533.951

CRITERIA FOR WAVE GROWTH

A. I. AKHIEZER and R. V. POLOVIN

Usp. Fiz. Nauk 104, 185–200 (June, 1971)

CONTENTS

1. INTRODUCTION

THE problem of stability of dynamic systems arises in a great variety of problems in physics and engineering.

If the parameters characterizing the system are lumped at individual points, so that the system is described by ordinary differential equations, then the stability problem, as a rule, can be solved on the basis of the linear approximation. For dynamic systems not with lumped but with distributed parameters, the situation becomes much more complicated, since such systems are described not by ordinary differential equations but either by partial differential equations or by integro-differential equations (these equations can also be nonlinear). The characteristic frequencies, which form a discrete spectrum for system with lumped parameters, may form a continuous spectrum for systems with distributed parameters. In this case the superpositions of the characteristic solutions will have the form not of sums but of integrals, the asymptotic behavior of which with increasing time t will not coincide, generally speaking, with the asymptotic behavior of the integrand. Therefore an investigation of the stability of systems with distributed parameters turns out to be much more complicated even in the linear approximation than an investigation of the stability of systems with lumped parameters.

The purpose of the present review is to develop a theory of stability of dynamic systems with distributed parameters that depend neither on the time nor on the coordinates, in the linear approximation.

We shall assume the dynamic system to be sufficiently extended. In this case the linearized equations for the quantities $u_\alpha \equiv u_\alpha(\mathbf{r}, t)$, characterizing the state of the system*, have solutions in the form of plane waves

$$u_\alpha = A_\alpha e^{i(\mathbf{kr}-\omega t)},$$

where **k** is the wave vector and ω the natural frequencies of the system.

If the system is described by partial differential equations, then there exists a definite relation between the frequency ω and the wave vector **k**. This relation is algebraic, i.e., there exists an equation

$$D(\mathbf{k}, \omega) = 0, \qquad (1.1)$$

where $D(\mathbf{k}, \omega)$ is a certain polynomial. This equation is called the dispersion equation.

If the initial equations describing the system are integro-differential equations, then the frequency, generally speaking, is not a definite function of the wave vector. However, such a functional dependence arises asymptotically at large values of t. Here again we obtain the dispersion relation (1.1), where $D(\mathbf{k}, \omega)$ is no longer a polynomial but a certain transcendental function. We shall henceforth confine ourselves to a consideration of only algebraic dispersion equations* and show that these equations make it possible to clarify the character of the instability in the linear approximation.

2. ABSOLUTE AND CONVECTIVE INSTABILITIES

If certain real values of the wave vector **k** correspond to certain complex values of the frequency ω with $\operatorname{Im}\omega > 0$, then a perturbation in the form of a plane monochromatic wave $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ will increase in time without limit, and the dynamic system under consideration will be unstable.

Actually, small perturbations do not have the form of individual plane monochromatic waves, but are wave packets, i.e., superpositions of plane monochromatic waves. On the other hand, the asymptotic behavior of a wave packet can greatly differ from the behavior of the individual waves. Namely, if the individual components in a wave packet increase without limit in time, then nonetheless the entire packet as a whole can remain bounded in a fixed point of space, since the perturbation can "drift" downstream. Therefore to clarify the character of the instability of a dynamic system it is necessary to solve the problem of the development of the initial perturbation.

We assume that the state vector $u = (u_1, u_2, \ldots, u_n)$ satisfies a system of partial differential equations with constant coefficients:

$$\sum_{\beta=1}^{n} P_{\alpha\beta}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u_\beta(x, t) = 0 \quad (\alpha = 1, 2, \ldots, n), \qquad (2.1)$$

where $P_{\alpha\beta}$ are certain polynomials in $\partial/\partial x$ and $\partial/\partial t$ with constant coefficients. We should find the solutions

*The aggregate of these quantities will be called the vector of state of the dynamic system.

*Transcendental dispersion equations are investigated in [1].

*For simplicity we confine ourselves here to the one-dimensional case; for an investigation of a many-dimensional case see [2].

of this system of equations, satisfying arbitrary initial conditions $u_1(x, 0), u_2(x, 0), \ldots, u_n(x, 0)$.

If in a wave packet $u(x, t)$ the perturbation at $x = \text{const}$ and $t \to \infty$ remains bounded in spite of the presence of components with $\operatorname{Im} \omega > 0$, (it usually tends in this case to zero)

$$\lim_{\substack{t\to\infty \\ x=\text{const}}} u(x, t) = 0, \tag{2.2}$$

then one speaks of convective or drift instability.

On the other hand, if the perturbation $u(x, t)$ increases without limit at fixed x and $t \to \infty$,

$$\lim_{\substack{t\to\infty \\ x=\text{const}}} u(x, t) = \infty, \tag{2.3}$$

then the instability is called absolute*.

Then, in an investigation of instability it suffices to check on the existence of complex frequencies in the dispersion equation $D(\mathbf{k}, \omega) = 0$, and it is also necessary to ascertain how the wave packet behaves at a fixed point of space as $t \to \infty$.

It is clear that in the case of absolute instability the existence of an equilibrium state $u = 0$ is impossible. Indeed, the growth of random perturbations (which always arise, at least because of thermal fluctuations), leads either to a transition to a stable state or to a destruction of the state under consideration. On the other hand, in the case of convective instability the equilibrium state $u = 0$ can exist. In this case there is established in the system a stationary (i.e., time-independent) perturbation level, corresponding to equilibrium thermal fluctuations at the input of the system.

If the system is unstable, then it can be used to generate oscillations. It can be stated that absolute instability is necessary for the generation.

If the system has convective instability, then the perturbation "drifts downstream"; this means that this form of instability corresponds to amplification and not to generation of oscillations, in other words, systems with convective instability can be amplifiers of oscillations. It must be borne in mind, however, that systems with convective instability can also be used to generate oscillations, if their input is coupled to their output; this gives rise to feedback, and the "drifting" perturbation is returned, i.e., that the instability in the system acquires an absolute character.

