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NONLINEAR WAVES

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1. INTRODUCTION

THE extent to which wave motion is widespread in nature need hardly be restated. Waves on the surface of a heavy liquid, elastic waves, electromagnetic waves, waves in a plasma—these are only the most outstanding representatives of an extensive family of waves in continuous media. The properties of such waves in the linear approximation have been investigated in sufficient detail. Actually, however, very frequently the amplitude of the wave is not small, and we are dealing with nonlinear wave motions. All of us are quite familiar with nonlinear waves on water, nonlinear waves in a plasma occur almost as frequently, and the discovery of lasers has led to a wide circle of nonlinear effects in optics. Nonlinear effects in wave motion have been the object of intense research in recent years, and many interesting physical phenomena have been discovered therein. This includes the already common effect of frequency multiplication, phenomena of self-focusing and self-contraction of wave packets, which have been most thoroughly investigated in optics, effects of stimulated scattering, which are called wave decays in plasma physics, stochastic wave interaction—weak turbulence, etc. In various fields of physics these effects, in spite of their being identical, are described essentially in somewhat different terms, and their analogy with related phenomena in other fields is not always indicated. Taking this circumstance into account, and also in view of the fact that a certain clarity has been attained in the understanding of nonlinear phenomena in wave motions and that the results are of interest to a sufficiently large group of physicists, we deemed it advantageous to compile the present review, which describes from a unified point of view and in a relatively simple form the main results obtained in this field. We shall also have, of course, occasion to repeat well-known things, but we shall attempt to describe them from a common point of view.

As we shall see below, the character of the nonlinear processes depends strongly on the dispersion, i.e., on the dependence of the phase velocity on the wave number. It is therefore natural to begin the analysis with the case of nondispersive media, and then take dispersion effects into account.

2. SIMPLE WAVES

2.1. Beam of Noninteracting Particles

As the simplest example of a nondispersive medium, let us consider a beam of noninteracting particles. The velocity of each particle of such a beam remains constant, so that

$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=0. \tag{2.1}$$

An aggregate of noninteracting particles is not, of course, a nonlinear system, but Eq. (2.1) has a nonlinear appearance and has, as we shall show, solutions having the characteristic properties of nonlinear waves.

Let us first consider small oscillations near a homogeneous beam with constant velocity v_0: $v = v_0 + v'$. Putting $v' \sim \exp(-i\omega t + ikx)$ and linearizing (2.1), we obtain a relation between the frequency ω and the wave vector of the perturbation:

$$\omega = kv_0. \tag{2.2}$$

We see therefore that in the linear approximation we deal with a nondispersive medium, $v_f = \omega/k$ = const. Let us assume now that the initial perturbation of the velocity is of the form $\sim \sin kx$. It is convenient to consider the evolution of this perturbation in a coordinate system that moves with velocity v_0, putting $v = v_0 + u$. In this system we have

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0. \tag{2.3}$$

Let us visualize the (x, u) phase plane (Fig. 1a). On this plane, the initial state of the beam is represented by the sinusoid 1. With time, all the phase points, including the points of the beam, move with a velocity proportional to the distance from the x axis, and the wave profile becomes distorted—the particles with $u > 0$ run ahead, and those with $u < 0$ lag the wave. This effect leads to a perturbation of the density—at the points 1 and 2, where the slope is increased, the parti-

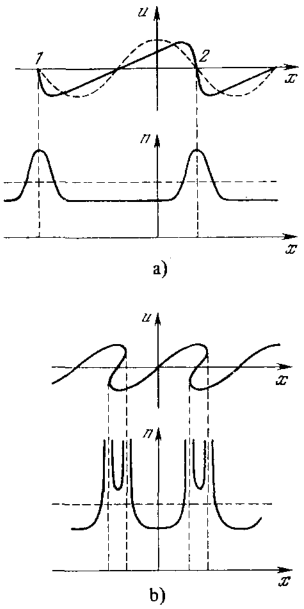


FIG. 1

cles become condensed and the density increases. This is the so-called "bunching" of the particles. It is precisely bunching of this type which is used for the generation of high-frequency oscillations in klystrons. The increase of the density continues until the derivative $\partial u/\partial x$, and with it also the density, becomes infinite at the points 1 and 2. This is followed by "breaking" of the wave u(x, t), and double the number of singularities appear in the density (Fig. 1b).

As $t \to \infty$, the turning points move apart, the number of opposing beams increases without limit, and the density again tends to a constant value with small "spikes." At each point of space there are very many beams with a great variety of velocities, so that one can speak approximately of a velocity distribution function f(u) in the form of a "table." Of course, the entire process is fully reversible—it suffices to reverse the directions of all the velocities, and oscillations will appear out of a many-stream state that is homogeneous at first glance: namely, the number of beams begins to decrease, "spikes" of the density appear, and, finally, the system arrives at the initial state. This again is followed by "breaking" and formation of a many-stream motion, but this time to the left and not to the right as in Fig. 1a.

Thus, a beam of non-interacting particles has many properties of a nonlinear system—it is subject to "wave breaking" and to generation of higher harmonics, and also to amplification of small density oscillations as a result of the nonlinear connection between the density and the velocity.

2.2. Simple Waves in a Gas

Let us now consider nonlinear waves in continuous media. We start with ordinary gasdynamics. The propagation of one-dimensional nonlinear acoustic waves is described by the gasdynamic equations

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + \frac{1}{mn}\frac{\partial p}{\partial z} = 0, \qquad (2.4)$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv) = 0. \qquad (2.5)$$

We shall assume that p is connected with n by the equation of state (isotherm or adiabat), and introduce the notation $c_s^2 = (1/m)dp/dn$ for the square of the velocity of sound.

Out of the tremendous number of nonlinear solutions of the gasdynamic equations, an important role is played by the so-called simple waves. These waves are a generalization of traveling linear waves—they propagate in one direction, and, just as in linear waves, the density n is uniquely determined by the value of v, i.e., n = n(v). Consequently, for simple waves Eqs. (2.4) and (2.5) can be written in the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + c_s^2\frac{1}{n}\frac{\partial n}{\partial v}\frac{\partial v}{\partial z} = 0, \qquad (2.6)$$

$$\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z}\right)\frac{dn}{dv} + n\frac{\partial v}{\partial z} = 0. \qquad (2.7)$$

Multiplying the first of these equations by dn/dv and subtracting from the second, we obtain after cancelling $\partial v/\partial z$

$$c_s^2\left(\frac{dn}{dv}\right)^2 = n^2. \qquad (2.8)$$

From this we get

$$c_s\frac{dn}{dv} = \pm n. \qquad (2.9)$$

Substituting this value in (2.6) we obtain

$$\frac{\partial v}{\partial t} + (v \pm c_s)\frac{\partial v}{\partial z} = 0. \qquad (2.10)$$

Thus, we obtain an equation of the same form as for a beam of noninteracting particles, the only difference being that now the nonlinear term is preceded by the factor $v \pm c_s$ in place of v.

