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# BOUND STATES AND SCATTERING IN A SYSTEM OF THREE PARTICLES

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## INTRODUCTION

THE main difficulties in modern nuclear theory are connected both with our insufficient knowledge of the nature of the nuclear interaction and with the many-particle character of the nuclear systems. Study of the nuclear interaction between nucleons—the simplest component particles of nuclei—is one of the central problems of nuclear theory. The presently available experimental data on nuclear interaction do not contradict the assumption of the two-particle character of nuclear forces. Nuclear theory is a nonrelativistic theory and is based on the assumption that nuclear interaction between nucleons can be described with the aid of a potential which depends in the general case on the spin and isospin state of the nucleons.

Although the interaction between nucleons is strong, nonetheless a system of two nucleons has only one bound state—a deuteron—characterized by an exceedingly low binding energy. By virtue of this circumstance it is impossible to determine the potential uniquely from the scattering of two nucleons in the nonrelativistic energy region. To eliminate the ambiguity in the determination of the potential, it is necessary to refine greatly the experimental data on nuclear scattering, or to make use of other data.

One can expect a study of nuclear systems consisting of three and more nucleons to be a major source of additional information on the interaction between nucleons. Indeed, the properties of a system of two nucleons at energies close to zero can be explained by means of forces with a zero radius. However, in the case of zero-radius forces, the binding energy of a system of three nucleons turns out to be infinitely large[111]. Therefore the very existence of triton nuclei and $He^3$ indicates that the nuclear forces are characterized by a finite radius of action.

Data on the scattering of nucleons by nucleons at low energies ($\lesssim 10$ MeV) make it possible to determine only the effective parameters characterizing the magnitude and radius of the interaction potential. These data, however, are insufficient for a complete determination of the form of the potential. Unlike the two-nucleon problem, the problem of motion of three (or more) nucleons turns out to be more sensitive to the form of the two-nucleon potential. Of considerable interest is the possibility of explaining the properties of three-nucleon systems and systems consisting of a large number of nucleons on the basis of two-particle forces determined from a solution of the two-nucleon problem.

Data on two-nucleon interaction can be obtained from the properties of nuclei containing more than two nucleons; for this purpose it is necessary to calculate the properties of such nuclei. In essence, this calculation means solution of the many-body problem; the simplest example of such a problem is that of the motion of three particles, a rigorous formulation of which and proof of existence of whose solution were obtained relatively recently.

Owing to the indicated difficulties, the properties of a system of three nucleons and other few-nucleon nuclei were investigated in detail with the aid of different variational methods. During the last three decades, a considerable number of variational calculations were performed using different potentials that agree with data on the two-nucleon interaction[112]. In the case of central forces, the value of the binding energy of the triton turned out to be strongly dependent on the potential and too high in comparison with the experimental value. The introduction of tensor forces leads to a qualitative improvement of the agreement with experiment. The most consistent variational calculations of the binding energy and of the wave function of the triton, using the Hamada-Johnston potential, which takes into account different details of the nuclear interaction (repulsion at small distances, tensor and spin-orbit forces) were carried out recently by Delves et al.[58]. The value of the binding energy of the triton, obtained in[58], turned out to be $6.7 \pm 1.0$ MeV. Variational methods were also used to estimate the lengths of scattering of a neutron by a deuteron. The Kohn variational method makes it possible to obtain an upper bound for the neutron-deuteron scattering length. The results of the calculations depend, however, on the form of the trial functions, and therefore the accuracy of the obtained values is low. For the Hamada-Johnston potential the value obtained for the doublet scattering length in[58] by the variational method is $^2A = 1.2 \pm 1.0$ F.

From the fundamental point of view, more interest attaches to calculations based on a direct solution of the equations of motion of the three-nucleon system. During the last decade, considerable progress was made in the formulation of the equations of the three-particle problem and in the development of methods of solving such equations.

As shown by Skronyakov and Ter-Martirosyan[28], the problem of the motion of three nucleons in the case of two-particle forces with zero radius of action reduces to a solution of one-dimensional integral equations. If the orbital angular momentum of the system of three nucleons is equal to zero, then at a summary spin $S = 3/2$ and isospin $T = 1/2$ there is one integral equation, and at $S = 1/2$ and $T = 1/2$ there are two integral equations. For the quartet spin state of three nucleons, the integral equation contains only one two-particle parameter—the triplet nucleon-nucleon scattering length. In the case of zero energy of the incident neutron, the authors of[28] were able, by solving the indicated equation, to calculate the quartet neutron-deuteron scattering length, which was found to be $^4A = 5.1$ F. For the doublet spin state of a system of three nucleons, the integral equations contain two two-particle parameters—the triplet and singlet scattering lengths. The homogeneous system of equations corresponding to the bound state of the three-nucleon system leads in this case to an infinitely large value of the binding energy. On the other hand, the inhomogeneous system of equations describing the neutron-deuteron scattering does not have a unique solution. (The ambiguity is connected with the existence of a solution of the corresponding homogeneous system of equations for any value of the energy of the neutron-plus-deuteron system). To eliminate this ambiguity, Danilov[14] introduced into the problem, in addition to the two-particle parameters, one additional parameter, namely the binding energy of the system of three nucleons. In this manner it was possible in[14] to obtain for the doublet length of neutron-deuteron scattering a value $^2A = 0.48$ F, which agrees well with the experimental value.

The formal theory of scattering and reactions in a system of three particles in the case of two-particle forces with a finite radius of action was developed by Faddeev[30,31]. Although the formalism of[30] did not get rid of the main difficulty of many-particle problems, connected with the fact that the complete Hamiltonian of the system does not break up into a sum of additive terms that depend on the coordinates of the individual subsystems, it was possible in[30], by separating the wave function of the system of three particles into individual terms, to obtain for them a system of integral equations admitting of a unique solution. The kernels of the obtained equations are expressed in terms of two-particle scattering amplitudes off the energy shell. An essential advantage of the Faddeev equations over the well known Lippman-Schwinger equation is that their kernels can be reduced to the Fredholm type, and it is therefore possible to use known methods for the solution of the indicated equations. A generalization of Faddeev's integral equations to the case of a system of three nucleons with allowance for the spin and isospin dependences of the interaction between the nucleons was carried out in[107]. A generalization of Faddeev's equations to the case of an arbitrary number of particles was carried out in[32,39].

The technique of summation of nonrelativistic Feynman diagrams for the three-value problem was developed by Komarov and Popova[17].

The Faddeev equations for the system of three particles are transformed in the general case, by expanding the wave function in terms of the angle functions and separating the angle variables, into an infinite system of two-dimensional integral equations. If the two-particle potential is characterized by a finite radius of action, then the interaction between each pair of particles is manifest only in a finite number of partial states. In this case it is necessary to take into account only a finite number of equations out of the infinite number of two-dimensional integral equations.

The problem simplifies greatly if the two-particle scattering amplitude has a separable form. In this case the indicated system of two-dimensional integral equations reduces to a system of one-dimensional integral equations[34,107]. The two-particle scattering amplitude has a separable form if the interaction between the particles is described by a separable non-local potential takes into account the finite radius of action of the forces between the nucleons and makes it possible to solve in explicit form the problem of motion of two nucleons. Such a potential is a rather good approximation of the local short-range potential for a system of two particles, if formation of not more than one bound state is possible in the system. The application of a separable potential for the description of the two-particle interaction has made it possible to develop a technique for calculating three-particle systems (see[27]). By numerically solving the one-dimensional integral equations, we obtained in[107] the binding energy and the wave function of the triton, and also the quartet and doublet scattering lengths of a zero-energy neutron by a deuteron. The problem of three nucleons was considered independently, using a separable potential, by Mitra[93].

The calculations with separable potentials pointed to an appreciable dependence of the triton binding energy and of the doublet neutron-deuteron scattering length on the form of the potential, and also on the magnitude of the singlet effective radius of interaction between the nucleons. A change of the value of the singlet effective radius from 2.4 to 2.7 F leads to a decrease of the triton binding energy by approximately 1 MeV and to an increase of the doublet length by approximately 0.7 F. The quartet neutron-deuteron scattering length turned out to be insensitive to the change of the form of the two-particle interaction, this being connected with the influence of the Pauli principle, which prevents simultaneous approach of all three nucleons to small distances from one another.

The calculated values of the doublet and quartet neutron-deuteron scattering lengths turned out to be close to one of the two possible experimental sets of scattering lengths which were known at that time, $^4A > {}^2A$. Subsequently, experimental measurements performed by Shapiro et al.[45] using polarized neutrons and deuterons, confirmed the correctness of this set.

The influence of the tensor forces on the properties of a system of three nucleons was investigated for separable potentials in[26,108,78,54,94,102]. It turned out

that an admixture of D state with weight of 4% decreases the triton binding energy by approximately 2 MeV and increases the doublet length by more than 1 F[26].

An investigation of the elastic scattering of a neutron by a deuteron at finite energies, and also of the reaction of the disintegration of the deuteron in the interaction with a neutron has been carried out by Amado et al.[42,40] and by Phillips[100,101]. The calculated differential cross sections in the case of a separable potential agree well with the experimental data. The three-particle model of the stripping reaction with a separable potential was considered in[43,104].

The Faddeev equations for a system of three particles can also be brought to a one-dimensional form in the case when the interaction is separable between at least two particles. The interaction with zero radius is a particular case of a separable interaction. Baz' et al.[9] investigated the three-body problem in the case when the interaction between two particles is characterized by a zero radius of action and the potential of interaction of each of these particles with a third infinitely heavy particle has the form of a square well.

The problem of motion of three particles the interaction between which is described by local potentials was recently considered on the basis of the Faddeev equations in a number of papers. We note first of all papers based on numerical solution of the two-dimensional integral equations. The direct solution of the two-dimensional homogeneous integral equation describing the bound state of three identical spinless particles was carried out by Osborn[72]. We note that such calculations are at the borderline of feasibility by modern computation techniques. The method of successive iterations of a two-dimensional integral equation was used by Malfliet and Tjon[91] to find the binding energy and the wave function of the triton. They used a local potential in the form of a superposition of two Yukawa potentials with repulsion at small distances. The triton binding energy turned out to be 8.3 MeV compared with the value 12.1 MeV obtained in the case of a purely attractive Yukawa potential with the same values of the low-energy parameters.

The two-dimensional integral equations of the three-nucleon problem can be reduced to one-dimensional equations by using a separable representation for the two-particle scattering amplitude, which is equivalent to a separable expansion of the matrix elements of the interaction potential. In the case of the potentials customarily used for the phenomenological description of two-nucleon interaction, which are not very singular at small distances and decrease rapidly at large distances, the two-particle scattering amplitude can be approximated well by the first few terms of the expansion.

Several methods were proposed for approximating the two-particle scattering amplitude by a series with separable terms[96,86,67,103,15,87]. In the method based on the use of the Hilbert-Schmidt theorem for symmetrical integral equations[18], the separable expansion of the two-particle scattering amplitude off the energy shell is expressed in terms of the eigenfunctions and eigenvalues of the kernel of the Lippmann-Schwinger integral equation. Such a separable representation was first used by Weinberg[115] to eliminate divergences of the Born series for the two-particle scattering amplitude. Faddeev has pointed out[61] the possibility of using the Hilbert-Schmidt method for solving the problem of the motion of three particles. By now, this approach has been used in a number of papers for the solution of both atomic and nuclear three-particle problems[50,65,109,76,81,49,84]. In the case of attractive potentials, each term of the separable expansion corresponds to a corresponding separable attraction potential. Allowance for the succeeding terms in this expansion means introduction of a stronger attraction. Therefore allowance for further corrections in the Hilbert-Schmidt expansion can change the results of the calculation only in one direction (increase the binding energy of the triton and decrease the doublet scattering length). In the case of a potential of rectangular form, the two-particle scattering amplitude is well approximated by two terms of the Hilbert-Schmidt expansion. Calculations with such potentials make it possible to estimate the contribution of the higher partial moments in the two-particle interaction to the values of the binding energy of the triton and the neutron-deuteron scattering length. This contribution turned out to be small and amounts to 0.2% for the triton binding energy and 0.5% for the doublet length[37,79]. The triton binding energy and the doublet neutron-deuteron scattering length in the case of a rectangular well agree better with the experimental values than do the values for the Hulthen potential, which has a smeared edge and a singularity at zero.

The Hilbert-Schmidt expansion for the two-particle scattering amplitude is unitary only if account is taken of an infinite number of terms. If we use the expansion of the matrix elements of the potential in eigenfunctions of the kernel of the Lippman-Schwinger equation at a fixed energy, then we obtain for the two-particle scattering amplitude a separable expansion which is unitary when account is taken of a finite number of terms. Such an expansion, called the unitary pole expansion, was used by Harms[69]. The unitary pole expansion can serve as a basis for the use of separable potentials, inasmuch as allowance for the first term of such an expansion (the unitary pole approximation) is equivalent to introduction of a separable Yamaguchi potential.

Another method of obtaining separability of the two-particle scattering amplitude, the so-called Bateman method[52,16] is based on replacing the matrix element of the potential by a sum of separable terms, the coefficients of which are chosen such that the sum coincides with the matrix element at definite values of the arguments. This method was used to solve the three-nucleon problem by Belyaev et al.[1-3,11]. In[3,11] the binding energy, the form factors, and the weight of the states of intermediate symmetry were calculated for the triton, and also the neutron-deuteron scattering lengths in the case of a Morse potential and a potential chosen in the form of a superposition of Yukawa potentials. The employed potentials took into account repulsion at small distances and were reconciled with data on the interaction of two nucleons at low and high energies. It turned out that the aggregate of the calculated quantities in the case of a superposition of Yukawa potentials is in better agreement with the experimental data than the values calculated for the Morse potential.

Simonov and Badalyan[23,7,25] proposed a method of solving the problem of the bound state of a system of

three nucleons, based on expansion of the wave function in six-dimensional angular harmonics and making it possible to reduce the problem to a solution of a system of one-dimensional differential equations. Devoted to the construction of a complete system of basis eigenfunctions for a system of three particles are the papers of Smorodinskiĭ et al.[20,21]. Filippov and Ovcharenko developed a method of expansion in terms of the reciprocal powers of the interaction parameter, which makes it possible to calculate the energy of the bound states of a system of three nucleons[33].

In the present survey, we develop a nonrelativistic theory of three-particle systems with paired interaction. The development is based on the Faddeev formalism using a separable representation for the two-particle interaction. We consider bound states and elastic scattering in a system of three identical spinless particles and in a system of three nucleons*.

## 1. NONRELATIVISTIC THEORY OF SYSTEMS CONSISTING OF THREE PARTICLES

### 1.1. The Lippman-Schwinger Equation

We consider a quantum-mechanical system consisting in the general case of several interacting particles. We shall assume that the Hamiltonian of the system H can be broken up into two parts:

$$H = H_0 + V, \tag{1.1}$$

where the first term $H_0$ describes the unperturbed motion of the system, and the second term V describes the interaction that vanishes when the interacting parts of the system come sufficiently far apart.

In the stationary formulation, the scattering problem reduces to finding the solution of the Schrödinger equation

$$(H - E)\,\Psi = 0 \tag{1.2}$$

with definite boundary conditions (E is the energy of the system). At infinity, the solution $\Psi$ should have the form of a sum of an incident wave $\Phi$, which is a solution of the unperturbed equation

$$(H_0 - E)\,\Phi = 0, \tag{1.3}$$

and a diverging scattered wave.

The solution of (1.2) satisfying the indicated boundary conditions can be represented in the form

$$\Psi = \Phi + G_0(E + i0)\,V\Psi, \tag{1.4}$$

where $G_0(z)$ is the Green's function of the unperturbed equation (1.3):

$$G_0(z) \equiv (z - H_0)^{-1}, \quad z = E + i0. \tag{1.5}$$

The rule for going around the pole in (1.5) corresponds to the choice of a diverging scattered wave in the asymptotic form of $\Psi$. The formal solution of the Schrödinger equation (1.4) is an integral equation and is usually called the Lippman-Schwinger equation[88].

*A number of recent reviews [27, 60, 92, 46, 59, 98] discussed the results of calculations on the basis of the Faddeev integral equations for three-particle systems with a nonlocal separable interaction. Unlike these papers, the present review considers the properties of systems with local interaction.

The amplitude of the probability of the transition of the system from a state asymptotically described by the function $\Phi$ into a state described by the function $\Phi'$ is determined by the expression $(\Phi', V\Psi)$ at $E' = E$.

The solution of (1.4) can be expressed directly in terms of the asymptotic function $\Phi$ with the aid of the Green's function of Eq. (1.2)

$$G(z) \equiv (z - H)^{-1}. \tag{1.6}$$

The Green's function G(z) satisfies the equation

$$G(z) = G_0(z) + G_0(z)\,VG(z). \tag{1.7}$$

By direct verification it is easy to check that the solution (1.4) can be represented in the form

$$\Psi = \lim_{\varepsilon \to 0} i\varepsilon G(E + i\varepsilon)\,\Phi. \tag{1.8}$$

Indeed, multiplying (1.7) with $z = E + i\epsilon$ by i, applying it to the function $\Phi$, and noting that $\lim_{\epsilon \to 0} i\epsilon G_0(E + i\epsilon)\Phi = \Phi$, we obtain (1.4).

Usually in considering the scattering problem one employs the transition operator T. We represent the Green's function G(z) in the form

$$G(z) = G_0(z) + G_0(z)\,T(z)\,G_0(z); \tag{1.9}$$

and then we obtain from (1.7) the following equation for the operator T(z):

$$T(z) = V + VG_0(z)\,T(z). \tag{1.10}$$

This equation, generally speaking, is equivalent to the Lippman-Schwinger equation (1.4). Indeed, applying the operator equation (1.10) to $\Phi$ and comparing the obtained equation with (1.4) multiplied from the left by V, we readily see that

$$T\Phi = V\Psi. \tag{1.11}$$

According to (1.11), the amplitude of the transition of the system from the state $\Phi$ into the state $\Phi'$ is directly determined by the matrix element of the operator T:

$$(\Phi', V\Psi) = (\Phi', T\Phi). \tag{1.12}$$

The solutions of the Schrödinger equation (1.2) for negative values of the relative-motion energy correspond to bound states of the system. The Schrödinger equation for bound states of the system can be written with the aid of the Green's function (1.5) in the form of a homogeneous integral equation

$$\Psi = G_0(E + i0)\,V\Psi. \tag{1.13}$$

The Lippman-Schwinger equation (1.4) in the presence of bound states of the system has in the general case no unique solutions[63]. In fact, even for a system consisting of two particles, the energy levels

$$E = \frac{K^2}{2M} + \varepsilon$$

(K—total momentum, M—total mass and $\epsilon$—energy of relative motion) are multiply degenerate, since different distributions of the energy between the internal motion and the motion of the system as a unit correspond to different states. Since the presence of a bound state of the system denotes the existence of a solution of the homogeneous equation (1.13) at a fixed value of E, the solution of the inhomogeneous equation (1.4) becomes multiple-valued. For a system consisting of two parti-

cles, the Lippman-Schwinger equation admits of a unique solution only on going over to the center of mass system; in this case K = 0 and the energy degeneracy is lifted at $E = \epsilon$.

An analogous ambiguity of the solutions of the Lippman-Schwinger equations takes place also for systems consisting of three and more particles. Since in this case we can have besides the bound state of the entire system also bound states of the subsystems consisting of smaller numbers of particles, the solutions of the inhomogeneous Lippman-Schwinger equations are always ambiguous. A unique solution is obtained only for the homogeneous equation in the case of a bound state of the entire system regarded in the center of mass system.

Skornyakov and Ter-Martirosyan have shown[28], using as an example a system of three particles with zero action radius, that the wave function of the system can be broken up into individual terms for which a coupled system of integral equations is obtained. A generalization to the case of systems of three particles with arbitrary radius of action was carried out by Faddeev[30,31]. Unlike the Lippman-Schwinger integral equations, the obtained integral equations admit of a unique solution.

### 1.2. Faddeev's Equations

The method of obtaining integral equations for systems of interacting particles will be considered with a system of three nonrelativistic spinless particles as an example. The unperturbed Hamiltonian $H_0$ is chosen to be the operator of the kinetic energy in the particles; the interaction V in the case of two-particle forces is represented in the form of a sum of three terms,

$$V = V_{12} + V_{23} + V_{31}, \tag{1.14}$$

where $V_{ij}$ characterizes the interaction between particles i and j, which decreases with increasing relative distance between them. We note that the kernel of the integral equation (1.10) is in this case singular because of the presence of δ functions expressing the conservation of the momentum of the particle that does not interact with the chosen pair.

The system transition operator T, in accord with (1.10), is naturally represented in the case of two-particle forces (1.14) likewise in the form of a sum

$$T(Z) = T^{(1)}(Z) + T^{(2)}(Z) + T^{(3)}(Z), \tag{1.15}$$

where the individual terms are conveniently defined by means of the equalities

$$T^{(k)}(Z) = V_{ij} + V_{ij} G_0(Z)\, T(Z), \qquad ijk = 123,\ 231,\ 312 \tag{1.16}$$

(Z = E + i0, E is the total energy of the three-particle system). Representing the operator T(Z) in the right-hand sides of (1.16) in the form (1.15), relations (1.16) can be regarded as a system of coupled operator equations defining the individual terms of (1.15). We note that the iteration series for $T^{(k)}(Z)$ from (1.16) contains both singular terms (of the type $V_{23}G_0(Z)V_{23}$, $V_{23}G_0(Z)V_{23}G_0(Z)V_{23}$, etc.), and terms in which the δ functions are eliminated by intermediate integration (of the form $V_{23}G_0(Z)V_{31}$ etc.). The obtained system (1.16) is obviously equivalent to (1.10) and therefore, like the Lippman-Schwinger equation, has no unique solution.

To eliminate the ambiguity, we reconstruct the system (1.16), assuming the two-particle operators $T_{ij}$ to be known. We determine the operators $T_{ij}$ with the aid of the equation

$$T_{ij}(Z) = V_{ij} + V_{ij} G_0(Z)\, T_{ij}(Z), \tag{1.17}$$

which is obtained from (1.10) if we neglect in the latter the interaction of the particles i and j with the third particle. We note that the right-hand side of Eq. (1.17) contains all the singularities of Eq. (1.16), and therefore the singularities in (1.16) can be eliminated with the aid of (1.17). Separating the diagonal part in (1.16) and inverting the two-particle operator $[1 - V_{ij}G_0(Z)]$, we rewrite the system of coupled equations for the individual terms of the transition operator T with the aid of (1.17) in the form

$$T^{(k)}(Z) = T_{ij}(Z) + T_{ij}(Z) G_0(Z) [T^{(i)}(Z) + T^{(j)}(Z)], \quad ijk = 123,\ 231,\ 312. \tag{1.18}$$

The obtained system of integral equations, unlike (1.16), has a unique solution. The iteration series for the second term of the right-hand side of (1.18) does not contain any singularities. Therefore the system of integral equations (1.18) can be solved by Fredholm methods*.

Taking into account the connection between the transition operator T and the Green's function G (1.9), we can obtain from (1.18) a system of equations for G. In accordance with the breakdown (1.15) we have

$$\left.\begin{aligned} G(Z) &= G_0(Z) + G^{(1)}(Z) + G^{(2)}(Z) + G^{(3)}(Z), \\ G^{(i)}(Z) &= G_0(Z)\, T^{(i)}(Z)\, G_0(Z), \qquad i = 1,\ 2,\ 3. \end{aligned}\right\} \tag{1.19}$$

The functions $G^{(i)}$ satisfy here the equations

$$G^{(k)}(Z) = G_{ij}(Z) - G_0(Z) + G_0(Z)\, T_{ij}(Z) [G^{(i)}(Z) + G^{(j)}(Z)], \quad ijk = 123,\ 231,\ 312, \tag{1.20}$$

where

$$G_{ij}(Z) = G_0(Z) + G_0(Z)\, T_{ij}(Z)\, G_0(Z).$$

On the basis of (1.20) we can easily obtain with the aid of (1.8) the corresponding equations for the determination of the wave function of the system Ψ.