In order to formulate criteria of absolute and convective instabilities, let us first connect the vector of state u_α with the Green's function $g(x, t)$ of Eqs. (2.1):

$$u_\alpha(x, t) = L_\alpha\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) g(x, t),$$

where $L_\alpha(\partial/\partial x, \partial/\partial t)$ is a certain differential operator connected by simple relations with the operators $P_{\alpha\beta}(\partial/\partial x, \partial/\partial t)$, and the function $g(x, t)$ satisfies the differential equation

$$D\left(-i\frac{\partial}{\partial x}, i\frac{\partial}{\partial t}\right) g(x, t) = \delta(x)\delta(t),$$

where D is a polynomial relative to $-i\partial/\partial x$, $i\partial/\partial t$, $D(-i\partial/\partial x, i\partial/\partial t) \equiv \det|P_{\alpha\beta}(\partial/\partial x, \partial/\partial t)|$, which enters in the left side of the dispersion equation (1.1).

Taking the Laplace transform with respect to time and the Fourier transform with respect to the coordinate in the last equation, we obtain

$$g(x, t) = \frac{1}{4\pi^2}\int_\Omega e^{-i\omega t}\, d\omega \int_{-\infty}^{\infty} \frac{e^{ikx}\, dk}{D(k, \omega)}, \tag{2.4}$$

where Ω is a straight line in the complex ω plane, parallel to the real axis and passing above all the singular points of the integrand.

We first integrate in (2.4) with respect to ω:

$$g(x, t) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\sum_{\alpha=1}^{n} \frac{e^{ikx - i\omega_\alpha(k)t}}{D_\omega[k, \omega_\alpha(k)]}\, dk,$$

where $D_\omega \equiv \partial D/\partial\omega$ and the summation is carried out over all the roots of the dispersion equation (1.1) $\omega = \omega_\alpha(k)$.

We put in this formula $x = 0$ and replace in the integrand the integration variable k by ω_α:

$$g(0, t) = -\frac{i}{2\pi}\sum_\alpha \int_{\Omega_\alpha} \frac{e^{-i\omega_\alpha t}\, d\omega_\alpha}{D_\omega[k(\omega_\alpha), \omega_\alpha]\frac{d\omega_\alpha}{dk}}, \tag{2.5}$$

where Ω_α is the contour described by the point ω_α in the complex plane when k runs along the real axis from $-\infty$ to $+\infty$. (Since we are considering the case of instability, certain real values of k correspond to $\operatorname{Im}\omega_\alpha > 0$.)

We now deform the integration contours Ω_α by dropping them downward: $\operatorname{Im}\Omega_\alpha \to -\infty$. The integral (2.5) is represented here as a sum of terms of the form $\exp(-i\omega_\beta t)$, where ω_β are the singular points of the integrand. The singular points are usually the branch points ω_β of the function $k(\omega)$.

If there is at least one branch point of the function $k = k(\omega)$ in the upper half-plane $\operatorname{Im}\omega_\beta > 0$ between the contour Ω_α and the real axis ω_α, then we have absolute instability. On the other hand, if there are no branch points of the function $k(\omega)$ for any value of α in the upper half-plane $\operatorname{Im}\omega_\beta > 0$, then the instability is convective[5].

Let us investigate by way of an example the dispersion equation

$$(\omega - kv_1)(\omega - kv_2) + m = 0,$$

the solution of which is

$$\omega_{1,2}(k) = \frac{1}{2}(v_1 + v_2)k \pm \frac{1}{2}\sqrt{(v_1 - v_2)^2 k^2 - 4m}. \tag{2.6}$$

If $m > 0$ and

$$-\frac{2\sqrt{m}}{|v_1 - v_2|} < k < \frac{2\sqrt{m}}{|v_1 - v_2|},$$

then the frequency ω will be complex. One of the values of ω has a positive imaginary part, meaning the presence of instability. To ascertain the character of this instability, we find the inverse function

$$k(\omega) = \frac{(v_1 + v_2)\omega \pm \sqrt{(v_1 - v_2)^2\omega^2 - 4v_1v_2m}}{2v_1v_2}.$$

We see that the function $k(\omega)$ has branch points at

$$\omega = \pm\frac{2\sqrt{v_1v_2m}}{v_1 - v_2}.$$

If $m > 0$ and $v_1v_2 > 0$, then the branch points lie on the real axis, i.e., the instability is convective. Let us show that at $m > 0$ and $v_1v_2 < 0$ one of the branch points

*The concepts of absolute and convective instability were introduced by Twiss [3] an by Landau and Lifshitz [4].

$$\omega_0 = \frac{2\sqrt{|v_1v_2|m}}{|v_1-v_2|}i$$

of the function $k(\omega)$ lies in the upper half-plane between the contour Ω_α and the real ω axis, i.e., at $m > 0$ and $v_1v_2 < 0$ there is absolute instability. To this end, we consider the contour Ω_1, define by formula (2.6) and locate it in the upper half-plane of ω.

Putting

$$\omega = \alpha + i\beta,$$

we obtain an equation for that part of the contour Ω_1, which lies in the upper half-plane

$$\beta = \sqrt{\left(\frac{v_1-v_2}{v_1+v_2}\right)^2\alpha^2 - m^2}.$$

Since at the branch point $\omega_0 = \alpha_0 + i\beta_0$ we have

$$\alpha_0 = 0, \quad \beta_0 = \frac{2\sqrt{|v_1v_2|m}}{|v_1-v_2|}$$

and $2\sqrt{|v_1v_2|} \le |v_1 - v_2|$ at $v_1v_2 < 0$, the branch point ω_0 lies between the contour Ω_1 and the real axis, i.e., there is absolute instability at $m > 0$ and $v_1v_2 < 0$.

