to elucidate their interconnections with the elastic scattering (in the restricted sense which we have specified). In this theory we make a natural separation of the inelastic processes into

*The index b means "binary" and the index i indicates the type of binary reaction; c_i does not depend on t_i.

peripheral and nonperipheral* in the very first equation. The results show that a characteristic feature of the peripheral collisions in the high energy region is the formation of pion bunches—fireballs.

The basic mathematical apparatus of the theory is the Bethe-Salpeter equation. It is the use of this equation that enables us to show a direct connection between the elastic and inelastic processes, and at the same time to separate the inelastic processes into peripheral and nonperipheral processes. It must be emphasized at once that our treatment will not be the ladder approximation often referred to as the Bethe-Salpeter method; we shall use the exact Bethe-Salpeter equation in the treatment of the general questions. Of course, it is an equation in the usual sense of the word only when its kernel and inhomogeneous term are specified. But some general properties of the scattering amplitude which do not follow from the Bethe-Salpeter equation (primarily, analyticity and unitarity) impose serious supplementary restrictions on the quantities that appear in it. The result is that despite the extremely general nature of the equation we can extract from it definite information about the properties of the interaction at high energies.**

In examining all the general consequences that follow from the combined use of the Bethe-Salpeter equation, analyticity, and unitarity, we shall for simplicity assume at first that the only particles involved in the interaction are identical neutral pseudoscalar particles of mass μ (for example, π^0 mesons†). We consider the process of elastic scattering of such particles with four-momenta p_1, p_2, p_3, p_4. The Bethe-Salpeter equation involves the scattering amplitude not only on the mass shell (with $p_i^2 = -\mu^2$), but also off of it with respect to two external momenta. We denote it by $A(s, t, p_1^2, p_3^2)$, where $s = -(p_1 + p_2)^2$, $t = -(p_1 + p_3)^2$. The Bethe-Salpeter equation can be written in the following form:

$$A(s, t, p_1^2, p_3^2) = \bar{A}(s, t, p_1^2, p_3^2) - \frac{i}{(2\pi)^4} \int d^4k_1 \bar{A}(s_1, t, p_1^2, p_3^2, k_1^2, k_2^2) \times A(s_2, t, k_1^2, k_2^2) D(k_1^2) D(k_2^2) \qquad (25)$$

here $s_1 = -(p_1 - k_1)^2$, $s_2 = -(p_2 + k_1)^2$, $k_2 = p_1 + p_3 - k_1$. $\bar{A}$ denotes the irreducible (in the t channel) part of the amplitude,‡ and $D(k_i^2)$ denotes the propagation function. The diagram form of Eq. (25) is shown in Fig. 1.

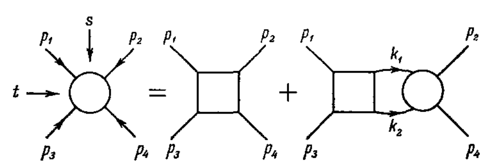

FIG. 1. Diagram representation of the Bethe-Salpeter equation.

*We shall give an explanation of these terms later.

**Speaking more exactly, it enables us to relate some properties of the interaction at high energies with the characteristics of processes at comparatively low energies. The situation is somewhat like that resulting from the sum rules (see Chapter VII, Sec. 2).

†In Chapter VIII we shall extend the treatment to actual processes of NN and πN collisions.

‡That is, the part that contains no two-particle intermediate states in this channel.

Let us expand both the amplitude A itself and its irreducible part $\bar{A}$ in terms of partial waves in the t channel*:

$$A(s_2, t, k_1^2, k_2^2) = \sum_{l=0}^{\infty} (2l+1) f_l(t, k_1^2, k_2^2) P_l(z_2), \qquad (26)$$

$$\bar{A}(s_1, t, p_1^2, p_3^2, k_1^2, k_2^2) = \sum_{l=0}^{\infty} (2l+1) \bar{f}_l(t, p_1^2, p_3^2, k_1^2, k_2^2) P_l(z_1), \qquad (27)$$

where z_1 and z_2 are the cosines of the scattering angles in the t channel with squared masses $(\mu^2, \mu^2, -k_1^2, -k_2^2)$ and $(-p_1^2, -p_3^2, -k_1^2, -k_2^2)$, respectively; that is,

$$z_1 = \frac{2ts_1 + t(p_1^2 + p_3^2 + k_1^2 + k_2^2) + (p_1^2 - p_3^2)(k_1^2 - k_2^2) + t^2}{[2t(k_1^2 + k_2^2) + (k_1^2 - k_2^2)^2 + t^2]^{1/2}[2t(p_1^2 + p_3^2) + (p_1^2 - p_3^2)^2 + t^2]^{1/2}}, \qquad (28)$$

and z_2 is obtained from z_1 if we replace s_1 with s_2 and set $p_1^2 = p_3^2 = -\mu^2$. Substituting (26) and (27) in (25) and using the orthogonality of the Legendre polynomials, we get the Bethe-Salpeter equation for the partial waves

$$f_l(t, p_1^2, p_3^2) = \bar{f}_l(t, p_1^2, p_3^2) - \frac{2i}{(2\pi)^3} \int_0^{\infty} q^2 dq \int_{-\infty}^{\infty} dq_0 \bar{f}_l(t, p_1^2, p_3^2, k_1^2, k_2^2) f_l(t, k_1^2, k_2^2) D(k_1^2) D(k_2^2), \qquad (29)$$

where

$$q = |\mathbf{k}_1|, \quad q_0 = (p_{1_0} + p_{3_0})/2 - k_{1_0} = (k_2^2 - k_1^2)/2t^{1/2}. \qquad (30)$$

Equations (25) and (26) are valid both for $t > 4\mu^2$ and for $t < 4\mu^2$. It is an important point, however, that for $t < 4\mu^2$ all of the singularities of the integrands in (25) and (29) are in the second and fourth quadrants of the q_0 plane. Therefore, we perform the Wick rotation[50] and use the invariant variables

$$r = -k_1^2 - k_2^2 - 2\mu^2, \quad r_0 = -p_1^2 - p_3^2 - 2\mu^2, \qquad (31)$$
$$v = k_2^2 - k_1^2, \quad v_0 = p_3^2 - p_1^2,$$

and write Eq. (30) for $t < 4\mu^2$ in the form

$$f_l(t, r_0, v_0) = \bar{f}_l(t, r_0, v_0) + \frac{4}{(4\pi)^3 |t|} \int dr\, dv \frac{[-t(t - 4\mu^2) + 2tr - v^2]^{1/2} \bar{f}_l(t, r_0, v_0, r, v) f_l(t, r, v)}{(r + i\varepsilon)^2 - v^2}, \qquad (32)$$

where the region of integration is determined by the condition

$$2tr - t(t - 4\mu^2) - v^2 \geqslant 0. \qquad (33)$$

For $t = 0$ the region of integration over v, and v itself, go to zero. In order to take the limit $t \to 0$ in Eq. (32), we must first separate out the kinematic factors of the functions f_1 and $\bar{f}_1$, which go to zero and infinity in this limit. Since for $q \to 0$ these partial amplitudes are proportional to q^l, it is convenient to introduce functions φ_1 and $\bar{\varphi}_1$ defined by the relations

$$\varphi_l = |t|^{l/2} [2tr - t(t - 4\mu^2) - v^2]^{-l/2} (t - 4\mu^2)^{-l/2} f_l, \qquad (34a)$$

$$\bar{\varphi}_l = |t|^l [2tr - t(t - 4\mu^2) - v^2]^{-l/2} [2tr_0 - t(t - 4\mu^2) - v_0^2]^{-l/2} \bar{f}_l. \qquad (34b)$$

These functions no longer contain the kinematic singularities, and therefore for $t \to 0$ they [like the propagation function $D(k^2)$] can be taken outside the sign of integration over v. Integrating the expression remaining under the integral sign with respect to v, we get the Bethe-Salpeter equation for the partial amplitude at the point $t = 0$,

*Both functions are expansible inside the Martin-Lehmann ellipse.

$$\varphi_l(p^2) = \bar{\varphi}_l(p^2) + \frac{2^{2l}\pi^{1/2}\Gamma(l+3/2)}{(2\pi)^3\Gamma(l+2)}\int_0^\infty \bar{\varphi}_l(p^2, k^2)\varphi_l(k^2)D^2(k^2)(k^2)^{l+1}dk^2, \quad (35)$$

where $p^2 = p_1^2 = p_3^2$ and $k^2 = k_1^2 = k_2^2$.

An important peculiarity of Eq. (35) is that in the region of integration the four-vector k is spacelike, $k^2 > 0$. It must be emphasized that the Bethe-Salpeter equation has this property only in the region $t \le 0$. It can be seen from (33) that for any value $t > 0$ the region of integration also contains time-like four-momenta $k_{1,2}^2$, indeed arbitrarily large such vectors. For spacelike values of the momenta the Watson-Sommerfeld formula (like the dispersion relations with respect to s) is certainly applicable if it can be used on the mass shell $k^2 = -\mu^2$, since in this region the imaginary part of the amplitude has no singularities.[51a] Therefore we can represent the imaginary parts of the amplitude and of its irreducible block at $t = 0$, $A_1(s_2) = (1/2i)[A(s_2 + i\epsilon) - A(s_2 - i\epsilon)]$ and $\bar{A}_1(s_1) = (1/2i)[A(s_1 + i\epsilon) - A(s_1 - i\epsilon)]$ in the following form:

$$A_1(z_2) = -\frac{i}{4}\int_{b-i\infty}^{b+i\infty} dl\,(2l+1)f_l P_l(z_2), \quad (36a)$$

$$\bar{A}_1(z_1) = -\frac{i}{4}\int_{b-i\infty}^{b+i\infty} dl\,(2l+1)\bar{f}_l P_l(z_1), \quad (36b)$$

where b = const. The inverse relations are of the form

$$f_l = \frac{2}{\pi}\int_{z_{\min}}^\infty A_1(z_2)Q_l(z_2)\,dz_2, \quad (37a)$$

$$\bar{f}_l = \frac{2}{\pi}\int_{z_{\min}}^\infty \bar{A}_1(z_1)Q_l(z_1)\,dz_1. \quad (37b)$$

The functions φ_1 and $\bar{\varphi}_1$ can be expressed similarly*:

$$\varphi_l(p^2, k^2) = (4pk)^{-l}\frac{2}{\pi}\int_{z_{\min}}^\infty A_1(z_2)Q_l(z_2)\,dz_2 = \frac{(4kp)^{1-l}}{\pi}\int_{z_{\min}}^\infty (z_2^2-1)^{1/2}\sigma(s_2, p^2, k^2)Q_l(z_2)\,dz_2, \quad (38a)$$

$$\bar{\varphi}_l(p^2, k^2) = (4pk)^{-l}\frac{2}{\pi}\int_{z_{\min}}^\infty \bar{A}_1(z_1)Q_l(z_1)\,dz_1 = \frac{(4kp)^{1-l}}{\pi}\int_{z_{\min}}^\infty (z_1^2-1)^{1/2}\bar{\sigma}(s_1, p^2, k^2)Q_l(z_1)\,dz_1, \quad (38b)$$

where

$$z_{\min} = (4\mu^2 + p^2 + k^2)/2pk, \quad p \equiv \sqrt{p^2}; \quad k \equiv \sqrt{k^2} = (\mathbf{k}^2 - k_0^2)^{1/2}.$$

Substituting the relations (38a) and (38b) (and also their inverses, which express A_1 and $\bar{A}_1$ in the form of integrals of φ and $\bar{\varphi}$) in Eq. (35),† we get the following expression for the imaginary part of the amplitude at $t = 0$:

$$A_1(s, p^2) = \bar{A}_1(s, p^2) + \frac{1}{8\pi^3[(s+p^2-\mu^2)^2+4p^2\mu^2]^{1/2}}\int \bar{A}_1(s_1, p^2, k^2) \times A_1(s_2, k^2)D^2(k^2)\,dk^2\,ds_1\,ds_2. \quad (39)$$

The region of integration in (39) is determined by the conditions

$$-k^2[-su + (p^2+\mu^2)^2] + (s_1 + p^2 + k^2)^2\mu^2 - p^2(s_2 + k^2 - \mu^2)^2 + (s_1 + p^2 + k^2)(s_2 + k^2 - \mu^2)(s + p^2 - \mu^2) \le 0,$$
$$s_{1,2} \ge 4\mu^2, \quad s^{1/2} \ge s_1^{1/2} + s_2^{1/2}. \quad (40)$$

*The inverse relations are analogous to (36a) and (36b).