If we use the equation of the adiabatic process

$$c_s^2 = \frac{1}{m}\frac{dp}{dn} = c_{s0}^2\left(\frac{n}{n_0}\right)^{\gamma-1},$$

then we get from (2.9)

$$c_s(v) = c_{s0} + \frac{\gamma-1}{2}v,$$

where c_{s0} is the velocity of the wave in the linear approximation and γ is the adiabatic exponent of the gas. Substituting this in (2.10), we obtain an equation for a simple wave propagating in the positive z direction, in the form

$$\frac{\partial v}{\partial t} + \left(c_{s0} + \frac{\gamma+1}{2}v\right)\frac{\partial v}{\partial z} = 0. \qquad (2.11)$$

If we change to a reference frame that moves with the speed of sound c_{s0} in the linear approximation ($z = x + c_{s0}t$) and introduce a new scale for the velocity of the perturbation $u = (\gamma + 1)v/2$, then we obtain exactly Eq. (2.3).

This leads to the well-known result that the velocity profile of large-amplitude acoustic waves should also be subject to steepening. In ordinary hydrodynamics, this steepening continues all the time, so long as the velocity v(z) remains single-valued. However, as soon as $\partial v/\partial z$ becomes infinite, a shock wave begins to form, on the front of which energy becomes dissipated as a result of viscosity.

2.3. The Burgers Equation

The influence of viscosity in the evolution of a simple wave can be taken into account in simplest fashion by adding the term $\mu(\partial^2 v/\partial z^2)$ in the right-hand side of (2.11). We then obtain in lieu of (2.3) the so-called Burgers equation[1]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2}, \qquad (2.12)$$

where μ is the damping coefficient of sound in the linear approximation.

It turns out that a general solution of the Burgers equation can be obtained in closed analytic form[2].

Indeed, if we put

$$u=-2\mu\frac{\partial}{\partial x}\ln\varphi(x, t), \qquad (2.13)$$

then we obtain for $\varphi(x, t)$ the heat-conduction equation $\partial\varphi/\partial t = \mu(\partial^2\varphi/\partial x^2)$, which has well-known solutions. They can be used to trace readily the evolution of any initial velocity profile. Let us consider, for example, the time variation of a perturbation $u_0(x) = u(x, t = 0)$ in the form of a certain pulse that is limited in x. Integrating Eq. (2.12) with respect to x, we easily verify that the area under the velocity profile

$$M=\int_{-\infty}^{+\infty} u(x, t)\,dx \qquad (2.14)$$

remains constant in time, i.e., it is an "integral of the motion." From (2.13) we see that M determines the "temperature" drop: $\varphi(+\infty)/\varphi(-\infty) = \exp(-M/2\mu)$. But at a given temperature drop, regardless of the profile of the transition layer, the solution of the heat-conduction equation tends as $t \to \infty$ to a self-similar solution corresponding to the presence, at the initial instant of time t = 0, of a sharp transition from one "temperature" $\varphi_1 = \varphi(\infty)$ to another temperature $\varphi_2 = \varphi(-\infty)$, at the point of their "tangency" x = 0. It follows therefore that as $t \to \infty$ the velocity profile u(x, t) will tend to a certain universal asymptotic value, determined only by the value of the constant M. In particular, as $\mu \to 0$, as can be readily shown, the profile u(x, t) for $M > 0$ tends to the value

$$u_{(\mu\to 0)}(x, t)=\begin{cases} x/t & \text{for}\quad 0<x<\sqrt{2Mt},\\ 0 & \text{for}\quad x<0,\ x>\sqrt{2Mt},\end{cases} \qquad (2.15)$$

i.e., it has the form of an expanding triangle with a shock wave on the leading part of the profile. When $M < 0$, the triangle is directed with its vertex downward, and the shock wave is produced on its trailing edge. The value of the jump in the shock wave is $(2M/t)^{1/2}$, i.e., it decreases like $t^{-1/2}$, and the width of the profile, to the contrary, increases in proportion to the square root of the time, so that the total area of the perturbation retains a constant value M (see Sec. 95 in the book[3]).

The Burgers equation also has a stationary solution describing a profile moving without deformation at constant velocity c. In fact, if we substitute

$$u=f(x-ct) \qquad (2.16)$$

in (2.12) then we obtain a second-order differential equation for f:

$$(f-c)\frac{df}{dx}=\mu\frac{d^2 f}{dx^2}. \qquad (2.17)$$

This equation has a solution, bounded at ∞, in the form

$$f(\xi)=u_0+\frac{\Delta u}{1+\exp[(\Delta u/2\mu)\,\xi]}, \quad c=u_0+\frac{\Delta u}{2}, \qquad (2.18)$$

where u_0 and Δu are constants. This solution represents a shock wave with a discontinuity Δu and a transition-region width $\delta = 2M/\Delta u$. As $\mu \to 0$, the width δ also tends to zero. With the aid of (2.13) it can be shown that any perturbation with a velocity discontinuity Δu tends asymptotically with time to this wave.

Thus we see that the Burgers equation describes quite fully the general picture of formation and structure of shock waves. For weak waves, this description is sufficiently accurate also quantitatively (see also[4]).

3. NONLINEAR WAVES IN WEAKLY DISPERSIVE MEDIA

3.1. Nonlinearity and Dispersion

The process of the steepening and breaking of waves, considered in Sec. 2.1, is essentially connected with the absence of dispersion (or dissipation, which can be regarded as "imaginary" dispersion). It is precisely because of the absence of dispersion that all small-amplitude waves with different wave numbers k propagate with identical velocity and are capable of interacting with one another for a long time, so that even a small nonlinearity sooner or later should lead to accumulation of the distortion. The influence of dissipation was explained in Sec. 2.3. We consider dispersion below, neglecting dissipative effects.