In deriving the criterion for the absolute instability, we have first integrated with respect to ω in (2.4). It is possible to obtain another form of this criterion by first integrating in (2.4) with respect to k. Using the residue theorem, we obtain

$$g(x, t) = \begin{cases} \frac{i}{2\pi}\int\limits_{\Omega}\sum\limits_{r}\frac{e^{ik_r(\omega)x-i\omega t}}{D_k[k_r(\omega), \omega]}d\omega & \text{for } x>0, \\ -\frac{i}{2\pi}\int\limits_{\Omega}\sum\limits_{l}\frac{e^{ik_l(\omega)x-i\omega t}}{D_k[k_l(\omega), \omega]}d\omega & \text{for } x<0, \end{cases} \tag{2.7}$$

where $D_k \equiv \partial D/\partial k$ and $k_r(\omega)$ and $k_l(\omega)$ are the roots of the dispersion equation, which at $\mathrm{Im}\,\omega \to +\infty$ lie respectively in the upper or in the lower half-plane,

$$\mathrm{Im}\,k_r(\omega) > 0, \quad \mathrm{Im}\,k_l(\omega) < 0 \quad (\mathrm{Im}\,\omega \to +\infty).$$

We shall say that the terms $k_r(\omega)$ in the upper sum of (2.7) describe waves propagating to the right, and the terms $k_l(\omega)$ in the lower sum describe waves propagating to the left.

The singular points of the integrand of (2.7) can be only branch points of the function $k(\omega)$. Not all branch points of $k(\omega)$, however, contribute to the integral (2.7). If the quantities $k_\alpha(\omega)$ and $k_\beta(\omega)$, which correspond to two waves propagating in the same direction, become equal at the branch point ω_0, then (inasmuch as in (2.7) we sum over r or over l) the point ω_0 will actually not be a branch point of the integrand. Therefore the criterion for the absolute instability is that the upper half-plane of ω, $\mathrm{Im}\,\omega > 0$, contain a branch point of the function $k(\omega)$ at which two branches of this function, corresponding to waves propagating in opposite directions, coincide[6]:

$$k_r(\omega) = k_l(\omega).$$

In the opposite case the instability will be convective.

Let us investigate, using this formulation of the criterion for absolute or convective instability, the type of instability of the system characterized by the dispersion equation

$$(\omega - kv_1)(\omega - kv_2) + m = 0 \quad (m > 0),$$

which we have already considered above.

If $v_1 > 0$ and $v_2 > 0$, then the imaginary part of the function

$$k(\omega) = \frac{(v_1+v_2)\,\omega \pm \sqrt{(v_1-v_2)^2\,\omega^2 - 4v_1v_2m}}{2v_1v_2}$$

will be positive as $\mathrm{Im}\,\omega \to +\infty$. Therefore, at $v_1 > 0$ and $v_2 > 0$, both waves propagate to the right, and consequently there will be no absolute instability, i.e., the instability will be convective.

On the other hand, if $v_1v_2 < 0$, then one of the waves $k(\omega)$ propagates to the right, and the other to the left. Then one of the branch points, namely

$$\omega = \frac{2\sqrt{|v_1v_2|m}}{|v_1-v_2|}i,$$

lies in the upper half-plane. Thus, when $v_1v_2 < 0$ we deal with absolute instability.

As already indicated, in convective instability a perturbation, while growing, drifts downstream at the same time. If s is the "drift" velocity of the packet, then obviously, in a reference frame moving with velocity s, the perturbation will increase without limit, i.e.,

$$\lim_{t\to\infty} u(st, t) = \infty. \tag{2.8}$$

In other words, convective instability becomes absolute in this reference frame. The velocity s is equal to[7]

$$s = \left(\frac{d\,\mathrm{Re}\,\omega(k)}{dk}\right)_{k=k_0}, \tag{2.9}$$

where k_0 is the value of the wave number at which

$$\frac{d\,\mathrm{Im}\,\omega(k)}{dk} = 0$$

(ω and k are connected by the dispersion equation (1.1). Expression (2.9) is a generalization of a well known expression for the group velocity to include waves that grow in time[8].

In the case of the simplest dispersion equation

$$(\omega - kv_1)(\omega - kv_2) + m = 0$$

the drift velocity s is the arithmetic mean of the velocities v_1 and v_2:

$$s = \frac{1}{2}(v_1 + v_2).$$

3. AMPLIFICATION AND NONTRANSMISSION OF OSCILLATIONS

We now proceed to consider amplification of oscillations.

Obviously, the oscillations capable of being amplified are those for which $\mathrm{Im}\,k < 0$ at real ω (the system is assumed semi-infinite, $x > 0$, the x axis is chosen such that the amplified waves move towards increasing values of x).

The conditions $\mathrm{Im}\,k < 0$ with real ω do not by themselves suffice for amplification of the oscillations. To verify this, let us consider a monochromatic wave $\exp[i(kx - \omega t)]$ propagating in a plane waveguide. In this case the dispersion equation can be readily seen to be

$$\omega^2 = k^2c^2 + a^2, \tag{3.1}$$

where c is the speed of light and $a = n\pi c/d$ (d is the distance between the planes of the waveguide and n is an integer). For $|\omega| < a$, Eq. (3.1) has two imaginary roots:

$$\left.\begin{aligned} k_1(\omega) &= \frac{i}{c}\sqrt{a^2-\omega^2}, \\ k_2(\omega) &= -\frac{i}{c}\sqrt{a^2-\omega^2}. \end{aligned}\right\} \qquad (3.2)$$

The root $k_1(\omega)$ corresponds to a wave with an amplitude that decreases exponentially at $x > 0$

$$u_1 = A_1 \exp\left(-\frac{x}{c}\sqrt{a^2-\omega^2} - i\omega t\right),$$

and the root $k_2(\omega)$ corresponds to a wave with an exponentially growing amplitude

$$u_2 = A_2 \exp\left(\frac{x}{c}\sqrt{a^2-\omega^2} - i\omega t\right).$$

In the region $x > 0$ this wave would lead to an exponentially growing oscillation energy, which is impossible, since in our problem there is no external energy source. Obviously, this solution corresponds actually not to a growing wave moving in the positive x direction, but to a damped wave moving in the opposite direction. If the energy source lies in the plane $x = 0$, then the growing solution must be discarded. It can be stated that in the present case, in spite of the fact that $\mathrm{Im}\, k < 0$, we have not amplification of the oscillations, but their nontransmission*.