In the system of three interacting particles it is possible to have both infinite motion of all three particles, and infinite motion of an individual particle relative to the two other particles that are in a bound state. The corresponding asymptotic wave functions of the system will be denoted by $\Phi_{123}$ and $\Phi_i$, where i = 1, 2, or 3. (An index next to a function denotes the number of an unbound particle in the system. All the particles are assumed to be different.)

Applying Eqs. (1.20) multiplied by iε to the functions $\Phi_{123}$ and taking (1.8) into account, we obtain the following equation for the determination of the wave function $\Psi_{123}$ of the system in the case of unbound motion of all three particles:

$$\left.\begin{aligned} \Psi_{123} &= \Phi_{123} + \Psi^{(1)}_{123} + \Psi^{(2)}_{123} + \Psi^{(3)}_{123}, \\ \Psi^{(1)}_{123} &= \Phi_{1(23)} - \Phi_{123} + G_0(Z)\, T_{23}(Z) [\Psi^{(2)}_{123} + \Psi^{(3)}_{123}], \\ \Psi^{(2)}_{123} &= \Phi_{2(31)} - \Phi_{123} + G_0(Z)\, T_{31}(Z) [\Psi^{(3)}_{123} + \Psi^{(1)}_{123}], \\ \Psi^{(3)}_{123} &= \Phi_{3(12)} - \Phi_{123} + G_0(Z)\, T_{12}(Z) [\Psi^{(1)}_{123} + \Psi^{(2)}_{123}], \quad Z = E + i0, \end{aligned}\right\} \tag{1.21}$$

*A generalization of Faddeev's equations to the case of an arbitrary number of particles was made in [32, 39].

where $\Phi_{123} = \lim_{\epsilon \to 0} i\epsilon G_{23}(E + i\epsilon)\Phi_{123}$ etc. The functions $\Phi_{1(23)}$ differ from $\Phi_{123}$ in that account is taken of the interaction between particles 2 and 3. It is easy to verify that the difference $\Phi_{1(23)} - \Phi_{123}$ is a diverging wave at large distances with respect to the relative coordinate between the particles 2 and 3.

In the case of scattering of an individual particle by two other particles that are in a bound state, we can obtain the following equations for the determination of the wave function of the system $\Psi_1$ (for the scattering of particle 1 by the bound state of particles 2 and 3)

$$\left.\begin{aligned} &\Psi_1 = \Psi_1^{(1)} + \Psi_1^{(2)} + \Psi_1^{(3)}, \\ &\Psi_1^{(1)} = \Phi_1 + G_0(Z)\,T_{23}(Z)\,[\Psi_1^{(2)} + \Psi_1^{(3)}], \\ &\Psi_1^{(2)} = G_0(Z)\,T_{31}(Z)\,[\Psi_1^{(3)} + \Psi_1^{(1)}], \\ &\Psi_1^{(3)} = G_0(Z)\,T_{12}(Z)\,[\Psi_1^{(1)} + \Psi_1^{(2)}], \quad Z = E + i0. \end{aligned}\right\} \quad (1.22)$$

Analogous systems of equations hold for the functions $\Psi_2$ and $\Psi_3$.

The system of three integral equations (1.22) can be reduced to a system of two integral equations for the functions $\Psi_1^{(I)} \equiv \Psi_1^{(1)}$ and $\Psi_1^{(II)} \equiv \Psi_1^{(2)} + \Psi_1^{(3)}$. It is easy to verify that the functions $\Psi^{(I)}$ and $\Psi^{(II)}$ satisfy the equations

$$\Psi_1^{(I)} = \Phi_1 + G_0(Z)\,T_{23}(Z)\,\Psi_1^{(II)}, \quad \Psi_1^{(II)} = G_0(Z)\,T_1(Z)\,\Psi_1^{(I)}, \quad (1.23)$$

where the operator $T_1$ describes the scattering of an individual particle by two others in the absence of interaction between them:

$$T_1(Z) = V_1 + V_1 G_0(Z)\,T_1(Z), \quad V_1 = V_{12} + V_{13}. \quad (1.24)$$

We note that both in the case of infinite motion of all particles and in the case of scattering of an individual particle by two others in the bound state, we have an inhomogeneous system of integral equations admitting of a unique solution. The integral equations for the system of three particles (1.21) and (1.22) were obtained by Faddeev[30]. The integral equations (1.23) were obtained by Baz' et al.[9]. The main advantage of the Faddeev equations over the Lippman-Schwinger equations is that the Faddeev integral equations have Fredholm kernels, and therefore it is possible to use known methods for the solution of the indicated equations.

In the case of a bound state of the entire system, it is possible to obtain in similar fashion the following homogeneous system of integral equations for the determination of the wave function $\Psi_0$:

$$\left.\begin{aligned} &\Psi_0 = \Psi_0^{(1)} + \Psi_0^{(2)} + \Psi_0^{(3)}, \\ &\Psi_0^{(1)} = G_0(Z)\,T_{23}(Z)\,[\Psi_0^{(2)} + \Psi_0^{(3)}], \\ &\Psi_0^{(2)} = G_0(Z)\,T_{31}(Z)\,[\Psi_0^{(3)} + \Psi_0^{(1)}], \\ &\Psi_0^{(3)} = G_0(Z)\,T_{12}(Z)\,[\Psi_0^{(1)} + \Psi_0^{(2)}]. \end{aligned}\right\} \quad (1.25)$$

The homogeneous system of equations (1.25) has solutions only at energies corresponding to bound states of the system. In the c.m.s. this energy is negative ($E < 0$). We note that the system of equations (1.25) can be obtained formally from the system (1.22) by putting in the latter $\Phi_1 = 0$.

### 1.3. Coordinates and Momenta in a System of Three Particles

We denote the masses of the three particles by $m_1$, $m_2$, and $m_3$; accordingly the radius vectors and the momenta of the particles are denoted by $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. In the description of the relative motion in a system consisting of three particles, it is convenient to choose the Jacobi coordinates

$$\boldsymbol{\rho}_1 = \mathbf{r}_1 - \frac{m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_2 + m_3}, \quad \mathbf{r}_{23} = \mathbf{r}_2 - \mathbf{r}_3, \quad \mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3), \quad (1.26)$$

where $M = m_1 + m_2 + m_3$ is the total mass of the system. In place of the relative coordinates $\boldsymbol{\rho}_1$ and $\mathbf{r}_{23}$ we can use the relative coordinates $\boldsymbol{\rho}_2$ and $\mathbf{r}_{31}$ or $\boldsymbol{\rho}_3$ and $\mathbf{r}_{12}$. Accordingly, it is convenient to introduce the Jacobi momenta

$$\mathbf{p}_1 = \frac{(m_2 + m_3)\,\mathbf{k}_1 - m_1(\mathbf{k}_2 + \mathbf{k}_3)}{M}, \quad \mathbf{k}_{23} = \frac{m_3\mathbf{k}_2 - m_2\mathbf{k}_3}{m_2 + m_3}, \quad \mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3. \quad (1.27)$$

In place of the relative momenta $\mathbf{p}_1$ and $\mathbf{k}_{23}$ we can choose $\mathbf{p}_2$ and $\mathbf{k}_{31}$ or $\mathbf{p}_3$ and $\mathbf{k}_{12}$.

The kinetic-energy operator of the system is written in the Jacobi coordinates $\mathbf{r}_{23}$, $\boldsymbol{\rho}_1$, and $\mathbf{R}$ in the form

$$H_0 = -\frac{1}{2\mu_{23}}\Delta_{23} - \frac{1}{2\mu_1}\Delta_1 - \frac{1}{2M}\Delta, \quad (1.28)$$

where $\mu_{23} = m_2m_3/(m_2 + m_3)$ and $\mu_1 = m_1(m_2 + m_3)/M$ are the reduced masses. In the momentum representation $H_0$ takes the form

$$H_0 = \frac{k_{23}^2}{2\mu_{23}} + \frac{p_1^2}{2\mu_1} + \frac{K^2}{2M}. \quad (1.29)$$

We present explicit expressions for the asymptotic functions. We denote the initial values of the momenta by zero indices. The asymptotic function $\Phi_{123}$, which describes the free motion of all the particles, can be written in the form

$$\Phi_{123} = e^{i\mathbf{p}_1^0\boldsymbol{\rho}_1 + i\mathbf{k}_{23}^0\mathbf{r}_{23} + i\mathbf{K}^0\mathbf{R}}, \quad E_{123} = \frac{p_1^{0^2}}{2\mu_1} + \frac{k_{23}^{0^2}}{2\mu_{23}} + \frac{K^{0^2}}{2M}. \quad (1.30)$$

We note that the function $\Phi_{123}$ is invariant against substitution of the Jacobi coordinates. The asymptotic function $\Phi_1$ takes the form

$$\Phi_1 = e^{i\mathbf{p}_1^0\boldsymbol{\rho}_1 + i\mathbf{K}^0\mathbf{R}}\varphi_{\kappa_{23}}(\mathbf{r}_{23}), \quad E_1 = \frac{p_1^{0^2}}{2\mu_1} - \frac{\kappa_{23}^2}{2\mu_{23}} + \frac{K^{0^2}}{2M}, \quad (1.31)$$

where $\varphi_{K_{23}}(\mathbf{r}_{23})$ is a solution of the equation

$$\left(-\frac{1}{2\mu_{23}}\Delta_{23} + V_{23} - \varepsilon_{23}\right)\varphi_{\kappa_{23}}(\mathbf{r}_{23}) = 0 \quad (1.32)$$

at a negative value of the energy of the relative motion $\epsilon_{23} = -\kappa_{23}^2/2\mu_{23}$, i.e., a function describing the bound state of the system of particles 2 and 3. Asymptotically the function $\Phi_{1(23)}$ is written in the form

$$\Phi_{1(23)} = e^{i\mathbf{p}_1^0\boldsymbol{\rho}_1 + i\mathbf{K}^0\mathbf{R}}\varphi_{k_{23}^0}(\mathbf{r}_{23}), \quad E_{1(23)} = E_{123}, \quad (1.33)$$

where $\varphi_{k_{23}^0}(\mathbf{r}_{23})$ is the solution of Eq. (1.32) for $\epsilon_{23} = k_{23}^{02}/2\mu_{23} > 0$, which has at infinity the form of a sum of a plane wave and a diverging spherical wave.

### 1.4. Momentum Representation

The Faddeev integral equations have the simplest form in the momentum representation. Each of the components of the wave function of the system $\Psi^{(i)}$ can be conveniently represented in the form of a function of the corresponding set of coordinates:

$$\Psi^{(i)} \equiv \Psi^{(i)}(\mathbf{r}_{jk}, \boldsymbol{\rho}_i, \mathbf{R}), \quad ijk = 123, 231, 312. \quad (1.34)$$

In the momentum representation, the function $\Psi^{(i)}$ is

defined with the aid of the equality

$$\Psi^{(i)}(\mathbf{k}_{jk}, \mathbf{p}_i, \mathbf{K}) = \int e^{-i\mathbf{k}_{jk}\mathbf{r}_{jk} - i\mathbf{p}_i\boldsymbol{\rho}_i - i\mathbf{K}\mathbf{R}} \Psi^{(i)}(\mathbf{r}_{jk}, \boldsymbol{\rho}_i, \mathbf{R})\, d\mathbf{r}_{jk}\, d\boldsymbol{\rho}_i\, d\mathbf{R}. \tag{1.35}$$

Inasmuch as the kinetic-energy operator in the momentum representation is a multiplication operator, the Green's function $G_0(z)$ is diagonal in the momentum representation. The operator of two-particle scattering $T_{jk}(z)$ is diagonal in the representation of the total momentum of the system and the momentum of the free particle $\mathbf{p}_i$:

$$\langle \mathbf{k}_{jk}\mathbf{p}_i\mathbf{K} | T_{jk}(Z) | \mathbf{k}'_{jk}\mathbf{p}'_i\mathbf{K}' \rangle = (2\pi)^6 \left\langle \mathbf{k}_{jk} \left| t_{jk}\left(Z - \frac{p_i^2}{2\mu_i} - \frac{K^2}{2M}\right) \right| \mathbf{k}'_{jk} \right\rangle \delta(\mathbf{p}_i - \mathbf{p}'_i)\, \delta(\mathbf{K} - \mathbf{K}'), \tag{1.36}$$

where $t_{ij}(z)$ is a two-particle t matrix defined by the integral equation

$$t(z) = V + V(z - H_0)^{-1} t(z),$$

or

$$\langle \mathbf{k} | t(z) | \mathbf{k}' \rangle = \langle \mathbf{k} | V | \mathbf{k}' \rangle + \int \frac{\langle \mathbf{k} | V | \mathbf{q} \rangle \langle \mathbf{q} | t(z) | \mathbf{k}' \rangle}{z - \frac{q^2}{2\mu}} \frac{d\mathbf{q}}{(2\pi)^3}, \quad z = \varepsilon + i0 \tag{1.37}$$

($\epsilon$ is the energy of the relative motion of two particles and $\mu$ is the reduced mass). In the general case Eq. (1.37) determines the t matrix off the energy shell ($k^2/2\mu \neq k'^2/2\mu \neq \epsilon$). We recall that the amplitude of the elastic scattering of the particles is expressed in terms of the t matrix on the energy shell:

$$f(\mathbf{k}, \mathbf{k}') = -\frac{\mu}{2\pi} \langle \mathbf{k} | t(\varepsilon + i0) | \mathbf{k}' \rangle, \quad \varepsilon = \frac{k^2}{2\mu} = \frac{k'^2}{2\mu}. \tag{1.38}$$

The two-particle t matrix $\langle \mathbf{k}|t(z)|\mathbf{k}'\rangle$ in the plane of complex energies z has singularities, namely poles, corresponding to a discrete spectrum, and a cut along the positive part of the real axis, generated by the continuous spectrum of the system of two particles. The explicit form of the indicated singularities can be obtained from the so-called spectral representation of the t matrix[89]:

$$\langle \mathbf{k} | t(z) | \mathbf{k}' \rangle = \langle \mathbf{k} | V | \mathbf{k}' \rangle + \sum_N \frac{g_N(\mathbf{k})\, g_N(\mathbf{k}')}{z + (\kappa_N^2/2\mu)} + \int \frac{\left\langle \mathbf{k} \left| t\left(\frac{q^2}{2\mu} + i0\right) \right| \mathbf{q} \right\rangle \left\langle \mathbf{q} \left| t\left(\frac{q^2}{2\mu} - i0\right) \right| \mathbf{k}' \right\rangle}{z - (q^2/2\mu)} \frac{d\mathbf{q}}{(2\pi)^3}, \tag{1.39}$$

where $g_N(\mathbf{k}) = (k^2 + \kappa_N^2/2\mu)\varphi_N(\mathbf{k})$, $N = \{n, l, m\}$, and $\kappa_N^2/2\mu$ and $\varphi_N(\mathbf{k}) \equiv \varphi_{nl}(k)Y_{lm}(\hat{\mathbf{k}})$ are the binding energy and the wave function of the bound state of the two particles in the momentum representation $\left(\int \varphi_N^2(k) \frac{d\mathbf{k}}{(2\pi)^3} = 1\right)$.

Since all operators entering in the Faddeev equations are diagonal in the K representation, the wave function $\Psi$ will contain the factor $\delta(\mathbf{K} - \mathbf{K}^0)$, which expresses the law of conservation of the total momentum of the system. Therefore all dependences on K can be completely eliminated by changing over to the c.m.s.

By way of an example let us consider in the momentum representation the Faddeev system of integral equations (1.22) in the case of scattering of a particle by two other particles that are in a bound state:

$$\Psi_1 = \Psi_1^{(1)}(\mathbf{k}_{23}, \mathbf{p}_1) + \Psi_1^{(2)}(\mathbf{k}_{31}, \mathbf{p}_2) + \Psi_1^{(3)}(\mathbf{k}_{12}, \mathbf{p}_3), \tag{1.40}$$

$$\Psi_1^{(1)}(\mathbf{k}, \mathbf{p}) = \Phi_1(\mathbf{k}, \mathbf{p}) + \left(Z - \frac{k^2}{2\mu_{23}} - \frac{p^2}{2\mu_1}\right)^{-1} \times \int \left[ \left\langle \mathbf{k} \left| t_{23}\left(Z - \frac{p^2}{2\mu_1}\right) \right| \frac{m_2}{m_2+m_3}\mathbf{p} + \mathbf{p}' \right\rangle \Psi_1^{(2)}\left(-\mathbf{p} - \frac{m_1}{m_3+m_1}\mathbf{p}', \mathbf{p}'\right) + \left\langle \mathbf{k} \left| t_{23}\left(Z - \frac{p^2}{2\mu_1}\right) \right| -\frac{m_3}{m_2+m_3}\mathbf{p} - \mathbf{p}' \right\rangle \Psi_1^{(3)}\left(\mathbf{p} + \frac{m_1}{m_1+m_2}\mathbf{p}', \mathbf{p}'\right) \right] \frac{d\mathbf{p}'}{(2\pi)^3},$$

$$\Psi_1^{(2)}(\mathbf{k}, \mathbf{p}) = \left(Z - \frac{k^2}{2\mu_{31}} - \frac{p^2}{2\mu_2}\right)^{-1} \times \int \left[ \left\langle \mathbf{k} \left| t_{31}\left(Z - \frac{p^2}{2\mu_2}\right) \right| \frac{m_3}{m_3+m_1}\mathbf{p} + \mathbf{p}' \right\rangle \Psi_1^{(3)}\left(-\mathbf{p} - \frac{m_2}{m_1+m_2}\mathbf{p}', \mathbf{p}'\right) + \left\langle \mathbf{k} \left| t_{31}\left(Z - \frac{p^2}{2\mu_2}\right) \right| -\frac{m_1}{m_3+m_1}\mathbf{p} - \mathbf{p}' \right\rangle \Psi_1^{(1)}\left(\mathbf{p} + \frac{m_2}{m_2+m_3}\mathbf{p}', \mathbf{p}'\right) \right] \frac{d\mathbf{p}}{(2\pi)^3},$$

$$\Psi_1^{(3)}(\mathbf{k}, \mathbf{p}) = \left(Z - \frac{k^2}{2\mu_{12}} - \frac{p^2}{2\mu_3}\right)^{-1} \times \int \left[ \left\langle \mathbf{k} \left| t_{12}\left(Z - \frac{p^2}{2\mu_3}\right) \right| \frac{m_1}{m_1+m_2}\mathbf{p} + \mathbf{p}' \right\rangle \Psi_1^{(1)}\left(-\mathbf{p} - \frac{m_3}{m_2+m_3}\mathbf{p}', \mathbf{p}'\right) + \left\langle \mathbf{k} \left| t_{12}\left(Z - \frac{p^2}{2\mu_3}\right) \right| -\frac{m_2}{m_1+m_2}\mathbf{p} - \mathbf{p}' \right\rangle \Psi_1^{(2)}\left(\mathbf{p} + \frac{m_3}{m_3+m_1}\mathbf{p}', \mathbf{p}'\right) \right] \frac{d\mathbf{p}'}{(2\pi)^3},$$

$$Z \equiv \frac{p_1^{0\,2}}{2\mu_1} - \frac{\kappa_{23}^2}{2\mu_{23}} + i0.$$

The function $\Phi_1(\mathbf{k}_{23}, \mathbf{p}_1)$ equals, according to (1.31),

$$\Phi_1(\mathbf{k}_{23}, \mathbf{p}_1) \equiv (2\pi)^3 \delta(\mathbf{p}_1 - \mathbf{p}_1^0)\, \varphi_{\kappa_{23}}(\mathbf{k}_{23}), \tag{1.41}$$

where $\varphi_{K_{23}}(\mathbf{k}_{23})$ is the wave function of the bound state of two particles with binding nnergy $\kappa_{23}^2/2\mu_{23}$ and $\mathbf{p}_1^0$ is the momentum of the relative motion of the system in the initial state.

If all three particles are the same and have zero spin and isotopic spin, then the total wave function of the system $\Psi$ should be symmetrical against permutation of any pair of particles; in this case

$$\Psi^{(1)}(\mathbf{k}, \mathbf{p}) = \Psi^{(2)}(\mathbf{k}, \mathbf{p}) = \Psi^{(3)}(\mathbf{k}, \mathbf{p}) \equiv \psi(\mathbf{k}, \mathbf{p}).$$

Therefore the wave function can be represented in the form

$$\Psi = \psi(\mathbf{k}_{23}, \mathbf{p}_1) + \psi(\mathbf{k}_{31}, \mathbf{p}_2) + \psi(\mathbf{k}_{12}, \mathbf{p}_3), \tag{1.42}$$

where $\psi(-\mathbf{k}, \mathbf{p}) = \psi(\mathbf{k}, \mathbf{p})$. In place of the system (1.40) we obtain in this case one integral equation for the function $\psi(\mathbf{k}, \mathbf{p})$:

$$\psi(\mathbf{k}, \mathbf{p}) = \varphi(\mathbf{k}, \mathbf{p}) + \left(Z - \frac{k^2}{m} - \frac{3}{4}\frac{p^2}{m}\right)^{-1} \int \left\{ \left\langle \mathbf{k} \left| t\left(Z - \frac{3}{4}\frac{p^2}{m}\right) \right| \frac{\mathbf{p}}{2} + \mathbf{p}' \right\rangle + \left\langle \mathbf{k} \left| t\left(Z - \frac{3}{4}\frac{p^2}{m}\right) \right| -\frac{\mathbf{p}}{2} - \mathbf{p}' \right\rangle \right\} \psi\left[\mathbf{p} + \frac{\mathbf{p}'}{2}, \mathbf{p}'\right] \frac{d\mathbf{p}'}{(2\pi)^3}, \tag{1.43}$$

where $\varphi(\mathbf{k}, \mathbf{p})$ is a function in terms of which the initial wave function of the system is expressed via symmetrization:

$$\Phi = \varphi(\mathbf{k}_{23}, \mathbf{p}_1) + \varphi(\mathbf{k}_{31}, \mathbf{p}_2) + \varphi(\mathbf{k}_{12}, \mathbf{p}_3).$$

In the case of a bound state of three particles $\varphi = 0$, $Z = E < 0$; in the case of scattering of one particle by the two others which are in the bound state, the function $\varphi(\mathbf{k}, \mathbf{p})$ is determined by an expression of the type (1.41).

The integral equation (1.43) determines the wave function of the system, which depends in the general case on six variables (two relative vectors). By expanding the wave function in terms of the angle functions and separating the angle variables, the three-dimensional integral equation (1.43) in the case of central interaction between the particles can be reduced to a system of two-dimensional integral equations[44,99,114].

### 1.5. Expansion in Partial Waves

Assuming that the interaction between the particles is central, it is convenient to expand the two-particle potential in terms of the partial components

$$\langle \mathbf{k} | V | \mathbf{k}' \rangle = \sum_l (2l+1) V_l(k, k') P_l(\cos\theta) = 4\pi \sum_{lm} V_l(k, k') Y_{lm}(\hat{k}) Y^*_{lm}(\hat{k}'), \quad (1.44)$$

where $Y_{lm}(\hat{k})$ and $Y_{lm}(\hat{k}')$ are spherical functions of the angles characterizing the directions of the vectors $\mathbf{k}$ and $\mathbf{k}'$; $\theta$ is the angle between the vectors $\mathbf{k}$ and $\mathbf{k}'$. The individual terms in (1.44) describe the interaction between the particles in states with different values of the orbital angular momentum $l$. An analogous expansion is best carried out also in the expressions for the t matrix and for the scattering amplitude

$$\langle \mathbf{k} | t(z) | \mathbf{k}' \rangle = \sum_l (2l+1) t_l(k, k'; z) P_l(\cos\theta), \quad (1.45)$$

$$f(\mathbf{k}', \mathbf{k}) = \sum_l (2l+1) f_l(k) P_l(\cos\theta). \quad (1.46)$$

From (1.37) we easily obtain the following integral equation for the determination of the partial component of the t matrix:

$$t_l(z) = V_l + V_l (z - H_0)^{-1} t_l(z) \quad (1.47)$$

or

$$t_l(k, k'; z) = V_l(k, k') + \frac{1}{2\pi^2} \int dq q^2 V_l(k, q) \frac{1}{z - (q^2/2\mu)} t_l(q, k'; z). \quad (1.48)$$

The partial scattering amplitude $f_l(k) = (1/k)e^{i\delta_l}\sin\delta_l$ ($\delta_l$ is the phase of the scattering at infinity) is connected with the partial component of the t matrix by the relation

$$f_l(k) = -\frac{\mu}{2\pi} t_l\left(k, k; \frac{k^2}{2\mu} + i0\right). \quad (1.49)$$

In the case of a system of three identical particles, we introduce the orbital momentum of the relative motion of the two particles 1 and the orbital momentum of the relative motion of the third particle and the center of mass of the two other particles $\lambda$. It is obvious that the total angular momentum of the system L is equal to the vector sum of 1 and $\lambda$:

$$\mathbf{L} = \mathbf{l} + \boldsymbol{\lambda}.$$

The corresponding wave function describing the state of the system with total angular momentum L can be chosen in the form

$$Y_{l\lambda LM}(\hat{k}, \hat{p}) = \sum_{m\mu} (l\lambda m\mu | LM) Y_{lm}(\hat{k}) Y_{\lambda\mu}(\hat{p}). \quad (1.50)$$

The functions $Y_{l\lambda LM}$ form a complete system of orthonormal functions.