In this case the phase velocity of waves with different k is not the same, and therefore dispersion can compete with nonlinearity if the wave amplitude is not very large. Higher harmonics, which are generated in the case of nonlinear distortion of the wave, will either overtake or lag the fundamental wave as a result of dispersion, depending on whether the group velocity increases or decreases with k. Consequently, the wave can "fall apart," even prior to breaking, into individual wave packets (which, generally speaking, are nonlinear), and no shock wave will be produced. In order to trace in greater detail the physics of this phenomenon without complicating the exposition with extraneous processes, we consider a weakly dispersive medium without dissipation.*

3.2. Waves in Shallow Water

Under natural conditions, there exists an object that is easily accessible for observation of wave propagation in weakly dispersive media, namely shallow water. We consider a layer of liquid of height h_0, poured over a solid surface, and consider gravitational waves with wavelength much larger than h_0 in such a layer. For simplicity we assume that the waves propagate along the x axis, so that the pressure p and the velocity v do not depend on the variable y. If the wavelength is large, i.e., the water is shallow, then the horizontal component of the velocity can be regarded as uniform over the height (independent of z), so that we have for $v_x = v$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0. \qquad (3.1)$$

Here the pressure p can be understood in the sense of its mean value over the height. It is obviously larger

*A medium is called weakly dispersive if the dispersion appears only at sufficiently large values of the wave number k.

where the height of the liquid is larger, by an amount $(h - h_0)\rho g$ compared with the pressure in the unperturbed layer. Thus, Eq. (3.1) takes the form

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{\partial h}{\partial x}=0. \qquad (3.2)$$

The height h, in turn, is determined by the continuity equation

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hv)=0, \qquad (3.3)$$

which expresses the fact that the rate of change of the height of the layer, $\partial h/\partial t$, is connected with the difference of the flows hv through infinitesimally close cross sections x and x + dx.

Equations (3.2) and (3.3) coincide in form with the equations of gas dynamics with $\gamma = 2$. This means that in the linear approximation, waves in shallow water have no dispersion, and the effect of steepening and turning of the waves should take place in a nonlinear wave. For a simple wave these equations reduce to the form

$$\frac{\partial v}{\partial t}+\left(\frac{3}{2}v\pm c_0\right)\frac{\partial v}{\partial x}=0, \qquad (3.4)$$

where $c_0 = \sqrt{gh_0}$ is the phase velocity of the small-amplitude wave. Changing over to a coordinate system that moves with velocity $\pm c_0$, and putting $u = 3v/2$, we again arrive at the equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0, \qquad (3.5)$$

which is common to a broad class of nondispersive media.

3.3. The Korteweg–de Vries Equation

The phase velocity of very long waves on shallow water does not depend on the wave number k and is simply equal to $\sqrt{gh_0}$. With increasing k, however, it should begin to change, so that at very large $k \gg h_0^{-1}$ it goes over into the relation $v_f = \sqrt{g/k}$ for gravitational waves on deep water. Since at large k the phase velocity decreases, and furthermore v_f is an even function of k, at small values of k it can be represented in the form

$$v_f=c_0\left(1-\frac{k^2}{k_0^2}\right), \qquad (3.6)$$

where $1/k_0$ determines the characteristic "dispersion length," for which the change of v_f becomes of the order of unity. For waves on shallow water, we have $k_0 = \sqrt{6/h_0}$.*

Let us attempt now to take into account the dispersion in the equation for simple waves. We consider for concreteness a wave propagating to the right, i.e., with $v_f > 0$. In the linear approximation we should obtain for such a wave, in a coordinate system moving with velocity c_0, in accordance with (3.6), a frequency $\omega = c_0k^3/k_0^2$, i.e., the corresponding equation in terms of the variables x and t should take the form

$$\frac{\partial u}{\partial t}+\frac{c_0}{k^2}\frac{\partial^3 u}{\partial x^3}=0. \qquad (3.7)$$

On the other hand, however, at a finite amplitude the equation should contain the nonlinear term $u(\partial u/\partial x)$. Thus, the complete equation describing nonlinear waves on shallow water should be of the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0, \qquad (3.8)$$

where β, which we shall call the dispersion parameter, is equal to c_0/k_0^2.

This equation was obtained by Korteweg and de Vries in 1895[5] for waves on water, and recently in[6-8] for waves in a plasma. From the foregoing considerations it is clear that the Korteweg–de Vries equation has a much wider range of applications–it describes "quasi-simple" waves for any medium with dispersion (3.6), which is customarily called a medium with negative dispersion. If the phase velocity increases with k, i.e., if at small k we have $v_f = c_0(1 + (k^2/k_0^2))$, then the medium is said to have positive dispersion.

For media with positive dispersion it would be necessary to reverse the sign of the last term in (3.8). But if at the same time we also make the substitutions $x \to -x$ and $u \to -u$, then we again obtain an equation of the type (3.8). Since x is reckoned in our case from the point c_0t, this means that in media with positive and negative dispersion the waves propagate with mirror symmetry relative to the point $x_0 = c_0t$, which moves with the velocity of the long-wave perturbations. By virtue of this, it suffices to consider only the case of a medium with negative dispersion, such as shallow water.

3.4. Waves in Plasma

Another example of a weakly dispersive medium is a plasma in a magnetic field. The long-wave perturbations in such a plasma propagate with a velocity that is independent of the wave number, and only at sufficiently high frequencies does dispersion appear. Accordingly, the propagation of such waves is described by the Korteweg–de Vries equation.

Let us consider, for example, the case of propagation of a magnetosonic wave in a plasma placed in a strong magnetic field.

If the wave propagates at an angle α to the strong magnetic field $\mathbf{H}_0$ (the energy of which greatly exceeds the thermal energy of the plasma $H_0^2/8\pi \gg nT$), and the angle α satisfies the relation

$$\alpha^2 \gg \frac{kc}{\omega_{0i}}, \quad \omega_{0i}=\frac{4\pi ne^2}{m_i} \qquad (3.9)$$

(this condition excludes waves propagating along the magnetic field and those close to them), then the quantity

$$h=H_z-H_0\sin\alpha$$

satisfies, under the condition $h/(H_0\sin\alpha) \ll 1$, the following equation:

$$h_t+\left(c_A+\frac{3}{2}hc_A\frac{\sin\alpha}{H_0}\right)h_x+\beta h_{xxx}=0, \qquad (3.10)$$

where

$$c_A=\frac{H_0}{\sqrt{4\pi\rho_0}}, \quad \beta=c_A\frac{c^2}{2\omega_{0i}^2}\left(\frac{m_e}{m_i}-\operatorname{ctg}^2\alpha\right), \qquad (3.11)$$

and c is the speed of light. All the remaining quantities describing the wave (the density ρ, the macroscopic velocities v_e and v_i, etc.) are expressed in terms of h. Thus, Eq. (3.10) is of the Korteweg–de Vries type.