In our case, the situation is very simple, and a clarification of the character of the propagating oscillations at $\mathrm{Im}\, k < 0$ entails no difficulty. There can arise, however, more complicated cases, when it is not so simple to differentiate between amplification and nontransmission. Such cases arise whenever the system contains external energy sources. By way of an example we present a system comprising a plasma and a beam of charged particles. In this case the beam supplies energy to the plasma continuously, and a special analysis is necessary to determine the character of the propagating oscillations.

Let us formulate mathematically the difference between amplification and nontransmission of oscillations.

We denote by $k_\beta \equiv k_\beta(\omega)$ ($\beta = 1, 2, \ldots, n$) different solutions of the dispersion equation with respect to k. It is clear that the vector of state $(u_1, u_2, \ldots, u_n)$ of the system can be represented in the form

$$u_\alpha(x, t) = \int_{-\infty}^{\infty} \sum_{\beta=1}^{n} b_{\alpha\beta}(\omega)\, e^{ik_\beta(\omega)x - i\omega t}\, d\omega, \qquad (3.3)$$

where the coefficients $b_{\alpha\beta}(\omega)$ are determined by the perturbations $u_\alpha(0, t)$ on the boundary of the region $x = 0$. The quantities $u_\alpha(0, t)$ can always be chosen such that the vector of state at all instants of time vanishes at $x < 0$:

$$u_\alpha(x, t) = 0, \quad x < 0. \qquad (3.4)$$

By so choosing the boundary perturbation of the state vector, we cause all the waves in the system to propagate in a definite direction, namely in the positive x direction.

We now fix the value of β corresponding to $\mathrm{Im}\, k_\beta < 0$. Two cases can arise here: either some of the quantities $b_{\alpha\beta}$ ($\alpha = 1, 2, \ldots, n$) differ from zero, or else they are all equal to zero. In the former case we have amplification of a wave with wave number $k = k_\beta$, and in the latter case we have nontransmission of this wave[9-11].

Thus, the condition $\mathrm{Im}\, k_\beta < 0$ is not sufficient for a wave with wave number $k = k_\beta$ to be amplified. It is necessary also that this wave be contained in the wave packet (3.3), which satisfies the condition (3.4).

To establish a criterion for amplification and nontransmission of oscillations, we choose the quantities $u_\alpha(x, t)$ in such a manner that all the waves propagate in the positive x direction, i.e., so as to satisfy the condition (3.4).

The imaginary part of $k(\omega)$ should then be positive at $\mathrm{Im}\, \omega \to +\infty$ for waves moving to the right. On the other hand, the condition for the spatial growth of the waves has obviously the form $\mathrm{Im}\, k(\omega) < 0$ (ω is real).

Thus, for wave amplification it is necessary that the imaginary part of the function $k(\omega)$ have opposite signs for $\mathrm{Im}\, \omega \to +\infty$ and $\mathrm{Im}\, \omega = 0$. On the other hand, if $\mathrm{Im}\, k(\omega)$ has the same sign for $\mathrm{Im}\, \omega \to +\infty$ and $\mathrm{Im}\, \omega = 0$, then nontransmission of the oscillations takes place[6].

Let us consider by way of an example a system with a dispersion equation

$$(\omega - kv_1)(\omega - kv_2) + m = 0.$$

By solving this equation with respect to k, we obtain

$$k_1(\omega) = \frac{(v_1+v_2)\,\omega + \sqrt{(v_1-v_2)^2\omega^2 - 4v_1v_2m}}{2v_1v_2},$$

$$k_2(\omega) = \frac{(v_1+v_2)\,\omega - \sqrt{(v_1-v_2)^2\omega^2 - 4v_1v_2m}}{2v_1v_2}.$$

If $v_1v_2m < 0$, then any real value of ω corresponds to a real value of k. In this case there is transmission of the oscillations. Indeed, as shown in Sec. 2, the system is absolutely stable at $m > 0$ and $v_1v_2 < 0$.)

On the other hand, if $v_1v_2m > 0$, then for

$$-\frac{2\sqrt{v_1v_2m}}{|v_1-v_2|} < \omega < \frac{2\sqrt{v_1v_2m}}{|v_1-v_2|} \qquad (3.5)$$

k becomes complex. If furthermore $v_1v_2 < 0$ and $v_1 + v_2 > 0$, then for two waves with wave vectors $k_1(\omega)$ and $k_2(\omega)$ we have at $\mathrm{Im}\, \omega = 0$ and $\mathrm{Im}\, \omega \to +\infty$ the inequalities

$$\mathrm{Im}\, k_1(\omega) < 0, \quad \mathrm{Im}\, k_2(\omega) > 0.$$

Since $\mathrm{Im}\, k_{1,2}(\omega)$ does not reverse sign on going from $\mathrm{Im}\, \omega = 0$ to $\mathrm{Im}\, \omega \to +\infty$, the system under consideration does not transmit oscillations in the frequency interval (3.5) in which $\mathrm{Im}\, k \neq 0$.

If the inequalities

$$v_1v_2m > 0, \quad v_1 > 0, \quad v_2 > 0$$

are satisfied, then both waves with wave numbers $k_1(\omega)$ and $k_2(\omega)$ have positive imaginary parts as $\mathrm{Im}\, \omega \to +\infty$. At $\mathrm{Im}\, \omega = 0$ we have the inequalities

$$\mathrm{Im}\, k_1(\omega) > 0, \quad \mathrm{Im}\, k_2(\omega) < 0.$$

Therefore the solution $k_1(\omega)$ corresponds to nontransmission of the oscillations, and the solution $k_2(\omega)$ to amplification of the oscillations.

