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STABILITY OF RELATIVISTIC ELECTRON BEAMS IN A PLASMA AND THE PROBLEM OF CRITICAL CURRENTS

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1. INTRODUCTION

In recent years, the attention of researchers has turned again to problems of strong-current electron accelerators. This is due, on the one hand, to the development of high-power electronics and the creation of well-emitting surfaces (electron beams, plasma cathode, etc.) and, on the other hand, to the increasing interest in the intense sources of x-rays and microwaves. A major role was played here also by the recently advanced ideas of the use of powerful relativistic electron beams, to initiate controlled thermonuclear reactions and to transmit energy over large distances.

It was shown already in Langmuir's first papers (see [1] and the literature cited there) that the main obstacle in the path of obtaining strong-current electron beams is the space charge, which limits the maximum current in the beams. To overcome this obstacle, Pierce[2] advanced the idea of compensating the charge of the electron beam by means of an ion background. He called attention to the fact that in compensated electron beams the current cannot increase without limit; at currents exceeding a certain critical value, electrostatic instabilities can develop in the system. However, confining himself to the case of infinitely heavy ions, Pierce could not prove that under these conditions the system is actually unstable. The physical nature of the instability of an electron beam passing through an ionic core was uncovered much later by Budker[3] and by Buneman,[4] who took into account the finite mass of the ions. It will be shown below that it is precisely this instability which determines the limiting current in a compensated electron beam. In the case of nonrelativistic beams, this current is only a few times larger than the vacuum limiting current determined by the space charge of the electrons of the beam. Experimental investigations,[5] however, did not confirm this theoretical conclusion fully. In some cases, when the electron beam was confined by a not very strong magnetic field, the limiting current turned out to be smaller than that determined from the condition for the development of the Buneman instability. In [5] it was noted correctly that the reason for the discrepancy between theory and experiment is the current-convective instability,[6,7] which under certain conditions can develop at currents smaller than required for the development of the Buneman instability.

Later, in [8,9], an attempt was made to systematize different types of instabilities that can develop in compensated electron beams, and the critical currents of such a system were determined from the conditions for the development of the instabilities. In spite of the qualitative agreement with experiment, the analysis carried out in these papers must be regarded as unsatisfactory. The point is that in [8,9] they used the theory of stability of a spatially-inhomogeneous plasma in the geometrical-optics approximation (the theory developed in [6,7]), which makes it possible to determine the critical currents in compensated beams relative to excitation of short-wave oscillations. A more rigorous analysis carried out in [10] has shown that the critical currents are actually determined by the excitation of long-wave oscillations with a wavelength larger than the transverse dimensions of the electron beam. To investigate the stability of the beams against such long-wave oscillations, it is necessary to solve the boundary-valve problem; this solution is given in Chs. 4 and 5 of the present review. For comparison, we present in Ch. 3 expressions for the limiting currents in uncompensated electron beams in the cases of both nonrelativistic and relativistic beam energies. It should be noted that, unlike the nonrelativistic electron beams, the limiting currents in which were discussed many times in the literature (see [1,11] and also in the review papers[12,13]), the question of the limiting currents of relativistic uncompensated beams was discussed, insofar as we know, only in [10, 14, 15]. This remark, however, pertains to electron beams kept from spreading by a strong longitudinal magnetic field greatly exceeding the magnetic field produced by the beam current. The question of limiting currents in partly or fully compensated relativistic electron beams balanced by the current's own field was discussed long ago.[16, 17] A review of the literature on limiting currents of relativistic beams, determined from the equilibrium conditions, was recently presented in [18] where, in particular, it is shown that in the presence of a strong longitudinal magnetic field the current in the electron beam is not limited by the equilibrium conditions. The limitation on the beam current is imposed by the requirement that it be stable.

As already noted above, the critical current in a nonrelativistic compensated beam, determined by the stability condition, can exceed the limiting current of an uncompensated beam by only a few times. The situation is different in the case of relativistic energies of the beam electrons. In Chs. 4 and 5 of the present review we show that the critical current in a relativistic compensated beam can exceed the vacuum current by a factor $(\mathcal{E}/mc^2)^2$, where $\mathcal{E}$ is the electron energy. It should be noted, however, that such an increase of the current is possible only under conditions when no current-convective instability develops in the system. The latter, generally speaking, can develop at very small currents, even smaller than the limiting current in an uncompensated beam.

At finite values of the magnetic field, it is quite difficult to produce conditions when the current-convective instability in the compensated beam does not develop. It is therefore difficult to make use of the aforementioned advantage of the compensated beam over the uncompensated one under real conditions. It is much easier, as shown in [19], to obtain large currents in overcompensated electron beams in the case when the beam passes through a denser plasma. The conditions for the development of the electron current-convective instability,[7, 20] which limits the critical current in uncompensated beams, are much more difficult. Questions of the interaction of a relativistic electron beam with a denser plasma and problems of critical currents in such a system are discussed in Chs. 6, 7, and 8 of the present review.

The theory of the interaction of electron beams with a plasma has been the subject of a large number of papers (see [21, 22] and the literature cited therein). Nonetheless, the presently available tremendous experimental material[23] cannot always be explained and compared quantitatively with the theory. From our point of view, a good approach [5, 20] has been noted in recent years in the experimental investigation of the interaction of beams with a plasma; this approach makes it possible to establish the correspondence between the experiments and the theory and consists in investigation of the interaction of a beam with a plasma produced by the beam itself by ionization of the gas and in the determination of the critical parameters (the critical current or the critical plasma density), at which instabilities arise in the system. The process of plasma formation is in this case slow compared with the characteristic times of development of the instabilities. Therefore, by regulating the gas pressure, the beam density, its energy, the magnitude of the longitudinal magnetic field, and the geometrical dimensions of the system, it is always possible to create conditions for the development of some single type of instability and to investigate it in detail. Such an experimental approach to a beam-plasma interaction calls for a corresponding change in the formulation of the theoretical research. It becomes the task of the theory to determine those critical parameters of the plasma and of the beam under which some type of collective interaction arises in a bounded system. It is precisely from this point of view that we present in the present review the theory of stability of electron beams in a plasma.

Finally, we point to one more advantage of a system with an overcompensated electron beam. As will be shown below (see Ch. 8 and also [24, 25]), when a relativistic electron beam interacts with a plasma, the relative loss of beam energy to excitation of the oscillations is of the order of $(\mathcal{E}/mc^2)(n_1/n_2)^{1/3}$, where n_1 and n_2 are the densities of the electrons in the beam and in the plasma, respectively. Under conditions when this quantity is small, the loss of beam energy and the resultant energy spread of the electrons are negligible, and in spite of the fact that the conditions for development of instability are satisfied in this system, the beam will pass through the plasma practically unchanged. In this case one should speak of critical currents in the system, to distinguish it from the case of strictly compensated beams, where the beam loses a considerable fraction of its energy as a result of the development of the Buneman instability[26] and undergoes considerable changes. Therefore the critical currents in compensating beams are simultaneously also limiting currents.

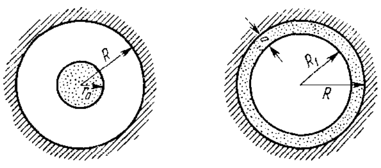
Finally, in Ch. 9 of the present review the developed theoretical concepts are compared with experiments[5, 9, 20, 27, 28] on the interaction of electron beams with the plasma produced by them.

2. FORMULATION OF PROBLEM AND INITIAL EQUATIONS

We consider an electron beam in an equipotential drift space, passing along the axis of a metallic waveguide of radius R and of longitudinal dimension $L \gg R$. The critical currents passed by such a system will be determined from the condition of the stability of the electron beam as it passes either through the compensating background of ions or through the denser plasma. We shall therefore formulate in the present chapter the fundamental equations describing the stability of such a beam. We investigated two possible geometries of the electron beam, as shown in Fig. 1 (cases (a) and (b)). In the first of them, a beam of radius $r_0 \leq R$ passes along the axis of the waveguide, and in the second it is "tubular" with thickness $a = R - R_1$. In the limits as $r_0 \to R$ and $R_1 \to 0$, the two cases coincide and describe an electron beam completely filling the cross section of the waveguide.

We note first that in order to prevent spatial splitting of the beam, the system must be placed in a sufficiently strong longitudinal magnetic field satisfying the condition

$$B_0^2/8\pi \gg n_1 mc^2\gamma, \tag{2.1}$$



a) b)

FIG. 1

where n_1 is the density of the electrons in the beam, m is the rest mass, $\gamma = (1 - u^2c^2)^{-1/2}$ (u is the electron velocity), and B_0 is the magnetic field intensity. When the condition (2.1) is satisfied, the energy of the magnetic field is much higher than the energy of the beam electrons, making it possible to confine ourselves in the investigation of the stability to consideration of only the potential (electrostatic) oscillations of the field in the system.

In addition, we shall assume that the magnetic field B_0 greatly exceeds the self-field of the current, i.e., in case (a)

$$B_0 \gg 2J/cr_0, \tag{2.2a}$$

and in case (b)

$$B_0 \gg \frac{2JR}{c(R^2-R_1^2)} \to \frac{J}{ca} \quad \text{for} \quad a \ll R. \tag{2.2b}$$

It will be shown below that the conditions (2.2) are equivalent to requiring that the Larmor radius of the electrons in the longitudinal magnetic field be small compared with the transverse dimensions of the beam, i.e., to the condition of transverse localization of the electrons.

In fields $B_0 \gtrsim 10^5$ Oe the conditions (2.1) and (2.2) are well satisfied for beams with energy $\mathcal{E} \lesssim (1-10)$ MeV and $r_0 \sim a \sim 1$ cm at currents $J \lesssim (1-5) \times 10^5$ A (or, equivalently, for $n_1 \lesssim (1-3) \times 10^{13}\ \mathrm{cm^{-3}}$).

Finally, it must be specially stipulated that the entire subsequent analysis of the interaction of the electrons beams with the plasma pertains to the stationary case, and therefore does not take into account the work necessary to produce the self-field of the beam current. This work, especially in the region of large currents and at relativistic electron energies, can greatly exceed the kinetic energy of the beam. It is consumed, however, only during the transient buildup of the current in the system; in the stationary regime, on the other hand, the energy of the magnetic field remains unchanged and no work is done. We note that in the system considered by us the magnetic energy of the current exceeds the kinetic energy of the beam under conditions when the following inequalities are satisfied:

$$J > \frac{4mc^3}{e}\frac{c}{u}(\gamma-1)\frac{1}{1+4\ln(R/r_0)} \tag{2.3a}$$

in case (a) and

$$J > \frac{4mc^3}{e}\frac{c}{u}(\gamma-1)\frac{R^2-R_1^2}{R^2+R_1^2} \tag{2.3b}$$

in case (b) (tubular beam). For electron beams that fill the cross section of the waveguide completely (i.e., $r_0 \approx R$ or $R_1 = 0$), at energies 1–10 MeV the inequality (2.3) is satisfied only for very large currents $J \gtrsim (1-15) \times 10^5$ A. However, if the beam does not fill the waveguide completely, then the magnetic energy of the current turns out to be larger than the kinetic energy of the beam particles even at relatively small currents, on the order of several dozen kiloamperes.

Having indicated the main limitations, let us formulate the equations describing the motion of the electron beam. Since we are interested in electron beams of high energy and in their stability against rapidly growing oscillations, we can neglect the thermal motion of the particles. To describe the considered system we therefore use the relativistic equations of two-fluid hydrodynamics of a cold plasma (see, for example, [29])

$$\left.\begin{aligned} &\frac{\partial n_\alpha}{\partial t} + \operatorname{div} n_\alpha \mathbf{v}_\alpha = 0, \\ &\left(\frac{\partial}{\partial t} + \mathbf{v}_\alpha\nabla\right)\frac{\mathbf{v}_\alpha}{\sqrt{1-\frac{v_\alpha^2}{c^2}}} = \frac{e_\alpha}{m_\alpha}\left\{-\nabla\Phi + \frac{1}{c}[\mathbf{v}_\alpha\cdot\mathbf{B}_0]\right\}, \\ &\Delta\Phi = -4\pi\sum_\alpha e_\alpha n_\alpha, \end{aligned}\right\} \tag{2.4*}$$

where α = e or i, and Φ is the potential of the electric field, which arises in the system only during the oscillations.

In the analysis of an uncompensated electron beam it is necessary to solve the nonlinear stationary system (2.4) with allowance for the space charge of the electrons, and to determine from the beam cutoff conditions the limiting current passed by the system (see Ch. 3).