We call attention to the characteristic behavior of the

*The right-hand side of (3.6) comprises the first two terms of the expansion of the exact expression $v_f = \sqrt{(g/k)\tanh(kh)}$ in powers of k (see, for example, [3]).

dispersion parameter β as a function of the angle α between the propagation directions and the magnetic field. When

$$\operatorname{ctg}\alpha < \left(\frac{m_e}{m_i}\right)^{1/2} \tag{3.12}$$

(almost "transverse" waves), the parameter β is positive and its order of magnitude is $\beta \sim c_A c^2/\omega_{0e}^2$. When $\cot\alpha > (m_e/m_i)^{1/2}$, the dispersion parameter reverses sign and the corresponding waves should be regarded as "oblique." In this case the parameter β is not only negative, but has essentially a different value, $\beta \sim c_A c^2/\omega_{0i}^2$. Accordingly, the dispersion length $\delta \sim k_0^{-1}$ for perpendicular and "oblique" propagation has different orders of magnitude:

$$\delta \sim \frac{c}{\omega_{0e}}, \quad \text{if} \quad \frac{\pi}{2} - \alpha < \left(\frac{m_e}{m_i}\right)^{1/2},$$

$$\delta \sim \frac{c}{\omega_{0i}}, \quad \text{if} \quad \frac{\pi}{2} - \alpha > \left(\frac{m_e}{m_i}\right)^{1/2}.$$

By way of another example, let us consider ion-acoustic waves in a plasma without a magnetic field.

Let us assume that the ion temperature is low compared with the electron temperature. In this case the plasma is described sufficiently well by the hydrodynamic equations with adiabatic exponent $\gamma = 1$:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{c_0^2 \nabla n}{n}, \quad \frac{\partial n}{\partial t} + \nabla(n\mathbf{v}) = 0, \tag{3.13}$$

where

$$c_0^2 = T_e/m_i, \tag{3.14}$$

and the deviations from quasineutrality ($n_e = n_i = n$) are neglected; this is legitimate if the electron Debye radius $D = \sqrt{T_e/4\pi n e^2}$ is negligibly small compared with the characteristic wavelength. Allowance for the finite character of the Debye radius in first nonvanishing approximation leads to the following equations:

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} + c_0^2 \frac{\nabla n}{n} + \frac{2c_0\beta}{n_0}\nabla(\Delta n) = 0, \\ &\frac{\partial n}{\partial t} + \nabla(n\mathbf{v}) = 0, \end{aligned} \tag{3.15}$$

where $\beta = c_0 D^2/2$.

In the linear approximation, Eqs. (3.15) lead to the dispersion equation

$$\omega(k) = c_0 k\left(1 - \frac{k^2 D^2}{2}\right), \tag{3.16}$$

the right-hand side of which constitutes the first two terms of the "exact" dispersion equation

$$\omega = \sqrt{\frac{T}{m_i}}\, k\,(1 + k^2 D^2)^{-1/2}.$$

In this case we can also write down a Korteweg–de Vries equation[9,10] in the form

$$\frac{\partial v}{\partial t} + (c_0 + v)\frac{\partial v}{\partial x} + \beta\frac{\partial^3 v}{\partial x^3} = 0, \tag{3.17}$$

and according to (3.16) the quantity $\beta = c_0 D^2/2$ is positive, and the dispersion length is of the order of the Debye radius.

3.5. Periodic Waves, Solitons

We consider first periodic solutions of the Korteweg–de Vries equation of the traveling-wave type $u = u(x - ct)$, where c is the phase velocity. For such waves we have $\partial u/\partial t = -c(\partial u/\partial x)$, so that (3.8) changes from a partial differential equation to an ordinary one.

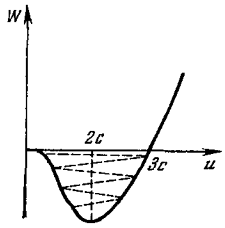


FIG. 2

It can be immediately integrated once to obtain

$$\beta\frac{d^2u}{dx^2} = a + cu - \frac{1}{2}u^2, \tag{3.18}$$

where a is the integration constant, which we can set equal to zero without loss of generality (this can always be done by changing over to a moving system of coordinates). Then Eq. (3.18) can be represented in the form

$$\beta\frac{d^2u}{dx^2} = -\frac{\partial W}{\partial u}, \tag{3.19}$$

where

$$W = -\frac{cu^2}{2} + \frac{u^3}{6}. \tag{3.20}$$

Equation (3.19) can be regarded as the equation of motion for a nonlinear oscillator—a material point of mass β, moving in a potential well W(u), with the coordinate x playing the role of the time. The potential energy W as a function of u is shown in Fig. 2. It vanishes when u = 0, u = 3c and reaches a minimum at u = 2c. In the case of oscillations about the minimum of the potential energy W(u), the wave is practically harmonic:

$$u = 2c + u_0 \exp\left\{i\sqrt{\frac{c}{\beta}}(x - ct)\right\}.$$

We see that u oscillates about the value 2c, i.e., in a coordinate system moving with velocity 2c, where the oscillations occur about a zero value, the wave propagates with velocity c to the left, as it should for negative dispersion. Since $k = \sqrt{c/\beta} = k_0\sqrt{c/c_0}$, the increment to the phase velocity is equal to $c_0 k^2/k_0^2$, as follows from the dispersion equation.

With increasing oscillation amplitude, the wave becomes more and more asymmetrical (as seen from the diagram for the potential energy); the particle will spend a longer time having a low velocity u, where the elasticity is smaller, and will jump faster through values with large u (see Figs. 2 and 3). Finally, when the amplitude increases to such an extent that values u = 0 become possible, solutions of the type of solitary waves or solitons appear, where only a single pulse propagates through the liquid. In this case the "point" u is situated "for an infinitely long time" at the position u = 0, and then "slides down" into the potential well W(u), reaching a value u = 3c, where W = 0, is reflected from it, and again returns to the position u = 0. The corresponding solution is of the form

$$u = \frac{a}{\operatorname{ch}^2\left(\frac{x - ct}{\Delta}\right)}. \tag{3.21}$$

The amplitude a and the width Δ of a solitary wave are given by the relations

$$a = 3c, \quad \Delta = 2\sqrt{\frac{\beta}{c}} = \frac{2}{k_0}\sqrt{\frac{c_0}{c}}. \tag{3.22}$$

We see therefore that the larger the soliton velocity c,

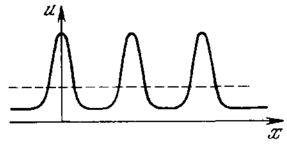


FIG. 3

the larger its amplitude and the smaller its width.

If the wave amplitude a is barely smaller than 3c, then the solution is in the form of pulses very close in shape to solitons, following one another periodically (Fig. 3).

Thus, with increasing oscillation amplitude, the phase velocity changes from negative to positive and increases to a value a/3 in solitary waves.

3.6. Evolution of Initial Perturbation

Let us now discuss the question of excitation of oscillations by initial perturbations of finite amplitude. For simplicity we confine ourselves to the one-dimensional case, assuming that the perturbation is infinitely extended and is homogeneous along the y axis.

