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DISLOCATION THEORY OF THE ELASTIC TWINNING OF CRYSTALS

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CONTENTS

TWINNING together with slip is one of the fundamental types of plastic deformation of crystals. The experimental investigation of twinning in a number of crystals can be carried out by very clean methods and is rather exactly characterized quantitatively. In connection with this, a detailed theoretical analysis of the twinning process, which can be carried out on the basis of the theory of dislocations, is of interest. At the present time a detailed and systematic investigation of twins has been completed for the stage of elastic twinning. The exceptional nature of the situation which is realized in the case of an elastic twin,—namely, the macroscopic pileup of dislocations of the same type which are in equilibrium with the external elastic field and also in equilibrium with the resistive forces from the side of the crystal lattice—has made it possible to create a quantitative dislocation theory of thin twins by using a comparatively simple model. Within the framework of this theory, it has been possible to describe the basic laws of elastic twinning and to obtain dependences which are accessible to direct experimental verification.

The theory of elastic twinning which is discussed below is semi-microscopic in nature since it contains two phenomenological parameters whose exact values do not follow from the theory itself. However, at the present time methods of determining these parameters of the theory by setting-up quantitative experiments are being proposed and realized. Therefore, one can assume that a relatively complete quantitative description of the plasticity of a crystal in the case of elastic twinning can be achieved at the level of dislocations. Since such a description is absent in the majority of cases of plastic deformation, but its derivation is the most important problem in the physics of tensile strength and plasticity, then it appears to us that an exposition of the fundamental theoretical and experimental results which are obtained by investigating elastic twinning would be extremely useful.

INTRODUCTION

In the simplest case, that defect in a crystalline lattice, which arises in connection with the simultaneous existence in the solid of two crystalline structures that are mirror images of each other (see Fig. 1), is called a twin. The plane AA′ shown in Fig. 1 (and any plane parallel to it) carries the name, twinning plane. It is natural that several equivalent systems of twinning planes may exist in a crystal.

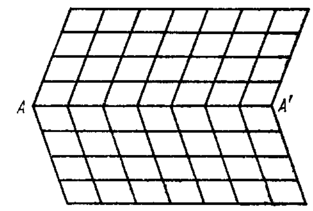


FIG. 1. Diagram showing the arrangement of the crystal planes at the coherent boundary of a twin.

We shall confine our attention to precisely this simplified definition of a twin since, on the one hand, its content is sufficient for a description of the qualitative properties of twinning in the general case but, on the other hand, it is comprehensive enough for twins in certain materials of the type of calcite ($CaCO_3$), sodium nitrate ($NaNO_3$), antimony, etc. One can find a complete crystallographic classification of twins in the monograph[1].

If one of the structures depicted in Fig. 1 occupies an insignificant part of the crystal's volume, then by convention this part is called the twinning layer whereas the remaining part of the sample is called the parent crystal. These two regions of the crystal are separated by the twin boundary. In that case when the twin boundary coincides with the twinning plane, in the same way as shown in Fig. 1, it is called a coherent boundary.

Twins may arise during the process of crystal growth, during the transition from one modification to another, and due to the influence of mechanical effects. In what follows we shall be interested in mechanical twinning, i.e., that process of plastic deformation in which part of the crystal acquires a twinning orientation due to the influence of an external load. For the experimental investigation of twinning, the most convenient crystals are those in which the observation of this process is not impeded by slip flowing in parallel. The classical object for the investigation of twinning is calcite, in which slip is actually not observed at room temperatures and under ordinary loadings.

A very important step in the development of twinning was the discovery by Garber[2] of the elastic twinning phase. Elastic twinning usually refers to that process of plastic deformation in which a twin, which appears in the crystal owing to the effect of an external load, changes its dimensions reversibly in connection with a change of the external load. With an increase of the loading the size of the twin increases, but with a reduction of the loading—it decreases, "escaping" from the crystal upon removal of the load. In order to avoid misunderstanding, we emphasize that the word "elastic" in the present case does not have any connection to the concept of an elastic deformation, since the process of twinning itself is one of the realizations of plastic deformation. The term "elastic twin" only reflects the reversible nature of the corresponding deformation of the crystal.

The essential factor in Garber's experiments[2-5] was the application of concentrated loads in order to create and maintain an elastic twin in the crystal. Prior to Garber's experiments, distributed loads which created an almost homogeneous elastic field in the crystal were usually used for twinning. Under such conditions it was possible to observe twins only in the stage of residual twinning layers (twinning layers which cross the entire crystal and remain in the crystal after the removal of the loading). At the present time it is clear (see Chapter 4 of the present review) that for the existence of an elastic twin it is necessary to have an inhomogeneous elastic field which falls off sufficiently rapidly inside the crystal. Precisely such a field appears as a result of the application of concentrated loads, which were also utilized in[2-5].

An external load concentrated in a small region on the surface of the sample creates a twin having the shape of a thin wedge-shaped lobe lying in the twinning plane (see Fig. 2).

If the load is created by a knife-edge, then the twin acquires the shape of a wedge which emerges onto the lateral faces of the crystal. Such a twin in a clear crystal causes interference, which is equivalent to the interference in a thin wedge (see Fig. 2 b).

If the length of the elastic twin is appreciably smaller than the thickness of the crystalline sample, then its dimensions increase continuously with an increase of the applied load. In samples of thickness 1 cm, twinning lobes up to several millimeters in length can be observed. Information about the thickness of an elastic twin can be estimated from its interference coloring (Fig. 2). Elastic twins of large lengths have thicknesses ranging from a few tenths of a micron up to several microns. Therefore, the ratio of the thickness of an elastic twin to its length is usually of the order of 10^{-4} to 10^{-3}. Garber's observations and also the detailed measurements of how the shape of an elastic twin depends on the magnitude of the external load, which were carried out by Obreimov and Startsev,[6] showed that an elastic twin remains very thin during the entire course of its growth in the crystal. This result is the very important experimental fact on which the expounded dislocation theory of twins is essentially based.

If the length of the twin becomes comparable with the thickness of the crystal, then the smooth dependence of its length on the load is violated, and a sudden transition of the twinning lobe into the residual lamella occurs.[3]

(a) (b)

FIG. 2. (a) Interference of the elastic twinning lobe in a calcite crystal. (b) Interference produced in an elastic twin in calcite by knife-edge loading.

In that case when the external load begins to decrease before the instant of formation of the residual twinning lamella, the twin usually shrinks in proportion to the reduction in the load. The reversible nature of the plastic deformation associated with twinning is a very specific property of this phenomenon. The physical reason for this reversibility is very simple: The interphase boundary between the twin and the parent crystal possesses a certain surface energy which generates surface-tension forces which act on the contour of the twin. It is precisely these forces which, after the removal of the external load, can reestablish the original shape of the crystal, having "expelled" the elastic twin from it.

The universal nature of the phenomenon of elastic twinning as the compulsory initial stage in the development of a twin is confirmed by the observed facts in regard to elastic twins in sodium nitrate,[7], antimony,[8] bismuth,[9] silicon iron,[10] zinc,[11] and also graphite, albite, and so forth (for more details, see[1]).

The discovery of the phenomenon of elastic twinning, described by a very simple connection between the external forces and the characteristic parameters of the twin, did not attract the attention of theorists. A macroscopic theory of twinning was constructed in the articles by Vladimirskii[12] and Lifshitz.[13,14] Lifshitz and Obreimov,[15] and also Frenkel' and Kontorova[16] analyzed the process of twinning at the atomic level. Stepanov[17] gave an explanation of twinning, having considered it as a mechanically-oriented process in an anisotropic crystal.

Vladimirskii[12] considered elastic twinning as the establishment of mechanical equilibrium in the system consisting of a crystal containing a twin and an external load. In article[12] essentially the small ratio of the thickness of the twin to its length was used, and an estimate of this ratio was given in terms of the macroscopic parameters. However, the most important fundamental result of[12] is the introduction of the concept of a twinning dislocation and the formulation of the dislocation model of a twin. Vladimirskii's model of a twinning dislocation, which he published in 1947, is presented in Fig. 3. This dislocation appears as if there were a separate step at the coherent boundary of the twin; therefore a certain collection of dislocations can ensure any arbitrary slope of the macroscopic boundary of the twin to the twinning plane. Unfortunately, in the foreign literature the opinion has been maintained that the model of a twinning dislocation was first proposed by Frank and van der Merwe[18] in 1949, and the work of Vladimirskii is usually not mentioned.

The general theory of planar macroscopic twins in an unbounded medium, making it possible to more completely describe the process of elastic twinning, was developed by I. M. Lifshitz[13,14] on the basis of the nonlinear theory of elasticity. In the articles by Lifshitz the local equilibrium of the twinning layers associated with given forces was studied, where the forces were applied to the interface between the parent crystal and the twin. The equation describing the profile of the twin was derived, and the following important result was obtained: The aperture angle at the end of a free twin must be equal to zero, i.e., the end of the twinning wedge must be "infinitely sharp."

But since the forces acting on the twin boundary were not specified in[13,14], a number of questions pertaining to the equilibrium shape of the twin remained unclear; in particular, the question concerning the ratio of the thickness of the twin to its length. The point is that the premises of[13,14] did not contain any physically small parameter which might determine the ratio of the twin's thickness to its length, but experiment always gives a small value for this ratio in the case of an elastic twin.

In the article by Lifshitz and Obreimov,[15] the process of twinning was subjected to a systematic investigation at the atomic level. Whereas a nonlinear relation between the stresses and the deformations was assumed in the macroscopic theory,[13,14] in[15] a conjecture was made concerning the nonlinear nature of the interatomic interaction forces, which reduced to the introduction of a "twinning force" (a certain special pair of forces). The process of twinning took place as the displacement of atomic steps at the twin boundary. The description of twinning in such terms is very similar to the use of an atomic model of twinning dislocations, although such concepts did not appear in in[15]. In spite of the fact that the atomic approach developed in[15] makes it possible to treat the phenomenon of twinning at a microscopic level, the complexity and inadequate investigation of the nature of the interatomic interaction forces does not permit one to obtain, within the framework of such an approach, quantitative relations governing the process of twinning. It has turned out to be possible to derive such relationships by using the dislocation model of twinning. But before we go on to the justification of the dislocation approach, one should say something about the useful theoretical conclusions which follow from the analysis of the one-dimensional model of twinning due to Frenkel' and Kontorova.[16] A profound analogy between the processes of twinning and slip is pointed out in their article. From the analysis given in[16] it follows that in both cases the deformation process is due to the gradual propagation of a finite displacement in the lattice. These displacements differ only in the type of travelling configurations of the atoms, by means of which the required rearrangement of the crystalline lattice is achieved. In modern language this actually means that the two indicated types of plastic deformation differ only by the type of corresponding dislocations.

Direct experimental proof of the dislocation composition of the twin boundary has been obtained relatively recently, only after a high level of development of the technique of selective etching had been achieved. The question of proving the dislocation structure of the boundary between the twin and the parent crystal arises because in principle a broad boundary region could exist in which a continuous transition would occur from the crystal lattice of the initial crystal to its twin modification. Precisely such a point of view was put forward in the article by Kontorova,[19] where the question of the thickness of the twin boundary is discussed in analogy with a consideration of the interface between the domains in a ferromagnetic substance. However, etching the surface of a calcite sample onto which the boundary of an elastic twin emerges,[20] proved that this boundary consists of extended coherent segments which are lightly etched in the form of narrow grooves and with isolated deep etching pits characteristic of the dislocations, occurring at distances of several microns from each other. The traces of these defects vanish after the emergence of the elastic twin from the crystal. Observation by using this same method of the etching figures characteristic of dislocations on the boundary of the residual twin were carried out in[21-23,87]. The most convincing proof of the

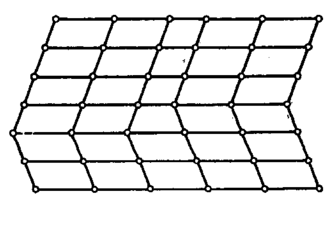


FIG. 3. The model of a twinning dislocation according to Vladimirski [12].

connection between these etching figures in calcite and twinning dislocations is obtained in the experiments[23] where, in particular, the displacement of these defects due to the influence of mechanical loads was observed. Direct experimental observations of twinning dislocations in antimony are described in article[24], where it was shown that the dislocations are spaced in the twin boundary at distances of the order of a micron. After the enumerated publications, the results of a number of researches became known in which the observation of twinning dislocations was carried out by other methods (electron microscope,[25] methods involving X-ray topography,[26] etc.). Thus, we consider that at the present time there is no doubt to the dislocation structure of a twin boundary.

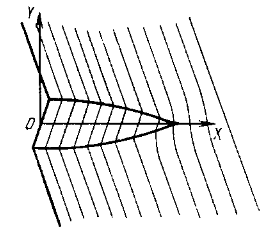
In the dislocation model the problem of the shape of the twin in an external elastic field reduces to the problem of the equilibrium of a certain accumulation of a single type of dislocations. In the case of a thin twin, one can talk about twinning dislocations which are located in a single slip plane. Then we arrive at the problem of a planar pile-up of dislocations, which has been repeatedly discussed in the literature.[27,28,29] But upon such a simplification of the problem, it becomes similar to another dislocation problem of plasticity, in particular, the problem of a slip band[30] or of thin fractures.[31] And what is more, it is found that the dislocation description of thin fractures[32] leads to the basic equations of the so-called force theory of fractures which was developed by Barenblatt (see the review article[33]) and which does not use any concepts involving dislocations at all. The similarity between the description of the plastic deformation associated with twinning or slip and the destruction of the crystal associated with the development of fracture is due to the following physical property. In a crystalline lattice with its enormous forces of binding between the atoms, the kinetics of the indicated inelastic deformations of the crystal certainly must include localization of the front of growth of the process into a small region and a certain gradualness of its propagation.

In virtue of the analogy between the dislocation models of a thin twin, incomplete bands of slip, and thin fracture, the distribution of the dislocations along such macroscopic defects obeys one and the same relations (only the Burgers vectors of the corresponding dislocations and the forces acting on the dislocation from the side of the crystal are different). And what is more, in some kind of sense an incomplete band of slip and a thin fracture are limiting cases of a thin twin. A discussion of this situation is contained in the review article[34] by one of the authors, and will be repeated at the appropriate places in the discussion which follows below.

1. THE MODEL OF A THIN TWIN

Let us consider an elastic twin near the surface of the crystal. The cross section of such a twinning wedge, where the trace of the twinning plane coincides with the X axis, is schematically shown in Fig. 4. The inclined parallel straight lines shown in the Figure indicate the orientation of the atomic planes in the twinning lamella and in the parent crystal. In that case

FIG. 4. A twin at the surface of a solid.



when the twinning is produced by the load on an infinitely long knife-edge, which acts on the crystal surface along a straight line parallel to the Z axis (perpendicular to the plane of the figure), the twin which arises is infinitely extended along the Z axis. The description of such a twin reduces to the specification of its profile in the XOY plane, and the corresponding mathematical problem about the equilibrium of such a twin reduces to a planar problem in the theory of elasticity. In connection with the latter, we shall call such a twin a planar twin. However, it should be kept in mind that usually the twins which arise near the surface of a crystal are created by a concentrated load, and therefore they are not planar. In connection with a certain concentration of stresses, the twin generally does not appear at the surface but deep inside the crystal.[35,36] The cross section of a twin located inside a crystal is shown in Fig. 5.

Proceeding to the dislocation description of a twin and trying to make the initial model clear, let us represent it by a "monatomic twinning lamella," where a macroscopic twin is realized by a collection of such lamella. A diagram of the cross section of such a lamella, made in the spirit of the generally accepted atomic dislocation scheme, is shown in Fig. 6. The monatomic twin is completed by a partial dislocation whose length passes through the cross-hatched region shown in the figure. The component of the Burgers vector **b** in the XOY plane is shown in Fig. 6, and its magnitude is obviously equal to $b = 2a \tan \alpha$ (where 2α is the twinning angle).

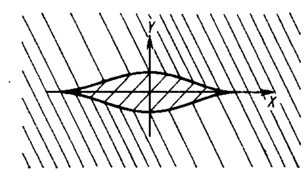
It is easy to see that a twin boundary can be realized by a certain collection of twinning dislocations of the type shown in Fig. 3, distributed along the contour of the twinning lamella (see Fig. 7).

Passing on to the conventional representation of dislocations, the twin in Fig. 4 can be replaced by the collection of twinning dislocations shown in Fig. 8. The thickness h of the twin, which it has when it emerges onto the surface, is equal to the product of the total number N of dislocations forming the twin times the distance a:

$$h = Na.$$

The magnitude δ of the step, which appears at the crystal surface in connection with twinning, is related to N and to the Burgers vector **b** in a similar fashion.

FIG. 5. A twin in an unbounded medium.



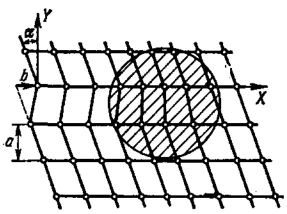


FIG. 6. Monatomic twinning lamella in a crystal.

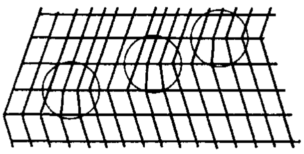


FIG. 7. Dislocation model of a twin boundary.

In the case of a thin twin, as was proved a long time ago by Vladimirskii,[12] the average distance between dislocations along the length of the twin is of the order of 10^{-6} m. Thus, this distance is approximately 10,000 times larger than the interatomic distance which separates the slip planes of neighboring twinning dislocations. It is obvious that the average distance between the dislocations, expressed in interatomic distances, determines the order of magnitude of the ratio of the twin's length to its thickness, i.e., the order of magnitude of the ratio L/h. Therefore, in the fundamental approximation with respect to the small parameter h/L, one can regard all dislocations as distributed in a single plane (the twinning plane).[37] The series of dislocations corresponding to a twin of length L is conventionally depicted in Fig. 9. The straight line, along which the dislocations are distributed and which is the trace of the twinning plane, is usually called the line of twinning.

We shall assume the twin to be planar, and the X axis coinciding with the twinning line is inclined at an angle θ to the surface of the sample (see Fig. 9). Such a twin is formed by the pile-up of straight dislocations, which are parallel to the Z axis. We assume that at a certain point $x = a_0$ on the surface of the sample, there is a source of straight twinning dislocations which is able, due to the effect of the external loading, to create the necessary number of dislocations. We shall regard the position of the source of dislocations as separated from the surface by a distance which, in order of magnitude, cannot be smaller than the step δ which is produced in connection with twinning at the surface of the crystal. The latter is associated with the fact that in the process of twin formation, the region where the twinning plane emerges onto the surface of the crystal may be subjected to distortions of a non-dislocation type. The linear dimensions of this region are of the order of δ. Since it is precisely in this region that

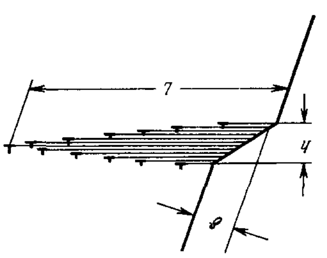


FIG. 8. A twin as a collection of dislocations.

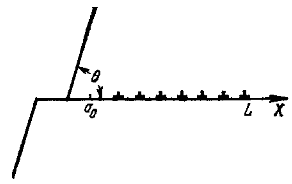


FIG. 9. The dislocation model of a thin twin.

nucleation of the twinning dislocations occurs, but we will not be interested in the mechanism of their formation, then it is sufficient to assume that the dislocations are "generated" in pairs at the point $x = a_0$, where $a_0 \gtrsim \delta$. In virtue of the law for the conservation of the Burgers vector, dislocations of the opposite sign are simultaneously produced, where the number of positive dislocations is equal to the number of negative ones. We shall assume that the external load is such that due to its influence, the negative dislocations emerge onto the surface of the crystal, creating a characteristic step on it, but the positive dislocations are displaced into the region $a_0 < x < L$ where they create a twinning wedge.

