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NONLINEAR THEORY OF THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A SOLID-STATE PLASMA AND IN A GASEOUS DISCHARGE

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CONTENTS

AT the present time nonlinear effects in semiconductors and gas-discharge plasmas are attracting more and more attention by theorists and experimentalists. The nonlinear effects which arise in the region of frequencies where temporal (frequency) and spatial dispersion are unimportant are extremely interesting, and also attenuation (solitons, shock waves, and so forth). Under these conditions the interaction of the harmonics, which are being "generated" as a consequence of the nonlinearity, is strong; as a consequence of this the wave far away from the generator, which was sinusoidal when emitted, acquires a nonsinusoidal shape.[1] Turbulences may also arise in the presence of an instability of one or another state of the plasma.[2]

In a semiconductor or gas-discharge plasma there is a wide frequency interval in which temporal dispersion plays a major role (see below for further details), as a consequence of which the temporal harmonics interact among themselves weakly. Nevertheless, if the damping is negligible, then this interaction may lead to a number of new phenomena, which are investigated by nonlinear optics.[3] The problems mentioned above have been widely discussed in monographs and review articles (some of which are cited by us).

The nonlinearities connected with the dissipation of electromagnetic waves occupy a special place. The review article by V. L. Ginzburg and A. V. Gurevich[4] is devoted to these nonlinearities. However, since its appearance a number of new results have been obtained, and these results are stated in the present article, which may be regarded as a continuation of the review article[4].

In a semiconductor or gas-discharge plasma, the nonlinear effects connected with the heating of the gas by the current carriers become important even for relatively small electric fields. The distribution function of the electrons, heated by a constant electric field, was found a rather long time ago.[5]

Upon heating the plasma by a variable electric field, the so-called effects of self-action appear. Interest in these effects is due to the experimental use of electromagnetic fields of large intensity. It was found that in connection with the investigation of the self-action of electromagnetic waves, one can also investigate those properties of a plasma which do not appear in weak fields.

By self-action we shall understand the change in the dielectric constant of the medium due to the influence of the waves propagating in it. This is connected with the fact that both the plasma of the current carriers in semiconductors and a gas-discharge plasma are effectively heated by a relatively weak electric field.[4,5] The dielectric constant depends on the temperature of the current carriers and, consequently, on the propagating field. Mathematically the problem of the propagation of electromagnetic waves in a medium reduces to the determination of the dependence of the current on the field, the substitution of this current into Maxwell's equations, and their solution. We shall follow this scheme.

I. FUNDAMENTAL EQUATIONS

1. The Kinetic Equation and the Equation of Balance

We shall describe the electrons in the plasma* with the aid of the distribution function f(**p**, **r**, t) where **p** denotes the momentum (quasimomentum) of the electron, **r** denotes its coordinate, and t is the time.

If the scattering of energy by the electrons on the scattering centers is quasielastic (and here we shall only consider such processes) and if the following condition on the inhomogeneity of the field is satisfied[4]

$$|\mathbf{E}| \gg \frac{\bar{v}}{\sqrt{\omega^2+\nu^2}}|\nabla\mathbf{E}| \qquad (1.1)$$

(**E** denotes the amplitude of the electric wave, ω is its frequency, $\bar{v}$ is the average thermal velocity of the electrons, and ν denotes the frequency of collisions involving momentum transfer between the electrons and the scattering centers), then one can represent the distribution function in the form[4-6]

$$f(\mathbf{p},\mathbf{r},t)=f_0(\varepsilon,\mathbf{r},t)+\chi(\varepsilon,\mathbf{r},t)\frac{\mathbf{p}}{p}, \qquad (1.2)$$

where $|\chi| \ll f_0$ (ϵ denotes the electron energy).

Here and below it is assumed that the dispersion law of the electrons is quadratic and isotropic, that is, $\epsilon = p^2/2m$.

One can obtain the following system of equations for f_0 and χ by using well-known methods (see[4-6]):

*In what follows a plasma containing a single type of carrier is considered (for the sake of definiteness, the carriers are assumed to be electrons).

$$\left.\begin{aligned} &\frac{\partial f_0}{\partial t}+\frac{p}{3m}\nabla_{\mathbf{r}}\chi+\frac{e}{3}\frac{1}{n(\varepsilon)}\frac{\partial}{\partial\varepsilon}\left\{\frac{p}{m}n(\varepsilon)\chi\mathbf{E}\right\}=\\ &=\frac{1}{n(\varepsilon)}\frac{\partial}{\partial\varepsilon}\left\{n(\varepsilon)\varepsilon^2\nu_e(\varepsilon)\left[\frac{1}{T}f_0(1-f_0)+\frac{\partial f_0}{\partial\varepsilon}\right]\right\}+S_0\{f_0,f_0\},\\ &\frac{\partial\chi}{\partial t}-\omega_H[\mathbf{h}\chi]+\nu(\varepsilon)\chi=-\frac{p}{m}\left[\nabla_{\mathbf{r}}f_0+e\mathbf{E}\frac{\partial f_0}{\partial\varepsilon}\right],\end{aligned}\right\}\quad (1.3)^*$$

where m is the electron mass, $n(\epsilon) = 4\sqrt{2}\pi m^{3/2}\epsilon^{1/2}$ is the density of states, $\mathbf{h} = \mathbf{H}/H$, $\mathbf{H}$ is the external constant magnetic field, $\omega_H = eH/mc$, e is the electron charge, c is the velocity of light in vacuum, $\nu(\epsilon)$ is the frequency of the collisions connected with momentum transfer between the electrons and the scattering centers, $\nu_e(\epsilon)$ is the frequency of the collisions connected with energy transfer, and $S_0\{f_0, f_0\}$ is the collision integral describing electron-electron collisions and having an order of magnitude $\nu_{ee}(\epsilon)f_0$. Formulas for the frequency $\nu_{ee}(\epsilon)$ of interelectron collisions are given in article[7] for the case of a nondegenerate electron gas, and in article[8] for the case of a degenerate electron gas.

Expressions for $\nu(\epsilon)$ and $\nu_e(\epsilon)$ in semiconductors are calculated in[5,9], and in a plasma—in[4].

If the collisions between the electrons and the scattering centers are quasielastic, then $\nu_e/\nu \sim \delta \ll 1$, where in a plasma $\delta \sim m/M^4$ (where M denotes the mass of the ion or neutral molecule), and in a semiconductor $\delta \sim (\hbar\omega_n/\epsilon)^2$ (ω_n denotes the frequency of the phonon on which energy relaxation occurs).

In the presence of several mechanisms for the transfer of energy and momentum we have

$$\nu_e(\varepsilon)=\sum_k \nu_{ek}(\varepsilon),\quad \nu(\varepsilon)=\sum_l \nu_l(\varepsilon), \tag{1.4}$$

where the summation over k runs over all mechanisms of energy transfer, and the summation over l runs over all mechanisms of momentum transfer.

The term $S_1\{f_0, \chi\}$ is omitted in the second equation of (1.3); this term describes the electron-electron collisions in the next approximation in δ. This is valid if the following inequality is satisfied: †[10]

$$\nu_{ee} \ll \omega. \tag{1.5}$$

If in addition to inequality (1.5) the relation**

$$\nu_{ee} \gg \nu_e, \tag{1.6}$$

is also satisfied, then, as follows from (1.3) (also see[4]), the symmetric part of the distribution function f_0 will be a Fermi distribution, with an effective temperature Θ. From inequalities (1.5) and (1.6) it follows that

$$\nu_e \ll \omega. \tag{1.7}$$

In this connection, to the zero-order approximation in ν_e/ω f_0 will not depend on the time‡ [13-15]. Thus

$$f_0(\varepsilon,\mathbf{r})=\left\{1+\exp\left[\frac{\varepsilon-\mu(\mathbf{r})}{\Theta(\mathbf{r})}\right]\right\}^{-1}, \tag{1.8}$$

where μ is the chemical potential.

The relation between the chemical potential μ, the temperature Θ, and the electron concentration N is given by the normalization condition

$$\frac{2}{h^3}\int_0^\infty f_0(\varepsilon,\mathbf{r})\,n(\varepsilon)\,d\varepsilon=N. \tag{1.9}$$

Condition (1.6) is satisfied for sufficiently large electron concentrations. However if relation (1.6) does not hold (the concentration is small), in order to calculate the kinetic coefficients one can as usual use the Fermi distribution function with a certain effective temperature Θ. In this connection the kinetic coefficients differ from those obtained upon a rigorous kinetic investigation by factors of the order of unity. This is connected with the fact that the kinetic coefficients, which are expressed in terms of the moments of the distribution function, are not sensitive to its explicit form.[16] In what follows, for the calculations we shall use f_0 in the form (1.8).

Confirmation of the conclusions reached above follows from a comparison of the results of article[17] with the results of articles[18,19].

As will be demonstrated below by direct calculations (also see[18]), the dielectric constant of a plasma does not depend on the time if f_0 does not depend on the time. Under this assumption the effects associated with frequency multiplication are not present, and monochromatic waves can propagate in the medium.