To determine the critical currents in compensated and overcompensated beams, the system of equations (2.4) is linearized and the stability problem is solved. In the stationary equilibrium state it is assumed here that the beam electrons move with velocity u relative to the resting ions or the dense plasma. The z axis is assumed directed along the waveguide axis and the beam velocity and the external magnetic field are parallel to the z axis. Since the critical currents in electron beams, as will be shown presently, are determined by excitation of the longest-wavelength oscillations in the system, the parameters of the beam and of the plasma (the dimensions of which we assume to coincide with the beam dimensions) can be regarded as homogeneous with a sharp boundary. Linearizing the system (2.4) with respect to small deviations from the equilibrium state and assuming for the non-equilibrium quantities a time and coordinate dependence in the form

$$f(r)\,e^{-i\omega t + il\varphi + ik_z z}, \tag{2.5}$$

where l and k_z are respectively the azimuthal and longitudinal wave numbers, we obtain for the potential of the perturbation field the following equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\varepsilon_\perp r\frac{\partial\Phi}{\partial r}\right) + \left(\frac{l}{r}\frac{\partial g}{\partial r} - \varepsilon_\perp\frac{l^2}{r^2} - \varepsilon_\parallel k_z^2\right)\Phi = 0. \tag{2.6}$$

We have introduced here the notation

$$\varepsilon_\perp = 1 - \sum \frac{\omega_L^2(1-\beta^2)^{1/2}}{\frac{(\omega-\mathbf{k}\mathbf{u})^2}{1-\beta^2} - \Omega^2},$$

$$\varepsilon_\parallel = 1 - \sum \frac{\omega_L^2(1-\beta^2)^{3/2}}{(\omega-\mathbf{k}\mathbf{u})^2},$$

$$g = \sum \frac{\omega_L^2\Omega}{\omega-\mathbf{k}\mathbf{u}}\left[\frac{(\omega-\mathbf{k}\mathbf{u})^2}{1-\beta^2} - \Omega^2\right]^{-1},$$

where $\omega_L^2 = 4\pi e^2 n/m$, $\Omega = eB_0/mc$, $\beta = u/c$ and the

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$^*[\mathbf{v}_\alpha \mathbf{B}_0] \equiv \mathbf{v}_\alpha \times \mathbf{B}_0$.

summation extends over all species of charged particles in the system.

Equation (2.6) should be supplemented with boundary conditions. On the free surface of the beam these conditions are obtained by integrating Eq. (2.6) over an infinitesimally thin layer near the surface, and take the form

$$\{\Phi\}_{z=r_0,\,R_1}=0, \tag{2.7}$$

$$\left\{\varepsilon_\perp \frac{\partial\Phi}{\partial r}+\frac{l}{r}\,g\Phi\right\}_{r=r_0,\,R_1}=0. \tag{2.8}$$

On the surface of the metallic waveguide, on the other hand, we have the obvious condition

$$\Phi\,|_{r=R}=0. \tag{2.9}$$

Solving Eq. (2.6) with allowance for the boundary conditions (2.8) and (2.9), we find the dispersion equation for small electrostatic oscillations of the system. In case (a), i.e., for a beam passing along the axis of the waveguide, this equation is written in the form

$$\varepsilon_\perp \frac{1}{J_l(i\alpha r_0)}\frac{dJ_l(i\alpha r_0)}{dr_0}+\frac{l}{r_0}\,g+f_l=0, \tag{2.10}$$

where

$$f_l=|k_z|\frac{I_l(|k_z|R)\,K_l'(|k_z|r_0)-K_l(|k_z|R)\,I_l'(|k_z|r_0)}{I_l(|k_z|r_0)\,K_l(|k_z|R)-I_l(|k_z|R)\,K_l(|k_z|r_0)}. \tag{2.11}$$

For a tubular beam (case (b)) the dispersion equation of the electrostatic oscillations is of the form

$$J_l(i\alpha R)\left\{I_l(|k_z|R_1)\left[i\alpha\varepsilon_\perp N_l'(i\alpha R_1)+\frac{l}{R_1}\,gN_l(i\alpha R_1)\right]\right.$$
$$\left.-|k_z|N_l(i\alpha R_1)\,I_l'(|k_z|R_1)\right\}+N_l(i\alpha R)\left\{|k_z|J_l(i\alpha R_1)\,I_l'(|k_z|R_1)\right.$$
$$\left.-I_l(|k_z|R_1)\left[i\alpha\varepsilon_\perp J_l'(i\alpha R_1)+\frac{l}{R_1}\,gJ_l(i\alpha R_1)\right]\right\}=0. \tag{2.12}$$

In (2.10)–(2.12), J_l, I_l, N_l, and K_l are Bessel functions, and

$$\alpha_1=k_z^2(\varepsilon_\parallel/\varepsilon_\perp). \tag{2.13}$$

We note that under conditions when $R_1 \to 0$ or $r_0 \to R$, i.e., when the beam fills the cross section of the waveguide completely, Eqs. (2.10) and (2.12) coincide and are written in the form

$$J_l(i\alpha R)=0. \tag{2.14}$$

For a system that is infinite in the longitudinal direction, relations (2.10), (2.12), and (2.14) constitute the dispersion equations that determine the oscillation spectra $\omega=\omega(\mathbf{k})$. Our problem consists of determining the parameters of the plasma and of the beam for which instability arises in the system, i.e., there appear among the roots of Eqs. (2.10), (2.12), and (2.14) solutions with $\mathrm{Im}\,\omega(\mathbf{k})>0$. It is precisely from this condition that we shall determine in Chs. 4 and 6 the critical currents in the beam and the critical densities of the plasma, corresponding to the onset of instability in this system.

On the other hand, if the waveguide has finite longitudinal dimensions, then relations (2.10)–(2.14) must be regarded as characteristic equations for the determination of the wave numbers k_z. Then the solution takes the form $\Phi=\sum_n C_n \exp\{ik_{zn}z\}$, where k_{zn} are the roots of the characteristic equation. To obtain the dispersion equation, this solution must be substituted into the boundary conditions on the ends of the waveguide, and the number of these boundary conditions must correspond to the number of roots k_{zn}. These boundary conditions will be discussed in Chs. 5 and 7 below, when we determine the beam and plasma critical parameters corresponding to the onset of instability in bounded systems.

3. LIMITING CURRENTS IN UNCOMPENSATED ELECTRON BEAMS

In uncompensated electron beams, the currents cannot exceed a value determined by the space charge of the electrons. Expressions for the limiting current can be obtained directly from (2.4) by considering a purely electronic beam. Taking into account the energy and momentum conservation laws

$$mc^2\gamma+e\Phi=mc^2=\text{const},\qquad env=j=\text{const}, \tag{3.1}$$

the Poisson equation for a cylindrically-symmetrical electron beam reduces to the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)=-\frac{4\pi j}{c}\left[1-\left(1-\frac{e\Phi}{mc^2}\right)^{-2}\right]^{-1/2}. \tag{3.2}$$

In writing out relations (3.1) and (3.2) we assumed that $\Phi=0$ at $v=0$. This is how the potential is reckoned in the literature* [1, 2, 11–14].

An analysis of Eq. (3.2) in the general case for an arbitrary ratio $e\Phi/mc^2$ is very complicated and is possible only by numerically solving this equation. However, such an analysis becomes much simpler in the limiting cases of nonrelativistic and relativistic beam energies. In the nonrelativistic limit, when $e\Phi \ll mc^2$, we obtain from (3.2)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)=-\frac{4\pi j}{\sqrt{2e\Phi/m}}. \tag{3.3}$$

This equation is valid inside the electron beam. In the region not occupied by the beam, the potential satisfies the homogeneous equation.

To determine the limiting current of the electron beam, the Poisson equation must be solved under the following boundary conditions:

$$\Phi|_{r=0}=\Phi_0,\quad \Phi|_{r=R}=\frac{mu^2}{2e},\qquad \{\Phi\}_{r=r_0,\,R_1}=\left\{\frac{\partial\Phi}{\partial r}\right\}_{r=r_0,\,R_1}=0. \tag{3.4}$$

The relation obtained in this case between the current in the beam and the value of the potential on the waveguide axis $J(\Phi_0)$ must be maximized with respect to Φ_0. The maximum value of the current $J(u)$ is the

*To avoid misunderstandings, we note that on entering the waveguide (the drift space) all the beam electrons have the same energy (and consequently the same velocity), which is determined by the accelerating system. We note, however, that over a length on the order of the radius of the waveguide the electrons are decelerated by the space charge of the beam, and at large distances $z \gg R$ there is established a stationary picture that is uniform in the longitudinal direction and not uniform radially, and is described by the system of equations (2.4). The considered particular solution of this system (3.1) and (3.2), which is homogeneous along the z axis, corresponds in essence to the presence of a sufficiently strong longitudinal magnetic field satisfying the inequalities (2.2).

sought limiting current of the uncompensated electron beam passed by the system in question. Since Eq. (3.3) is nonlinear, the solution of such a problem, strictly speaking, is possible only numerically. It is precisely in this manner that the limiting current was determined for a nonrelativistic electron beam passing in drift space along the axis of a metallic waveguide.[11] We present here an approximate analytic expression for the current passed by such a system, obtained from the particular solution of the equation

$$\Phi = \left(\frac{9}{4} J \sqrt{\frac{m}{2e}}\right)^{2/3} \left(\frac{r}{r_0}\right)^{4/3}$$

(for this solution $\Phi_0 = 0$):

$$J_0 = \frac{8}{9} \frac{mu^3}{4e} \frac{1}{\left[1 + \frac{4}{3} \ln (R/r_0)\right]^{3/2}}. \quad (3.5)$$

The numerical analysis carried out in [11] gives the following values for the limiting current at different ratios R/r_0:

$$J_0 = \frac{mu^3}{4e} \begin{cases} 0.88 & \text{for } R \approx r_0, \\ 0.31 & \text{for } R = 2{,}2 r_0, \\ 0.14 & \text{for } R = 10 r_0. \end{cases} \quad (3.6)$$

A comparison of these values with (3.5) shows that this formula gives, with a sufficient degree of accuracy, the limiting current up to $R/r_0 \sim 10$.

At large values of the ratio R/r_0, when $\ln (R/r_0) \gg 1$, formula (3.5) becomes inaccurate. In this case it is possible to obtain an exact analytic expression for the limiting current of the beam for all electron energies, assuming the potential inside the beam to be homogeneous and equal to Φ_0. The second boundary condition (3.4) must in this case be replaced by

$$\Phi|_{r=R} = \frac{\mathscr{E} - mc^2}{e} = \frac{mc^2}{e}(\gamma - 1), \quad (3.7)$$

which is valid for all electron energies. Here $\gamma = \mathscr{E}/mc^2 = 1/\sqrt{1-(u^2/c^2)}$. By determining the current passed by the system under these conditions, and by maximizing it with respect to Φ_0, we obtain an expression for the limiting current in the beam

$$J_0 = \frac{mc^3}{e} \frac{1}{2 \ln (R/r_0)} (\gamma^{2/3} - 1)^{3/2}. \quad (3.8)$$

In the nonrelativistic limit, when $\gamma \approx 1 + u^2/2c^2$, this expression takes the form

$$J_0 = \frac{2}{3\sqrt{3}} \frac{mu^3}{4e} \frac{1}{\ln (R/r_0)}. \quad (3.9)$$

In the opposite limit of ultrarelativistic energies of electrons in the beam, when $\gamma \gg 1$, we obtain from (3.8)

$$J_0 = \frac{\mathscr{E} c}{e} \frac{1}{2 \ln (R/r_0)}. \quad (3.10)$$

For an ultrarelativistic electron beam, the limiting current can be obtained for arbitrary values of the ratio R/r_0. The point is that in this case the Poisson equation (3.2) with allowance for the inequality $e\Phi \gg mc^2$ reduces to the linear equation

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r}\right) = -\frac{4\pi j}{c}, \quad (3.11)$$

which can readily be solved analytically and exactly. As a result we obtain for the limiting current the expression[10, 14]

$$J_0 = \frac{\mathscr{E} c}{e} \frac{1}{1 + 2 \ln (R/r_0)}. \quad (3.12)$$

When $\ln (R/r_0) \gg 1/2$ this expression goes over into (3.10).

Finally, we present an interpolation formula for the limiting current in an electron beam passing in the drift space along the axis of the metallic waveguide, one that generalizes formula (3.6)–(3.12):

$$J_0 = \frac{mc^3}{e} \frac{1}{1 + 2 \ln (R/r_0)} (\gamma^{2/3} - 1)^{3/2}. \quad (3.13)$$

At high electron energies, $\gamma \gg 1$, this formula is exact, but at nonrelativistic energies it coincides with good accuracy with (3.6).

From a comparison of formulas (3.5)–(3.13) we see that with increasing electron energy the growth of the limiting current in the beam slows down. At nonrelativistic energies we have $J_0 \sim \mathscr{E}^{3/2}$, whereas in the ultrarelativistic limit $J_0 \sim \mathscr{E}$. This circumstance limits considerably the limiting current in vacuum systems at relativistic beam energies. Thus, as $\mathscr{E} \sim 1$–10 MeV and $R \sim r_0$, the vacuum current changes in the range $J_0 \sim (3\text{–}30) \times 10^4$ A. When $R > r_0$, the limiting current passed by the vacuum system turns out to be even smaller.