The wave function of the system of three particles $\psi(\mathbf{k}, \mathbf{p})$ depends not only on the vectors $\mathbf{k}$ and $\mathbf{p}$ but also on the vector of relative momentum of the system in the initial state $\mathbf{p}_0$. Let us assume that the angular momentum of the bound state of the system of two particles from which the third particle is scattered is equal to zero ($l_0 = 0$). Then by virtue of the scalar character of the wave function $\psi(\mathbf{k}, \mathbf{p})$ the expansion of this function in terms of the functions (1.50) takes the form

$$\psi(\mathbf{k}, \mathbf{p}) \equiv \psi(\mathbf{k}, \mathbf{p}; \mathbf{p}_0) = \sum_{l\lambda LM} \psi_{l\lambda L}(k, p; p_0) Y_{l\lambda LM}(\hat{k}, \hat{p}) Y^*_{LM}(\hat{p}_0). \quad (1.51)$$

Substituting the expansions (1.51) and (1.45) in (1.43) and using the orthonormalization of the functions (1.50), we obtain for the expansion coefficients $\psi_{l\lambda L}$ the following system of integral equations:

$$\psi_{l\lambda L}(k, p; p_0) = (2\pi)^3 \varphi_{l0}(k) \delta_{l0}\delta_{\lambda L} \frac{\delta(p - p_0)}{p^2} + 2\Delta_l \left(Z_p - \frac{k^2}{m}\right)^{-1} \sum_{l'\lambda'} \frac{1}{2\pi^2} \int_0^\infty dp' \int_{|p - \frac{p'}{2}|}^{p + \frac{p'}{2}} dk' \frac{k'p'}{p} \times t_l(k, Q; Z_\nu) K^{(L)}_{l\lambda, l'\lambda'}(p, p'; k') \psi_{l'\lambda' L}(k', p'; p_0), \quad (1.52)$$

where $\Delta_l = \frac{1}{2}\{1 + (-1)^l\}$, $\mathbf{Q}^2 = \mathbf{k}'^2 + \frac{3}{4}\mathbf{p}'^2 - \frac{3}{4}\mathbf{p}^2$ and

$$K^{(L)}_{l\lambda, l'\lambda'}(p, p'; k') \equiv 2 \int do_p do_{p'} Y^*_{\lambda L 0}\left(\widehat{\frac{p}{2} + p'}, \hat{p}\right) \times \delta\left\{\cos\theta - \frac{k'^2 - p^2 - \frac{1}{4}p'^2}{pp'}\right\} Y_{l'\lambda' L 0}\left(\widehat{p + \frac{p'}{2}}, \hat{p}'\right). \quad (1.53)$$

We have thus reduced the integral equation (1.43) for the function $\psi(\mathbf{k}, \mathbf{p})$ to an infinite system of two-dimensional integral equations for the expansion coefficients $\psi_{l\lambda L}(\mathbf{k}, \mathbf{p})$.

Substituting in (1.53) the explicit form of the functions $Y_{l\lambda L_0}$ and integrating with respect to the angles, we can obtain the kernels of the integral equations $K^{(L)}_{l\lambda, l'\lambda'}(\mathbf{p}, \mathbf{p}'; \mathbf{k}')$ in the form

$$K^{(L)}_{l\lambda, l'\lambda'}(p, p'; k') = (4\pi)^{\frac{3}{2}} (2\lambda' + 1)^{-\frac{1}{2}} \sum_{mm'} (-1)^{l+l'-m-m'} (lLm, -m | \lambda 0) \times (l'Lm', -m | \lambda' m' - m) Y^*_{lm}(\vartheta, 0) Y_{l'm'}(\vartheta', 0) Y_{\lambda' m - m'}(\theta, 0), \quad \left|p - \frac{p'}{2}\right| \leq k' \leq p + \frac{p'}{2}, \quad (1.54)$$

where the angles $\theta$, $\vartheta$ and $\vartheta'$ are determined by the expressions

$$\cos\theta = \frac{k'^2 - p^2 - \frac{1}{4}p'^2}{pp'}, \quad \cos\vartheta = \frac{k'^2 - \frac{1}{2}p^2 - \frac{1}{4}p'^2}{p\sqrt{k'^2 + \frac{3}{4}(p'^2 - p^2)}}, \quad \cos\vartheta' = \frac{k'^2 + p^2 - \frac{1}{4}p'^2}{2pk'}. \quad (1.55)$$

We note that

$$K^{(0)}_{00, 00} = 1. \quad (1.56)$$

In the case of the bound state of the system of three particles, the wave function can also be expanded in terms of the angle functions (1.50). If the system in the bound state is characterized by a total angular momentum L and a projection M, then this expansion takes the form

$$\psi_{LM}(\mathbf{k}, \mathbf{p}) = \sum_{l\lambda} \psi_{l\lambda L}(k, p) Y_{l\lambda LM}(\hat{k}, \hat{p}). \quad (1.57)$$

The expansion coefficients $\psi_{l\lambda L}(\mathbf{k}, \mathbf{p})$ are determined by the system of integral equations (1.52) if we set the inhomogeneous terms in the latter equal to zero.

Owing to the presence of the factor $\Delta_l$ in Eqs. (1.52), the components $\psi_{l\lambda L}$ with odd $l$ are equal to zero, this being due to the symmetry of the wave function $\Psi$ against permutations of any pair of particles.

For short-range potentials, the elements of the two-particle t matrix $t_l(\mathbf{k}, \mathbf{k}'; z)$ decrease rapidly with in-

creasing $l$ (at small k or k′ the elements $t_l(k, k'; z)$ are proportional to $k^l k'^l$, at large k the components $t_l$ are small for all $l$; in addition, the contribution of large k to the equations is suppressed by the factor $(Z_p - k^2/m)^{-1}$). Therefore the summation over $l$ in the obtained equations (meaning summation over $\lambda$ for specified L) can be limited to a finite number of terms, and consequently the systems of integral equations become finite.

The systems (1.52) become much simpler if the two-particle t matrix has a separable form. In this case the systems (1.52) can be reduced to systems of one-dimensional integral equations which admit of a numerical solution.

### 1.6. Method of Harmonic Polynomials in the Three-body Problem

Another approach in the solution of the problem of the bound state of three-particle system, based on introducing harmonic polynomials, was developed by Simonov and Badalyan[23,7,25]. In this case the Schrödinger differential equation, which defines a system wave function that depends on six relative coordinates, reduces to an infinite system of equations for functions of one variable.

To illustrate the method of harmonic polynomials, let us consider the simplest case of three identical spinless particles. In the c.m.s., the components of the vectors $\mathbf{r}_{23}/\sqrt{2}$ and $\sqrt{2/3}\boldsymbol{\rho}_1$, describing the relative motion of the three particles, are best considered as components of a certain single vector in six-dimensional space, the square of the modulus of which is equal to

$$\rho^2 = \frac{1}{2} r_{23}^2 + \frac{2}{3} \rho_1^2 = r_1^2 + r_2^2 + r_3^2 = \frac{1}{3}(r_{23}^2 + r_{31}^2 + r_{12}^2). \tag{1.58}$$

In the spherical coordinate system of the six-dimensional space, the kinetic energy of the system (1.28) takes the form

$$H_0 = -\frac{1}{2m}\left[\frac{1}{\rho^5}\frac{\partial}{\partial\rho}\left(\rho^5\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\Delta_\Omega\right], \tag{1.59}$$

where $\Delta_\Omega$ is the angular part of the six-dimensional Laplace operator. (The letter $\Omega$ denotes all the angle variables of the six-dimensional vector $\mathbf{r}_{23}/\sqrt{2}$, $\sqrt{2/3}\boldsymbol{\rho}_1$.)

The wave function $\Psi$, which is a solution of the Schrödinger equation, will be represented in the form of an expansion in a complete system of orthonormal eigenfunctions $u_{Kn}(\Omega)$ of the operator $\Delta_\Omega$, called harmonic polynomials:

$$\left.\begin{aligned} &\Delta_\Omega u_{Kn}(\Omega) = -K(K+4)\,u_{Kn}(\Omega), \\ &\int u^*_{\tilde{K}\tilde{n}}(\Omega)\,u_{K'n'}(\Omega)\,d\Omega = \delta_{KK'}\delta_{nn'}. \end{aligned}\right\} \tag{1.60}$$

This expansion is of the form

$$\Psi = \sum_{Kn} \varphi_{Kn}(\rho)\,u_{Kn}(\Omega). \tag{1.61}$$

The quantum number K characterizes the total angular momentum in six-dimensional space and can assume arbitrary positive integer values. The remaining quantum numbers are denoted by the letter n. The aggregate n contains L—the total orbital angular momentum of the relative motion of the particles, M—the projection of the total angular momentum, $\nu$—a quantum number characterizing the symmetry of the polynomial against permutation of the coordinates of the particles, and an additional quantum number, which is necessary in the case of the higher values of L and K. If the total orbital angular momentum of the system is equal to zero, L = 0, then the quantum numbers K and $\nu$ determine completely the set of orthonormal functions. Harmonic polynomials for a system of three particles were constructed in[23,21].

Substituting the function $\Psi$ in the form (1.61) in the Schrödinger equation for three particles with interaction in the form (1.14) and using relations (1.59) and (1.60), we obtain for the coefficients of the expansion (1.61) the following system of ordinary differential equations:

$$\left[\frac{d^2}{d\rho^2} + \frac{5}{\rho}\frac{d}{d\rho} - \frac{K(K+4)}{\rho^2} + \frac{2mE}{\hbar^2}\right]\varphi_{Kn}(\rho) = \sum_{\tilde{K}\tilde{n}} v_{Kn,\tilde{K}\tilde{n}}(\rho)\,\varphi_{\tilde{K}\tilde{n}}(\rho), \tag{1.62}$$

where

$$v_{Kn,\tilde{K}\tilde{n}}(\rho) = \frac{2m}{\hbar^2}\int d\Omega u^*_{Kn}(V_{12} + V_{23} + V_{31})\,u_{\tilde{K}\tilde{n}}. \tag{1.63}$$

The system (1.62) consists of an infinite number of coupled differential equations and is exact.

It is easily seen that for paired oscillator potentials of the type

$$V_{ij} = \gamma r_{ij}^2 \tag{1.64}$$

the matrix elements (1.63) have a diagonal form

$$v_{Kn,\tilde{K}\tilde{n}}(\rho) = \frac{6m}{\hbar^2}\gamma\rho^2\delta_{K\tilde{K}}\delta_{n\tilde{n}}. \tag{1.65}$$

and the system (1.62) splits into independent differential equations for each of the functions $\varphi_{Kn}(\rho)$. In this case K is an exact quantum number.

In the general case, the nondiagonal matrix elements (1.63) differ from zero and K is not an integral of the motion. However, as shown in[5,7,25], in the case of short-range paired potentials the main contribution to the binding energy and to the wave function of the triton is made by the term of the expansion (1.61) with the minimum value K = 0. The role of the higher values of K in the system (1.62) is immaterial both because of the smallness of the nondiagonal matrix elements compared with the diagonal ones, and because with increasing K the effective attractive potentials $v_{Kn,\tilde{K}\tilde{n}}(\rho)$ are cancelled by the terms of the centrifugal potential energy $K(K+4)/\rho^2$.

The eigenvalue of the energy of the system, calculated by the method of harmonic polynomials, has a variational property: it is the upper limit for the exact value of the energy. This can easily be verified by choosing as the trial function an approximate wave function in the form (1.61), satisfying the system (1.62) when account is taken of a finite number of terms in (1.61) and (1.62).

A generalization of the method of harmonic polynomials to the case of four and more particles is considered in[6,29,24,8,10].

## 2. SEPARABLE REPRESENTATION OF TWO-PARTICLE t MATRIX

### 2.1. Separable Representation of Two-particle t Matrix and Reduction of the Integral Equations for a System of Three Particles to One-dimensional Form

The two-particle t matrix $t_l(k, k'; z)$ is called separable if it can be represented in the form of a sum of a

finite number of terms, each of which is the product of a function of k by a function of k′. Such a form is assumed by the t matrix in the case of a separable two-particle potential

$$V_l^{[N]}(k,k') = -\sum_{n=1}^{N} g_{nl}(k)\, g_{nl}(k'). \tag{2.1}$$

The explicit form of the two-particle t matrix can easily be obtained by solving Eq. (1.48). In the case of one (N = 1) term in (2.1) the t matrix is determined by the expression

$$t_l(k,k';z) = -g_l(k)\, g_l(k')\left(1 - \frac{1}{2\pi^2}\int_0^\infty dq q^2 \frac{g_l^2(q)}{z-(q^2/2\mu)}\right)^{-1} \tag{2.2}$$

The potential (2.1) in the general case is nonlocal. The use of non-local potentials of the form (2.1) in three-particle problems has made it possible to develop a calculation procedure for three-particle systems[107,27]. We note that the zero-radius potential used in[28] can be regarded as separable (2.1) with $l = 0$, in which the functions g do not depend on k.

In the general case, real physical potentials are local and do not have the form (2.1). However, short-range potentials, which are usually employed for phenomenological description of interactions between nucleons, can be uniformly approximated by separable expressions of the type (2.1), leading to a separable representation for the two-particle t matrix. Indeed, using expression (2.1), we can obtain the solution of the Lippman-Schwinger equation in the form

$$t_l(k,k';z) = -\sum_{nn'} g_{nl}(k)\,\tau_{nn',l}(z)\, g_{n'l}(k'), \tag{2.3}$$

where $\tau_{nn',l}$ represents the elements of the matrix inverse to the matrix

$$(\tau^{-1})_{nn',l} \equiv \delta_{nn'} + \frac{1}{2\pi^2}\int_0^\infty dq q^2 \frac{g_{nl}(q)\, g_{n'l}(q)}{z-(q^2/2\mu)}. \tag{2.4}$$

With the aid of the separable representation of the two-particle t matrix (2.3) the system of two-dimensional integral equations for three particles could be reduced to a system of one-dimensional equations. For example, in the case of three identical spinless particles, substitution of the expression for the t matrix (2.3) in (1.52) makes it possible to represent the function $\psi_{l\lambda L}$ in the form

$$\psi_{l\lambda L}(k,p;p_0) = (2\pi)^3\left\{\varphi_{l0}(k)\, p^{-2}\delta(p-p_0)\,\delta_{l0}\delta_{\lambda L} - \Delta_l\left(Z_p - \frac{k^2}{m}\right)^{-1}\sum_{nn'} g_{nl}(k)\,\tau_{nn',l}(Z_p)\, A_{n'l\lambda L}(p,p_0)\right\} \tag{2.5}$$

and to obtain for the functions $A_{nl\lambda L}$ the following system of one-dimensional integral equations:

$$A_{nl\lambda L}(p,p_0) = W^0_{nl\lambda L,\,l_0\lambda L}(p,p_0) + \sum_{n'n''l'\lambda'}\int_0^\infty dp' p'^2 W_{nl\lambda L,\,n'l'\lambda' L}(p,p')\,\tau_{n'n'',l'}(Z_{p'})\, A_{n''l'\lambda' L}(p',p_0), \tag{2.6}$$

where we have introduced the notation

$$W^0_{nl\lambda L,\,l_0\lambda L}(p,p_0) = \frac{\Delta_l}{\pi^2}\int_{\left|p-\frac{1}{2}p_0\right|}^{p+\frac{1}{2}p_0} dk' \frac{k'}{pp_0} K^{(L)}_{l\lambda,l_0\lambda}(p,p_0;k')\, g_{nl}\left(\sqrt{k'^2+\frac{3}{4}p_0^2-\frac{3}{4}p^2}\right)\varphi_{l0}(k'). \tag{2.7}$$

$$W_{nl\lambda L,\,n'l'\lambda' L}(p,p') = \frac{\Delta_l\Delta_{l'}}{\pi^2}\int_{\left|p-\frac{1}{2}p'\right|}^{p+\frac{1}{2}p'} dk' \frac{k'}{pp'} K^{(L)}_{l\lambda,l'\lambda'}(p,p';k')\,\frac{g_{nl}(Q)\, g_{n'l'}(k')}{(k'^2/m)-Z_{p'}}.$$

There are different possible methods of approximate factorization of the potential, and consequently of the two-particle t matrix. In the Bubnov-Galerkin method an approximate factorization is obtained by choosing the solution of the Lippman-Schwinger equation in the form of a linear combination of a certain system of linearly independent functions[115]. In the Bateman method[52,16,11] the potential $V_l(k,k')$ is directly approximated by the separable potential $V_l^{[N]}(k,k')$, which is chosen in such a way that the functions $V_l$ and $V_l^{[N]}$ coincide on 2N straight lines ($k = q_j$, $k' = q_j$, i, j = 1, 2, ..., N). The Bateman separable approximation for the potential $V_l(k,k')$ can be represented[77] in the form (2.1), where

$$g_{nl}(k) = \frac{G_{nl}(k,q_n)}{\sqrt{-G_{nl}(q_n,q_n)}}, \quad G_{1l}(k,k') = V_l(k,k'),$$

$$G_{nl}(k,k') = G_{n-1,l}(k,k') - \frac{G_{n-1,l}(k,q_{n-1})\, G_{n-1,l}(q_{n-1},k')}{G_{n-1,l}(q_{n-1},q_{n-1})}. \tag{2.8}$$

We note that the separable approximation of Noyes-Kowalski[98,86] for the two-particle t matrix

$$t_l(k,k';z) \approx F_l(k,q)\, t_l(q,q;z)\, F_l(k',q), \qquad F_l(k,q) \equiv \frac{t_l(k,q;z)}{t_l(q,q;z)}, \quad q \equiv \sqrt{2\mu z} \tag{2.9}$$

is in essence the Bateman approximation of the t matrix by one term. For the values of the variable momenta on the energy shell k = q or k′ = q ($z = E + i0$, $E > 0$) the approximate expression (2.9) for the t matrix coincides with the exact one. An approximation of the type (2.9) for negative energies was considered by Kowalski[86].

In the case when the solution of the problem of the bound state of two particles can be found in explicit form, it is convenient to use for the approximation of the two-particle t matrix by a separable expression a method based on the application of the Hilbert-Schmidt theorem for the solution of symmetrical integral equations, which we shall henceforth call the Hilbert-Schmidt method[61,50,65,109,76,81,49,84,19].

A comparison of different separable representations for the two-particle t matrix at positive and negative values of the energy z is carried out in[103,87].

## 2.2. The Hilbert-Schmidt Expansion for a Two-particle t Matrix

The Lippman-Schwinger integral equation (1.48), which defines the two-particle t matrix, can be reduced with the aid of the similarity transformation to the symmetrical form

$$\bar{t}_l(z) = \bar{V}_l(z) + \bar{V}_l(z)\,\bar{t}_l(z), \tag{2.10}$$

where

$$\bar{V}_l(z) = -(H_0-z)^{-\frac{1}{2}} V_l (H_0-z)^{-\frac{1}{2}}, \quad \bar{t}_l(z) = -(H_0-z)^{-\frac{1}{2}} t_l(z)\,(H_0-z)^{-\frac{1}{2}}. \tag{2.11}$$

The idea of the Hilbert-Schmidt method consists in using the eigenfunctions of the kernel of the integral equation (2.10), i.e., the solutions of the equation

$$\bar{V}_l(z)\,\bar{g}_{nl}(z) = \eta_{nl}(z)\,\bar{g}_{nl}(z). \tag{2.12}$$

The eigenvalues $\eta_{nl}(z)$ (n—quantum numbers characterizing the eigenvalues in decreasing order of absolute magnitude) and the eigenfunctions $\bar{g}_{nl}(z)$ depend on z as

a parameter. The eigenfunctions $\bar{g}_{nl}(z)$ are chosen in Hilbert space, i.e., it is assumed that they have a finite norm

$$(\bar{g}_{nl}(z),\ \bar{g}_{n'l}(z)) = \delta_{nn'}. \tag{2.13}$$

In concrete calculations it is more convenient to use the eigenfunctions $g_{nl}(z)$ of the kernel of the nonsymmetrized integral equation (1.47), i.e., the eigenfunctions of the operator $V_l G_0(z)$:

$$V_l G_0(z)\, g_{nl}(z) = \eta_{nl}(z)\, g_{nl}(z). \tag{2.14}$$

Since the operators $V_l G_0(z)$ and $\bar{V}_l(z)$ are connected by the similarity transformation, their eigenvalues correspond, and the eigenfunctions $g_{nl}(z)$ and $\bar{g}_{nl}(z)$ are connected by the relation

$$g_{nl}(z) = (H_0 - z)^{\frac{1}{2}} \bar{g}_{nl}(z). \tag{2.15}$$

In the momentum representation, Eq. (2.14) can be written in the form

$$\int_0^\infty V_l(k, k') \left(z - \frac{k'^2}{2\mu}\right)^{-1} g_{nl}(k', z)\, \frac{k'^2\, dk'}{2\pi^2} = \eta_{nl}(z)\, g_{nl}(k, z). \tag{2.16}$$

The orthonormalization condition of the eigenfunctions $g_{nl}(k, z)$ then becomes

$$\int_0^\infty g^*_{n'l}(k, z^*)\, g_{nl}(k, z) \left(\frac{k^2}{2\mu} - z\right)^{-1} \frac{k^2\, dk}{2\pi^2} = \delta_{nn'}. \tag{2.17}$$

Although the system of eigenfunctions $g_{nl}(z)$ of the kernel of the Lippman-Schwinger integral equation (1.47) is not complete, nevertheless the solution $t_l(z)$ and the free term of the integral equation $V_l$ can always be represented in the form of expansions in terms of these functions.

We write down the solution of the integral equation (1.48) in the momentum representation in the form of the series

$$t_l(k, k'; z) = \sum_n a_{nl}(k', z)\, g_{nl}(k, z). \tag{2.18}$$

The coefficients $a_{nl}$ can easily be obtained by substituting (2.18) in (1.48) and using Eq. (2.16) and the orthonormalization condition (2.17). As a result we obtain the following separable representation for the two-particle t matrix:

$$t_l(k, k'; z) = -\sum_n \frac{\eta_{nl}(z)}{1 - \eta_{nl}(z)}\, g_{nl}(k, z)\, g^*_{nl}(k', z^*). \tag{2.19}$$

Similarly, with the aid of formulas (2.16) and (2.17) it is easy to obtain a separable representation for the interaction potential

$$V_l(k, k') = -\sum_n \eta_{nl}(z)\, g_{nl}(k, z)\, g^*_{nl}(k', z^*). \tag{2.20}$$

The expansion for the t-matrix (2.19) can be obtained by directly solving the Lippman-Schwinger equation (1.48), using an expansion for the potential (2.20) in which the value of the parameter z is chosen equal to the value of z in Eq. (1.48).