Assume that at the initial instant t = 0 there is produced a certain perturbation of velocity with amplitude u_0 and width of the order Δ. If $u_0 \ll c_0$, then the perturbation can be regarded as weakly nonlinear. In this case the perturbation breaks up rapidly, before nonlinearity has a chance to manifest itself, into two perturbations constituting simple waves traveling in opposite directions with velocity $\approx c_0$. It suffices therefore to trace the slow evolution of only one simple wave, i.e., it suffices to investigate the nonstationary solutions of the Korteweg–de Vries equation. We see therefore, in particular, why simple waves are more important than any other solution, even the most general nonlinear solutions.

Nonstationary solutions of the Korteweg–de Vries equations were first investigated numerically[11,13] and an analytic theory for them was constructed only later[14] (see also[15]). But the qualitative picture can be described quite simply with the aid of the following simple considerations.

Let us assume first that the initial perturbations coincide exactly in form with the soliton, i.e.,

$$u(x, t=0) = u_0 \operatorname{sech}^2\left(\frac{x-x_0}{\Delta}\right).$$

Then this perturbation could propagate like a soliton if its amplitude were connected with the velocity by the relation

$$u_0\Delta^2 = 12\beta = \frac{12c_0}{k_0^2} = \text{const}, \tag{3.23}$$

which follows from (3.22). In other words, the dimensionless quantity

$$\sigma = \sqrt{\frac{u_0}{c_0}}(k_0\Delta) = \Delta\sqrt{\frac{u_0}{\beta}} \tag{3.24}$$

is equal to $\sigma_S = \sqrt{12}$ for the soliton. But σ^2 is proportional to the amplitude, and can therefore be regarded as the nonlinearity parameter of the wave: at small values of σ^2 the perturbation has a very small amplitude and can be regarded as "almost" linear, at $\sigma^2 = 12$ a solitary wave is produced, and at $\sigma^2 \gg 12$ the amplitude is so large that the solution has a form that differs significantly from a traveling stationary wave.

It is easy to see why in a weakly-dispersive medium the nonlinearity index is precisely the product of the amplitude by the square of the width, and not the seemingly more natural ratio of the wave amplitude u_0 to the characteristic phase velocity c_0. The reason is that in the absence of dissipation a one-dimensional wave of arbitrarily small amplitude is nonlinear in a nondispersive medium—it must sooner or later "break." It is precisely the dispersion that prevents this from happening, and therefore the amplitude u_0 should be compared not with the phase velocity c_0 of the long-wave perturbations, but with the increment $c_0k^2/k_0^2 \sim c_0(k_0\Delta)^2$, which is connected with the dispersion, so that the nonlinearity index is the quantity $(u_0/c_0)(k_0\Delta)^2$.

If the initial perturbation does not coincide in profile with the soliton, but has the form of a pulse of width Δ and amplitude u_0, then the nonlinearity parameter can again be assumed to be the quantity σ. When $\sigma \ll \sigma_S$ we deal again with an almost linear perturbation. The character of propagation of the waves due to such a perturbation was already considered above—the long-wave part of the spectral expansion of such a perturbation in powers of k propagates with a velocity close to c_0, and the short-wave components lag the main pulse.

Great interest attaches to the case of a strongly nonlinear perturbation $\sigma \gg \sigma_S$, when the width Δ is large. During the first stage of the evolution of such a perturbation, the dispersion does not play any role and its behavior is determined by the nonlinearity. This means that a steepening of the leading front should occur in the pulse, and the latter has a tendency to break. However, when high harmonics appear, dispersion comes into play, and should "separate" the perturbations with different wavelengths. Therefore, after the lapse of a sufficiently long interval, the perturbation should "tumble"—break up into individual groups analogous to groups of waves in the linear case.

Each group can be set in correspondence with its own "local" value of the parameter $\sigma = \sigma_L$. If $\sigma_L < \sigma_S$, then the group is transformed in final analysis into a weakly nonlinear wave packet. When $\sigma_L > \sigma_S$, the group should again break up into smaller oscillations. This will be accompanied by formation of pulses, where σ_L differs little from the critical value σ_S. It is natural to expect such pulses to become transformed into solitons with time. Thus, in the course of time an initial pulse with $\sigma > \sigma_S$ should become transformed into a certain number of solitons and a weakly-nonlinear wave packet.

All the solitons move with velocity $c > c_0$, and the larger the soliton amplitude u_0, the larger this velocity, while the wave packet, which spreads in time and decreases in amplitude, lags the point $x = c_0t$.

This picture agrees fully with the numerical calculations. Figure 4, for example, shows the results of a numerical solution of the Korteweg–de Vries equation for an initial pulse with $\sigma = 10$. This solution was obtained in[12]. We see that the perturbation is broken up into four solitons and a short-wave packet of small amplitude. It is easy to note that the vertices of the solitons lie on a single straight line. This fact has a simple explanation. The point is that the velocity of a soliton, as we know, is equal to $c = u_0/3$, i.e., it is proportional to the soliton amplitude. Therefore the distance $\Delta x = ct = u_0t/3$ traversed by the soliton during the

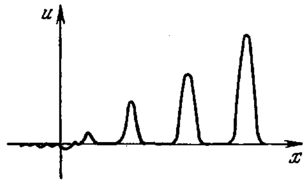


FIG. 4

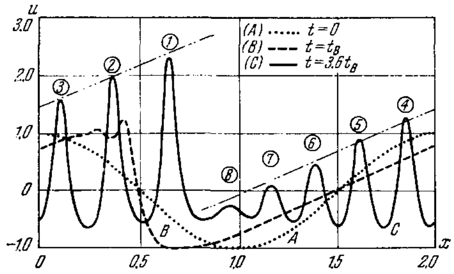


FIG. 5

time t from the point $x = c_0t$ corresponding to the "initial" position of the perturbation is proportional to u_0. In other words, all the solitons have emerged from a single point and therefore the distance traversed by them is proportional to the velocity.

For a more general initial distribution of the solitons, the picture becomes more complicated. But if the amplitude changes smoothly in space, i.e., $u_0 = u_0(x)$, then from the condition that the velocity of the individual soliton be constant we obtain

$$\frac{du_0}{dt} = \frac{\partial u_0}{\partial t} + \frac{u_0}{3}\frac{\partial u_0}{\partial x} = 0, \tag{3.25}$$

i.e., we again obtain a nonlinear equation of the type of the equation for a simple wave. This means that the envelope wave for the solitons should evolve in time like a simple wave—it should become steeper. No breaking occurs in this case; a soliton merely emerges in front and moves with maximum velocity.