Much larger currents can be reached in tubular electron beams under conditions when $R \gg a$, i.e., when the beam thickness a is many times smaller than the radius of the waveguide. In this limit, the Poisson equation (3.2) reduces, by making the substitution $r = R - x$ (with $R \gg x$) to the planar equation

$$\frac{\partial^2 \Phi}{\partial x^2} = -\frac{4\pi j}{c}\left[1 - \left(1 - \frac{e\Phi}{mc^2}\right)^{-2}\right]^{-1/2}, \quad (3.14)$$

which admits of an exact analytic solution. Satisfying the boundary conditions (3.4) and (3.7), we ultimately obtain for the limiting current in a tubular electron beam at $R \gg a$ the expression[15]

$$J_0 = \frac{mc^3}{e} (\gamma^{2/3} - 1)^{3/2} \frac{R}{a}. \quad (3.15)$$

We see that the limiting current in an uncompensated tubular electron beam exceeds by R/a times the limiting current in the vacuum system when the waveguide cross section is completely filled with the beam. This result seems physically obvious. Actually, when the waveguide cross section is completely filled, the limiting current in the electron beam does not depend on the radius of the beam. It is important, however, that the beam touches the metallic walls of the waveguide. Therefore, by locating beams with radius $a \ll R$ near the surface of the waveguide it is possible to increase the current in the system by a factor R/a. At electron energies $\mathscr{E} \sim 1$–10 MeV and $R/a \sim 10$, the limiting current in the tubular beam turns out to be of the order of $J_0 \approx (3\text{–}30) \times 10^5$ A.

It should be noted that the ratio R/a in a tubular beam cannot be arbitrarily large. The point is, as indicated above, that neglect of the self-field of the beam current (of condition (2.2)) imposes the limitation that the Larmor radius of the electrons be small compared with the transverse dimensions of the beam. Indeed, by assuming for the current in a relativistic electron beam the estimate $J_0 \sim (mc^3/e)\gamma(R/a)$ (at $R \approx a$ this estimate is also suitable for a beam completely filling the waveguide cross section), we get from conditions (2.2)

$$a \gg \frac{mc^2}{eB_0}\gamma\frac{R}{a} = \frac{c}{\Omega_e}\gamma\frac{R}{a},$$

where $\Omega_e = eB_0/mc$. In the electron energy range $\mathscr{E} \sim 1\text{–}10$ MeV of interest to us and at $a \sim 1$ cm the ration $R/a \approx 10$ can be attained only in sufficiently strong magnetic fields $B_0 \gtrsim 10^5$ Oe.

The above-noted increase of the limiting current by a factor R/a in a tubular beam takes place under conditions when the cavity inside the beam is evacuated or when the beam is contained between two coaxial metallic cylinders. If the cavity inside the beam is filled with plasma, then, as shown in [13], the limiting current in the tubular beam is no larger than in the case of a solid beam. In the system considered by us, with a strong longitudinal magnetic field, the plasma is produced only in the region occupied by the electron beam, and cannot diffuse outside the limits of the tubular beam (during the short time of the pulse duration). Therefore the cavity inside the beam is actually a vacuum.

In conclusion, we emphasize once more that the expressions given above for the limiting currents in uncompensated electron beams, like all the results of the present review, are valid only in the presence of a strong longitudinal magnetic field satisfying the conditions (2.2). Only in this case does the self-field of the current play no important role in the formation of the beam. The beam electrons are contained and guided by the external longitudinal magnetic field. On the other hand, if conditions (2.2) are not satisfied, then the trajectories of the electrons and the equilibrium of the beams as a whole are determined essentially by the self-field of the beam. It is obvious that the first integrals of motion in the form (3.1) likewise become meaningless in this case, together with the formulas given above for the limiting currents of the uncompensated electron beams. Expressions for the limiting currents in electron beams with allowance for the influence of the magnetic field of the current on the motion of the electrons can be found in [1, 12, 16–18].

4. CRITICAL CURRENTS IN COMPENSATED UNBOUNDED ELECTRON BEAMS

Let us consider now an electron beam whose charge is compensated by ions. As already noted, the current can be larger in such compensated beams than in uncompensated ones. However, even in compensated beams the current cannot be arbitrarily large and is limited by the condition of the development of electrostatic instabilities. The minimum value of the current at which instability occurs in the beam will henceforth be called critical. An expression for the critical current can be obtained from an analysis of relations (2.10)–(2.14), which for the here-considered case of an electron beam unbounded in the longitudinal direction constitute the dispersion equation of the oscillations.

We investigate first the stability of an electron beam filling completely the cross section of the waveguide, i.e., $R \approx r_0$. It is easy to show that in this case the dispersion equation (2.14) reduces to

$$k_z^2\left[1 - \frac{\omega_{Le}^2}{\gamma^3(\omega - k_z u)^2} - \frac{\omega_{Li}^2}{\omega^2}\right] + \frac{\mu_{sl}^2}{R^2}\left(1 - \frac{\omega_{Li}^2}{\omega^2 - \Omega_i^2}\right) = 0, \qquad (4.1)$$

where $n_e = n_i = n_1$, and μ_{sl} are the roots of the Bessel function $J_l(\mu_{sl}) = 0$. In deriving (4.1) we took into account the inequality (2.1) which makes it possible to confine ourselves to an investigation of the stability of the beam in the frequency region $(\omega - k_z u)^2 \ll \Omega_e^2(1 - \beta^2) = \Omega_e^2/\gamma^2$.

In the analysis of Eq. (4.1) it is necessary to distinguish between two limiting cases. If $(\mu_{sl}^2\gamma^3/k_z^2R^2) \times (m/M) \ll 1$, then the unstable solutions of this equation lie in the region $\omega < k_z u$, with

$$\omega^2 = \frac{b \pm \sqrt{b^2 - 4a\omega_{Li}^2\Omega_i^2}}{2a}, \qquad (4.2)$$

where

$$a = 1 - \frac{\omega_{Le}^2}{\gamma^3 k_z^2 u^2} + \frac{\mu_{sl}^2}{k_z^2 R^2},$$

$$b = a\Omega_i^2 + \omega_{Li}^2 + \mu_{sl}^2\frac{\omega_{Li}^2}{k_z^2 R^2}.$$

From the instability condition ($\omega^2 < 0$ when $a < 0$) we obtain the current in the beam at which an oscillation mode with specified wave numbers k_z, l, and μ_{sl} is excited:[3, 19]*

$$J = \frac{mu^3}{4e}\gamma^3(k_z^2R^2 + \mu_{sl}^2). \qquad (4.3)$$

In the opposite limiting case when $(\mu_{sl}^2\gamma^3/k_z^2R^2) \times (m/M) \gg 1$, the unstable solutions of Eq. (4.1) lie in the region $\omega \approx k_z u$. Writing $\omega = k_z u + \gamma_0$, we obtain for the determination of the small increment γ_0

$$\gamma_0^2 = \frac{\omega_{Le}^2 k_z^2 R^2}{\mu_{sl}^2\gamma^3\left(1 - \frac{\omega_{Li}^2}{k_z^2u^2 - \Omega_i^2}\right)}. \qquad (4.4)$$

We see therefore that the oscillations that can grow ($\gamma_0^2 < 0$) are those for which $\omega_{Li}^2 > k_z^2u^2 \geq \Omega_i^2$. The current necessary to excite oscillations with a specified value of ω is given by

$$J = \frac{mu^3}{4e}\frac{M}{m}k_z^2R^2. \qquad (4.5)$$

The critical current in a compensated electron beam with radius $r_0 = R$, filling completely the cross section of the waveguide, is determined by minimizing expressions (4.3) and (4.5) and corresponds to the minimum current in the beam, at which Buneman instability occurs in the system. It is equal to

$$J_{cr} = \min\begin{cases}\frac{mu^3}{4e}(2{,}4)^2\gamma^3, \\ \frac{mu}{4e}\frac{m}{M}R^2\Omega_e^2.\end{cases} \qquad (4.6)$$

In the case of strong magnetic fields, when $\Omega_e > (2.4u\gamma^{3/2}/R)\sqrt{M/m}$, the critical current passed by the system considered by us is determined by the first expression of (4.6); in the opposite limiting case it coincides with the second expression. It should be noted here that, strictly speaking, this statement is valid practically for unbounded systems, the length of which is

$$L \gg \left(\frac{\pi u}{\Omega_i}, \frac{\pi R}{2.4}\sqrt{\frac{M}{m}}\frac{1}{\gamma^{3/2}}\right).$$

The influence of finite longitudinal dimensions of the system on the critical currents in compensated elec-

*We note that in deriving (4.3) we have neglected terms of order $(m/M)^{1/3}\gamma \ll 1$. We shall henceforth neglect such terms throughout.

tron beams will be discussed in the next chapter.

Expressions (4.3), (4.5), and (4.6) are valid not only for the case when the beam fills the waveguide completely and $r_0 \approx R$, but also in the case when there is a small gap satisfying the condition $2.4 \ln (R/r_0) \ll 1$ between the metallic walls of the waveguide and the beam. If the gap is sufficiently large, however, so that the inverse condition is satisfied (more accurately $\ln (R/r_0) \gg 1$), then Eq. (2.10) takes the form

$$1-\frac{\omega_{Li}^2}{\omega(\omega+\Omega_i)}-\frac{\omega_{Le}^2}{\Omega_e(\omega-k_z u)}+\frac{k_z^2 r_0^2}{2l(l+1)}\left[1-\frac{\omega_{Le}^2}{\gamma^3(\omega-k_z u)^2}-\frac{\omega_{Li}^2}{\omega^2}\right]+\frac{r_0}{l}f_l=0, \quad (4.7)$$

where

$$f_l=\begin{cases}\frac{1}{r_0}\frac{1}{\ln(R/r_0)} & \text{for } l=0,\\ \frac{l}{r_0} & \text{for } l\neq 0.\end{cases} \quad (4.8)$$

Equation (4.7) is valid for long-wave oscillations, when $\alpha r_0 \ll 1$ and $k_z R \ll 1$. It is precisely such oscillations which determine the critical currents in the system considered by us.

For axially-symmetric modes ($l = 0$) we obtain from (4.7)

$$1+\frac{2}{k_z^2 r_0^2 \ln(R/r_0)}-\frac{\omega_{Le}^2}{\gamma^3(\omega-k_z u)^2}-\frac{\omega_{Li}^2}{\omega^2}=0. \quad (4.9)$$

It is easy to show that unstable solutions of this equation (with $\mathrm{Im}\,\omega > 0$) appear only if

$$\omega_{Le}^2>\gamma^3\left(k_z^2u^2+\frac{2u^2}{r_0^2\ln(R/r_0)}\right),$$

and the instability has a two-stream character (is of the Buneman type[3,4]) and the frequency of the excited oscillations lies in the region $\omega \lesssim (m/M)^{1/3}\gamma k_z u$. The minimum current necessary for the excitation of these oscillations is then determined by the expression

$$J_0=\frac{mu^3}{4e}\gamma^3\frac{2}{\ln(R/r_0)}. \quad (4.10)$$

On the other hand, if $l \neq 0$, then the unstable oscillations will appear both in the frequency region $\omega \ll k_z u$ and at $\omega \gg k_z u$. For oscillations in the region $\omega \ll k_z u$ the dispersion relation (4.7) reduces to the quadratic equation

$$\left[2+\frac{\omega_{Le}^2}{k_z u\Omega_e}-\frac{r_0^2\omega_{Le}^2}{\gamma^3u^2 2l(l+1)}\right]\omega(\omega+\Omega_i)-\omega_{Li}^2=0, \quad (4.11)$$

the roots of which are

$$\omega=\frac{1}{2}\left[-\Omega_i\pm\sqrt{\Omega_i^2+4\omega_{Li}^2\left(2+\frac{\omega_{Le}^2}{k_z u\Omega_e}-\frac{r_0^2\omega_{Le}^2}{\gamma^3u^2 2l(l+1)}\right)^{-1}}\right]. \quad (4.12)$$

From the condition for the instability of the oscillations ($\mathrm{Im}\,\omega > 0$) we obtain the following expression for the current corresponding to the excitation of an axially-asymmetrical mode with specified k_z and l:

$$J_l=\frac{mu^3}{e}l(l+1)\gamma^3\frac{1}{1+\frac{2l(l+1)u\gamma^3}{|k_z|\Omega_e r_0^2}}. \quad (4.13)$$

This instability has a current-convective character[6,7] and develops only at finite values of the longitudinal magnetic field, when the second term in the denominator of (4.13) differs from zero. As $k_z \to 0$, according to this expression, we get $J_l \to 0$. Actually, however, k_z cannot be arbitrarily small; the oscillations turn out to be unstable only if $4k_z u \geq \Omega_i$. At the minimum values of k_z, the current J_l tends to

$$J_l=\frac{mu}{8e}\frac{m}{M}r_0^2\Omega_e^2. \quad (4.14)$$

It is obvious that the critical current for the excitation of long-wave oscillations in the frequency region $\omega \ll k_z u$ corresponds to the smaller of the expressions (4.10) and (4.14), i.e.,

$$J_{cr}=\min\{J_0, J_l\}. \quad (4.15)$$

It follows therefore that in strong magnetic fields, when

$$\Omega_e>\frac{2u\gamma^{3/2}}{r_0}\sqrt{\frac{M}{m\ln(R/r_0)}},$$

the critical current is determined by excitation of axially-symmetrical oscillation modes because of the development of the Buneman instability; in the opposite case, the decisive factors are the axially-asymmetrical modes and the current-convective instability.