Since the heating of the electromagnetic field is damped, the electron temperature Θ will decrease in the direction of propagation of the wave, i.e., a gradient appears in the electron temperature. Therefore, in addition to the variable field, an electrostatic field also appears (thermoeffect), which is associated with this gradient.*)

Thus, one can represent the electric field $\mathbf{E}$ in the form of the sum of the constant field $\mathbf{E}_c$ due to the gradient of the electron temperature and the variable field $\sim e^{-i\omega t}$. Below we shall denote the amplitude of this field by the letter $\mathbf{E}$. Correspondingly one can represent χ in the form of the sum of the following quantities: χ_c, which does not depend on the time, and $\chi_U e^{-i\omega t}$. With the aid of χ the electric current $\mathbf{j}$ and the heat flux $\mathbf{Q}$ are determined by the following formulas:[22,23]

$$\mathbf{j}=\frac{16\pi me}{3h^3}\int_0^\infty \varepsilon\chi(\varepsilon)\,d\varepsilon,\quad \mathbf{Q}=\frac{16\pi m}{3h^3}\int_0^\infty \varepsilon^2\chi(\varepsilon)\,d\varepsilon. \tag{1.10}$$

We find χ from the second equation of the system (1.3), we substitute it into the first equation of this system, and integrate the resulting equation over the momentum, having multiplied it by 1 and by ϵ. Assuming that the magnetic field is directed along the direction of propagation of the electromagnetic wave,† it is convenient to change (see[17]) to normal waves possessing circular polarization, $E = E_x \pm iE_y$. It is assumed everywhere below that the wave $E = E_x - iE_y$ is propagating in the plasma. Then we obtain a system consisting of two transport equations (we note that the system which

*$[\mathbf{h}\chi] \equiv \mathbf{h} \times \chi$.

†Nonfulfillment of this condition leads to an error of the order of unity in the numerical factors of the kinetic coefficients. [6]

**For a nondegenerate electron gas, in connection with the scattering of energy via acoustic phonons, inequality (1.6) goes over into the criterion of Fröhlich and Paranjape. [7]

‡In that case when the wave is circularly polarized, f_0 does not depend on the time for any arbitrary relation between ω and ν. [11,12]

*The investigation of such effects is of independent interest. Here we shall not consider them, referring to articles [20-21].

†The results are given in articles [17,18] for an arbitrary orientation of the magnetic field.

is thus obtained follows from the condition for solvability of the system of equations (1.3)[16]):

$$\left.\begin{aligned} &\operatorname{div} \mathbf{j}_c = 0, \\ &\operatorname{div} \mathbf{Q}_c + NT\nu_e(\vartheta)(\vartheta - 1) = \sigma(\vartheta)\, u^2; \end{aligned}\right\} \tag{1.11}$$

here the following notation has been introduced (see[23]): $\mathbf{j}_c$ and $\mathbf{Q}_c$ denote the densities of electric current and of the heat flux, which are connected with the thermal e.m.f., where

$$\left.\begin{aligned} &\mathbf{j}_c = e^2 J_{10}\mathbf{E}'_c - eJ_{11}\nabla \ln \vartheta + \mathbf{h}\,(e^2 J_{30}\mathbf{E}'_c - J_{31}\nabla \ln \vartheta,\ \mathbf{h}), \\ &\mathbf{Q}_c = eJ_{11}\mathbf{E}'_c - J_{12}\nabla \ln \vartheta + \mathbf{h}\,(eJ_{31}\mathbf{E}'_c - J_{32}\nabla \ln \vartheta,\ \mathbf{h}), \\ &\mathbf{E}'_c = \mathbf{E}_c - \frac{\vartheta}{e}\nabla \frac{\mu}{\vartheta}, \\ &J_{lj} = -\frac{16\sqrt{2}\,\pi\omega_H^{l-1} m^{\frac{1}{2}} T^{j+\frac{3}{2}}}{3h^3} \int_0^\infty \frac{dx\, x^{j+\frac{3}{2}}}{\nu^l\left[1 + \left(\frac{\omega_H}{\nu}\right)^2\right]} \frac{df_0}{dx}, \end{aligned}\right\} \tag{1.12}$$

$\vartheta = \Theta/T$ is the dimensionless electron temperature,

$$\sigma(\vartheta) = -\frac{8\sqrt{2}\, m^{\frac{3}{2}} T^{\frac{3}{2}} \omega_0^2}{3N} \int_0^\infty \frac{\nu(x)\, x^{\frac{3}{2}}}{(\omega_H - \omega)^2 + \nu^2(x)} \frac{df_0}{dx}\, dx \tag{1.13}$$

is the high-temperature conductivity, $\omega_0^2 = 4\pi e^2 N/m$ is the Langmuir frequency, $x = \epsilon/T$, and u is the absolute value of the electric field.

As the calculations show,[18] the following formulas hold for $\nu(x)$ and $\nu_e(\vartheta)$:

$$\nu(x) = \nu_0(T)\, x^{-q}, \quad \nu_e(\vartheta) = \nu_{0e}(T)\, \vartheta^{r-1}. \tag{1.14}$$

The values of r, q, $\nu_0(T)$, and $\nu_{0e}(T)$ are determined by the mechanism responsible for the scattering of the electrons (see[9,10] and also see the table).

The second equation of (1.11) admits a simple physical interpretation. The term on the right-hand side of the equation describes the liberation of heat due to the heating of the electron gas by the field of the wave; the second term on the left describes the transfer of heat from the electron subsystem to the lattice, and the first term on the left describes the heat flux in the electron gas. Thus, this equation is the equation of energy balance for the electron subsystem.

Let us transform the equation of balance, having noted beforehand that in the one-dimensional case which is being considered here, $\mathbf{j}_c$, $\mathbf{Q}_c$, ϑ, and the other quantities only depend on a single coordinate (on z).

Let us eliminate the static field $\mathbf{E}_c$ from the expression for the heat flux $\mathbf{Q}_c$. In a longitudinal magnetic field $E_{cx} = E_{cy} = 0$. The z-component of the static current j_c is therefore equal to zero. This follows from the first equation of (1.11) and from the absence of a static current at infinity (we recall that there is no heating at infinity). We substitute the thermomagnetic field E_{cz} found from these conditions into the second equation of (1.12) and, by taking the one-dimensional nature of the problem into consideration, we obtain the following relation for $Q_{cz} = Q_z$:

$$Q_z = -T\varkappa(\vartheta)\frac{d\vartheta}{dz}, \tag{1.15}$$

where $\kappa(\vartheta)$ denotes the coefficient of electronic thermal conductivity, and moreover

$$\varkappa(\vartheta) = \lambda_0 \vartheta^{1+q}, \tag{1.16}$$

and $\lambda_0 = 4\Gamma(5/2 + q)NT/3\pi^{1/2}m\nu_0$ is the thermal conductivity of the electrons in a weak field.[18,23]

For a completely degenerate electron gas

$$\varkappa(\vartheta) = \frac{\pi^2 NT}{3\nu(\epsilon_0)\, m}\vartheta, \quad \nu(\epsilon_0) = \nu_0(T)\left(\frac{\epsilon_0}{T}\right)^{-q}, \tag{1.17}$$

where ϵ_0 denotes the Fermi energy.

Finally, with Eq. (1.15) taken into consideration, we obtain the equation for the electron temperature from the second equation of (1.11):

$$T\frac{d}{dz}\varkappa(\vartheta)\frac{d\vartheta}{dz} - NT\nu_e(\vartheta)(\vartheta - 1) = -\sigma(\vartheta)\, u^2. \tag{1.18}$$

2. Maxwell's Equation and the Boundary Conditions

Maxwell's equation for the amplitude $E = E_x - iE_y$ has the form[18]

$$\frac{d^2E}{dz^2} + k^2\varepsilon(\vartheta)\, E = 0, \tag{2.1}$$

where

$$\operatorname{Im} \varepsilon(\vartheta) = \frac{4\pi i\sigma(\vartheta)}{\omega},$$

$$\varepsilon(\vartheta) = \varepsilon_0 - \frac{8\sqrt{2}\,\pi}{3}\frac{m^{\frac{3}{2}} T^{\frac{3}{2}}\omega_0^2}{N\omega}\int_0^\infty \frac{x^{\frac{3}{2}}}{\omega_H - \omega - i\nu(x)}\frac{df_0}{dx}\, dx; \tag{2.2}$$

here ϵ_0 denotes the dielectric constant of the lattice (in the plasma $\epsilon_0 = 1$), and $k = \omega/c$.

Maxwell's equation is nonlinear in virtue of the dependence of the dielectric constant on Θ. However, for semimetals and degenerate semiconductors $\partial f_0/\partial\epsilon = -\delta(\epsilon - \epsilon_0)$ (δ is the delta-function) correct to $(\Theta/\epsilon_0)^2 \ll 1$, and the dielectric constant does not depend on Θ.[18] In this connection the fields are described by the formulas of the linear theory.

Now let us formulate the boundary conditions on the problem. Let the plasma (semiconductor or gas-discharge) occupy the half-space $z > 0$; a plane monochromatic wave of frequency ω, coming from infinity ($z = -\infty$), is incident on the plasma at right angles to the interface ($z = 0$). Incidence of the wave at some angle (other than 90°) is not of any interest since the results in this case differ from the results obtained below by a redefinition of certain constants.[24] For simplicity we assume that the region of space $z < 0$ is filled with a linear nondissipative medium with an index of refraction $n = 1$. Then for $z < 0$ the wave will have the form

$$E = E_0\left(e^{ikz} + Re^{-ikz}\right), \tag{2.3}$$

where E_0 is the amplitude of the incident wave, c is the velocity of light in vacuum, and R is the coefficient of reflection. We shall also assume that the temperature of this medium coincides with the lattice temperature T.