It only makes sense to talk about a macroscopic twin when the number of twinning dislocations which are produced is large and the length of the twin is appreciable (in an experiment the number of dislocations per centimeter of twin's length frequently reaches the order of magnitude of 10^4 for a twin length of the order of a few millimeters). But in such a case the distribution of the dislocations can be described by a density which is a continuous function of x and which is subsequently denoted by $\rho(x)$. The twin's thickness $h(x)$ at a certain point x is related to the density of the dislocations in the obvious manner:

$$h(x) = a \int_x^L \rho(\xi)\, d\xi. \tag{1.1}$$

From Eq. (1.1) it follows that the function $\rho(x)$ must always satisfy the condition

$$\int_{a_0}^{L} \rho(x)\, dx = \frac{\delta}{b} = \frac{h}{a} = N, \tag{1.2}$$

where, as we have already mentioned, N denotes the total number of twinning dislocations of a single sign, and δ denotes the total shift along the line of twinning at the surface of the solid.

The distribution of the dislocations along an elastic twin cannot be arbitrary, but is determined by the condition for equilibrium of the crystal containing the twin under the influence of the external loads. It is quite clear that such an equilibrium distribution is determined in the form of the solution of a certain variational problem. However, it is easy to formulate the condition for equilibrium of the twin without formulating a variational principle, but by using very simple physical considerations. In fact, it is well known that the effect on a dislocation in a crystal can be described in terms of the forces acting on it. Therefore, the condition for equilibrium of the system of dislocations can be written down in the form of setting all of the forces acting on each dislocation equal to zero. In connection with the formulation of such a condition, one should bear in mind that two types of forces act on a dislocation in a crystal: a force of elastic origin (the

Peach-Koehler force) which takes into consideration the elastic fields created by both the external loads and by the remaining dislocations in the crystal, and a force of inelastic origin (of the type of the Peierls force or the surface-tension force) which hinders the free "pile-up" of dislocations in the crystal.

Being primarily interested in qualitative physical results, we shall analyze those situations which lead to the simplest mathematical expressions, but also possess sufficient generality. Proceeding along this path, first of all we make an assumption about whether all of the dislocations in the twin are either purely edge dislocations or purely screw dislocations. Then the Burgers vector of the dislocations under consideration will have only a single component ($b = b_x$ for edge dislocations and $b = b_z$ for screw dislocations). Further, since to some extent the free movement of the dislocations under consideration can occur only in the twinning plane, then it is sufficient to consider the components of the forces of interest to us only along the X axis. In what follows, the projection of the force along the X axis will be denoted without any subscript indicating the axis.

The elastic force, acting per unit length on a certain dislocation on the part of the external field and the other dislocations forming the twin, is given by (see the Peach-Koehler formula in[32])

$$f^{\,el}(x) = b\sigma^e(x) + b\int_{a_0}^{L} \sigma^0(x, \xi)\,\rho(\xi)\,d\xi, \tag{1.3}$$

where $\sigma(x)$ is the appropriate component of the elastic stress tensor, and where $\sigma^e(x)$ denotes the stresses created by the external loads and $\sigma^0(x, \xi)$ denotes the stress created at the point x on the line of twinning by an individual dislocation located at the point ξ on this same line.

The integral in Eq. (1.3) is to be understood in the sense of a principal value which, on the one hand, excludes taking into account the self-interaction of the dislocations under consideration, but on the other hand, it makes the indicated integration reasonable, since the function $\sigma^0(x, \xi)$ has a singularity when its arguments coincide (it has a singularity of the type of the singularity in the Green's function).

In an unbounded homogeneous crystal one can always write

$$\sigma^0(x, \xi) = \frac{B}{x-\xi}\,.$$

where the coefficient B is of the order of magnitude of the product of the shear modulus μ times the magnitude of the Burgers vector ($B \sim \mu b$), and its specific value is determined by the anisotropy of the medium.[38] In an isotropic medium we have the following results for edge and screw dislocations, respectively:

$$B_{\text{edge}} = \frac{\mu b}{2\pi(1-\nu)} \qquad B_{\text{screw}} = \frac{\mu b}{2\pi},$$

where ν is Poisson's ratio.

For dislocations in a finite crystal we have

$$\sigma^0(x, \xi) = B\left[\frac{1}{x-\xi} + K(x, \xi)\right],$$

where the function $K(x, \xi)$ of two variables has a very cumbersome form; however, it does not have a singularity at $x = \xi$. In the case of a semi-finite homogeneous crystal, its expression can be obtained on the basis of the results of article[39] for an isotropic medium or article[40] for an anisotropic medium.

The force of inelastic origin consists of two essentially different parts. In the first place, any arbitrary dislocation (both a twinning dislocation and a perfect dislocation) experiences a force of retardation which is similar to the force of dry friction. This force, which is important even in a perfect crystal and which is due to the discrete nature of the structure of the crystal lattice, was first analyzed by Peierls[41] and then by Nabarro.[42]

If the twins are being propagated in a defect crystal, then the effective force of retardation acting on the dislocation includes, in addition to the Peierls force, a dragging force due to the defects distributed in the sample. The defects turn out to have both a direct effect on the dislocations, impeding their passage around their intersection, and so forth, and also a resistive effect which is described by their elastic fields. In order to emphasize the fact that the described force has terms which differ from the Peierls force, in what follows we shall simply call it a friction force. In equilibrium the magnitude and direction of this force depends on the direction of motion of the dislocations prior to equilibrium, since it includes the dissipative force which is always directed against the motion. Therefore, the form of the force of inelastic origin depends, to a considerable extent, on the process of twin formation.* We shall assume that it is directed against the motion, and in the limit of an infinitesimal velocity it is equal to a constant, nonvanishing quantity.

In the second place, the force of inelastic origin includes the surface-tension force which acts on the twinning dislocation from the side of the parent crystal along the twinning plane perpendicular to the line of dislocation. This force is due to the fact that a twinning dislocation, in contrast to a perfect dislocation, generates a certain two-dimensional stacking defect during its motion (the "monatomic twinning lamella" shown in Fig. 6), whose formation is associated with the additional surface energy. It is obvious that only the dislocations which are located at the end of the twin experience the effect of such a force. In fact, the addition of a single dislocation to that part of the twin whose width is of macroscopic dimensions actually does not change the area of the interface between the parent and the twinned crystal, and does not significantly change the surface energy. At the same time, the addition of a single dislocation at the tip of the twin, where the dividing boundaries are separated from each other by several atomic layers, may appreciably change the corresponding surface energy. This assumption has been confirmed by a qualitative investigation.[43] In[43] it is shown that the interphase surface energy in the twin decreases substantially with an increase of the number of atomic layers passing into the twinning state. In particular, it turns out that a triple-layer twin can already be practically regarded as one possessing two coherent twinning boundaries,

*As we will see below, the dependence of this force on the nature of the motion prior to equilibrium is the reason for the hysteresis which occurs during the process of elastic twinning.

whereas a single-layer twin is a much more strongly disturbed region of the crystal with a higher defect energy.

The difference in the nature of the crystal deformations generated at the tip of the twin (by the leading dislocations in the pile-up) and the dislocations on the twin boundary can be discerned by a comparison of the diagram shown in Fig. 6, corresponding to a leading dislocation (a dislocation of the Shockley type), and the diagram shown in Fig. 3 for a twinning dislocation.

We shall take the indicated properties of the surface tension into account by assuming that this part of the force of inelastic origin vanishes everywhere outside of a small neighborhood at the end of the twin and abruptly increases in magnitude at the end, attaining a certain large finite value at the very end of the twin. The introduction of such a force is analogous to the introduction of a "modulus of coupling" at the ends of a thin fracture,[33] but it has a different physical character.

The condition for the equilibrium of a dislocation at the point x, under the influence of all the forces enumerated above, is equivalent to the requirement

$$^{\mathrm{el}}(x)+f^{\mathrm{inel}}(x)=0, \quad f^{\mathrm{inel}}(x)=bS(x), \tag{1.4}$$

where S(x) denotes the stresses on the line of twinning, which are equivalent to the presence of forces of inelastic origin.

After isolating the characteristic singular part of the kernel $\sigma^0(x, \xi)$, relation (1.4) may be written in the form

$$\int_{a_0}^{L}\frac{\rho(\xi)\,d\xi}{\xi-x}-\int_{a_0}^{L}K(x,\xi)\rho(\xi)\,d\xi=\frac{1}{B}[\sigma(x)+S(x)]\equiv\omega(x). \tag{1.5}$$

If the field of the external loads and the dependence of the forces of inelastic origin on the coordinate x are considered to be known, then relation (1.5) may be regarded as the equation for the determination of the function $\rho(x)$ from the given function $\omega(x)$. We note that it is a singular integral equation.

Thus, an analysis of the density $\rho(x)$ of the distributions, and hence of the profile of the twin, reduces to an investigation of the mathematical problem represented by Eq. (1.5). The kernel of the equation under consideration has a singularity of the same type as a Cauchy singular kernel; therefore, one can carry out a qualitative investigation of its solution in the general case in considerable detail.[44]

2. CERTAIN CHARACTERISTIC PROPERTIES OF THE PROFILE OF A TWIN

We have already mentioned that the profile of a planar twin is completely described by the function $\rho(x)$. Therefore, a qualitative investigation of its shape actually reduces to an analysis of the equation for equilibrium, whose derivation was completed in the previous Chapter.

The explicit form of the equation for the equilibrium of a twin in a certain crystalline sample of a certain shape is obtained by the substitution of the specific expression for $K(x, \xi)$ into Eq. (1.5). The kernel $K(x, \xi)$ essentially depends on the anisotropy of the medium, the angle θ (see Fig. 9), and on the distance of the twinning dislocations from the surface of the crystal.*

We cannot give a closed analytic expression for the solution of Eq. (1.5) for any arbitrary kernel $K(x, \xi)$. However, even in the general case one can reach a number of physically important qualitative conclusions, pertaining to the characteristic properties of the equilibrium distribution $\rho(x)$ of the dislocations.

First of all we note that for $L - a_0 \ll L \sim a_0$, when the source of the dislocations is located far inside the crystal and the stress field of the dislocations is practically the same as the stress field in an unbounded crystal, then Eq. (1.5) goes over into the equation for the equilibrium of a thin twin, which was first derived by Lifshitz:[13]

$$\int_{a_0}^{L}\frac{\rho(\xi)\,d\xi}{\xi-x}=\omega(x). \tag{2.1}$$

The definition of the function $\omega(x)$ is given in Eq. (1.5). It should only be noted that in the manner of writing down Eq. (2.1) described here, the force $\omega(x)$ has a quite specific form and, in particular, it explicitly includes the forces of inelastic origin which were not included in the equations of[13,14].

The important result of[13], concerning the shape of the twin, was the conclusion that the tip of a free twinning wedge has a zero aperture angle (the opposite boundaries of the twin touch each other at its end). A similar conclusion can be naturally reached on the basis of Eq. (1.5) even in the general case, i.e., for a twin in a crystal sample of arbitrary shape and for arbitrary anisotropy. It is sufficient to note that the twin can increase its length upon an increase of the external loading only in that case when the retardation forces acting on its tip are bounded. But in such a case, the quantity $\omega(L)$ must be finite and therefore the first term on the left hand side of Eq. (1.5),

$$\int_{a_0}^{L}\frac{\rho(\xi)\,d\xi}{\xi-L}$$

must also be bounded at the corresponding point. In order to satisfy this requirement it is necessary that $\rho(\xi) \to 0$ as $\xi \to L$. From the physical meaning of the function $\rho(x)$ defined by relation (1.1), it follows that the condition $\rho(L) = 0$ makes the tip of the twin infinitely sharp, i.e., it requires that the aperture angle of the profile at the end of the twin should be equal to zero. In this case the profile of the end of the twin is schematically shown in Fig. 10 a.

*For reference purposes we present a short list of articles devoted to the explicit writing down of Eq. (1.5) and to the investigation of the properties of the shape of the twin based on this equation in a number of special cases. A twin in an unbounded isotropic medium was considered in [45,46], and a twin in an unbounded anisotropic medium was treated in [47]. A twin perpendicular to the plane surface of an isotropic solid is investigated in detail in [48,49,50]. A twin whose twinning plane is located at an arbitrary angle to the surface of an isotropic solid is considered in [51]; equations are also derived there for the case when the twin is located at a certain depth under the surface of the isotropic solid. Article [40] is devoted to a twin at the surface of an anisotropic medium. A twin in an isotropic plane-parallel slab is considered in [52,53], and a twin near the boundary of two anisotropic media having different moduli of elasticity is treated in [54].

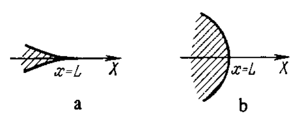


FIG. 10. Shape of the end of an elastic twin deep inside a crystal: a) end is free; b) end is blocked.

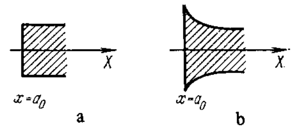


FIG. 11. Shape of the twin near its emergence onto the crystal surface. a) Surface of the crystal is free; b) there is an obstruction on the surface which prevents the emergence of the dislocations upon unloading.

The same considerations pertain to the point $x = a_0$; if the twin is restrained in the crystal by only the force of the stresses applied to the surface ($\omega(a_0)$ is finite), then $\rho(a_0) = 0$. Usually all twinning dislocations are "generated" on the surface of the solid (at the point $x = a_0$, which is the only source of dislocations) and under an increasing load they are only displaced along the X axis. Then in the interval $a_0 < x < L$ there are only dislocations of a single sign ($\rho(x) \geq 0$) and the condition $\rho(a_0) = 0$ means that the twin emerges onto the surface of the solid in the form of a plane-parallel lamella (see Fig. 11a).

If the twin is free, i.e., if the forces restraining it are finite at both ends of the twin ($\rho(a_0) = \rho(L) = 0$), then the question of its length remains open. In order to formulate the mathematical condition which determines the length of the twin, it is simplest to utilize certain formal results of the theory of singular integral equations. It is well known[44] that an equation of the type (1.5) has a solution which vanishes at both ends of the interval (a_0, L) only in that case when the right hand side of this equation, $\omega(x)$, satisfies the following special orthogonality condition:

$$\int_{a_0}^{L} \omega(x)\,\rho_0(x)\,dx = 0, \qquad (2.2)$$

where $\rho_0(x)$ denotes the solution of the homogeneous equation which is the adjoint equation to (1.5).* The physical meaning of condition (2.2) is quite simple[32]— it expresses the requirement of boundedness of the stresses produced in the crystal by the free twin.

Since the function $\rho_0(x)$ is uniquely determined by the kernel of the integral equation (naturally, to within an arbitrary factor), then relation (2.2) is the condition for determining the length of the twin. In the simplest case of a twin in an unbounded homogeneous medium, this condition takes the following form:[32]

$$\int_{a_0}^{L} \frac{\omega(x)\,dx}{\sqrt{(L-x)(x-a_0)}} = 0. \qquad (2.3)$$

Up till now the subject under discussion has been free elastic twins. However, conditions may arise in the crystal such that the twin's growth is stopped by certain barriers in the depths of the sample. We shall call such obstacles, which stop the slip of the twinning dislocations, "stoppers." The following can serve as some of the simplest examples of a two-dimensional "stopper": the residual twinning lamella of a different twinning system,[55,36] a crystalline grain boundary,[56] and an interphase boundary. In such a case, an increase of the external loading does not lead to a change in the length of the twin, its growth having reached the "stopper." Therefore, the length of the twin may be regarded as fixed for practically any arbitrary external loadings. In such a situation, relation (1.2) determines the total number of twinning dislocations of the same sign which are produced by a given external load (or the thickness of the twin for its emergence onto the surface).

Since in this case condition (2.2) is generally not satisfied, then from the theory of singular integral equations[44] it follows that $\rho(L)$ becomes unbounded. One can take the "stopper" into account in Eq. (1.5) by introducing a certain force, concentrated at the point $x = L$, as a retarding force. The presence of such a point force leads to a singularity in the function $\rho(x)$ at the indicated point ($\rho(L) = \infty$). The geometrical meaning of this property of the function $\rho(x)$ reduces to the fact that the aperture angle of the twin's profile is equal to 180° at its end (see Fig. 10b).

Of course, the contour of the twin shown in Fig. 10b and also the literal formulation of the assertion on which it is constructed must be understood in the conventional sense. The proposed theory of thin twins starts from the assumption that the average distance between neighboring dislocations is much larger than the magnitude of the Burgers vector ($b\rho(x) \ll 1$). Formally this assumption is violated in the immediate neighborhood of the "stopper," and a consistent investigation of the problem in general should be based on an analysis of the equilibrium of a discrete series of dislocations which are distributed in parallel twinning planes. A similar problem was discussed in the article by Eshelby, Frank, and Nabarro[27] in regard to the pile-up of dislocations in a single slip plane which is restrained by a "stopper." From the results of[27] it follows that for a large number of dislocations in the pile-up, the distribution of practically all of the dislocations (except for a few dislocations near the "stopper" itself) differs very little from that distribution which follows from a continuous treatment. But since the macroscopic theory cannot pretend to give an exact determination of the coordinates of the dislocations which are nearest to the "stopper," the corresponding conclusions are consistent within the framework of applicability of the macroscopic approach. It is true that the situation is complicated by the fact that in the case of twinning the dislocations occur in parallel "slip" planes (they form a "different-tiered" pile-up). But even in this case one can have confidence that right down to distances for which $b\rho(x) \sim 1$ the qualitative behavior of the function $\rho = \rho(x)$ will not differ from that which we predict on the basis of the analysis of the equations for the equilibrium of a thin twin. Thus, the expounded theory "does not bear the responsibility" for the exact shape of the contour of the twin shown in Fig. 10b only in the immediate vicinity of the end of the twin, where $b\rho(x) > 1$.

A different situation may occur, namely, it may come about that upon a certain external loading the

*The homogeneous equation, which is the adjoint to the integral equation $\int_{a_0}^{L} K(x,\ \xi)\,\rho(x)\,dx = 0$, has the form $\int_{a_0}^{L} K(\xi,\ x)\,\rho_0(x)\,dx = 0$.

nucleation of the dislocations at the point $x = a_0$ ceases. This fixes the thickness of the twin at its emergence onto the surface, or what is the same thing, the value of the quantity δ. If for any reason at all a "stopper" for the twinning dislocations is produced on the surface of the solid, then upon unloading a concentrated force appears at the point $x = a_0$, and this force restrains the dislocations from emerging from the crystal. Then a singularity of the function $\rho(x)$ appears at this point, and the profile of the twin will have the shape indicated on Fig. 11b in this region.

We shall not present the cumbersome expressions for the function $\rho(x)$, which are obtained by the selection of different solutions of Eq. (1.5) corresponding to the presence or absence of "stoppers" at the points $x = a_0$ and $x = L$ (one can find, for example, the appropriate expressions in[32]). As an example of such an analysis of the shape of a twin during its growth process, we only consider the simplest case when the entire qualitative description may be illustrated by closed analytic expressions. Let us assume that the twin, whose line of twinning is perpendicular to the plane surface of the crystal, is formed by screw dislocations. Then the kernel $K(x, \xi)$ has an extremely simple form

$$K(x, \xi) = \frac{1}{x-\xi} - \frac{1}{x+\xi}, \qquad (2.4)$$

which corresponds to the appearance of a force of mirror reflection for a screw dislocation near the surface of the crystal. For such a kernel Eq. (1.5) reduces to the following singular equation:

$$\fint_{a_0}^{L} \frac{\rho(\xi)\,\xi\,d\xi}{\xi^2 - x^2} = \frac{1}{2}\,\omega(x), \qquad (2.5)$$

whose solution can be written down explicitly without any difficulty.