In the expansions (2.19) and (2.20), the rate of decrease of the terms with increasing n is determined by the rate of decrease of the eigenvalues $\eta_{nl}(z)$ with increasing n [the eigenfunctions $g_{nl}(k, z)$ are bounded functions of n]. If we separate in the expansion (2.19) the interaction potential in the form (2.20), then we obtain for the two-particle t matrix the following formula:

$$t_l(k, k'; z) = V_l(k, k') - \sum_n \frac{\eta^2_{nl}(z)}{1 - \eta_{nl}(z)}\, g_{nl}(k, z)\, g^*_{nl}(k', z^*). \tag{2.21}$$

The series in this expansion converges more rapidly than (2.19), since the eigenvalues $\eta_{nl}$ which decrease with n enter quadratically in (2.21). Calculations performed in[105,66] have shown that allowance for the Born term $V_l$ and the first term from the sum (2.21) (the so-called quasi-Born approximation) is a very good approxi-approximation both for the consideration of bound states and for the consideration of the scattering of two particles (even at sufficiently low energies). Expression (2.21), however, is not separable. Therefore in the consideration of three-particle systems we shall use the expansion for the t matrix in the form (2.19), confining ourselves to allowance for the first few terms.

The expansion (2.19) can easily be generalized to the case when the potential contains repulsion at small distances[80,64]. In this case there exist for the operator $V_l G_0(z)$ eigenfunctions $g^a_{n_a l}$ which correspond at $z < 0$ to positive eigenvalues $\eta^a_{n_a l} > 0$, and eigenfunctions $g^c_{n_c l}$ which correspond at $z < 0$ to negative eigenvalues $\eta^c_{n_c l} < 0$ ($n_a, n_c = 1, 2, 3, \ldots, |\eta^c_{1l}| > |\eta^c_{2l}| > |\eta^c_{3l}| > \ldots$). The functions $g^a_{n_c l}$ and $g^c_{n_c l}$ form an orthonormal system. In the case of a potential with repulsion, the separable expansion for the t matrix takes the form[80]

$$\begin{aligned} t_l(k, k'; z) = &-\sum_{n_a=1}^{\infty} \frac{\eta^a_{n_a l}(z)}{1 - \eta^a_{n_a l}(z)}\, g^a_{n_a l}(k, z)\, g^{a*}_{n_a l}(k', z^*) \\ &-\sum_{n_c=1}^{\infty} \frac{\eta^c_{n_c l}(z)}{1 - \eta^c_{n_c l}(z)}\, g^c_{n_c l}(k, z)\, g^{c*}_{n_c l}(k', z^*). \end{aligned} \tag{2.22}$$

The functions $g^c_{n_c l}$ can be regarded as eigenfunctions of the operator $-V_l G_0(z)$ corresponding to positive eigenvalues $-\eta^c_{n_c l}(z)$.

The expansion (2.20) for the potential $V_l(k, k')$ takes place at an arbitrary value of z; the total sum in (2.20), naturally, does not depend on z. What does depend on the parameter z, however, is the rate of convergence of the sum in (2.20). With increasing z, the convergence of the expansion (2.20) [just as that of the expansion for the t matrix (2.19)] becomes worse. Indeed, as $|z| \to \infty$ the eigenvalues $\eta_{nl}(z) \to 0$ and each term in the expansions (2.19) and (2.20) tends to zero. However, the sums of the series (2.19) and (2.20) remain finite ($t_l(k, k'; z) \to V_l(k, k')$).

It is possible to obtain for the two-particle t matrix a separable expansion with better convergence than that of (2.19) by starting from the expansion for the potential (2.20) with a fixed value of the parameter $z = z_0$, without equating it to the variable z in the Lippman-Schwinger equation (1.48). The parameter $z_0$ is best chosen to be negative, $z_0 \leq 0$, for in this case the eigenvalues $\eta_{nl}(z_0)$ and the eigenfunctions $g_{nl}(k, z_0)$ are real. Thus, we choose

$$V_l(k, k') = -\sum_n \eta_{nl}(z_0)\, g_{nl}(k, z_0)\, g_{nl}(k', z_0). \tag{2.23}$$

When $z_0 \leq 0$, each term in (2.23) is a real and symmetrical operator. The sum (2.23) with a finite number of terms (N terms) is a separable potential of rank N. The

corresponding two-particle t matrix takes the form (2.3) with form factors

$$g_{nl}(k) = \sqrt{\eta_{nl}(z_0)}\, g_{nl}(k, z_0). \tag{2.24}$$

The hermiticity of the separable potential (2.23) with account taken of any number of terms ensures unitarity for the corresponding t matrix in all orders of the approximation. [The Hilbert-Schmidt expansion (2.19) is unitary only if an infinite number of terms is taken into account in the sum.]

In the case when the system of two particles can be in the bound state with a binding energy $\epsilon$, it is convenient to set the parameter $z_0$ equal to the energy of the bound state, $z_0 = -\epsilon$. With such a choice of $z_0$, the binding energy and the wave function coincide with the exact ones already in the first approximation, and do not change when the succeeding terms are taken into account in the expansion (2.23).

The expansion (2.23) justifies the use of separable potentials. Thus, the first term in the expansion (2.23) with a form factor in the form $g_{10}(k, -\epsilon)$ is equivalent to the separable Yamaguchi potential[116], the parameters of which are fixed by the choice of the parameters for the initial local potential V(r). Introduction of the succeeding terms in the expansion (2.23) makes it possible to obtain further improvement of the approximation of the local potential by a sum of separable ones.

Separable expansions for the potential (2.23) and for the t matrix (2.3) with $g_{nl}(k)$ in the form (2.24) were first considered by Harms[69] and designated the unitary pole expansion. The accuracy of approximation of the two-particle t matrix by one term of this expansion was investigated for different potentials in the papers of Levinger et al.[87,55,70].

### 2.3. Properties of the Eigenvalues and Eigenfunctions

Let us consider the main properties of the eigenvalues and the eigenfunctions used in the Hilbert-Schmidt method. We note first that the kernel of the symmetrized integral equation (2.11) is hermitian only for real negative values of z. From the hermiticity of the kernel $\overline{V}_l(z)$ it follows in this case that the eigenvalues $\eta_{nl}(z)$ with $z < 0$ are always real. In the remaining cases the operator $\overline{V}_l(z)$ is not hermitian (although it remains symmetrical as before) and its eigenvalues $\eta_{nl}(z)$ are complex.

In place of the eigenfunctions g(z) it is convenient to use the functions $\psi(z)$, which are connected with the functions g(z) by the relation

$$\psi(z) = G_0(z)\, g(z) \tag{2.25}$$

and satisfy the equation

$$G_0(z)\, V\psi(z) = \eta(z)\, \psi(z). \tag{2.26}$$

[The operators $G_0(z)V$ and $VG_0(z)$ are connected with each other by the similarity transformation.] The solutions of Eq. (2.26) should be square-integrable. From the condition for the existence of such solutions at arbitrary values of z (with the exception of values lying on the real positive axis) one determines the eigenvalues $\eta(z)$.

Equation (2.26) can be rewritten in the form of the Schrödinger equation

$$\left\{H_0 + \frac{V}{\eta(z)} - z\right\} \psi(z) = 0, \tag{2.27}$$

which contains the generalized potential $V/\eta(z)$, which depends on z. At real negative values of z this generalized potential is hermitian, and the solution of Eq. (2.27), $\psi(z)$, can be regarded as the wave function of the bound state of the system with binding energy $-z$. Consequently, when $z < 0$ the eigenvalue $\eta(z)$ determines the number by which the potential V must be divided in order for the system to have a bound state with a specified binding energy $-z$. At $\eta(z) = 1$, Eq. (2.27) goes over into the usual Schrödinger equation, which admits of solutions that decrease at infinity only for definite negative values of the energy $z = -\epsilon$. Obviously, these values correspond to bound states of the system and are determined from the condition

$$\eta(-\varepsilon) = 1. \tag{2.28}$$

The wave function of the bound state of the system in the momentum representation can be written in the form

$$\varphi_{nl}(k) = N_{nl}\, \frac{g_{nl}(k, -\varepsilon_{nl})}{(k^2/2\mu) + \varepsilon_{nl}}, \tag{2.29}$$

where $N_{nl}$ is a normalization constant and $g_{nl}(k, -\epsilon_{nl})$ is an eigenfunction of Eq. (2.14). Thus, out of the entire set of wave functions $\psi_{nl}(z)$ with $z < 0$ the only physical wave functions describing the bound states of the system are the functions with such values of n and $z = -\epsilon_{nl}$ at which the eigenvalues $\eta_{nl}(z)$ become equal to unity. In the case of potentials with definite signs, the eigenvalues $\eta(z)$ for real negative values of z are of definite sign, namely, for attraction potentials ($V < 0$) the eigenvalues are positive, $\eta(z) > 0$, and are increasing functions of z; for repulsion potentials ($V > 0$), the eigenvalues are negative, $\eta(z) < 0$, and are decreasing functions of z ($(1/\eta_{nl}(z))\, d\eta_{nl}(z)/dz > 0$ when $z < 0$). According to (2.28), it follows therefore that a bound state with negative energy $z = -\epsilon_{nl}$ is possible only if

$$\eta_{nl}(0) \geqslant 1. \tag{2.30}$$

In the case of complex values of z, solutions of (2.26) are possible only for complex values of $\eta(z)$. If the eigenvalue $\eta_{nl}(z)$ at a complex energy $z_0$ with small positive imaginary part is such that

$$\operatorname{Re}\eta_{nl}(z_0) \approx 1 \text{ and } \operatorname{Im}\eta_{nl}(z_0) \ll 1, \tag{2.31}$$

then the corresponding wave function describes a quasi-discrete resonant or else a virtual state of the system.

In the coordinate representation, the determination of the eigenfunctions $\psi_{nl}(r, z) \equiv u_{nl}(r, z)/r$ and of the eigenvalues $\eta_{nl}(z)$ reduces to a solution of the differential equation

$$\left\{-\frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{l(l+1)}{2\mu r^2} + \frac{V(r)}{\eta_{nl}(z)} - z\right\} u_{nl}(r, z) = 0, \tag{2.32}$$

with boundary conditions

$$u_{nl}(r, z) \sim r^{l+1}, \quad r \to 0, \tag{2.33}$$

$$u_{nl}(r, z) \sim e^{iqr}, \quad r \to \infty, \tag{2.34}$$

where q is defined by the equation

$$q = \sqrt{2\mu z}, \quad \operatorname{Im} q > 0.$$

Let us examine the Schrödinger equation with potential gV(r), in which for convenience we have separated

the coupling constant. The solution of such an equation $\varphi_l(r, q, g)$ at an energy $z = q^2/2\mu$, satisfying the boundary condition (2.33) can be represented in the form

$$\varphi_l(r, q, g) \sim f_l(-q, g) f_l(r, q, g) - f_l(q, g) f_l(r, -q, g), \quad (2.35)$$

where $f_l(\pm q, g)$ is the Jost function and $f_l(r, \pm q, g)$ are the solutions of the Schrödinger equation with boundary conditions at infinity

$$\lim_{r\to\infty} e^{\pm iqr} f_l(r, \pm q, g) = 1. \quad (2.36)$$

We recall that the Jost function $f_l(q, g)$ is connected with the corresponding solution $f_l(r, q, g)$ by the relation

$$f_l(q, g) \equiv \lim_{r\to 0} (2l+1) r^l f_l(r, q, g). \quad (2.37)$$

Obviously, the solution of Eq. (2.32) can be written in the form (2.25), where g should be taken to mean the generalized coupling constant $g/\eta(z)$. Since the proper solution of (2.32), unlike (2.35), should contain at infinity only a diverging wave, the dependence of $\eta$ on z can be obtained from the condition that the coefficient of the converging wave vanish

$$f_l\left(-q, \frac{g}{\eta}\right) = 0. \quad (2.38)$$

The roots of these equations determine the eigenvalues $\eta_{nl}(z)$.

As is well known[75], the Jost function $f_l(-q, g)$ coincides with the Fredholm determinant $D_l(z, g)$ of the integral equation (2.10). Noting that the Fredholm determinant $D_l(z, g)$ is represented in terms of the kernel of the equation $\overline{V}_l(z)$ in the form of an infinite series

$$D_l(z, g) \equiv 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int_0^{\infty} dk_1 \ldots \int_0^{\infty} dk_m \begin{vmatrix} \overline{V}_l(k_1, k_1; z) & \overline{V}_l(k_1, k_2; z) & \ldots & \overline{V}_l(k_1, k_m; z) \\ \overline{V}_l(k_2, k_1; z) & \overline{V}_l(k_2, k_2; z) & \ldots & \overline{V}_l(k_2, k_m; z) \\ \ldots & \ldots & \ldots & \ldots \\ \overline{V}_l(k_m, k_1; z) & \overline{V}_l(k_m, k_2; z) & \ldots & \overline{V}_l(k_m, k_m; z) \end{vmatrix}, \quad (2.39)$$

and using the expansion (2.20), we get

$$D_l(z, g) = \prod_n (1 - \eta_{nl}(z)). \quad (2.40)$$

Thus, the Jost function $f_l(-q, g)$ is expressed directly in terms of the eigenvalues $\eta_{nl}(z)$:

$$f_l(-q, g) = \prod_n (1 - \eta_{nl}(z)). \quad (2.41)$$

We note that the relation (2.20) is linear in g and $\eta_{nl}(z)$ and it is easy to verify that

$$f_l\left(-q, \frac{g}{\eta}\right) = \prod_n \left(1 - \frac{\eta_{nl}(z)}{\eta}\right), \quad (2.42)$$

where $\eta$ is an arbitrary parameter. Equating the left side of (2.42) to zero in accordance with (2.38), we actually obtain an equation for finding the eigenvalues $\eta_{nl}(z)$.

Relation (2.41) enables us to find the connection between the phase for the scattering of two particles $\delta_l(q)$, in terms of which the component of the two-particle t matrix is expressed on the energy shell, and the eigenvalues $\eta_{nl}(E + i0)$ at $E = q^2/2\mu > 0$. Indeed, the scattering phase shift $\delta_l(q)$ is an argument of the Jost function $f_l(q, g)$. Therefore, recognizing that $\delta_l(q)$ is an odd function of q, and using (2.41), we find

$$\delta_l(q) = -\sum_n \arg(1 - \eta_{nl}(E + i0)). \quad (2.43)$$

The Jost function $f_l(-q, g)$ for complex values of g is an entire function of g for any value of q or z. Therefore the eigenvalues $\eta_{nl}(z)$ at a fixed value of the energy z, determined by the zeroes of the Jost function (2.38), form a discrete set; only a finite number of eigenvalues $\eta_{nl}(z)$ will lie outside a circle of finite radius. Owing to the discreteness of the eigenvalues $\eta_{nl}(z)$, they can be renumbered with the aid of the integer number n (in order of decreasing absolute magnitude of $\eta_{nl}$), as has indeed been assumed implicitly thus far.

The analytic properties of the Jost function $f_l(-q, g)$ in the plane of the complex energy z are determined by the behavior of the potential V(r). We shall assume that the singularity of the potential at the point r = 0 is weaker than $r^{-2}$, and that at infinity the potential decreases sufficiently rapidly (for example, more rapidly than $r^{-3}$). In this case the Jost function $f_l(-q, g)$ is analytic and has no singularities on the physical sheet of the Riemann surface of the complex z, i.e., on the entire complex z plane, with the exception of a cut along the real positive axis. The analytic properties of the eigenvalues $\eta_{nl}(z)$ follow from the analytic properties of the Jost function $f_l(-q, g)$. Since the Jost function $f_l(-q, g)$ is analytic in both variables and in the general case $\partial f_l/\partial g \neq 0$, Eq. (2.38) has a unique solution $\eta_{nl}(z)$, which is an analytic function of z in the region of analyticity of the function $f_l(-q, g)$. Thus, the eigenvalues $\eta_{nl}(z)$ of the Hilbert-Schmidt operator are, like the Jost function $f_l(-q, g)$, analytic functions without singularities on the entire complex z plane, with the exception of a cut along the real positive axis.

From the reality of $\eta_{nl}(z)$ at $z < 0$ it follows that the Schwartz reflection principle holds for complex z

$$\eta_{nl}^*(z) = \eta_{nl}(z^*). \quad (2.44)$$

We note that at real negative values of z it is possible to make the functions $g_{nl}(k, z)$ real by a suitable choice of the phase factor. In this case the reflection principle also holds for the eigenfunctions

$$g_{nl}^*(k, z) = g_{nl}(k, z^*). \quad (2.45)$$

As noted above, the eigenvalues $\eta_{nl}(z)$ are real only on the real negative axis. Therefore, taking into account the sign-definite character of the imaginary part of $\eta_{nl}(z)$ in the upper and lower half-planes of z and taking into account the inequality $\eta_{nl}(z)\, d\eta_{nl}(z)/dz > 0$, which holds for real negative values of z, we find that in the upper half of the z plane

$$\operatorname{Im}\eta_{nl}(z) > 0 \quad \text{for} \quad V < 0 \quad \text{and} \quad \operatorname{Im}\eta_{nl}(z) < 0 \quad \text{for} \quad V > 0. \quad (2.46)$$

In the lower half of the z plane the imaginary part of $\eta_{nl}(z)$ has the opposite sign.

Using the independence of the trace of an operator of the choice of the representation, we find

$$\sum_{n=1}^{\infty} (\eta_{nl}(z))^p = \operatorname{Sp}(G_{0l}(z) V)^p, \qquad p = 1, 2, 3, \ldots \quad (2.47)$$

For p = 1 we have

$$\sum_{n=1}^{\infty} \eta_{nl}(z) = -2i\mu q \int_0^{\infty} j_l(qr) V(r) h_l^{(1)}(qr) r^2 dr, \quad q = \sqrt{2\mu z}. \quad (2.48)$$

The integral in the right-hand side of (2.48) converges

if the potential is characterized by a finite radius of action and the singularity of the potential at the point r = 0 is weaker than $r^{-2}$. To ensure the convergence of the series in the left side of (2.48) in this case, the eigenvalues $\eta_{nl}(z)$ should decrease with increasing n like $n^{-\gamma}$, where $\gamma > 1$. For $z < 0$, the dependence of $\eta_{n0}(z)$ on n for large values of n can easily be determined in the quasiclassical approximation, using the Bohr-Sommerfeld quantization rule

$$\sqrt{2\mu}\int_0^{\rho}\sqrt{-\frac{V(r)}{\eta_{n0}(z)}+z}\,dr = \pi n, \tag{2.49}$$

where $\rho$ is determined from the condition $V(\rho) = z\eta_{n0}(z)$. For sufficiently large n

$$|z|\,\eta_{n0}(z) \ll V_0 \tag{2.50}$$

($V_0$ is the depth of the potential), and therefore the integration in (2.49) can be continued to infinity; it is possible to neglect the second term under the square-root sign. As a result we obtain

$$\eta_{n0}(z) \approx \frac{2\mu V_0 \tilde{R}^2}{\pi^2 n^2}, \tag{2.51}$$

where $\tilde{R} = \int_0^{\infty}\sqrt{-V(r)/V_0}\,dr$ is the effective radius of action of the potential. Using (2.51), we can rewrite (2.50) in the form

$$\sqrt{-2\mu z}\,\tilde{R} \ll \pi n. \tag{2.52}$$

## 2.4. Eigenvalues and Eigenfunctions for a Hulthen Potential and for a Square Well

By way of illustration, we present explicit expressions for the eigenvalues and eigenfunctions in the case of a Hulthen potential and a square potential well.

a) Hulthen potential

$$V(r) = -V_0(e^{r/R}-1)^{-1}, \quad V_0 \equiv \frac{g}{2\mu R^2}. \tag{2.53}$$

At small distances ($r \ll R$) the potential (2.53) is characterized by a singularity of the type $r^{-1}$, and at large distances ($r \gg R$) it decreases exponentially.

We confine ourselves to spherically-symmetrical states ($l = 0$, the index $l$ is omitted). For the Hulthen potential, the Jost function takes the form

$$f(-q, g) = \prod_{n=1}^{\infty}\left(1-\frac{g}{n(n-2iqR)}\right). \tag{2.54}$$

The zeroes of the Jost function (2.54) determine the energies of the bound states:

$$E_n = -\frac{\varkappa_n^2}{2\mu}, \quad \varkappa_n = \frac{1}{2R}\left(\frac{g}{n}-n\right), \quad n = 1, 2, \ldots \tag{2.55}$$

The corresponding functions of the bound states are

$$\varphi_n(k) = \frac{N_n}{k^2+\varkappa_n^2}\sum_{\nu=1}^{n} a_{n\nu}\frac{\nu(\nu+2\varkappa_n R)}{(kR)^2+(\nu+\varkappa_n R)^2},$$
$$a_{n\nu} = (-1)^{\nu+1}\frac{n!}{\nu!\,(n-\nu)!}\frac{\Gamma\left(\frac{g}{n}+\nu\right)\Gamma\left(\frac{g}{n}-n+1\right)}{\Gamma\left(\frac{g}{n}\right)\Gamma\left(\frac{g}{n}-n+\nu+1\right)}. \tag{2.56}$$

Knowing the Jost function, we can easily find in accord with (2.38) the eigenvalues of the kernel of the Lippman-Schwinger equation

$$\eta_n(z) = \frac{g}{n(n-2i\sqrt{2\mu z}\,R)}, \quad \operatorname{Im}\sqrt{z} \geqslant 0. \tag{2.57}$$

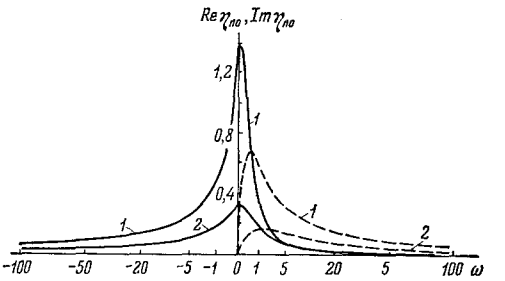


Fig. 1. Dependence of Re $\eta_{n0}$ (solid curves and Im $\eta_{n0}$ (dashed curves) ofi the parameter $\omega \equiv 2\mu z R^2$ for the Hulthen potential (g = 1.403). The numbers next to the figures indicate the values of n.

The corresponding eigenfunctions, which are solutions of (2.16), are of the form

$$\left.\begin{aligned} g_n(k, z) &= C_n(z)\sum_{\nu=1}^{n} A_{n\nu}(z)\frac{g}{\eta_\nu(z)}\frac{1}{k^2R^2-2\mu z R^2+\frac{g}{\eta_\nu(z)}}, \\ A_{n\nu}(z) &= (-1)^{\nu+1}\prod_{\sigma=1}^{n}\frac{n-\sigma+1}{n+\sigma-1}\frac{\eta_\sigma(z)}{\eta_{n+\sigma-1}(z)}, \\ C_1^2(z) &= \frac{\pi g R}{\mu\eta_2(z)}, \quad C_n^2(z) = \frac{\pi g^{2n-1}Rn}{\mu(n!)^4}\left[\eta_{2n}(z)\prod_{\nu=1}^{n-1}\eta_\nu^2(z)\right]^{-1}. \end{aligned}\right\} \tag{2.58}$$

The functions (2.58) are normalized in accordance with the condition (2.17).