It is assumed below that the characteristic distance L over which the field changes is much larger than the

Scattering Mechanism	r	q
Semiconductors		
Acoustic vibrations	3/2	−1/2
Optical vibrations, $T < T_d$	1	0
Optical vibrations, $T > T_d$	−1/2	−1/2
Piezo-acoustic vibrations	1/2	1/2
Polar semiconductors, scattering by optical vibrations, $T > T_d$	−1/2	−3/2
Neutral impurities	—	0
Charged impurities	—	3/2
Dipole impurities	—	1/2
Plasma		
Scattering by ions	−1/2	3/2
Scattering by molecules	3/2	−1/2

Debye radius $d \sim \bar{\epsilon}/(4\pi e^2 N)^{1/2}$. As is well known from the theory of plasmas, upon fulfillment of this inequality one can regard the plasma as quasineutral. For a semiconductor containing carriers of a single sign this means that at any arbitrary point the charge density of the electrons (or holes) is equal to the equilibrium density unless processes of the type of impact ionization, changes in the recombination coefficient in the field, etc. are taken into consideration.

From here it follows that the concentration of carriers in an impurity semiconductor does not depend on the coordinates. In a plasma the last assertion is valid provided the duration Δt of the pulse is much shorter than the time Δt_N for establishing the concentration.[25]

The case when the concentration depends on the coordinates, which corresponds, upon fulfillment of the condition for quasineutrality $L \gg d$, to intrinsic semiconductors and plasma with $\Delta t \gg \Delta t_N$, will not be considered here, just as semiconductors containing two types of carriers will not be considered for $\Delta t \ll \Delta t_N$. Also the case of a layered-inhomogeneous plasma will not be considered. Readers who are interested in these questions are referred to[25-29].

We assumed above that the wave in the plasma is circularly polarized. For this to occur, the polarization of the wave incident on the half-space must also be circular. We note that in contrast to the linear theory, where by the superposition of normal waves one can satisfy the boundary conditions for arbitrary polarization of the incident wave, for the case of nonlinear propagation the superposition of normal waves is not a solution of Maxwell's equation. Therefore, if the polarization of the incident wave does not coincide with the polarization of one of the normal waves in the plasma, the picture becomes complicated.[30] We shall not dwell on this question here since the results obtained below remain qualitatively valid even for arbitrary polarization of the incident wave.[30]

Since the plasma is semi-bounded, it is necessary to add to Eqs. (1.18) and (2.1) the boundary conditions at the interface $z = 0$ and for $z \to \infty$.

The boundary conditions for the field have the usual form:

$$E(-0) = E(+0), \quad \frac{\partial E(-0)}{\partial z} = \frac{\partial E(+0)}{\partial z}. \qquad (2.4)$$

In the presence of attenuation

$$\lim_{z\to\infty} E(z) \to 0, \qquad (2.5)$$

i.e., the heating of the electron gas is absent at infinity; therefore

$$\lim_{z\to\infty} \vartheta(z) \to 1. \qquad (2.6)$$

If the wave incident on the half-space has the form (2.3), then from the boundary conditions (2.4) one obtains the following expressions for R and for the coefficient of transmission P:

$$P = \frac{E(+0)}{E_0} = \frac{2\zeta}{1+\zeta}, \quad R = \frac{\zeta-1}{\zeta+1}; \qquad (2.7)$$

here ζ denotes the surface impedance which is defined by the formula

$$\zeta^{-1} = -\frac{ic}{\omega}\frac{1}{E(+0)}\frac{\partial E(+0)}{\partial z}. \qquad (2.8)$$

For $|\zeta| \ll 1$

$$P = 2\zeta, \quad R = -1 + 2\zeta. \qquad (2.9)$$

The boundary condition for the electron temperature on the plane $z = 0$ is obtained in the following way: let us integrate Eq. (1.18) term by term with respect to z from $0 - \xi$ to $0 + \xi$ and then let ξ tend to zero. Assuming that no surface conductivity is present, we have[31]

$$\left.\begin{array}{c} T\varkappa(\vartheta)\dfrac{d\vartheta}{dz} = \eta(\vartheta)(\vartheta-1)|_{z=+0}, \\ \eta = \lim\limits_{\xi\to 0} NT\displaystyle\int_{-\xi}^{+\xi}\nu_e(\vartheta)\,dz. \end{array}\right\} \qquad (2.10)$$

In order to obtain (2.10) it was considered that no heat fluxes are present in the thermostat ($Q(z \leq 0) = 0$). The sign of η is chosen such that the heat fluxes in the sample were directed outwards. The right-hand side of Eq. (2.10) describes the inelastic surface mechanism for the absorption of energy, and η characterizes its effectiveness. Thus, for $\eta = 0$ (see the second formula of (2.10)) no specific surface mechanisms exist for the absorption of energy, but as $\eta \to \infty$ this mechanism becomes so strong that $\vartheta(+0) = 1$, that is, $\Theta(+0) = T$.

II. THE NORMAL SKIN EFFECT

3. Propagation of Weakly-damped Electromagnetic Waves

As has already been indicated above, two mechanisms exist for the removal of energy from the electrons: the thermal conductivity which is described by the first term on the left-hand side of Eq. (1.18), and the transfer of energy to the lattice, which corresponds to the second term on the left-hand side of this same equation. The ratio of these terms is of the order of l_e^2/L_Θ^2, where L_Θ is the characteristic length over which the temperature changes, and

$$l_e \approx \frac{\bar{v}}{\sqrt{\nu\nu_e}} \qquad (3.1)$$

is the energy mean free path, characterizing the transfer of energy to the lattice.

One can neglect the thermal conductivity provided $l_e^2/L_\Theta^2 \ll 1$. In this connection, as follows from Eq. (1.18), the relation between the temperature and the field is local and consequently $L_\Theta \sim L$.

If the inequality

$$\frac{l_e^2}{L_\Theta^2} \sim \frac{l_e^2}{L^2} \ll 1, \qquad (3.2)$$

is satisfied, then we shall talk about the normal skin effect.

By the anomalous skin effect we mean that situation when the inequality

$$l \ll L \leqslant l_e \qquad (3.3)$$

is satisfied ($l = \bar{v}/\nu$ denotes the mean free path connected with momentum transfer), because the case $L \lesssim l$ is actually not realized in semiconductors.*) However, the inequality (3.3) may be satisfied since from the defini-

*Upon fulfillment of the inequality $L \lesssim l$, the principal role is played not by the effects connected with the heating of the electron gas but by the so-called striction effects. [25,32]

tion of l_e it follows that $l/l_e \sim \delta^{1/2} \ll 1$. From Eq. (1.18) it follows that for $L \lesssim l_e$ the gradient term is of the same order as the remaining terms, and the relation between the electron temperature and the amplitude of the wave is nonlocal.

Let us consider the propagation of weakly damped waves, for which the wavelength is much smaller than the attenuation length. In this case the nonlinearity due to the dissipative part of the dielectric constant will be small. This smallness is guaranteed by the fulfillment of one of the other inequalities:

$$\frac{\nu}{|\omega-\omega_H|} \ll 1, \quad \frac{\omega_0^2}{\omega\,|\omega_H-\omega+i\nu|} \ll 1. \tag{3.4}$$

We note that the second inequality is only due to the smallness of the concentration and may be satisfied for any arbitrary relation between $|\omega - \omega_H|$ and ν. In what follows, for this case we shall assume $|\omega - \omega_H| \ll \nu$.* For small attenuation, the dielectric constant can be written as follows:

$$\left.\begin{aligned} \varepsilon &= \varepsilon_0 + \frac{\omega_0^2}{\omega(\omega_H-\omega)} + \frac{4i\Gamma\left(\frac{5}{2}-q\right)\omega_0^2\nu_0}{3\pi^{1/2}\omega(\omega_H-\omega)^2}\vartheta^{-q} \quad \text{for} \quad \frac{\nu}{|\omega_H-\omega|} \ll 1; \\ \varepsilon &= \varepsilon_0 + \frac{4i\Gamma\left(\frac{5}{2}+q\right)\omega_0^2}{3\pi^{\frac{1}{2}}\omega\nu_0}\vartheta^{q} \quad \text{for} \quad \frac{\omega_0^2}{\omega\nu} \ll 1. \end{aligned}\right\} \tag{3.5}$$

If the nonlinearity is small, then one can use the methods developed in the monograph[33] in order to solve Maxwell's equation. Let us consider the case when the first of the relations (3.5) is satisfied. We shall seek the solution in the form

$$E = u(z)\,e^{i(knz-\omega t)}, \quad n = \sqrt{\varepsilon_0 - \frac{\omega_0^2}{\omega(\omega-\omega_H)}}. \tag{3.6}$$

In virtue of the assumption about the smallness of the attenuation, u(z) changes slowly in comparison with the exponential. One can write down the following shortened equation for u(z) (see[34]):**

$$\frac{du}{dz} + \xi_0\vartheta^{-q}u = 0; \tag{3.7}$$

here

$$\xi_0 = \frac{2\Gamma\left(\frac{5}{2}-q\right)\omega_0^2\nu_0}{3\pi^{1/2}c(\omega_H-\omega)^2 n}$$

is the attenuation in the linear theory.