We shall assume that during its growth the free elastic twin first of all encounters a "stopper" at the point $x = a_2$ on the line of twinning (Fig. 12). It may happen that under the influence of the external load, the dislocation source at the point $x = a_0$ is able to emit a number of dislocations, not exceeding a certain finite value N. Then, provided that

$$\int_{a_0}^{L} \rho(x)\,dx < N$$

the length of the twin is determined by the relation (2.2) and the function $\rho(x)$ itself has the form

$$\rho(x) = -\frac{2}{\pi^2}\sqrt{(L^2 - x^2)(x^2 - a_0^2)} \fint_{a_0}^{L} \frac{\omega(\xi)\,\xi\,d\xi}{(\xi^2 - x^2)\sqrt{(L^2-\xi^2)(\xi^2 - a_0^2)}}. \qquad (2.6)$$

One should expect that with increasing load the length of the twin will increase (later we will verify the validity of this assertion). Let us assume that the source of the dislocations is "exhausted" before the end of the twin reaches the "stopper." Then upon a further increase of the external load for this twin, as well as for any arbitrary free twin, conditions of the type (1.2) and (2.2) must be satisfied for a given value of N. One can easily reason out that when the value of a_0 is fixed, these mathematical conditions generally cannot be satisfied simultaneously. This also implies

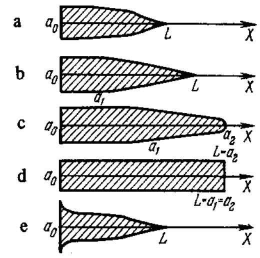


FIG. 12. Evolution of the shape and dimensions of an elastic twin in a crystal associated with loading (Figs. a to d) and with unloading (Fig. e): a) Freely growing twin; b) growth of the twin after the dislocation source on the surface of the crystal ceases to operate; c) the end of the twin encounters a "stopper" (a_2 is the point where the "stopper" is located); d) the shape of the jammed twin in the limit of an infinitely large loading; e) an obstruction ("stopper") on the surface prevents the emergence of the twin from the crystal upon unloading.

that on the parts of the twinning line near the source ($x = a_0$) the dislocations must be completely absent:

$$\rho(x) = 0, \qquad a_0 < x < a_1,$$

where the point $x = a_1$ determines the right-hand boundary of the indicated interval. Therefore, in the interval ($a_0 < x < a_1$) the twin has the shape of a plane-parallel lamella (see Fig. 12b). To the right of the point $x = a_1$ the density $\rho(x)$ of the dislocations is determined by a formula of the type (2.6) with the value a_0 replaced by a_1. The quantity a_1 and the length L of the twin are now determined by formulas of the type (1.2) and (2.2) in which one should again replace a_0 by a_1.

After the end of the twin reaches the "stopper" ($L = a_2$), the length of the twin becomes fixed, but its end is rounded off* (see Fig. 12c), where the function $\rho(x)$ has the form

$$\rho(x) = -\frac{2}{\pi^2}\sqrt{\frac{x^2 - a_1^2}{L^2 - x^2}} \fint_{a_1}^{L} \sqrt{\frac{L^2 - \xi^2}{\xi^2 - a_1^2}}\,\frac{\omega(\xi)\,\xi\,d\xi}{\xi^2 - x^2}. \qquad (2.7)$$

It is easy to verify that in connection with a strong increase of the external loading the point $x = a_1$ approaches the end of the twin, $x = L$. In fact, let us assume that $\sigma(x) \gg S(x)$ at all points of the twin, and the external force is proportional to a certain parameter P: $\sigma(x) = P\tau(x)$. Then $\omega(x) \approx (P/B)\tau(x)$, and from the equation of equilibrium (of the type (1.5) with the lower limit of integration equal to a_1) it follows that $\rho(x) = P\chi(x)$, where the function $\chi(x)$ is the solution of this equation for equilibrium with the right-hand side equal to $\tau(x)/B$. In this case the condition of the type (1.2) takes the form

$$\int_{a_1}^{L} \chi(x)\,dx = \mathrm{const}/P. \qquad (2.8)$$

With an increase of P the left hand side of Eq. (2.8) must decrease, tending to zero. But since $\chi(x) > 0$ and $\chi(L)$ is unbounded, a_1 will inevitably approach L. Thus, in the limit of infinitely large values of P the twinning wedge is completely transformed into the

*In connection with the twin contour shown in Fig. 12c, it is necessary to repeat the same remark which has made in connection with Fig. 10b, namely, formula (2.7) describes this contour only in the range of values x where $b\rho(x) \lesssim 1$.

plane-parallel lamella* shown in Fig. 12d. Usually the so-called residual twinning lamella has such a shape.

Now if at this stage, when the twin has still not been converted into a plane-parallel lamella, the external loading is reduced (unloading), then from the stage shown in Fig. 12c we again arrive at the stage shown in Fig. 12b. The subsequent behavior depends on the type of defect which cuts off the action of the dislocation source. If this defect is equivalent to some kind of "stopper," then the twinning dislocations cannot escape from the crystal and for sufficiently small loads its shape takes the form shown in Fig. 12e, and the function $\rho(x)$ is given by the expression

$$\rho(x) = -\frac{2}{\pi^2}\sqrt{\frac{L^2-x^2}{x^2-a_0^2}}\int_{a_0}^{L}\sqrt{\frac{\xi^2-a_0^2}{L^2-\xi^2}}\,\frac{\omega^*(\xi)\,\xi\,d\xi}{\xi^2-x^2}, \qquad (2.9)$$

where $\omega^*(x)$ is a function which differs from $\omega(x)$ by the change of the forces of inelastic origin associated with unloading (for more details about this, see Chapter 4).

In concluding our analysis of the equation for equilibrium, we note that always when the function $S(x)$ is "sufficiently good" the behavior of $\rho(x)$ at the ends of the twin in the general case depends very little on the shape of the function $\omega(x)$. At the free end of the twin ($x = x_1$) the density of the dislocations falls off like the square root of the distance to the point $x = x_1$:

$$\rho(x) \backsim \sqrt{|x-x_1|}, \qquad (2.10)$$

but for an obstruction at the point $x = x_2$ it increases without limit according to the law

$$\rho(x) \backsim \frac{1}{\sqrt{|x-x_2|}}. \qquad (2.11)$$

Recalling the restrictions which were imposed on the domain of applicability of formula (2.11), let us attempt to estimate the length of the interval near the end of a "trapped" twin, where this formula is invalid. Let us consider the most "dangerous" case from this point of view, namely, the case of a homogeneous external load: $\sigma(x) = P = \text{const}$. Then the density of the dislocations near the "stopper" is given by[32]

$$b\rho(x) = \frac{2P(1-\nu)}{\mu}\sqrt{\frac{x-a_0}{|x_2-x|}} \sim \frac{P}{\mu}\sqrt{\frac{L}{|x_2-x|}}. \qquad (2.12)$$

As we mentioned above, the critical distances are those for which $b\rho(x) \sim 1$. It follows from Eq. (2.12) that these distances correspond to a distance from the end of the twin whose order of magnitude is given by

$$|x_2-x| \sim \left(\frac{P}{\mu}\right)^2 L.$$

If one takes $P \sim 10^{-3}\,\mu$, which even exceeds the loadings which are customarily applied to a crystal undergoing twinning, then it turns out that $|x_2 - x| \sim 10^{-6}$ L. Twins having lengths of up to several millimeters are usually studied; for them the region of nonvalidity of formula (2.11) turns out to be smaller than the resolving ability of the optical instruments which are applicable in this case.

The results cited above, pertaining to the shape of the twin, can be compared with the experimental data obtained by Soldatov and Startsev[36] in connection with their investigation of the shape of twins in bismuth. It is observed that if the twin is moving freely in the crystal, without encountering any obstacles, then it has the shape of a wedge which is very greatly elongated in the direction of the motion and is very thin at the end (see Fig. 13a). If the twin encounters an obstacle during its motion in the depths of the crystal, in particular, an obstacle in the form of a twinning lamella of a different orientation, then the increase in its length ceases. The thickness of the twin rapidly increases and a characteristic rounding-off occurs at the "nose" of the twin; in regard to its shape this is reminiscent of the semicircle associated with the boundaries of a twin (see Figs. 13b and 13c). Numerical differentiation of the experimental data on the thickness of the twin according to the formula which is the inverse of Eq. (1.2) makes it possible to determine the function $\rho(x)$. For the case of a trapped twin, the obtained dependence[36] is described by a formula of the type (2.11) with a small amount of error.

In our opinion the described results of[36] indicate good agreement between the dislocation description of twins and their observed properties; in particular, these results confirm the theoretical predictions about the shape of a growing twin which encounters an obstacle.*

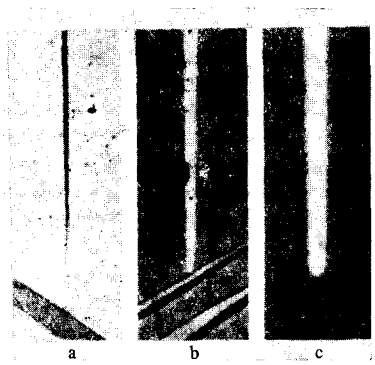


FIG. 13. The shape of the end of a twin in a bismuth crystal. [36] a) Free twin; b) and c) correspond to trapped twins.

*The conclusion about the infinitely large value of P at which the conversion of the twin into a plane-parallel lamella occurs is, of course, related to the assumption that all of the twinning dislocations are located in a single plane. If it is taken into consideration that the dislocations are located in neighboring atomic planes, separated by a distance a from each other, then it turns out that even for finite stresses of the order of $\sigma \sim \mu b/a$ the twinning dislocations will be aligned in a wall one above the other, being restrained only by the planar surface of the "stopper." The latter assertion has been confirmed by a direct machine calculation. [57] The fact that the distance between the slip planes of neighboring dislocations is finite plays an important role in connection with the encounter between the twin and the "stopper," whose dimension d in the direction perpendicular to the twinning plane is small. In this case, with an increase of the external loading the twin's thickness h may exceed the value of d, after which the twinning dislocations are able to pass "over the obstruction." The process of a twin passing around a "stopper" of finite dimensions was experimentally observed in article [58].

*In principle one might imagine that a "trapped" twin remains wedge-like, with an increase of the aperture angle of the wedge in proportion to the loading.

3. WHY AN ELASTIC TWIN IS THIN

Many of the characteristic features of the elastic twinning process are essentially determined by the forces of inelastic origin.* The special role of these forces can be best illustrated in a discussion of the question of the ratio of the thickness of an elastic twin to its length. In particular, we shall now verify that only the inclusion of forces of inelastic origin in the equation for equilibrium of the twin makes it possible to understand why a free twin of finite length can exist in a field of elastic stresses of constant sign, and why such a twin always remains thin.

In order to discuss this question it is necessary to derive an expression for the thickness and length of an elastic twin. It is clear that closed formulas for these quantities can be obtained only in certain of the most simple cases. One of these cases corresponds to a plane twin produced by screw dislocations, for which the kernel $K(x, \xi)$ is given by formula (2.4). Using this kernel, let us write down an equation of the type (2.5) in a somewhat different form:

$$\int_{a_0}^{L} \frac{\rho(\xi)\,d\xi}{x-\xi} - \int_{a_0}^{L} \frac{\rho(\xi)\,d\xi}{x+\xi} = \omega(x). \qquad (3.1)$$

It is interesting to note that such an equation will determine the equilibrium for a twin which is created in an unbounded crystal by dislocations whose source is located in the interval $(-a_0, a_0)$ near the origin of coordinates. For this it is sufficient to assume that the external load is an even function of x, but two nascent dislocations of the opposite sign simultaneously appear at the points $x = \pm a_0$.

In order to specify the function $\omega(x)$, which is defined by relation (1.5), we shall assume that the twin is growing, i.e., it is produced by a monotonically increasing force. In this case, as mentioned in Chapter 1, the forces of inelastic origin, which are described by the function $S(x)$, have the form

$$S(x) = -S_0 - S_s(x), \quad S_0 = \text{const}. \qquad (3.2)$$

The first term on the right hand side of Eq. (3.2), namely S_0, determines the magnitude of the frictional force which a twinning dislocation experiences in a crystal. In order of magnitude

$$S_0 \sim \sigma_s, \qquad (3.3)$$

where σ_S denotes the "microscopic" limit of the yield of the material with regard to twinning, that is, the starting stress, beginning with which the external load displaces an individual twinning dislocation. The second term in Eq. (3.2), $S_S(x)$, describes the surface-tension force, which differs from zero only in a small neighborhood ϵ near the end of the twin. If $\epsilon \ll L$, then this force does not depend on the length of the twin. In order to emphasize the fact that $S_S(x)$ does not depend on the length of the twin, let us write $S_S(x) = A(L - x)$, where the function $A(\xi)$ monotonically decreases with increase of its argument from a certain maximum value S_S^0 to zero over a small interval $(0 < \xi < \epsilon)$.

The solution of Eq. (3.1) or (2.3) for a free twin, when $\rho(a_0) = \rho(L) = 0$, is given by formula (2.4) where the half-length L of the twin is determined from condition (2.3). Since we shall only be interested in those solutions of Eq. (3.1) which tend to zero at the point $x = a_0$, in order to simplify the following analysis one can set the small quantity a_0 equal to zero ($a_0 = 0$). Then Eq. (3.1) reduces to

$$\int_{-L}^{L} \frac{\rho(\xi)\,d\xi}{\xi - x} = \frac{1}{B}[\sigma(x) + S(x)] \qquad (3.4)$$

and its solution (2.6) takes the form

$$\rho(x) = \rho_0(x) + \rho_n(x), \qquad (3.5)$$

$$\rho_0(x) = -\frac{1}{B\pi^2}\sqrt{L^2 - x^2}\int_{-L}^{L} \frac{\sigma(\xi)\,d\xi}{(\xi - x)\sqrt{L^2 - \xi^2}}, \qquad (3.6a)$$

$$\rho_n(x) = \frac{1}{B\pi^2}\sqrt{L^2 - x^2}\int_{-L}^{L} \frac{S_n(\xi)\,d\xi}{(\xi - x)\sqrt{L^2 - \xi^2}}. \qquad (3.6b)$$

Condition (2.2) is now replaced by the condition

$$F(L) = S_0 + J(L), \qquad (3.7)$$

where the following notation is being used:

$$F(L) = \frac{1}{\pi}\int_{-L}^{L} \frac{\sigma(x)\,dx}{\sqrt{L^2 - x^2}}, \qquad J(L) = \frac{1}{\pi}\int_{-L}^{L} \frac{S_n(x)\,dx}{\sqrt{L^2 - x^2}}. \qquad (3.8)$$

Condition (3.7) is a trancendental equation which determines the length of the twin. Let us note certain properties of the functions $F(L)$ and $J(L)$ appearing in this equation, said properties following from their definitions given by Eqs. (3.8). Let us assume that the function $\sigma(x)$ is positive for all values of x and integrable over an infinite interval,* i.e., we assume that a nonvanishing integral

$$\Phi_0 = \frac{1}{\pi}\int_{-\infty}^{\infty} \sigma(x)\,dx, \qquad (3.9)$$

exists which, to within a factor, determines the total external force acting in the twinning plane (taken per unit length along the Z axis). If x_0 denotes the distance over which the function $\sigma(x)$ decreases substantially, then it is obvious that in order of magnitude

$$\Phi_0 \sim x_0\sigma(0). \qquad (3.9a)$$

From Eqs. (3.7) and (3.8) it follows that asymptotically for large values of L

$$F(L) = \frac{\Phi_0}{L} \quad \text{as} \quad L \to \infty. \qquad (3.10)$$

As to $J(L)$, then from the properties of the function $S_S(x)$ it follows that

$$J(L) = \frac{2}{\pi}\int_{0}^{L} \frac{A(L-x)\,dx}{\sqrt{L^2 - x^2}} \approx \frac{M}{\sqrt{L}}, \qquad M = \frac{\sqrt{2}}{\pi}\int_{0}^{\infty} \frac{A(x)\,dx}{\sqrt{x}}, \qquad (3.11)$$

for $L \gg \epsilon$, where M is a constant which does not depend on the length of the twin.†

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*Here the following analogy exists with electrostatics: In the same way that taking account of the forces of nonelectrical origin is frequently decisive for the description of the equilibrium of a system of electric charges, in the theory of elasticity the forces of inelastic origin often play a decisive role in connection with the investigation of the equilibrium of a system of dislocations.

*Any arbitrary elastic field created by a concentrated load possesses this property.

†The constant M differs from the modulus of coupling in article [33] by the factor $1/(\pi\sqrt{2})$, and from the definition adopted in articles [52,59,60] by the factor $1/\pi$.

From the definition (3.11) of the modulus M, it follows that its order of magnitude is given by

$$M \sim \sqrt{\varepsilon} S_{\text{s}}^{0}. \tag{3.12}$$

S_S^0 characterizes the stress which is required in order to move the leading dislocation (the partial dislocation schematically shown in Fig. 6), and in order of magnitude it is given by

$$S_{\text{s}}^{0} \sim \frac{\beta}{b}, \tag{3.13}$$

where β denotes the energy per unit area of the stacking defect associated with this partial dislocation. For an estimate one can assume that it is equal to the doubled value of the surface energy of a coherent twinning boundary (for example, for calcite by using the results of[59,60] we obtain $\beta \sim 2\alpha \sim 70$ erg/cm^2 and $S_S^0 \sim 7 \times 10^3$ kg/cm$^2 \sim 3 \times 10^{-2}\,\mu$). Apparently in the majority of cases $S_S^0 \sim 10^{-1}$ to $10^{-2}\,\mu$.

Since the function $S_S(x)$ differs from zero only at the end of the twin, in all of the calculations where we are interested in the properties of the twin at points far removed from its ends, this function can be replaced by a concentrated function having the proper normalization. It is easy to see that this change can be realized with the aid of a δ-function of the following type:

$$\begin{aligned} S_{\text{s}}(x) &= \frac{\pi}{\sqrt{2}} M \sqrt{L-x}\,\delta(L-x) \quad \text{for} \quad x>0, \\ S_{\text{s}}(x) &= \frac{\pi}{\sqrt{2}} M \sqrt{L+x}\,\delta(L+x) \quad \text{for} \quad x<0. \end{aligned} \tag{3.14}$$

By using Eq. (3.11) one can write Eq. (3.7) in the form

$$F(L) = S_0 + \frac{M}{\sqrt{L}}. \tag{3.15}$$

For known values of S_0 and M and for a given function $\sigma(x)$, relation (3.15) is the equation for the determination of the length of the twin. Thus, the solution of (3.5) and (3.6) together with this equation makes it possible, for known external stresses and known forces of inelastic origin, to completely determine the mechanically stable shape of an elastic twin.* However, it is important for us to call attention to a certain physical distinction between Eqs. (3.4) and (3.15), which is related to the different degree of particularization of the function $S_S(x)$. In order to describe the profile of the twin (in any event near its ends), it is necessary to know the exact form of the function $S_S(x)$ which appears in the integrand in Eq. (3.6b). But the form of this function is essentially determined by the nature of the interatomic interaction forces. This property places the function $S_S(x)$ under special conditions within the framework of our theory. In particular, one can hardly propose any kind of macroscopic experiment for the determination of the form of this function. Therefore, one can only obtain some kind of information about $S_S(x)$ by means of the analysis of microscopic models of twinning dislocations. Unfortunately, the exact form of the potentials of the interatomic interactions is not known to us, and it is impossible to regard such a method for the determination of $S_S(x)$ as sufficiently reliable.

On the other hand, Eq. (3.15) for the length of the twin includes the forces of inelastic origin only in the form of the two parameters S_0 and M. Since the length of the twin is its macroscopic characteristic, the quantities S_0 and M in Eq. (3.15) can be regarded as phenomenological parameters. Therefore, one can propose a method for the experimental determination of the quantities S_0 and M. The setting-up of a quantitative experiment under conditions which are as close as possible to those treated in the theory was proposed and realized in articles[59-61] where the values of S_0 and M were determined for calcite ($S_0 \approx 0.2$ to 0.3 kg/cm^2, $M \approx 1$ kg/cm$^{3/2}$). In our opinion, the possibility of an experimental determination of the major parameters of the model is the major merit of the semi-microscopic theory of elastic twins.