The development of current-convective instability also determines the character of the long-wave oscillations in the frequency region $\omega \gg k_z u$, in which the dispersion equation (4.7) takes the form

$$2+\frac{\omega_{Li}^2}{\omega(\omega+\Omega_i)}\left(\frac{\omega}{\Omega_i}+\frac{k_z u}{\omega}\right)-\frac{\omega_{Le}^2\gamma^{-3}k_z^2r_0^2}{\omega^2\cdot 2l(l+1)}=0. \quad (4.16)$$

It is easily seen that Eq. (4.15) can have solutions corresponding to unstable oscillations only if $\omega \ll \Omega_i$ (and consequently $\Omega_i \gg k_z u$), with

$$\frac{\omega^2}{\Omega_i^2}=\frac{M}{m}\frac{k_z^2r_0^2}{2l(l+1)}\left(1+2\frac{\Omega_i^2}{\omega_{Li}^2}\right)^{-1}\left[1-\frac{2l(l+1)u}{\Omega_e|k_z|r_0^2}\right]. \quad (4.17)$$

The unstable modes are those for which $2l(l+1)u > \Omega_e|k_z|r_0^2$. This inequality does not depend on the density, and it can be assumed that such an instability develops at arbitrarily small currents in the beam. Actually, however, the current is bounded from below by the value

$$J \geqslant \frac{mu}{e}\Omega_e|k_z|ur_0^2. \quad (4.18)$$

In infinitely long systems, the quantity $|k_z|$ can tend to zero, and therefore the current (4.18) (together with the critical current) turns out to be arbitrarily small. In systems with finite length this is not the case.

For oscillations that are short-wave in the radius, when $\alpha r_0 \gtrsim 1$ and $k_z R \ll 1$, the roots of Eq. (2.10) coincide with a good degree of accuracy with the roots of the equation $J_l'(i\alpha r_0) = 0$. Then the entire analysis of the dispersion relation is similar to that presented above and leads to currents similar to (4.3) and (4.5) for the excitation of such oscillations, where μ_{sl} are the nonzero roots of the equation $J_l'(\mu_{sl}) = 0$. The critical current for the excitation of short-wave oscillations is in this case analogous to (4.6) with the factor $(2.4)^2$ replaced by $(3.9)^2$.

From the foregoing analysis of the stability of compensated relativistic electron beams passing through the drift space along the axis of a metallic waveguide, it follows that the critical currents for the excitation of different oscillation modes under conditions when the waveguide is completely filled with the beam are always larger than in the case of incomplete filling. A similar situation obtains, as shown in Ch. 3, for uncompensated beams with the same geometry. On the other hand, in the case of tubular electron beams in the absence of charge compensation, the limiting current is larger than in a homogeneous beam completely filling the waveguide. It is therefore of interest to determine the critical currents in tubular beams under conditions of compensation of the electron charge by ions.

The dispersion equation for electrostatic oscillations in a compensated tubular beam of unlimited length (2.12) under conditions when $R - R_1 = a \ll R$ and $\alpha R \gg 1$, reduces to the form

$$k_z^2\left[1-\frac{\omega_{Le}^2\gamma^{-3}}{(\omega-k_zu)^3}-\frac{\omega_{Li}^2}{\omega^2}\right]+\frac{\pi^2\left(s+\frac{1}{2}\right)^2}{a^2}\left(1-\frac{\omega_{Li}^2}{\omega^2-\Omega_i^2}\right)=0, \quad (4.18')$$

where $s = 0, 1, 2, \ldots$. This equation is completely analogous to Eq. (4.1) and describes the interaction of the electron beam with the charge-compensating ions (i.e., it contains only an instability of the Buneman type). Therefore the analysis of expression (4.1) remains in force also for Eq. (4.18′). As a result we obtain for the critical current upon excitation of short-wave (in the radial direction, $a\alpha \gtrsim 1$) oscillations in a compensated tubular beam, the expression[15] (compare with (4.6))

$$J_{cr}=\min\begin{cases}\frac{\pi^2}{2}\frac{mu^3}{4e}\gamma^3\frac{R}{a},\\ \frac{mu}{4e}\frac{m}{M}\Omega_e^2\cdot 2Ra.\end{cases} \quad (4.19)$$

From a comparison of this expression with (4.6) it follows that in the case of strong magnetic fields, when the upper expressions are minimal in formulas (4.6) and (4.19), the critical current for the excitation of axially-symmetrical modes in a tubular electron beam exceeds by R/a times the corresponding current of the homogeneous beam completely filling the waveguide,* in exactly the same manner as for uncompensated beams. On the other hand, if the magnetic field is not very strong, then the indicated gain in the magnitude of the critical current does not exist, and it may even turn out that the critical current in the compensated tubular beam is smaller than in the homogeneous beam.

Let us consider now (radially) long-wave oscillations in the tubular beam, $\alpha R \ll 1$. The dispersion equation (2.12) for such oscillations under the condition $k_z a \ll 1$ is written in the form

$$1-\frac{\omega_{Li}^2}{\omega^2-\Omega_i^2}-\frac{la}{R}\left[\frac{\omega_{Li}^2\Omega_i}{\omega(\omega^2-\Omega_i^2)}-\frac{\omega_{Le}^2}{\Omega_e(\omega-k_zu)}\right]=0. \quad (4.20)$$

This equation, unlike (4.18′), contains a current-convective instability due to excitation of axially asymmetrical modes with $l \neq 0$. It is possible both in the frequency region $\omega > \Omega_i$ and for $\omega < \Omega_i$. In the frequency region $\omega > \Omega_i$ the unstable modes are those for which $\omega < k_z u$, with

$$\omega^2=\frac{\omega_{Li}^2}{1-\frac{la}{R}\frac{\omega_{Le}^2}{\Omega_e|k_z|u}}. \quad (4.21)$$

The minimum current necessary for the excitation of such oscillations coincides with the lower expression of (4.19). Therefore the critical current in a compensated tubular beam, with allowance for the possible development of the high-frequency current-convective instability ($k_z u > \omega > \Omega_i$) does not change and is determined by formula (4.19).

*We note that its conclusion is valid only in the case when the cavity inside the beam is not filled with plasma. If the cavity is filled with plasma then the critical current in the compensated tubular beam for excitation of axially symmetrical oscillations is of the same order as in the solid beam. [13]

The situation is different for the low-frequency current-convective instability when $\omega < \Omega_i$. From (4.20) it follows that

$$\omega=\frac{k_zu}{2}\pm\sqrt{\frac{k_z^2u^2}{4}+\frac{la}{R}\frac{\omega_{Li}^2}{\Omega_i}\frac{k_zu}{1+\omega_{Li}^2/\Omega_i^2}}. \quad (4.22)$$

The condition for the occurrence of instability (Im $\omega > 0$) leads in this case to the following expression for the current:[15]

$$J=\frac{mu}{4e}\frac{\Omega_e|k_z|u\cdot 2Ra}{4l\frac{a}{R}-\frac{|k_z|u}{\Omega_i}}. \quad (4.23)$$

In systems of unlimited length $|k_z| \to 0$ and consequently the current $J \to 0$. As a result, in a tubular beam the critical current, which in infinitely long systems is determined by formula (4.23), also turns out to be arbitrarily small. In systems with finite length, as will be shown in the next chapter, the critical current in the beam is bounded from below.

5. INFLUENCE OF FINITE LONGITUDINAL DIMENSIONS OF THE SYSTEM ON THE CRITICAL CURRENTS IN ELECTRON BEAMS

For systems that are limited in the longitudinal direction, relations (2.10) and (2.12) are characteristic equations determining the wave numbers k_{zn}. The solution of the field equation (2.6) $\Phi = \sum_n C_n \exp(ik_{zn}z)$ should satisfy the boundary conditions on the ends of the waveguide, i.e., at $z = 0$ and $z = L$. If the ends of the waveguide are metallic, then, obviously,

$$\Phi|_{z=0,\,L}=0. \quad (5.1)$$

These conditions are sufficient if the characteristic equations reduce to quadratic equations. Frequently, however, these equations turn out to be of higher order in k_z and it becomes necessary to use additional boundary conditions. We can choose as such conditions the Pierce boundary conditions[2]

$$\tilde{v}_{z1}=\frac{e}{m}(1-\beta^2)^{3/2}\sum_n\frac{k_{zn}}{\omega-k_{zn}u}C_ne^{ik_{zn}z}|_{z=0}=0,$$

$$\tilde{n}_1=\frac{e}{m}\sum_n C_ne^{ik_{zn}z}\frac{k_{zn}^2(1-\beta^2)^{3/2}}{(\omega-k_{zn}u)^2}\bigg|_{z=0}=0, \quad (5.2)$$

where $\tilde{v}_{z1}$ and $\tilde{n}_1$ are the perturbed values of the velocity and density of the beam electrons in the oscillations. In the cases considered below, conditions (5.1) and (5.2) are sufficient.

We start the investigation of the stability of a compensated electron beam in bounded systems with the case of a beam completely filling the waveguide, $r_0 \approx R$. The characteristic equation (2.14) reduces in this case to Eq. (4.1). In the frequency region $\omega < k_z u$ this equation determines two values $k_{z1,2} = \pm k_n$. The boundary conditions (5.1) give for k_n the following values:* $k_n = \pi n/L$ at $n = 1, 2, 3, \ldots$. As a result, the frequencies of the excited oscillations and the current necessary for their excitation are determined by formulas (4.2) and (4.3), in which $k_z \to k_n$. It follows therefore that if the length of the system satisfies the condition

$$\frac{L}{r_0}<\frac{\pi}{2.4}\sqrt{\frac{M}{m}}\frac{1}{\gamma^{3/2}}, \quad (5.3)$$

*We note that conditions (5.2) are in this case satisfied automatically.

then the critical current in a homogeneous compensated beam is determined by the expression

$$J_{\text{cr}} = \frac{(2,4)^2 mu^3}{4e(1-\beta^2)^{3/2}}. \tag{5.4}$$

If the system is sufficiently long so that a condition inverse to (5.3) is satisfied, then unstable oscillations occur in the system at frequencies $\omega \gg \pi u/L$. The characteristic equation (4.1) determines in this case two roots $k_{z1,2} = \omega/u \pm k_n$, and the boundary conditions (5.1) lead to the following dispersion equation:

$$k_n = \frac{\omega}{u} \frac{r_0 \omega_{Le}}{u\mu_{sl}\gamma^{3/2}} \left(1 - \frac{\omega_{Li}^2}{\omega^2 - \Omega_i^2}\right)^{-1} = \frac{\pi n}{L}. \tag{5.5}$$

From an analysis of this equation and from the condition for the occurrence of the instability (Im $\omega > 0$) we determine the critical currents passed by the system in the case under consideration:

$$J_{\text{cr}} = \max \begin{cases} 2,4 \dfrac{\pi}{4} \dfrac{mu^3 r_0}{eL} \sqrt{\dfrac{M}{m}}\, \gamma^{3/2}, \\ \dfrac{mu}{4e} \dfrac{m}{M} r_0^2 \Omega_e^2. \end{cases} \tag{5.6}$$

Formulas (5.4) and (5.6) determine the critical currents in systems of arbitrary length. In particular, in the limit of an unbounded waveguide ($L \to \infty$) they correspond to the results obtained in the preceding chapter, if it is recognized that the critical current in the system is determined by the smaller of the expressions (5.4) and (5.6). This remark pertains also to systems of finite length under conditions when the parameters are such that the critical current is determined by the first expression of (5.6).

Let us consider now the case of an incompletely filled waveguide, when $r_0 \ll R$. The characteristic equation (2.10) for long-wave oscillations reduces in this case to (4.7). Just as in the case of an unbounded waveguide, let us analyze this equation separately for modes with $l = 0$ and for modes with $l \neq 0$.

For axially-symmetrical modes ($l = 0$) in the frequency region $\omega \ll k_z u$, the only one in which the existence of instability is possible, Eq. (4.7) is quadratic and determines two roots $k_{z1,2} = \pm k_n$, which assume, when the boundary conditions (5.1) are taken into account, the discrete values $k_n = \pi n/L$, where $n = 1, 2, \ldots$ (we note that conditions (5.2) are satisfied automatically in this case). As a result we obtain for the spectrum of the oscillation frequencies

$$\omega^2 = \omega_{Li}^2 \left[1 - \frac{\omega_{Le}^2}{\gamma^3 k_n^2 u^2} - \frac{2}{\ln(R/r_0)}\right]^{-1}. \tag{5.7}$$

From the condition for the occurrence of instability in the system ($\omega^2 < 0$) we obtain in this case the following expression for the minimum current necessary to excite the axially-symmetrical modes:

$$J_0 = \frac{mu^3}{4e} \frac{2}{\ln(R/r_0)} \gamma^3. \tag{5.8}$$

In deriving this expression we took into account our main premise concerning the length of the system, $L/r_0 > \pi\sqrt{\ln(R/r_0)/2}$. Owing to this condition, formula (5.8) coincides exactly with (4.10), which is obtained for the case of an unbounded waveguide, as expected.