In the same approximation the equation of balance appears as follows:

$$\sigma_0\vartheta^{-q}u^2 = NT\nu_{0e}\vartheta^{r-1}(\vartheta-1), \tag{3.8}$$

where

$$\sigma_0 = \frac{4\Gamma\left(\frac{5}{2}-q\right)\nu_0\omega_0^2}{3\pi^{3/2}(\omega_H-\omega)^2}$$

is the conductivity in the linear theory.

In order for Eq. (3.8) to determine ϑ as a single-valued monotonic function of u, the inequality $r + q > 0$ must be satisfied. In what follows this inequality will be regarded as satisfied (as well as the inequality $r - q > 0$).*

The system of equations (3.7) and (3.8) has been investigated in a series of articles (see[13,34,35]). Numerical methods for its solution were developed in articles[34,35]. Here we shall be interested in the derivation of expressions for $\vartheta(z)$ and u(z) and the investigation of their asymptotic behavior.

Eliminating u(z) from Eqs. (3.7) and (3.8), we obtain the following equation for ϑ:

$$\{(r+q)\vartheta^{q} - (r+q-1)\vartheta^{q-1}\}\frac{d\vartheta}{dz} + 2\xi_0(\vartheta-1) = 0, \tag{3.9}$$

whose solution has the appearance:

$$-2\xi_0 z = \int\limits_{\vartheta_0}^{\vartheta} \frac{[(r+q)\vartheta-(r+q-1)]}{\vartheta-1}\vartheta^{q-1}\,d\vartheta, \tag{3.10}$$

where $\vartheta_0 = \vartheta(+0)$. The value of ϑ_0 is related to $u_0(u(+0))$ by Eq. (3.8), where it is necessary to express u_0 in terms of the amplitude E_0 of the incident wave. From Eqs. (2.3), (2.7), and (3.6) we find this relation and the coefficient of reflection:

$$u_0 = \frac{2E_0}{1+n}, \quad R = \frac{1-n}{1+n} + \frac{2i\xi_0}{(1+n)^2}\vartheta_0^{-q}. \tag{3.11}$$

For the important case of helical waves ($\omega_H \gg \omega$, $\omega_0^2/\omega\omega_H \gg 1$),† assuming that $\vartheta_0 \gg 1$ we have

$$\vartheta_0 = \left(\frac{16\Gamma\left(\frac{5}{2}-q\right)\nu_0\omega|E_0|^2}{3\pi^{\frac{3}{2}}\omega_H\nu_{0e}NT}\right)^{\frac{1}{r+q}}. \tag{3.12}$$

In this connection

$$n = \frac{\omega_0}{(\omega\omega_H)^{\frac{1}{2}}}, \quad \xi_0 = \frac{2\Gamma\left(\frac{5}{2}-q\right)\omega_0\nu_0\omega^{\frac{1}{2}}}{3\pi^{\frac{1}{2}}c\,\omega_H^{\frac{3}{2}}}. \tag{3.13}$$

In the absence of any magnetic field

$$\left.\begin{aligned} \vartheta_0 &= \left(\frac{8cn^{\frac{1}{2}}\xi_0|E_0|^2}{\pi\nu_{0e}(1+n)^2NT}\right)^{\frac{1}{r+q}}, \quad n = \left(\varepsilon_0 - \frac{\omega_0^2}{\omega^2}\right)^{\frac{1}{2}}, \\ \xi_0 &= \frac{2\Gamma\left(\frac{5}{2}-q\right)\omega_0^2\nu_0}{3\pi^{\frac{1}{2}}c\omega^2 n}. \end{aligned}\right\} \tag{3.14}$$

Equation (3.10) determines ϑ as an implicit function of z and simultaneously with (3.7) determines u(z) in parametric form. It has not been possible to evaluate the integral in (3.10) for arbitrary values of r and q. Therefore let us consider two regions: the region immediately adjacent to the surface $z = 0$, where in this region we confine our attention to the case of strong heating, $\vartheta(z) \gg 1$, and the region in the depths of the sample where $\vartheta(z) - 1 \ll 1$ (the region of weak heating).

In the region where $\vartheta(z) \gg 1$, by neglecting unity in comparison with ϑ in Eq. (3.10), we find the following result for ϑ:

$$\vartheta = \vartheta_0\left(1 - \frac{2q}{r+q}\xi_0\vartheta_0^{-q}z\right)^{1/q}. \tag{3.15}$$

As $z \to \infty$ the field is attenuated; therefore the temperature must tend to unity. Let us find the asymptotic behavior of ϑ for large values of z. To this end we isolate the divergent part from the integral in (3.10), having re-

*The resonance case $\omega = \omega_H$, $\nu/\omega_H \ll 1$ associated with strong attenuation is considered in Sec. 4 of the present review, and for the case of weak attenuation, it is considered in [18].

**Here and below it is assumed for simplicity that there is one mechanism for the scattering of energy and one mechanism for the scattering of momentum.

*The cases $r \pm q < 0$ will be considered separately (see Sec. 5).

†Helical waves of large amplitude have been investigated in article [18].

written (3.10) in the form

$$-2\xi_0 z = \int_{\vartheta_0}^{\vartheta} \left[\frac{[(r+q)\vartheta-(r+q-1)]}{\vartheta-1}\vartheta^{q-1} - \frac{1}{\vartheta-1} \right] d\vartheta + \ln\frac{\vartheta-1}{\vartheta_0-1}. \quad (3.16)$$

For $\vartheta \to 1$ one can replace the upper limit on the integral in Eq. (3.16) by unity since the integral converges. Finally we obtain the following expression for $\vartheta(z)$ as $z \to \infty$ (assuming that $\vartheta_0 \gg 1$)

$$\vartheta = 1 + \vartheta_0 S_\vartheta e^{-2\xi_0 z}, \quad (3.17)$$

where

$$S_\vartheta = \exp\left\{ \int_1^{\vartheta_0} \left[\frac{[(r+q)\vartheta-(r+q-1)]}{\vartheta-1}\vartheta^{q-1} - \frac{1}{\vartheta-1} \right] d\vartheta \right\} \quad (3.18)$$

is the so-called self-action factor for the temperature.[18] After the temperature is found, E(z) is obtained from Eqs. (3.6) and (3.7). For $\vartheta(z) \gg 1$ we have

$$E = PE_0 e^{iknz}\left(1 - \frac{2q}{r+q}\xi_0\vartheta_0^{-q} z\right)^{\frac{r+q}{2q}}, \quad (3.19)$$

and for $\vartheta(z) - 1 \ll 1$

$$E = PE_0 S_E \exp\{iknz - \xi_0 z\}, \quad (3.20)$$

where S_E is the self-action factor for the field,[34]

$$S_E = \exp\left\{ \frac{1}{2}\int_1^{\vartheta_0} \frac{[(r+q)\vartheta-(r+q-1)]}{\vartheta(\vartheta-1)}[\vartheta^q - 1]\, d\vartheta \right\}, \quad (3.21)$$

and also for $\vartheta_0 \gg 1$

$$S_E \sim \begin{cases} \exp\left\{\frac{r+q}{2q}\vartheta_0^q\right\} \gg 1 & \text{for } q > 0, \\ \vartheta_0^{-\frac{r+q}{2}} \ll 1 & \text{for } q < 0. \end{cases} \quad (3.22)$$

This factor has the following physical meaning: for large values of z the temperature ϑ of the electron gas is close to unity, i.e., the nonlinearity is essentially absent. However, the wave "remembers" that it went through a strongly nonlinear region near z = 0. The self-stress factor takes this fact into consideration. From Eqs. (3.20) and (3.22) it is seen that for q > 0 the self-stress factor increases the field for large values of z in comparison with the linear case, but for q < 0 it decreases the field. This is connected with the fact that $\nu(\epsilon) \sim \epsilon^{-q}$ and for negative values of q the quantity $\nu(\epsilon)$, and consequently also the attenuation, increase with increasing values of ϵ and the field is attenuated more rapidly than in the linear theory; for q > 0 the converse situation occurs.

The characteristic distance over which the field decreases in the region of strong heating is given by $L \sim L_0\vartheta_0^q$ ($L_0 \sim \xi_0^{-1}$ is the attenuation depth in the linear theory), i.e., the conclusions reached above remain valid.

For q = 0 the expressions for the field given by Eqs. (3.19) and (3.20) together with (3.21) go over into the usual formula of the theory of the linear skin effect with exponential damping of the field.