Returning to the question of the length and thickness of the twin, let us return to Eq. (3.15), considering the function $\sigma(x)$ to be positive for all values of x and integrable. First let us assume that forces of inelastic origin are not present. Then from the positiveness of the function F(L) and from the form (3.10) of its asymptotic behavior, it follows that for $S(x) = 0$ the only possible solution of Eq. (3.15) corresponds to $L = \infty$, i.e., to an infinitely long twin.

Thus, we arrive at the following assertion: If the forces of inelastic origin were not present, then a free, stable twin of finite dimensions could not exist in a solid under the influence of external loadings of constant sign.

It is, of course, possible to imagine an alternating field of elastic stresses which might stabilize the twin even in the absence of forces of inelastic origin.* However, in connection with the study of elastic twins the case of a distribution of stresses, all having the same sign, is the case of most interest since the possibility of a twin being in equilibrium under such conditions is of fundamental importance.

Since the length of the twin would be infinite for $S(x) = 0$, by virtue of the continuity of the functions appearing in Eq. (3.15) the finite equilibrium length of an elastic twin in a crystal is large when the forces of inelastic origin are reasonably small.

Keeping in mind the last conclusion, let us consider the case when the forces of inelastic origin are very small, that is, when S(x) and therefore also the right-hand side of Eq. (3.15) are vanishingly small quantities. Then the length L of the twin is very large and, as follows from Eqs. (3.15) and (3.10), it is determined by the equation

$$\frac{\Phi_0}{L} = S_0 + \frac{M}{\sqrt{L}}. \tag{3.16}$$

It is clear that Eq. (3.16) is valid for $L \gg \epsilon$.

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*If $S_s(x) \to 0$ then Eq. (3.4) gives the distribution of the perfect dislocations along an incomplete slip band, and Eq. (3.15) makes it possible to determine the length of the pile-up in an external field. As $S_0 \to 0$ Eq. (3.4) goes over into the equation which describes the shape of a thin fracture, and Eq. (3.15) actually coincides with the basic equation for the force theory of thin fractures, [33] which determines the length of the fracture in an external field.

*Such a situation occurs in electrostatics, where a specifically selected electric field can ensure the equilibrium of an arbitrary system of charges.

First of all let us estimate the relative role of the two forces represented by the first and second terms on the right-hand side of Eq. (3.16), having constructed the ratio

$$\gamma=\frac{S_0\sqrt{L}}{M}=\varkappa\sqrt{L},\quad \varkappa=\frac{S_0}{M}. \tag{3.17}$$

Since in principle a rather simple analysis of Eq. (3.16) for arbitrary values of γ entails cumbersome calculations, we shall confine the investigation to the two limiting cases $\gamma \gg 1$ and $\gamma \ll 1$.

When $\gamma \gg 1$, in order to obtain the half-length of the twin it is sufficient to set $M = 0$ in Eq. (3.16) (we note that $M = 0$ occurs, for example, for an incomplete slip band):

$$L=\frac{\Phi_0}{S_0}\sim x_0\frac{\sigma(0)}{\sigma_s}. \tag{3.18}$$

From Eq. (3.18) it is seen that the compulsory condition $L \gg x_0$ can be realized only for $\sigma(0) \gg \sigma_s$, that is, only for a very large concentration of stresses at the point where the dislocation source is located.

For $\gamma \ll 1$ the half-length of the twin is given by

$$L=\left(\frac{\Phi_0}{M}\right)^2. \tag{3.19}$$

as is clear from Eq. (3.16). In this case the concentration of stresses which is required in order to create a twin with $L \gg x_0$ must be such that $\sigma(0) \gg M/\sqrt{x_0}$.

Formulas (3.18) and (3.19) explicitly confirm the conclusions reached above to the effect that the length of an elastic twin in a crystal is large if the forces of inelastic origin are sufficiently small. However, the length of the twin itself is not the complete characteristic of its shape, and the ratio of the length of a twin to its thickness is also of physical interest.

Let us consider the thickness $h(0)$ of the twin at its center, having represented it, in analogy with Eq. (3.5), in the form of the sum

$$h(0)=h_0(0)+h_s(0). \tag{3.20}$$

The first and second terms on the right-hand side are defined in terms of ρ_0 and ρ_s, respectively, according to formula (1.1).

Before carrying out the calculations of h_0 and h_s, let us call attention to the following very important property: The thickness of a twin is completely determined by the number of dislocations emitted by the source. Since the concentration of external stresses in the neighborhood of the source of dislocations is always large, their role in this region is appreciably greater than the role of the forces of inelastic origin. In connection with this, the thickness of the twin at its center is practically completely described by the term h_0. Quantitative estimates carried out in article[46] confirm this conclusion. Therefore, in order to estimate $h(0)$ it is sufficient to calculate h_0. Using Eqs. (1.1) and (3.5), let us represent h_0 in the form

$$h_0=\frac{La}{B\pi^2}\int_{-1}^{1}\frac{\sigma(L\xi)\,d\xi}{\sqrt{1-\xi^2}}\,\fint_0^1\frac{\sqrt{1-x^2}\,dx}{x-\xi}.$$

We again recall that the condition $L \gg x_0$ is assumed to be satisfied for the twins we are considering. But, in this connection, asymptotically one finds

$$h_0=\frac{La}{B\pi^2}\int_{-1}^{1}\sigma(L,\,\xi)\,d\xi\,\fint_0^1\frac{dx}{x-\xi}\approx-\frac{2La}{B\pi^2}\int_0^1\sigma(L\xi)\ln\xi\,d\xi.$$

Therefore, in order of magnitude one has

$$h_0\sim\frac{a}{B}\ln\left(\frac{L}{x_0}\right)\Phi_0\sim\frac{\ln\left(\frac{L}{x_0}\right)}{\left(\frac{L}{x_0}\right)}\frac{\sigma(0)}{\mu}L. \tag{3.21}$$

Since the explicit dependence of the length of the twin on the effective forces can be obtained only in the limiting cases (3.18) and (3.19), we shall confine our investigation to precisely these cases.

When $\gamma \gg 1$, then from Eqs. (3.18) and (3.21) it follows that the ratio of the thickness of the twin to its length is of the following order of magnitude:

$$\frac{h}{L}\sim\frac{\sigma_s}{\mu}\ln\left(\frac{\sigma(0)}{\sigma_s}\right). \tag{3.22}$$

The estimate (3.22) indicates that, for a vanishingly small value of σ_S/μ, the ratio h/L tends to zero since $\sigma(0)$ is always less than μ. As to the dependence of h/L on the external stresses $\sigma(0)$ at the point where the dislocation source is located, one should expect a weak logarithmic increase of the ratio h/L with increasing values of $\sigma(0)$.

In the case $\gamma \ll 1$, by using Eqs. (3.19) and (3.21) we obtain

$$\frac{h}{L}\sim\frac{1}{\mu}\frac{M^2}{\Phi_0}\ln\left(\frac{\Phi_0}{\sqrt{x_0}M}\right). \tag{3.23}$$

From the estimate (3.23) it follows that for a vanishingly small value of $1/\mu(M^2/\Phi_0)$ the ratio h/L tends to zero, and it also follows that in this case the value of h/L must decrease with an increasing value of the external load.

Let us combine the conclusions reached in the two limiting cases into a single statement: If the function $\sigma(x)$ has a constant sign and the forces of inelastic origin are small, then the ratio of the twin's thickness to its length is a small quantity which tends to zero together with the vanishing of the forces of inelastic origin.*

Concluding our discussion of the question about the ratio of the thickness of an elastic twin to its length, we consider it necessary to indicate that the first estimates of h/L within the framework of the macroscopic theory were made by Vladimirskii, who expressed the value of this ratio in terms of the external loading and the constants of the crystal (the modulus of elasticity and the lattice constant). The work of Cooper[62,63] should also be mentioned, in which the thickness of the twin and also the ratio h/L were estimated for a very simplified model of twinning.

In connection with the experimental investigation of elastic twins,[2-11,36,64-66] the small value for the ratio of its thickness to its length has always been mentioned (for example, $h/L \sim 10^{-4}$ for calcite and $h/L \sim 10^{-3}$

*The estimate proposed by Friedel [67] for the ratio of the length of a twin, which is stabilized by only an external elastic field, to its length is not of interest from the point of view of the problem under discussion, since it implicitly assumes that the elastic field is described by an alternating function of the coordinates, and in such a case the length of the twin is determined by the characteristic distance over which the sign of this function changes.

for bismuth and antimony). However, it would appear to us to be desirable to direct attention not toward the determination of the numerical value of this quantity but instead toward an explanation of its tendency to vary with increasing length of the twin, which was done in article[65]. The corresponding experiment was carried out with twins for which the condition $\gamma \ll 1$ is satisfied. But in this case, by combining Eqs. (3.19) and (3.23) we have

$$\frac{h}{L} \sim \frac{1}{\mu} \frac{M^2}{M\sqrt{L}} \ln\left(\sqrt{\frac{L}{x_0}}\right) \sim \frac{1}{2\mu} \frac{M}{\sqrt{L}} \ln \frac{L}{x_0}.$$

Thus, the ratio h/L should decrease with increasing length of the twin. Precisely this result of the theory is confirmed in article[65], where it is shown that with an increase of the loading the twin's length increases more rapidly than its thickness.

In order to estimate the ratio of the thickness of the twin to its length we shall use the fact that h is determined by the external elastic forces. But it is necessary to keep in mind that the shape of the twin essentially depends on the forces of inelastic origin, in particular, on $S_S(x)$. And what is more, one can verify that for materials having a large surface tension, for which one can neglect the force of friction ($S_0 = 0$), the shape of the end of a long twin is completely determined by the surface-tension force.

From Eq. (3.6a) it follows that the law governing the decrease of $\rho_0(x)$ at the end of the twin is described by the function

$$\rho_0(x) = \frac{C_0}{B} \sqrt{L^2 - x^2}, \quad C_0 = \text{const} \sim \frac{\Phi_0}{L^2}. \tag{3.24}$$

Similarly, from Eq. (3.6b) it follows that at the end of the twin we have

$$\rho_S(x) = \frac{C_S}{B} \sqrt{L^2 - x^2}, \quad C_S = \text{const} \sim \frac{1}{\varepsilon} \frac{M}{\sqrt{L}}, \tag{3.25}$$

therefore for $\Delta x \lesssim \epsilon$ one obtains

$$\frac{\rho_0(x)}{\rho_S(x)} = \frac{C_0}{C_S} \sim \left(\frac{\varepsilon}{L}\right) \frac{\Phi_0}{M\sqrt{L}}. \tag{3.26}$$

Thus, for very long twins ($L \to \infty$) the contribution of the density $\rho_S(x)$ plays the fundamental role, and the profile of the twin near its end actually depends only on $\rho_S(x)$. It has the same shape as the end of a thin fracture,[68] and may be represented schematically by the curve shown in Fig. 14. On the basis of formula (1.1), the dependence of the thickness h(x) of the twin on the coordinate is given by the expression

$$h(x) = \frac{2}{3} \frac{aC_S}{B} \sqrt{2L}\,(L - x)^{3/2}, \quad L - x \leqslant \varepsilon. \tag{3.27}$$

The dashed line shown in Fig. 14 represents the curve

$$y(x) = \text{const} \cdot \sqrt{L^2 - x^2}, \tag{3.28}$$

which joins with the true profile of the twin for $L - x \gtrsim \epsilon$. Therefore, the shape of the end of the twin is actually determined only by the surface-tension force and does not depend on the external loading or on the length of the twin (it is thanks to this that the possibility of introducing the constant M arises).

Finally, by starting from physical considerations about the surface-tension force on the twin boundary, it should be possible to indicate the order of magnitude of ϵ. Let us define ϵ as the distance from the end of the twin, at which the twin's thickness is such that the opposite surfaces of the twin cease to influence each other by means of the molecular-interaction forces. Let us denote this thickness by h_k, and we shall utilize (3.27), the estimate of C_S in (3.25) and (3.12). Then we find

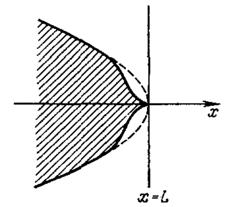


FIG. 14. Diagram showing the profile of the twin on the two outside end segments.

$$\varepsilon \sim \frac{\mu}{S_S^0} h_k. \tag{3.29}$$

If we assume $h_k \sim 10^{-7}$ cm and take $S_S^0 \sim 10^{-1}$ to $10^{-2}\,\mu$, then for ϵ we obtain the estimate: $\epsilon \sim 10^{-6}$ to 10^{-5} cm.

Although the estimate (3.29) is very rough, it indicates that strictly speaking the conclusions about the shape of the end of the twin were reached at the limit of validity of the macroscopic theory which we have been using. And what is more, since the size of ϵ has a semi-microscopic character, in any experiment investigating the macroscopic shape of the twin, a profile will be observed which is close to the one described by the curve (3.28). Precisely such a profile was observed in article[69] for wedge-shaped twins in α-Fe. The observation of a "beak" at the end of the twin requires a very large resolving power of the instruments which are being used to study the profile of the twin.

In the present section we have discussed in detail why a twin is thin, considering the example of a twin created by knife-edge loading, i.e., produced by straight dislocations. However, in connection with the analysis of many questions of twinning in a crystal, in particular, the problem of the nucleation of twins, it is necessary to analyze twins of a different shape. But no matter what shape the free elastic twin has, the ratio of its thickness to its length should obey the same qualitative relationships which have been expounded above. The investigation of an axisymmetric twin in article[46], where it is shown that the ratio of the thickness of such a twin to its radius is small when the forces of inelastic origin are reasonably small, can serve as an illustration of this statement.

4. THE HYSTERESIS ASSOCIATED WITH ELASTIC TWINNING

In the present chapter we shall study the growth of a twin in an unbounded crystal associated with a nonmonotonic dependence of the external loading on the time. The change of the loading will be assumed to be quasistatic, i.e., taking place infinitely slowly and at each moment of time completely described by the dependence of the stresses on a certain external parameter. Let us assume that the function $\sigma(x)$ depends on the parameter P such that with a variation of P from 0 to P_0 the value of $\sigma(x)$ increases monotonically from 0 to $\sigma_P(x) = \sigma(P_0, x)$. Then an increase of P will correspond to an increase of the loading, and a decrease of P corresponds to the process of unloading. In the simplest case the stresses $\sigma(x)$ are proportional to the external load;* then $\sigma(P, x) = P\tau(x)$ and $F(L) = PG(L)$ where $G(L)$ only depends on the length of the twin and is determined by the first formula in (3.8), in which one should substitute $\tau(x)$ in place of $\sigma(x)$.

It appears to us that the most significant indication of the characteristics of the growth of a twin is the variation of its length, which one can analyze by means of a graphical solution of the transcendental equation (3.7). The right-hand side of (3.7) is always a monotonically decreasing function of L, having the maximum value $S^* = S_0 + S_g^0$ at $L = 0$ and asymptotically approaching S_0 as $L \to \infty$ (see Fig. 15, curves 1a and 1b). Therefore, the type of solution of Eq. (3.7) is essentially determined by the form of the function $G(L)$. Let us assume that $G(L)$ is a monotonically decreasing function of its argument, decreasing more slowly than $J(L)$. The latter assumption corresponds to the conditions which are usually encountered in an experiment. The point is that an appreciable decrease of $G(L)$ always occurs for macroscopic values of L. This is associated with the small rate of decrease of the function $\tau(x)$ with depth, due to the macroscopic nature of the creation by external loading. As to the function $J(L)$, its fundamental decrease occurs for $L \sim \epsilon$.

First we shall assume that P increases, and let us consider Fig. 15, on which the graphical solution of Eq. (3.7) is shown schematically. For very small values of P the graph of the function $F(L)$ lies below curve 3; therefore Eq. (3.7) does not have any solutions. With increasing values of P, the graph of the function $F(L)$ touches curve 3 for a certain value $P = P_{min}$ at the point (P_{min}, L_{min}). A solution of Eq. (3.7) appears at this point. Upon a further increase of P the curves under consideration intersect at two points, which indicates the splitting of the solution of Eq. (3.7) into two solutions (L_1 and L_2 shown in Fig. 15). The solution L_1 decreases with increasing P; therefore, as mentioned in[33], it corresponds to a solution which is unstable with regard to an infinitesimal change in the external loading on the twin. The solution L_2 corresponds to a stable twin, but nevertheless in our method for the creation of a twin, such a twin does not appear. In fact, as long as $F(0) \equiv \sigma(0) < S^*$ the external force applied to the dislocation at the point where the dislocation source is located is smaller than the total force of retardation. Therefore, under the influence of the elastic field the dislocations cannot be "separated" from their source (we recall that nucleation of the dislocations is assumed to be inactive and their multiplication consists in the emission of dislocations from the source due to the external stresses). Therefore, a twin of length L_2 may arise only by means of fluctuations. But if the changes of the parameter P under consideration take place during the course of a finite time interval, then the fluctuation mechanism for the formation of a macroscopic twin cannot be taken into account. The possibility of the formation of a twin cannot be realized up to the value $P = P^*$, which is determined from the condition $P^* G(0) = S^*$. At this point the solution $L_1(P)$, corresponding to the unstable twin, disappears and the condition $\sigma(0) = S^*$ is satisfied, i.e., a sufficient condition for the formation of the twin is fulfilled, thanks to which a twin of finite length (let us denote it by L^*) suddenly appears. With a further increase of P ($P > P^*$) only one solution of Eq. (3.7) remains (the length L_4 indicated on Fig. 15), and the twin smoothly increases its length with increasing load.

A graph of the dependence of L on P for the case under consideration is shown in Fig. 16. The lower segment of the curve $L = L(P)$ for $P < P^*$ corresponds to the unstable twin, and the upper segment of this curve for $P < P^*$ corresponds to the length of the twin which is produced only by means of fluctuations (therefore, the corresponding part of the curve is also represented by a dashed line).

The dependence $L(P)$ will be somewhat different if it is assumed that the monotonic function $F(L)$ decreases with increasing values of its argument more rapidly than $J(L)$. Such a situation is certainly realized for $S_g \ll S_0$ and for any noticeable decrease of $\sigma(x)$, i.e., in particular for the generation of dislocations ($S_g = 0$) due to the influence of a concentrated load. Then, for the creation of the twin it is also necessary to exceed the threshold load P^*; in this case Eq. (3.7) only has a single solution (the stable twin) and the twin starts to grow with zero length.

An investigation of the limiting case of homogeneous loading is of special physical interest. If $\sigma(x) \equiv \sigma(0) = \text{const}$, then $F(L) = \sigma(0)$ and for $S_0 < \sigma(0) < S^*$ Eq. (3.7) has only one solution ($L = L_3$ on Fig. 15), which corresponds to an unstable twin. As soon as the parameter P reaches the value P^*, this solution vanishes. However, formally the solution $L = \infty$ still exists. Therefore, in accordance with the analysis presented above, we may conclude that at $P = P^*$ a

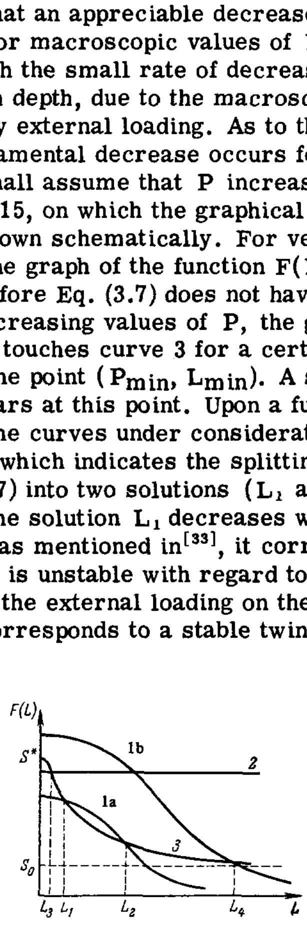


FIG. 15. Determination of the length of a twin by means of a graphical solution of Eq. (3.7). 1—The curve F(L) for the case of an external field which falls off monotonically in the depths of the crystal (Curve 1a corresponds to $P < P^*$, curve 1b corresponds to $P > P^*$); 2—The curve F(L) in the case of a homogeneous external field; 3—The curve $S_0 + J(L)$.