For axially asymmetrical modes ($l \neq 0$) and for $k_z^2 r_0^2 \ll 1$, the roots of the characteristic equation (4.7) are equal to

$$k_{z1,2} = a \pm b, \tag{5.9}$$

where

$$a = \frac{1}{u}\left[2\omega + \frac{\omega_{Li}^2(\omega - \Omega_i)}{2\Omega_i(\omega + \Omega_i)}\right]\left[2 - \frac{\omega_{Li}^2}{\omega(\omega+\Omega_i)} - \frac{r_0^2\omega_{Le}^2}{2l(l+1)u^2\gamma^3}\right]^{-1},$$

$$b = \left\{\left[2\omega + \frac{\omega_{Li}^2(\omega-\Omega_i)}{2\Omega_i(\omega+\Omega_i)}\right]^2 \frac{1}{u^2} - \frac{\omega^2}{u^2}\left(2 + \frac{\omega_{Li}^2}{\Omega_i(\omega+\Omega_i)}\right) \times \right.$$
$$\left. \times \left[2 - \frac{\omega_{Li}^2}{(\omega+\Omega_i)\omega} - \frac{r_0^2\omega_{Le}^2}{2l(l+1)u^2\gamma^3}\right]\right\}^{1/2} \left[2 - \frac{\omega_{Li}^2}{\omega(\omega+\Omega_i)} - \frac{r_0^2\omega_{Le}^2}{2l(l+1)u^2\gamma^3}\right]^{-1}.$$

The boundary conditions (5.1) lead in this case to a dispersion equation $b = k_n = \pi n/L$, which in the limit $\omega \ll k_n u$ gives the following spectrum of the low-frequency oscillations:

$$\omega = -\frac{\Omega_i}{2} \pm \left\{\frac{\Omega_i^2}{4} \pm \omega_{Li}^2\left[2 - \frac{r_0^2\omega_{Le}^2}{2l(l+1)u^2\gamma^3}\left(1 + \frac{ul(l+1)\gamma^3}{k_n\Omega_e r_0^2}\right)\right]^{-1}\right\}^{1/2}. \tag{5.10}$$

From the condition for the occurrence of instability (Im $\omega > 0$), which has the character of current convection, we obtain in this case the minimum current necessary for the excitation of the axially-asymmetrical modes with frequency $\omega \ll k_n u$ in a compensated electron beam (this current corresponds to excitation of the modes with $l = 1$):

$$J_l = \frac{mu^3}{e}\gamma^3 l(l+1)\left[1 + \frac{ul(l+1)\gamma^3}{k_n\Omega_e r_0^2}\right]^{-1}. \tag{5.11}$$

This formula is similar to formula (4.13), obtained for excitation of low-frequency oscillations ($\omega < k_z u$) in an unbounded waveguide. In the case of sufficiently long systems, when $I > (\pi/2)\Omega_e r_0^2/u\gamma^3$, it leads to the following dependence of the minimum current necessary for excitation of axially-asymmetrical modes on the beam parameters:

$$J_l = \frac{\pi}{L}\frac{mu^2}{e}\Omega_e r_0^2. \tag{5.12}$$

With increasing system length L, the current J_l decreases. This formula, however, is valid only so long as $L < 2\pi u/\Omega_i$. In longer systems, when $L > 2\pi u/\Omega_i$, the low-frequency modes of axially-asymmetrical oscillations ($\omega \ll k_n u$) turn out to be stable. The equation $b = k_n = \pi n/L$ in long systems has unstable solutions for $\omega_{Li}^2 > \Omega_i k_n u$, and the frequency of the excited oscillations is $\omega = 2k_n u$. The minimum current necessary for the occurrence of such instabilities is (compare with (4.18))

$$J_l = \frac{mu}{4e}\Omega_e \frac{\pi u}{L} r_0^2. \tag{5.13}$$

We see therefore that in long systems the critical current can be arbitrarily small and is bounded from below by the longitudinal dimension of the system.

Summarizing all the foregoing, we arrive at the following expression in the limit of long-wave oscillations for the critical current in a compensated electron beam in a waveguide of finite length in the case $R \gg r_0$:

$$J_{\text{cr}} = \min\{J_0, J_l\}, \tag{5.14}$$

where J_0 is determined by formula (5.8) and J_l is the larger of expressions (5.11) and (5.12).

As to the short-wave oscillations, just as in the case of an unbounded waveguide, the critical current for their excitation is determined by an expression similar to (5.6) with (2.4) replaced by (3.9). The true critical current in the system is obviously determined by the smaller of the expressions (5.6) and (5.14).

The analysis of the stability and the determination of the critical currents in a compensated tubular beam of finite length are perfectly analogous. Thus, in sufficiently short systems, when

$$\frac{L}{a} < 2\sqrt{\frac{M}{m}}\frac{1}{\gamma^{3/2}},$$

only short-wave oscillations for which $\alpha R \ll 1$ and $\omega \ll |k_z|u$ can be unstable. The characteristic equation (4.18), which is perfectly analogous to (4.1) yields, with allowance for the boundary conditions (5.1), $k_{z1,2} = \pm k_n = \pi n/L$; for the critical current we obtain the expression (compare with (5.4))

$$J_{cr} = J_1 = \frac{\pi^2}{2}\frac{R}{a}\frac{mu^3}{4e}\gamma^3. \tag{5.15}$$

In longer systems satisfying the condition $2\sqrt{M/m} \times \gamma^{-3/2} < L/a < \pi u/a\Omega_i$, the unstable oscillations are those with $\omega > \pi u/L$, and the characteristic equation (4.18) determines two roots $k_{z1,2} = \omega/u \pm k_n$. In exactly the same manner as above, we have in this case for the critical current in a compensated tubular beam

$$J_{cr} = \min\{J_1, J_2\}, \tag{5.16}$$

where (compare with (5.6))

$$J_2 = \max\begin{cases}\frac{\pi^2}{4}\frac{mu^3}{e}\frac{R}{L}\sqrt{\frac{M}{m}}\gamma^{3/2},\\ \frac{mu}{4e}\frac{m}{M}\Omega_e^2 2Ra.\end{cases} \tag{5.17}$$

Finally, if the system is sufficiently long, so that $L > \pi u/\Omega_i$, then the critical current in the compensated electron beam is determined by the relation

$$J = \min\{J_1, J_2, J_3\}, \tag{5.18}$$

where

$$J_3 = \frac{mu^3}{4e}\frac{M}{m}\frac{2\pi^2 Ra}{L^2}. \tag{5.19}$$

The currents J_1 and J_2 are due to excitation of axially-symmetrical modes, i.e., they are connected with the development in the system of two-stream instability of the Buneman type; the current J_3, on the other hand, is connected with the development of current-convective instability in the tubular beam, accompanied by excitation of axially-asymmetrical modes. We see that in very long systems it is precisely this instability which determines the critical current.

In conclusion, we present quantitative estimates of the critical currents in compensated electron beams and compare them with the limiting currents that can be obtained in vacuum systems. We confine ourselves here to systems of finite length, for which $L < \pi u/\Omega_i$.

In the case of sufficiently heavy ions with atomic weight larger than 100, this inequality is satisfied up to $L \sim 10^3$ cm even in very strong magnetic fields $B_0 \sim 10^6$ Oe. It is precisely systems of such length that are of greatest interest in real cases.

Under conditions when the radius of the electron beam coincides with the radius of the waveguide, i.e., $r_0 \approx R$, the critical current upon compensation of the charge of the beam is determined by formulas (5.4) and (5.6). A comparison of these formulas with (3.6), (3.12), and (3.13), which determine the limiting currents in uncompensated electron beams, shows that when $L/R < \sqrt{M/m}\,\gamma^{-3/2}$ the main advantage of compensated beams lies in the region of relativistic electron energies. In the nonrelativistic region, the critical current in the compensated beam is only six times larger than the limiting current J_0 passed by the vacuum system, whereas in the relativistic region this ratio can be quite large, $J_{cr}/J_0 \approx (\mathscr{E}/mc^2)^2 \approx \gamma^2$. This circumstance uncovers great possibilities for obtaining strong-current electron beams by compensating the charge of the electrons with an ion background. Thus, at $\mathscr{E} \sim 5$ MeV the current in the compensated beam can be of the order of $J_{cr} \sim 10^7$ A, which is 100 times larger than the limiting vacuum current J_0 for such a beam. It should be noted, however, that such a current can be obtained only in sufficiently short systems, when $L/R \lesssim (\sqrt{M/m})/25 \lesssim 20$. In longer systems, as seen from (5.6), the critical current in the compensated electron beam is smaller and when $L/R > \sqrt{M/m}\,\gamma^{-3/2}$ we have $J_{cr}/J_0 \approx 2(R/L)\sqrt{M\gamma/m}$. We therefore obtain for $\mathscr{E} \sim 5$ MeV, $\sqrt{M/m} \sim 400$, and $L/R \approx 10^2$ the value $J_{cr}/J_0 \approx 25$.

Under conditions when the beam does not fill the waveguide completely and $\ln(R/r_0) \gg 1$, the indicated advantages of the compensated beam over the uncompensated one can occur only in the presence of a sufficiently strong longitudinal magnetic field, satisfying the requirement

$$\Omega_e > \frac{u}{r_0}\frac{L}{2\pi r_0}\frac{\gamma^3}{\ln(R/r_0)}. \tag{5.20}$$

When this inequality is satisfied, the critical current in the compensated beam is determined by formula (5.8) and $J_{cr}/J_0 \approx (\mathscr{E}/mc^2)^2 = \gamma^2$. For a beam with E ~ 5 MeV, $r_0 \sim 1$ cm, $R/r_0 \sim 10$ and $L/r_0 \sim 100$, the inequality (5.20) is satisfied only in very strong magnetic fields $B_0 \gtrsim 10^7$ Oe, which can be obtained only in pulsed systems. In weaker fields, the critical current in a compensated electron beam is determined by formula (5.12), with $J_{cr}/J_0 \sim (2\pi r_0/L)(\Omega_e r_0/c)\ln(R/r_0)$. On the other hand, if $B_0 < 10^5$ Oe, then the critical current in the beam turns out to be smaller than the vacuum current. The reason for this decrease of the current in the beam is the current-convective instability.

A similar situation also occurs in tubular electron beams. In relatively short systems and in the presence of strong magnetic fields, as seen from formulas (5.15)–(5.19), compensated tubular beams, as well as uncompensated ones, have an appreciable advantage over homogeneous beams: it is possible to attain in them currents larger by a factor R/a. When $R/a \approx 10$ and $\mathscr{E} = 5$ MeV, the maximum current in the compensated tubular beam can be of the order of 10^8 A. In long systems and relatively weak magnetic fields, however, this advantage of tubular currents is not obtained, owing to the development of current-convective instability. Moreover, the critical current in sufficiently long compensated tubular beams may even turn out to be smaller than in homogeneous uncompensated beams.

6. INTERACTION OF UNBOUNDED RELATIVISTIC ELECTRON BEAMS WITH A PLASMA

We have verified above the advantages of compensated electron beams over uncompensated ones when it comes to obtaining large currents. Under real conditions, however, it is very difficult to make use of these advantages, for this calls for placing the system in a very strong magnetic field. Otherwise there can develop in the compensated beam a current-convective instability corresponding to excitation of axially-asymmetrical modes and, in essence, cancelling out the aforementioned advantages.

More promising in practice for the obtaining of strong-current electron beams under real conditions, from our point of view, are overcompensated beams, or in other words systems with relatively dense plasma through which an electron beam is made to pass. The question of the critical parameters of such a system, under which electrostatic instabilities develop in a relativistic overcompensated electron beam, is discussed in [19].* The exposition that follows is based on this reference.