We note that the amplified perturbation satisfies the causality condition

$$\lim_{\substack{x\to+\infty \\ t=\mathrm{const}}} u_\alpha(x, t) = 0. \qquad (3.6)$$

*The concepts of amplification and nontransmission (reflection) of oscillations were introduced by Twiss [9].

This condition can be regarded as a definition of amplification[12,13]. The condition (3.6) enables us to discard waves moving to the left and replace by the same token the boundary conditions (3.4).

To discard waves moving to the left, we can also use one property of the Fourier-Laplace transformation

$$u_\alpha(k, p) = \int_0^\infty e^{-pt}\, dt \frac{1}{2\pi} \int_{-\infty}^\infty e^{-ikx} u_\alpha(x, t)\, dx,$$

which follows from the boundary condition (3.4) and from the obvious condition

$$u_\alpha(x, t) = 0 \quad \text{if} \quad t < 0. \tag{3.7}$$

From these conditions it follows[6,14] that the Fourier-Laplace transformation $u_\alpha(k, p)$ should be an analytic function of both variables at $\mathrm{Im}\, k < 0$ and $\mathrm{Re}\, p > 0$.

An essential fact is that the wave packet as a whole is amplified in space, i.e.,

$$\lim_{x\to+\infty} u_\alpha\left(x, \frac{x}{s'}\right) = \infty, \tag{3.8}$$

where s' is a certain quantity characteristic of the system. Namely,

$$s' = \frac{1}{\left(\frac{d\,\mathrm{Re}\,k}{d\omega}\right)_{\omega=\omega_0}}, \tag{3.9}$$

where ω_0 is the value of the frequency at which

$$\frac{d\,\mathrm{Im}\,k}{d\omega} = 0.$$

The quantity s' is a generalization of the group velocity to the case of waves growing in space.

In the case of the dispersion equation

$$(\omega - kv_1)(\omega - kv_2) + m = 0$$

s' is equal to the harmonic mean of the velocities v_1 and v_2:

$$\frac{1}{s'} = \frac{1}{2}\left(\frac{1}{v_1} + \frac{1}{v_2}\right).$$

Since the arithmetic mean is larger than the harmonic mean when $v_1 > 0$ and $v_2 > 0$, the group velocity of wave that grow in time exceeds the group velocity of waves that grow in space:

$$s > s'.$$

4. STURROCK'S RULES

The practical application of the wave instability and amplification criteria formulated in the preceding sections entails in general great difficulties, since the determination of the branch points of the functions in regions bounded by the real axes of the planes ω_α and the contours Ω_α (which themselves must be determined) and the determination of the direction of propagation of the waves are complicated and laborious problems.

Considerable simplifications occur when the dispersion equation $D(k, \omega) = 0$ is an algebraic equation with real coefficients, which breaks up in the region of large values of $|k|$, or, which is the same, in the regions of large values of $|\omega|$, into a product of factors in the form $\omega - vk$, where v is a certain constant different from zero. (This assumption corresponds to the condition that the signal propagation velocity be finite*.) For systems with a dispersion equation of this kind, the location of the branch points, and consequently also the character of the instability, can be established from the general form of the curve representing the dispersion equation in the (k, ω) plane.

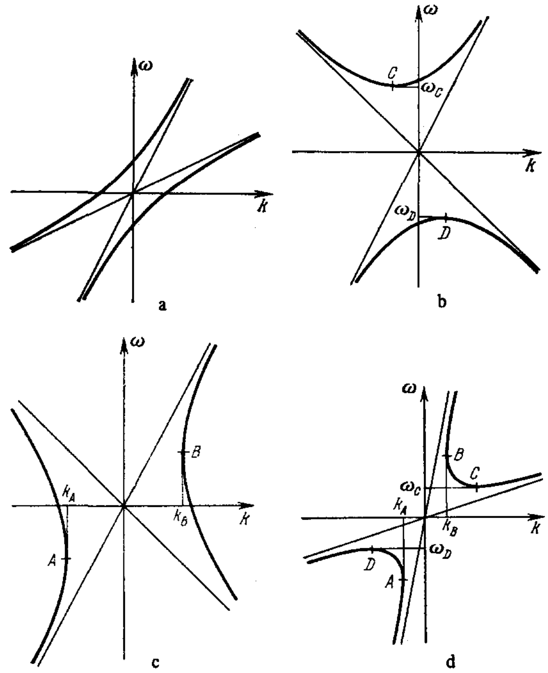

FIG. 1. Typical dispersion curves. a) Stability, transmission; b) stability, nontransmission; c) absolute instability, transmission; d) convective instability, amplification.

Four typical dispersion curves are shown in Fig. 1. Figures 1a and 1b correspond to stable systems, while Figs. 1c and 1c to unstable systems. The difference between Figs. 1d and 1d lies in the fact that in the former case the asymptotes of the dispersion curves are inclined in different directions, and in the latter case in the same direction. It can be shown that Fig. 1c corresponds to absolute instability and Fig. 1d to convective instability.

Thus, the instability will be absolute or convective, depending on whether the asymptotes are inclined in opposite directions or in the same direction (Sturrock's first rule[12]).

Let us see now how to distinguish between amplification and non-transmission of the oscillations by means of the dispersion curve.

Figures 1a and 1c correspond to transmission of oscillations, while Figs. 1b and 1d correspond to either amplification or nontransmission of oscillations. The difference between Figs. 1b and 1d lies in the fact that in the former case the asymptotes are inclined in different directions, and in the latter in the same direction. It can be shown that Fig. 1b corresponds to nontransmission of oscillations and Fig. 1d to amplification of oscillations.

*In other words, we assume in this section that the system of differential equations (2.1) is of the hyperbolic type. We note that the criteria obtained in the two preceding sections are valid for a broader class of differential-equation systems, for which the Cauchy problem is correct [2], and in particular for systems of the parabolic type [15].