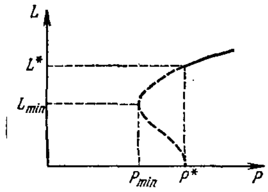


FIG. 16. Dependence of the length of the twin on the loading in the initial stage of twinning.

*A similar situation arises when the application of an external load to the surface of the crystal does not cause any appreciable crumpling of the latter.

twin of infinite length suddenly arises. The physical meaning of such an assertion reduces to the fact that at $P = P^*$ a twinning lamella appears which passes through the entire crystal. Thus, an equilibrium stable twin of finite length cannot be produced in the crystal by a homogeneous field of stresses. In order to create such a twin, a certain concentration of stresses is required, creating an elastic field which falls off sufficiently rapidly with distance. This conclusion is confirmed by the experimental results of Garber.[4,70]

We have already mentioned that usually the twins which are observed in an experiment are not planar and they have the form of "platelets," produced by a loading which is concentrated in a small region on the surface of the crystal.[2-7,20,65,66] The shape of such twins is close to the shape of half of a very thin circular lens, and in the simplest case it may be represented in the form of a cluster of round dislocation loops. An analysis of the growth of a twin in such a model[46] shows that in this case one can repeat all of the basic derivations with regard to the characteristic properties of twin formation.

Summing up the results of the investigation of the creation of the twin under an increasing load, let us note the following important points. In the first place, in any event the twin can arise only when the loading exceeds a certain threshold value. In the second place, the initial stage of growth of the twin is completely determined by the nature of the external loading. In the third case, a stable twin of finite dimensions can be produced only in connection with a concentration of the stresses in the region where the source of the twinning dislocations is located. A homogeneous stress field cannot create a twin of finite length.

Let us compare these conclusions with the experimental results with regard to the formation of elastic twins.

The necessity of reaching the threshold value of the loading for the formation of the twin was mentioned in the experimental articles[2-4,66] in connection with the elastic twinning of calcite and in article[9] in connection with twinning in metals. The role of the concentration of the load in elastic twinning was indicated by Garber[2-4] and was subsequently mentioned by all of the investigators who have studied elastic twinning. In fact, it is precisely due to the application of a concentrated load that Garber was able to observe and study elastic twinning, whereas before the experiments[2] the twinning in calcite had, in the majority of cases, been studied in a homogeneous stress field, when elastic twins of finite length could not be observed. The dependence of the initial stage of growth of the twin on the nature of the external load is illustrated, in our opinion, by the results of the experiments[65]. In this work it was actually shown that the length of the abruptly generated twin increases with increasing size of the region of the significant decrease of the function $F(L)$ which in turn is related to the degree of concentration of the external loading.

After the formation of the twin, the nature of its subsequent growth with increasing load is determined by the form of the function $F(L)$. If $F(L)$ is a monotonic function of L, then the length of the twin will smoothly increase with increasing P. A sudden elongation of the twin may be observed in connection with a nonmonotonic dependence of $F(L)$ on its argument. In this case all of the qualitative conclusions for twins coincide with the results of the theory of fracture.[33] Therefore we shall not discuss them in detail.

Now let us go on to an investigation of the behavior of the twin upon unloading, i.e., upon decreasing the parameter P. Let us assume that the increase of the external loading was discontinued when the value of the parameter $P = P_m$. Then the equilibrium density $\rho_m(x)$ of the dislocations along the twin and its length L_m at the end of the loading process will be determined by Eqs. (3.5) and (3.7), respectively, in which one should set $P = P_m$.

The change of the dimensions and shape of the twin upon unloading will occur at the expense of displacements of its constituent dislocations in a direction opposite to the direction of their motion during loading. But since the sign of the external stresses $\sigma(x)$ does not change upon a reduction in the value of P, the reverse motion of the dislocations is possible only as a result of their interaction and the effect of the surface-tension force. However, it should be kept in mind that the presence of the force of "dry friction," directed against a possible displacement of a dislocation, hinders the displacement of the dislocations during the initial period of unloading, when the decrease in the external force is very small and that part of the interaction force which is uncompensated by this decrease is less than S_0. Therefore, initially the motion of a certain dislocation essentially depends on the ratio of the forces acting on it from the side of the external field and from the side of the remaining dislocations.

To a considerable extent the analysis of the behavior of the dislocations during the unloading process is determined by the nature of the external stresses. We shall confine our attention to the simplest case when the stresses $\sigma(x)$ are monotonically decreasing functions of $|x|$ and, as usual, are directly proportional to the parameter P, that is, $\sigma(x) = P\tau(x)$.

As long as all of the dislocations are stationary, their effect on a unit dislocation at the point x is determined by the stress $\psi_0(x)$, an expression for which follows from the equilibrium equation (3.4):

$$\psi_0(x) = B \int_{-L_0}^{L_0} \frac{\rho_m(\xi)\, d\xi}{x-\xi} = S_0 + S_s(x) - P_m \tau(x). \quad (4.1)$$

The total sum of the stress $\varphi(x) = \varphi(P, x)$ acting on the dislocation from the side of the remaining dislocations, the surface tension, and the external fields is given by the following expression, at the beginning of the unloading process:

$$\varphi(x) = \psi_0(x) + \sigma(x) - S_s(x) = S_0 - (P_m - P)\tau(x). \quad (4.2)$$

In order for the dislocations to begin to move, the magnitude of the corresponding force must exceed the force of friction, even if only at a single point: $|\varphi(x)| \geq S_0$. The feasibility of achieving this condition is determined by the value of $\sigma_m(0)$ and by the relation between S_0 and S_s^0, and depending on the relative values of these quantities there may be three different cases.

Case 1. $S_s^0 < S_0$ and $\sigma_m(0) < 2S_0$. In this case, as follows from Eq. (4.2),

$$-S_0 < \varphi(x) < S_0,$$

and therefore the force $\varphi(x)$ cannot exceed the force of retardation for any value of the parameter P. Therefore, not a single dislocation is displaced during the unloading process, and the shape of the twin after unloading remains the same as it was at the end of the loading process. Thus, a twin possessing a small surface energy, produced by a comparatively small external force, does not change its shape and dimensions after the removal of the loading.

Case 2. $M < \sqrt{L_m}\, S_0$ and $\sigma_m(0) > 2\, S_0$. In this case, as long as the largest value of the absolute magnitude of the force $\varphi(P, x)$ is smaller than S_0, all of the dislocations will be found at their old positions, since at no single point does the force applied to the dislocation exceed the retardation force S_0. Upon a further decrease of P, the behavior of the dislocations depends on the form of the function $\varphi(P, x)$ and, in particular, it depends on the position of the largest (in absolute magnitude) negative value of $\varphi(x)$. In our case this point will correspond to the minimum of $\varphi(P, x)$ for $x = 0$.

For the value of the parameter $P = P_K$, determined from the condition $\varphi(P_K, 0) = -S_0$, i.e., for $P_K = P_m - 2S_0/\tau(0)$, at the center of the twin the force $\varphi(0)$ is comparable with the retardation force S_0, and upon a further decrease of P exceeds it, causing a displacement of the dislocations in the central part of the twin. We note that the difference $P_m - P_K$ does not depend on the length of the twin, and for a given type of loading it is a fixed quantity.

The possibility of the motion of the dislocations for $P > P_K$ in a certain neighborhood of the point $x = 0$ leads to their redistribution. Let this redistribution encompass the interval $(-x_0, x_0)$, outside of which the dislocations remain fixed. Then in the interval $x_0 < |x| < L_0$ the density of the dislocations is described as usual by the function $\rho_0(x)$, but in the interval $(-x_0, x_0)$ one can easily obtain the following expression for the density of dislocations:[45]

$$\rho(x) = \rho_0(x) + \frac{P_m - P}{\pi} \sqrt{x_0^2 - x^2} \int_{-x_0}^{x_0} \frac{\tau(\xi)\, d\xi}{(\xi - x)\sqrt{x_0^2 - \xi^2}}, \tag{4.3}$$

where the quantity x_0 is determined from an orthogonality condition analogous to (3.7), namely

$$\int_{-x_0}^{x_0} \frac{\tau(x)\, dx}{\sqrt{x_0^2 - x^2}} = \frac{2\pi S_0}{P_m - P}. \tag{4.4}$$

One can easily see that under our assumptions about the function $\tau(x)$, Eq. (4.4) defines a monotonically decreasing function $x_0 = x_0(P)$. Formally, the limiting value $x_0(0)$ at the end of the loading process (for $P = 0$) is given by the equation

$$\int_{-x_0(0)}^{x_0(0)} \frac{\tau(x)\, dx}{\sqrt{x_0^2 - x^2}} = \frac{2\pi S_0}{P_m}. \tag{4.5}$$

But Eq. (4.5) has meaning only in the case when $x_0(0) \le L_m$. Comparing Eq. (4.5) with (3.7) and by taking into account the assumed properties of the function $\sigma(x)$ ($\tau'(x) \le 0$ for $x \ge 0$), one can easily verify that for $M < \sqrt{L_m}\, S_0$ the limiting value $x_0(0)$ is always less than L_m.

Thus, when $M < \sqrt{L_m}\, S_0$ a twin remains in the medium even after complete unloading (a "trapped" twin), and the distribution of the dislocations along the twin is substantially different from (3.5). To be sure, at the end of the twin $(x_0(0) < |x| < L_m)$ this distribution as usual is described by the function $\rho_m(x)$; however, in its center part the density of the dislocations is given by expression (4.3) for $P = 0$, when $x_0 = x_0(0)$.

Therefore, if the twin is produced by the action of a force of very large magnitude, so that $\sqrt{L_m}\, S_0 > M$, then after unloading its length is unchanged, the shape of the twin near the ends $(x_0(0) < |x| < L_m)$ remains unchanged, but the thickness of its center part is decreased.

The difference between the function $\rho(x)$ associated with the unloading and the density of dislocations associated with loading implies that *twinning exhibits hysteresis*. In the present case it is convenient to illustrate the hysteresis in the dependence of the thickness h of the central part of the twin on P for an increase and subsequent decrease of the external force.

As was shown above, for an increase of P from 0 to P^* the twin is absent $(h = 0)$; at $P = P^*$ the twin is created with, let us assume, a thickness h^*, and with a subsequent increase of P its thickness in the simplest case increases monotonically (see Fig. 17), reaching the value h_m at $P = P_0$. Upon a change of P in the opposite direction (from P_m to 0) the thickness h remains constant for values $P_K < P < P_m$ and *monotonically decreases (in the simplest case*) in proportion to the further decrease of* P. At the end of the loading process, the thickness of the twin differs from zero by a certain definite amount Δh which is smaller than h_m (see Fig. 17).

An important conclusion follows from the nature of the hysteresis considered above: A plane twin inside a solid (formally—a twin inside an unbounded medium) remains inside the crystal even after the removal of the stresses, provided that it possesses a sufficiently large length so that $\sqrt{L_m}\, S_0 > M$. This conclusions is always valid for slip (for perfect dislocations $S_s = 0$).

The question may arise as to whether a frictional force is distributed along the twin which is "trapped" in the crystal. It should be noted that up till now we have only specified the limiting values of the friction force, considering them to be equal to the quantities $\pm S_0$ (depending on the direction of the quasistatic motion of the dislocation). However, if the dislocation is fixed and the forces applied to it are not known beforehand, then the friction force is not determined, al-

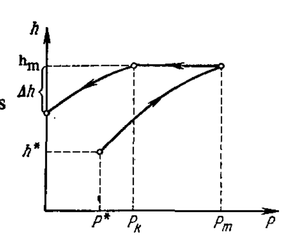


FIG. 17. Dependence of the thickness h of the central part of the twin on P upon loading and unloading.

*This will always occur for a monotonically decreasing function $\sigma(x)$.

though it is bounded: $|s| \leq S_0$. The force s and its distribution $s = s(x)$ along the twin are completely determined by the condition for equilibrium of the dislocations under the action of the remaining forces. In the case of a twin which is trapped in the crystal (when $\sigma(x) = 0$) under the unloading process described above, the force $s(x)$ can easily be determined on the basis of Eqs. (3.4) and (4.3).

Case 3. $M > \sqrt{L_m} S_0$. In this case, for a certain value of the parameters P, Eqs. (4.4) for $x_0 = x_0(P)$ has the solution $x_0(P_1) = L_m$ corresponding to that instant when, in the returning motion, the dislocations arrive along the entire length of the twin. During the subsequent decrease of P, the density of the dislocations along the twin is described by formula (3.5), where the half-length L of the twin is determined by the solution of the transcendental equation

$$F(L) = \frac{M}{\sqrt{L}} - S_0. \tag{4.5'}$$

A simple graphical analysis of Eq. (4.5′) shows that the twin disappears ($L = 0$) for $P = P_2 = S_g^0 - S_0/\tau(0)$. It is easy to see that the difference between the loadings at which the twin appears and disappears is determined by the relation $P^* - P_2 = 2S_0/\tau(0)$ and does not depend on the length of the twin. It is interesting to note that this quantity exactly coincides with the length of the hysteresis segment on the curve showing the dependence of the twin's thickness on the load $P^* - P_2 = P_m - P_K$ (see Figs. 17 and 18). The magnitude of the hysteresis segment $\Delta P = P_m - P_1$ on the curve $L = L(P)$ (see Fig. 18) significantly depends on the length of the twin. In the simplest case, when $F(L) = PG(L)$, the following expression for this quantity follows from Eqs. (3.15) and (4.5):

$$\Delta P = \frac{2S_0}{G(L)}. \tag{4.6}$$

Since the lengths of stable equilibrium twins correspond to regions where the function $G(L)$ is decreasing, the value ΔP of the hysteresis region increases with the length L of the twin. For example, from Eqs. (4.6) and (3.10) it follows that for twins of long lengths, ΔP increases in direct proportion to L.

Thus, in the case $M > \sqrt{L_m} S_0$ the hysteresis associated with twinning may also be characterized by a different dependence of the length L of the twin on the parameter P for loading and unloading. After the creation of the twin at $P = P^*$, an increase in the length of the twin occurs with an increase of P, where the length reaches the value L_m for $P = P_m$ (Fig. 18). When the parameter P decreases, then at first the redistribution of the dislocations is not associated with a change in the length of the twin ($P_1 < P < P_m$) and only for $P < P_1$ does the twin become contracted, disappearing under the load corresponding to $P = P_2$.

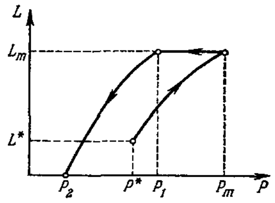
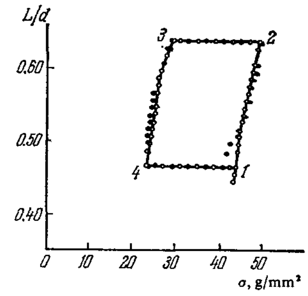


FIG. 18. The hysteresis loop on the curve showing the dependence of the length L of a twin on the external load P.

Therefore, a relatively short twin having a large surface energy ($M > \sqrt{L_m} S_0$) vanishes after the removal of the load which produced it. In comparing the different cases, which we have been analyzing, for the behavior of the twin upon unloading with the experimental observations, we wish to emphasize one important result of the expounded theory. If the process of contraction of the twin has started during the process of unloading, then such a twin inevitably escapes from the crystal upon the complete removal of the loading (it exists in the form of an elastic field which decreases monotonically with depth). The "wedging" of the twin after its partial contraction,[8] which has been noted in experimental work, might be due to the appearance of additional obstacles (associated with the elastic fields of the glide dislocations which generate the accomodation band, or other types of obstructions) near the base of the twin.

Quantitative measurements of the hysteresis associated with elastic twinning in calcite were carried out in article[60]. The corresponding experimental curve is shown in Fig. 19, showing the dependence of the twin's length (in terms of a convenient scale) on the macroscopic loading. The 1—2 segment of the curve corresponds to the loading stage (an increase of the load from σ_1 to σ_2 took place). After reaching the value σ_2 a decrease of the loading occurred, during which the length of the twin remained unchanged (the segment 2—3 shown of Fig. 19) down to the value σ_3 for the loading. The amount of hysteresis, $\Delta\sigma = \sigma_2 - \sigma_3$, associated with the given method of loading turned out to be appreciable, which made it possible to carry out the measurements with sufficient accuracy. The further reduction of the loading was accompanied by a decrease in the length of the twin (the segment 3—4 shown on Fig. 19). After the reduction of the loading to the value σ_4, at which the length of the twin coincided with its initial value, the unloading process was discontinued. Subsequent loading (from σ_4 to σ_1) generated the second horizontal segment on the diagram under consideration (the interval 4—1 shown on Fig. 19), which closes the hysteresis loop indicated on the graph by the open circles. In order to verify the reproducibility of the results, the cycle of measurements was repeated. It was found that the second hysteresis loop (it is indicated on Fig. 19 by the black circles) agrees with the first loop to within an accuracy of tenths of a percent, which indicates that the reproducibility of the experiment is quite satisfactory. From Fig. 19 it is seen that $\sigma_2 - \sigma_3 > \sigma_1 - \sigma_4$; this is experimental confirmation of the assertion made above about the depend-

FIG. 19. Dependence of the relative length of the twin on the macroscopic load. [60] d denotes the width of the crystal; the open circles correspond to the experimental points on the first hysteresis loop, and the black circles represent experimental points on the repeated loop.



ence of the amount of hysteresis on the length of the twin.*

The hysteresis associated with twinning is completely due to the presence of the force of retardation, which has the nature of a force of dry friction; therefore the experimental observation[60] of the hysteresis loop is a very reliable confirmation of the existence of such a type of force. We are forced to make such a comment in connection with the fact that certain investigators (see, for example,[71]) ascribe to the force of retardation the nature of a *viscous friction* (the force of viscous friction vanishes together with the velocity of the dislocation, and it cannot lead to hysteresis under quasistatic conditions).

The analysis of hysteresis carried out by us was based on an investigation of the length of a mechanically stable twin, which corresponds to a certain instantaneous state associated with a quasistatic process. We have verified that one and the same macroscopically determined state (specified by the value of P and, for example, the temperature of the crystal) correspond to at least two limiting values for the length of the twin (for infinitely slow loading, when the force of friction is given by $s(x) = -S_0$, and for infinitely slow unloading when $s(x) = S_0$). We have already mentioned above that in the general case the only requirement imposed on the function $s(x)$ is that its absolute value be bounded, $|s(x)| \leq S_0$; therefore, depending on the loading conditions and, in particular, on the alternation of the periods of loading and unloading, twins of different shapes and lengths can be generated, corresponding to different functions $s(x)$. In this case, when the length L of a symmetric twin appreciably exceeds the distance over which the external loading falls off substantially, it is determined as the solution of the equation

$$\frac{\Phi_0}{L} = \frac{M}{\sqrt{L}} - \int_{-L}^{L} \frac{s(x)\,dx}{\sqrt{L^2 - x^2}}. \qquad (4.7)$$

If it is assumed that $M \gg \sqrt{L}\, S_0$, then in the linear approximation with respect to $\sqrt{L}\, S_0/M$ the solution of Eq. (4.7) can be written as

$$L = L_0\left[1 + \frac{2\sqrt{L_0}}{M} \int_{-L_0}^{L_0} \frac{s(x)\,dx}{\sqrt{L_0^2 - x^2}}\right], \qquad L_0 = \left(\frac{\Phi_0}{M}\right)^2. \qquad (4.8)$$

Thus,

$$L_0\left(1 - \frac{\Delta L}{L_0}\right) \leqslant L \leqslant L_0\left(1 + \frac{\Delta L}{L_0}\right), \qquad \frac{\Delta L}{L_0} = \frac{2S_0\sqrt{L_0}}{M}, \qquad (4.9)$$

where the longest length corresponds to $s(x) = S_0$ and the shortest length corresponds to $s(x) = -S_0$. The limiting values of $L = L_0(1 \pm \Delta L/L)$ as a function of the parameter P are schematically shown on Fig. 20 in the form of the curves 1 and 2. The hatched region shown on Fig. 20 consists of the points (P, L) representing the mechanically stable states of the twin. The

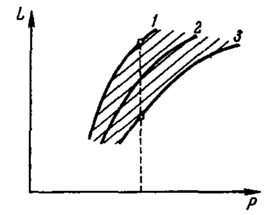


FIG. 20. The relation between the length L of a twin and the external loading P as a function of the magnitude and sign of the friction force s(x). 1–s(x) - const = $-s_0$; 2–s(x) = const = 0; 3–s(x) = const = s_0.

intersection of this region with the straight line P = const determines the range of the possible equilibrium lengths for a fixed value of the parameter P. But the length of the twin is a macroscopic characteristic of the system; therefore, the presence of a certain range for its equilibrium values indicates that the mechanically stable twins being analyzed do not correspond to thermodynamical equilibrium of the crystal.