We consider first the interaction of an unbounded electron beam with a plasma. To analyze this problem, we start from Equations (2.10) and (2.12), which constitute in the present case the dispersion equations of the electrostatic oscillations. When $R \approx r_0$, i.e., in the case of a plasma completely filling the waveguide, Eq. (2.10) reduces to

$$k_z^2\left[1-\frac{\omega_{L1}^2}{\gamma^3(\omega-k_zu)^2}-\frac{\omega_{L2}^2}{\omega^2}\right]+\frac{\mu_{sl}^2}{R^2}\left(1-\frac{\omega_{L2}^2}{\omega^2-\Omega_e^2}\right)=0, \qquad (6.1)$$

where ω_{L1} and ω_{L2} are the Langmuir frequencies of the beam and plasma electrons, the densities of which in the laboratory frame are respectively n_1 and n_2. In writing down Eq. (6.1) we took into account the condition (2.1), which allows us to confine ourselves to an investigation of the stability of the electron beam in the frequency region $(\omega - k_zu)^2 \ll \Omega_e^2/\gamma^2$. It is easily seen that when $n_1 \ll n_2\gamma^3$ the solutions of (6.1) corresponding to unstable oscillations of the system can lie only in the frequency region $\omega \approx k_zu$. For long-wave oscillations with $k_zu < \Omega_e$ the condition for the occurrence of instability is written in the form

$$n_2 \geqslant 3\cdot 10^{-10}\left(k_z^2+\frac{\mu_{sl}^2}{R^2}\right)u^2\left\{\left[1+\left(\frac{n_1}{n_2}\right)^{1/3}\frac{1}{\gamma}\right]^3-\frac{\mu_{sl}^2u^2}{r_0^2\Omega_e^2}\right\}^{-1}. \qquad (6.2)$$

For short-wave oscillations with $k_zu > \Omega_e$, on the other hand, the instability sets in if†

$$n_2 \geqslant 3\cdot 10^{-10}k_z^2u^2\left\{1+\left[\frac{n_1\gamma^{-3}}{n_2(1+\mu_{sl}^2/k_z^2R^2)}\right]^{1/3}\right\}^{-3}. \qquad (6.3)$$

The minimum value of the plasma density n_2 corresponding to the start of the instability in the system will be called critical. Obviously, to determine the critical density of the plasma, expressions (6.2) and (6.3) must be minimized with respect to the wave numbers k_z and μ_{sl}, and the smallest minimum determined in this case (minimum minimorum) will be the sought critical density[19]

$$n_{2\,\mathrm{cr}} = \min\begin{cases} 3\cdot 10^{-10}\dfrac{\mu_{00}^2u^2}{R^2}\left\{\left[1+\left(\dfrac{n_1}{n_2}\right)^{1/3}\dfrac{1}{\gamma}\right]^3-\dfrac{\mu_{00}^2u^2}{R^2\Omega_e^2}\right\}^{-1}, \\ 3\cdot 10^{-10}\Omega_e^2\left\{1+\left[\dfrac{n_1\gamma^{-3}}{n_2\left(1+\dfrac{\mu_{00}^2u^2}{R^2\Omega_e^2}\right)}\right]^{1/3}\right\}^{-1}, \end{cases} \qquad (6.4)$$

where $\mu_{00} = 2.4$. According to these expressions, the dependence of the critical density of the plasma on the magnetic field is quite strong and can even have an almost "resonant" character at sufficiently high beam densities; at low beam densities, when $(n_1/n_2)^{1/3}\gamma^{-1} \ll 1$, no resonance appears. When the critical density (6.4) is reached, a two-stream instability develops in the plasma and the fundamental modes $s = l = 0$ are excited with frequency $\omega \approx k_zu \lesssim \Omega_e$ and with growth increment $\gamma_0 \lesssim (n_1/2n_2)^{1/3}k_zu\gamma^{-1}$. The wave number k_z can vary in this case in a rather wide range, from $k_{z\,\min}$, which is determined by the length of the system (see Ch. 7) up to μ_{00}/R. Therefore the frequencies of the excited oscillations lie in the range $\pi u/L \lesssim \omega \lesssim \omega_{L2}$.

Formula (6.4) remains valid also in the case when there is a gap between the plasma and the metallic jacket, but the gap is sufficiently small so that $2.4\ln(R/r_0) \ll 1$. In the opposite limit, when $R \gg r_0$, more accurately $\ln(R/r_0) \gg 1$, the gap exerts a noticeable influence on the character of the interaction of the beam with the plasma. The critical density of the plasma is determined in this case by the excitation of radially long-wave oscillations for which $\alpha r_0 \ll 1$ and Eq. (2.10) takes the form

$$1-\frac{k_z^2r_0^2}{k_z^2r_0^2+2l(l+1)}\left[\frac{\omega_{L2}^2}{\omega^2}+\frac{\omega_{L1}^2}{\gamma^3(\omega-k_zu)^2}\right]- -\frac{2l(l+1)}{k_z^2r_0^2+2l(l+1)}\left[\frac{\omega_{L2}^2}{\omega\Omega_e}+\frac{\omega_{L1}^2}{\Omega_e(\omega-k_zu)}-\frac{r_0}{l}f_l\right]=0. \qquad (6.5)$$

In writing down this equation it was assumed also that $k_zR \ll 1$, for it is precisely the long-wave oscillations which determine the critical plasma density. From (6.5) we obtain the following expression for the minimum plasma density, at which excitation of symmetrical modes with $l = 0$ takes place:

$$n_{20} = 3\cdot 10^{-10}\frac{u^2}{r_0^2}\frac{2}{\ln(R/r_0)}\left\{1+\left(\frac{n_1}{n_2}\right)^{1/3}\frac{1}{\gamma}\right\}^{-3}. \qquad (6.6)$$

An analysis of (6.5) for asymmetrical modes ($l \neq 0$) is more complicated, and it can be carried out analytically only provided one can neglect terms of the order of $(n_1/n_2)^{1/3}\gamma^{-1}$. Under this condition, we obtain for the minimum plasma density corresponding to the start of the excitation of the mode with given l and k_z the expression[19]

$$n_{2l} = 3\cdot 10^{-10}\frac{u^2}{r_0^2}\left[4l(l+1)+\eta^2\frac{\omega_{L1}^2r_0^2}{u^2\gamma^3}\right](1+2\eta)^{-1}, \qquad (6.7)$$

where

$$\eta = \frac{l(l+1)u}{|k_z|r_0^2\Omega_e}.$$

Minimizing (6.7) with respect to l and k_z, we obtain the critical plasma density for the excitation of asymmetrical modes

$$n_{2\,\mathrm{cr}} = 5\cdot 10^{-5}\frac{u}{r_0}\sqrt{n_1}\,\gamma^{3/2}. \qquad (6.8)$$

In the derivation of this formula it was assumed that $\omega_{L1}^2r_0^2 \ll 4u^2\gamma^{-3}$, for only under this condition can the critical density determined by formula (6.8) be smaller than (6.6). This requirement can be expressed more accurately in the form

$$\frac{u}{r_0} > 10^5\sqrt{n_1}\,\gamma^{3/2}\ln R/r_0. \qquad (6.9)$$

If this inequality is satisfied, the critical density of the plasma corresponds to the development of beam-drift

*The existence of a critical plasma density for the excitation of symmetrical modes by a nonrelativistic electron beam was point out earlier in [30].

†It should be noted that the excitation of cyclotron oscillations, when $\omega \approx k_zu \approx \Omega_e$, is possible at plasma densities exceeding the values (6.3).

instability in the system with excitation of the mode with $l = 1$ and $k_z = \omega_{L1}\gamma^{3/2}/\sqrt{2}r_0\Omega_e$, and in accordance with (6.8) it increases with increasing beam density. In the opposite case the critical plasma density is determined by (6.6) and corresponds to excitation of axially-symmetrical modes in the system*

We note that formulas (6.6) and (6.8) remain valid also for a finite but sufficiently long waveguide. If it is recognized that the frequency of the excited oscillations is of the order of $\omega \approx k_z u$, and the increment is $\gamma_0 \lesssim (n_1/n_2)^{1/3} k_z u \gamma^{-1}$, then in order for formula (6.8) to be valid it is necessary to have

$$\frac{L}{r_0} \gg \frac{\sqrt{2}\,\Omega_e}{\omega_{L1}\gamma^{1/2}} \left(\frac{2n_2}{n_1}\right)^{1/3} \approx 1{,}5\cdot 10^{-6} \left(\frac{u}{r_0}\right)^{1/3} \frac{\Omega_e}{n_1^{2/3}}. \qquad (6.10)$$

Formula (6.6), on the other hand, is valid if the weaker condition

$$\frac{L}{r_0} \gg \left(\frac{2n_2}{n_1}\right)^{1/3} \gamma. \qquad (6.11)$$

is satisfied. The same inequality determines the applicability of formula (6.4) for the case $r_0 \approx R$. It should be noted that the inequalities (6.10) and (6.11) are rough estimates. The influence of the finite longitudinal dimensions of the waveguide on the critical plasma density is investigated more rigorously in the next chapter. Nonetheless, it follows even from these inequalities that the conditions for the development of a beam-drift instability, (6.9) and (6.10), are very difficult to realize. In practice, the critical density of the plasma is always determined by the development of the two-stream instability with excitation of axially-symmetrical oscillation modes.

It is even more difficult to excite axially-asymmetrical modes in a tubular electron beam interacting with a dense plasma. This can be verified easily by analyzing the dispersion equation (2.12) for such a beam for instability. For short-wave oscillations, $\alpha R \gg 1$, Eq. (2.12) takes the form

$$k_z^2 \left[1 - \frac{\omega_{L1}^2 \gamma^{-3}}{(\omega - k_z u)^2} - \frac{\omega_{L2}^2}{\omega^2}\right] + \frac{\pi^2 \left(s + \frac{1}{2}\right)^2}{a^2} \left(1 - \frac{\omega^2}{\omega^2 - \Omega_e^2}\right) = 0, \qquad (6.12)$$

where $s = 0, 1, 2, \ldots$. This equation is perfectly analogous to (6.1). Therefore the entire analysis given above remains in force also in the present case. Namely, for the critical density we have in analogy with (6.4)

$$n_{2\,\mathrm{cr}} = \min \begin{cases} 3\cdot 10^{-10} \dfrac{\pi^2 u^2}{4a^2} \left\{\left[1 + \left(\dfrac{n_1}{n_2}\right)^{1/3} \dfrac{1}{\gamma}\right]^3 - \dfrac{\pi^2 u^2}{4\Omega_e^2 a^2}\right\}^{-1}, \\ 3\cdot 10^{-10}\Omega_e^2 \left[1 + \left(\dfrac{n_1}{n_2\gamma^3}\right)^{1/3} \left(1 + \dfrac{\pi^2 u^2}{4a^2\Omega_e^2}\right)^{-1/3}\right]^{-3}. \end{cases} \qquad (6.13)$$

Expressions (6.13) correspond to the development of two-stream instability and to excitation of axially-symmetrical modes in an over-compensated tubular beam. Axially asymmetrical modes can be excited in principle in the long-wave limit $\alpha R \ll 1$, when Eq. (2.12) takes the form

$$1 - \frac{\omega_{L2}^2}{\omega^2 - \Omega_e^2} + \frac{la}{R} \left[\frac{\omega_{L2}^2 \Omega_e}{\omega(\omega^2 - \Omega_e^2)} - \frac{\omega_{L1}^2}{\Omega_e(\omega - k_z u)}\right] = 0. \qquad (6.14)$$

It is easy to prove that this equation has unstable solutions only if $\omega_{L2} \geq k_z u > \Omega_e$, and consequently the critical density for the development of the beam-drift instability is always larger than (6.13).

7. STABILITY OF BOUNDED ELECTRON BEAMS IN A PLASMA

The analysis presented in the preceding chapter is suitable, strictly speaking, only for an infinitely long waveguide. Only in this case are relations (2.10) and (2.12) the dispersion equations of the oscillations. In the case of a bounded waveguide, these relations are characteristic equations and determine the set of possible wave numbers k_{zn} of the system oscillations. To obtain the dispersion equations in this case, the solutions of the field equation $\Phi = \sum_n C_n \exp(ik_{zn}z)$ must be substituted in the boundary conditions (on the ends of the waveguide), the number of which should correspond to the number of roots of the characteristic equations. The conditions for the solvability of the systems of algebraic equations obtained in this case constitute the dispersion equations of the oscillations.