†Analogous calculations are given in detail in [57b].

Accordingly we see that if the amplitude can be put in the form of a Watson-Sommerfeld integral, at $t = 0$ it is possible to get from the Bethe-Salpeter equation for the complete amplitude an analogous equation (39) for the imaginary part of the amplitude.

Using the optical theorem, which states that

$$A_1(s, p^2) = \sigma(s, p^2)[(s + p^2 - \mu^2)^2 + 4\mu^2p^2]^{1/2}, \quad (41a)$$

where σ is the total interaction cross section, and defining the quantity $\bar{\sigma}$ by the relation

$$\bar{A}_1(s, p^2, k^2) = \bar{\sigma}(s, p^2, k^2)[(s + p^2 + k^2)^2 - 4p^2k^2]^{1/2}, \quad (41b)$$

we get from (39) an equation for the cross section $\sigma(s, p^2)$

$$\sigma(s, p^2) = \bar{\sigma}(s, p^2) + \frac{1}{8\pi^3}\int dk^2\,ds_1\,ds_2 D^2(k^2)\bar{\sigma}(s_1, p^2, k^2)\sigma(s_2, k^2) \times H(s, s_1, s_2, p^2, k^2), \quad (42)$$

where

$$H = \frac{[(s_1 + p^2 + k^2)^2 - 4p^2k^2]^{1/2}[(s_2 + k^2 - \mu^2)^2 + 4\mu^2k^2]^{1/2}}{(s + p^2 - \mu^2)^2 + 4\mu^2p^2}. \quad (43)$$

For specific channels with production of n particles Eq. (42) can be written in the form of a system of equations**

$$\left.\begin{aligned} &\sigma_2(s, p^2) = \bar{\sigma}_2(s, p^2), \\ &\sigma_n(s, p^2) = \bar{\sigma}_n(s, p^2) + \frac{1}{8\pi^3}\sum_{m=1}^{n-1}\int dk^2\,ds_1\,ds_2 D^2(k^2) \\ &\qquad \times \bar{\sigma}_m(s_1, p^2, k^2)\sigma_{n-m}(s_2, k^2)H. \end{aligned}\right\} \quad (44)$$

Accordingly, we have established a connection between the inelastic processes described by Eqs. (42) and (44) (the meaning of these equations will be explained below) and the elastic scattering which they cause, which has partial amplitudes in the cross channel described by Eqs. (29) and (35).

We note also that that in the appropriate regions of the kinematic variables s and t it is easy to obtain from these equations the Mandelstam spectral function ($s > 4\mu^2$, $4\mu^2 < t < 16\mu^2$), the nonrelativistic Bethe-Salpeter equation ($s \to 4\mu^2$) in which $\bar{A}$ can be interpreted as a potential, and the equation of the quasipotential approximation[52] (cf.[23]).

V. PHYSICAL MEANING OF THE EQUATIONS AND FUNDAMENTAL ASSUMPTIONS

Let us now analyze the physical meaning of the various terms that appear in Eq. (42), and also the way it corresponds to the equation of the multiperipheral model.

The integral term in (42) represents the total cross section σ^P of the peripheral interaction. This term refers to all processes caused by the exchange of one meson. In fact, the transition from the equation (25) for the elastic amplitude to the equation (39) for its imaginary part at $t = 0$, and then to the equation (42) for the total cross sections, corresponds in diagram language to the transition from the diagrams of Fig. 1 to those of Fig. 2. It can be seen from this that strictly speaking the expression for the cross section

**$\bar{\sigma}_1$ and σ_1 denote form-factors.

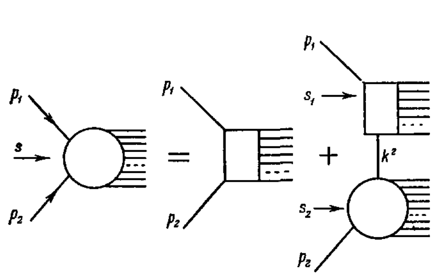

FIG. 2. Diagram representation of the equation for the total cross sections.

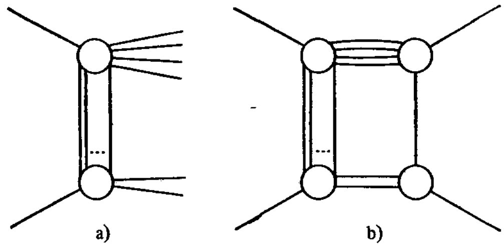

FIG. 4. a) The many-meson process; b) its interference with the one-meson process.

of the peripheral interaction, as used in the first papers on the one-meson approximation,[18,19] is incorrect. It was actually of the form of (42), but with the $\bar{\sigma}$ in the integrand taken to be equal to the total cross section σ.

What is the meaning of the quantity $\bar{\sigma}$? Equation (42) is similar to the equation which Amaldi et al.[22] took as the basis of the multiperipheral model. In this model $\bar{\sigma}$ was assumed simply equal to the cross section of the interaction in the low-energy region (below the threshold of inelastic processes), and this led to an asymptotic decrease of the total cross section with increasing energy. The use of the Bethe-Salpeter equation allows us to interpret the quantity $\bar{\sigma}$ more accurately. According to the definition of the function $\bar{A}$ which we have given, we can assert that $\bar{\sigma}$ is the sum of the cross sections of all processes with more than one meson and the contribution from the various interference terms. In the theory we have developed, as in a number of related schemes (for example, the multiperipheral and many-reggeon models) it is extremely important that the irreducible part is positive. It is this fact that allows us to associate the irreducible block with the cross section of the nonperipheral (more-than-one-meson) inelastic interactions and to interpret it as the formation of a fireball.

That the irreducible block is positive is due to the magnitudes and signs of the interference terms. The point is that besides the diagrams obtained by squaring those shown in Figs. 3, a and 4, a, which of course are positive, the irreducible block also includes contributions from diagrams of the type of Figs. 3, b and 4 b, which describe the interference of the amplitudes for inelastic processes. The sum of the contributions can become negative only if the interference terms are negative and exceed the main contribution.

If the interference terms are even of the order of magnitude of the contribution of the main process (for example, half or a third of it), but the kernal remains positive, then all of our assertions (both mathematical and interpretative) remain valid.

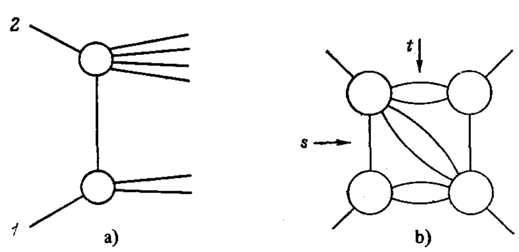

FIG. 3. a) The one-meson process; b) its interference diagram in the elastic scattering.

The question of the magnitude and sign of the interference terms cannot be solved in the framework of the original scheme (nor of models related to it). Therefore in discussing it we can only rely on intuitive physical arguments, which are to be regarded as less rigorous. Even on this level, however, it is not sufficiently discussed in the literature. Most of the papers tacitly take it for granted as a matter of course that the interference terms are unimportant. It seems to us that this is true, but nevertheless the question deserves discussion.

We must distinguish two types of interference terms. The first is the interference between amplitudes of two one-meson processes, and is often called in exchange interference. It appears in cases in which some of the particles produced in the first process at a certain point of the diagram come out in the c.m.s. at the same angles and with the same momenta as particles produced at a different point of the second process (see Fig. 3, b). The contribution of such processes can be estimated on the basis of kinematic considerations.

If the relative velocity of the blocks is very large and the Lorentz factor $\bar{\gamma} \gg 1$, then this sort of interference is small. This is so because in the c.m.s. of the blocks the particles produced in the different blocks are strongly collimated and fly out in different directions with no overlap of their angular distributions.

On the other hand, if the relative velocity of the blocks is small, $\bar{v} \ll 1$ and the Lorentz factor is small, $\bar{\gamma} - 1 \ll 1$, then the interference terms can be of the same order of magnitude as the main contribution.

Our case is intermediate between these two situations. In fact, calculating $\bar{\gamma}$ in order of magnitude from simple kinematical considerations[51b] we have

$$\gamma_0 = [1 + (s_0/4k^2)]^{1/2},$$

while

$$\bar{\gamma} = 2\gamma_0^2 - 1$$

(here γ_0 is the Lorentz factor of the blocks in their c.m.s., s_0 is the square of the "mass" of a block, and k^2 is the square of the four-momentum transfer between the blocks).

It follows from this that $\bar{\gamma}$ is large only to the extent that $s_0 > k^2$. In actual cases, according to Chapter VII, Sec. 2, $s_0 \approx 5k^2$ and $\bar{\gamma} \approx 3.5$. The relative velocities of the blocks are of the order of the speed of light, but not extremely relativistic. Then the angular and momentum distributions of the secondary particles produced in different blocks overlap only partially. It is clear that there cannot be any complete extinction of the main contribution in such a case. Accordingly, the first type of interference cannot lead in our case to a negative irreducible part.

The interference terms of the second type are due to diagrams of the form of Fig. 4, b. The interference is between inelastic processes of peripheral (right-hand part) and many-meson (left-hand part) interactions.

It is clear that the interference terms of this type can be large only if the quantum numbers, multiplicities, and also the angular and momentum distributions of secondary particles in the peripheral and the many-meson processes coincide.

However, it was shown in[53] that even when these characteristic quantities coincide interference is possible only if an odd number of pions is exchanged in the many-meson process. Although we cannot give a rigorous proof that such an interference is small, we know of no demonstration that the contrary situation can actually occur.*

In summary, we conclude: that the irreducible part is positive seems to be an altogether natural condition (although it has not been rigorously proved).

We have gone into the question of the absence of interference between one-meson and many-meson processes only because it is basic to the whole further development of the proposed scheme. That there is no completely destructive interference is a main assumption of the following theory. The arguments we have given provide evidence that this assumption is not unjustified.

If we accept this assumption, then the one-meson (or peripheral) processes and the processes (nonperipheral) with more than one meson are practically independent of each other. The quantity $\overline{\sigma}$ in (42) is the total cross section of all the processes with more than one meson and therefore is positive; i.e., $\overline{\sigma} \geq 0$. We shall also call $\overline{\sigma}$ the cross section of the nonperipheral interaction. Accordingly, we are assuming that the total cross section is the sum of two positive terms, the cross sections of the peripheral and the nonperipheral interactions. We emphasize once again that since we have no rigorous proof of this assertion it is one of the additional postulates or hypotheses which we adopt as fundamental to our theory.

VI. SOME GENERAL PROPERTIES OF THE PROCESSES

Let us examine a number of general consequences of our equations in the framework of the assumption we have made.

1. The Pomeranchuk Pole

To begin with, it follows from (38b) that $\overline{\varphi}_1(p^2, k^2) \geq 0$ for $l > l_0$ is the position of the furthest-right singularity of the function $\overline{\varphi}_1$. Since the analogous assertion always holds for the function φ_1 [because by (38a) its sign, in the part of the plane to the right of all its singularities, is the same as the sign of the total cross section σ], Eq. (12) cannot have solution if, independently of p^2 and k^2, the positions of the furthest-right singularities of the functions φ_1 and $\overline{\varphi}_1$ coincide and both functions go to infinity at the singularity, Therefore at the singular point $l = \alpha$ of the function φ_1 the function $\overline{\varphi}_l$ is always finite if $\varphi_\alpha \to \infty$, so that for $l = \alpha$ the inhomogeneous equation (35) becomes a homogeneous equation:

$$R_\alpha(p^2) = \frac{2^{2\alpha}\pi^{1/2}\Gamma(\alpha+3/2)}{(2\pi)^3\Gamma(\alpha+2)} \int\limits_0^\infty \frac{(k^2)^{\alpha+1}}{(k^2+\mu^2)^2} \overline{\varphi}_\alpha(p^2, k^2) R_\alpha(k^2)\, dk^2, \quad (45)$$

where $R_\alpha(p^2)$ is the coefficient of the singular factor in the function $\varphi_1(p^2)$ at $l = \alpha$ (see the Appendix).