The qualitative difference between mechanically stable and thermodynamically stable twins was first pointed out by Lifshitz.[13] However, in the macroscopic theory[13] the parameters and relationships are not present which would permit one to derive a quantitative description of the difference between the shapes of a twin in thermodynamical equilibrium and those of twins in mechanical equilibrium. Therefore, we shall follow the expounded dislocation theory of twins, and we shall explain which twin, out of all of those which are in mechanical equilibrium under a given external loading, is the one in thermodynamical equilibrium.

Thermodynamical equilibrium of an elastic medium *containing a thin twin corresponds to the minimum of* the quantity

$$\mathscr{E} = E_{\mathrm{el}} - \oint_{\Sigma} \sigma_{ik}^0 u_i\, d\Sigma_k + E_s,$$

where E_{el} denotes the elastic deformation energy, E_s denotes the surface on the twinning plane, σ_{ik}^0 denotes the external loading on the surface of the solid, and the integration in the surface integral is taken over the surface $\sum$ of the elastic solid.

It is easy to verify that[72]

$$E_{\mathrm{el}} - \oint_{\Sigma} \sigma_{ik}^0 u_i\, d\Sigma_k = \frac{1}{2}\int \sigma_{ik}^e e_{ik}^e\, d\Omega + \frac{1}{2}\int \sigma_{ik}^t e_{ik}^t\, d\Omega \qquad (4.10)$$
$$- bH \int_{-L}^{L} dx\, \rho(x) \int_{0}^{x} \sigma_{xy}^e(x')\, dx' - \int_{\Sigma} \sigma_{ik}^e u_i^e\, d\Sigma_k,$$

where the superscript e, just as earlier, denotes the external field, and the superscript t denotes the field created by the twin itself, that is, by those dislocations which generate it. The length of the dislocation along the Z axis is denoted by H.

It is a characteristic feature that only the displace*ments created by the external stresses on the surface* of the solid appear in the integrand of the surface integral in Eq. (4.10).

The surface energy E_s, generating the "force" $S_s(x)$, is determined by the relation

$$BE_s = H \int_{-L}^{L} dx\, \rho(x) \int_{0}^{x} S_s(x')\, dx'. \qquad (4.11)$$

Since the field of the external stresses is assumed to be fixed, then the part of the quantity E of interest to us is given by

*Let us recall that hysteresis in regard to the length of an elastic twin was observed by Garber. [4] Detailed investigations of this effect were then carried out by Williams and Cahn. [66] However, the application of a concentrated loading, and the uncontrollable changes in the conditions of contact at the surface of the sample associated with this, do not permit one to isolate the hysteresis in a pure form, and these factors affect the reproducibility of the experiment. Therefore, in our opinion, the methods used by the authors of articles [4,66] do not give the possibility of making quantitative measurements of the hysteresis.

$$\Delta\mathscr{E} = \frac{1}{2}\int \sigma^t_{ik}\varepsilon^t_{ik}\,d\Omega - H\int_{-L}^{L} dx\,\rho(x)\int \frac{\sigma(x')\,dx'}{B} + E_s. \quad (4.12)$$

The first term in Eq. (4.12) determines the interaction energy of the dislocations which generate the twin; therefore

$$B\cdot\frac{1}{2}\int \sigma^t_{ik}\varepsilon^t_{ik}\,d\Omega = -\frac{H}{2}\int_{-L}^{L} dx\int_{-L}^{L} dx'\ln|x-x'|\,\rho(x)\rho(x'). \quad (4.13)$$

The second term in (4.12) determines the energy of the dislocations in the external stress field.

Substituting (4.11) and (4.13) into (4.12), we obtain ΔE as a functional of the density $\rho(x)$ of the distribution of the dislocations. The function $\rho(x)$, giving the minimum of this functional, satisfies the condition

$$\int_{-L}^{L} d\xi\,\rho(\xi)\ln|x-\xi| + \int_{0}^{x} d\xi\,[\sigma(\xi)-S_s(\xi)] = 0. \quad (4.14)$$

Let us differentiate (4.14) with respect to x:

$$\int_{-L}^{L}\frac{\rho(\xi)\,d\xi}{\xi-x} = \frac{1}{B}[\sigma(x)-S_s(x)]. \quad (4.15)$$

The condition (4.15) is a singular integral equation with respect to $\rho(x)$ and determines the density of the dislocations along the twin which is in thermodynamical equilibrium. This equation formally agrees with Eq. (3.4) if one sets $S_0 \equiv 0$ in the latter. Therefore, all of the formulas which we previously derived and discussed also pertain to the case of the twin in thermodynamical equilibrium, provided one sets $S_0 \equiv 0$ in them. In particular, formula (4.7) is replaced by

$$PG(L) = \frac{M}{\sqrt{L}}, \quad (4.16)$$

and the graph of the function $L = L(P)$ for all values of P is schematically represented by curve 3 in Fig. 20.

On the (P, L) diagram the quasistatic processes involving a change of the mechanical state of the twin will be represented by the following curves, consisting of two different segments. Upon loading, the first segment of the curve for the process is a straight line, passing from the point corresponding to an arbitrary initial state (the point 0 on Fig. 21 a) and terminating on the lower limiting curve (point A on Fig. 21 a). During this part of the process, the twin's length does not change. A further increase of the loading leads to an increase in the twin's length, which is described by the motion along the lower limiting curve (the segment AQ on Fig. 21 a). Similarly, for the unloading process the curve consists of the straight line segment OM

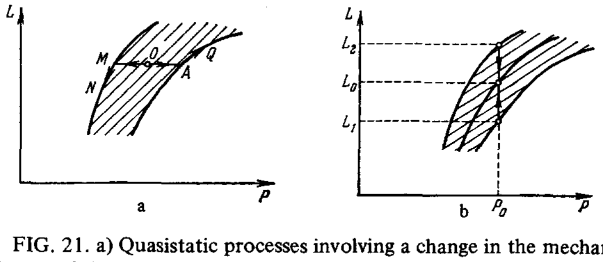


FIG. 21. a) Quasistatic processes involving a change in the mechanical state of the twin. b) Determination of the thermodynamically stable length of a twin under a fixed external loading.

Table I. Determination of the thermodynamically stable length of an elastic twin

Number of the twin	Initial length	Final length	Duration of the experiment, in hours
1	0.33	0.25	51
2	0.45	0.35	170
3	0.48	0.45	96
4	0.62	0.60	83

(L = const) and the segment MN on the upper limiting curve (see Fig. 21 a).

The curves for establishing the length of a twin in thermodynamical equilibrium, corresponding to a fixed external loading, have a different appearance. If a mechanically stable twin, existing in the state (P_0, L_1) during the loading process (see Fig. 21 b), is granted the possibility to go into the state of thermodynamical equilibrium, then its length L will increase with time. The other situation was observed in experiment[60], namely, the establishment of thermodynamical equilibrium in the unloading stage, when the initial state corresponded to the point (P_0, L_2) shown in Fig. 21 b. After completion of the process of partial quasistatic unloading, the loading was fixed and maintained for a long time interval. The observations were carried out as long as the daily change in the length of the twin associated with (the variation of) the room temperature was not comparable to the error in the measurements of the length. In the experimental results which are shown in Table I, it is easy to see a characteristic tendency toward a contraction of the twin. A small heating-up of the crystal led to an intensification of the process involving the establishment of the thermodynamically stable length of the twin.

By comparing the data shown in Table I with a typical experimental graph shown in Fig. 19, one can conclude that the ageing time in the described experiment was not sufficient to complete the process of establishing the thermodynamically stable length of the twin. In fact, under the condition $S_0\sqrt{L} \ll M$, which corresponds to the experimental setup, it follows from Eq. (4.9) that the thermodynamically stable length is given by $L_0 = (1/2)(L_1 + L_2)$ (see Fig. 21 b). Even a very roughly estimated arrangement of the data of Table I onto the graph shown in Fig. 19 shows that the length of the twin at the end of the experiment still appreciably exceeded the value of L_0 predicted by the theory.

Finally, let us point out that a twin which is trapped in the crystal after complete unloading (cases 1 and 2 of hysteresis which were discussed above) must emerge from the crystal during the process of the establishment of thermodynamical equilibrium. This theoretical result may explain the experimental facts related to the annealing of twins which have been trapped in $NaNO_3$,[73] bismuth,[74] and antimony[75] (of course, the process which is being observed in these experiments may be complicated by the overcoming of obstructions).

5. A TWIN IN A LAMINA. THE EXPERIMENTAL DETERMINATION OF THE PARAMETERS OF THE THEORY

The qualitative properties, discussed in the preceding Sections, of the behavior of a twin associated with the variation of the external loading have been analyzed for the example of a twin in an unbounded crystal (or a twin near the surface of a crystal occupying half-space). However, in an experiment one usually has to deal with a twin whose length is comparable to the dimensions of the crystalline sample. Therefore, in order to make a quantitative comparison of the experimental results with the theoretical conclusions, it is necessary to have the corresponding formulas for twins in finite crystals, i.e., for those situations which are as close as possible to the actual experimental conditions. But, in addition to such a purely applied side of the question, the investigation of twins in a finite crystal has important fundamental value for the study of qualitatively new properties of twins, said properties not being present for twins in an infinite crystal.

In order to clarify the last remark, let us briefly discuss the problem of the stability of a twin in a crystal. In[50] it was shown that a twin near the plane surface of a semifinite crystal is always stable, provided that its length appreciably exceeds the dimensions of the region of application of the external loading. This is caused by the fact that at such a depth the stress field already falls off monotonically inside the crystal, as a consequence of which there is a smooth and monotonic increase of the length of the twin with increasing load. But an increase of the length with increasing values of the external loading is an indication of the stability of the twin. However, as long ago as the work[12,13] it was noted that in that case when the length of the twin becomes comparable with the thickness of the crystal, its stability is destroyed. It is clear that a rigorous determination of the boundaries of stability for a twin in a finite crystal can be made only on the basis of a quantitative analysis of the corresponding problem.

As the simplest model of a finite crystal, let us consider a plane-parallel lamina, i.e., a crystal bounded by two parallel free planes. We shall regard the twin in the lamina as planar, produced by a collection of screw dislocations, the twin being perpendicular to the surface and one of its ends emerging onto the surface (see Fig. 22).

Such a twin must be counterbalanced by the surface force, which is directed in parallel to each dislocation line, and which does not vary along it (in the theory of elasticity, the corresponding deformed state is called an antiplane deformation). The choice of the coordinate system is indicated on Fig. 22. In particular, this choice assumes that the dislocation lines are straight lines parallel to the Z axis. The problem of the equilibrium of such a twin was completely solved in article[52], where the explicit form of the transcendental equation which determines the length of the twin was derived in the isotropic approximation. Confining our attention to the case of an isotropic medium, let us present the equation for equilibrium which was derived in[52], and which determines the density of the dislocations along the twin:

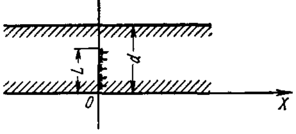


FIG. 22. Model of an elastic twin in a lamina.

$$\int_0^L \frac{\rho(\xi)\sin\frac{\pi\xi}{d}\,d\xi}{\cos\frac{\pi\xi}{d}-\cos\frac{\pi y}{d}} = -\frac{2d}{\mu b\pi}\left[\sigma_{xz}^{e}(y)-S_0-S_s(y)\right], \tag{5.1}$$

where b is the Burgers vector, d is the thickness of the lamina, and μ is the shear modulus.

By making the substitution $x = \cos(\pi\xi/d)$, Eq. (5.1) can be reduced to a simple integral equation with a Cauchy kernel; therefore its solution can be obtained in general form. If one is interested in the bounded solutions of this equation, i.e., if we consider only free twins, then one can easily obtain a relation which is equivalent to the orthogonality condition (3.7); without changing the form in which Eq. (3.7) is written down, let us present the new expressions for the functions F(L) and J(L):

$$F(L)=\frac{1}{d}\int_0^L \frac{\sigma_{xz}(y)\sin\frac{\pi y}{d}\,dy}{\sqrt{\left(1-\cos\frac{\pi y}{d}\right)\left(\cos\frac{\pi y}{d}-\cos\frac{\pi L}{d}\right)}}, \tag{5.2a}$$

$$J(L)=\sqrt{\frac{\pi}{2d}\operatorname{ctg}\frac{\pi L}{2d}}\,M, \tag{5.2b}$$

where the parameter M has the same meaning as before.

Let us consider the transcendental equation (3.7) containing the function (5.2) from the point of view of its utilization for the experimental determination of the basic parameters of the theory, S_0 and M.

If the elastic fields are proportional to the parameter P ($\sigma_{xz}(y) = P\tau(y)$), then we may write down an equation analogous to (3.15), namely

$$PG(L)=S_0+\sqrt{\frac{\pi}{2d}\operatorname{ctg}\frac{\pi L}{2d}}\,M, \tag{5.3}$$

where G(L) is determined by formula (5.2a), in which $\sigma_{xz}(y)$ should be replaced by $\tau(y)$.

In that case when the external loadings are known (i.e., the PG(L)) and the length L of the twin corresponding to these loadings is known, then from Eq. (5.3) one can obtain information about S_0 and M. Thus, relations of the type (5.3) can be directly used for the determination of the parameters of the theory, S_0 and M.

However, let us start with a discussion of the qualitatively new phenomena which are unique for twins in a lamina. First of all, let us turn our attention to the behavior of the function F(L) in the neighborhood of L = d. One can easily verify that the condition $F'(d) = 0$ always holds when the external stresses are continuous at the point where the line of twinning emerges onto the opposite boundary (y = d) of the crystal slab.

We also note that, from Eq. (5.2) it follows that

$$F(d)=\frac{1}{d}\int_0^d \sigma_{xz}(y)\,dy=\frac{Q_z}{d}, \tag{5.4}$$

where the integral determines the total force on the twinning plane, and Q_Z is equal to the z-component of the total force which is applied to the external surface of the crystal from one side of the twinning plane (if the twinning plane is taken as the plane $x = 0$, then we are talking about the "right-hand" side, $x > 0$). The last part of Eq. (5.4) is obtained on the basis of the static equations of equilibrium, from which it also follows that the total force applied to the external surface of the sample from the other side of the twinning plane ($x < 0$) differs from Q_Z only by its sign.

It is interesting to note that, in that case when all of the external stresses directed along the Z axis are distributed on a single side of the twinning plane, then $F(d) = 0$.

As to the function $J(L)$, its behavior in the neighborhood of $L = d$ is also of interest. One can easily see that for $d - L \ll d$ one can write

$$J(L) = \frac{\pi M}{2d}\sqrt{d-L}, \tag{5.5}$$

from which it follows that $J(d) = 0$ and $J'(d) = -\infty$.

Having some general ideas about the functions $F(L)$ and $J(L)$, let us go on to an analysis of the growth of a twin. Under small external stresses (small P), as long as $L \ll d$, the nature of twin growth will be the same as in a semi-finite medium, and the properties of twin growth which are characteristic for a finite crystal appear only for $L \sim d$.

Let us assume that during the loading process the condition $Q_Z < S_0 d$ is always satisfied (in order to realize such a condition, it is sufficient to assume that the curve $F(L)$ at any arbitrary instant of loading would cross the axis of abscissas at a single point). Then, from an elementary analysis of the graphs shown in Fig. 23 it follows that the equilibrium length of the twin turns out to always be smaller than the thickness of the sample ($L < d$). In the simplest case, when $F(L) = PG(L)$ and $G(L)$ is an alternating function, one can easily confirm the existence of a maximum length L_{max}, which cannot be exceeded for any finite loading. It is obvious that L_{max} is the smallest root of the equation $G(L) = 0$. It may turn out that $L_{max} = d$ (as will occur, for example, when the external forces are applied from a single side of the twinning plane). In this connection, under the influence of the increasing values of the loading, the twin will approach arbitrarily close to the opposite surface of the crystal, without ever reaching it for any finite value of the external force.

If $Q_Z > S_0 d$ starting with a certain loading and if the curve $F(L)$ does not intersect the axis of abscissas, then for a sufficiently large external force the twin passes through the entire lamina (it "jumps" through the entire sample). In this situation the question of how the concluding phase of crossing the crystal is achieved is of fundamental interest. The latter problem is directly related to the question of the stability of an elastic twin in a finite crystal. The physical reason for the possible loss of the twin's stability has a simple interpretation in terms of dislocations. The presence of the opposite crystal surface generates a force which attracts the dislocations toward it and under certain conditions the effect of this force may turn out to be decisive for the dislocations at the end of the twin. The twin ceases to continuously follow after the increase of the loading, and upon a further increase of the external force it abruptly increases its length to $L = d$.

In the same way as in the force theory of fracture,[33] the formal criterion for the stability of a twin with respect to infinitesimal changes of the external load reduces to the requirement

$$\frac{dL}{dP} > 0. \tag{5.6}$$

It is easy to show that if the length of the twin can increase up to $L = d$, then requirement (5.6) must be violated in any case for twin lengths which differ very slightly from the thickness of the sample ($d - L \ll d$). From the condition $F'(d) = 0$ it follows that for $d - L \ll d$ the graph of the function $F(L)$ has the form of a horizontal straight line, $F(L) = F(d)$, which intersects the curve $S_0 + J(L)$ for values of $F(d)$ slightly exceeding the values S_0. Using the behavior (5.5) of the function $J(L)$ in the region under discussion, one can easily determine the dependence of the twin's length on the loading:

$$L = d - 4\left(\frac{d}{\pi M}\right)^2 [F(P, d) - S_0]^2. \tag{5.7}$$

Consequently, if $(dF/dP) > 0$ then for $F(d) > S_0$, instead of condition (5.6) we obtain the opposite inequality $dL/dP < 0$, showing that a sufficiently long twin necessarily loses its stability, quite independently of the method of its formation.

An examination of the graphical solution of Eq. (5.3), shown in Fig. 24, may serve as an illustration of what has been described above. Let us assume that $F(L)$ is a monotonically decreasing function. Then a region exists in which Eq. (5.3) has two solutions, L_1 and L_2 (curve 1). The second of these solutions does not satisfy the condition for stability. With increasing load, the length L_1 of the stable twin increases. Finally, contact between the graphs of $F(L)$ and $[J(L) + S_0]$ occurs upon reaching a certain value, $P = P_K$, of the loading parameter (see curve 2). At the point of contact $L = L_K$, the derivative dL/dP vanishes as one can easily verify. The solution of Eq. (5.3) vanishes upon a further increase of the loading, that is, the possibility

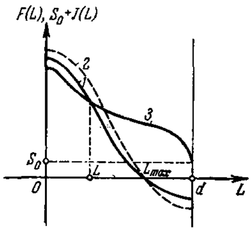


FIG. 23. The graphical solution of Eq. (5.3) for $Q_Z < S_0 d$. 1, 2 denote the curves F(L) for different values of P; 3 is the curve of $S_0 + J(L)$.