When the parameter z changes from $-\infty$ to 0, the eigenvalues $\eta_n(z)$ are real and change from 0 to $g/n^2$. The bound states correspond to values of z for which $\eta_n(z) = 1$. With further change of z from 0 to $\infty + i0$ (along the upper edge of the cut), the eigenvalues $\eta_n(z)$ become complex and describe on the complex plane semicircles of radius $g/2n^2$ with centers lying on the real axis at the points $g/2n^2$. The real and imaginary parts of the eigenvalues $\eta_n(z)$ are shown in Fig. 1 for the energy parameter z ranging from $-\infty$ to $+\infty + i0$, for the case g = 1.4 (corresponding to interaction of the nucleons in the triplet spin state).

b) Square potential well

$$V(r) = \begin{cases} -V_0, & r < R, \\ 0, & r > R, \end{cases} \quad V_0 \equiv \frac{g}{2\mu R^2}. \tag{2.59}$$

In the case of a square well, the Jost function f(-q, g) takes the form

$$f(-q, g) = e^{iqR}\left\{\cos\sqrt{g+q^2R^2} - i\frac{qR}{\sqrt{g+q^2R^2}}\sin\sqrt{g+q^2R^2}\right\}. \tag{2.60}$$

The energies of the bound states are determined by the roots of the transcendental equation

$$\sqrt{g-\varkappa^2R^2}\operatorname{ctg}\sqrt{g-\varkappa^2R^2} = -\varkappa R. \tag{2.61}$$

The corresponding wave functions of the bound states are equal to

$$\varphi_n(k) = \frac{N_n}{k^2+\varkappa_n^2}\frac{\cos kR + (\varkappa_n/k)\sin kR}{g-(\varkappa_n^2+k^2)R^2}. \tag{2.62}$$

The eigenvalues $\eta_n(z)$ in the case of a square potential well can be found by solving the transcendental equation

$$\sqrt{\frac{g}{\eta}+q^2R^2}\operatorname{ctg}\sqrt{\frac{g}{\eta}+q^2R^2} = iqR. \tag{2.63}$$

At small z, the eigenvalues $\eta_n(z)$ can be obtained in explicit form:

$$\eta_n(z) \approx \frac{4g}{(2n-1)^2\pi^2}\left\{1+\frac{8i}{(2n-1)^2\pi^2}qR+\frac{4}{(2n-1)^2\pi^2}\left(1-\frac{20}{(2n-1)^2\pi^2}\right)(qR)^2\right\},$$
$$q \equiv \sqrt{2\mu z}, \quad |q|\,R \ll 1. \tag{2.64}$$

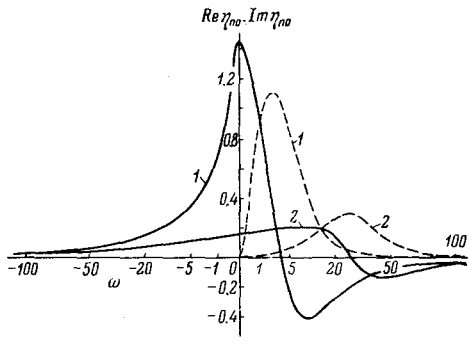


Fig. 2. Dependence of Re $\eta_{n0}$ (solid curves) and Im $\eta_{n0}$ (dashed curves on the parameter $\omega \equiv 2\mu z R^2$ for a square well ( g = 3.608). The numbers next to the curves indicate the values of n.

The eigenfunctions $g_n(k, z)$ for a square well are determined by the formula

$$\left.\begin{aligned} g_n(k, z) &= C_n(z)\,\frac{g}{\eta_n(z)}\,\frac{\cos kR - i\,(q/k)\sin kR}{(g/\eta_n(z)) + (q^2-k^2)\,R^2}, \\ C_n^2(z) &= \frac{4\pi R}{\mu}\,\frac{1+q^2R^2\,(\eta_n(z)/g)}{(g/\eta_n(z)) - iqR}. \end{aligned}\right\} \qquad (2.65)$$

The real and imaginary parts of the eigenvalues $\eta_n(z)$ for a square well with z varying along the real axis from $-\infty$ to $+\infty + i0$ are shown in Fig. 2.

## 2.5. Convergence of Separable Hilbert-Schmidt Expansion

Let us investigate the convergence of the Hilbert-Schmidt expansion for the two-particle t matrix using negative values of z as an example. If $z < 0$, then the kernel $\overline{V}_l(k, k'; z)$ of the integral equation (2.10) is symmetrical and real. For short-range potentials having a singularity weaker than $r^{-2}$ as $r \to 0$, the kernel and the free term of the integral equation (2.10) are square-integrable:

$$\int_0^\infty \int_0^\infty dk\,dk' \,|\overline{V}_l(k, k', z)|^2 < \infty. \qquad (2.66)$$

In this case it can be shown that on going over to a finite integration interval in the Lippman-Schwinger equation by a suitable change of variables, the kernel of the resultant symmetrized equation is continuous and bounded in the chosen interval. In the case of positive eigenvalues $\eta_{nl} > 0$ (attractive potentials) the series (2.20) for such a kernel [and consequently also the series (2.19) for the t matrix] converges absolutely and uniformly with respect to both variables k and k′ (the Mercer theorem[18]).

For attractive potentials $V(r) < 0$, each kernel of the expansion (2.20) is a separable attraction potential. Therefore further refinement of the expansion, i.e., the use of a larger number of terms in the Hilbert-Schmidt expansion, corresponds to introduction of a stronger attraction.

In the expansion (2.19) of the t matrix, the rate of decrease of the terms with increasing n is determined by the rate of decrease of the eigenvalues $\eta_{nl}(z)$ with n (the eigenfunctions are bounded functions of n). Therefore the convergence of the Hilbert-Schmidt expansion increases with weakening of the singularity of the potential and with decrease of its range. At z = 0, the eigenvalues $\eta_{n_0}(0)$ for n = 1, 2, 3, and 4 are related like

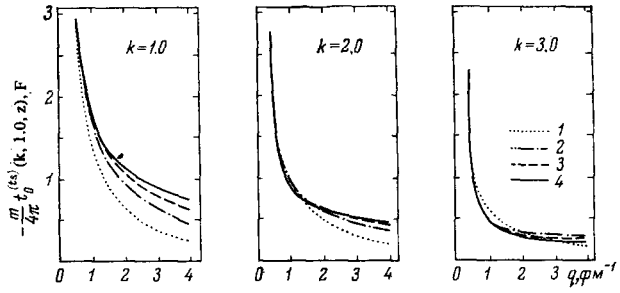


Fig. 3. Dependence of the partial two-particle t matrix ($l = 0$) on the parameter $q \equiv \sqrt{m|z|}$ ($z < 0$) for three values of the momentum k ($k' = 1.0\ F^{-1}$) in the case of a Hulthen potential. The numbers next to the curves indicate the order of the approximation (the number of terms in the separable expansion (2.19) taken into account in the calculation). The parameters (a) of Table II for the triplet-singlet and spin-isospin state were used.

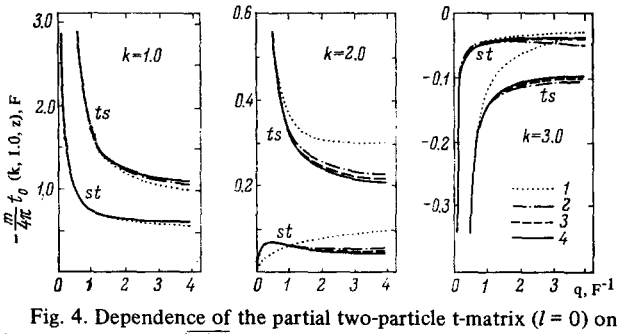


Fig. 4. Dependence of the partial two-particle t-matrix ($l = 0$) on the parameter $q \equiv \sqrt{m|z|}$ ($z < 0$) in the case of a square-well potential. We used the values of the parameters (a) for the triplet-singlet state (curves ts). See the caption of Fig. 2.

1 : 1/4 : 1/9 : 1/16 in the case of a Hulthen potential and like 1 : 1/9 : 1/25 : 1/49 in the case of a square well. [At large n, the eigenvalues $\eta_{n_0}(0)$ for all the short-range potentials decrease like $n^{-2}$.]

In the case of a square well, the dependence of the eigenvalues $\eta_{nl}(0)$ with $l > 0$ on n and $l$ can readily be obtained in explicit form by using the conditions for the appearance of levels with zero energy

$$j_{l-1}\left(\sqrt{\frac{g}{\eta_{nl}(0)}}\right) = 0$$

and the asymptotic expansion of the spherical Bessel function. As a result we have

$$\eta_{nl}(0) \approx \frac{4g}{\pi^2\,(2n+l-1)^2}, \quad (l-1)\,(l-\pi) < 2\pi n. \qquad (2.67)$$

In a system of two nucleons, owing to their identity, only even triplet-singlet and singlet-triplet spin-isospin states are possible. For such states the largest contribution to the separable expansion of the t matrix is made by the terms with n = 1, $l = 0$; n = 2, $l = 0$; n = 1, $l = 2$, etc.; the corresponding eigenvalues in decreasing order are $\eta_{10}(0) \sim 1$; $\eta_{20}(0)$, $\eta_{12}(0) \sim 1/9$; $\eta_{30}(0)$, $\eta_{22}(0)$, $\eta_{14}(0) \sim 1/25$ etc. In spite of the fact that the eigenvalues $\eta_{20}$ and $\eta_{12}$ are of the same order of magnitude, the contribution of the term with n = 1, $l = 2$ to the expansion of the two-particle t matrix $t_1(k, k'; z)$ at small k or k′ is much smaller than the contribution of the term with n = 2, $l = 0$. [Owing to the finite radius of action of the forces, the functions $g_{nl}(k, z)$, and consequently also the partial components $t_l(k, k'; z)$ at small k or k′ decrease rapidly with increasing $l$, $g_{nl}(k, z) \sim k^l$.]

The convergence of the expansions of the two-particle t matrix (2.19) for different values of the momentum k and negative values of z ($q \equiv \sqrt{m|z|}$) in the case $l = 0$ is shown for the Hulthen potential in Fig. 3 and for the square well in Fig. 4. As follows from Fig. 4, in the case of a square well the values of the two-particle t matrix calculated with the aid of (2.19) with allowance for two terms, turn out to be very close to the exact values.

## 3. BOUND STATES AND SCATTERING IN A SYSTEM OF THREE IDENTICAL SPINLESS PARTICLES

### 3.1. Separable Expansion of Two-particle t Matrix and One-dimensional Integral Equations for a System of Three Identical Spinless Particles

We now use the introduced separable expansion of a two-particle t matrix to solve the three-particle problem. We stop first to discuss the simplest case of three identical spinless particles. We start with the system of two-dimensional integral equations (1.52). With the aid of the Hilbert-Schmidt expansion (2.19) for the two-particle t-matrix we can reduce the system (1.52) to a system of one-dimensional integral equations. Indeed, substituting (2.19) in (1.52) and using expression (2.29) for the wave function of two particles in the bound state, we can represent the function $\psi_{l\lambda L}$ in the form

$$\psi_{l\lambda L}(k, p; p_0) = (2\pi)^3 \sum_n N_{10} \frac{g_{nl}(k, Z_p)}{\frac{k^2}{m} - Z_p} \left\{ \delta_{nl}\delta_{l0}\delta_{\lambda L} \frac{\delta(p-p_0)}{p^2} + \tau_{nl}(Z_p)\, a_{nl\lambda L}(p, p_0) \right\}, \tag{3.1}$$

where the functions $a_{nl\lambda L}(p, p_0)$ satisfy the following system of integral equations:

$$a_{nl\lambda L}(p, p_0) = U_{nl\lambda L,\, 10LL}(p, p_0; Z) + \sum_{n'l'\lambda'} \int_0^\infty U_{nl\lambda L,\, n'l'\lambda' L}(p, p'; Z)\, \tau_{n'l'}(Z_{p'})\, a_{n'l'\lambda' L}(p', p_0)\, p'^2 dp', \tag{3.2}$$

$$U_{nl\lambda L,\, n'l'\lambda' L}(p, p'; Z) \equiv \frac{\Delta_l \Delta_{l'}}{2\pi^2} \int_{-1}^{1} dy K^{(L)}_{l\lambda, l'\lambda'}(p, p'; y) \frac{g_{nl}(Q, Z_p)\, g_{n'l'}(Q', Z_{p'})}{\frac{1}{m}(p^2 + p'^2 + pp'y) - Z},$$

$$Q = \sqrt{\tfrac{1}{4}p^2 + p'^2 + pp'y}, \quad Q' = \sqrt{p^2 + \tfrac{1}{4}p'^2 + pp'y}. \tag{3.3}$$

(In the determination of the quantities $U_{nl\lambda L, n'l'\lambda' L}$ we used the property $\Delta_l^2 = \Delta_l$.) The factor $\Delta_l$ vanishes for odd values of $l$, and therefore all the components $a_{nl\lambda L}$ with odd $l$ vanish. The function $K^{(L)}_{l\lambda, l'\lambda'}(p, p'; y)$, which enters in (3.3), is determined by expression (1.54), in which the angles $\theta$, $\vartheta$ and $\vartheta'$ are expressed in terms of the variable y by means of the relations

$$\cos\theta = y, \quad \cos\vartheta = \frac{\frac{1}{2}p + p'y}{\sqrt{\frac{1}{4}p^2 + p'^2 + pp'y}}, \quad \cos\vartheta' = \frac{p + \frac{1}{2}p'y}{\sqrt{p^2 + \frac{1}{4}p'^2 + pp'y}}. \tag{3.4}$$

The amplitude of elastic scattering of a particle by a system of two particles that are in the bound state is equal to

$$f(p, p_0) = \left[ \frac{p^2 - p_0^2}{4\pi} \int \varphi(k)\, \Psi_1(k, p; p_0) \frac{dk}{(2\pi)^3} \right]_{p=p_0}. \tag{3.5}$$

Using formulas (1.42), (1.51), and (3.1) we can show that the partial amplitude $f_L(p_0, p_0)$ is expressed directly in terms of the quantity $a_{10LL}$ at $p = p_0$:

$$f_L(p_0, p_0) = \frac{2\pi m}{3} \left[ \frac{d\eta_{10}(z)}{dz} \right]^{-1}_{z=-\varepsilon_{10}} a_{10LL}(p_0, p_0). \tag{3.6}$$

Thus, the wave function of the system of three particles $\psi_{l\lambda L}$ is represented in terms of the functions $a_{nl\lambda L}(p, p_0)$ which are in essence the amplitudes for the scattering of the particle off the energy shell by a system of two other particles in the bound state with quantum numbers $n_0 = 1$ and $l_0 = 0$ (the final state is described by the numbers $nl$). The functions $U_{nl\lambda L, n'l'\lambda' L}$ play the role of the matrix elements of the effective interaction potential of the particle with the system of two particles in the bound state, and the functions $\tau_{nl}(Z_p)$ are the propagation functions of the free particle and of the pair of interacting particles in the bound state with quantum numbers $nl$. The effective potential depends on the energy and is nonlocal.

In the case of zero energy of the incident particle ($p_0 = 0$) the free term of (3.2) equals, in accord with (3.3) and (1.54)

$$U_{nl\lambda L,\, 10LL}(p, 0; -\varepsilon_{10}) = \delta_{L0}\delta_{l\lambda} U_{nll_0,\, 1000}(p, 0; -\varepsilon_{10}), \tag{3.7}$$

and consequently the only nonvanishing amplitudes are $a_{nl\lambda L}(p, 0)$ with zero total orbital angular momentum $L = 0$ and with $l = \lambda$:

$$a_{nl\lambda L}(p, 0) = \delta_{L0}\delta_{l\lambda} a_{nll_0}(p, 0). \tag{3.8}$$

The integral equations determining the functions $a_{nll_0}(p, 0) \equiv a_{nl}(p, 0)$ are

$$a_{nl}(p, 0) = U_{nl,\, 10}(p, 0; -\varepsilon_{10}) + \sum_{n'l'} \int_0^\infty U_{nl,\, n'l'}(p, p'; -\varepsilon_{10})\, \tau_{n'l'}\left(-\varepsilon_{10} - \frac{3}{4}\frac{p'^2}{m}\right) a_{n'l'}(p', 0)\, p'^2 dp', \tag{3.9}$$

where

$$U_{nl,\, n'l'}(p, p'; Z) \equiv U_{nll_0,\, n'l'l'0}(p, p'; Z) = \frac{\Delta_l \Delta_{l'}}{2\pi^2} (2l+1)^{1/2}(2l'+1)^{1/2}(-1)^{l+l'} \int_{-1}^{1} P_l(\cos\vartheta) \frac{g_{nl}(Q, Z_p)\, g_{n'l'}(Q', Z_{p'})}{\frac{1}{m}(p^2 + p'^2 + pp'y) - Z} \times P_{l'}(\cos\vartheta')\, dy. \tag{3.10}$$

The scattering length, i.e., the amplitude for elastic scattering at zero energy with minus sign, is determined by the expression

$$A = -\frac{2\pi}{3} m \left[ \frac{d\eta_{10}}{dz} \right]^{-1}_{z=-\varepsilon_{10}} a_{10}(0, 0).$$

We write down also the integral equations for the bound state of a system of three particles with total orbital angular momentum L and projection M. Using the separable expansion (2.19) of the two-particle t matrix, we can represent the functions $\psi_{l\lambda L}$ satisfying the homogeneous system of equations corresponding to (1.52) in the form

$$\psi_{l\lambda L}(k, p) = \sum_n \frac{g_{nl}(k, Z_p)}{\frac{k^2}{m} - Z_p} \tau_{nl}(Z_p)\, a_{nl\lambda L}(p). \tag{3.11}$$

For the partial amplitudes $a_{nl\lambda L}(p)$ we obtain a homogeneous system of one-dimensional integral equations

$$a_{nl\lambda L}(p) = \sum_{n'l'\lambda'} \int_0^\infty U_{nl\lambda L,\, n'l'\lambda' L}(p, p'; Z)\, \tau_{n'l'}(Z_{p'})\, a_{n'l'\lambda' L}(p')\, p'^2 dp'. \tag{3.12}$$

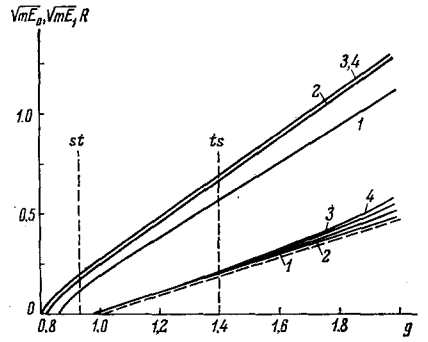


Fig. 5. Dependence of the quantities $\sqrt{mE_0}R$ and $\sqrt{mE_1}R$ ($E_0$ and $E_1$ are the binding energies of the system of three identical spinless particles in the ground and first-excited states with L = 0) on the effective depth of the two-particle interaction g in the case of a Hulthen potential. The numbers next to the curves indicate the order of the approximation. The dashed curve pertains to the system of two particles (the dependence of $\sqrt{m\epsilon}R$ on g, $\epsilon$ is the binding energy of two particles in the ground state with $l = 0$). The values of g are marked for the triplet-singlet (ts) and the singlet-triplet (st) state of two nuclei.

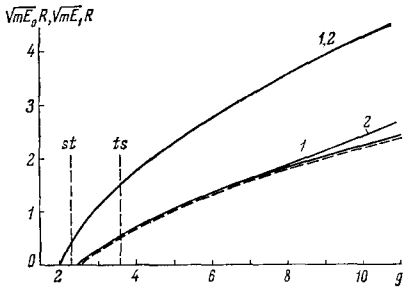


Fig. 6. Dependence of the quantities $\sqrt{mE_0}R$ and $\sqrt{mE_1}R$ on the effective depth of the two-particle interaction g in the case of a square-well potential. See the caption to Fig. 5.

If the total orbital angular momentum of the system is equal to zero, L = 0, then

$$\left.\begin{aligned} U_{nl\lambda 0,\, n'l'\lambda' 0}(p,\, p';\, Z) &= \delta_{l\lambda}\delta_{l'\lambda'} U_{nl,\, n'l'}(p,\, p';\, Z), \\ a_{nl\lambda 0}(p) &= \delta_{l\lambda} a_{nl}(p), \end{aligned}\right\} \qquad (3.13)$$

and Eqs. (3.12) are reduced to the form

$$a_{nl}(p) = \sum_{n'l'} \int_0^\infty U_{nl,\, n'l'}(p,\, p';\, Z)\, \tau_{n'l'}(Z_{p'})\, a_{n'l'}(p')\, p'^2\, dp' . \qquad (3.14)$$

The systems of one-dimensional integral equations (3.9) and (3.14) admit of a numerical solution.

### 3.2. Binding Energy and Scattering Length

We present the results of the numerical solution of the obtained systems of integral equations (3.9) and (3.14) in the case when the interaction between particles is described by a Hulthen potential[109,76] or by a square-well potential[81,37]. We calculated the binding energies for a system of three identical particles in the ground and first-excited states with total angular momentum equal to zero (L = 0), $E_0$ and $E_1$ and the scattering length A of one particle by a system of two particles in the bound state. To study the convergence of the solution when a separable expansion is used for the two-particle t matrix, we took into account different numbers of terms in the t matrix expansion.

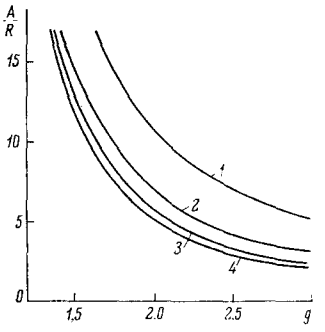


Fig. 7. Dependence of the scattering length A of one particle by a system of two particles in the bound state on g in the case of three identical spinless particles for a Hulthen potential. The numbers next to the curves indicate the order of the approximation.

Figures 5 and 6 show plots of the quantities $\sqrt{mE_0}R$ and $\sqrt{mE_1}R$ on the effective depth of the two-particle interaction g for a Hulthen potential and for a square well. The dependence of the scattering length A on g for the Hulthen potential is shown in Fig. 7. In the calculation of the functions shown in Figs. 5–7, we took into account the interaction between the pairs of particles only in the S states ($l = 0$). Different curves were obtained with allowance for different numbers of terms in the separable expansion (2.19) [the numbers next to the curves indicate the order of the approximation, i.e., the number of terms of expansion (2.19) taken into account in the calculation].

In the case of a Hulthen potential, curves 3 and 4 for the binding energy of the ground state of the system of three particles practically coincide; consequently, we can confine ourselves in the calculation to only three terms in the separable expansion (2.19). At large values of the effective depth g, curve 4 goes over asymptotically into a straight line, i.e., the calculated dependence of $\sqrt{E_0}$ on g agrees with the linear dependence

$$\sqrt{mE_0}\,R = Cg - \tfrac{3}{4}C^{-1}, \qquad (3.15)$$

obtained for the Hulthen potential in the strong-coupling approximation in[22], at a value C = 1.03. In the case of a square well the convergence of the solution using separable expansion (2.19) is even better than in the case of the Hulthen potential. As follows from Fig. 6, the values of the binding energy of the three particles $E_0$ calculated with allowance for one and two terms in the expansion (2.19) are very close.

The calculations also point to the existence, besides the bound ground S-state of the three-particle system, of an excited S-state for all values of the parameter g at which the formation of the bound ground S-state of the two particles is possible. For a large interval of the parameter g, the binding energy of the first excited three-particle state $E_1$ slightly exceeds the binding energy $\epsilon$ of the ground state of two particles. An excited three-particle state is possible in the case of paired potentials of different form (separable Yamaguchi potential[41,77], exponential potential[72], Hulthen potential[76,84], square well[37]).