As already noted above, in the case when the waveguide is completely filled with a plasma, when $r_0 \approx R$, and also for excitation of symmetrical oscillations with $l = 0$ at $R \gg r_0$ in a tubular beam, the influence of the boundary conditions on the ends of the waveguide can be neglected if the condition (6.11) is satisfied. The critical plasma density is then determined by formulas (6.4), (6.6), and (6.13), respectively. The condition (6.11) is satisfied in real experiments as a rule with a large margin; therefore, following [19], we confine ourselves below to an investigation of the influence of the boundedness of the waveguide on the excitation of asymmetrical modes with $l \neq 0$ at $R \gg r_0$, when the characteristic equation is of the form (6.5).* This equation determines four roots k_{zn}, and therefore, generally speaking, it is necessary to satisfy four boundary conditions on the ends of the waveguide at $z = 0$ and L. However, if it is recognized that the most appreciable influence is exerted by the boundary conditions on the long-wave oscillations with wavelength comparable with the longitudinal dimension, we can change over in (6.5) to the limit $k_z^2 r_0^2 \ll 1$. If we note in addition that the critical density of the plasma can be smaller than that defined by formula (6.6) and can correspond to excitation of the mode with $l = 0$ only if $\eta \gg 1$, then we obtain from (6.5) a quadratic equation the roots of which are equal to

$$k_{z1,2} = p \pm \sqrt{p^2 - 4q}, \qquad (7.1)$$

where

$$p = \left[2\omega u\left(2 - \frac{\omega_{L2}^2}{\omega\Omega_e}\right) - u\frac{\omega_{L1}^2}{\Omega_e}\right]\left[\left(2 - \frac{\omega_{L2}^2}{\omega\Omega_e}\right)u^2 - \frac{\omega_{L1}^2 r_0^2}{\gamma^3 2l(l+1)}\right]^{-1},$$

$$q = \left[\omega^2\left(2 - \frac{\omega_{L2}^2}{\omega\Omega_e}\right) - \omega_{L1}^2\frac{\omega}{\Omega_e}\right]\left[u^2\left(2 - \frac{\omega_{L2}^2}{\omega\Omega_e}\right) - \frac{\omega_{L1}^2 r_0^2}{\gamma^3 2l(l+1)}\right]^{-1}.$$

This circumstance enables us to confine ourselves, in the solution of the problem, to the obvious boundary conditions (5.1) on the metallic ends of the waveguide,

*In formulas (6.6) and (6.8) there is no resonant dependence of the critical plasma density on the magnetic field, which is characteristic of a completely filled waveguide (see formula (6.4)). Such a dependence can remain also in the presence of a gap, provided $\ln(R/r_0) \lesssim 1$. In this case, however, the dispersion equation (2.10) must be analyzed numerically. The resonant dependence can appear also under conditions when the plasma radius exceeds the beam radius.

*The finite length of the waveguide for axially symmetrical modes is manifest only in the fact that the frequency of the excited oscillations is limited, $\omega \gtrsim \pi u/L$.

which lead to the following dispersion equation: k_1 $k_{z1} - k_{z2} = 2k_n = 2\pi n/L$, where $n = 1, 2, \ldots$, or, what is the same,

$$\omega^3 + 3p_1\omega + 2q_1 = 0, \qquad (7.2)$$

where

$$3p_1 = -\frac{4k_n^2 u^4 l\,(l+1)\,\gamma^3}{\omega_{L1}^2 r_0^2}, \quad 2q_1 = \frac{2k_n^2 u^4 l\,(l+1)\,\omega_{L2}^2\gamma^3}{\Omega_e \omega_{L1}^2 r_0^2}.$$

From the condition for the occurrence of complex roots of Eq. (7.2), corresponding to unstable oscillations, we obtain the plasma density at which an asymmetrical mode with specified wave numbers n and l can be excited in the system:

$$n_2 \geqslant 10^{-14}\frac{|k_n|\,u^2\Omega_e\sqrt{l\,(l+1)}\,\gamma^{3/2}}{r_0\sqrt{n_1}}. \qquad (7.3)$$

From this we have for the minimum density corresponding to excitation of the mode with $l = n = 1$

$$n_{2\mathrm{cr}} = 10^{-14}\frac{\sqrt{2}\,\pi u^2\Omega_e\gamma^{3/2}}{Lr_0\sqrt{n_1}}. \qquad (7.4)$$

With increasing longitudinal dimension of the system, as seen from (7.4), the quantity n_{2cr} decreases and tends to zero as $L \to \infty$. Actually, however, the decrease of the critical density of the plasma with increasing L continues until expression (7.4) becomes equal to (6.8), which is valid for the case of an unbounded waveguide. Further increase of the waveguide length no longer influences the critical density of the plasma; the waveguide becomes unbounded, by virtue of which we get the results of the preceding chapter. On the other hand, expression (7.4) is valid only under conditions when it is smaller than (6.6), for in the opposite case excitation of a mode with $l = 0$ will occur in the system. Thus, the dependence of the critical density of the plasma on the magnetic field which is contained in formula (7.4) can appear in a rather narrow region lying between the values (6.8) and (6.6), or, equivalently, when

$$10^{-4}\frac{\Omega_e\gamma^{3/2}}{\sqrt{n_1}}\ln(R/r_0) < \frac{L}{r_0} < 10^{-9}\frac{u\Omega_e}{n_1 r_0}. \qquad (7.5)$$

We note that the inequalities (7.5) can be satisfied only under the condition (6.9). This confirms the idea advanced above concerning the difficulty of observing beam-drift instability in overcompensated electron beams.

8. CRITICAL CURRENTS OF RELATIVISTIC ELECTRON BEAMS IN A PLASMA

The foregoing analysis of the interaction of a relativistic electron beam of low density with a plasma makes it possible to estimate the critical currents in beams that can be passed through a plasma filling a waveguide. Greatest interest attaches, naturally, to the transport of the beam over large distances. We shall therefore discuss this question with an infinitely long waveguide as an example.

It follows from formulas* (6.4), (6.6), and (6.13) that an electron beam can be unstable in the plasma at practically arbitrarily small current in the beam, provided the plasma density exceeds the critical value n_{2cr}. On the other hand, when $n_1 \ll n_2\gamma^3$ and $n_2 < n_{2cr}$, the electron beam in the plasma is always stable, and the dependence of n_{2cr} on the current in the beam is weak. It is precisely this circumstance that makes it possible to estimate the upper limit of the critical current of a relativistic electron beam in a plasma. It is equal to

$$J_{\mathrm{cr}} < eun_{2\mathrm{cr}}\gamma^3 S_0, \qquad (8.1)$$

where S_0 is the beam cross sectional area (for a cylindrical beam $S_0 = \pi r_0^2$, where $r_0 \leq R$; for a tubular beam $S_0 = \pi(R^2 - R_1^2)$, $S_0 = 2\pi Ra$), and n_{2cr} is determined, depending on the geometry of the beam, by formulas (6.4), (6.6), and (6.13). As a result we obtain for the critical current of an electron beam in a plasma the following limit:

$$J_{\mathrm{cr}} < \frac{mu^3}{4e}\gamma^3\xi; \qquad (8.2)$$

where ξ is a geometrical factor, equal to $\xi = \mu_{00}^2 = (2.4)^2$ in the case when the waveguide is completely filled with a plasma, $\xi = 2/\ln(R/r_0)$ when $R \gg r_0$, and $\xi = (\pi^2/2)R/a$ for a tubular beam. In writing down (8.2) it was assumed that the system is placed in a sufficiently strong longitudinal magnetic field, so that $\Omega_l \gg u/\Delta$, where Δ is the transverse dimension (radius or thickness) of the electron beam. In addition, we have neglected terms of order $(n_1n_2)^{1/3}\gamma^{-1}$, which is valid if the current in the beam is much smaller than the limit (3.2); by at least one order of magnitude. Even under this limitation, the currents in electron beams that can be passed through a dense plasma turn out to be quite impressive. In such overcompensated beams it is possible to obtain currents that are smaller by only one order of magnitude than the maximum attainable currents in compensated beams. Thus, at $\mathscr{E} = 5$ MeV in an overcompensated beam completely filling the waveguide, it is possible to have without special difficulty a current $J \sim 10^6$ A. In a tubular beam this current can be R/a times larger.

It is important to note that strongly overcompensated beams have one essential advantage over compensated beams. The point is that even if an instability arises in such a system, the beam energy loss (the conversion of the directional energy of the beam into energy of electrostatic oscillations) and the associated energy spreading of the beam can be negligible if the plasma density is high enough. Indeed, from the foregoing analysis it follows that, as a rule, single-mode instabilities should develop in low-density beams under conditions when the plasma density is close to critical.* The interaction of the beam with the wave and the growth of the latter continues in this case until the wave amplitude reaches the critical value corresponding to the capture of the beam electrons by the wave. From this condition we obtain an

*It was already noted above that the conditions for the development of beam-drift instability with excitation of axially-asymmetrical modes is very difficult to realize. We therefore confine ourselves to an examination of the consequences ensuing from ordinary two-stream instability with excitation of axially-symmetrical modes.

*This is indeed the case for axially-asymmetrical modes, since the necessary plasma densities corresponding to excitation of different modes of such oscillations, according to (7.3), differ greatly from each other. As to the symmetrical modes, the plasma densities necessary for their excitation generally speaking differ little from each other and the appearance of single-mode instability should be expected only in relatively short systems, when $L/r_0 \sim (2n_2/n_1)^{1/3}\gamma$ (see also [25]).

estimate for the amplitude of the potential of the steady-state wave upon development of instability in the plasma-beam system:

$$e\Phi_0' = \frac{mu'^2}{2}, \qquad (8.3)$$

where Φ_0' and u′ are the amplitude of the potential and the beam-electron velocity in a coordinate system connected to the wave. In this coordinate system, the beam electrons are nonrelativistic because of the condition $\omega \approx k_z u$. On going over to the laboratory coordinate system, taking into account the fact that in this system the field of the wave is purely potential, we obtain

$$\Phi_0 = \frac{m}{2e}\left(\frac{\omega}{k_z} - u\right)^2 \gamma^3, \qquad (8.4)$$

Recognizing further that $E_0 = -\nabla\Phi_0$ and that the maximum growth increment of the oscillations is $\gamma_{0\,max} = (\omega - k_z u)_{max} \approx (n_1/2n_2)^{1/3}\gamma^{-1}k_z u$, we obtain after simple calculations the relative fraction of the energy transferred by the beam to the electrostatic wave:[25]*

$$\frac{E^2}{8\pi n_1 mc^2\gamma} = \frac{1}{2^{4/3}}\frac{u^2}{c^2}\left(\frac{n_1}{2n_2}\right)^{1/3}\gamma. \qquad (8.5)$$

This ratio, as can be readily understood, also characterizes the smearing of the beam upon development of instability in the plasma. Therefore, under conditions when the right-hand side of (8.5) is small, the beam energy loss and its energy smearing are negligibly small.† To the contrary, in compensated beams (when $n_1 \approx n_2$) there is no such small parameter, and the development of the instabilities in such a beam causes its total destruction. In this sense it can be assumed that the critical currents obtained above for compensated beams are in essence limiting currents, whereas in overcompensated beams they are only critical; in the latter it is possible to attain large currents without significantly destroying the beam. This is precisely the advantage of overcompensated beams compared with compensated ones.

Finally, attention should be called to the fact that the right-hand side of (8.5) increases with increasing electron energy and can become of the order of unity in the relativistic region. This means that the energy of the oscillations excited in the plasma can become of the order of the energy of the beam itself, i.e., practically the entire beam energy goes over, upon development of the instability, into the energy of the microwave oscillations. This, too, is an essential advantage of relativistic electron beams over nonrelativistic ones when used to construct microwave generators and amplifiers for the centimeter and millimeter bands. The powers of such sources can be very large. Indeed, at an electron energy 10 MeV and a current $J \sim 10^5$ A (the current is smaller than the vacuum value, see Ch. 3) the power of a microwave generator based on the interaction of such a beam with a plasma can exceed in accordance with (8.5) 10^{10} W even at $n_1/n_2 \lesssim 10^{-4}$.

It is possible to vary the spectrum of the microwaves excited by the beam in the plasma over a rather wide range. It follows from the results of Ch. 7 that in a system of finite length there occurs excitation of oscillations with a spectrum $\omega = (\pi n/L)u$, where n = 1, 2, ... are the numbers of the various modes. If it is recognized that two-stream instabilities are convective[22] and therefore intensify only the modes for which $\gamma_{0\,max} \approx (n_1/2n_2)^{1/3}\pi nu/\gamma L > u/L$, it is easy to determine the region of the frequencies of the oscillations excited by the beam,*

$$\gamma\left(\frac{2n_2}{n_1}\right)^{1/3}\frac{u}{L} < \omega < \frac{\mu_0 u}{r_0} \approx \omega_{L2}, \qquad (8.6)$$

where $\mu_0 = 2.4$ for a cylindrical beam and $\mu_0 = \pi/2$ for a tubular beam, (in the latter case r_0 should be taken to be the beam thickness a). We see therefore that under conditions when $\gamma(2n_2/n_1)r_0/L \ll 1$, the microwave spectrum is broad; on the other hand, if this ratio is of the order of unity, then the spectrum turns out to be narrow and one can speak of development of practically single-mode two-stream instability in the plasma.

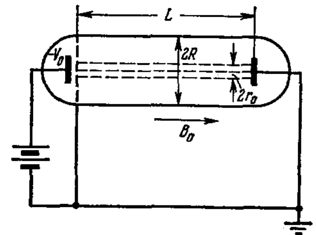

FIG. 2

9. COMPARISON OF THEORY WITH EXPERIMENT

The interaction of beams of charged particles with a plasma has been the subject of a large number of experimental investigations. A splendid review of these investigations was recently given by Ya. B. Faĭnberg.[23] The most profitable, from our point of view, are investigations of the interaction of an electron beam with a plasma produced by the beam itself as a result of ionization of the residual gas in the chamber, aimed at determining the critical parameters under which instabilities arise in the system. Such an approach of investigating plasma-beam interaction was proposed by M. V. Nezlin[5,9] and developed in [20,27,28]. The experimental setup used in these investigations is shown schematically in Fig. 2. The electron beam emitted from an incandescent cathode at a potential $-V_0$ is accelerated in the gap between the cathode and a grounded grid (gap I), passes through drift space II, and is gathered by a collector. The electron beam ionizes the residual gas, producing a plasma in the chamber and by the same token compensating the charge of the beam. The process of gas ionization is slow compared with the characteristic times of instability development in the system, making it possible to investigate the behavior of the electron beam during different stages of its compensation.