Furthermore, if we use the unitarity relation in the s channel, then, as we shall show below, it follows at once from what we have said that the vacuum Pomeranchuk pole $\alpha p(t)$ with $\alpha p(0) = 1$ is incompatible with Eq. (35).[54] It is not hard to verify that this contradiction is removed only if the partial amplitude in the t channel, $f_l(t)$ [or $\varphi_l(t)$], has for $t = 0$ a singularity at the point $l = 1$ which is weaker[50] than $\varphi_1 \sim 1/(l-1)\ln^{1/2}(l-1)$, or else if the singularity is located to the left of the point $l = 1$, i.e., if $\alpha p(0) < 1$.[54,55] This means, of course, that at asymptotically large energies the total cross section must decrease:

$$\sigma < \text{const}/[\ln\ln(s/\mu^2)]^{1/2}. \quad (46)$$

To prove this we first assume the opposite, i.e., that there exists a vacuum pole with $\alpha p(0) = 1$. Then, since the entire two-particle contribution to the unitarity relation in the s channel is contained in $\overline{A}$, for $s \to \infty$ the quantity $\overline{\sigma}$ must decrease at least logarithmically: $\overline{\sigma}(s, p_1^2, p_3^2) \geq \sigma_{el} \geq \text{const}/\ln s$, where σ_{el} is the elastic scattering cross section.* It is easy to see that in this case $\overline{\varphi}_1 \to \infty$, and consequently Eq. (35) has no solution. This can also be verified directly by substituting $\overline{\sigma} \geq \text{const}/\ln s$ and $\sigma = \sigma(p^2) = \text{const.}$ in the integral term of Eq. (42); then when we integrate we find that its increase with the energy is now slower than $\ln\ln s$, contrary to our initial assumption.†

Accordingly, the existence of a Pomeranchuk pole with $\alpha p(0) = 1$ is in contradiction with our equations (42) and (35). In order for them to have a solution it is necessary that the total cross section decrease, although this decrease can indeed be extremely weak.

Thus on the mass shell there can be a pole at the point $l = 1$ for $t = 0$ only if its trajectory depends on the external masses, or else if it ceases to be a pole when we go off of the mass shell, i.e., there is a change in the nature of the singularity. Strictly speaking there can be no pole of the Regge type, depending only on t, at this point.

Though this conclusion has been reached as a general consequence of the equations considered here, nevertheless it is clear that actually the difference in nature between the maximum admissible singularity and a

*It might be supposed that there could be a strong interference between the amplitude of a one-meson inelastic process and that of diffraction production. Here, however, interference is forbidden, as was shown in [53], owing to the conservation of G parity.

*In fact, unless there is complete destructuve interference the total contribution of all the other intermediate states to $\overline{\sigma}$ cannot be negative.

†The physical meaning of this contradiction is actually the same as in [21], where it was shown that if the quantities $\overline{\sigma}$ and σ are independent of the energy the integral term in (42) increases logarithmically with the energy.

pole can show up only at ultrahigh energies, when $\ln \ln (s/\mu^2) \gg 1$. Therefore if the actual behavior of the total cross section is close to the limit of what is allowed, then in either the phenomenological treatment of experimental data or in the theoretical discussion of inelastic processes there is no point in considering this difference, at all reasonable energy values (up to energies at which the difference causes a large drop in the cross section). In what follows we shall use the Pomeranchuk pole with $\alpha_P(0) \approx 1$, with the understanding that this is not justified at ultrahigh energies.

2. The Character of the Leading Singularity

We shall now show that the leading singularity of the partial amplitude φ_l must be a moving singularity. To do so we determine the slope γ of the trajectory $\alpha_P(t)$ for this singularity.[56] By a method given in detail in the Appendix, we readily find

$$\gamma \equiv \left(\frac{d\alpha(t)}{dt}\right)_{t=0} = \frac{9}{(4\pi)^4 R_1(-\mu^2)} \int_0^\infty \int_0^\infty \bar{\varphi}_1'(p^2, k^2) R_1(k^2) R_1(p^2) \times [p^2 k^2 D(p^2) D(k^2)]^2 \, dp^2 \, dk^2, \quad (47)$$

where

$$R_l(p^2) = \varphi_l(p^2)(l - \alpha(0)), \qquad \bar{\varphi}_1' = (d\bar{\varphi}_1/dt)_{t=0}.$$

$R_1(p^2)$ is defined as the correctly normalized (see below) solution of the homogeneous equation [see (A.4)]:

$$R_1(p^2) = \frac{3}{(4\pi)^2} \int_0^\infty \bar{\varphi}_1(p^2, k^2)[k^2 D(k^2)]^2 R_1(k^2) \, dk^2. \quad (45a)$$

In the integrand in (47) all of the functions are positive definite: $R_1 > 0$, since according to (38a) the sign of R_1 is the same as that of the total cross section; $\bar{\varphi}_1'(0, p^2, k^2) > 0$, since according to (38b) the sign of $\bar{\varphi}_1'(0)$ is the same as that of $\partial \bar{A}_1/\partial t$; and the last factor is positive.* Consequently, the slope of the leading trajectory is positive, i.e., the inelastic processes described by the integral term in (42) lead to moving singularities in the elastic partial amplitude, and the diffraction peak in the shadow elastic scattering caused by these processes becomes narrower with increasing energy. The physical meaning of this result was explained in[58], where it was shown that the increase of the interaction range (narrowing of the peak) is closely connected with the increase of the number of blocks in the iterative solution of Eq. (42), and the coefficient γ is of the order of magnitude of the reciprocal of the perpendicular component of the momentum transferred from one block to another. We shall calculate its numerical value in what follows.

3. The Physical Meaning of the Leading Moving Singularity

The question is often asked: what sort of inelastic processes is it that lead to the manifestation of the exchange of the leading moving singularity (the vacuum reggeon)† in the elastic scattering? In the framework of our present approach the answer to this question is clear. As we have shown earlier, the inclusion of the one-meson diagrams in the inelastic processes on the basis of the Bethe-Salpeter equation leads to the appearancy of a moving pole in the l plane of the partial wave in the cross channel for the corresponding elastic diffraction process. Furthermore, according to our assumption that there is no completely destructive interference and that therefore the function $\bar{\varphi}_1(0, p^2, k^2)$ is positive definite, there is no such pole in the irreducible part $\bar{\varphi}_1(t, p^2, k^2)$. We note that it is precisely these one-meson diagrams that lead, in the study of the elastic amplitude in the cross channel, below the threshold for production of new particles, to the unitarity of the scattering amplitude and to the appearance of a branch point of the amplitude at $t \equiv 4\mu^2$. At the same time we know that the irreducible part $\bar{\varphi}_1(t, p^2, k^2)$ is regular with respect to t at $t = 4\mu^2$, while for the Pomeranchuk pole this point must be a branch point.

Therefore it seems to us natural to picture the "physical structure" of the vacuum reggeon as entirely due to the one-meson inelastic processes.*** Meanwhile the diffraction from nonperipheral inelastic processes can lead to an important contribution to the elastic scattering which is not described by the exchange of a vacuum reggeon.

We particularly emphasize that the interconnection of the elastic and inelastic processes comes about by means of the unitarity in the direct (s) channel, and not in the cross (t) channel [transition from Eq. (39) to Eq. (42)].

4. The Asymptotic Value of the Total Cross Section

The homogeneous equation (45) determines the function $R_1(p^2)$ only up to a normalizing factor. It was derived, however, as the limit of an inhomogeneous equation whose solution is unique and determines the value of the cross section of the peripheral interactions in the asymptotic region according to (38a) and (41a). The correct normalization can be found if we consider Eq. (35) at the point $l = 1 + \epsilon$ ($\epsilon \to 0$) (for details see[59,60a] and the Appendix). The result is that the asymptotic value of the total cross section of the peripheral interactions can be expressed in terms of the function $R_1(p^2)$ in the form

$$\sigma^P = 2\pi^3 R_1^2(-\mu^2) \Big/ \int_0^\infty\!\!\int p^2 D^2(p^2) K_1'(p^2, k^2) R_1(p^2) R_1(k^2) \, dp^2 \, dk^2, \quad (48)$$

where

$$K_1' = \left(\frac{\partial K_l}{\partial l}\right)_{l=1}, \qquad K_l(p^2, k^2) = \frac{\sqrt{\pi}\,\Gamma(l + 3/2)}{(2\pi)^3 \Gamma(l+2)} \frac{k^2 \bar{f}_l(p^2, k^2)}{(k^2 + \mu^2)^2}. \quad (49)$$

The total cross section is naturally given by

$$\sigma_{tot} = \bar{\sigma} + \sigma^P. \quad (50)$$

*We can easily verify that $\partial \bar{A}_1/\partial t$ is positive by repeating for it the well known proof [57a] that the quantity $[\partial A(s, \partial/\partial t]_{t=0}$ is positive.

†Here we shall take no account of the slight difference between this singularity and a pole, nor of the possible shift of the singularity, as discussed earlier.

*We note that in principle one can imagine models in which there exist simultaneously two vacuum poles of the function $\bar{\varphi}_1$, which are complex conjugates for $t > 4\mu^2$, exactly cancel each other for $t = 4\mu^2$, and for $t < 4\mu^2$ lead to some sort of compensation of the pole caused by the integral term in the Bethe-Salpeter equation. However, in the first place, for this we should have to assume that the interference is very important at high energies, which, as we have explained, seems to us unlikely; and in the second place, on the present phenomenological level of the theory it scarcely makes any sense to complicate the model before the simplest versions have encountered any contradictions.

5. The Differential Distributions

It is also not hard to derive the mass distributions of the irreducible blocks and their distribution in the squares of the momentum transfers in peripheral processes (see Fig. 2). To do so we examine the integral term in Eq. (42). Again including only the contribution from the pole at $l = 1$, we replace $\sigma(s_2, k^2)$ by $\sigma^P(k^2) = R_1(k^2)\sigma^P/R_1(-\mu^2)$ and recall that the effective values of s_2 are much larger than the quantity k^2. Integrating over s_2, we get the following distributions:

$$\frac{d\sigma^P}{ds_1} = \frac{\sigma^P}{4\pi^3 R_1(-\mu^2)} \int_0^\infty dk^2 \frac{(k^2)^2 D^2(k^2)\overline{A}_1(s_1, -\mu^2, k^2) R_1(k^2)}{\{s_1 + k^2 - \mu^2 + [(s_1 + k^2 - \mu^2)^2 + 4k^2\mu^2]^{1/2}\}^2}, \tag{51}$$

$$\frac{d\sigma^P}{dk^2} = \frac{\sigma^P (k^2)^2 D^2(k^2)}{4\pi^3} \frac{R_1(k^2)}{R_1(-\mu^2)} \int_{4\mu^2}^\infty ds_1 \frac{\overline{A}_1(s_1, -\mu^2, k^2)}{\{s_1 + k^2 - \mu^2 + [(s_1 + k^2 - \mu^2)^2 + 4k^2\mu^2]^{1/2}\}^2}. \tag{52}$$

It is seen that the concrete form of these distributions depends on the behavior of the function $\overline{A}_1(s_1, -\mu^2, k^2)$ and the function $R_1(k^2)$ which is associated with it by (46). Nevertheless we can state that these distributions have a maximum, since they are integrable (i.e., they go to zero when s_1 and k^2 simultaneously go to infinity for $s \to \infty$), and they go to zero at the points $s_1 = 4\mu^2$ and $k^2 = k^2_{\min} \to 0$.

VII. THE FUNDAMENTAL PARAMETERS AND THE PROPERTIES OF PERIPHERAL INTERACTIONS

1. The Irreducible-block Model and Its Parameters

We now attempt a qualitative estimate of the effective values of s_1 and k^2 that play the main part in these distributions. First we note that the kernels of *the equations* (35) *and* (42) *are positive.* Therefore in the iterative solution of these equations each iteration will also be positive.* Since the sum of these iterations (the total cross section) exists and is finite for $s \to \infty$, the iteration series must converge. This means that the kernel $\overline{\sigma}(s_1, p^2, k^2)$ must be a decreasing function of its arguments as they go to infinity:

$$\lim_{s_1, p^2, k^2 \to \infty} \overline{\sigma}(s_1, p^2, k^2) = 0. \tag{53}$$

To look for the approximate numerical values of the quantities s_0 and $k_0^2 = p_0^2$ at which there must be an effective cutoff of the cross section $\overline{\sigma}(s, p^2, k^2)$, we first make a very crude qualitative estimate. Later (see VII.2) we shall do this more correctly, but we believe that this sort of estimate is very useful as a guide to understanding the relations of the various parameters.