The values of the binding energy and the scattering length $E_0$ and A corresponding to the parameters of the interaction potential of the two identical spinless particles coinciding with the interaction parameters of the two nucleons in the triplet-singlet spin-isospin state are listed in Table I. The numbers 1, 2, 3, and 4 denot-

**Table I.** Binding energy of three particles and scattering length of particle by a system of two particles in the bound state in the case when the particles are identical and spinless

| | Hulthen potential (triplet-singlet parameters (a)) | | | | Square-well potential (triplet-singlet parameters (c)) | | |
|---|---|---|---|---|---|---|---|
| | $l = 0$ | | | | $l = 0$ | | $l = 0.2$ |
| | 1 | 2 | 3 | 4 | 1 | 2 | 1 |
| $E_0$, MeV | 18.37 | 25.74 | 27.13 | 27.41 | 20.42 | 20.64 | 20.44 |
| A, F | 20.68 | 14.92 | 13.53 | 12.85 | 29.23 | 28.34 | 29.14 |

**Table II.** Parameters for a Hulthen potential and for a square well, obtained from the data on the interaction of two nucleons at low energies ($v_0 \equiv mV_0/h^2$)

| | $v_{0\,ts}$, $F^{-2}$ | $R_{ts}$, F | $v_{0st}$, $F^{-2}$ | $R_{st}$, F | Two-nucleon parameters |
|---|---|---|---|---|---|
| Hulthen potential | 1.8509 | 0.8708 | 1.3493 | 0.8317 | (a) |
| Square-well potential | 0.8513 | 2.043 | 0.3390 | 2.586 | (a) |
| | 0.8178 | 2.093 | 0.3770 | 2.457 | (b) |
| | 0.7945 | 2.131 | 0.3380 | 2.590 | (c) |

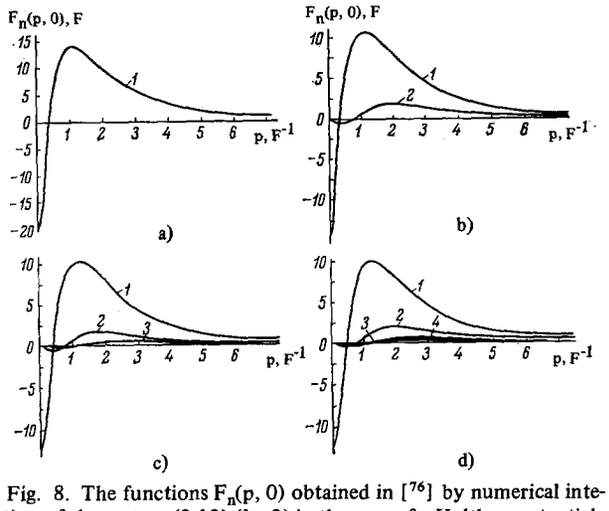


Fig. 8. The functions $F_n(p, 0)$ obtained in [76] by numerical integration of the system (3.12) ($l = 0$) in the case of a Hulthen potential with allowance for one (a), two (b), three (c) and four (d) terms in the expansion of the t matrix (2.19). The numbers next to the curves indicate the values of n. The parameters (a) of Table II for the triplet-singlet state of two nucleons were used.

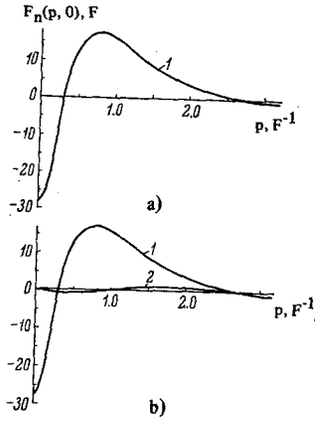


Fig. 9. The functions $F_n(p, 0)$ obtained in [37] by numerical integration of the system (3.12) ($l = 0$) in the case of a square well with allowance for one (a) and two (b) terms in the expansion (2.19). The numbers next to the curves indicate the values of n. The parameters (c) of Table II for the triplet-singlet state of two nucleons were used.

ing the columns of the table indicate the numbers of terms in the expansion (2.19) which were included in the calculation. (The two-nucleon parameters are given in Table II.)

Figures 8 and 9 show the components of the succeeding approximations of the amplitude

$$F_n(p, 0) \equiv \frac{\pi}{2} p^2 \tau_{n0} \left( -\frac{\varkappa^2}{m} - \frac{3}{4}\frac{p^2}{m} \right) a_{n0}(p, 0)$$

for a Hulthen potential (four approximations) and for a square well (two approximations). The given relations offer evidence of good convergence of the method.

To estimate the contribution of the interactions in the states with $l \neq 0$, the binding energy $E_0$ and the scattering length A were calculated in[37] for a square-well potential wi allowance for the interaction in the states with $l = 0$ an... $l = 2$. [To simplify the calculations, only the first term n = 1 was taken into account in the separable expansion (2.19).] The results of the calculations are given in Table I in the column marked $l = 0, 2$. As expected, the contribution of the interaction to the state with $l = 2$ turned out to be negligible.

The results shown in Figs. 5–7 confirm the previously made statement that refinement of the separable expansion, i.e., allowance for a large number of terms in (2.19) (as well as allowance of the interactions in states with higher orbital angular momenta) leads only to an increase of the attraction and consequently to an

increase of the binding energy of the three particles $E_0$ and $E_1$ and to a decrease of the scattering length A for a particle scattered by two others in a bound state. Therefore allowance for further corrections in the expansion can change the results only in one direction.

The performed calculations point to a strong dependence of the properties of the three-particle system on the form of the two-particle interaction even in the case of sufficiently short-range forces, unlike the two-particle system, the properties of which are practically insensitive to the form of the two-particle interaction. Thus, we see that the binding energy and the scattering length of one particle by two other particles in the bound state can differ noticeably in the case of a three-particle system for two-particle interactions described by a Hulthen potential and by a square well the parameters of which are chosen such that the binding energy of the two-particle system, the scattering length, and the radius of the effective interaction are the same.

## 4. BOUND STATES AND SCATTERING IN A SYSTEM OF THREE NUCLEONS

### 4.1. Symmetry of Wave Function of a System of Three Nucleons

The total wave function of a system of three nucleons should be antisymmetrical against permutations of the spatial, spin, and isospin coordinates of any pair of nucleons. In the case of central forces, the nuclear interaction between the nucleons (i and j) is characterized by a potential

$$V_{ij} = \sum_{\nu=1}^{4} V^{(\nu)}(r_{ij})\, P_{ij}^{(\nu)}(\sigma, \tau), \tag{4.1}$$

where $P_{ij}^{(\nu)}(\sigma, \tau)$ is the operator of projection in the $\nu$-spin-isospin state. (The values $\nu$ = 1, 2, 3, and 4 denote respectively the triplet-singlet ts, triplet-triplet tt, singlet-triplet st, and singlet-singlet ss spin-isospin states of the two nucleons.)

In a system of three nucleons the interaction between which is described by (4.1), the total spin of the system S, the total isotopic spin T, and their projections $M_S$ and $M_T$ are conserved quantities. The total spin and the total isotopic spin S and T of a system of three nucleons can assume two values: 1/2 and 3/2. According to[12], in the case of central forces (4.1) the wave function $\Psi^{ST}$ as a function of the values of S and T can be represented in the form

$$\left.\begin{aligned}
\Psi^{\frac{1}{2}\frac{1}{2}} &= \Psi'\xi^a - \Psi^a\xi'' + \Psi'\xi'' - \Psi''\xi', \\
\Psi^{\frac{3}{2}\frac{1}{2}} &= \Psi'\chi^a\zeta'' - \Psi''\chi^a\zeta', \\
\Psi^{\frac{1}{2}\frac{3}{2}} &= \Psi'\chi''\zeta^a - \Psi''\chi'\zeta^a, \\
\Psi^{\frac{3}{2}\frac{3}{2}} &= \Psi^a\chi^a\zeta^a;
\end{aligned}\right\} \tag{4.2}$$

Here $\Psi^S$ and $\Psi^a$ are the symmetrical and antisymmetrical spatial functions. The functions $\Psi'$ and $\Psi''$ transform upon permutation of the spatial coordinates in accordance with a two-dimensional irreducible representation of the permutation group $S_3$[12]. The spin function $\chi^S$ (the isospin function $\zeta^S$) is symmetrical against the permutation of the spins (isotopic spins) of the three nucleons. The spin functions $\chi'$ and $\chi''$ (the isospin functions $\zeta'$ and $\zeta''$) transform upon permutation of the spin (isospin) coordinates of the nucleons in a manner similar to $\Psi'$ and $\Psi''$. The spin-isospin functions $\xi^a$, $\xi^S$, $\xi'$, and $\xi''$ are equal to

$$\left.\begin{aligned}
\xi^a &= \frac{1}{\sqrt{2}}(\chi'\zeta'' - \chi''\zeta'), \quad \xi^s = \frac{1}{\sqrt{2}}(\chi'\zeta' + \chi''\zeta''), \\
\xi' &= \frac{1}{\sqrt{2}}(\chi'\zeta'' + \chi''\zeta'), \quad \xi'' = \frac{1}{\sqrt{2}}(\chi'\zeta' - \chi''\zeta'').
\end{aligned}\right\} \tag{4.3}$$

If the projection of the total spin of the system $M_S$ is equal to 1/2, then the spin functions are determined by the expressions

$$S = \frac{1}{2}: \begin{cases} \chi' = \frac{1}{\sqrt{2}}\alpha(1)\{\alpha(2)\beta(3) - \beta(2)\alpha(3)\}, \\ \chi'' = \frac{1}{\sqrt{6}}\alpha(1)\{\alpha(2)\beta(3) + \beta(2)\alpha(3)\} - \sqrt{\frac{2}{3}}\beta(1)\alpha(2)\alpha(3); \end{cases} \tag{4.4}$$

$$S = \frac{3}{2}: \chi^s = \frac{1}{\sqrt{3}}\{\alpha(1)\alpha(2)\beta(3) + \alpha(1)\beta(2)\alpha(3) + \beta(1)\alpha(2)\alpha(3)\}, \tag{4.5}$$

where $\alpha$ and $\beta$ are the spin wave functions of the individual nucleon, corresponding to spin projections 1/2 and −1/2. We note that the function $\chi'$ corresponds to a singlet spin state of the pair of nucleons 2 and 3, and the function $\chi''$ to a triplet state of the same pair of nucleons. The isospin functions $\zeta'$ and $\zeta''$ have a similar form.

The wave functions $\Psi^{ST}$ at T = 1/2 describe the bound state of a system of two neutrons and a proton (the triton $H^3$) or the scattering of a neutron by a deuteron (n-d scattering). In the investigation of the possibility of a bound state in a system of three neutrons $n^3$, it is necessary to consider the wave functions $\Psi^{ST}$ with T = 3/2. If the system consists of two protons and one neutron (the $He^3$ nucleus, p-d scattering), then the conserved quantities are S, $M_S$, and $M_T$ (Coulomb interaction between the protons violates conservation of the total isotopic spin T). In this case the wave function of the system can be represented in the form of a superposition of functions with T = 1/2 and T = 3/2.

We shall henceforth assume that the spatial functions $\Psi^a$, $\Psi^S$, $\Psi'$ and $\Psi''$ depend on the momenta, i.e., a Fourier transformation has been carried out over all the spatial coordinates. The symmetry properties of the functions against permutations remain unchanged in this case.

### 4.2. Electromagnetic Form Factors of the Nuclei $H^3$ and $He^3$

An analysis of the experimental data on the scattering of electrons by nuclei is best carried out by introducing two momentum-transfer functions—the charge and magnetic form factors, which represent Fourier transformations of the spatial distributions of the electric charge and magnetic moment of the nucleus. In the case of the nuclei $H^3$ and $He^3$, the charge and the magnetic form factors $F_c(q)$ and $F_m(q)$ are determined by the expressions[106]

$$zF_c(q) = \int e^{i\mathbf{qr}}(\Psi, \rho_c\Psi)\, d\mathbf{r}, \quad \mu F_m(q) = \int e^{i\mathbf{qr}}(\Psi, \rho_m\Psi)\, d\mathbf{r}, \tag{4.6}$$

where z and $\mu$ are the charge and magnetic moment of the corresponding nucleus. [The parentheses in (4.6) denote integration with respect to the internal relative

coordinates.] The charge and magnetic-moment density operators $\rho_c$ and $\rho_m$ are given by

$$\left.\begin{aligned}\rho_c &= \sum_{i=1}^{3}\left[\tfrac{1}{2}(1+\tau_{iz})\,\varphi_c^p(\mathbf{r}-\mathbf{r}_i)+\tfrac{1}{2}(1-\tau_{iz})\,\varphi_c^n(\mathbf{r}-\mathbf{r}_i)\right],\\ \rho_m &= \sum_{i=1}^{3}\left[\tfrac{1}{2}(1+\tau_{iz})\,\mu_p\varphi_m^p(\mathbf{r}-\mathbf{r}_i)+\tfrac{1}{2}(1-\tau_{iz})\,\mu_n\varphi_m^n(\mathbf{r}-\mathbf{r}_i)\right]\sigma_{iz},\end{aligned}\right\} \tag{4.7}$$

where the functions $\varphi_c$ and $\varphi_m$ describe the spatial distribution of the charges and of the magnetic moments of the individual nucleons. The Fourier transformations of the functions $\varphi_c(r)$ and $\varphi_m(r)$ are usually called the form factors of the nucleons

$$f_c(q)=\int e^{iqr}\varphi_c(r)\,d\mathbf{r},\quad f_m(q)=\int e^{iqr}\varphi_m(r)\,d\mathbf{r}. \tag{4.8}$$

For the proton and the neutron, these form factors are normalized in the following manner:

$$f_c^p(0)=1,\quad f_c^n(0)=0,\quad f_m^p(0)=f_m^n(0)=1. \tag{4.9}$$

Substituting in (4.7) the wave functions for the nuclei $H^3$ and $He^3$ in the form

$$\Psi^{H^3}\equiv\Psi^{\frac{1}{2}\frac{1}{2}}\text{ and }\Psi^{He^3}\equiv\Psi^{\frac{1}{2}\frac{1}{2}}+\Psi^{\frac{1}{2}\frac{3}{2}}$$

and using formulas (4.7) for the charge and magnetic-moment density operators, we can readily obtain explicit expressions for the form factors of $H^3$ and $He^3$. In the case of $H^3$ we have

$$\begin{aligned}F_c^{H^3}(q)&=f_c^p(q)\,F_0^{H^3}(q)+2f_c^n(q)\,F_L^{H^3}(q),\\ \mu_{H^3}F_m^{H^3}(q)&=\mu_p f_m^p(q)\,G_0^{H^3}(q)+\tfrac{2}{3}\mu_n f_m^n(q)\,[G_0^{H^3}(q)-G_L^{H^3}(q)],\end{aligned} \tag{4.10}$$

where

$$\left.\begin{aligned}F_0^{H^3}(q)&=\int e^{iqr_1}[(\Psi^s+\Psi'')^2+(\Psi'-\Psi^a)^2]\,d\tau,\\ F_L^{H^3}(q)&=F_0^{H^3}(q)-3\int e^{iqr_1}(\Psi^s\Psi''-\Psi^a\Psi')\,d\tau,\\ G_0^{H^3}(q)&=F_0^{H^3}(q)-\tfrac{4}{3}\int e^{iqr_1}(\Psi'-\Psi^a)^2\,d\tau,\\ G_L^{H^3}(q)&=F_L^{H^3}(q)-\tfrac{10}{3}\int e^{iqr_1}\left(\Psi'^2+\tfrac{8}{5}\Psi'\Psi^a+\Psi^{a2}\right)d\tau.\end{aligned}\right\} \tag{4.11}$$

Formulas (4.10) express the charge and the magnetic form factors of the nucleus $H^3$ in terms of the form factors of the nucleons $f_c(q)$ and $f_m(q)$ and the structure form factors (4.11), which are determined completely by the wave function of the nucleus. The form factors $F_0(q)$ and $F_L(q)$ describe the spatial distribution of the nucleons in the nucleus in the proton (unpaired) and neutron (paired) states, and the functions $G_0(q)$ and $G_L(q)$ characterize the distribution of the average value of the projection of the proton and neutron spin on the z axis. The normalization of the structure form factors is determined by the normalization of the wave function of the nucleus. If the total wave function is normalized to unity, then

$$F_0^{H^3}(0)=F_L^{H^3}(0)=1, \tag{4.12}$$

and with allowance for the conditions for the normalization of the form factors of the nucleons (4.9) we have also $F_c^{H^3}(0)=1$. We note that the magnetic form factors $G_0^{H^3}(q)$ and $G_L^{H^3}(q)$ are not normalized to unity. We denote the weights of the symmetrical and antisymmetrical states and the state with intermediate symmetry by $P^s$, $P^a$, and $P'$. Then

$$G_0^{H^3}(0)=1-\tfrac{2}{3}(P'+2P^a),\quad G_L^{H^3}(0)=1-\tfrac{5}{3}(P'+2P^a). \tag{4.13}$$

From formula (4.10) at q = 0, with allowance for the normalization conditions (4.9), (4.13), and the condition $F_m^{H^3}(0)=1$ it is easy to obtain the following expression for the magnetic moment of the triton:

$$\mu_{H^3}=\mu_p+\tfrac{2}{3}(\mu_n-\mu_p)(P'+2P^a). \tag{4.14}$$

Analogously, in the case of the nucleus $He^3$ we obtain for the electromagnetic form factors the following formulas:

$$\left.\begin{aligned}2F_c^{He^3}(q)&=f_c^n(q)\,F_0^{He^3}(q)+2f_c^p(q)\,F_L^{He^3}(q),\\ \mu_{He^3}F_m^{He^3}(q)&=\mu_n f_m^n(q)\,G_0^{He^3}(q)+\tfrac{2}{3}\mu_p f_m^p(q)\,[G_0^{He^3}(q)-G_L^{He^3}(q)],\end{aligned}\right\} \tag{4.15}$$

where

$$\left.\begin{aligned}F_0^{He^3}(q)&=\int e^{iqr_1}[(\Psi^s+\Psi''-\overline{\Psi}'')^2+(\Psi'+\overline{\Psi}'-\Psi^a)^2]\,d\tau,\\ F_L^{He^3}(q)&=F_0^{He^3}(q)-3\int e^{iqr_1}[\Psi^s(\Psi''-\overline{\Psi}'')+(\Psi'\overline{\Psi}'-\Psi''\overline{\Psi}'')-\\ &\qquad-\Psi^a(\Psi'+\overline{\Psi}')]\,d\tau,\\ G_0^{He^3}(q)&=F_0^{He^3}(q)-\tfrac{4}{3}\int e^{iqr_1}(\Psi'+\overline{\Psi}'-\Psi^a)^2\,d\tau,\\ G_L^{He^3}(q)&=F_L^{He^3}(q)-\tfrac{10}{3}\int e^{iqr_1}(\Psi'+\overline{\Psi}'-\Psi^a)^2\,d\tau-\\ &\qquad-3\int e^{iqr_1}[2\Psi^s\overline{\Psi}''-2(\Psi'\overline{\Psi}'-\Psi''\overline{\Psi}'')-\\ &\qquad-(\overline{\Psi}'^2-\overline{\Psi}''^2)+2\Psi^a(2\Psi'+\overline{\Psi}')]\,d\tau.\end{aligned}\right\} \tag{4.16}$$

In (4.16) the functions $\Psi^S$, $\Psi'$, $\Psi''$, and $\Psi^a$ pertain to the state with total isotopic spin T = 1/2, and the functions $\overline{\Psi}'$ and $\overline{\Psi}''$ to the state T = 3/2. The form factors (4.16) are normalized in the following manner:

$$\left.\begin{aligned}F_0^{He^3}(0)&=F_L^{He^3}(0)=1,\\ G_0^{He^3}(0)&=1-\tfrac{2}{3}\left(P'+2P^a+\overline{P}'+4\int\Psi'\overline{\Psi}'\,d\tau\right),\\ G_L^{He^3}(0)&=1-\tfrac{5}{3}\left(P'+2P^a+\overline{P}'+4\int\Psi'\overline{\Psi}'\,d\tau\right);\end{aligned}\right\} \tag{4.17}$$

$\overline{P}'$ is the weight of the state with T = 3/2. The magnetic moment of $He^3$ is determined by the expression

$$\mu_{He^3}=\mu_n+\tfrac{2}{3}(\mu_p-\mu_n)\left(P'+2P^a+\overline{P}'+4\int\Psi'\overline{\Psi}'\,d\tau\right). \tag{4.18}$$

If we neglect in the wave function of $He^3$ the admixture of the state with T = 3/2 ($\overline{\Psi}'=\overline{\Psi}''=0$), then the formulas for the structure form factors (4.16) coincide with the corresponding formulas (4.11) for the nucleus $H^3$.

Expanding the form factors in formulas (4.10) and (4.15) in series in $q^2$, we express the rms radii of the charge distributions for the nuclei $H^3$ and $He^3$ in terms of the rms radii of the distributions for the unpaired and paired nucleons $R_0$ and $R_L$ and the rms radii of the charge distributions for the proton $r_c(p)$ and neutron $r_c(n)$:

$$\left.\begin{aligned}R_c^2(H^3)&=R_0^2(H^3)+r_c^2(p)+2r_c^2(n),\\ R_c^2(He^3)&=R_L^2(He^3)+r_c^2(p)+\tfrac{1}{2}r_c^2(n).\end{aligned}\right\} \tag{4.19}$$

In similar fashion we can also obtain expressions for the rms radii of the distributions of the magnetic moments of the nuclei $H^3$ and $He^3$.

### 4.3. Integral Equations for a System of Three Nucleons

The Faddeev equations can easily be generalized to the case of a system of three nucleons with allowance

for the spin dependence of the interaction between them[34,107]. To this end it is necessary to take into account in Eqs. (1.40), which describe an arbitrary system of three particles, the operator character of the two-particle t matrix in the spin-isospin space of the nucleons, and to take into account the dependence of the wave function of the system on the spin and isospin variables. Owing to the identity of the nucleons, the system of equations (1.40) reduces to a single equation

$$\Psi(\mathbf{k}, \mathbf{p}) = \Phi(\mathbf{k}, \mathbf{p}) + \left(Z_p - \frac{k^2}{m}\right)^{-1} \int \left\{ \left\langle \mathbf{k} \left| t_{23}(Z_p) \right| -\frac{1}{2}\mathbf{p} + \mathbf{p}' \right\rangle \Psi^{(2)}\left(-\mathbf{p} - \frac{1}{2}\mathbf{p}', \mathbf{p}'\right) + \left\langle \mathbf{k} \left| t_{23}(Z_p) \right| -\frac{1}{2}\mathbf{p} - \mathbf{p}' \right\rangle \Psi^{(3)}\left(\mathbf{p} + \frac{1}{2}\mathbf{p}', \mathbf{p}'\right) \right\} \frac{d\mathbf{p}'}{(2\pi)^3}, \tag{4.20}$$

where the functions $\Psi^{(2)}$ and $\Psi^{(3)}$ differ from the function $\Psi \equiv \Psi^{(1)}$ by cyclic permutation of the spatial, spin, and isospin coordinates of the nucleons.

In the case of a central interaction (4.1), the two-nucleon t matrix which enters in (4.20) can be represented in the form

$$\langle \mathbf{k} | t_{ij}(z) | \mathbf{k}' \rangle = \sum_{\nu=1}^{4} \langle \mathbf{k} | t^{(\nu)}(z) | \mathbf{k}' \rangle P_{ij}^{(\nu)}(\sigma, \tau), \tag{4.21}$$

where $t^{(\nu)}(z)$ is the eigenvalue of the t matrix in a definite spin-isospin state $\nu$. Projecting Eq. (4.20) on the possible spin-isospin states of the system of three nucleons, we can obtain a system of integral equations for the spatial functions $\Psi^a$, $\Psi^c$, $\Psi'$, and $\Psi''$.