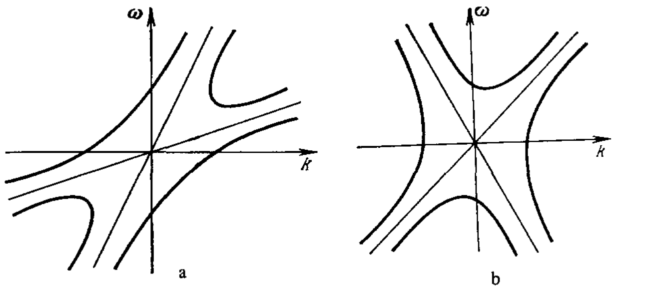

FIG. 2. Dispersion curve of two-beam tube. a) Beam velocities parallel; b) beam velocities antiparallel.

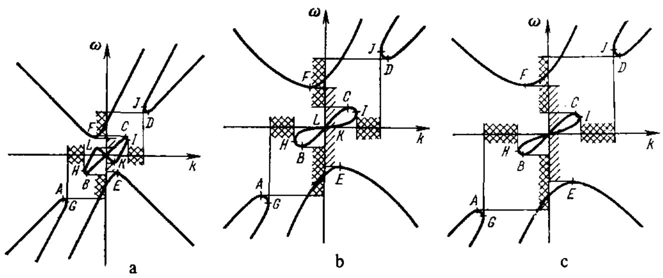

FIG. 3. Reduction of a dispersion equation to a second-order polynomial. a) Small ξ; b) $\xi = \xi_0$; c) $\xi = 1$.

We note that in the case when the dispersion equation is a polynomial of second degree

$$(\omega - kv_1)(\omega - kv_2) + m = 0,$$

all four cases shown in Figs. 1a–d are realized. Figure 1a corresponds to $v_1 > 0$, $v_2 > 0$, $m < 0$; Fig. 1b to $v_1v_2 < 0$, $m < 0$; Fig. 1c to $v_1v_2 < 0$, $m > 0$; and finally, Fid. 1d, to $v_1 > 0$, $v_2 > 0$, $m > 0$.

Thus, if the asymptotes of the dispersion curve are inclined in different directions, then nontransmission of the oscillations takes place, and if the asymptotes are inclined in the same direction, then amplification of the oscillations takes place (Sturrock's second rule*).

We see that the same dispersion curve can correspond to both the instability problem and to the wave-amplification problem, whereas if absolute instability obtains, then there is also transmission of the oscillations. On the other hand if the system is convectively unstable, then it can also be used to amplify oscillations. The particular possibility that is realized depends, naturally, on the concrete physical formulation of the problem.

Let us illustrate the foregoing rule using as an example a two-beam tube[18], the dispersion equation of which is given by

$$\frac{\omega_1^2}{(\omega - ku_1)^2} + \frac{\omega_2^2}{(\omega - ku_2)^2} = 1, \tag{4.1}$$

where

$$\omega_{1,2}^2 = \frac{4\pi e^2 n_{1,2}}{m_{1,2}},$$

$n_{1,2}$ and $u_{1,2}$ are the densities and velocities of the particles of both types, and $m_{1,2}$ are the masses of the particles. This equation corresponds to the dispersion curve shown in Fig. 2a, if the velocities u_1 and u_2 have the same direction, and to the dispersion curve shown in Fig. 2b if the velocities u_1 and u_2 are oppositely directed. It is seen immediately from these figures that when the signs of u_1 and u_2 are the same there is an amplification band and a band of convective instability, and when the signs of u_1 and u_2 are opposite there is a nontransmission band and a band of absolute instability.

The use of Sturrock's rules leads to difficulties in the case when the instability bands or the bands of amplification and nontransmission overlap. In this case it is necessary to introduce in the dispersion equation a certain parameter ξ, variation of which makes it possible to break up the polynomial $D(k, \omega)$ into a product of linear factors $\omega - v_jk - a_j$. For concreteness we shall assume that the value of the parameter $\xi = 1$ corresponds to the initial equation, and $\xi = 0$ corresponds to the breakdown of the polynomial into factors.

Obviously, at $\xi = 0$, the dispersion curve will be a set of straight lines. We shall assume that at each point there intersect not more than two straight lines and that when ξ varies in the interval $0 < \xi \leq 1$ the topological character of the dispersion curves remains unchanged.

At small ξ, the instability (or amplification) bands will lie near the points of intersection of the lines $\omega - v_jk - a_j = 0$, into which the dispersion curve breaks up at $\xi = 0$. On the other hand, since it is assumed that the straight lines intersect pairwise, at small values of ξ the dispersion curves should obviously be similar to the curves shown in Fig. 1.

It can be shown that in the case of continuous increase of the parameter ξ, the character of the instability cannot change. This makes it possible to determine the character of the instability of the initial dispersion equation corresponding to the value of the parameter $\xi = 1$.

Let us consider by way of an example the dispersion equation

$$\frac{\omega_b^2}{(\omega - ku_b)^2} + \frac{\omega_p^2}{\omega^2 - k^2v_p^2} = 1, \tag{4.2}$$

which in the case

$$\omega_b < \omega_p, \quad v_p < u_b < v_p\sqrt{1 + \frac{\omega_b^2}{\omega_p^2}} \tag{4.3}$$

corresponds to the dispersion curve shown in Fig. 3c.

We shall assume ω_p^2 to be variable and replace ω_p^2 by $\omega_p^2\xi$:

$$\frac{\omega_b^2}{(\omega - ku_b)^2} + \frac{\omega_p^2\xi}{\omega^2 - k^2v_p^2} = 1. \tag{4.4}$$

At $\xi = 0$ Eq. (4.4) breaks up into four linear equations

$$\omega - ku_b = \pm\omega_b, \quad \omega = \pm kv_p,$$

and the dispersion curve degenerates into four straight lines.