Let $\overline{\sigma}$ have the very simple form

$$\overline{\sigma}(s, p^2, k^2) = \sigma_0 \theta(s_0 - s)\,\theta(k_0^2 - k^2)\,\theta(k_0^2 - p^2), \tag{54}$$

where

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Substituting (54) in (38b), we get the following expression for $\overline{\varphi}_1$:

$$\overline{\varphi}_1(p^2, k^2) \approx \frac{\sigma_0}{3\pi} \ln(z_0/z_{min})\,\theta(k_0^2 - k^2)\,\theta(k_0^2 - p^2), \tag{55}$$

where

$$z_0 = (s_0 + p^2 + k^2)/2kp, \qquad z_{\min} = (4\mu^2 + k^2 + p^2)/2kp.$$

By substituting (55) in (45) we can find a connection between the parameters σ_0, s_0, and k_0^2. To get it in explicit form, we use a well known approximate relation, according to which the trace of the kernel must be approximately equal to unity, i.e.,

$$\frac{\sigma_0}{16\pi^3} \int_0^{k_0^2} \ln\left(\frac{s_0 + 2k^2}{4\mu^2 + 2k^2}\right) \left(\frac{k^2}{k^2 + \mu^2}\right)^2 dk^2 \approx 1. \tag{56}$$

Using the fact that it is values $k^2 \gg \mu^2$ that are important (see below), we get

$$\frac{\sigma_0 k_0^2}{16\pi^3}\left[\ln\left(\frac{s_0}{2k_0^2}\right) + 1 + \frac{2k_0^2}{s_0}\right] \approx 1 \quad \text{for} \quad 2k_0^2 < s_0, \tag{57a}$$

$$\frac{\sigma_0 s_0}{32\pi^3}\left[\ln\left(\frac{2k_0^2}{s_0}\right) + 1 + \frac{s_0}{2k_0^2}\right] \approx 1 \quad \text{for} \quad s_0 < 2k_0^2, \tag{57b}$$

$$3\sigma_0 k_0^2/32\pi^3 \approx 3\sigma_0 s_0/64\pi^3 \approx 1 \quad \text{for} \quad 2k_0^2 \approx s_0. \tag{57c}$$

Since the meaning of σ_0 is the mean value of the cross *section of nonperipheral interactions for* $s < s_0$, *and* thus actually also in the resonance region (see below), it is natural to assume that $\mu^{-2} \lesssim \sigma_0 \lesssim 3\mu^{-2}$. Then from one of our formulas, say (57c), we find that

$$4.3\ \text{GeV}^2 \leqslant 2k_0^2 \approx s_0 \leqslant 13\ \text{GeV}^2 \tag{58}$$

If the quantities k_0^2 and s_0 are very different, then according to (56) the larger s_0 the smaller k_0^2, and conversely. Accordingly, a new large parameter of the dimensions of energy squared appears in the theory. It is important that this result has been obtained only from our justified requirement that the cross section for peripheral processes be finite* [that Eq. (45) be solvable].

2. A More Accurate Determination of the Parameters

We can estimate the parameter s_0 more accurately[60b] if we make use of the hypothesis of duality. We shall make the usual assumption that the imaginary part of the elastic scattering amplitude consists of two contributions of different types—a contribution corresponding to the leading vacuum singularity, which does not fit into the framework of the usual dual scheme, and the remainder after subtraction of this contribution, to which we can apply the duality hypothesis. As already noted, the first contribution is completely contained in the peripheral part of the amplitude. As for the second part, in the spirit of the duality hypothesis it can be represented as a sum of resonances in the s channel. Therefore, in order to calculate its integral contribution, which is all that is of interest for Eq. (39) at large energies [or, what is the same thing, for Eq. (45)], it is sufficient to substitute instead of the actual amplitude $\overline{A}_1$ the sum of the contributions of the corresponding Regge trajectories. This procedure, however, can be

*Moreover, as we shall show, each term of the iteration series is an observable quantity and has a clear physical meaning. For finite energy the iteration series contains a finite number of terms, which increases without bound as the energy goes to infinity.

**If the cross section at the large energies is only approximately constant, i.e., if the vacuum singularity is located not right at the point $l = 1$, but near it, [55] or even if the singularity is not a pole, [54] this does not affect our conclusion, but can only have a slight effect on the numerical value of this parameter, with no change in its order of magnitude.*

justified only for integration up to some finite energy. The point is that at a high enough energy, when the two-pion exchange is really "reggeized," the contribution of all trajectories that have a nonvanishing two-pion vertex* is already contained in the peripheral term, so that to use them in the duality condition for the nonperipheral term would now be to count the diagrams over again. It is reasonable to suppose that in the sense indicated the boundary s_0 between "low" and "high" energies is located where the clearly marked resonance structure of the cross section of the nonperipheral interactions disappears and it becomes of the order of the geometric cross section, i.e., where $\bar{\sigma}(\bar{s}_0) \approx \mu^{-2}$. For $s > s_0$ we shall assume that this cross section can be described as the tail, perhaps of a large number, but still of a finite number of resonances that exist in the region $s < s_0$, i.e., that it behaves asymptotically like s^{-1}.† Joining these two forms at the boundary, i.e., at $s = s_0$, we arrive at the following expression for $\bar{A}_1$ on the mass shell (in the sense of the sum rules!):

$$\bar{A}_1(s_1) = \bar{\sigma}(s_0)\,[s_0(s_0-4\mu^2)]^{1/2}\left\{\left(\frac{s_1-4\mu^2}{s_0-4\mu^2}\right)^{\alpha}\theta(s_0-s_1)+\frac{s_0}{s_1}\,\theta(s_1-s_0)\right\}, \tag{59}$$

where α is the effective value of the position of the pole at $t = 0$. The possibility of such a "one-pole" form is justified by the fact that in nonperipheral $\pi\pi$ scattering the P' or f trajectory must predominate, and it is natural to regard all of the other contributions as small corrections. Accordingly we may assume that $\alpha \approx 0.5$–0.6. Incidentally, it will be seen further on that the result (i.e., the value of s_0) is not very sensitive to the choice of the exact value of α. When we use (38b) and (59) to calculate the function $\bar{A}_1$ in explicit form and substitute it in (45), we arrive at the following equation [we shall discuss the effect of the fact that (45) actually involves the function $\bar{A}_1$ off the mass shell at the end of this section]:

$$R_1(p^2) = \frac{\sigma(s_0)\,s_0^2}{16\pi^3(1+\alpha)}\int_0^\infty dk^2 R_1(k^2)\left\{\frac{F(2,\,1+\alpha;\,2+\alpha;\,-s_0/(k^2+p^2+4\mu^2))}{(k^2+p^2+4\mu^2)^2}\right.$$
$$\left.+(1+\alpha)\left[\frac{1}{(k^2+p^2)^2}\ln\left(1+\frac{k^2+p^2}{s_0}\right)-\frac{1}{k^2+p^2}\,\frac{1}{s_0+k^2+p^2}\right]\right\}, \tag{60}$$

where $F(2, 1+\alpha, 2+\alpha; x)$ is the hypergeometric function. It is clear that we can determine the quantity s_0 from the condition for solvability of this equation. For an approximate and incomplete calculation we equate the trace of the kernel of Eq. (60) to unity and recall that according to arguments given earlier $\bar{\sigma}(s_0) \approx \mu^{-2}$. The result is ‡

$$s_0/\mu^2 \approx 32\pi^3\alpha/[1+\alpha-(\pi\alpha/\sin\pi\alpha)\,(4\mu^2/s_0)^{\alpha}]. \tag{61}$$

When we substitute in (61) $\alpha \approx 0.5$–0.6, i.e., a value close to that corresponding to the P' trajectory, we see that $s_0 \approx 7$–$8\ (\text{GeV})^2$. At the same time it is clear that the average value $\bar{\sigma}_0$ of the quantity $\sigma(s)$ in the range $s < s_0$ is much larger than μ^{-2}. Using the expression (59) and the fact that $\bar{\sigma}(s_0) \approx \mu^{-2}$, we readily verify that $\sigma_0 \approx 3\mu^{-2}$. Knowing the values of s_0 and σ_0, we can use (57a) to find the value of k_0^2—the effective parameter of the cutoff with respect to k^2. We get $k_0^2 \approx 1.2$–$1.5\ (\text{GeV})^2$. One can arrive at the same result[60b] by directly analyzing Eq. (60).

Here it is appropriate to point out that this estimate of s_0 agrees well with the estimates that are obtained as the result of analyses of the experimental data on elastic scattering on the basis of finite-energy sum rules[61] (FESR) and the interference model.[62] It is well to discuss the latter method in a bit more detail. If in first approximation we represent the elastic scattering amplitude as the sum of the contributions of resonances in the s channel and the contribution from the exchange of a vacuum pole in the t channel, then a direct analysis of the experimental data on πp scattering at energies of the order of 2 GeV and higher shows[62] that the irregularities of the amplitude caused by the resonances in the direct channel show up strongly only at energies $E_{\text{lab}} \lesssim 3$–$4$ GeV. This means, first, that the entire nonperipheral interaction can be described fundamentally by resonances in the direct channel, and second, that the maximum masses of the resonances are of the order of 2.5–3 GeV. We have arrived at this conclusion by relying on the results of comparison of the theory with the experimental data on πp scattering. Unfortunately, it is so far impossible to analyze the $\pi\pi$ interaction in a similar way. However, we see no grounds for expecting a large qualitative difference between the main properties of these interactions. Accordingly, it is natural to expect that a valid upper bound on s_0 is $s_0 \lesssim 10\ (\text{GeV})^2$.

Let us now consider what the effect on these results must be when one takes into account the dependence of the function $\bar{A}_1$ on p^2 and k^2. Since for $p^2, k^2 > 0$, i.e., in the spacelike region, the amplitude $\bar{A}_1$ must be a decreasing function of the variables p^2, k^2, it is clear that including this dependence can lead only to an increase of the value of s_0. At the same time we see no reasons to expect a very strong decrease of the function $\bar{A}_1$ in the range $p^2, k^2 \ll k_0$, since none of the ways now known to approach the description of the strong interactions, including that expounded here, involves a corresponding parameter.* Accordingly, the value of the quantity s_0 which we have obtained is a lower bound, and the value of k_0^2 is therefore an upper bound, but the true values should not differ much from these estimates. Arguments based on the interference model also indicate that this is so.

It also follows from the treatment we have given that in a certain sense the vacuum singularity can be included in the framework of the dual scheme, if we let it correspond to the multiple production of all possible sets of resonances in the s channel, unlike the nonvacuum trajectories, which are usually taken to correspond to the set of single resonances. In this connection it is not surprising that its properties are essen-

*At any rate this is a property of all nonexotic trajectories with positive G parity, and in particular of the P′ and f trajectories, in which we shall be primarily interested.

†It must be pointed out that because of the presence of branch points the kernel also contains another contribution, which decreases much more slowly with increasing s, (see [54] and Sec. 1 of Chapter VI). This contribution, important in principle, is apparently numerically small and unimportant for the questions treated here.

‡Actually this approximation is very good within the limits of accuracy to which the analysis in question can pretend. [60b]

*Evidently all the experimental data show reliably that the quantity μ^2 cannot be such a parameter.

tially different from the universal properties of all the other trajectories.

3. Fireballs

We shall now explain the physical meaning of our results. For this purpose let us examine the iterative solution of Eq. (25). In terms of diagrams it is of the form shown in Fig. 5, where the terms correspond to the first, second, third, and so on, iterations of (25). The transition from (25) to the equation for the imaginary parts, (39), and that for the total cross sections (42), corresponds to letting the particles in the intermediate states be on the mass shell, so that we go from Fig. 5 to the diagrams of Fig. 6, which are topologically the same as those considered in the AFST model.[22] To the blocks in these diagrams there correspond irreducible parts described by the quantities $\bar{\sigma}(s_i, p^2, k^2)$. It can be seen from the diagrams that the production of the particles can be regarded as occurring in the "blocks", i.e., in centers of emission which are connected with each other by only one meson line. The physical meaning of the parameters s_0 and k_0^2 is that they describe the effective squared masses of the blocks and the squares of four-momentum transfers from block to block. It follows from what we said earlier that the "masses" of the blocks must remain bounded for $s \to \infty$ (though indeed rather large, of the order of $s_0^{1/2}$), as must also the squares of the four-momenta transfered between blocks.

Physically the blocks are separated because of the comparatively large value of their relative γ factor (this question was discussed in Chapter V).