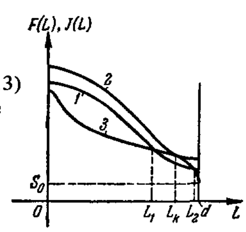


FIG. 24. Graphical solution of Eq. (5.3) for $Q_Z > S_0 d$. 1, 2—curves of F(L); 3—the curve of $S_0 + J(L)$.

zero); the upper surface of the lamina is free for $x > x_a$, and a force of magnitude P parallel to the Z axis is applied to its lower surface at the point $x = x_c > x_a$. Such boundary conditions are similar to the situation realized in the experiment, when the crystal is held in a clamp and an external force is applied to it by means of a thin rod which is glued onto its surface.

The required (by us) component of the stress tensor as a function of y for x = 0 has the form

$$\sigma_{zx} = A \frac{e^{\frac{\pi x_c}{d}} \cos\left(\frac{\pi y}{d} - \varphi\right) - \cos\frac{\pi\varphi}{d}}{\sqrt[4]{\operatorname{ch}\frac{2\pi x_a}{d} - \cos\frac{2\pi y}{d}}\left(\operatorname{ch}\frac{\pi x_c}{d} - \cos\frac{\pi y}{d}\right)}, \tag{5.10}$$

where

$$\varphi = \frac{1}{2}\operatorname{arctg}\frac{\sin\frac{2\pi y}{d}}{\cos\frac{2\pi y}{d} - \exp\left(-\frac{2\pi|x_a|}{d}\right)},$$

$$A = \frac{P}{d}\frac{\sqrt{e^{\frac{2\pi x_c}{d}} e^{\frac{2\pi x_a}{d}} - 1}}{2^{5/4}\sqrt{e^{\frac{\pi|x_a|}{d}} e^{\frac{\pi x_c}{d}}}}.$$

We see that the stress state in the present case can be represented in the form $\sigma_{zx}(y) = P\tau(y)$, that is, the elastic field is proportional to the loading parameter P. Earlier we verified that such a dependence of the elastic stresses on P considerably simplifies the theoretical analysis of the evolution of the twin.

In connection with the performance of the corresponding experiment, it was possible to create an elastic twin consisting of straight screw dislocations, and then to ensure its stability and growth under the influence of a distributed load, which was applied according to a scheme similar to the one shown in Fig. 27. In order to do this, the sample of calcite 1 was cemented into the clamp 2, as shown in Fig. 27. The rod 3 for application of the load was glued on at a small distance from the clamp. Before the experiment a concentrated load, sufficient for the formation of the elastic twin 4 consisting of straight segments of twinning screw dislocations, was applied with the aid of a special set-up. Under the appropriate load P, the twin being produced became stabilized.* Thus, in the experiment it was actually possible to create conditions corresponding to the scheme shown in Fig. 26.

For complete agreement of the experimental situation with the calculated stressed state, it is necessary to demand that the thickness of the sample must be appreciably smaller than its dimensions along the Z axis (see Fig. 26). Unfortunately, this requirement could not be fulfilled. However, estimates[80] indicate that the experimental errors caused by this, and also those arising in connection with the measurements of P and L, are altogether of the order of magnitude of 10%.

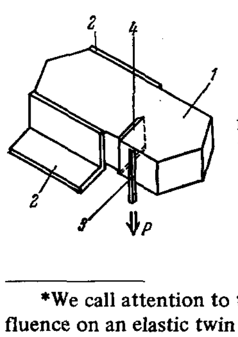


FIG. 27. Diagram showing loading and fastening of the crystal: [59] 1—calcite crystal, 2—clamps, 3—rod, 4—twin.

*We call attention to the fact that the retention and exertion of influence on an elastic twin in a crystal with the aid of a distributed load represents an interesting experimental problem which, so far as we know, was first solved in [59].

The described method of exerting influence on the twin was used in quantitative experiments with regard to the determination of the phenomenological parameters S_0 and M.

First let us describe the determination of the quantities S_0 and M, based on measurements of the dependence of the length of the twin on the load. The scheme used for their determination was as follows. The experimentally obtained values of the load P and the twin lengths L corresponding to them were first substituted into Eq. (5.11), and then the result of the calculation was substituted into (5.2a). After this had been done, relation (5.3) was written down for each pair of values P and L. A systematic analysis of a large number of the thus obtained relations for different twins enables one to rather accurately determine S_0 and M. The results for two crystals are given in Table II.

The use of experimental data about hysteresis is another method which allows one to determine S_0 and M. In fact, the combination of Eqs. (5.3) (for the instant immediately preceding unloading) and (4.6) can be regarded as a system permitting one to completely determine the parameters S_0 and M for a known stress state and for the experimentally determined hysteresis interval ΔP. The corresponding experiments and calculations were carried out in article[60]. The results are presented in Table III. The values of the parameters S_0 and M, obtained from an analysis of the L(P) diagram for the same twins, are presented in parallel.

The calculation of a specific stress state usually presents certain technical difficulties; therefore it is useful to have some relationship which is only slightly sensitive to the stress state. By eliminating G(L) from Eqs. (4.6) and (5.3) we can obtain the following relation between the parameters S_0 and M:

$$\varkappa = \frac{S_0}{M} = \frac{\Delta P}{2P - \Delta P}\sqrt{\frac{\pi}{2d}\operatorname{ctg}\frac{\pi L}{2d}}, \tag{5.11}$$

which does not depend on the distribution of the stresses and can be determined in experiments with hysteresis. By comparing the value of κ, obtained without calculation of the stressed state from experiments on hysteresis, with the value of κ obtained after an analysis of the L = L(P) diagram on the basis of the calculated distribution of the elastic stresses, one can verify the validity of the formulas calculated in article[59].

In the case of crystal No. 1 we have, respectively, 0.26 $cm^{-1/2}$ and 0.28 $cm^{-1/2}$, and for crystal No. 2 we

Table II. Values of S_0 and M (from the dependence of the length of the twin on the value of the load)

Number of the crystal	S_0, kg/cm²	M, kg/cm$^{3/2}$
1	0.50±0.05	1.00±0.12
2	0.31±0.01	1.03±0.02

Table III. Values of the parameters S_0 and M (determined from hysterisis and from the dependence of the length of the twin on the load)

Number of the crystal	From hysteresis		From the dependence L(P)	
	S_0, kg/cm²	M, kg/cm$^{3/2}$	S_0, kg/cm²	M, kg/cm$^{3/2}$
1	0.27±0.02	1.04±0.06	0.28±0.05	1.00±0.06
2	0.49±0.09	0.93±0.09	0.49±0.17	0.92±0.17

have, respectively, 0.53 cm$^{-1/2}$ and 0.53 cm$^{-1/2}$. A comparison of the cited values of κ for each crystal shows that the calculation of the stress state corresponds rather well to the conditions of the experiment.

It is important to note that the spread in the experimental values of M does not exceed the experimental errors. For S_0 the spread in the experimental values from crystal to crystal turns out to be larger than the experimental error. This may be due to differences in the defect structure of the different samples. In order to verify this conjecture, measurements of S_0 and M were made in crystals which necessarily differ very strongly in regard to the number of defects.[81] In crystals in which the density of perfect glide dislocations varied from 10^2 cm^{-2} to 10^4 cm^{-2}, the differences in the values of M did not exceed the experimental errors. As to the quantity S_0, it varied correspondingly within the limits from 0.3 kg/cm² to 1 kg/cm².*

As long as we are talking about what determines the value of S_0, we note that defects which arise in the process of elastic twinning also give a contribution to the force of friction (the appearance of such defects has been observed in antimony[82] and calcite[83]). In[83] it is shown that during the process of multiple repetitions of the loading-unloading cycles the number of defects which are produced increases, and this is accompanied by an increase in the area of the hysteresis loop.†

In comparatively pure single crystals, where the initial density of the perfect dislocations ~10^2 cm^{-2}, the value of S_0 is practically unchanged during such a cycling process. And what is more, estimates show that the average internal stresses from the defects present in such crystals are substantially smaller than the measured value of S_0. Therefore, one can conjecture that the measured value of S_0 in these crystals only slightly exceeds the value of the lattice force of friction on a twinning dislocation.*

The utilization of experimental data about the loss of stability of an elastic twin is the third method enabling us to determine S_0 and M. The set of Eqs. (5.8) for the critical length and (5.3) for the length of the twin at the instant of its loss of stability for the experimentally measured values of P_K and L_K give the values of S_0 and M. Such an approach was used in article[60], where the critical length turned out to be equal to 0.85 to 0.90 of the thickness of the sample. The values of S_0 and M calculated on the basis of an analysis of the loss of stability are presented in Table IV, where the values obtained for the same twin by using the other methods are also given.

In principle the parameter M can also be determined from relation (4.16) for a known stress state of the crystal and the experimentally determined thermodynamically stable length of the twin. However, as mentioned in Chapter 4 the attempt to use this method for quantitative measurements is related to the necessity of maintaining the crystal under a fixed load for an extremely long ageing period.†

The very important circumstance, which we regard as necessary to call attention to, is the fact that three different independent physical experiments give very similar values for the quantities S_0 and M. On this basis we are justified in regarding them as the constants of the real crystal. Thereby the dislocation theory under consideration is freed from model parameters and can be used for a reliable quantitative description of those processes of plastic deformation which are realized by means of elastic twinning.

In conclusion we note that one does not usually use the parameter M as the constant of the material characterizing its tendency toward twinning, but rather a different macroscopic quantity is employed—namely, the coefficient α of surface tension associated with the interphase between the twin boundary and the parent crystal. The relation between M and α for twins can be derived in exactly the same way as the relation between the modulus of coupling and the coefficient for the free surface of a crystal in the force theory of

Table IV. Values of S_0 and M (determined from the critical length, from hysterisis, and from the dependence L(P))

Method of measurement	S_0, kg/cm²	M, kg/cm$^{3/2}$
From the L(P) dependence	0.22	1.01
From hysteresis	0.37	0.87
From the critical length	0.29	0.89

*Such an increase is confined to that which must be generated by the elastic fields of glide dislocations which are uniformly distributed in the crystal with the appropriate densities.

†The observation of defects which arise in connection with elastic twinning in calcite is also reported in the recent work by Kaga and Gilman, [88] and the authors regard these defects as the cause of the hysteresis associated with elastic twinning. In fact, as is shown in [83], in the presence of a large density of perfect dislocations in a crystal an appreciable part of the hysteresis loop is due to the defects which arise in connection with elastic twinning. At the same time, in very good crystals of calcite one is able to make up to 30 cycles of loading–unloading, and the change in the hysteresis loop is found to be within the limits of the experimental errors. It appears to us that this testifies to the fact that in good crystals the hysteresis is due to the lattice force of friction, which corresponds to theoretical ideas.

*Additional evidence in support of the assertion about the lattice nature of the friction force in a crystal containing a small number of defects is obtained in article [89] in connection with measuring the temperature dependence of S_0. The nature of the observed temperature dependence and the qunatitative characteristics turned out to be in agreement with those calculated theoretically for the Peierls model of the force of retardation. [90,91]

†Perhaps the process involving the establishment of the thermodynamically stable length can be accelerated by some kind of weak cyclic action; however, as far as we know such experiments have not been out.

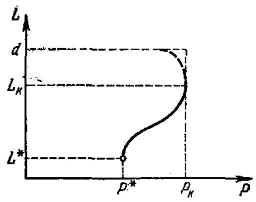


FIG. 25. Dependence of the length of a twin on the loading in a finite crystal.

of the existence of a static equilibrium twin disappears. Since at the instant when $P = P_K$ the twin already had a length L_K, but all of the dislocations generating it experience the effect of an external force along the direction of its growth for $P < P_K$, the subsequent dynamical behavior of the twin must lead to the result that the twin "jumps" through the entire crystal.

Therefore, a certain critical length L_K always exists for the twins under consideration, this length being determined by the distribution of the external forces, and upon reaching this length the stability of the static equilibrium twin is lost. A schematic graph of the dependence $L = L(P)$ is shown in Fig. 25, on which the appearance of the critical length L_K is quite evident. This diagram was constructed for the case shown in Fig. 24.

In the general case the critical value L_K is determined as the largest root of the equation

$$\frac{dL}{dP} = 0. \tag{5.8}$$

In that case when $F(L) = PG(L)$, condition (5.8) is transformed into the following equation for the determination of the critical length of a stable twin:

$$G(L)\,J'(L) = [J(L) + S_0]\,G'(L). \tag{5.9}$$

Thus, only two possibilities can be realized in a finite crystal. If $F(L)$ is an alternating function, then a maximum possible length exists which cannot be exceeded for any finite value of P. However, if $F(L)$ is a function of constant sign, then for sufficiently large values of P (such that $F(d) > S_0$) a loss of stability occurs with a subsequent abrupt transformation of the elastic twin into a residual twinning lamella. A qualitative analysis shows that this rule holds even in the case of elastic twins in a finite crystal with an anisotropic general form,[52] and it also holds for elastic twins near the interface between media possessing different moduli of elasticity.[54] Therefore, the theoretical conclusions expounded above can be used in order to analyze the experimental data concerning the behavior of elastic twins near the boundaries of residual twinning lamella, grain boundaries, and so forth. The loss of the stability of an elastic twin near a residual twinning lamella has been experimentally observed in calcite.[55] The interaction of a twin with a grain boundary is described in[56]. It also appears to us that the loss of the stability of twins in crystals may be one of the causes leading to the discontinuous nature of plastic deformation in polycrystals, when the latter phenomenon is primarily realized by means of twinning (see, for example,[76,77]).

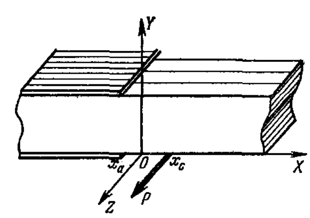
In concluding the discussion of the question of stability, we note that our entire investigation is also applicable to the so-called fractures of longitudinal shear, the development of which was previously analyzed in article[78]. An approximate solution of the problem of the behavior of a fracture of general form near the surface of a solid was previously obtained in article[79].

A somewhat different situation, involving the stability of a twin produced by screw dislocations, must arise in that special case when the twinning plane is strictly parallel to the surfaces of the infinitely extended lamina. It turns out[53] that for such an orientation of the twinning plane, the critical length of the twin is absent; however, the critical value P_K of the loading remains and as the loading approaches this value the twin continuously increases its length, the length tending to infinity as $P \to P_K$.

One should say a few words about the hysteresis associated with twinning in a finite crystal. One can easily verify that all of the qualitative conclusions of the previous section (chapter 4) can also be derived for a twin near the surface of an isotropic solid[50] and for a twin in a lamina.[52] However, the value of the interval $\Delta P = P_m - P_1$ on the hysteresis curve (see Fig. 18) essentially depends on the shape and dimensions of the crystal in which the twinning occurs. It is also clear that it only makes sense to talk about hysteresis in the case of a twin whose length is smaller than the critical length or the maximum possible length.

Finally, let us go on to a description of the experimental determination of the phenomenological parameters of the theory, namely, the quantities S_0 and M. In order to determine these parameters, one should use the theoretical relations connecting S_0 and M with the experimentally observable characteristics of an elastic twin. Its mechanical-equilibrium and critical lengths may be regarded as such quantities, and also the magnitude of the hysteresis interval ΔP on the curve showing the dependence of the twin's length on the loading. Integrals of the function $\sigma(x)$ enter into some of the formulas relating the characteristic lengths of the twin to the quantities S_0 and M; in order to calculate these integrals it is necessary to know the distribution of the elastic stresses along the twinning plane. In other words, in order to have the possibility of making a quantitative interpretation of the experimental results it is necessary to know the stress state of the sample. This stress state must be found by solving the corresponding problem in the theory of elasticity with boundary conditions which are close to those actually realized in the experiment. An exact solution of the problem of the antiplane deformation of a lamina under the following boundary conditions (see Fig. 26) was obtained in article[59]. For $x < x_a$ the upper and lower surfaces of the lamina are clamped together (their displacements are exactly equal to

FIG. 26. Schematic drawing of the antiplane deformation of a lamina.



fracture.[33,32] In the case of a twin produced by screw dislocations we have

$$\alpha = \frac{\pi^3 M^2}{4\mu}. \tag{5.12}$$

If we substitute the values for M given above and the value for the shear modulus along the appropriate plane in calcite into Eq. (5.12), we obtain $\alpha \approx 35$ erg/cm^2. An estimate of the coefficient α according to the approximate formulas given in the article by Vladimirskii,[12] using the experimental data given in[59,60], leads to values of α of the same order of magnitude.[80]

6. THE THEORY OF I. M. LIFSHITZ AND A DISLOCATION DESCRIPTION OF TWINS OF FINITE THICKNESS

In the preceding sections we have repeatedly mentioned the theory of Lifshitz and have turned our attention to the specification and development of formulations of a number of the conclusions of this theory in terms of dislocations. Since the phenomenological theory of twins proposed by Lifshitz[13,14] does not use any concepts about the twinning boundary, then it is important to verify that the dislocation description of a twin leads to results which are in agreement with the conclusions of the macroscopic theory. Simultaneously we wish to point out certain specific results which take the finite thickness of the twin into account.

In this connection, we regard it as useful to give a brief exposition of the theory of twins of finite thickness. The theory proposed by Lifshitz is based on the concept of a specific nonlinear dependence of the stresses σ_{ik} on the deformations u_{ik} in a crystal which is undergoing twinning. Such a dependence is due to the fact that in such a crystal two equilibrium states exist, corresponding to the twinned and the ordinary states of the crystal, and differing by a shearing deformation equal to the angle of twinning. For simplicity we shall assume that the medium is isotropic or possesses cubic symmetry (in the latter case, the axes of the Cartesian coordinate system are assumed to be directed along the symmetry axes of fourth order). Just as earlier, let the X axis coincide with the trace of the twinning plane. Then the graph of the dependence of σ_{xy} on u_{xy} has the form shown schematically in Fig. 28a, where α is the angle of twinning. However, the requirement of mechanical stability with respect to infinitesimal displacements (for example, those associated with thermal vibrations) leads to the result that the states of the crystal associated with

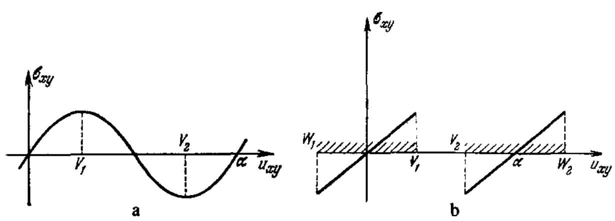
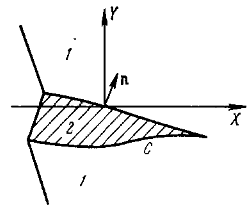


FIG. 28. Nonlinear dependence of the shearing stresses on the deformations in a crystal undergoing twinning. [13] a) Diagram showing the exact dependence; b) Simplified dependence used for small deformations.

FIG. 29. Twin of finite thickness inside a crystal: 1–parent crystal; 2–twin.



shearing deformations $V_1 < u_{xy} < V_2$ are not realized (they are unstable). The part of the crystal in which u_{xy} reaches the critical value V_1 goes over into the twinning state $u_{xy} > V_2$. Thus, the region of the twin (region 2 shown in Fig. 29) will be separated from the remaining crystal by a boundary of discontinuities of the deformation tensor, and according to what has been stated above we have

$$u_{xy}^{(1)} < V_1, \tag{6.1a}$$
$$u_{xy}^{(2)} > V_2, \tag{6.1b}$$

where the indices (1) and (2) refer, respectively, to the parent crystal and to the twin.

For our purposes, instead of the graph shown in Fig. 28a it is sufficient to consider the scheme consisting of two straight-line segments (see Fig. 28b), where the first line corresponds to an elastic deformation of the parent crystal:

$$\sigma_{xy} = 2\mu u_{xy}, \qquad \mu = \lambda_{1212},$$

and the second line corresponds to an elastic deformation of the twinned material

$$\sigma_{xy} = 2\mu(u_{xy} - \alpha),$$

where α denotes the twinning angle (in our model $\alpha = \tan^{-1}(b/2a)$).