At small ξ, the dispersion curve has been the form shown in Fig. 3a. As seen from this figure, the system described by the dispersion equation (4.4) has at small ξ a convective instability. (The convective-instability bands correspond to the wave-number intervals (k_G, k_H) and (k_I, k_J); k_M denotes the wave number corresponding to the point M.)

*These rules were obtained by Sturrock [12], but the proof contained inaccuracies, the elimination of which was the subject of [16]. A heuristic derivation of Sturrock's rules, connected with the concept of characteristics, is given in [11] (see also [17]).

If we solve the problem of amplifying oscillations in a system described by the dispersion equation (4.4) at small ξ, then it is seen directly from Fig. 3a that there are two amplification bands in the frequency intervals (ω_A, ω_B) and (ω_C, ω_D), and also two nontransmission bands in the frequency intervals (ω_F).

With increasing ξ, the dispersion curve (4.4) becomes deformed, but its topological character remains unchanged. When the parameter ξ reaches the value

$$\xi_0 = \frac{\omega_b^2}{\omega_p^2} \frac{v_p^2}{u_b^2},$$

the tangent to one of the branches of the dispersion curve, passing through the origin, becomes horizontal (see Fig. 3b). The two nontransmission bands (ω_E, ω_K) and (ω_L, ω_F) then merge into one band (ω_E, ω_F). It follows from the inequalities (4.3) that such a merging of the nontransmission bands occurs at $\xi_0 < 1$.

With further increase of ξ, the dispersion curve assumes the form shown in Fig. 3c. Since the curves in Figs. 3a, b, and c are topologically equivalent, the conclusion drawn concerning the character of the instability at small values of ξ remains valid also at $\xi = 1$, i.e., for the initial dispersion equation (4.2).

From a comparison of Figs. 3a, b, and c we can conclude that the initial system, described by the dispersion equation (4.2), has convective instability in the wave-number intervals (k_G, k_B) and (k_I, k_J). In addition, the system under consideration has two amplification bands in the frequency intervals (ω_A, ω_B) and (ω_C, ω_D), and also a nontransmission band (ω_E, ω_F).

5. GLOBAL INSTABILITY

So far, in the investigation of the instability of dynamic systems, we assumed them to extend to infinity and disregarded therefore the presence of boundaries. Yet the presence of a boundary can be very important, owing to reflection of waves from it. This can give rise to feedback between the "input" and "output" of the system, and as a result a convectively unstable system can behave as if it were absolutely unstable. An essential fact is that the effective absolute instability of this kind (it is called global instability*) will take place in the limiting case of infinitely extended systems, and this conclusion does not depend on the concrete form of the boundary conditions.

In order to clarify the concept of global instability, we recall that the natural oscillations in bounded systems result from a superposition of waves traveling in different directions. The frequencies of these waves are discrete, and the system will be stable in the case when at least one of the frequencies has a positive imaginary part.

As already noted in Sec. 2, to distinguish between waves traveling to the right and to the left it is necessary to determine the sign of the imaginary part of the function $k = k(\omega)$ at $\mathrm{Im}\,\omega \to +\infty$; if in this case $\mathrm{Im}\,k > 0$, then the wave travels to the right, and if $\mathrm{Im}\,k < 0$, then the wave travels to the left. In accord with this definition, we denote the wave numbers of the waves travelling to the right by $k_r(\omega)$, and those traveling to the left by $k_l(\omega)$. We note that these functions are solutions of the dispersion equation $D(k, \omega) = 0$ for an unbounded system.

Let now ω represent the natural frequency of the bounded system and let $\min \mathrm{Im}\, k_r(\omega)$ and $\max \mathrm{Im}\, k_l(\omega)$ denote respectively the smallest value of $\mathrm{Im}\,k$ for waves traveling to the right and the largest value of $\mathrm{Im}\,k$ for waves traveling to the left. It is clear that at a finite value of ω the quantity $\min \mathrm{Im}\, k_r(\omega)$ need not necessarily be positive, nor need $\max \mathrm{Im}\, k_l(\omega)$ be negative.

To obtain an equation for the natural frequency ω, we assume that on the left end of the system ($x = -L$) there are excited all the waves with wave numbers $k_r(\omega)$ and $k_l(\omega)$. Then only waves with wave numbers $k_r(\omega)$ will move to the right. On reaching the right end of the system ($x = L$), the largest amplitude at large L will be possessed by the wave with the wave number corresponding to the smallest value of Im $\mathrm{Im}\, k_r(\omega)$. If the amplitude of this wave was equal to unity at $x = -L$, then at $x = L$ its amplitude will be equal to

$$\exp[-2L \min \mathrm{Im}\, k_r(\omega)].$$

When this wave is reflected, waves with wave numbers $k_l(\omega)$ will be produced on the right end of the system, and will move to the left. When the left end of the system ($x = -L$) is reached, the largest amplitude will be possessed by the wave with the wave number corresponding to $\max \mathrm{Im}\, k_l(\omega)$. Its amplitude at $x = -L$ will be

$$T_+ \exp(-2L \min \mathrm{Im}\, k_r + 2L \max \mathrm{Im}\, k_l),$$

where T_+ is the coefficient of transmission of the wave with $\min \mathrm{Im}\, k_r$ into the wave with $\max \mathrm{Im}\, k_l$ on the right end of the system. When the wave with $\max \mathrm{Im}\, k_l$ is reflected from the left end of the system, a wave with $\min \mathrm{Im}\, k_r$ is again produced and has at $x = -L$ the amplitude

$$T_+ T_- \exp(-2L \min \mathrm{Im}\, k_r + 2L \max \mathrm{Im}\, k_l),$$

where T_- is the coefficient of transformation of the wave with $\max \mathrm{Im}\, k_l$ into a wave with $\min \mathrm{Im}\, k_r$ on the left end of the system.