An example of the process of crumbling of a wave into solitons may be found in the evolution of a sinusoidal perturbation of large amplitude, which is shown in Fig. 5. A numerical solution of this problem was obtained by Zabusky and Kruskal, from whose paper[11] Fig. 5 was borrowed. This figure shows only one period—the entire picture should be periodically continued on both sides. We see that at first the sinusoidal perturbation evolves like a simple wave—it increases its slope and has a tendency to break. However, before the actual break, dispersion comes into play and the first soliton begins to separate itself. Gradually the entire perturbation breaks up into a set of solitons, which themselves form a "simple wave" and whose amplitudes lie on one straight line for not too large a period of time (curve C). This is followed by formation of the "multistream" state—the solitons with large amplitude overtake the slower ones, so that intersection of the soliton trajectories takes place.

It should be borne in mind that the solitons can be produced only in the case when the initial perturbation x(u, 0) has a positive amplitude (in a medium with negative dispersion). If the initial perturbation has a negative amplitude everywhere, then it cannot give rise to solitons. In this case it evolves into a nonlinear wave "tail" corresponding to periodic solutions (see below).

All the foregoing pertains to media with positive dispersion, with the following modification: the solitons correspond in such media not to "crests," but to "troughs," and all the solitons move with a velocity smaller than c_0, while the wave "tail" of small amplitude moves with a velocity larger than c_0. It would therefore be more appropriate to call it not a "tail" but a "precursor." In media with positive dispersion, solitons are produced only at a negative perturbation amplitude.

3.7. Conservation Laws

The Korteweg–de Vries equation can be written in divergence form:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2} + \beta u_{xx}\right) = 0, \tag{3.26}$$

which has the form of the law for the conservation of the "momentum" $I_1 = \int_{-\infty}^{\infty} u(x, t)dx$. Multiplying both sides of the Korteweg–de Vries equation by u and u^2, we obtain after simple calculations two more conservation laws, of which the first reflects the "energy" conservation

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left[\frac{u^3}{3} + \beta\left(uu_{xx} - \frac{u_x^2}{2}\right)\right] = 0, \\ &\frac{\partial}{\partial t}\left(\frac{u^3}{3} - \beta u_x^2\right) + \frac{\partial}{\partial x}\left[\frac{u^4}{4} + \beta\left(u^2u_{xx} + 2u_tu_x\right) + \beta^2 u_{xx}^2\right] = 0. \end{aligned} \tag{3.27}$$

It turns out that this does not complete the number of conservation laws. It was shown in[16] that the Korteweg–de Vries equation corresponds to an infinite number of conserved quantities (invariants) $I_m = \int_{-\infty}^{\infty} Q_m$ (x, t)dt, whose densities $Q_m(t)$ satisfy relations of the form

$$\frac{\partial Q_m(x, t)}{\partial t} + \frac{\partial P_m(x, t)}{\partial x} = 0 \tag{3.28}$$

and are polynomials of β, u, and of the derivatives u_x, u_{xx}, u_{xxx}, ... the derivatives with respect to t can be eliminated with the aid of the Korteweg–de Vries equation). Let us consider now the more general structure of the densities Q [u] as "functionals" of u. If we arrange the terms in $Q_m[u]$ in order of increasing powers of β, then the term not containing β is always proportional to u^m; using the fact that Q_m is determined accurate to a constant factor, it is convenient to write this term in the form u^m/m. Further, the quantity $Q_m[u]$ contains terms with β^k (k = 1, 2, ..., m − 2), and the coefficients of β^k have the form of certain polynomials of u, u_x, u_{xx}, ..., the general structure of which can be established from dimensionality considerations.

The general program for obtaining the numerical coefficients of the individual terms in $Q_m[u]$ is quite cumbersome and will not be considered here (see[16] concerning this question). We present only the first two terms in the expansion of $Q_m[u]$ in powers of the parameter β:

$$Q_m[u] = \frac{u^m}{m} - \beta\frac{(m-1)(m-2)}{2}u_x^2u^{m-3} + O(\beta^2). \tag{3.29}$$

3.8. Analytic Relations

Let us consider now the most important analytic results characterizing the solutions of the Korteweg–

de Vries equation u(x, t). We write the initial condition in the form

$$u(x, 0) = u_0\varphi\left(\frac{x}{\Delta}\right), \quad (3.30)$$

where u_0 and Δ are the characteristic amplitude and width of the initial perturbation, and $\varphi(\xi)$ is a dimensionless function describing its profile (in this section it is assumed throughout that $\varphi(\xi) \to 0$ as $\xi \to \pm\infty$.

Changing over to the dimensionless variables

$$\xi = \frac{x}{\Delta}, \quad \tau = \frac{tu_0}{\Delta}, \quad \eta(\xi, \tau) = \frac{u}{u_0}, \quad (3.31)$$

we obtain the Korteweg–de Vries equation and the initial condition in the form

$$\frac{\partial\eta}{\partial\tau} + \eta\frac{\partial\eta}{\partial\xi} + \frac{1}{\sigma^2}\frac{\partial^3\eta}{\partial\xi^3} = 0, \quad \eta(\xi, 0) = \varphi(\xi), \quad (3.32)$$

where

$$\sigma = \Delta\left(\frac{u_0}{\beta}\right)^{1/2}. \quad (3.33)$$

It follows from (3.32) that solutions with identical σ and $\varphi(\xi)$ should be similar to each other. In particular, the number of solitons produced as a result of the evolution of the initial perturbation, the ratio of their amplitudes, etc. are all determined uniquely by the quantity σ and by the form of the initial profile, characterized by the function $\varphi(\xi)$. Therefore the quantity σ can be called the similarity parameter[12].

As shown in[14], the amplitudes of the solitons produced by an initial perturbation that attenuates as $x \to \pm\infty$ are determined by the eigenvalues of a certain Strum-Liouville boundary-value problem (or, using the language of quantum mechanics, by the energy levels in a certain potential well). Let us discuss this question in greater detail.

We consider the Schrödinger equation

$$\Psi_{\xi\xi}(\xi;\ \tau) + \frac{\sigma^2}{6}[E(\tau) + \eta(\xi, \tau)]\Psi(\xi;\ \tau) = 0, \quad (3.34)$$

where the potential $-\eta(\xi, \tau)$, the eigenvalues $E(\tau)$, and the wave functions $\Psi(\xi, \tau)$ depend on the time τ as a parameter, and this dependence is determined by the fact that $\eta(\xi, \tau)$ satisfies the Korteweg–de Vries equation in the form (3.32), under the initial condition $\eta(\xi, 0) = \varphi(\xi)$. (The role of the quantity $2m/\hbar^2$ is played in this case by $\sigma^2/6$.)