These properties of the emission centers—their bounded masses of a characteristic size (several GeV), and the order of magnitude of the k^2 connecting neighboring centers—coincide with the properties of the fireballs, the clumps of pion matter with bounded mass of which we spoke in the Introduction (Chapter I). Therefore hereafter we shall use the term "fireball" to denote such centers of the emission of particles. Here it must be kept in mind that in the scheme under consideration the emission centers do not arise as independent objects, with properties not depending on the interaction. On the contrary, the properties of the fireballs are completely determined by the peripheral character of the process, namely by the fact that the interaction between them is due to the exchange of one meson. Therefore it would be correct to choose a term defining the process as a whole, and not its separate features. In other words, we cannot define what a fireball is independently of the way it is "prepared." It would be more logical to speak of processes of the fireball type. Nevertheless, following an established tradition, we shall hereafter use the term "fireball", though keeping in mind its inadequacy.

The quantity $\bar{\sigma}$ which describes an irreducible block comprises, in particular, the total cross section for elastic scattering; that is, we must also consider, for example, the diagram shown in Fig. 7, a. If an elastic scattering is due to the exchange of a reggeon, then this diagram is equivalent to that shown in Fig. 7, b. Joining the pions with the nearer of the colliding particles, we see that this process can also be simultaneously interpreted as diffractive production of particles (an inelastic process without exchange of quantum numbers between the blocks). Since, however, the elastic cross section is much smaller than the inelastic, we shall still use the term "fireball."

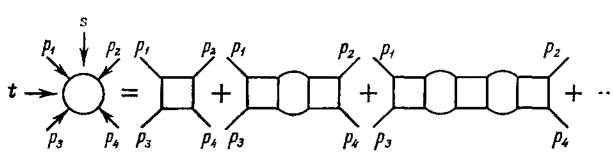

FIG. 5. Iterative solution of the Bethe-Salpeter equation.

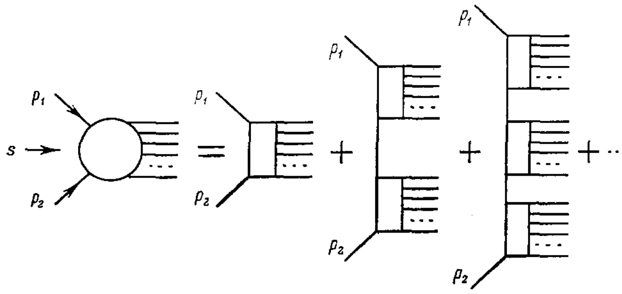

FIG. 6. Iterative solution of the equation for the inelastic processes.

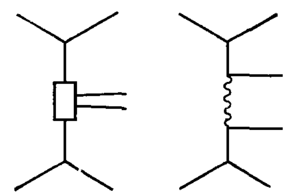

FIG. 7. Contribution of elastic processes to an irreducible block.

Thus one of the main propositions of our theory, following from rather general assumptions, is that in peripheral interactions there must arise massive centers of particle emission—fireballs.

We also note here that from the fact that the mass of a fireball and the square of the four-momentum imparted to it are bounded it follows that the number of fireballs will increase logarithmically with increasing energy at high energies. Here all of the calculations are analogous to those done in[22] (for more details see[23]).

4. The Spin of Fireballs

This is also a proper place to discuss the question as to what sort of angular momenta contribute to the fireball-blocks. For this we note that a wide range of energies must contribute to the integral in (42), and in this entire region the cross section σ_0 of the nonperipheral reactions must be large:

$$\sigma_0 s_0 \sim 50\pi^3 \gg 1. \tag{62}$$

This estimate was obtained above on the assumption that $\sigma_0 \sim 3/\mu^2 = \text{const}$; in the case of an energy-dependent σ_0 the integral

$$\int_{4\mu^2}^{s_0} \sigma_0(s)\, ds \approx \frac{s_0}{\mu^2} \approx 50\pi^3 \gg 1. \tag{63}$$

must satisfy the analogous condition.

It follows that the number of partial waves taking part in the formation of the block must also be large.

This assertion is based on the following arguments. The largest partial cross section with angular momentum l allowed by the unitarity condition is

$$(\sigma_0^l)_{\max} = \pi(2l+1)/p^2 = (2l+1)\cdot 4\pi/(s-4\mu^2), \tag{64}$$

i.e., for given l it falls off with increasing s. At the same time the range of the strong interactions is finite and given by $r_0 \sim \mu^{-1}$. Therefore a strong interaction characterized by the angular momentum l can occur provided the momenta $|p|$ of the colliding particles are sufficiently large: $|p| r_0 \gtrsim l$, that is, if $|p| \gtrsim 1\ \mu$. For smaller s the interaction practically does not occur, and for larger s it falls off according to (64). Let us estimate the number of partial waves that must contribute effectively to the cross section in order that the necessary condition (63) be satisfied. In the range from $|p|_{\min} = 1\ \mu$ to $|p|_{\max} = [(s_0/4) - \mu^2]^{1/2}$ the contribution of the wave with a given l* will be

$$\int_{4\mu^2}^{s_0} (\sigma_0^l)_{\max}\, ds \approx 4\pi(2l+1)\ln\left(\frac{s_0}{4l^2\mu^2}\right) \qquad \left(0 < l \leqslant \frac{s_0^{1/2}}{2\mu}\right). \tag{65}$$

Even this maximum value, for any l in the range from 1 to $s_0^{1/2}/2\mu$, is smaller than the value $50\ \pi^3$ which follows from (63). Setting $s_0 \sim 10\ \text{GeV}^2$ in (65), we see that the inequality

$$16\pi^3 = \int_{4\mu^2}^{s_0} \sigma_0\, ds \leqslant \sum_{l=1}^{l_{\max}} \int_{4\mu^2}^{s_0} (\sigma_0^l)_{\max}\, ds = \sum_{l=1}^{l_{\max}} 4\pi(2l+1)\ln\left(\frac{s_0}{4l^2\mu^2}\right)$$
$$\approx \sum_{l=1}^{l_{\max}} 4\pi(2l+1)\ln\left(\frac{120}{l^2}\right). \tag{66}$$

must hold. This inequality is satisfied only for $l_{\max} \gtrsim 3$, i.e., when at least three partial cross sections contribute to the effective cross section. These estimates show that the presence of a set of waves is essential in order to satisfy the conditions (62) and (63). Actually the effective number of partial waves must be larger, since replacing the partial cross sections by their maximum values decidedly increases the right side of the inequality, because ordinarily none of the partial cross sections attains the limit set by the unitarity condition (cf. e.g.,[63]). In practice the number of partial waves contributing to the cross section is of the order of $L \sim s_0^{1/2}\ \mu \sim 10$. However, the important point here is that a fireball cannot be characterized by a given angular momentum.

It follows from these considerations that a fireball in the form in which it appears in the theory can certainly not be associated with any specific boson resonance, since there is no resonance with a given angular momentum l that can give the required contribution to the integral (63). Moreover, according to the duality hypothesis (see Chapter VII, Sec. 2) a fireball can be thought of as the set of all resonances, "weighted" according to the Bethe-Salpeter dynamical equation.

*We do not consider the s wave here. Including it would not change our conclusions, but would make all of the calculations more cumbersome.

5. The Preasymptotic Behavior of the Cross Sections

Having written the kernel of Eq. (42) in the form (54), we can determine the asymptotic value of the cross section for peripheral interactions according to (48). For $k_0^2 = s_0$ Eq. (47) can be rewritten in the form

$$R_1(p^2) = \frac{\sigma_0}{16\pi^3} \int_0^{k_0^2 = s_0} dk^2\, [k^2 D(k^2)]^2 \ln\left(\frac{s_0 + p^2 + k^2}{4\mu^2 + p^2 + k^2}\right) R_1(k^2). \tag{67}$$

The eigenfunctions R_1 and the eigenvalues σ_0 have been calculated with a computer for three different values of s_0 in Eq. (67). σ^P and σ_{tot} were determined from (48) and (49). The results are shown in the table.

Values of the Cross Sections for Various Choices of the Parameter s_0

$s_0 = 250\ \mu^2$			$s_0 = 400\ \mu^2$			$s_0 = 600\ \mu^2$		
σ_0	σ_∞^P	σ_{tot}	σ_0	σ_∞^P	σ_{tot}	σ_0	σ_∞^P	σ_{tot}
57	66	66	35.4	43	43	23	30	30

We see that the parameter s_0 must lie in the range $5 \lesssim s_0 \lesssim 12\ \text{GeV}^2$ if $\sigma_0 \sim (3-1)\mu^{-2}$. The most important result is that at asymptotically high energies the total cross section is found to be larger than its value at low energies for all these values of $s_0 (\sigma^P > \sigma_0)$. Therefore there must be a preasymptotic range of energies where this cross section increases. We shall show that this increase is monotonic. To do so we differentiate (42) with respect to s. At large energies $[(s_0/s)^2 \ll 1]$ we get

$$\frac{d\sigma(s, p^2)}{ds} = \frac{d\bar{\sigma}(s, p^2)}{ds} + \frac{1}{8\pi^3 (s + p^2 - \mu^2)^2} \int_{k^2_{\min}}^{k^2_{\max}} \frac{dk^{2*}}{(k^2 + \mu^2)^2}$$
$$\times \int_{4\mu^2}^{s_{1\max}} ds_1 \bar{\sigma}(s_1, p^2, k^2)\, [(s_1 + p^2 + k^2)^2 - 4p^2k^2]^{1/2}$$
$$\times \int_{s_0}^{s_{2\max}} ds_2\, (s_2 + k^2 - \mu^2)^2 \frac{d\sigma}{ds_2}(s_2, k^2). \tag{68}$$

In the model considered $d\bar{\sigma}/ds \equiv 0$ for $s > s_0$. Equation (68) is a homogeneous Volterra equation with a positive definite kernel. Therefore the eigenfunctions $d\sigma/ds$ must be of definite sign. A numerical solution of Eq. (42) showed that independently of s_0 the total cross section begins to increase at $s \gtrsim 150\ \text{GeV}^2$; that is, the derivative $d\sigma/ds$ is positive in this region. Consequently, it must be positive in the entire region in question, so that the cross section of the peripheral processes increases monotonically.

In our model this also leads to a preasymptotic increase of the total cross section. In the general case, however, we have not been able to prove this with the given approach. Simple considerations of the predominance of the peripheral processes allow us to suppose that the increase will also be found for the total cross section.

We note that in the Regge model of weak coupling[64] the conclusion that there is a preasymptotic increase has been proved precisely for the total cross section. In that case it comes from the fact that the contributions from the branch points in the l plane decrease with increasing energy, and that the main correction to the asymptotically constant total cross section is negative. One can also set a lower limit on the size of the correction term. It turns out that in the preasymptotic region the cross section behaves in the following way:[64]:

$$\sigma_t(s) \approx \sigma_{tot}(\infty)\,[1-(B/\ln s)], \tag{69}$$

where $B > \sigma_{tot}(\infty)/32\pi\alpha'$, and α' is the slope of the leading trajectory.

On this matter there are interesting experimental results obtained in[65], which indicate that the total cross sections for interaction of cosmic-ray particles with nuclei increase in the energy range from $2\cdot 10^{10}$ to 10^{12} eV.

6. The Average Multiplicity

In the model under consideration the average number of irreducible blocks (fireballs) at high energies is determined in exactly the same way as in[22]:

$$\mathfrak{N} = (d\alpha/d\lambda_\alpha)_{\lambda_\alpha=1,\ \alpha=1} \ln(s/2\overline{k^2}), \tag{70}$$

where λ_α is an additional variable parameter introduced by replacing $\overline{\varphi}_\alpha$ by $\lambda_\alpha\overline{\varphi}_\alpha$ in Eq. (32), where f_α is replaced by φ_α according to (34a), and (34b) (for more details see Appendix). The quantity $d\alpha/d\lambda_\alpha$ can be found by the method expounded in the Appendix, and is given by

$$(d\alpha/d\lambda_\alpha)_{\lambda_\alpha=\alpha=1} = \sigma^P\mu^2/16\pi^3 R_{10}^2(-\mu^2), \tag{71}$$

where the index 0 indicates that R_1 is normalized to unity. The average multiplicity $\overline{N}$ in peripheral processes is $\overline{N} = \mathfrak{N}\overline{n}$, where $\overline{n}$ is the average number of particles produced in the decay of a fireball.

For various values of the parameter s_0 the numerical coefficients in (71) are as follows:

$$\overline{N} = \left\{ \begin{array}{lll} 6.0\,\lg(s/2\overline{k^2}) & \text{for} & s_0 = 5\ \text{GeV}^2 \\ 6.8\,\lg(s/2\overline{k^2}) & \text{for} & s_0 = 8\ \text{GeV}^2 \\ 7.7\,\lg(s/2\overline{k^2}) & \text{for} & s_0 = 12\ \text{GeV}^2 \end{array} \right\} \tag{72}$$

Accordingly, the average multiplicity in peripheral processes increases logarithmically with the energy,[22] the coefficients in the model considered being given by (72).