We now let L go to infinity. Then the expression written out for the amplitude of the wave at $x = -L$ will differ from zero if the following condition is satisfied

$$\min \mathrm{Im}\, k_r(\omega) = \max \mathrm{Im}\, k_l(\omega), \tag{5.1}$$

which together with the dispersion equation $D(k, \omega) = 0$ determines the natural frequencies of a sufficiently long system. They correspond to a certain line (perhaps non-connected) on the complex ω plane. We note that in place of the spectrum of the natural oscillations we obtained a continuous line, since we have taken the limit as $L \to \infty$: each point of this line is the limit point of the discrete natural frequencies as $L \to \infty$.

If this line has points lying in the upper half-plane ($\mathrm{Im}\,\omega > 0$), then the system will be globally unstable.

It can be shown that a system possessing absolute instability will always be globally unstable. As to a convectively unstable system, we shall presently show that it can be either globally stable or globally unstable.

*The concept of global instability was introduced by Kulikovskii [19].

By way of an example let us consider a system whose dispersion equation is

$$3\omega^2 - 4\omega k + k^2 + 1 = 0.$$

This dispersion equation corresponds to two waves 1 and 2 with wave numbers

$$k_{1,2}(\omega) = 2\omega \pm \sqrt{\omega^2 - 1}.$$

It is easily seen that the system in question is convectively unstable. Indeed, putting Im $\omega \to +\infty$, we obtain Im $k_1 > 0$ and Im $k_2 > 0$. Therefore both waves 1 and 2 propagate to the right, i.e., in this case there are no branch points at which the wave numbers of the waves moving in opposite directions become equal.

Let us ascertain now whether this system is globally unstable.

Since both waves move to the right, no equation of the type (5.1) is obtained in this case, i.e., the system is globally stable.

We now consider a second example. Let the dispersion equation be

$$(3\omega^2 - 4\omega k + k^2 + 1)(\omega + k) = 0.$$

In this case three waves 1, 2, 3 are produced with wave numbers

$$k_{1,2} = 2\omega \pm \sqrt{\omega^2 - 1}, \quad k_3 = -\omega.$$

Waves 1 and 2 move to the right and wave 3 to the left.

It is clear that the system is convectively unstable, since the wave numbers of the first two waves do not differ from the wave numbers k_1 and k_2 considered in the preceding example (the third wave does not lead to instability).

However, unlike the preceding example, now the system will be globally unstable, and the instability is due to the existence of the third wave. To verify this, we define the natural frequencies in accordance with Eq. (5.1):

$$\mathrm{Im}\, k_{1,2}(\omega) = \mathrm{Im}\, k_3(\omega). \tag{5.2}$$

Putting $\omega = \alpha + i\beta$, we obtain

$$\mathrm{Im}\, k_{1,2} = 2\beta \pm \sqrt{\frac{\sqrt{(\alpha^2-\beta^2-1)^2 + 4\alpha^2\beta^2} - (\alpha^2 - \beta^2 - 1)}{2}},$$

and from (5.2) it follows that

$$\alpha^2 + 17\beta^2 - 1 = \sqrt{(\alpha^2-\beta^2-1)^2 + 4\alpha^2\beta^2}. \tag{5.3}$$

Squaring both sides of this equation, we obtain the equation of the line (5.2):

$$8\alpha^2 + 72\beta^2 = 9. \tag{5.4}$$

As follows from (5.3), the condition $\alpha^2 + 17\beta^2 \geq 1$ should be satisfied (satisfaction of this condition corresponds to a positive sign in front of the radical).

It is easily seen that the ellipse (5.4) lies entirely in the region $\alpha^2 + 17\beta^2 \geq 1$. This means that all the points of the line (5.4) satisfy Eq. (5.2), and since part of the ellipse (5.4) lies in the upper half-plane ($\beta > 0$), the system under consideration remains globally unstable.

Thus, a dynamic system having convective instability can be either globally stable or globally unstable.

[1] H. Weizner, Phys. Fluids 5, 933 (1962).

[2] K. B. Dysthe, Nucl. Fusion 6, 215 (1966).

[3] R. Q. Twiss, Proc. Phys. Soc. B64, 654 (1951).

[4] L. D. Landau and E. M. Lifshitz, Mikhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1954, p. 141 [Addison-Wesley, 1958].

[5] Ya. B. Faĭnberg, V. I. Kurilko, and V. D. Shapiro, Zh. Tekh. Fiz. 31, 633 (1961) [Sov. Phys.-Tech. Phys. 6, 459 (1961)].

[6] R. J. Briggs, Electron-stream Interaction with Plasmas, The Massachusetts Inst. of Technol. Press, Cambridge, Massachusetts, 1964.

[7] G. Convert, Ann. de Radioélectricité 19, 300 (1964).

[8] M. Feix, Nuovo Cimento 27, 1130 (1963).

[9] R. Q. Twiss, Phys. Rev. 84, 448 (1951).

[10] J. R. Pierce, Bell. Syst. Tech. J. 30, 626 (1951).

[11] A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, Kollektivnye kolebaniya v plazme (Collective Oscillations in a Plasma), Atomizdat, 1964.

[12] P. A. Sturrock, Phys. Rev. 112, 1488 (1959).

[13] A. M. Fedorchenko, Radiotekhhinka i élektronika 7, 1455 (1962).

[14] P. Rolland, Phys. Rev. 140, 776 (1965).

[15] A. M. Fedorchenko, Zh. Tekh. Fiz. 39, 1549 (1969) [Sov. Phys.-Tech. Phys. 14, 1163 (1970)].

[16] R. V. Polovin, ibid. 31, 1220 (1961) [6, 889 (1962)].

[17] G. F. Filimonov, Radiotekhnika i élektronika 7, 75 (1959).

[18] P. A. Sturrock, Phys. Rev. 117, 1426 (1960).

[19] A. G. Kulikovskiĭ, Prikl. Mat. Mekh. 30, 148 (1966).

Translated by J. G. Adashko