Since in the assumed model a discontinuity of the elastic stresses arises on the interface between the twin and the parent crystal, the equilibrium of the ordinary and twinning phases can exist only in that case when a certain surface force is distributed along the indicated boundary. If $x = x(s)$ and $y = y(s)$ are the parametric equations for the contour of the twin (s denotes the length measured along the contour), then according to[13] in the two-dimensional problem in the theory of elasticity this force has components $f_x = f_0(dx/ds)$ and $f_y = -f_0(dy/ds)$, where $f_0 = 2\alpha\mu$. In order to determine the stresses on a twin inside a crystal within the framework of such an approach, it is necessary to solve the two-dimensional problem in the theory of elasticity for the stresses produced in a medium by the concentrated forces f.

Let us use the general formulas of the theory of elasticity for an anisotropic solid,[84] specifying the stress tensor in the case when the XOY plane is a symmetry plane of the crystal. If the force generating the elastic stresses has the form $f(f_x, f_y, 0)$, then with the aid of functions of a complex variable one can represent the tensor σ_{ik} in the following way:

$$\sigma_{i1}(x, y) = -\frac{\partial \varphi_i}{\partial y}, \qquad \sigma_{i2}(x, y) = \frac{\partial \varphi_i}{\partial x}, \tag{6.2}$$
$$\varphi_i = 2\,\mathrm{Re} \sum_{\alpha=1}^{2} p_{i\alpha} \Phi_\alpha(z_\alpha) \qquad (i = 1, 2),$$

where $z_\alpha = x + im_\alpha y$. In the case of cubic crystals, the m_α are real positive numbers determined by the moduli of elasticity of the medium. The functions $\Phi_\alpha(z)$ are certain functions of the complex variable z,[84] and the matrix $p_{i\alpha}$ has the form

$$\| p_{i\alpha} \| = \begin{pmatrix} -im_1 & -im_2 \\ 1 & 1 \end{pmatrix}.$$

In what follows it will be necessary for us to use the derivatives $F_\alpha(z) = [d\Phi_\alpha(z)/dz]$, since it is precisely in terms of these derivatives that the components σ_{ik} are expressed. In the case of a force f, which is concentrated at the point (ξ, η), the function $F_\alpha(z)$ has the form[84]

$$F_\alpha(z) = \frac{1}{4\pi} \sum_{k=1}^{2} \left(\frac{N_{\alpha k}}{C_k}\right) \frac{f_k}{z - \zeta_\alpha},$$

where $\zeta_\alpha = \xi + im_\alpha \eta$, $N_{\alpha k}$ is some completely determined matrix of the second rank, and the numbers C_k $(k = 1, 2)$ are given by

$$C_1 = -\operatorname{Im} \sum_{\alpha=1}^{2} p_{1\alpha} N_{\alpha 1}, \qquad C_2 = -\operatorname{Im} \sum_{\alpha=1}^{2} p_{2\alpha} N_{\alpha 2}.$$

Let us utilize the specific form of the "twinning force" f and let us consider the effect of all of the forces concentrated along the contour c. Then, in the case of a twin the function $F_\alpha(z)$ has the form

$$F_\alpha(z) = \frac{f_0}{4\pi} \left\{ \frac{N_{\alpha 1}}{C_1} \oint_c \frac{d\xi}{z - \zeta_\alpha} - \frac{N_{\alpha 2}}{C_2} \oint_c \frac{d\eta}{z - \zeta_\alpha} \right\}, \tag{6.3}$$

where the integral is taken around the closed contour of the twin, which is situated in an infinite crystal.

It turns out that from the general formula (6.3) one can derive a very important conclusion pertaining to the nature of the sharp bend in the contour of a twin at corner points. Let us assume that there is a corner point on the contour c at $z = 0$, where the slopes of the tangents to the curve c are equal to θ_1 and θ_2 (see Fig. 30 where the contour c is being traversed from left to right). Let us clarify how the elastic stresses created by the twin behave upon approaching a corner point. In order to do this, it is sufficient to analyze the behavior of the functions $F_\alpha(z)$ as $z \to 0$. Let us represent this function in the form

$$F(z) = \oint_c \frac{\psi(\zeta)\, d\zeta}{z - \zeta}, \tag{6.4}$$

where the piecewise-continuous function $\psi(\zeta)$ is defined by the relation

$$\psi(\zeta) = \left[A + B\left(\frac{d\eta}{d\xi}\right)\right] \frac{1}{\frac{d\zeta}{d\xi}}, \tag{6.5}$$

in which the constant factors A and B are simply determined in terms of the constants in formula (6.3), $\zeta = \xi + im\eta(\xi)$, and the dependence $\eta = \eta(\xi)$ is the equation of the contour c. It is obvious that the limiting values of the function $\psi(\zeta)$ associated with approaching the corner point from the left and from the right do not coincide, since the derivatives $d\eta/d\xi$ which determine the slopes of the rays which are tangent to the contour c at the corner point do not coincide.

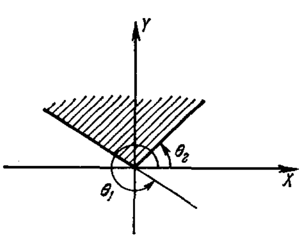


FIG. 30. Corner point on the boundary of a twin.

In the theory of singular integrals of the Cauchy type,[44,85] it is shown that in the presence of a discontinuity in the function $\psi(\zeta)$ at the point $\zeta = 0$ the function F(z) has a logarithmic singularity at the point $z = 0$ and its singular part is given by

$$F(z) = [\psi(+0) - \psi(-0)] \ln z. \tag{6.6}$$

If it is taken into consideration that

$$\left(\frac{d\eta}{d\xi}\right)_{-0} = \operatorname{tg}\theta_1, \qquad \left(\frac{d\eta}{d\xi}\right)_{+0} = \operatorname{tg}\theta_2,$$

then on the basis of expression (6.6) for the singular part of the function F(z) one can derive the following expression for its form:

$$F(z) = \left(A - \frac{i}{m} B\right) e^{i(\varphi_1 - \varphi_2)} \sin(\varphi_1 - \varphi_2) \ln z, \tag{6.7}$$

where

$$\operatorname{tg}\varphi_1 = m \operatorname{tg}\theta_1, \qquad \operatorname{tg}\varphi_2 = m \operatorname{tg}\theta_2.$$

Upon the inclusion of the logarithmic singularity (6.7) in the expression for the elastic stresses, it is found that the stresses retain this singularity. However, an unbounded increase of σ_{xy} and hence of the tensor u_{xy} contradicts the condition (6.1) for mechanical stability of the twin. Therefore, the singularity which we have been considering in the function F(z) must not appear, which is possible only for $\varphi_1 = \varphi_2$ or for $\varphi_1 - \varphi_2 = \pm\pi$. The condition $\varphi_1 = \varphi_2$ is trivial and implies the absence of a corner point, but the condition $\varphi_1 - \varphi_2 = \pm\pi$ means that the corner point is a cuspidal point. This implies that at such a point the tip of the twin must have zero aperture angle. Thus, the conclusion about the shape of the tip of an "untrapped" twin, which was discussed in Chapter 2, is not related to the assumption that the thickness of the twin is small. The zero aperture angle at the end of the twin is a general property of any free twins.

Now let us go on to a comparison of the Lifshitz theory with the dislocation theory of twinning.[47] As mentioned in Chapter 1, the twin depicted in Fig. 29 can be represented by a set of dislocations distributed around its contour (see Fig. 8). If the distribution of the dislocations is regarded as continuous and one introduces the density g(s) of this distribution along the contour of the twin, then it is not difficult to write down the relation between g(s) and the thickness of the twin at a given point:

$$h(x) = -a \int_{x(s) > x} g(s)\, ds, \tag{6.8}$$

where, as before, the OX axis is the trace of the twinning plane and a denotes the interatomic distance in the direction perpendicular to the twinning plane. It is easy to see that Eq. (6.8) is a generalization of formula (1.1) to the case of a twin of finite thickness. For the limiting case of a thin twin, Eq. (6.8) coincides with Eq. (1.1) if one sets $\rho(x) = 2g(x)$.

Let us show, by following[47], that the stresses on a twin inside a crystal, calculated on the basis of formula

(6.3), coincide with those created by the dislocations distributed around the contour of the twin. It is natural that we carry out the proof under the same assumptions which were used to derive the results[13], namely, let us assume that the medium possesses cubic symmetry and the coordinate axes are directed along the fourth-order symmetry axes. We shall regard the twin as infinitely extended along the Z axis and produced by a set of twinning edge dislocations, whose axes are located along its contour.

For the case of a single dislocation with a Burgers vector **b**, which is located at the point (ξ, η) in an infinite medium, the functions $F_\alpha(z)$ have the form[38]

$$F_\alpha(z) = \frac{1}{4\pi}\sum_{k=1}^{2}\left(\frac{M_{\alpha k}}{D_k}\right)\frac{b_k}{z-\zeta_\alpha}, \tag{6.9}$$

where $M_{\alpha k}$ is the matrix which is the inverse of the matrix $p_{k\alpha}$, and the numbers D_k $(k = 1, 2)$ are given by

$$D_1 = -\operatorname{Im}\sum_{\alpha=1}^{2} N_{\alpha 1}^{-1} M_{\alpha 1}, \qquad D_2 = -\operatorname{Im}\sum_{\alpha=1}^{2} N_{\alpha 2}^{-1} M_{\alpha 2},$$

and the symbol $N_{\alpha k}^{-1}$ denotes the matrix which is the inverse of $N_{\alpha k}$. Since the twin is produced by edge dislocations whose Burgers vectors are directed along the x axis, then $b_k = b\delta_{k1}$. Therefore, in the dislocation model a twin corresponds to the following expression for the function $F_\alpha(z)$:

$$F_\alpha(z) = \frac{bM_{\alpha 1}}{4\pi D_1}\oint_c \frac{g(s)\,ds}{z-\zeta_\alpha(s)}, \tag{6.10}$$

where $\zeta_\alpha(s) = \xi(s) + im_\alpha\eta(s)$ and $\xi = \xi(s)$, $\eta = \eta(s)$ are the parametric equations for the contour of the twin.

Now let us somewhat modify formula (6.3), bearing in mind its comparison with Eq. (6.10). We shall use the fact that $\int d\zeta/z - \zeta = 0$ for z outside of the contour c, and also the fact that $\eta'(s) = ag(s)$, and let us transform Eq. (6.3) to the form

$$F_\alpha(z) = -\frac{\mu b}{4\pi}\left(\frac{N_{\alpha 1}}{C_1}im_\alpha + \frac{N_{\alpha 2}}{C_2}\right)\oint_c \frac{g(s)\,ds}{z-\zeta_\alpha(s)}. \tag{6.11}$$

If the explicit form of the matrices $p_{i\alpha}$ and $N_{i\alpha}$ is taken into consideration (see[84,38]), then one can show that the coefficients standing in front of the integrals in (6.10) and (6.11) are equal.

Since the formulas for the stresses are fundamental for the analysis of the twinning process, the agreement between expressions (6.10) and (6.11) proves the equivalence of the two approaches. Thus, the dislocation description of an elastic twin is exact from the point of view of the theory of elasticity for arbitrary thickness of the twin.

Having formulas (6.3) or (6.10) for the determination of the elastic stresses coming from the dislocations on the contour of the twin, one can write down the condition for mechanical equilibrium, which in fact determines the equation for the contour of the twin. We shall not begin to analyze this equation in the general case, but let us make the limiting transition to the case of a thin twin and let us trace how the basic formulas of the theory of thin twins are derived. Let $\eta = \eta_1(x)$ be the equation for the upper boundary of the twin, and let $\eta = \eta_2(x)$ be the equation for the lower boundary (for such a definition to hold, $\eta_1(x) \geq \eta_2(x)$). The thickness of the twin at each point is determined by the obvious equation $h(x) = \eta_1(x) - \eta_2(x)$, and the position of its average length, whose equation we write in the form $y = \eta(x)$, is given by the condition $\eta(x) = (1/2)(\eta_1(x) + \eta_2(x))$. For the case of a thin twin, one can regard the quantities $h(x)$ and $\eta(x)$ as small. Then the stresses on the upper and lower boundaries of the twin can be determined by substituting (6.10) or (6.11) into (6.2) and by the subsequent expansion in powers of the small quantities $\eta(x)$ and $h(x)$.[86] Retaining the first two terms of such an expansion and taking into account all of the forces acting on the dislocations, we also obtain the equations for equilibrium of the dislocations on each of the boundaries of the twin:

$$\left.\begin{aligned} -a\tau_0 \int_{-L}^{L}\frac{\rho(t)\,dt}{t-x} + \tau_1\{h(x)\eta_2''(x) - 2a\rho(x)\eta'(x)\} \\ = \sigma^e(x, 0) + \eta_1(x)\left(\frac{d\sigma^e}{dy}\right)_0 + S_1(x), \\ -a\tau_0 \int_{-L}^{L}\frac{\rho(t)\,dt}{t-x} - \tau_1\{h(x)\eta_1''(x) - 2a\rho(x)\eta'(x)\} \\ = \sigma^e(x, 0) + \eta_2(x)\left(\frac{d\sigma^e}{dy}\right)_0 + S_2(x), \end{aligned}\right\} \tag{6.12}$$

where, as before, $a\rho(x) = -h'(x)$, and the constants τ_0 and τ_1, which are of the order of magnitude of μ, are respectively given by

$$\tau_0 = \frac{2\alpha\lambda_{1111}}{\pi(m_1+m_2)}, \quad \tau_1 = \alpha\lambda_{1111}.$$

$S_1(x)$ and $S_2(x)$ are the forces of inelastic origin on the upper and lower boundaries of the twin, respectively, (the difference between the forces of inelastic origin on each of the boundaries may be caused by the presence of obstructions on any of the boundaries, by nonidentical conditions for the nucleation of dislocations on each of the boundaries, etc.). The function $\sigma^e(x, y)$ determines the distribution of the inhomogeneous external stresses near the twinning plane.

The solution of the system of equations (6.12) for given external stresses $\sigma^e(x, 0)$, their gradients $\partial\sigma^e(x)/\partial y$, and forces of inelastic origin $S_1(x)$ and $S_2(x)$ enables us to reestablish the shape of the twin, which is determined by the two functions $h(x)$ and $\eta(x)$.

If the forces of inelastic origin are the same on both boundaries of the twin, and the asymmetry of the shape of the twin is due only to the gradient of the external stresses, then in the first approximation one can neglect terms of order h^2. Then, as the basic approximation in the parameter h/L, we obtain the equation

$$-a\tau_0 \int_{-L}^{L}\frac{\rho(t)\,dt}{t-x} = \sigma^e(x, 0) + S(x),$$

which agrees with the initial equation (1.5) for the dislocation theory of thin twins. This equation actually determines the thickness of the twin at each point x.

The equation of the second-order approximation is obtained from (6.12) by calculating the half-difference between the conditions on the upper and lower boundaries:[14]

$$\tau_1[h(x)\eta''(x) + 2\rho(x)\eta'(x)] = h(x)\left(\frac{d\sigma^e}{dy}\right)_0.$$

In this equation the functions $\rho(x)$ and $h(x)$ are assumed to be known from the solution of the first-order approximate equation, and consequently it determines $\eta(x)$, i.e., the kink of the twin in the inhomogeneous stress field. One can easily bring this equation into the form

$$\eta''(x)+2\left[\frac{d}{dx}\ln h(x)\right]\eta'(x)=-q(x), \qquad q(x)=\frac{a}{2b\mu}\left(\frac{d\sigma^e(x,0)}{dy}\right)_0. \tag{6.13}$$

It is clear that the solution of this equation can be obtained by quadratures, since

$$\eta'(x)=-\frac{1}{h^2(x)}\int_L^x q(x')h^2(x')\,dx'. \tag{6.14}$$

In writing down Eq. (6.14) we have assumed the natural, in our opinion, condition $\eta'(L)=0$. In fact, as we saw earlier, $\rho(L)=0$ for a free twin and since the kink of average length $\eta=\eta(x)$ near the end of the twin is only related to a redistribution of the dislocations on the upper and lower boundaries of the twin near its end, then one should set $\eta'(L)=0$.

Near the end of the twin (for example, in the neighborhood $L-x\ll L$) the explicit coordinate dependence $\eta=\eta(x)$ can be derived since, on the one hand one can assume $q(x)=q(L)=\text{const}$, and on the other hand the function $h=h(x)$ is known in this region. Using formulas (3.27) and (3.28) one can easily obtain

$$\left.\begin{aligned}\eta'(x)&=\frac{1}{4}q(L)(L-x), && x<\varepsilon,\\ \eta'(x)&=\frac{1}{2}q(L)(L-x), && \varepsilon\ll x\ll L.\end{aligned}\right\} \tag{6.15}$$

On the basis of Eqs. (6.15) we conclude that near the right-hand end of the twin the equation for its average length has the form

$$\eta(x)=\eta(L)+\text{const}\,(L-x)^2,$$

where the constant factor changes by a factor of two during the transition from the region $x<\epsilon$ to the region $\epsilon\ll x\ll L$.

In concluding this section, we wish to clarify the specific physical meaning of certain quantities which appear as parameters in the phenomenological theory proposed by Lifshitz. The stresses in the elastic medium around the twin associated with a given external loading are determined by the conditions on the boundary of the twin, where they are related to the forces of inelastic origin in a quite definite way. Upon approaching the twin boundary from the side of the parent crystal we obtain (see relation (1.4))

$$u_{xy}^{(1)}=\frac{\sigma_{xy}}{2\mu}=-\frac{S(x)}{2\mu}. \tag{6.16}$$

By using the property $-S_0\le S(x)$ which was discussed earlier for the force $S(x)$ of inelastic origin during loading, we can verify that a stable elastic twin can exist in the medium provided that the deformations satisfy the conditions

$$\left.\begin{aligned}&W_1<u_{xy}^{(1)}<V_1,\\ &V_1=\frac{S_s^0+S_0}{2\mu}\approx\frac{S_s^0}{2\mu},\quad W_1=-\frac{S_0}{2\mu}.\end{aligned}\right\} \tag{6.17}$$

Going on to an analysis of the conditions inside the twin, we note that in the case of a free twin of small thickness the inclination of the profile of the twin to the twinning plane is also small. In this case the condition $\sigma_{xy}^{(1)}=\sigma_{xy}^{(2)}$ is satisfied on the boundary between the parent crystal and the twin; therefore, one can derive the relation

$$u_{xy}^{(2)}=\alpha-\frac{S(x)}{2\mu},$$

from which it follows that

$$\left.\begin{aligned}&V_2<u_{xy}^{(2)}<W_2,\\ &V_2=\alpha-\frac{S_0}{2\mu},\quad W_2=\alpha+\frac{S_s^0+S_0}{2\mu}\approx\alpha+\frac{S_s^0}{2\mu}.\end{aligned}\right\} \tag{6.18}$$

It is interesting to note that the quantities V_1 and V_2, determined by formulas (6.17) and (6.18), do not satisfy the condition $V_1=\alpha-V_2$ which was proposed in article[13]. And what is more, one would expect that $V_1\gg\alpha-V_2$. This difference in the values of V_1 and $\alpha-V_2$ becomes quite obvious and natural if the nature of the surface-tension force is taken into account, which was not done in[13]. Just as in[13], in our model the quantity V_1 determines the onset of twinning associated with loading (the twin begins to form at $u_{xy}^{I}(0)=V_1$), and the quantity V_2 determines the onset of "detwinning" associated with unloading (the twin begins to decrease for $u_{xy}^{(2)}(0)=u_{xy}^{(1)}(0)+\alpha=V_2$). The physical interpretation of the difference between V_1 and $\alpha-V_2$ in the presence of the surface-tension forces consists in the fact that these forces "expel" the twinning wedge from the crystal; therefore, they oppose the formation of the twin, thus increasing the value of V_1, and they favor the "emergence" of the twin from the crystal, decreasing the value of $\alpha-V_2$.

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