The quantities $\overline{n}(s_0)$ are obtained from calculations with the statistical theory[31]: $\overline{n}(s_0) \approx 2(s_0/m^2)^{1/4}$. The best description[66] of the data on $\overline{N}(s)$ obtained with accelerators and with cosmic rays is given by using the following values of the parameters s_0 and $\overline{k^2}$ in (72): $s_0 \approx 8$ GeV^2, $\overline{k^2} \approx 2$ GeV^2 (Fig. 8).

The models we have considered are of course only a first approximation to reality, enabling us to understand the main features of the processes and to estimate their parameters. To carry out a comparison with experiment it is necessary, first, to examine the experimentally observed processes in proton-proton and pion-proton collisions, and, second, to examine more realistic models for $\overline{\sigma}$ (for example, to take into account the structures of low-energy resonances, and so on). We shall now discuss the problems that arise in the use of the simplest model for real processes.

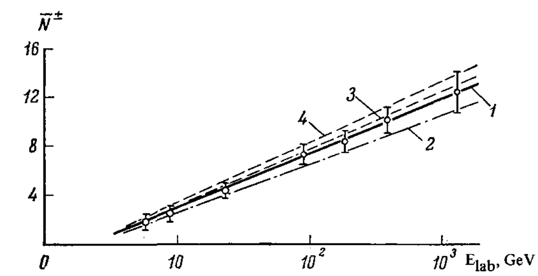

FIG. 8. Comparison of the theoretical predictions about the multiplicity with the experimental data. Since the experimental data include only the charged pions, for the comparison the right member of (72) has been multiplied by 2/3. A relative lowering of the theoretical estimates may be due to the fact that some of the pions are produced as the result of the decay of isobars. The straight line 1 is drawn through the experimental points; the calculated curves 2–4 correspond to values s_0 = 5, 8, and 12 GeV^2.

VIII. THE INTERACTION OF PROTONS WITH PROTONS AND PIONS

We have been considering the case of the interaction of neutral pseudoscalar particles. From the experimental point of view, however, the main interest is in the interaction of protons with protons and pions at high energies. If we assume, as is now customary, that at high energies the interaction does not depend on the isospin state of the colliding particles, the effects of spin rotation are negligibly small, and all of the amplitudes without spin reversal are equal, then the pp and πp scatterings will each be characterized by a single amplitude. Therefore in the study of such processes we must consider the following system of equations for the partial amplitudes[23] [besides Eq. (35) for the $\pi\pi$ scattering]:

$$\varphi_{\pi\pi\to p\bar{p}}(l, p^2) = \overline{\varphi}_{\pi\pi\to p\bar{p}}(l, p^2)$$
$$+\left\{\begin{array}{l} \dfrac{2^{2l}\pi^{1/2}\Gamma\left(l+\frac{3}{2}\right)}{(2\pi)^3\,\Gamma(l+2)} \displaystyle\int_0^\infty dk^2\,(k^2)^{l+1} D^2(k^2)\,\overline{\varphi}(l, p^2, k^2)\,\varphi_{\pi\pi\to p\bar{p}}(l, k^2), \qquad (73) \\ \dfrac{2^{2l}\pi^{1/2}\Gamma\left(l+\frac{3}{2}\right)}{(2\pi)^3\,\Gamma(l+2)} \displaystyle\int_0^\infty dk^2\,(k^2)^{l+1} D^2(k^2)\,\varphi(l, p^2, k^2)\,\overline{\varphi}_{\pi\pi\to p\bar{p}}(l, k^2), \qquad (74) \end{array}\right.$$

where $\varphi \equiv \varphi_{\pi\pi\to\pi\pi}$, and

$$\varphi_{p\bar{p}\to p\bar{p}}(l) = \overline{\varphi}_{p\bar{p}\to pp}(l) + \frac{2^{2l}\pi^{1/2}\Gamma\left(l+\frac{3}{2}\right)}{(2\pi)^3\,\Gamma(l+2)} \int_0^\infty dk^2\,(k^2)^{l+1} D^2(k^2)\,\overline{\varphi}_{\pi\pi\to p\bar{p}}(l, k^2)\,\varphi_{\pi\pi\to p\bar{p}}(l, k^2). \tag{75}$$

In a corresponding way we can also rewrite the system of equations for the total cross sections, associated with Eq. (42). In reality only (35) and (73) are equations, and (74) and (75) are simply relations between the amplitudes of the different processes (the irreducible blocks are regarded as given).

We shall now present the results that are obtained on the basis of these ideas in the treatment of a concrete process of the one-fireball type in pp collisions.[66] As we shall show below, processes of this type will appear in the energy range $30 \lesssim E_{lab} \lesssim 500$ GeV. For this process we have the three diagrams shown in Fig. 9. According to the optical theorem the contribution to the total cross section

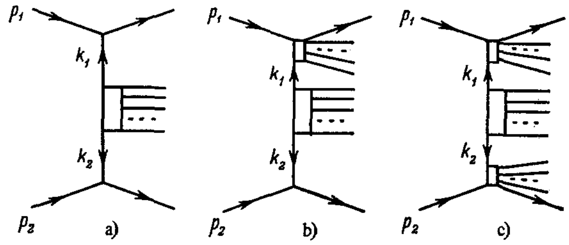

FIG. 9. Diagrams for the calculation of pp scattering.

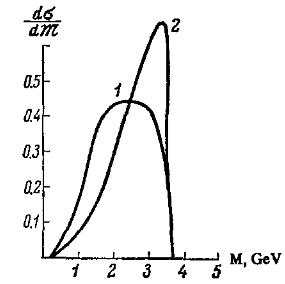

FIG. 11. Mass distribution of the fireballs.

caused by each of these diagrams is determined by the second iteration of the Bethe-Salpeter equation and can be written in the form

$$\sigma_{pp}^{(2)}(s)=\frac{4}{(8\pi^3)^2 s}\int\frac{ds_1\,ds_2\,dk_1^2\,dk_2^2\,ds_3\,ds_4}{(k_1^2+\mu^2)^2(k_2^2+\mu^2)^2}R(s_1,k_1^2)\,\bar{\sigma}_{\pi p}(s_1,k_1^2)$$
$$\times R^{-1}(s_2,k_1^2)\,R(s_3,k_1^2,k_2^2)\,\bar{\sigma}_{\pi\pi}(s_3,k_1^2,k_2^2)\,R(s_4,k_2^2)\,\bar{\sigma}_{\pi p}(s_4,k_2^2), \quad (76)$$

where

$$s_1=-(p_1+k_1)^2,\quad s_2=-(p_2-k_1)^2,\quad s_3=-(k_1+k_2)^2,\quad s_4=-(p_2+k_2)^2, \quad (77)$$

$$R(s_i,p^2,k^2)=[(s_i+p^2+k^2)^2-4p^2k^2]^{1/2},\quad R(s_i,k^2)\equiv R(s_i,-\mu^2,k^2), \quad (78)$$

and the regions of integration are determined by the relation (49) applied successively to the appropriate sets of variables. It is clear that the quantities to be substituted for $\bar{\sigma}_{\pi p}(s_i, p^2, k^2)$ in (76) are given in case a by the expression

$$\bar{\sigma}_{\pi p}(s_i,p^2,k^2)=\pi G^2k^2\delta(s-m^2)/[(s_i+p^2+k^2)^2-4p^2k^2]^{1/2}, \quad (79)$$

while in case b one of the quantities $\bar{\sigma}_{\pi p}$ must be determined according to (79) and the other must be set equal to twice the cross section for the nonperipheral πp interaction, since this diagram must be included twice, and, finally, in case c both factors $\bar{\sigma}_{\pi p}$ are equal to the cross section for the nonperipheral πp interaction. In the concrete calculations it has been assumed that this last cross section is equal to σ_0 in the region $s < s_{1,4} < 7\ \text{GeV}^2$ and is zero outside this region; that is, it has been assumed that practically the entire πp interaction is due to the region of the resonances (i.e., $\bar{\sigma}_{\pi\pi}$ and $\sigma_{\pi p}$ are quantities of the same order of magnitude).

On this basis the cross sections of these processes were calculated as functions of the energy (Fig. 10), and also the following characteristics of the processes at the energies where the cross sections have their maxima, i.e., at 40, 70, and 250 GeV:

1. The mass distribution of the fireballs, which is obtained if we omit the integration over s_3 in (76)

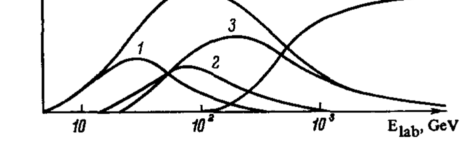

FIG. 10. Energy dependences of the cross sections for the processes shown in Fig. 9, a, b, c (curves 1, 2, 3, respectively). The total cross section is shown as curve 4. The cross section for two-fireball processes is shown by the solid curve without a number. The dashed line shows approximately the asymptotic value of the cross section for peripheral processes.

(Fig. 11). It must be emphasized that the specific shape of this distribution is very sensitive to the particular model. For example, the sharp upper limit of this distribution is altogether due to the fixing of $\bar{\sigma}$ in the form (54), and cannot be justified outside the framework of this model.

2. The distribution with respect to the square of the momentum transfer k_i^2, which is obtained if we omit the integration over the appropriate k_i^2 in (76). (Fig. 12).

3. The distributions of the inelasticity coefficients and the average values of the γ factors of the nucleons and isobars in the c.m.s. The inelasticity coefficient K of a nucleon is defined as

$$K=(s_2-m^2)/s, \quad (80)$$

and the inelasticity coefficient R of an isobar is defined as the fraction of the original energy of the nucleon that goes into the production of secondary particles, excluding those that are produced as the result of decay of the isobar:

$$R=(s_2-s_1)/s. \quad (81)$$

The corresponding γ factors are defined by the formulas: for a nucleon

$$\gamma_N=(s+m^2-s_2)/2ms^{1/2}=(1-K)\,s^{1/2}/2m \quad (82)$$

and for an isobar

$$\gamma_R=(s+s_1-s_2)/2(ss_1)^{1/2}=\frac{1-R}{2}\left(\frac{s}{s_1}\right)^{1/2}. \quad (83)$$

The distributions of the inelasticity coefficients are shown in Fig. 13, and the corresponding average values $\bar{\gamma}$ of the γ factors can be found easily by using Eqs. (80)–(83).

One finds that $\bar{\gamma}_N \approx 2.5$ at $E_{lab} = 40$ GeV, $\bar{\gamma}_N \approx 3.6$ at $E_{lab} = 70$ GeV, and $\bar{\gamma}_R \approx 4$ at $E_{lab} = 250$ GeV.

4. The average value $\bar{\gamma}_f$ of the γ factor of the fireball was calculated; it is given by the expression

$$\gamma_f=\left\{\frac{s_2+s_3-s_4}{2\sqrt{s_2}}+\left[\frac{(s+s_2-s_1)^2-4ss_2}{s+s_2-s_1}\cdot\frac{(s-s_1-s_2)^2-4s_1s_2}{4s}\right]^{1/2}\right.$$
$$\left.\times\frac{2(k_1^2-s_3-k_2^2)\,s_2+(s_2-k_1^2+p_2^2)(s_2+s_3-s_4)}{[(s_2+k_1^2+p_2^2)^2-4k_1^2p_2^2]^{1/2}\,[(s_2-s_3-s_4)^2-4s_3s_4]^{1/2}}\right\}\frac{s+s_2-s_1}{2(ss_2s_3)^{1/2}}. \quad (84)$$

Here the average values, as found from the calculations

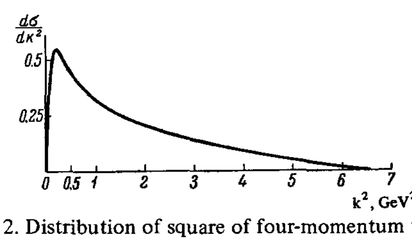

FIG. 12. Distribution of square of four-momentum transfer.

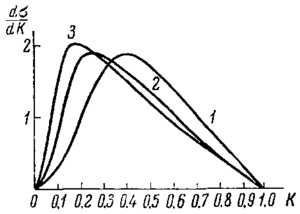

FIG. 13. Distribution of the inelasticity coefficients.

already mentioned, were substituted for the quantities s_i, k_i^2, and p_i^2.