In that case when the second system of inequalities (3.4) is satisfied, all of the results are obtained by replacing q by −q in Eqs. (3.7) through (3.22). In this connection, we obtain

$$n = \sqrt{\varepsilon_0}, \quad \xi_0 = \frac{2\Gamma\left(\frac{5}{2}+q\right)\omega_0^2}{3\pi^{\frac{1}{2}} c\nu_0}, \quad L = L_0\vartheta_0^{-q},$$

$$\vartheta_0 = \left(\frac{4\Gamma\left(\frac{5}{2}+q\right)\omega_0^2 |E_0|^2}{3\pi^{\frac{3}{2}}\nu_0\nu_{0e}NT} \right)^{\frac{1}{r-q}}. \quad (3.23)$$

The relation $\vartheta_0 \gg 1$ was used above. For this to hold, the amplitude of the incident wave must be much larger than the plasma field:[4]

$$|E_0| \gg u_p, \quad (3.24)$$

where

$$u_p = \begin{cases} \frac{T}{l_{0e}e}\left(\frac{\omega_0^2\omega_H}{\nu_0^2\omega}\right)^{\frac{1}{2}} & \text{for helical waves,} \\ \frac{T}{l_{0e}e} & \text{for } \omega_H = 0;\ \frac{\omega_0^2\vartheta^q}{\nu_0\omega} \ll 1. \end{cases}$$

From the definition of helical waves it follows that the plasma field for them is much larger than the one associated with the propagation of an electromagnetic wave in the absence of a magnetic field. The normal skin effect is realized, for example, in InSb with $N \sim 10^{16}$ cm^{-3}, $\nu \sim 3\times10^{11}$ sec^{-1} with T = 10°K for helical waves. In this connection, at the frequency $\omega = 10^{11}$ sec^{-1} (H = 10^3 Oe), one has

$$\omega = 10^{11}\,\text{sec}^{-1} (H = 10^3\,\text{Oe})\ l_e \sim 10^{-2}\,\text{cm},\ L \sim 10^{-1}\,\text{cm}.$$

4. The Propagation of Electromagnetic Waves in Media with Strong Damping

If the imaginary part of the dielectric constant $\epsilon(\vartheta)$ is much larger than the real part, then the electromagnetic wave will be strongly damped. From (2.2) it follows that this case is realized upon fulfillment of one of the two systems of inequalities:

$$\left.\begin{array}{l} \omega = \omega_H, \quad \frac{\nu}{\omega_H} \ll 1, \quad \frac{\omega_0^2\vartheta^q}{\omega_H\nu_0} \gg \varepsilon_0 \\ \text{or} \\ \omega_H = 0, \quad \frac{\nu}{\omega} \gg 1, \quad \frac{\omega_0^2\vartheta^q}{\omega\nu_0} \gg \varepsilon_0. \end{array}\right\} \quad (4.1)$$

The first system corresponds to cyclotron resonance in media possessing large electron concentrations, and the second corresponds to low-frequency waves.

Upon fulfillment of the inequalities (4.1), the following expressions for the dielectric constant are obtained from Eq. (2.2):

$$\varepsilon = \frac{4i\Gamma\left(\frac{5}{2}+q\right)\omega_0^2\vartheta^q}{3\pi^{\frac{1}{2}}\omega_H\nu_0} \quad \text{for } \omega = \omega_H, \quad \frac{\nu}{\omega_H} \ll 1,$$

$$\varepsilon = \frac{4i\Gamma\left(\frac{5}{2}+q\right)\omega_0^2\vartheta^q}{3\pi^{\frac{1}{2}}\omega\nu_0} \quad \text{for } \omega_H = 0, \quad \frac{\nu}{\omega} \gg 1.$$

If the first system of inequalities in (4.1) holds, then the equation of balance and Maxwell's equation take the following form:

$$\left.\begin{array}{l} \sigma_0\vartheta^q u^2 = NT\nu_{0e}\vartheta^{r-1}(\vartheta - 1), \\ \frac{d^2E}{dz^2} + 2i\xi_0^2\vartheta^q E = 0, \end{array}\right\} \quad (4.2)$$

where

$$\xi_0^2 = \frac{2\Gamma\left(\frac{5}{2}+q\right)\omega_H\omega_0^2}{3\pi^{\frac{1}{2}}c^2\nu_0}, \quad \sigma_0 = \frac{4\Gamma\left(\frac{5}{2}+q\right)\omega_0^2}{3\pi^{\frac{3}{2}}\nu_0}. \quad (4.3)$$

One is able to solve the system of equations (4.2) only in the region $\vartheta(z) \gg 1$.[17,18] Assuming that $\vartheta(z) \gg 1$, from the first equation in (4.2) we determine ϑ:

$$\vartheta = \left[\frac{4\Gamma\left(\frac{5}{2}+q\right)\omega_0^2 u^2}{3\pi^{\frac{3}{2}}\nu_0\nu_{0e}NT}\right]^{\frac{1}{r-q}},$$

or

$$\vartheta = \vartheta_0\left(\frac{u}{u_0}\right)^{\frac{2}{r-q}}, \tag{4.4}$$

where

$$\vartheta_0 = \left[\frac{4\Gamma\left(\frac{5}{2}+q\right)\omega_0^2 u_0}{3\pi^{\frac{3}{2}}\nu_0\nu_{0e}NT}\right]^{\frac{1}{r-q}}. \tag{4.5}$$

Substituting ϑ from (4.4) into the second equation in (4.2), we have

$$\frac{d^2E}{dz^2} + 2i\xi_0^2\vartheta_0^q\left(\frac{u}{u_0}\right)^{\frac{2q}{r-q}} E = 0. \tag{4.6}$$

We recall that $u = |E|$.

We shall seek the solution of Eq. (4.6) in the form

$$E = E_0(1+\varkappa z)^{-(\alpha+i\beta)}, \tag{4.7}$$

where E_0 denotes the value of E(z) for z = +0.

Substituting (4.7) into (4.6), equating the powers of the exponents associated with $(1 + \kappa z)$, and then separating the real and imaginary parts, we obtain equations for the determination of α, β, and κ. Finally, for E(z) we find

$$E = 2\zeta E_0\left(1 + \frac{q}{[(2r-q)(r-q)]^{\frac{1}{2}}}\,\frac{\omega}{c|\zeta|}z\right)^{-\frac{r+q}{q}+\frac{i}{q}[r(r-q)]^{\frac{1}{2}}}. \tag{4.8}$$

In the derivation of (4.8) it is assumed that $|\zeta| \ll 1$. The smallness of ζ follows from: the penetration depth in the resonant case is small in comparison with the wavelength in vacuum, and $|\zeta|$ is of the order of their ratio. The expression for ζ is obtained from the boundary conditions (2.9). It is only necessary to keep in mind that u_0, which appears in ϑ_0, must be expressed in terms of E_0. Finally for ζ and ϑ_0 we have

$$\zeta = 2^{-\frac{2q}{r}}\left(\frac{r}{r-q}\right)^{\frac{r-q}{4r}}\left(\frac{\omega_H\nu_0}{\omega_0^2}\right)^{-\frac{q}{2r}}\left(\frac{e|E_0|l_{0e}}{T}\right)^{-\frac{q}{r}} \tag{4.9}$$

$$\times|\zeta_0|\exp\left\{-i\,\mathrm{arctg}\left(\frac{r-q}{q}\right)^{\frac{1}{2}}\right\},$$

$$\zeta_0 = \left(\frac{4\Gamma\left(\frac{5}{2}+q\right)}{3\pi^{1/2}}\right)^{-\frac{1}{2}}\left(\frac{\omega_H\nu_0}{\omega_0^2}\right)^{\frac{1}{2}}$$

$\exp\left\{-i\frac{\pi}{4}\right\}$ is the value of ϑ for $\vartheta = 1$,

$$\vartheta_0 = 2^{\frac{4}{r}}\left(\frac{r}{r-q}\right)^{\frac{1}{2r}}\left(\frac{\omega_H\nu_0}{\omega_0^2}\right)^{\frac{1}{r}}\left(\frac{e|E_0|l_{0e}}{T}\right)^{\frac{2}{r}}. \tag{4.10}$$

After E(z) has been determined (see Eq. (4.8)), the dependence of ϑ on z is obtained from Eq. (4.4):

$$\vartheta = \vartheta_0\left(1 + \frac{q}{[(2r-q)(r-q)]^{\frac{1}{2}}}\,\frac{\omega}{c|\zeta|}z\right)^{-\frac{2}{q}}. \tag{4.11}$$

If the second system of relations in (4.1) is satisfied, then expressions for ζ and ϑ_0 are obtained from Eqs. (4.9) and (4.10) by replacing ω_H by ω.

For the derivation of the equation of balance in the case of the normal skin effect, we neglect the term containing the derivative with respect to the coordinate, which corresponds to neglecting the spatial derivatives in the kinetic equation. The condition $L \gg l_e$ must be satisfied in order to justify this neglect. In the case of strong nonlinearity, $L \sim k^{-1}|\zeta|$, as follows from Eq. (4.8). Hence follows the inequality for the impedance: $|\zeta| \gg kl_e$. On the other hand, from the resonance condition it follows that $|\zeta| \ll 1$. Thus, the value of the impedance is bounded, both from above and from below, by the conditions for the applicability of the theory. For strong nonlinearity the penetration depth of the wave deeply into the sample is given by $L \sim L_0\vartheta_0^{-q/2}$ (where $L_0 = 1/\xi_0$ is the penetration depth in the linear theory). For negative values of q, $L \gg L_0$, but for positive values of q, $L \ll L_0$.

The plasma field associated with cyclotron resonance acquires the form $u_p = T\omega_0/el_{0e}\omega_H^{1/2}\nu_0^{1/2}$. In InSb with the following parameters: $m = 10^{-28}$ g, $N = 10^{15}$ cm^{-3}, $\nu = 10^{10}$ sec^{-1}, $\omega = \omega_H = 10^{11}$ sec^{-1}, one finds $l_e \sim 10^{-3}$ cm and $L \sim 10^{-2}$ cm.