In place of the functions $\Psi^a$, $\Psi^s$, $\Psi'$, and $\Psi''$, which transform upon permutation of any pair of particles in accordance with the irreducible representations of the group of permutations $S_3$, it is convenient to change over to the functions $\psi_\nu(\mathbf{k}_{ij}, \mathbf{p}_k)$ corresponding to definite spin-isospin states $\nu = 1, 2, 3$, and 4 of the pair of particles ij. The functions $\psi_\nu(\mathbf{k}_{ij}, \mathbf{p}_k)$ are even if $\nu = 1$ and 3 and odd if $\nu = 2$ and 4 relative to permutation of the particles i and j ($\mathbf{k}_{ij} \to \mathbf{k}_{ji} = -\mathbf{k}_{ij}$):

$$\psi_\nu(-\mathbf{k}, \mathbf{p}) = (-1)^{\nu-1} \psi_\nu(\mathbf{k}, \mathbf{p}). \tag{4.22}$$

In the case $S = 1/2$ and $T = 1/2$, the spatial functions $\Psi^a$, $\Psi^s$, $\Psi'$, and $\Psi''$ are expressed in terms of four functions $\psi_\nu$ with $\nu = 1, 2, 3$, and 4; in the case $S = 1/2$ and $T = 3/2$, the spatial functions $\Psi'$ and $\Psi''$ are expressed in terms of two functions, $\psi_2$ and $\psi_3$; in the case $S = 3/2$ and $T = 1/2$, the spatial functions $\Psi'$ and $\Psi''$ are expressed in terms of two functions $\psi_1$ and $\psi_2$; finally, in the case $S = 3/2$, $T = 3/2$ the spatial function $\Psi^a$ is expressed in terms of one function $\psi_2$:

$$\left.\begin{aligned}
&S = \tfrac{1}{2},\ T = \tfrac{1}{2}:\ \Psi^a = \frac{1}{\sqrt{2}}\{\psi_2(1) + \psi_4(1) + \psi_2(2) + \psi_4(2) + \psi_2(3) + \psi_4(3)\},\\
&\Psi^s = \frac{1}{\sqrt{2}}\{\psi_1(1) + \psi_3(1) + \psi_1(2) + \psi_3(2) + \psi_1(3) + \psi_3(3)\},\\
&\Psi' = -\frac{1}{\sqrt{2}}\Big\{\psi_2(1) - \psi_4(1) - \frac{1}{2}[\psi_2(2) - \psi_4(2) + \psi_2(3) - \psi_4(3)] + \frac{\sqrt{3}}{2}[\psi_1(2) - \psi_3(2) - \psi_1(3) + \psi_3(3)]\Big\},\\
&\Psi'' = -\frac{1}{\sqrt{2}}\Big\{\psi_1(1) - \psi_3(1) - \frac{1}{2}[\psi_1(2) - \psi_3(2) + \psi_1(3) - \psi_3(3)] - \frac{\sqrt{3}}{2}[\psi_2(2) - \psi_4(2) - \psi_2(3) + \psi_4(3)]\Big\};
\end{aligned}\right\} \tag{4.23}$$

$$\left.\begin{aligned}
&S = \tfrac{1}{2},\ T = \tfrac{3}{2}:\ \Psi' = \psi_2(1) - \frac{1}{2}[\psi_2(2) + \psi_2(3)] - \frac{\sqrt{3}}{2}[\psi_3(2) - \psi_3(3)],\\
&\Psi'' = -\psi_3(1) - \frac{\sqrt{3}}{2}[\psi_2(2) - \psi_2(3)] + \frac{1}{2}[\psi_3(2) + \psi_3(3)];
\end{aligned}\right\} \tag{4.24}$$

$$\left.\begin{aligned}
&S = \tfrac{3}{2},\ T = \tfrac{1}{2}:\ \Psi' = \psi_2(1) - \frac{\sqrt{3}}{2}[\psi_1(2) - \psi_1(3)] - \frac{1}{2}[\psi_2(2) + \psi_2(3)],\\
&\Psi'' = -\psi_1(1) + \frac{1}{2}[\psi_1(2) + \psi_1(3)] - \frac{\sqrt{3}}{2}[\psi_2(2) - \psi_2(3)];
\end{aligned}\right\} \tag{4.25}$$

$$S = \tfrac{3}{2},\ T = \tfrac{3}{2}:\ \Psi^a = \psi_2(1) + \psi_2(2) + \psi_2(3). \tag{4.26}$$

To abbreviate the notation we have used the symbol

$$\psi_\nu(k) \equiv \psi_\nu(\mathbf{k}_{ij}, \mathbf{p}_k), \qquad ijk = 123, 231, 312.$$

The systems of functions $\psi_\nu$, generally speaking, differ for different spin-isospin states of the system of three nucleons, i.e., for different values of S and T.

The system of integral equations for the functions $\psi_\nu(\mathbf{k}, \mathbf{p})$ at arbitrary values of the spin and isospin of the system of three nucleons can be represented in the form

$$\psi_\nu(\mathbf{k}, \mathbf{p}) = \varphi_\nu(\mathbf{k}, \mathbf{p}) + \left(Z_p - \frac{k^2}{m}\right)^{-1} \sum_{\nu'=1}^{4} \left\{ \left\langle \mathbf{k} \left| t^{(\nu)}(Z_p) \right| -\frac{1}{2}\mathbf{p} + \mathbf{p}' \right\rangle + (-1)^{\nu+1} \left\langle \mathbf{k} \left| t^{(\nu)}(Z_p) \right| -\frac{1}{2}\mathbf{p} - \mathbf{p}' \right\rangle \right\} c_{\nu\nu'} \psi_{\nu'}\left(\mathbf{p} + \frac{1}{2}\mathbf{p}', \mathbf{p}'\right) \frac{d\mathbf{p}'}{(2\pi)^3}, \quad \nu = 1, 2, 3, 4, \tag{4.27}$$

where the matrices $c_{\nu\nu'}^{(ST)}$ for different spin-isospin states of the system of three nucleons are determined by the expressions

$$\left.\begin{aligned}
&c_{\nu\nu'}^{(\frac{1}{2}\frac{1}{2})} = \frac{1}{4}\begin{Vmatrix} 1 & \sqrt{3} & 3 & -\sqrt{3} \\ \sqrt{3} & -1 & -\sqrt{3} & -3 \\ 3 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & -1 \end{Vmatrix},\\
&c_{\nu\nu'}^{(\frac{1}{2}\frac{3}{2})} = \frac{1}{2}\begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -\sqrt{3} & 0 \\ 0 & -\sqrt{3} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix},\\
&c_{\nu\nu'}^{(\frac{3}{2}\frac{1}{2})} = \frac{1}{2}\begin{Vmatrix} -1 & -\sqrt{3} & 0 & 0 \\ -\sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}, \qquad c_{\nu\nu'}^{(\frac{3}{2}\frac{3}{2})} = -\delta_{\nu 2}\delta_{\nu' 2}.
\end{aligned}\right\} \tag{4.28}$$

In considering the bound states of a system of three nucleons, the free term in (4.27) must be set equal to zero. We note that the integral equation for the system of three identical spinless particles (1.43) coincides with (4.27) if we put in the latter

$$c_{\nu\nu'} = \delta_{\nu 1}\delta_{\nu' 1}. \tag{4.29}$$

In the derivation of (4.27) we have neglected the Coulomb interaction, and therefore Eqs. (4.27) are suitable only for the description of a system consisting of three neutrons or of two neutrons and one proton*. In a system consisting of two neutrons and a proton, a bound

*Integral equations for a system of two protons and one neutron in the bound state (the $He^3$ nucleus) were obtained in [36]. Veselova [113] proposed a method of obtaining integral equations for the wave function describing the scattering of a charged particle by a bound state of two other charged particles.

state (the triton) is possible, as is scattering of a neutron by the two other particles in the bound state. In the latter case, the free term in (4.27) must be chosen in the form

$$\varphi_\nu(\mathbf{k}, \mathbf{p}) = (2\pi)^3 \varphi(\mathbf{k})\, \delta(\mathbf{p}-\mathbf{p}_0)\, \delta_{\nu 1}, \tag{4.30}$$

where $\varphi(\mathbf{k})$ is the wave function of the ground state of the deuteron. Since two neutrons do not form a bound system, in the case of three neutrons it is possible to consider only the bound-state problem.

### 4.4. Scattering of a Neutron by a Deuteron

Let us stop to discuss in greater detail the problem of the scattering of a neutron by a deuteron. Since the isotopic spin of a deuteron is equal to zero, the total isospin T of the system is equal to 1/2, and the ordinary spin can assume the values 1/2 and 3/2 (doublet and quartet states). Taking into account the invariant character of the spatial functions $\psi_\nu(\mathbf{k}, \mathbf{p})$ against rotations, we expand the functions $\psi_\nu(\mathbf{k}, \mathbf{p})$ in terms of the angle functions in the form (1.51). For the coefficients of such an expansion, the functions $\psi^{(\nu)}_{l\lambda \mathrm{L}}(\mathrm{k}, \mathrm{p}; \mathrm{p}_0)$, just as in the case of three spinless particles, we obtain a system of two-dimensional integral equations

$$\psi^{(\nu)}_{l\lambda L}(k, p; p_0) = (2\pi)^3 \varphi_{10}(k) \frac{\delta(p-p_0)}{p^2} \delta_{l0}\delta_{\lambda L}\delta_{\nu 1} + \frac{\Delta^{(\nu)}_l}{\pi^2\left(Z_p - \frac{k^2}{m}\right)} \sum_{\nu' l' \lambda'} c_{\nu\nu'} \int_0^\infty dp' \int_{|p-p'/2|}^{p+p'/2} dk' \frac{k'p'}{p} t^{(\nu)}_l(k, Q; Z_p) \times K^{(L)}_{l\lambda, l'\lambda'}(p, p'; k')\, \psi^{(\nu')}_{l'\lambda' L}(k', p'; p_0), \tag{4.31}$$

where the functions $K^{(\mathrm{L})}_{l\lambda, l'\lambda'}$ are determined as before by the expression (1.54) and

$$\Delta^{(\nu)}_l \equiv \tfrac{1}{2}[1 + (-1)^{l+\nu+1}] \tag{4.32}$$

($p_0$ is the relative momentum of the neutron and the deuteron in the initial state).

Using the separable expansion for the two-particle t matrix (2.19) and expression (2.29) for the wave function of the deuteron, we can represent the function $\psi^{(\nu)}_{l\lambda \mathrm{L}}$ in the form

$$\psi^{(\nu)}_{l\lambda L}(k, p; p_0) = N_{10}(2\pi)^3 \sum_n \frac{g^{(\nu)}_{nl}(k, Z_p)}{\frac{k^2}{m} - Z_p} \left\{ \frac{\delta(p-p_0)}{p^2} \delta_{n1}\delta_{l0}\delta_{\lambda L}\delta_{\nu 1} + \tau^{(\nu)}_{nl}(Z_p)\, a^{(\nu)}_{nl\lambda L}(p, p_0) \right\}, \tag{4.33}$$

where we have introduced the notation

$$\tau^{(\nu)}_{nl}(z) = \eta^{(\nu)}_{nl}(z)\,[1 - \eta^{(\nu)}_{nl}(z)]^{-1}. \tag{4.34}$$

The functions $a^{(\nu)}_{n l \lambda \mathrm{L}}(\mathrm{p}, \mathrm{p}_0)$ are determined by the following system of one-dimensional integral equations:

$$a^{(\nu)}_{nl\lambda L}(p, p_0) = c_{\nu 1} U^{(\nu, 1)}_{nl\lambda L, 10 L L}(p; p_0; Z) + \sum_{\nu' n' l' \lambda'} c_{\nu\nu'} \int_0^\infty dp' p'^2 U^{(\nu\nu')}_{nl\lambda L, n'l'\lambda' L}(p, p'; Z)\, \tau^{(\nu')}_{n'l'}(Z_{p'})\, a^{(\nu')}_{n'l'\lambda' L}(p', p_0), \tag{4.35}$$

where the effective potential U is determined by the expression

$$U^{(\nu, \nu')}_{nl\lambda L, n'l'\lambda' L}(p, p'; Z) = \frac{\Delta^{(\nu)}_l \Delta^{(\nu')}_{l'}}{2\pi^2} \int_{-1}^{1} dy K^{(L)}_{l\lambda, l'\lambda'}(p, p'; y) \frac{g^{(\nu)}_{nl}(Q, Z_p)\, g^{(\nu')}_{n'l'}(Q', Z_{p'})}{\frac{1}{m}(p^2 + p'^2 + pp'y) - Z}. \tag{4.36}$$

It follows from (4.34)–(4.36) that the components $a^{(\nu)}_{n l \lambda \mathrm{L}}$ with $\nu$ and $l$ of equal parity are equal to zero. This is directly connected with the Pauli principle, which forbids, for a pair of nucleons, triplet-singlet and singlet-triplet spin-isospin states ($\nu$ = 1 and 3) with odd $l$ and triplet-triplet and singlet-singlet states ($\nu$ = 2 and 4) with even $l$.

The amplitudes of elastic scattering of a neutron by a deuteron in the quartet and doublet spin states $f^{3/2}$ and $f^{1/2}$ are determined by the formulas

$$\left.\begin{aligned} f^{3/2}(\mathbf{p}, \mathbf{p}_0) &= -\frac{m}{3\pi} \int e^{-i\mathbf{p}\rho} \left(\varphi(\mathbf{r})\, \chi^3 \xi', \left[V_{31}\left(-\tfrac{1}{2}\mathbf{r} - \rho\right) + V_{12}\left(-\tfrac{1}{2}\mathbf{r} + \rho\right)\right] \Psi^{\frac{3}{2}\frac{1}{2}}(\mathbf{r}, \rho; \mathbf{p}_0)\right) d\mathbf{r}\, d\rho, \\ f^{1/2}(\mathbf{p}, \mathbf{p}_0) &= -\frac{m}{3\pi} \int e^{-i\mathbf{p}\rho} \left(\varphi(\mathbf{r})\, \chi'' \xi', \left[V_{31}\left(-\tfrac{1}{2}\mathbf{r} - \rho\right) + V_{12}\left(-\tfrac{1}{2}\mathbf{r} + \rho\right)\right] \Psi^{\frac{1}{2}\frac{1}{2}}(\mathbf{r}, \rho; \mathbf{p}_0)\right) d\mathbf{r}\, d\rho. \end{aligned}\right\} \tag{4.37}$$

[The indices 23, 31, and 12 in the right-hand side of (4.37) number the particles.] The partial amplitudes $f^{3/2}_{\mathrm{L}}$ and $f^{1/2}_{\mathrm{L}}$ are directly expressed in terms of the functions $a^{(1)}_{10\mathrm{LL}}(\mathrm{p}_0, \mathrm{p}_0)$ for the corresponding spin states:

$$f_L(p_0, p_0) = \frac{2\pi}{3} m \left(\frac{d\eta^{(1)}_{10}(z)}{dz}\right)^{-1}_{z=-\varepsilon_{10}} a^{(1)}_{10LL}(p_0, p_0). \tag{4.38}$$

The neutron-deuteron scattering length in a definite spin state is determined by the expression

$$^{2S+1}A = -f^S_0(0, 0). \tag{4.39}$$

### 4.5. Bound State of a System of Three Nucleons

We now consider a system of three nucleons in a bound state with total orbital angular momentum L and projection M. We separate in the wave function of the system the angle part

$$\psi_{\nu LM}(\mathbf{k}, \mathbf{p}) = \sum_{l\lambda} \psi^{(\nu)}_{l\lambda L}(k, p)\, Y_{l\lambda LM}(\hat{k}, \hat{p}). \tag{4.40}$$

Substituting (4.40) in (4.27), we obtain for $\psi^{(\nu)}_{l\lambda \mathrm{L}}$ a homogeneous system of two-dimensional integral equations, having the form of the system (1.52) without the inhomogeneous term. Using the separable expansion for $t^{(\nu)}_l(\mathrm{k}, \mathrm{Q}; \mathrm{Z}_\mathrm{p})$ we can represent the functions $\psi^{(\nu)}_{l\lambda \mathrm{L}}$ in the form

$$\psi^{(\nu)}_{l\lambda L}(k, p) = \sum_n \frac{g^{(\nu)}_{nl}(k, Z_p)}{\frac{k^2}{m} - Z_p} \tau^{(\nu)}_{nl}(Z_p)\, a^{(\nu)}_{nl\lambda L}(p). \tag{4.41}$$

The partial components $a^{(\nu)}_{n l \lambda \mathrm{L}}$ are determined by the homogeneous system of one-dimensional integral equations

$$a^{(\nu)}_{nl\lambda L}(p) = \sum_{\nu' n' l' \lambda'} c_{\nu\nu'} \int_0^\infty dp' p'^2 U^{(\nu, \nu')}_{nl\lambda L, n'l'\lambda' L}(p, p'; Z)\, \tau^{(\nu')}_{n'l'}(Z_{p'})\, a^{(\nu')}_{n'l'\lambda' L}(p'). \tag{4.42}$$

This system of integral equations admits of solutions for strictly defined values of Z, which determine the energies of the bound states of the system.

### 4.6. Binding Energy of the Nuclei $H^3$ and $He^3$

We consider first the results of calculations of the binding energy of three-nucleon nuclei, which were car-

**Table III.** Binding energy of triton and scattering lengths of neutron on a deuteron

| | Hulthen Potential | | Square-well potential | | | Two-Nucleon Parameters |
|---|---|---|---|---|---|---|
| | $l = 0$ | | $l = 0$ | | $l = 0, 2$ | |
| | 1 | 2 | 1 | 2 | 1 | |
| $E_T$, MeV | 7.91 | 11.46 | 9.07 | 9.20 | — | (a) |
| | — | — | 9.39 | 9.53 | — | (b) |
| | — | — | 8.95 | 9.08 | 8.97 | (c) |
| $^2A$, F | −0.19 | −2.88 | 0.512 | 0.451 | — | (a) |
| | — | — | 0.204 | 0.145 | — | (b) |
| | — | — | 0.545 | 0.487 | 0.542 | (c) |
| $^4A$, F | 6.403 | 6.361 | 6.284 | 6.279 | — | (a) |
| | — | — | 6.315 | 6.310 | — | (b) |
| | — | — | 6.338 | 6.333 | 6.338 | (c) |

ried out under the assumption that the interaction between the nucleons is described by paired central potentials characterized by two parameters which determine the magnitude and action radius of the forces. The values of the parameters of the two-nucleon potentials in the triplet-singlet and singlet-triplet spin-isospin states can be determined by using experimental data on the scattering of nucleons by nucleons at low energies and the binding energy of the neutron plus proton system. In the case of central interaction, the values of the triplet-singlet parameters are determined from the length of the scattering of the neutron by the proton, $a_{ts}$, and the binding energy of the deuteron $\epsilon_d$. Since there is no singlet bound state of a system of two nucleons, the singlet-triplet parameters are determined from the scattering length $a_{st}$ and the magnitude of the effective radius $r_{0st}$. We note that the parameters $a_{st}$ and $r_{0st}$ for the singlet-triplet spin-isospin state are known with lesser accuracy than the triplet-singlet parameters. Even less is known concerning the interaction between the nucleons in the odd states (in the triplet-triplet and singlet-singlet spin-isospin states), than concerning the interaction in even states. At sufficiently low energies of relative motion of the two nucleons, the interaction potentials $V_{tt}$ and $V_{ss}$ can be neglected compared with the potentials $V_{ts}$ and $V_{st}$.

The available data on two-nucleon interaction are discussed in the review of Noyes[97].

We present several sets of values of two-nucleon parameters, which were used in[109,76,81,79,37] in the calculation of the three-particle binding energies and scattering lengths. According to[38]

$$a_{ts} = 5.378 \text{ F}, \quad \varepsilon_d = 2.225 \text{ MeV}, \quad a_{st} = -23.69 \text{ F}, \quad r_{0st} = 2.7 \text{ F} \quad \text{(a)}$$

The parameters obtained on the basis of measurements of Houk and Wilson[71] are equal to

$$a_{ts} = 5.405 \pm 0.006 \text{ F}, \quad a_{st} = -23.728 \pm 0.013 \text{ F}, \quad \varepsilon_d = 2.225 \text{ MeV} \quad r_{0st} = 2.56 \pm 0.10 \text{ F}. \quad \text{(b)}$$

The results of measurements of Houk and Wilson[71] and Koester[82] lead to the values

$$a_{ts} = 5.425 \pm 0.004 \text{ F}, \quad a_{st} = -23.714 \pm 0.013 \text{ F}, \quad \varepsilon_d = 2.225 \text{ MeV}, \quad r_{0st} = 2.704 \pm 0.095 \text{ F}. \quad \text{(c)}$$

The ground state of a system consisting of two neutrons and one proton, in the case of central forces, is the state with total orbital angular momentum equal to zero, L = 0 (triton $H^3$, S = 1/2, T = 1/2). The values of the triton binding energy $E_T$, calculated by the method of separable expansion for a Hulthen potential[109,76] and for a square well[81,37], are given in Table III (the notation is the same as in Table I). The values of the parameters of the two-nucleon potentials used in these calculations are given in Table II.

In the case of a Yukawa potential, the values of $E_T$, calculated by Efimov[15] on the basis of the Bubnov-Galerkin method with the parameters (a) for $a_{ts}$, $\epsilon_d$, and $a_{st}$, but with other values of $r_{0st}$, turned out to equal $E_T$ = 11.65 MeV ($r_{0st}$ = 2.21 F) and $E_T$ = 10.83 MeV ($r_{0st}$ = 2.5 F).

In[5], the triton binding energy was calculated in the case of a square well on the basis of the method of harmonic polynomials. For comparison with the results of[5], the value of $E_T$ was calculated in[81] by the method of separable expansion with the same parameters as in[5]. (The triplet parameters used in[5] coincide with the values of $a_{ts}$ and $\epsilon_d$ of set (a), and the singlet parameters were taken from the data on pp scattering[38]: $a_{st}$ = 16.83 F, $r_{0st}$ = 2.74 F). The value of $E_T$, calculated for the indicated set of parameters, turned out to be $E_T$ = 8.71 MeV when account is taken of only the first term in the expansion of (2.19), and $E_T$ = 8.84 MeV if two terms are taken into account in (2.19) (n = 1 and 2, $l$ = 0). Comparison of the quantity $E_T$ = 8.84 MeV obtained in[81], with the value $E_T$ = 8.43 MeV of[5] points to a better convergence of the method of separable expansion compared with the convergence of the method of harmonic polynomials. (When both methods are used, further corrections to the binding energy can only increase the value of $E_T$.)

Central attractive potentials reconciled with the data on the interaction of two nucleons at low energies lead to overestimates of the binding energy of the triton compared with the experimental value $E_T$ = 8.482 MeV. The values of $E_T$ obtained for potentials of different form differ greatly from each other. The binding energy $E_T$ assumes smaller values for less extended and less singular potentials. The binding energy of the triton for a potential of rectangular form (which changes for different values of the two-nucleon parameters in the interval from 8.84 to 9.53 MeV) agrees with the experimental value better than the energies for other poten-

tials. The binding energy $E_T$ decreases quite strongly with increasing singlet effective radius $r_{0st}$. The dependence of $E_T$ on the two-particle length $a_{st}$ is quite weak[108,78].

The difference in binding energy between the triton and the nucleus $He^3$ (the experimental binding energy of $He^3$ is $E_{He^3} = 7.718$ MeV, $\Delta E \equiv E_T - E_{He^3} = 0.764$ MeV) is well explained if account is taken of the Coulomb interaction between the protons in the $He^3$ nucleus. In[5], the Coulomb energy for the $He^3$ nucleus was calculated in the case of a square-well potential on the basis of the method of harmonic polynomials in first order of perturbation theory. When the indicated parameters were used for the two-nucleon interaction, the value of $\Delta E$ turned out to be 0.789 MeV. As we have already noted, the Coulomb interaction between the nucleons leads to an admixture of a state with $T = 3/2$ in the wave function of $He^3$. A direct solution of the system of differential equations with allowance for the harmonics with $K = 0$ and $K = 2$ leads[35] in this case, for a square-wave potential, to closely similar values of the difference $\Delta E$ [$\Delta E = 0.777$ MeV in the case of parameters (b) and $\Delta E = 0.751$ MeV in the case of parameters (c)].

The correct behavior of the amplitude of scattering of two nucleons on the energy shell at low energies can be ensured by describing the interaction between the nucleons by means of a potential containing two parameters, which are determined by specifying two experimental quantities—the scattering length and the effective radius. To find the two-particle scattering amplitude on the energy shell in a large interval of energies, it is necessary to use more complicated potentials, characterized by a larger number of parameters. As is well known, data on nucleon-nucleon scattering in the high-energy region point to the existence of a stronger repulsion between the nucleons at small distances. Introduction of a repulsive core in the two-nucleon potential makes it possible to explain the reversal of the sign of the S scattering phase at high energies. A potential with repulsion at small distances and with subsequent short-range attraction should contain at least three parameters. Introduction of the radius of the repulsive core—an additional parameter characterizing the repulsion—makes it possible to describe correctly the scattering amplitude not only at low energies but also at high ones. In order for the potential with repulsion to describe data on the interaction of two nucleons at low energies, the attractive part of the potential should be deeper and its radius smaller than the corresponding quantities for a purely attractive potential[38].