At first the electron beam is uncompensated, and the

*We note that an analogous result is also obtained in the quasilinear theory of beam relaxation in a plasma [24].

†It should be noted that the indicated small parameter also ensures smallness of the amplitudes of the higher harmonics of the electrostatic field upon development of two-stream instability [25].

*Recognizing that the excitation of axially-asymmetrical modes in overcompensated beams is difficult, we confine ourselves in the estimates to a consideration of only symmetrical modes.

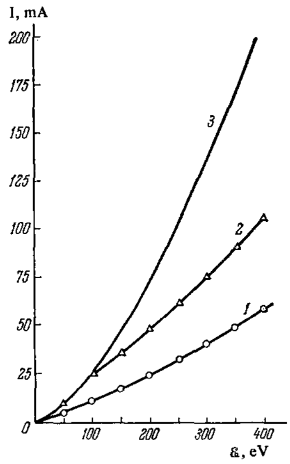

FIG. 3. Dependence of the critical current of a compensated beam on the electron energy at L = 100 cm, R = 5 cm, and r_0 = 0.5 cm. Curve 2 corresponds to a magnetic field B_0 = 4000 G, curve 1 to B_0 = 600 G, and curve 3 was calculated from formula (5.8).

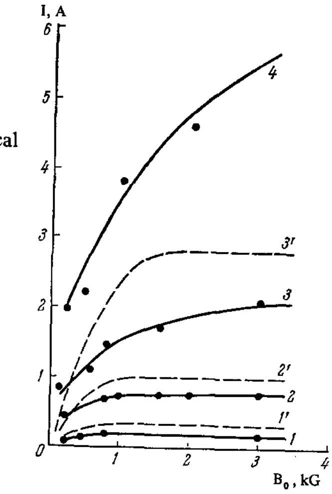

FIG. 4. Dependence of the critical current of a compensated beam on the magnetic field at L = 100 cm, R = 7.5 cm, and r_0 = 1.5 cm. Experimental curves 1–4 (solid lines) correspond to electron energies 0.5, 1, 2, and 4 keV. The dashed lines represent the theoretical curves.

transmitting ability (more accurately, the limiting current) of the system in question is determined by formulas of Ch. 3 of the present review.

During this stage, a comparison of the theory with experiment showing good quantitative agreement between them was carried out both in [5,9,20,28] and in numerous earlier studies* (see the reviews[1,12]).

Further on, during the course of ionization of the gas, the electron beam captures the produced ions and becomes neutralized. The critical currents in such a compensated beam were experimentally investigated in [5,9], where it was shown for the first time that the critical current in a compensated electron beam is determined by the development of the Buneman instability (accompanied by excitation of axially-symmetrical modes) and current-convective instability (with excitation of axially-asymmetrical modes). The transition from the first type of instability to the second is accompanied by a change in the dependence of the critical current on the beam energy (Fig. 3, taken from [5]), as predicted by the theory expounded in Ch. 5 (see formulas (5.8), (5.11), and (5.14)).

From the curves of Fig. 3 we see that at low energies $J_{cr} \sim u^3 \sim \mathscr{E}^{3/2}$, and at high energies $J_{cr} \sim u^2 \sim \mathscr{E}$. The larger the longitudinal magnetic field, the later (in energy) the transition from the first law to the second, in full accord with the theory.

It was already mentioned in the introduction that the attempt made in [9] to compare the theory with experiment is unsatisfactory, since it was based on the theory of stability of a compensated electron beam relative to short-wave oscillations described in the geometrical-optics approximation[6,7] (see also [8]). As shown in Chs. 4 and 5 of the present review, the critical currents in electron beams are determined by the development of long-wave oscillations. In this connection, P. S. Strelkov and A. Shkvarunets[31] again carried out careful experiments on critical currents in compensated electron beams, for the purpose of comparing the experimental results with the theory developed above. Their comparison of experiment with theory is shown in Fig. 4. We see that the agreement is good not only qualitatively but also quantitatively, thus indicating that the premises of the theory are correct.

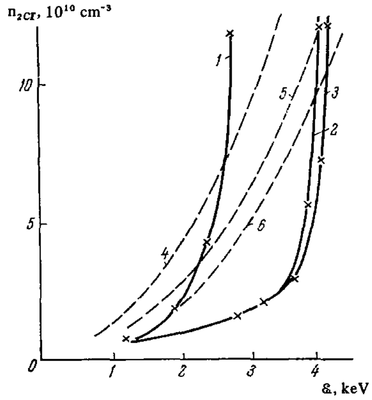

FIG. 5. Dependence of the critical density of the plasma on the beam energy at L = 60 cm, R = 0.65 cm, r_0 = 0.5 cm, B_0 = 1200 G. Experimental curves 1–3 (solid lines) correspond to currents 0.13, 0.4, and 0.7 A. The dashed lines represent the theoretical curves.

In conclusion, let us discuss experiments on the interaction of electron beams with a denser plasma,[20,27,28] also carried out with apparatus represented schematically by Fig. 2. Under conditions when the current in the electron beam is smaller than either the critical current in the compensated beam or the limiting current in the uncompensated beam (this can be attained by suitably choosing the cathode heating conditions), no instability develops in the system during the neutralization stage. The beam continues to ionize the residual gas in the chamber and produces a plasma whose density is larger than the density of the beam electrons, $n_2 \gg n_1$. When the plasma density reaches a certain critical value n_{2cr}, as shown in Chs. 6 and 7 of the present review, high-frequency two-stream and beam-drift instabilities develop in the system. These instabilities cannot lead to an interruption of the current (see Ch. 8) and are experimentally manifest in the form of powerful microwave radiation from the plasma and in a sharp growth of gas ionization under the influence of the fluctuation fields.[20,27,28] The sharp increase of the gas

*It should be noted that all the investigations of the limiting currents of electron beams were carried out for the system considered by us only at nonrelativistic electron energies. The experiments described in [18] pertain to a different system; in particular, they employed no external beam-containing longitudinal magnetic field. The beams investigated in [5,9,20,27,28] were likewise nonrelativistic.

Table I. Limiting currents in uncompensated beams

Cylindrical beam	Tubular beam
$J_0=\frac{17}{1+2\ln(R/r_0)}(\gamma^{2/3}-1)^{3/2}$, kA	$J_0=17\frac{R}{a}(\gamma^{2/3}-1)^{3/2}$, kA

Critical currents in compensated beams

Cylindrical beam		Tubular beam
$\ln R/r_0 \ll 1$	$\ln R/r_0 \gg 1$	
$\frac{L}{r_0}<\frac{\pi}{2,4}\sqrt{\frac{M}{m}}\gamma^{-3/2}$ $J_{cr}=(2,4)^2\frac{mu^3}{4e}\gamma^3$	$L<\frac{2\pi u}{\Omega_i}$ $J_{cr}=\min\begin{cases}\frac{mu^3}{4e}\frac{2}{\ln R/r_0}\gamma^3\\ \frac{mu^3}{e}\gamma^3\frac{l(l+1)}{1+\frac{ul(l+1)L}{\pi\Omega_e r_0^2}}\gamma^3\\ \frac{mu}{4e}\frac{m}{M}r_0^2\Omega_e^2\end{cases}$	$\frac{L}{a}<2\sqrt{\frac{M}{m}}\gamma^{-3/2}$ $J_{cr}=J_1=\frac{\pi^2}{2}\frac{R}{a}\frac{mu^3}{4e}\gamma^3$
$\frac{\pi}{2.4}\sqrt{\frac{M}{m}}\gamma^{-3/2}<\frac{L}{r_0}<$ $<\frac{2,4\pi u^2}{r_0^2\Omega_e^2}\left(\frac{M}{m}\gamma\right)^{3/2}$ $J_{cr}=\frac{2,4\pi mu^3}{4e}\frac{r_0}{L}\gamma^{3/2}$		$2\sqrt{\frac{M}{m}}\gamma^{-3/2}<\frac{L}{a}<\frac{\pi u}{a\Omega_i}$ $J_{cr}=\min\{J_1, J_2\}$ $J_2=\max\begin{cases}\frac{\pi^2 mu^3}{4e}\frac{R}{L}\sqrt{\frac{M}{m}}\gamma^{3/2}\\ \frac{mu}{4e}\frac{m}{M}\Omega_e^2\,2Ra\end{cases}$
$\frac{L}{r_0}>2,4\pi\frac{u^2}{r_0^2\Omega_e^2}\left(\frac{M}{m}\gamma\right)^{3/2}$ $J_{cr}=\min\begin{cases}\frac{(2,4)^2 mu^3}{4e}\gamma^3\\ \frac{mu}{4e}\frac{m}{M}r_0^2\Omega_e^2\end{cases}$	$L>\frac{2\pi u}{\Omega_i}$ $J_{cr}=\min\begin{cases}\frac{mu^3}{4e}\frac{2}{\ln R/r_0}\gamma^3\\ \frac{mu}{4e}\Omega_e\frac{\pi u}{L}r_0^2\end{cases}$	$\frac{L}{a}>\frac{\pi u}{a\Omega_i}$ $J_{cr}=\min\{J_1, J_2, J_3\}$ $J_3=\frac{mu^3}{4e}\frac{M}{m}\frac{2\pi^2 Ra}{L^2}$

Table II. Critical plasma densities for overcompensated beams

$\ln R/r_0 \ll 1$	$\ln R/r_0 \gg 1$
$n_{2cr}=\min\begin{cases}\frac{(2,4)^2u^2}{3\cdot10^9 r_0^2}\left\{\left[1+\left(\frac{n_1}{n_2}\right)^{1/3}\times\frac{1}{\gamma}\right]^3-\frac{(2,4)^2u^2}{r_0^2\Omega_e^2}\right\}^{-1}\\ \frac{\Omega_e^2}{3\cdot10^9}\left\{1+\left(\frac{n_1}{n_2}\right)^{1/3}\frac{1}{\gamma}\times\left[1+\frac{(2,4)^2u^2}{r_0^2\Omega_e^2}\right]^{-1/3}\right\}^{-3}\end{cases}$	$\frac{L}{r_0}<\frac{10^{-4}\Omega_e\gamma^{3/2}}{\sqrt{n_1}}\ln(R/r_0)$ $n_{2cr}=\frac{u^2}{3\cdot10^9 r_0^2}\frac{2}{\ln R/r_0}\times$ $\times\left[1+\left(\frac{n_1}{n_2}\right)^{1/3}\frac{1}{\gamma}\right]^{-3}$
Tubular beam	$\frac{10^{-4}\Omega_e}{\sqrt{n_1}}\gamma^{3/2}\ln R/r_0<\frac{L}{r_0}<10^{-9}\frac{u\Omega_e}{n_1}$ $n_{2cr}=4,4\cdot10^{-14}u^2\Omega_e\gamma^{3/2}(Lr_0\sqrt{n_1})^{-1}$
$n_{2cr}=\min\begin{cases}\frac{\pi^2}{4}\frac{u^2}{3\cdot10^9a^2}\left\{\left[1+\left(\frac{n_1}{n_2}\right)^{1/3}\times\frac{1}{\gamma}\right]^3-\frac{\pi^2u^2}{4\Omega_e^2a^2}\right\}^{-1}\\ \frac{\Omega_e^2}{3\cdot10^9}\left\{1+\left(\frac{n_1}{n_2}\right)^{1/3}\times\frac{1}{\gamma}\left[1+\frac{\pi^2u^2}{4\Omega_e^2a^2}\right]^{-1/3}\right\}^{-3}\end{cases}$	$\frac{L}{r_0}>10^{-9}\frac{u\Omega_e}{n_1}>10^{-4}\frac{\Omega_e\gamma^{3/2}}{\sqrt{n_1}}\ln(R/r_0)$ $n_{2cr}=5\cdot10^{-5}\frac{u}{r_0}\sqrt{n_1}\gamma^{3/2}$

ionization was used to determine the critical plasma density in [28], from which the results shown in Fig. 5 were borrowed. We see that the dependence of n_{2cr} on the electron energy at different values of the longitudinal magnetic field and the beam current, under conditions when the waveguide is completely filled with plasma, is well described by the theoretical formulas obtained in Ch. 7. In the case of incomplete filling of the waveguide with plasma, the agreement between theory and experiment has a qualitative character.[28]

All the experiments described above pertain to cylindrical beams passing along the axis of a metallic waveguide. Experiments with tubular beams are just now beginning. Also projected are experiments with relativistic electron beams. We are convinced that these experiments, as well as those described above, will lead to new progress in the theory and will uncover a way of obtaining ultrastrong electron beams of high power.

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