5. PROPAGATION OF ELECTROMAGNETIC WAVES IN A PLASMA IN CONNECTION WITH A NONUNIQUE DEPENDENCE OF THE ELECTRON TEMPERATURE ON THE AMPLITUDE OF THE FIELD

In connection with specific mechanisms for the transfer of energy and momentum ($r \pm q < 0$—superheating mechanisms), the electron temperature may become a triple-valued function of the field[37-40] (an S-shaped dependence). Such a situation can be realized in a plasma[39-40] and in *n*-InSb[41] (r = 1/2, q = 3/2). This is associated with the fact that for these mechanisms the effect of runaway of the electrons[42] occurs in the absence of interelectron collisions, and the frequent interelectron collisions totally play the role of a restraining mechanism.[16,38]

Thus, the results of this Section are valid only for a strong interelectron interaction ($\nu_{ee} \gg \nu_e$). The S-shaped dependence of the temperature on the field is obtained in the following way: for $r \pm q < 0$ the function $\vartheta(u)$, determined from Eqs. (3.8) or (4.2), will have the form shown in Fig. 1. However, at sufficiently large temperatures new scattering mechanisms become important, as a consequence of which the function $\vartheta(u)$ is deformed into the form shown in Fig. 2 (for more de-

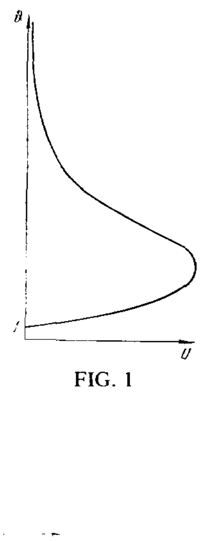

FIG. 1

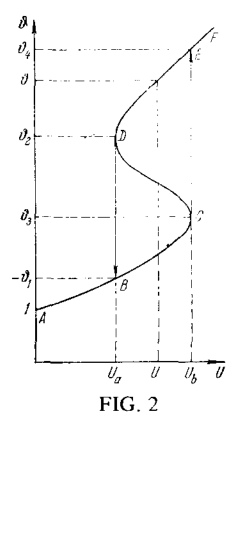

FIG. 2

tails, see[38]). From Fig. 2 it is clear that in the presence of superheating mechanisms, the equation of balance has roots which decrease with increasing values of u (the falling branch). Similarly[43] one can show that the falling branch is unstable with respect to small perturbations. Only those branches of the curve $\vartheta(u)$ are stable where ϑ increases with increasing values of u, that is, $d\vartheta/du > 0$.

For convenience let us rewrite the equation of balance in the form

$$D(\vartheta) = u^2, \tag{5.1}$$

where $D(\vartheta) = NT[\nu_e(\vartheta)(\vartheta - 1)/\sigma(\vartheta)]$.

Differentiating (5.1) with respect to ϑ, we obtain the following criterion for stability:

$$\frac{dD(\vartheta)}{d\vartheta} > 0. \tag{5.2}$$

The transition from stable branches to unstable branches occurs at temperatures satisfying the condition $du/d\vartheta = 0$, or what amounts to the same thing,

$$\frac{dD(\vartheta)}{d\vartheta} = 0. \tag{5.3}$$

The field u corresponding to the transition of the temperature from one branch to the other is determined from Eq. (5.1) upon substituting the roots of Eq. (5.3) into it.

Let us consider the change of the electron temperature as a function of the amplitude u_0 of the electromagnetic wave at the boundary, which in turn is determined by the amplitude E_0 of the incident field.

In connection with the adiabatic growth of the field from zero until $u_0 < u_b$ (see Fig. 2), the electron temperature as a function of the field in the interior of the sample is described by the lower branch AC. For $u_0 = u_b$ the temperature ϑ_0 of the electron gas on the boundary is changed by a jump from the value ϑ_3 to the value ϑ_4, omitting the unstable part of the curve CD. With a further increase of u_0, ϑ_0 will be moved to the right along the curve DF.

Owing to dissipation, with increasing distance z from the boundary of the sample the electric field is damped to zero as $z \to \infty$. At the same time the electron temperature will drop, tending to unity. If $u_0 > u_b$, then at a certain point $z = a\vartheta(z)$ it changes by a jump from ϑ_2 to ϑ_1, which leads to a discontinuity in the dielectric constant at this point. The electromagnetic wave is reflected from the point of discontinuity of the dielectric constant. Thus, in the case being investigated the plasma behaves like a lamina of thickness a. It is known[44] that the coefficient of reflection R from a lamina in vacuum is an oscillating function of its thickness. The thickness of the "lamina" must be found from the equation $u(a, E_0) = u_a$, which determines the field at the point of breakdown. Thus, a is a function of E_0 and u_a and therefore R oscillates with variation of E_0. As is well-known,[44] the oscillations of the coefficient of reflection with the thickness are determined by the factor e^{2ikna}. The calculation carried out in article[45] completely confirms the qualitative conclusions reached above, and moreover, if the wave is weakly damped then R has the form*)

$$R = \frac{1-n}{1+n} - \frac{i\alpha}{n}\left\{\frac{\Phi(\vartheta_0)}{(1+n)^2} + \frac{u_a^2\Phi^{1/2}(\vartheta_2)}{4E_0^2\Phi^{1/2}(\vartheta_0)}[\Phi(\vartheta_1) - \Phi(\vartheta_2)]e^{2ikna}\right\}. \tag{5.4}$$

In connection with the derivation of Eq. (5.4), it was assumed that the dielectric constant can be represented in the form

$$\varepsilon(\vartheta) = n^2 + i\alpha\Phi(\vartheta), \tag{5.5}$$

where the smallness of the parameter $\alpha \ll 1$ corresponds to weak damping. From article[45] it follows that $a(E_0)$ must be determined from the condition $u(a, E_0) = u_a$ cited above. To the zero-order approximation in α, u(z) for the case of weak damping is described by formula (3.7). Changing to the notation of the present section, for a we have

$$a = \frac{n}{k\alpha}\int_{\vartheta_2}^{\vartheta_0} \Phi^{-1}(\vartheta)\frac{d\ln D(\vartheta)}{d\vartheta}d\vartheta. \tag{5.6}$$

The period δE_0 of the oscillations of the coefficient of reflection in the amplitude of the incident electromagnetic wave E_0 must be determined from the relation

$$2kn[a(E_0 + \delta E_0) - a(E_0)] = 2\pi.$$

Since for the case of weak damping the period of the oscillations of the coefficient of reflection $\delta E_0 \ll E_0$, we obtain the following formula for δE_0:

$$\delta E_0 = \frac{\pi\alpha\Phi(\vartheta_0)}{n^2(d\ln D(\vartheta_0)/dE_0)}. \tag{5.7}$$

We note that the period of oscillations of the coefficient of reflection does not depend on the field u_a at the point of breakdown.

For strong damping the quantity δE_0 is of the order of or larger than E_0 and thus does not depend on u_a. In this case the amplitude of the oscillating term substantially decreases with increase of the field by an amount of the order of the period of the oscillations, in contrast to Eq. (5.4) where the amplitude does not change. From what has been said it follows that it is convenient to observe the oscillations of the coefficient of reflection with the field for small damping of the wave.

Owing to finite fluctuations of the temperature, the collapse from the upper branch to the lower branch may occur not at the point $u = u_a$ but at any arbitrary point of the interval $u_a \leq u \leq u_b$ (see Fig. 2). However, the results remain valid even in this case, provided u_a in Eq. (5.4) is understood as the field at the point of breakdown.

III. THE ANOMALOUS SKIN EFFECT

6. Small Currents on the Interface

As mentioned in Sec. 3, the anomalous skin effect corresponds to $l \ll L \lesssim l_e$. We shall consider the case of a strong anomalous skin effect, $l \ll L \ll l_e$.

Let us transform the equation of balance (2.18), having made the following change of variables:

$$w = \int_0^{\vartheta} \varkappa(\vartheta)\,d\vartheta \Big/ \int_0^1 \varkappa(\vartheta)\,d\vartheta. \tag{6.1}$$

*In article [45] the factor $\Phi^{1/2}(\theta_2)/\Phi^{1/2}(\theta_0)$ was erroneously omitted.

Then Eq. (2.18) takes the form

$$\frac{d^2w}{dz^2} - \delta^2 Q(w) = -\delta^2 P(w) u^2, \tag{6.2}$$

where

$$\delta^2 = \frac{N\nu_{0e}}{\int_0^1 \varkappa(\vartheta)\, d\vartheta}, \quad P(w) = \frac{\sigma(w)}{NT\nu_{0e}}, \quad Q(w) = \frac{\nu_e(w)}{\nu_{0e}} [\vartheta(w) - 1]. \tag{6.3}$$

We note that $\delta \sim l_e^{-1}$. In this connection the boundary conditions (2.6) and (2.10) are written as follows:

$$\left.\frac{dw}{dz}\right|_{z=0} = \gamma[\vartheta(w) - 1], \quad w|_{z\to\infty} \to 1, \tag{6.4}$$

where $\gamma = \eta/T\int_0^1 \kappa(\vartheta)d\vartheta$.

In this section we shall investigate the case $\gamma \ll \xi$. This corresponds to a small heat transfer across the boundary, when the energy from the electrons is largely transferred to the lattice.

If $\gamma \ll \xi$, then one can use the method of successive approximations in order to solve the equation of balance, neglecting the right-hand side of Eq. (6.2) in the zero-order approximation and taking it into account afterwards as a perturbation.