The influence of the short-range repulsion on the value of the binding energy of three nucleons was investigated in a number of papers both for separable[110,53,74] and for local[91,3,11,48,80] potentials. In[80] the triton binding energy was calculated on the basis of the separable expansion (2.22) for a local potential with infinite repulsion and with attraction in the form of a square well. The potential parameters were determined from the low-energy data (c) and from the dependence of the S phase on the energy in the interval from 0 to 400 MeV[90]. The values of the radii of the repulsive core turned out to be 0.186 F in the case of a triplet spin state and 0.180 F in the case of a singlet state. The calculated binding energy of the triton is equal to $E_T = 8.8$ MeV[80]. This value differs very little from the value $E_T = 9.1$ MeV obtained for a purely attractive square well.[81]

Calculations performed on the basis of the Bateman separable expansion[11,2,3] led in the case of a Morse potential[57]

$$V(r) = -V_0\left(2e^{-\frac{r-c}{R}} - e^{-2\frac{r-c}{R}}\right) \tag{4.43}$$

to the following values of the triton binding energy: $E_T = 9.12$ MeV ($a_{st} = -23.68$ F, $r_{0st} = 2.44$ F) and $E_T = 8.10$ MeV ($a_{st} = -17$ F and $r_{0st} = 2.80$ F). In the case of a superposition of Yukawa attraction and repulsion potentials[91]

$$V(r) = -V_{01}\frac{e^{-\frac{r}{R_1}}}{r} + V_{02}\frac{e^{-\frac{r}{R_2}}}{r} \tag{4.44}$$

a direct solution of the homogeneous system of two-dimensional integral equations (4.31) leads to a value $E_T = 8.3$ MeV[91] and a solution on the basis of the Bateman expansion ($N = 4$) leads to the value $E_T = 8.56$ MeV[3] ($a_{ts} = 5.45$ F, $r_{0ts} = 1.8$ F, $a_{st} = -23.3$ F, $r_{0st} = 2.8$ F). [In the case of a pure attractive Yukawa potential ($V_{02} = 0$) corresponding to the same values of the two-nucleon parameters we have $E_T = 12.1$ MeV[91].] We note that the values of the binding energy of the triton for different potentials correctly describing the phases of the two-nucleon scattering in a large energy interval are quite close to one another. Allowance for the tensor forces in the two-nucleon interaction can greatly influence the results. Thus, in the case of a separable potential[26,108,78,54,94,102], the introduction of tensor forces, for which the weight of the D wave in the ground state of the deuteron amounts to 4%, leads to a decrease of the triton binding energy by 15–20%. As already noted, in the case of two-particle forces that do not depend on the spins and lead to the existence of a bound S state of two particles, it is possible to have for a system of three particles, besides the ground bound state ($L = 0$), also an excited state ($L = 0$). Allowance for the spin dependence of the forces and for the Pauli principle weakens the two-particle interaction, as a result of which the excited state of the system of three nucleons ($S = 1/2$, $T = 1/2$, $L = 0$) turns out to be impossible. As shown in[5], there are also no excited states of the triton with other quantum numbers. A number of theoretical and experimental works point to the nonexistence of a bound state of a system of three neutrons[5,27,47,59].

### 4.7. Wave Functions of $H^3$ and $He^3$

Owing to the dependence of the two-nucleon interaction on the spin and on the isotopic spin, the wave function of the ground state of the triton ($L = 0$, $S = 1/2$, $T = 1/2$) is presented in the form of a superposition of spatial functions have different symmetries against permutations of the particle coordinates $\Psi^S$, $\Psi'$ and $\Psi''$, $\Psi^a$. Since the summary orbital angular momentum is equal to zero in the ground state, the weight of the symmetrical state $\Psi^S$ (S state) is close to unity. The magnitude of the admixture of the state of the intermediate symmetry (S′ state) is determined by the difference of

**Table IV.** Wave functions of the nuclei $H^3$ and $He^3$, calculated in[35] on the basis of the method of harmonic polynomials with allowance for K = 0 and 2 (square well, parameters (c))

| ρ, F | $H^3$ | | $He^3$ | | |
|---|---|---|---|---|---|
| | $\rho^{\frac{1}{2}}\chi_{00}^{\frac{1}{2}}(\frac{1}{2})(\rho)$ | $10\,\rho^{\frac{1}{2}}\chi_{21}^{\frac{1}{2}}(\frac{1}{2})(\rho)$ | $\rho^{\frac{1}{2}}\chi_{00}^{\frac{1}{2}}(\frac{1}{2})(\rho)$ | $10\,\rho^{\frac{1}{2}}\chi_{21}^{\frac{1}{2}}(\frac{1}{2})(\rho)$ | $100\,\rho^{\frac{1}{2}}\chi_{21}^{\frac{1}{2}}(\frac{3}{2})(\rho)$ |
| 0.125 | 0.00224 | $0.424\cdot10^{-5}$ | 0.00217 | $0.110\cdot10^{-4}$ | $-0.670\cdot10^{-4}$ |
| 0.525 | 0.0711 | 0.00199 | 0.0693 | 0.00259 | −0.00606 |
| 1.025 | 0.309 | 0.0383 | 0.303 | 0.0421 | −0.0389 |
| 1.525 | 0.584 | 0.210 | 0.576 | 0.218 | −0.0885 |
| 2.025 | 0.693 | 0.481 | 0.688 | 0.492 | −0.125 |
| 2.525 | 0.639 | 0.519 | 0.638 | 0.534 | −0.140 |
| 3.025 | 0.525 | 0.445 | 0.528 | 0.464 | −0.139 |
| 4.025 | 0.308 | 0.261 | 0.314 | 0.279 | −0.113 |
| 5.025 | 0.168 | 0.137 | 0.174 | 0.151 | −0.0778 |
| 6.025 | 0.0891 | 0.0696 | 0.0942 | 0.0790 | −0.0496 |
| 8.025 | 0.0247 | 0.0175 | 0.0269 | 0.0211 | −0.0177 |
| 10.025 | 0.00726 | 0.00405 | 0.00731 | 0.00515 | −0.00503 |

the interactions in the triplet-triplet and singlet-triplet spin-isospin states. Calculations lead to the following values of the admixture of the S′ state: P′ = 1.28% (square well[5]), P′ = 4.7% (Morse potential[3], $r_{0st}$ = 2.44 F), and P′ = 2% (superposition of Yukawa potentials[3,91]). The magnitude of the admixture of the antisymmetrical state $P^a$ is determined by the difference of the potentials acting in the odd states $(1/2)(V_{ss} - V_{tt})$, and is very small ($P^a = 0.003\%$[58]).

In the case of the nucleus $He^3$, the weight of the additional S′ state at T = 3/2 amounts to $\overline{P}' = 0.001\%$ (square well[5,35]).

The wave functions of the nuclei $H^3$ and $He^3$ differ insignificantly from each other. Table IV gives the values of the components of the wave functions of $H^3$ and $He^3$ in the case of a two-nucleon nuclear potential of rectangular form (parameters (c)) with allowance for the Coulomb interaction between the protons, calculated in[35] on the basis of the method of harmonic polynomials with allowance for the harmonics K = 0 and 2 (the corresponding values of the binding energies of $H^3$ and $He^3$ turned out to be $E_T$ = 8.45 MeV and $E_{He^3}$ = 7.70 MeV). The weights for the states with intermediate symmetry for the functions of Table IV turned out to be

$$H^3\text{: } P' = 1.01\%; \quad He^3\text{: } P' = 1.1\%, \quad \overline{P}' = 0.001\%.$$

According to (4.19), the charge rms radii of the nuclei $H^3$ and $He^3$ are expressed directly in terms of the rms radii of the nucleon distributions in the nuclei and the rms radii of the charge distribution of the nucleons. Assuming the charge rms radii of the nucleons to be $r_c(p) = (0.84 \pm 0.04)$ F and $r_c(n) = 0$, the following values were obtained in[4] for the charge root-mean-squared radii of the nuclei $H^3$ and $He^3$ (the potential is chosen in the form of a square well):

$$R_c(H^3) = (1.7 \pm 0.04)\ \text{F}, \quad R_c(He^3) = (1.8 \pm 0.04)\ \text{F}.$$

The corresponding experimental values are[56]

$$R_c(H^3) = (1.70 \pm 0.05)\ \text{F}, \quad R_c(He^3) = (1.87 \pm 0.05)\ \text{F}.$$

The influence of the Coulomb interaction between the protons on the quantity $R_c$ ($He^3$) is very small. Using the wave function of $He^3$ calculated without allowance for the Coulomb repulsion between the protons, the rms radius $R_c$ ($He^3$) decreases by approximately 1% compared with the radius calculated on the basis of the function obtained with allowance for the Coulomb interaction[35]. The difference between the values of $R_c$ ($H^3$) [or $R_c$ ($He^3$)], obtained using different sets of two-particle parameters (b) and (c), amounts to approximately 2.5% of their value[35] (the charge radii decrease with increasing $r_{0st}$).

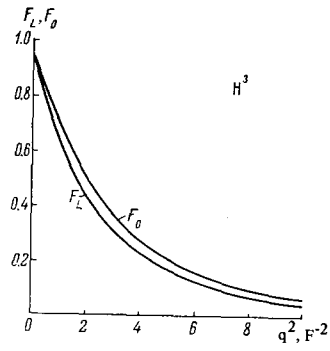


FIG. 10. Form factors of the distributions of the paired and unpaired nucleons in the $H^3$ nucleus, calculated for the case of a square-well potential [35].

The form factors of the distributions of the paired and unpaired nucleons in the $H^3$ nucleus, $F_L$ and $F_0$, calculated on the basis of the functions of Table IV, are shown in Fig. 10. The difference between the form factors $F_L$ and $F_0$ is due to the admixture of the S′ state (P′ = 1.01%). The charge form factors of the nuclei $F_c^{He^3}(q)$ and $F_c^{He^3}(q)$ were calculated for a potential in the form of a square well by Badalyan[4] and for a Morse potential and for a superposition of Yukawa potentials, containing repulsion at small distances, by Akhmadkhodzhaev, Belyaev, and Wrzecionko[3]. The calculated form factors $F_c^{H^3}$ and $F_c^{He^3}$ are in satisfactory agreement with the experimental values[56], but the theoretical values exceed the experimental ones somewhat.

In concluding this section, let us stop to discuss the magnetic moments of the nuclei $H^3$ and $He^3$. According to (4.14) and (4.18), the magnetic moments of the nuclei $H^3$ and $He^3$ in the case of central forces satisfy the inequalities $\mu_{H^3} < \mu_p$ and $\mu_{He^3} > \mu_n$. Allowance for the tensor forces makes these inequalities even stronger[12]. However, the experimental data lead to a reversed sign of the inequality. This indicates that the values of the magnetic moments and the magnetic form factors of three-nucleon nuclei depend strongly on the structure of the nucleons, due to the existence of exchange meson currents.

### 4.8. Deuteron-neutron Scattering Lengths

The scattering of a neutron by a deuteron in the limiting case of zero energy is characterized by two parameters—the quartet and doublet scattering lengths $^4A$ and $^2A$, corresponding to the two possible values of the total spin of the system S = 3/2 and S = 1/2 (T = 1/2). By numerical solution of the one-dimensional integral equations (4.35) in the case of zero energy of the incident neutron ($p_0$ = 0) it was possible to calculate the values of the lengths $^4A$ and $^2A$ for a number of two-

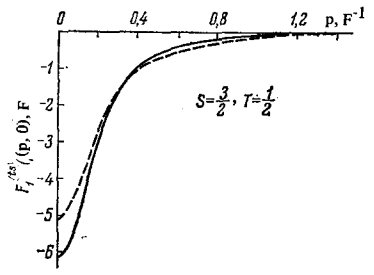


Fig. 11. The functions $F^{(ts)}(p, 0)$ obtained in [76, 81] by numerical integration of the system (4.35) ($l = 0$) for n-d scattering in the quartet state (S = 3/2, T = 1/2) in the case of Hulthen potentials and a square well with allowance for one term in the expansion (2.19). The curves for the Hulthen potentials and the square well practically coincide and are shown in the figure by the solid line. Allowance for the next terms in (2.19) leads to insignificant corrections. The dashed line corresponds to the function $F_l^{(ts)}$ calculated in the approximation of a zero radius of action of the forces [28]. The values of the parameters (a) of Table II were used.

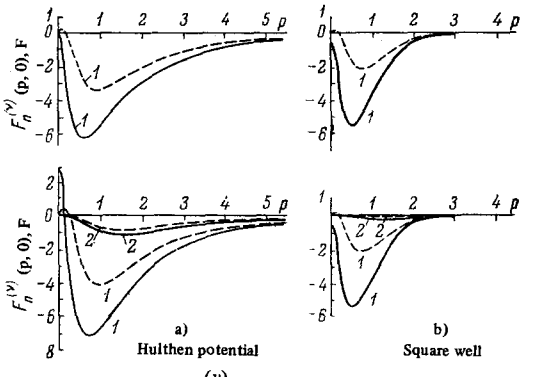


Fig. 12. The functions $F_n^{(\nu)}(p, 0)$ ($\nu$ = ts, st) for n-d scattering in the doublet state (S = 1/2, T = 1/2) calculated in [76, 81] in the case of Hulthen potentials (a) and a square well (b) with allowance for one and two terms in the expansion (2.19) for each spin state of the two nucleons ($l = 0$). The numbers next to the figures indicate the values of n. The values of the parameters (a) of Table II were used.

nucleon potentials. The results of the calculations of the n-d scattering for a Hulthen potential[109,76] and for a square-well potential[81,37], are given in Table III. The components of the function $F_n^{(\nu)}(p, 0) = (\pi/2)p^2\tau_{n_0}^{(\nu)} \times (-\epsilon_{10} - 3p^2/4m)a_{n\,000}^{(\nu)}(p, 0)$ for the quartet and doublet states are shown in Figs. 11 and 12.

We note that the system of integral equations (4.35) with coefficients (4.28), which describes the scattering in the quartet state, depends only on the triplet-singlet spin-isospin two-nucleon parameters, the values of which are well defined. The quartet length $^4A$ turned out to be practically independent of the form of the two-nucleon interaction. For the Hulthen potential, the value of $^4A$, obtained with allowance for four terms in (2.19), is equal to 6.336 F[109] (the parameters (a)). For a rectangular well the quartet length $^4A$ is somewhat smaller: $^4A$ = 6.279 F (in the case of the parameters (a) with allowance for two terms in (2.19)). Allowance for the repulsion between the nucleons at small distances does not lead to an appreciable change in the quartet length. In the case of a square well with infinite repulsion (using two attractive terms and one repulsive term, $n_a = 1$ and 2, $n_c = 1$, in the expansion (2.22)), the quartet length turned out to be $^4A$ = 6.33 F[80]. The quartet scattering length in the case of the Morse potential (4.43) is equal to $^4A$ = 6.35 F, and in the case of the potential (4.44) it is equal to $^4A$ = 6.37 F[11]. This insensitivity of the quartet length to the change of the form of the two-nucleon interaction is due to the Pauli principle, as a result of which the incident neutron does not penetrate inside the deuteron if the spins of all the particles are directed parallel to one another. Thus, the interaction of the neutron with the deuteron reduces to an effective repulsion[13] [the principal matrix element of the n-d interaction in (4.34) with $\nu = \nu' = 1$ corresponds to the repulsion potential, inasmuch as for the quartet spin state the coefficient $c_{11}$ is negative, $c_{11} = -1/2$]. Therefore even in the approximation of zero radius of action of the forces a sufficiently good value[28] $^4A$ = 5.1 F is obtained for $^4A$, although it is found by using only one two-particle parameter, namely the triplet scattering length $a_{ts}$ = 5.378 F.

The values of the doublet length $^2A$ obtained for the Hulthen potential and a square well (see Table III) differ greatly from each other. Just like the triton binding energy $E_T$, the doublet length $^2A$ depends significantly on the form of the two-nucleon potential. The length $^2A$ is also sensitive to the change of the singlet effective radius $r_{0st}$. In the case of a potential with repulsion, the following values are obtained for the doublet scattering length: $^2A$ = 0.76 F (square well with infinite repulsion[80]), $^2A$ = 0.54 F and 1.33 F (Morse potential for the respective cases $r_{0st}$ = 2.44 F, $a_{st} = -23.68$ F and $r_{0st}$ = 2.80 F, $a_{st} = -17$ F[11]), and $^2A$ = 1.15 F (superposition of Yukawa potentials, $r_{0st}$ = 2.80 F, $a_{st} = -23.3$ F[11]).

The calculations of the n-d scattering lengths for the local potentials, including the tensor interaction between nucleons, have not yet been carried out. In the case of separable potentials, allowance for the tensor forces leads to a considerable decrease of the doublet length[108].

Let us compare the calculated values of the quartet and doublet scattering lengths with the experimental values. During the last 20 years, several experiments were organized on the determination of the lengths $^2A$ and $^4A$. The first to measure the total cross section for the scattering of a zero-energy neutron by a deuteron, $\sigma = (4\pi/3)[2(^4A)^2 + (^2A)^2]$ were Fermi and Marshall[62]:

$$\sigma = (3.44 \pm 0.06) \text{ b} . \qquad (4.45)$$

Hurst and Alcock[73] determined the ratio of the doublet and quartet lengths from the scattering of thermal neutrons in ortho and para deuterium:

$$^2A/^4A = 0.12 \pm 0.04. \qquad (4.46)$$

Using the values of (4.45) and (4.46), it is possible to obtain two sets of possible values of the n-d scattering lengths[73]:

$$^4A = (6.38 \pm 0.06) \text{ F}, \quad ^2A = (0.7 \pm 0.3) \text{ F}, \qquad (4.47)$$

$$^4A = (2.6 \pm 0.2) \text{ F}, \quad ^2A = (8.26 \pm 0.12) \text{ F}. \qquad (4.48)$$

The experiments of Shapiro et al. at Dubna[45], using

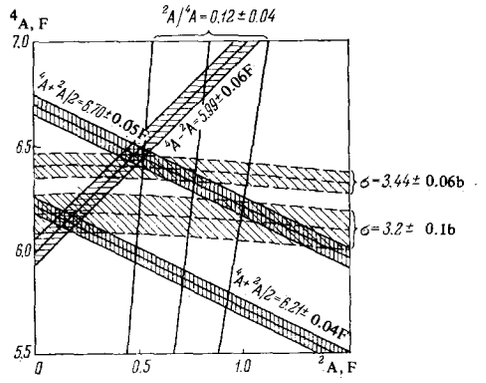


Fig. 13. Experimental data for the determination of the quartet and doublet lengths of n-d scattering.

polarized neutrons and polarized deuterons, have shown that

$$^4A > {}^2A,$$

and consequently have made it possible to discard the second set of scattering lengths (4.48).

Measurements of the cross sections for the scattering of neutrons by ortho and para deuterium, carried out by Nikitin et al.[95], led to the following values for the scattering lengths:

$$^4A = (6.47 \pm 0.14)\ \text{F}, \quad ^2A = (0.57 \pm 0.14)\ \text{F}. \tag{4.49}$$

Measurement of the cross section of the incoherent scattering of a neutron by deuterium bound in the $D_2O$ molecule, $\sigma_{incoh} = 2\pi(^4A - {}^2A)^2$, carried out by Gissler[68], has made it possible to determine the difference between the quartet and doublet scattering lengths:

$$^4A - {}^2A = 5.99 \pm 0.06\ \text{F}. \tag{4.50}$$

The coherent scattering length $A_{coh} = {}^4A + {}^2A/2$ was measured by Bartolini et al.[51] (by the method of specular reflection of neutrons) and by Koester and Ungerer[83] (with small-angle scattering of neutrons by a mixture of powdered solid and liquid). Using the coherence length

$$A_{coh} = 6.21 \pm 0.04\ \text{F}, \tag{4.51}$$

obtained by Bartolini et al.[51] and the length difference (4.50), it is possible to obtain the following values for the quartet and doublet lengths[112,51]:

$$^4A = 6.13 \pm 0.04\ \text{F}, \quad ^2A = 0.15 \pm 0.05\ \text{F}. \tag{4.52}$$

The set of lengths (4.52) corresponds to a total cross section $\sigma = 3.15 \pm 0.04$ b, which agrees with the value

$$\sigma = 3.2 \pm 0.1\ \text{b}, \tag{4.53}$$

obtained by extrapolating to zero energy the experimental values of the total cross section of n-d scattering[112], but is somewhat smaller than the total cross section obtained by Fermi and Marshall[62]. We note also that the ratio of the lengths from the set (4.52) ($^2A/^4A = 0.024 \pm 0.009$) differs greatly from the ratio measured by Hurst and Alcock[73].

Koester and Ungerer[83] obtained for the coherent length the value

$$A_{coh} = 6.70 \pm 0.05\ \text{F}, \tag{4.54}$$

which exceeds the value obtained by Bartolini et al.[51]. The value (4.54) and the value (4.50) for the length difference leads to the following values of the quartet and doublet lengths of n-d scattering:

$$^4A = 6.46 \pm 0.05\ \text{F}, \quad ^2A = 0.47 \pm 0.07\ \text{F}. \tag{4.55}$$

These values correspond to $\sigma = 3.51 \pm 0.06$ b and $^2A/^4A = 0.07 \pm 0.01$.

Figure 13 shows plots of $^4A$ against $^2A$, obtained on the basis of the data on the total cross section (4.45) and (4.53), the coherent length (4.51) and (4.54), the length difference (4.50), and the length ratio (4.46). The intersection of any two curves in Fig. 13 determines the values of the quartet and doublet lengths. The available two experimental values of the total cross section $\sigma$ and the two experimental values of the coherent length $A_{coh}$ lead to two possible sets of the lengths $^4A$ and $^2A$. One of them corresponds to the values of (4.52) obtained by Van Oers and Seagrave[112] and by Bartolini et al.[51], and the other corresponds to the results of Hurst and Alcock (4.47), Nikitin et al. (4.49), and also the data of Gissler and Koester and Ungerer (4.55). The values (4.52) correspond to the total cross section of Van Oers and Seagrave (4.53), but do not agree with the ratio $^2A/^4A$ of Hurst and Alcock (4.46). The values of $^4A$ and $^2A$ in (4.47), (4.49), and (4.55) lead to the total cross section (4.45) of Fermi and Marshall, and their ratios are close to the result of[73]. For final experimental determination of the lengths $^4A$ and $^2A$ it is necessary to have additional, more accurate experiments. In particular, it is necessary to refine the value of the ratio of the lengths $^2A/^4A$.

Let us compare the calculated values of the quartet and doublet lengths with the experimental values. We note first that the quartet length $^4A$ for all the potentials in question is close to the experimental values and agrees best with the value of $^4A$ from the second system of experimental lengths (4.47), (4.49), and (4.55).

The experimental data on the doublet length are less accurate. The experimental values of the doublet length are found in the interval of values from 0.1 to 1.0 F. The values of $^2A$ calculated both in the case of a square well (see Table III) and in the case of a potential with repulsion[11,80] fall in this interval. In this connection, it is of interest to carry out the simultaneous comparison with experiment of the doublet length $^2A$ and of the triton binding energy $E_T$, the value of which is well known ($E_T = 8.482$ MeV). In the case of a Hulthen potential and square well, the binding energy $E_T$ is too high. A decrease of $E_T$, which can be attained by increasing, for example, the singlet effective radius $r_{ost}$, corresponds to an increase of $^2A$. In the case of a square well with parameters (c), a value of $E_T$ equal to the experimental value is reached at $r_{ost} \sim 2.9$ F. The doublet length $^2A$ increases in this case to approximately 1 F. Close values were obtained in[11] for the potential (4.44), which contains repulsion: $E_T = 8.56$ MeV, $^2A = 1.15$ F ($r_{ost} = 2.80$ F, $a_{st} = -23.3$ F). As shown by calculations with separable potentials[26,108,78,27], allowance for the tensor interaction between the nucleons leads to the same effect for $E_T$ and $^2A$ as an increase of the singlet effective radius. Thus, the calculated doublet length corresponding to the experimental value of the binding energy of the triton, like the quartet length,

agrees best with the values of the experimental lengths (4.47), (4.49), and (4.55).

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