By considering simultaneously the equations (3.32) and (3.34) we can prove, first of all, that the eigenvalues of Eq. (3.34) do not depend on the time, i.e., $E(\tau) = E$, where E are the eigenvalues of the equation

$$\Psi_{\xi\xi}(\xi;\ 0) + \frac{\sigma^2}{6}[E + \varphi(\xi)]\Psi(\xi;\ 0) = 0. \quad (3.35)$$

It turns out, further, that the discrete spectrum of Eq. (3.35) completely determines the amplitude, and consequently also the velocities of all the solitons that are produced from the initial perturbation, namely, the soliton amplitudes a_r (r = 1, 2, ..., N) satisfy the relations

$$a_r = -2u_0E_r, \quad (3.36)$$

where E_r are the eigenvalues of the discrete spectrum.

Moreover, knowing the asymptotic behavior of the wave functions $\Psi(\xi; 0)$ as $\xi \to \pm\infty$ (for both the discrete and the continuous parts of the spectrum), we can also determine the asymptotic forms of the wave functions $\Psi(\xi; \tau)$ satisfying Eq. (3.34). On the other hand, knowing the asymptotic behavior of $\Psi(\xi; \tau)$ as $\xi \to \pm\infty$, we can, using the methods used in the solution of the inverse problem of scattering theory[17,18], reconstruct the "potential" $-\eta(\xi, \tau)$ of Eq. (3.34) for any τ. To this end it is necessary to solve a certain integral equation, which is now linear and which is given in[14]. Unfortunately, the latter is quite complicated in form and its exact solution for arbitrary τ is in general unknown. Therefore, for example, the described method encounters great difficulties when attempts are made to use it to investigate the character of "tails," etc.

Nonetheless, even formula (3.36) can yield very valuable information concerning solitons produced (at sufficiently large τ) from an initial perturbation.

In particular, if $\varphi(\xi) \leq 0$ for all ξ, then the solitons cannot arise, no matter what the value of σ. On the other hand, if the initial perturbation $\varphi(\xi)$ is of alternating sign, then the solitons appear only at sufficiently large σ. Finally, if $\int_{-\infty}^{\infty}\varphi(\xi)d\xi > 0$, then the Schrödinger equation always has a discrete spectrum, i.e., at least one soliton is produced from an initial perturbation with positive area of the profile*. In the latter case (at sufficiently small values of σ), the eigenvalue of the Schrödinger equation can be obtained by perturbation theory:

$$E \approx -\frac{\sigma^2}{24}\left[\int_{-\infty}^{\infty}\varphi(\xi)\,d\xi\right]^2$$

(see for example,[20], Sec. 45). Accordingly, the amplitude of the soliton is approximately equal to

$$a \approx \frac{u_0\sigma^2}{12}\left[\int_{-\infty}^{\infty}\varphi(\xi)\,d\xi\right]^2. \quad (3.37)$$

The condition for applicability of perturbation theory has in this case the form

$$\sigma^2 \ll \sigma_s^2, \quad (3.38)$$

where $\sigma_s = \sqrt{12}$ is the value of the similarity parameter for the soliton.

Let us consider now the case when $\sigma \gg 1$ and the number N of the produced solitons is large. In this case they can be characterized by a distribution function F(a), which determines the number of solitons dN having an amplitude in the interval (a, a + da):

$$dN = F(a)\,da. \quad (3.39)$$

The function F(a) can easily be obtained by calculating the level density in the potential well by the WKB method. As a result we obtain*

$$F(a) = \frac{\sigma}{4\pi}\left(\frac{1}{3u_0}\right)^{1/2}\int_M \frac{d\xi}{\sqrt{2u_0\varphi(\xi) - a}}, \quad (3.40)$$

where the region of integration M contains those values of ξ where

$$2u_0\varphi(\xi) > a. \quad (3.41)$$

From (3.40) it follows, in particular, that the ampli-

*These results were obtained earlier in [19] from other considerations.

†Formula (3.40) was first derived in [15] on the basis of the conservation laws (in a somewhat different form: by introducing into (3.40) a new integration variable $z = \varphi(\xi)$, we obtain expression (6) of [15]).

tudes of the solitons do not exceed double the maximum of the initial perturbation:

$$F(a)=0 \quad \text{for} \quad \frac{a}{2} > \max(u_0\varphi(\xi)). \qquad (3.42)$$

Integrating (3.40), we obtain an asymptotic expression for the total number of solitons[21]

$$N=\int_0^\infty F(a)\,da = \frac{\sigma}{\pi\sqrt{6}}\int_{\varphi(\xi)>0}\sqrt{\varphi(\xi)}\,d\xi \qquad (3.43)$$

(The integration region contains only those values of ξ for which $\varphi(\xi) > 0$). Thus, at large σ the number of solitons is determined only by the region where the initial perturbation is positive.

We present a few more asymptotic relations which make it possible to compare the relative roles of the solitons and the "tails" at large σ*. Denoting the asymptotic values of the invariants for the solitons and for the "tails" (at $\sigma \gg 1$) by $I_m^{(s)}$ and $I_m^{(t)}$, respectively, we obtain[21]

$$\left.\begin{aligned} I_m^{(s)} &= \frac{1}{m}\int_{u(x,0)>0} dx\,u^m(x,0),\\ I_m^{(t)} &= \frac{1}{m}\int_{u(x,0)<0} dx\,u^m(x,0). \end{aligned}\right\} \qquad (3.44)$$

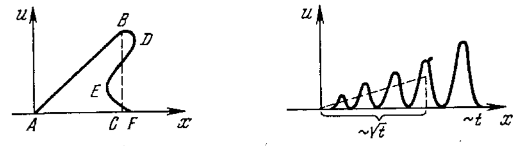
Thus, the analytic values of the invariants for the "tails" $I_m^{(t)}$ are determined only by those regions where the initial perturbation is negative. If $u(x, 0) \geq 0$ for all values of x, then it follows from (3.44) that $I_m^{(t)} = 0$, so that in this case one can state (with asymptotic accuracy at large values of σ) that the perturbation breaks up completely into solitons. The results of numerical calculations given in[12,22,23] show that this result, and also formula (3.43), are in many cases quite accurate at relatively small values of σ.

Finally, let us stop to discuss one more aspect of the results of[22], which are described in this section. Since it follows from the definition (3.33) of the number σ that the limiting case $\sigma \gg 1$ can be realized at finite u_0 and Δ and at small β, it follows that the formulas obtained above determine the asymptotic solution of the Korteweg—de Vries equation at small values of the parameter β. Let us consider this solution under the condition $\varphi(\xi \geq 0)$ and compare it with the solution of the equation for a simple wave in hydrodynamics $u_t + uu_x = 0$ under the same initial conditions, and also with the asymptotic solution of the Burgers equation (2.12) as $\mu \to 0$ (the latter has a triangular profile with a shock wave on the front).