The physical meaning of this consists in the following. From (6.2) it follows that the characteristic distance over which the electron temperature falls off is δ^{-1}. In virtue of the large anomalous nature of the skin effect, the field is damped considerably faster. Thus, the right-hand side of Eq. (6) plays the role of surface sources of heat, and for the solution of the equation of balance in the zero-order approximation, one can neglect it. It is necessary to take the right-hand side into consideration in the next approximation in order to satisfy the boundary conditions on the plane z = 0.

In connection with the solution of Maxwell's equations in the immediate vicinity of the boundary ($z \ll l_e$) one can replace the quantity w by w_0, since w essentially doesn't change over distances of the order of L ($L \ll l_e$). After this Maxwell's equation is a linear equation with constant coefficients, which one can easily solve, and moreover the field E near the boundary is given by

$$E = u_0 \exp\{iknz - \xi(w_0) z\}; \tag{6.5}$$

here $\xi(w_0)$ is the value of the coefficient of attenuation for z = 0.

In virtue of what has been said above, we shall seek the solution of Eq. (6.2) in the form

$$w = w' + w'', \quad w'' \ll w', \tag{6.6}$$

where w' is the solution of Eq. (6.2) without anything appearing on the right-hand side. The equation for w' is solved in quadratures:

$$-\sqrt{2}\,\delta z = \int_{w_0'}^{w'} dw \left[\int_1^{w} dw\, Q(w)\right]^{-\frac{1}{2}}. \tag{6.7}$$

The boundary conditions at infinity were taken into consideration in the derivation of (6.7).

The equation

$$\frac{d^2w''}{dz^2} - \delta^2 P(w') u^2 = 0. \tag{6.8}$$

is obtained for w'' correct to within quantities $\sim \delta/\xi \ll 1$ ($L/l_e \ll 1$). Correct to within terms of the order of δ/ξ, one can replace w' in Eq. (6.8) by w_0'. Substituting the expression for u = |E| from (6.5) into (6.8) and solving it, we find

$$w'' = -\frac{\delta^2 P(w_0') |E_0|^2}{4\xi^2(w_0')} e^{-2\xi(w_0')z}. \tag{6.9}$$

We note that although $w''/w' \sim \delta/\xi \ll 1$, dw''/dz is of the same order as dw'/dz.

From the boundary conditions for w, the equation for the determination of w_0' is obtained from (6.4) for z = 0

$$\frac{\delta^2 P(w_0') |E_0|^2}{2\xi(w_0')} = \sqrt{2}\,\delta \left[\int_1^{w_0'} Q(w)\, dw\right]^{\frac{1}{2}} + \gamma[\vartheta(w_0') - 1]. \tag{6.10}$$

We note that if $\gamma \ll \delta$, in formula (6.10) one can neglect the last term; however if $\gamma \gg \delta$, then the first term on the right becomes unimportant.

If there is only one mechanism for the scattering of energy and one for the scattering of momentum, then from Eq. (6.1) it follows that*)

$$w = \vartheta^{2+q}. \tag{6.11}$$

Returning to the variable ϑ, in the region where $\vartheta(z) \gg 1$ we have

$$\vartheta = \vartheta_0' \left\{[1 - (2+q-r)\delta(\vartheta_0') z]^{\frac{2}{2+q-r}} - \frac{\delta(\vartheta_0')}{\xi(\vartheta_0')} e^{-2\xi(\vartheta_0')z}\right\}, \tag{6.12}$$

where

$$\delta(\vartheta_0') = [2(2+q)(2+q+r)]^{-\frac{1}{2}} \delta \vartheta_0'^{\frac{r-q-2}{2}}.$$

In virtue of the condition $\delta/\xi \ll 1$ the second term in (6.12) is much smaller than the first, and we neglect it both in the expression for the temperature and in the calculation of the fields.†) Then ϑ_0' coincides with the temperature ϑ_0 of the electrons on the surface.

Now let us consider the region $\vartheta(z) - 1 \ll 1$. Proceeding in analogy to what was done in Section 3 [see Eqs. (3.16)–(3.18)], we find

$$\vartheta = 1 + \vartheta_0 S_\vartheta \exp\left\{-\frac{\delta z}{(2+q)^{1/2}}\right\}, \tag{6.13}$$

where

$$S_\vartheta = \exp\left\{-\frac{1}{\sqrt{2}} \int_{\vartheta_0}^{1} \left[\vartheta^{1+q}\left(\frac{\vartheta^{2+q+r}-1}{2+q+r} - \frac{\vartheta^{1+q+r}-1}{1+q+r}\right)^{-\frac{1}{2}} - \frac{\sqrt{2}}{\vartheta - 1}\right] d\vartheta\right\}. \tag{6.14}$$

The evaluation of the integral in (6.14) leads to the following estimates for S_ϑ for $\vartheta_0 \gg 1$:

$$\left.\begin{aligned} &S_\vartheta \sim \exp\left\{\frac{[2(2+q+r)]^{\frac{1}{2}}}{2+q-r} \vartheta_0^{\frac{2+q+r}{2}}\right\} \gg 1 \quad \text{for} \quad 2+q-r > 0; \\ &S_\vartheta \sim 1 \quad \text{for} \quad 2+q-r < 0. \end{aligned}\right\} \tag{6.15}$$

After the temperature $\vartheta(z)$ of the electron gas is found, Maxwell's equation becomes a linear equation with a coefficient which depends on z. As a consequence of the inequality $\delta(\vartheta_0) \ll \xi(\vartheta_0)$ the temperature as a function of z varies slowly in comparison with the electric field, thanks to which one can use the WKB method in order to solve Maxwell's equation.

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*For all known scattering mechanisms 2 + q > 0.

†Taking this term into account during the calculation of the fields leads to corrections $\sim(\delta/\xi)^3$. [45] We note, however, that in connection with the investigation of thermomagnetic effects, when derivatives of the temperature play the major role, the second term inside the curly brackets in (6.12) is essential.

In the case of the anomalous skin effect, the solution may be written down in general form for an arbitrary dependence of $\epsilon(\vartheta)$ on ϑ. However, it becomes visible only in connection with a number of simplifications, when certain restrictions are imposed on the frequency and on the magnetic field.

For large values of z in the region where $\vartheta(z) - 1 \ll 1$, the field has the form [see Eq. (3.20)]

$$E(z) = PE_0 S_E \exp\{iknz - \xi_0 z\}, \tag{6.16}$$

where S_E denotes the self-stress factor for the field, having the same meaning as in the case of the normal skin-effect.

If the first inequality in (3.4) is satisfied, by expanding $\epsilon(\vartheta)$ in powers of ν and solving Maxwell's equation by the WKB method, we obtain the following formula for the field:

$$E(z) = PE_0 \exp\left\{iknz - \xi_0 \int_0^z \vartheta^{-q}\, dz\right\}. \tag{6.17}$$

In the region where $\vartheta(z) \gg 1$, we have

$$E = PE_0 \times \exp\left\{iknz + (2-q-r)^{-1} \frac{\xi_0 \vartheta_0^{-q}}{\delta(\vartheta_0)}\left[(1-(2+q-r)\,\delta(\vartheta_0)\,z)^{\frac{2-q-r}{2+q-r}} - 1\right]\right\}. \tag{6.18}$$

Expressions for S_E in this and the subsequent cases are cited in article[18]. Here we shall confine our attention to estimates of S_E for $\vartheta_0 \gg 1$ for different relations between r and q. We cite, for example, the expressions for S_E only in two cases:

$$\left.\begin{aligned} &S_E \sim \exp\left\{-\frac{[2(2+q)(2+q+r)]^{\frac{1}{2}}}{2-q-r}\frac{\xi_0}{\delta}\vartheta_0^{\frac{2-q-r}{2}}\right\} \ll 1 \\ &\text{for } q<0,\ 2-q-r>0, \\ &S_E \sim \exp\left\{\frac{[2(2+q)(2+q+r)]^{\frac{1}{2}}}{2+q-r}\frac{\xi_0}{\delta}\vartheta_0^{\frac{2+q-r}{2}}\right\} \gg 1 \\ &\text{for } q>0,\ 2+q-r>0. \end{aligned}\right\} \tag{6.19}$$

Let us present expressions for the temperature on the boundary, which enters into Eqs. (6.17)–(6.19). From Eq. (6.10), assuming for simplicity that $\gamma = 0$, we have

$$\left.\begin{aligned} &\vartheta_0 = \left\{\frac{16\sqrt{2}(2+q+r)^{\frac{1}{2}}\Gamma\left(\frac{5}{2}-q\right)\nu_0^2 e^2 |E_0|^2 l_{0e} L_0}{3\pi^{\frac{1}{2}}(1+n)^2\omega^2 T^2}\right\}^{\frac{2}{2+q+r}} \\ &\text{in the case } \omega_H = 0 \text{ and} \\ &\vartheta_0 = \left\{\frac{16\sqrt{2}(2+q+r)^{\frac{1}{2}}\Gamma\left(\frac{5}{2}-q\right)\omega\nu_0^2 e^2 |E_0|^2 l_{0e} L_0}{3\pi^{\frac{1}{2}}\omega_H \omega_0^2 T^2}\right\}^{\frac{2}{2+q+r}} \end{aligned}\right\} \tag{6.20}$$

for helical waves.