Let us briefly discuss these results.

As was to be expected from simple kinematic considerations, as the energy increases the one-fireball processes of types a, b, and c replace each other successively, having their maxima at the respective energies 40, 70, and 250 GeV. With further increase of the energy they must be replaced by a two-fireball process. According to the results of preliminary calculations the maximum of the cross section of the two-fireball process without excitation of the nucleons comes at an energy of the order of 250 GeV. The total cross section of the one-fireball processes attains a value of the order of σ_0 (curve 4 in Fig. 10), so that at the energies in question these processes must make the main contributions to the total cross section for pp interaction. At the same time we see that at present accelerator energies (with the exception of the Serpukhov accelerator, at which experiments are only beginning) the fireball interaction mechanism is inappreciable. Meanwhile the analysis of the experimental data obtained at energies ~ 20 GeV already indicates the present of a small fraction of events which can be interpreted as manifestations of the one-fireball mechanism of particle production.[8,9] As for experiments with cosmic rays, as has already been mentioned they also permit us to speak of the existence of one and two-fireball (and perhaps even three-fireball) processes, although of course the accuracy of these data is far from that which is necessary for a definite conclusion on this question.

From the mass distributions it can be seen that the average mass of a fireball turns out to be of the order of 3 GeV and is in qualitative agreement with experimental results.[1-9] At the same time it is clear that at low energies, i.e., near the "threshold" for these processes, the masses of the fireballs must be somewhat smaller.

The results of theoretical calculations given here show that the distribution of k^2 is practically independent of the size of the total energy of the interaction and has a maximum at $k^2 \approx 0.5$ GeV^2, but the effective values are somewhat larger, of the order of $1-2$ GeV^2. Unfortunately, the details of this distribution are rather sensitive to the choice of a model.

The average value of the inelasticity coefficient K which is obtained from the distributions shown in Fig. 13 is of the order of 0.4 and is practically independent of the energy. This also agrees with the data from cosmic-ray experiments.

Finally, the value found for the average γ factor of a fireball, $\bar{\gamma}_f \approx 1.15$, is also not in contradiction with the experimental data, according to which[2,5] $\bar{\gamma}_f$ is about $1.1-1.2$.

Accordingly we can say that all of the results obtained on the basis of the fireball model for processes of the one-fireball type are as a whole in agreement with the very scanty experimental data available at present.

IX. THE CONNECTION BETWEEN DIFFERENT METHODS FOR DESCRIBING INELASTIC PROCESSES AT HIGH ENERGIES

As already pointed out, the method described above for theoretically describing inelastic interactions is suitable for application only in the energy range where the parameters used are stable.* At the same time there are many other models of inelastic processes at high energies. The ones most often used are the following: 1) the statistical (thermodynamic)[12,30,31] 2) the hydrodynamic method,[11] 3) the method of uncorrelated jets,[26,27,42] 4) the many-reggeon method.[24,25,67,68]

The various models claim to describe processes of different types and in different energy ranges. Therefore it is interesting to consider briefly a possible interconnection of all these models.

In accordance with the postulates on which the first three models are based, they should most readily be applied to describe nonperipheral interactions in the framework of the Bethe-Salpeter equation. Furthermore it evidently makes sense to apply the statistical model to interactions at rather low energies, i.e., to describe the decay of a fireball ($s \lesssim 10$ GeV^2). The hydrodynamical model can be applied only at very high energies ($s \gtrsim 200$ GeV^2). Therefore it has nothing to do with the decay of fireballs, and at such energies it can describe only the inhomogeneous term in the Bethe-Salpeter equation.

The region of applicability of the model of uncorrelated jets is still not really clear. An important point, however, is that in the form developed in[27] and[45] it is a certain extension of the statistical model, in which the decrease of the cross sections $\bar{\sigma}$ with increase of virtuality, which was assumed ad hoc in the model considered above [Eq. (54)] appears as the result of the limited momentum transfers in nonperipheral interactions.

The connection between our present scheme and the Regge-pole model must be discussed in more detail.

In what follows we shall mean by the many-reggeon scheme a description of inelastic processes in which there is in general exchange of arbitrary reggeons between groups of particles.

If we consider the exchange of vacuum reggeons only, such processes are different from the multiperipheral processes. A consistent analysis of them has been made in[24]. However, the region of phase volume accessible to them is small. Experimentally they appear as diffraction inelastic processes with small cross sections. Owing to this there have been attempts[25,67,68] to extend many-reggeon theory by considering the exchange of other reggeons and the production of particles in groups, in order to apply the method in the

*Its application at energies of the order of 10 GeV is possible for the phenomenological treatment of experimental data, with no claim to derive predictions for other energies (since here the parameters can depend on the energy).

entire phase space. Such an extension, however, leads in practice to a transition from the many-reggeon scheme into the multiperipheral scheme in the main part of the phase space.

1. The Role of Meson Exchange

The interrelation of the multiperipheral scheme and the Regge pole model manifests itself in the topological equivalence of the diagrams describing inelastic processes.[22,24] The main difference between the diagrams in the many-reggeon model and those shown in Fig. 6 is that the exchange involves not a pion, but some sort of reggeon. Therefore in the corresponding analytic expression there corresponds to an internal line, for example to a particle with positive signature, not the propagator $D(k^2)$, but a factor $I(t)$:

$$D(k^2) \to |I(t)| (\pi\alpha'(0)/2) = (\pi\alpha'(0)/2) \sin^{-1}(\pi\alpha_i(t)/2),$$

$\alpha_i(t)$ is the trajectory of the i-th Regge pole (the index i stands for both the order number of the reggeon in the chain and for the character of its trajectory).

Besides this, typical Regge factors of the form $z^{\alpha_i(t)}$ appear, where the quantities z are the corresponding cosines of scattering angles in the t channel. For example, for the one-reggeon diagram topologically equivalent to Fig. 3, a, with formations of groups of particles with "masses" $s_1^{1/2}$ at the upper point and $-s_2^{1/2}$ at the lower point, and with momentum transfer $|t|$, the quantity z is of the form

$$z \approx (2|t|s/s_1s_2) - 1. \tag{85}$$

The Regge approach, i.e., the introduction of factors of the type $s^{\alpha_i(t)}$, is justified when the quantity $|z|$ is large: $|z| \gg 1$. In this case the exchange of a vacuum trajectory is singled out.[24]

It can be seen from (85) that $|z|$ is large only if the total energy $s^{1/2}$ not only is much larger than the "masses" $s_{1,2}^{1/2}$ of the blocks that are produced, but also offsets the influence of the small ratio of the momentum transfer to the mass of a block. The region of phase volume for this is small.

It has been shown in[69] that in the main part of the phase space for inelastic processes the quantity z is of the order of unity. In particular, for "forward" inelastic scattering, i.e., at the boundary of the phase volume, we have $|t|_{min} \approx s_1s_2/s$ and $|z| = 1$.

Therefore in the main part of the phase volume the Regge factors "do not prefer" any particular Regge trajectory. At the same time the signature factors single out (numerically!) precisely the pion trajectory, since at the poles they reduce to propagators, i.e., for small $|t|$ they single out the pion pole* as the one nearest to the physical region.[69] Consequently it is precisely in the main part of the inelastic processes that the many-reggeon theory should be applicable.

In the inelastic processes (unlike the elastic processes) the exchange of vacuum reggeons is not of primary importance.[69,24,68] They are important only where $|z| \gg 1$, or for processes of resonance production (with masses $s_1^{1/2}$ and $s_2^{1/2}$ independent of the energy in the one-reggeon scheme), even if $|z| \sim 1$. This last assertion is based on the study of the four-dimensional type of reggistics in[70,71], where an expansion of the amplitude for the process in question in terms of irreducible representations of the four-dimensional rotation group was carried out. Then on the assumption that a Lorentz pole exists (i.e., the set of a main pole and all its daughter poles in the l plane) it can be shown that the asymptotic form of the amplitude for such a quasielastic process is given by the usual Regge formula

$$T \sim I(t)\, g(t, s_1)\, g(t, s_2)\, [s/(s_1s_2)^{1/2}]^{\alpha(t)}. \tag{86}$$

Near the boundary of the phase volume ($|z| \sim 1$) this is due to the behavior of the residues of the partial amplitude, and not to the asymptotic behavior of the spherical functions.[60]

However, neither the region $|z| \gg 1$ nor the processes of resonance production can give an asymptotically constant contribution to the total cross section,* and therefore the exchange of vacuum reggeons is not the dominant mechanism in inelastic interactions at high energies.

2. The Grouping of Particles

The contribution to the cross section for processes with exchange of a reggeon and production of only one particle in each irreducible block is thus even smaller. This is the reason that in comparing a concrete phenomenonological version of the reggeon scheme with experiment in the energy range from 5 to 16 GeV it has been necessary to assume[25] that the particles come out from the points of a diagram in groups (clusters) with relatively small total energies in their c.m.s., and that their disintegration is determined by the statistical model. The clusters can be regarded as fireballs that are not completely formed owing to insufficiently high initial energy (the parameter does not attain the value s_0).

3. The Correspondence between the Equations

We would like to emphasize that inclusion of the grouping of particles and the exchange of different reggeons is absolutely unavoidable in attempts at formal application of the many-reggeon approach in the main part of the phase volume, and actually corresponds to a transition to the multiperipheral description.

This can most intuitively be seen from the fact that one can write[72,73] a single multiperipheral equation which, in the main part of the phase volume of inelastic processes, goes over into the equation of multiperipheral processes described in the framework of the Bethe-Salpeter equation, and which also, in the region of applicability of the Regge approach, goes over into the equation for the many-reggeon diagrams that are described by means of the equation of Chew, Goldberger, and Low.[67] This single equation is of the

*For example, at $t = 0$ the ratio of the squares of the signature factors for the kaon and pion trajectories is of the order of $(\mu/m_K)^4 \sim 10^{-2}$.

*With increasing energy the cross sections for these processes fall off at least as $\ln^{-1} s$ (cf. [69,24]).

form

$$B(k_1, p_a, p_b) = \\ = \bar{B}(k_1, p_a, p_b) + \frac{1}{4\pi^4}\int d^4k_3 \bar{A}_1(k_1, k_3) D^2(k_3^2) R^2(k_3, k_1, p_a) B(k_3, k_1, p_b), \quad (87)$$

where B denotes the imaginary part of the elastic scattering amplitude with one integration not yet done (or the imaginary part of the "amplitude" for scattering of a reggeon by a particle), defined by the relation

$$A_1(p_a, p_b) = \frac{1}{4\pi^4}\int d^4k_1 \bar{A}_1(p_a, k_1) D^2(k_1^2) B(k_1, p_a, p_b). \quad (88)$$

This equation holds both in the multiperipheral and in the many-reggeon schemes, but in the former case $\bar{A}_1$ and D^2 are interpreted respectively as the imaginary part of the irreducible block of the elastic scattering amplitude and as the square of the propagator in the Bethe-Salpeter equation (39), and in the latter case, as the squares of the vertex part and of the signature factor in the Chew-Goldberger-Low equation.* R denotes the typical Regge factor

$$R = (z_{a3})^{\alpha(k_1^2)}, \quad (89)$$

where $\alpha(k_1^2)$ is the Regge trajectory that gets exchanged ($\alpha \equiv 0$ in the case of the Bethe-Salpeter equation), and $\bar{B}(k_1, p_a, p_b) = \bar{A}(k_1, p_b) \times R^2(k_1, p_a, p_b)$. The notation for the momenta is clear from Fig. 14 (for $t = 0$ we have $k_1^2 = k_2^2$, $k_3^2 = k_4^2$, etc.). The quantity $|z_{a3}| = (2|t|s_{a3}/s_{a1}s_{13}) - 1$ is the cosine of the scattering angle ($p_a \to k_3$) in the t channel. Sometimes, without justification, people take instead of z_{a3} the quantity s_{a3}/s_0 with $s_0 = \text{const}$.