If the second inequality of (3.4) is satisfied, then in absolutely similar fashion for $\vartheta(z) \gg 1$ we find

$$E = PE_0 \exp\left\{ikz + \frac{1}{2+3q-r}\frac{\xi_0}{\delta(\vartheta_0)}\vartheta_0^q \times\left[(1-(2+q+r)\,\delta(\vartheta_0)\,z)^{\frac{2+3q-r}{2+q-r}} - 1\right]\right\}. \tag{6.21}$$

The value of ϑ_0 is determined by the formula

$$\vartheta_0 = \left\{\frac{4\sqrt{2}(2+q+r)^{\frac{1}{2}}\Gamma\left(\frac{5}{2}+q\right)e^2|E_0|^2 l_{0e} L_0}{3\pi^{1/2}T^2}\right\}^{\frac{2}{2+q+r}}. \tag{6.22}$$

Upon fulfillment of inequalities (4.1), when the imaginary part of the dielectric constant becomes much larger than the real part, for the field we have[6,8]

$$E = 2\zeta E_0\left(\frac{\vartheta_0}{\vartheta}\right)^{\frac{q}{4}}\exp\left\{-(1-i)\,\xi_0\int_0^z \vartheta^{\frac{q}{2}}\, dz\right\}. \tag{6.23}$$

In the region where $\vartheta(z) \gg 1$, by substituting the value of $\vartheta(z)$ from (6.12) into (6.23), we obtain

$$E = \frac{2\zeta E_0(2+q+r)^{-1}}{[1-(2+q-r)\,\delta(\vartheta_0)\,z]^{\frac{q}{2}}} \times \exp\left\{\frac{1-i}{2+2q-r}\frac{\xi_0}{\delta(\vartheta_0)}\vartheta_0^{\frac{q}{2}}\left[(1-(2+q-r)\,\delta(\vartheta_0)\,z)^{\frac{2+2q-r}{2+q-r}} - 1\right]\right\}. \tag{6.24}$$

For cyclotron resonance ($\omega = \omega_H$) we find

$$\vartheta_0 = \left[\left(\frac{32(2+q+r)^{\frac{1}{2}}\omega_H^2 \nu_0^2}{\omega_0^4}\right)^{\frac{1}{2}}\frac{e^2|E_0|^2 l_{0e} L_0}{T^2}\right]^{\frac{1}{1+q+(r/2)}}. \tag{6.25}$$

The results for the case of the low-frequency waves associated with circular polarization of the incident field are obtained from Eq. (6.25) by replacing ω_H by ω. The relation between ξ_0 and ζ is given by the formula

$$\zeta = \frac{1-i}{2}\frac{\omega}{c\xi_0}\vartheta_0^{-\frac{q}{2}} \ll 1. \tag{6.26}$$

We note that in virtue of the conditions imposed on r and q ($r + q > 0$), the dependence of ϑ_0 on E_0 in all cases is such that ϑ_0 increases with increasing values of E_0. The condition for the anomalous skin effect is satisfied for $m = 10^{-29}$ g, $\nu = 10^{10}$ sec^{-1}, $N = 10^{15}$ cm^{-3}, and $\omega = \omega_H = 10^{11}$ sec^{-1}. In this case $l_e \sim 3 \times 10^{-2}$ cm and $L \sim 10^{-3}$ cm.

7. The Anomalous Skin Effect Associated with Large Currents on the Interface

Let us investigate the limiting case of large heat transfer on the boundary, $\gamma \gtrsim \xi$, associated with weak damping of the electromagnetic wave.

Here it is necessary to distinguish two spatial regions ($z \ll \delta^{-1}$). In the region immediately adjacent to the boundary, where one can neglect the energy transferred to the lattice ($z \ll \delta^{-1}$), the shortened Maxwell's equation and the equation of balance are written as follows:[46]

$$\left.\begin{aligned} &\frac{d^2 w}{dz^2} + \delta^2 P(w)\, u^2 = 0, \\ &\frac{du}{dz} + \xi(w)\, u = 0. \end{aligned}\right\} \tag{7.1}$$

Taking into account that $\xi(w) = 2\pi\sigma(w)/\omega\sqrt{\epsilon_r}$ (ϵ_r is the real part of the dielectric constant), and that the expression for P(w) [see Eq. (6.3)] is written as follows: $P = P_0\xi(w)$ ($P_0 = nc/2\pi NT\nu_{0e}$), we may easily solve the system (7.1). The answer has the following form:[46]

$$\left.\begin{aligned} &-2z = \int_{w_0}^{w} dw\left[\int_{w_\infty}^{w}\xi(w)\, dw\right]^{-1}, \\ &u = \left[\frac{4}{\delta^2 P_0}\int_{w}^{w_\infty}\xi(w)\, dw\right]^{\frac{1}{2}}, \end{aligned}\right\} \tag{7.2}$$

where w_∞ is the limiting temperature to which w(z) tends for $z \gg \xi^{-1}$. If it is assumed that there is a single mechanism for the scattering of momentum, then $\xi(w) = \xi_0 w^{\pm\frac{q}{2+q}}$ (see Section 6). Upon the fulfillment of the first condition in (3.4), in the expressions for ξ it is necessary to maintain the upper sign, but if the second condition in (3.4) is satisfied, then the lower sign holds.

Returning to the variable ϑ and integrating over ϑ in Eq. (7.2), we obtain

$$\left.\begin{aligned} \frac{2\xi_0 z}{2+q\mp q} &= \int\limits_{\vartheta_0}^{\vartheta} \frac{\vartheta^{(1+q)}\,d\vartheta}{\vartheta_\infty^{2+q\mp q}-\vartheta^{2+q\mp q}}, \\ E &= \left[\frac{8\pi T\lambda_0\xi_0}{(2+q\mp q)\,cn}\right]^{\frac{1}{2}} (\vartheta_\infty^{2+q\mp q}-\vartheta^{2+q\mp q})^{\frac{1}{2}} e^{iknz}. \end{aligned}\right\} \qquad (7.3)$$

The first formula determines $\vartheta(z)$ in implicit form, and the system of equations (7.6) determines E(z) in parametric form. ϑ_0 must be determined from the boundary condition (6.4) for z = 0, which gives

$$\frac{2\xi_0 T\lambda_0}{(2+q\mp q)}(\vartheta_\infty^{2+q\mp q}-\vartheta_0^{2+q\mp q}) = \eta(\vartheta_0-1). \qquad (7.4)$$

Taking the fact that $E(+0) = 2E_0/(1+n)$ into account, we obtain one more relation between ϑ_0 and ϑ_∞ [see the second formula in (7.3)]:

$$\frac{2(2+q\mp q)\,cn\,|E_0|^2}{\pi(1+n)^2 T\lambda_0\xi_0} = \vartheta_\infty^{2+q\mp q}-\vartheta_0^{2+q\mp q}. \qquad (7.5)$$

We can determine ϑ_0 and ϑ_∞ from Eqs. (7.4) and (7.5):

$$\left.\begin{aligned} \vartheta_0 &= \frac{1+4nc\,|E_0|^2}{\pi(1+n)^2\eta}, \\ \vartheta_\infty &= \vartheta_0\left[1+\frac{2(2+q\mp q)\,nc\,|E_0|^2}{\pi(1+n)^2 T\lambda_0\xi_0}\vartheta_0^{-(2+q\mp q)}\right]^{\frac{1}{2+q\mp q}} \end{aligned}\right\} \qquad (7.6)$$

For $z \gg \xi^{-1}$ the boundary conditions cease to be "felt" and the expressions for the field and for the temperature are given by formulas (6.7) and (6.17).

We note that the expression for ϑ_∞ coincides with the value of ϑ_0 determined by formula (6.10), if there the first term to the right is omitted (this is valid since $\gamma \gg \delta$).

Thus, there is an interval $z(\xi)^{-1} \ll z \ll \delta^{-1}$ where the expression for the temperature from (7.3) coincides with the temperature determined by formula (6.7). Thus, if in the first formula of (7.3) by ϑ_∞ one understands $\vartheta(z)$ from (6.7), then formula (7.3) will describe the dependence $\vartheta(z)$ over the entire range of variation of z. In fact, for $z \ll \delta^{-1}$ $\vartheta(z)$ from Eq. (6.7) reduces to a constant equal to ϑ_∞. On the other hand, for $z \gg \xi^{-1}$ we obtain the following expression for $\vartheta(z)$ from (7.3):

$$\vartheta = \vartheta_\infty - S_\vartheta \exp\{-2\xi_0\vartheta_\infty^{\mp q} z\}, \qquad (7.7)$$

where

$$S_\vartheta = (\vartheta_\infty-\vartheta_0)\exp\left\{\int\limits_{\vartheta_0}^{\vartheta_\infty}\left[\frac{\vartheta_\infty^{\mp q}\vartheta^{1+q}}{(2+q\mp q)(\vartheta_\infty^{2+q\mp q}-\vartheta^{2+q\mp q})}-\frac{1}{\vartheta_\infty-\vartheta}\right]d\vartheta\right\},$$

i.e., ϑ reduces to ϑ_∞, which we replace by $\vartheta(z)$ from Eq. (6.7).

In conclusion we note that effects which are closely related with the "self-action" were not considered in the present review, that is, effects involving the nonlinear interaction of waves. These effects are discussed in articles[18,47-50].

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