The dependence on p_a comes into Eq. (87) only through the factor R, but $R \equiv 1$ in the case of the Bethe-Salpeter equation. Therefore in the multiperipheral scheme D does not depend on p_a. The meaning of this is that the correlation length in the peripheral chain is much smaller in the case of exchange of an elementary pion than for the exchange of a Regge trajectory. This fact is very important in going over to the corresponding equations for the partial amplitudes. The analysis of these equations in[73] showed that the condition for their solvability with an asymptotically constant total cross section is the relation

$$\nu + 2\alpha(k^2) < 1, \quad (90)$$

if the asymptotic form of $\bar{A}_1$ is†

$$\bar{A}_1(s_i) \sim s_i^{\nu} \quad \text{for } s_i \to \infty. \quad (91)$$

This makes clear the reasons for the nonselfconsistency of the Pomeranchuk trajectory. It was shown in[54] that the condition (90) was violated in the Bethe-Salpeter equation because the value $\nu = 1$ is not permissible for $\alpha \equiv 0$ (see Chapter VI, Sec. 1). In the Chew-Goldberger-Low model, on the other hand, it was assumed that $\nu \equiv -1$ (the δ-function model for $\bar{A}_1$) but that $\alpha(k^2)$ went to unity at the point $k^2 = 0$ (the Pomeranchuk trajectory). Since for $k^2 > 0$ we have everywhere $\alpha(k^2) < 1$, there are three ways to remove the difficulty in this case:

*In the concrete model considered in [67] the choice was $\bar{A}_1(p_a, k_1) = g^2(p_a^2, k_1^2)\delta_+((p_a - k_1)^2 + m^2)$.

†The case with $\nu < -1$ is equivalent to the case $\nu = -1$.

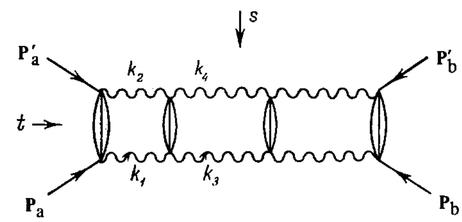

FIG. 14. Elastic scattering caused by multiperipheral (or many-reggeon) processes.

1) by excluding the point $k^2 = 0$ from consideration, requiring that the coupling constant between vacuum reggeons and the particle go to zero at that point[24] (weak coupling model[64]);

2) by assuming that the vacuum trajectory passes below the point $l = 1$, i.e., $\alpha_P(0) < 1$)[54,55,67];

3) by assuming that the singularity at this point is weaker than a pole[54] (cf. Chapter VI, Sec. 1).

A second important conclusion which can be drawn from an analysis of the equation for the partial amplitudes[73] is that after integration over the Treiman-Yang angle[74] the whole difference between the many-reggeon and multiperipheral approaches drops out for any choice of R (among those we have indicated), because the dependence on s_{a3} reduces simply to a dependence on the product $s_{a1} \cdot s_{13}$, i.e., to the choice of definite form-factors at the vertices of the multiperipheral chain.

Consequently, the equation actually reduces to the Bethe-Salpeter equation, despite the fact that formally it takes into account the "reggeization" of the particle that is exchanged.

4. Diffraction Inelastic Processes

For processes in which the masses s_1 and s_2 are fixed Eq. (86) leads to a number of interesting conclusions. For large s the total cross section for the processes caused by exchange of the i-th Regge trajectory is given by[69]

$$\sigma_i \approx C_i |I_i(0)|^2 s^{2(\alpha_i(0)-1)} \ln^{-1} s, \quad (92)$$

where the C_i are constants. It can be seen from (92) that the contributions from all the nonvacuum trajectories, for which $\alpha_i(0) < 1$, fall off with increasing s according to power laws, whereas the exchange of a vacuum reggeon $[\alpha_V(0) = 1]$ leads to a cross section for production of one or two resonances which falls off only logarithmically with increasing energy.

It is, however, interesting to note that the coefficient in (92) is the quantity $|I_i(0)|^2$, which in the case of the pion trajectory is much larger than for the other trajectories (where it is of the order of unity):

$$|I_\pi(0)|^2 \approx 4/\pi^2\alpha_\pi'^2\mu^4 \approx 10^3 \gg 1.$$

Therefore in principle there could be a situation in which at not too high energies pion exchange could be more important for some resonance-production processes than the exchange of a vacuum reggeon. This may be the explanation of the fact that diffraction inelastic processes have appeared clearly only at energy $E_{lab} \sim 20$ GeV. At lower energies it turns out[75] that an analysis of the isospin relations, even for reactions such as $p\pi \rightarrow N\pi\pi$, in which few particles are produced, leads to the conclusion that vacuum-reggeon exchange is not always the main mechanism. Pion exchange must also be taken into account in reactions of the type of $pp \rightarrow pp\pi^+\pi^-$ at energy 16 GeV.[76]

In conclusion we emphasize once again that an analysis of essentially inelastic processes with an attempt to apply the many-reggeon scheme to them has shown that in this case the many-reggeon scheme definitely comes close to the method based on the Bethe-Salpeter equation, since one has to recognize, first, the important role of meson trajectories, and second, the necessity of "grouping" the final particles, both of which follow in a natural way from the Bethe-Salpeter equation.

X. CONCLUSION

We have examined the question of the role of inelastic processes at high energies and of the methods for describing them. By means of the unitarity principle we have shown that inelastic processes predominate at high energies to such an extent that they determine the elastic scattering at any angle. Therefore a theoretical description of elastic scattering is possible only after one has understood the nature of the inelastic processes. At the same time it has turned out that all of the proposed theoretical models of inelastic processes lead to an excessively wide diffraction cone of the elastic shadow scattering. This is evidently due to the fact that in these models one has not taken into account the relative phases of the matrix elements of the inelastic interactions, which play a decisive part in the formation of the diffraction cone.

It is interesting to note that attempts to interpret the elastic scattering in this range of angles by means of Regge poles have led to a similar difficulty: the main contribution to the width of the diffraction cone must be ascribed to the residues at the poles, and not to the term which determines the pole trajectory itself.

The interconnection of these problems can be understood through a simultaneous study of elastic and inelastic processes by means of the Bethe-Salpeter equation. With such an approach we succeed in showing that a knowledge of the absolute values of the matrix elements for the inelastic processes, or, more exactly, of the total cross sections, allows us to study the analytic structure of the elastic scattering amplitude. At the same time light is thrown on many features of inelastic processes at high energies.

Our theory of inelastic processes is based on an exact relation of quantum field theory, the Bethe-Salpeter equation, with the assumption of the absence of interference (we have considered this question in detail in Chapter V). The theory is internally self-consistent, satisfactorily describes the available experimental data on inelastic processes obtained with cosmic rays, and connects the properties of the inelastic process with those of the elastic scattering amplitude. Furthermore the amplitude for elastic shadow scattering has the correct analytic properties and is connected with the inelastic processes by the unitarity condition.

The theory provides a natural explanation of a feature of the inelastic process which in our opinion is extremely interesting—the formation of fireballs. Among the general consequences of the theory are limits on the masses of fireballs and on the squares of the four-momentum transfers, a logarithmic increase of the number of fireballs with the energy, the appearance of a moving singularity in the elastic scattering amplitude owing to the existence of processes of the fireball type, and the conclusion that the assumption of the existence of a Pomeranchuk vacuum pole with $\alpha_P(0) = 1$ at ultrahigh energies is not internally self-consistent. Moreover, the theory leads to a natural connection between peripheral and nonperipheral processes and the automatic appearance of a new energy scale associated with the fireball mass.

Along with this it must be pointed out that the theory is still very crude; the parameters that appear in it are still not fixed accurately enough. In other words, in its present state the theory can predict qualitative effects, but is not yet able to give accurate quantitative results. This level of development has so far been enough for comparison with the data obtained with cosmic rays, but it is inadequate for detailed comparison with more accurate experiments with accelerators. At present, however, there are no such exact data in the required energy range. The point is that the theory was developed for, and is suitable for, the description of processes at very high energies. It seems to us senseless to use it for the description of experiments at energies of the order of 10 GeV. The parameters of the theory become stable and cease to be energy-dependent at energies at which at least one fireball can be produced. According to our estimates this requires at least some tens of GeV. For comparison of the theoretical results with experiments in this energy range it is necessary, first, to get more accurate values of the parameters, and second, to state the theoretical information in a form convenient for comparison with a specific experiment. By refining the parameters to the extent allowed by the basic assumptions and the supplementary conditions, one can obtain models which lead to definite quantitative predictions. The first very simple attempts of this kind are described here in Chapters VII and VIII, At present, after the startup of the Serpukhov accelerator, there has come to be a possibility of realizing this program.

After the parameters have been refined and a concrete model has been chosen with the aid of accelerator data, it will be possible to return to experiments with cosmic rays and make a number of predictions with greater definiteness that at present. Only after this will it be possible to make a detailed comparison between experiment and theory over a wide range of energies. In particular, the predictions obtained can then be verified with the accelerators in the range 200—300 GeV which are now planned.

In conclusion we take occasion to express our gratitude to E. L. Feĭnberg for important comments and to V. N. Akimov for interesting discussions.

APPENDIX

The asymptotic value of the total cross section, the slope of the diffraction cone, and the multiplicity of a process can be determined by a single method, which we shall now explain. We consider Eq. (32), along with (34a) and (34b), and write it in the most general form

$$\varphi_l(t)=\bar{\varphi}_l(t)+\lambda_l c_l \bar{\varphi}_l(t)\otimes\varphi_l(t). \tag{A.1}$$

Here, in order to simplify the writing of the formulas, we have omitted from the arguments the dependence on the external masses, have defined

$$c_l\bar{\varphi}_l(t)\otimes\varphi_l(t)=\frac{1}{16\pi^3 t^{l+1}}\int dr\,dv\,\frac{[t(t-4\mu^2)-2tr-v^2]^{l+1/2}}{r^2+v^2}\bar{\varphi}_l(t, r, v)\,\varphi_l(t, r, v) \tag{A.2}$$

and have introduced an additional factor λ_1, which for Eq. (32) is identically equal to unity, but which for the present we regard as an additional free parameter.

Let us write $\varphi_1(t)$ in the form

$$\varphi_l(t)=R_l(t)/[l-\alpha(t)],$$

where $\alpha(t)$ does not depend on the masses, and R_1 is regular at $l = \alpha(t)$. Then, expanding $R_1(t)$, $\alpha(t)$, and $\bar{\varphi}_1(t)$ for small t, we have from (A.1)

$$R_l+tR_l'=[l-\alpha-\gamma t]\,[\bar{\varphi}_l+t\bar{\varphi}_l']+\lambda_l c_l[\bar{\varphi}_l+t\bar{\varphi}_l']\otimes[R_l+tR_l'], \tag{A.3}$$

where, for example, $R_1 \equiv R_1(0)$, and so on, primes denote derivatives with respect to t at t = 0, and $\gamma = d\alpha(t)/dt\,|_{t=0}$.*

Equating terms of the same order in t and considering all of them at the point $l = \alpha$, we have

$$\begin{cases} R_\alpha=\lambda_\alpha c_\alpha\bar{\varphi}_\alpha\otimes R_\alpha, & \text{(A.4)}\\ R_\alpha'=-\gamma\bar{\varphi}_\alpha+\lambda_\alpha c_\alpha\bar{\varphi}_\alpha'\otimes R_\alpha+\lambda_\alpha c_\alpha\bar{\varphi}_\alpha\otimes R_\alpha'. & \text{(A.5)}\end{cases}$$

Differentiating (A.4) with respect to α, we get

$$\frac{dR_\alpha}{d\alpha}=\frac{d\lambda_\alpha}{d\alpha}c_\alpha\bar{\varphi}_\alpha\otimes R_\alpha+\lambda_\alpha\frac{d}{d\alpha}[c_\alpha\bar{\varphi}_\alpha]\otimes R_\alpha+\lambda_\alpha c_\alpha\bar{\varphi}_\alpha\otimes\frac{dR_\alpha}{d\alpha}. \tag{A.6}$$

It can be seen from these formulas that the kernels of the integral equations (A.4), (A.5), and (A.6) are identical. But (A.4) is a homogeneous equation, while (A.5) and (A.6) are inhomogeneous equations. In order for all of these equations to have solutions, it is necessary and sufficient that the inhomogeneous terms of Eqs. (A.5) and (A.6) be orthogonal to the solutions of the equation adjoint to (A.4). From this one easily finds the values of γ and $d\lambda_\alpha/d\alpha$ used in Eqs. (46) and (69). The asymptotic value of the total cross section σ^P, Eq. (48), is obtained by exactly the same method, if we consider (A.3) for t = 0, $\lambda_1 = 1$ and $l = 1 + \epsilon$ ($\epsilon \to 0$). Equating terms of the same order in ϵ, we get an inhomogeneous equation for $(dR_1/dl)_{l=1}$. The condition that its inhomogeneous term be orthogonal to the solution of the equation adjoint to (A.4) leads to Eq. (48).

*The contribution of the term with c_l' is small, and therefore we neglect it.

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Translated by W. H. Furry