The solution of the equation $u_t + uu_x = 0$ is represented by the curve ABDEF in Fig. 6a, and the solution of the Burgers equation by the curve ABC. Under identical initial conditions, the areas of the curves are equal, since the momentum conservation follows from the equation $u_t + uu_x = 0$ and from the Burgers equation.

The solution of the Korteweg—de Vries equation for $\beta \to 0$ and $\varphi(\xi) \geq 0$ consists, according to the foregoing, of a number of solitons the number of which increases

*Although the "tail" does spread out in the course of time, its contributions to the momentum, energy, and to other invariants $I_m = \int^\infty Q_m[u]\,dx$ remain constant (after the solitons have "broken away" from it). This raises the question of the relative magnitudes of these contributions.



a) b)

FIG. 6

like $\beta^{-1/2}$ and whose widths decrease like $\beta^{1/2}$. This solution is shown in Fig. 6b.

As $\beta \to 0$, the invariants of the Korteweg—de Vries equation assume according to (3.29) the form

$$I_m=\frac{1}{m}\int_{-\infty}^{+\infty} u^m(x,t)\,dx \qquad (3.45)$$

and consequently they coincide with the invariants of the equation $u_t + uu_x = 0$, which has an infinite number of conservation laws in the form

$$\frac{\partial}{\partial t}\left(\frac{u^m}{m}\right)+\frac{\partial}{\partial x}\left(\frac{u^{m+1}}{m+1}\right)=0. \qquad (3.46)$$

Thus, under the same initial conditions, the profile ABDEF and the profile shown in Fig. 6b have in the limiting case $\beta \to 0$ not only identical areas but also other invariants of the type (3.45).

3.9. Shock Waves in Dispersive Media

We have assumed above that there are no dissipative processes, i.e., that their role is much smaller than that of dispersion. If there is some dissipation, for example viscosity, then all the waves discussed above are weakly damped, and second-derivative terms appear in the Koretweg—de Vries equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=\mu\frac{\partial^2 u}{\partial x^2} \qquad (3.47)$$

(this equation can be called the Korteweg—de Vries—Burgers equation).

We consider again a traveling stationary wave in the form $u = u(x - ct)$. For such a wave, the equation (3.47) can be integrated once with respect to x:

$$\beta\frac{\partial^2 u}{\partial x^2}-\mu\frac{\partial u}{\partial x}=-cu+\frac{u^2}{2}=-\frac{\partial W}{\partial u}. \qquad (3.48)$$

We see therefore that we again obtain an equation for a nonlinear oscillator, but this time with damping. Accordingly, the oscillations of the equivalent oscillator with potential W(u) will be damped; therefore in place of a periodic wave there will arrive an asymmetrical train of waves (Fig. 7a). We see that after passage of such a train, the state of the medium changes, for the medium behind the train moves with a certain velocity u_0. This means that we obtain a jump, namely a shock wave, but with oscillating structure[24,25]. At low viscosity, the first waves of this structure are close to solitons, and if the viscosity is large compared with the dispersion then we arrive at an ordinary shock wave with a monotonic increase of u from 0 to u_0.*

Since the minimum of the potential W is reached at u = 2c, the difference between the limiting values $u(\infty)$

*The critical value of the parameter μ, at which the transition from an oscillatory to a monotonic structure of the shock wave takes place, is determined by the expression $\mu = \sqrt{4\beta c}$.

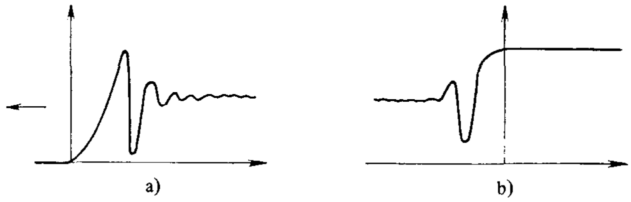


FIG. 7

= 0 and u(−∞) = 2c turns out to be connected with the velocity of the wave by the relation

$$c = \frac{\Delta u}{2}, \qquad (3.49)$$

where Δu = u(−∞) − u(∞). Comparing this expression with (2.18), we see that c coincides with the velocity of the shock wave without dispersion. In a coordinate system where the medium is at rest, the velocity of the shock wave is equal to $(c_0 + \Delta u)/2$, and the corresponding Mach number is

$$M = 1 + \frac{\Delta u}{2c_0}. \qquad (3.50)$$

This picture pertains to media with negative dispersion, when the largest soliton travels with maximum velocity, and the oscillating "tail" remains behind the front. In media with positive dispersion, to the contrary, the oscillatory structure is ahead of the wave front, as shown in Fig. 7b. Both types of shock waves were observed experimentally in a plasma (see, for example[26]) and also in nonlinear transmission lines (electromagnetic shock waves)[25].

4. SELF-FOCUSING AND SELF-CONTRACTION OF WAVE PACKETS

4.1. Self-focusing

We have become acquainted above, in sufficient detail, with nonlinear waves in dispersive media, but have confined ourselves at all times to the one-dimensional case, when all the quantities depended only on one coordinate x and on the time t. To obtain a more complete representation of the dynamics of nonlinear wave processes, it is necessary to forgo the one-dimensionality limitation and to change over to the general case of three-dimensional or at least two-dimensional waves. But before we proceed to the rather complicated general case, we shall consider a simpler class of problems, when the waves differ little from one-dimensional, i.e., when we deal with a wave whose amplitude and phase vary slowly in space and in time. In this case we encounter two extremely interesting nonlinear processes—self-focusing and self-contraction of wave packets. Let us consider first self-focusing.

The phenomenon of self-focusing was predicted by Askar'yan[27] from very simple considerations. Let us assume that a powerful laser beam propagates in an optically transparent medium. Owing to a large number of effects (nonlinear polarizability, electrostriction, heating, etc.) such a beam changes slightly the refractive index of the medium. If this change is positive, i.e., the medium becomes optically denser, then the beam itself produces something in the way of a lens, which

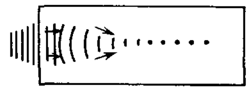


FIG. 8

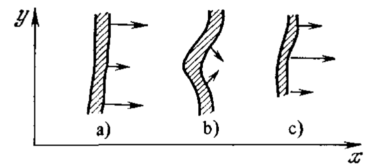


FIG. 9

will focus the beam. In other words, the central part of the wave front lags somewhat behind the peripheral regions, and the wave becomes convergent (Fig. 8).

This reasoning, the physics of which is quite clear, may still turn out to be insufficiently justified if account is taken of the fact that self-focusing is a nonlinear and furthermore rather slow process, and this raises the question whether stronger effects of nonlinear wave breaking (in the language of optics—frequency multiplication) occur beforehand. However, the analysis in the preceding chapter shows that the dispersion readily prevents the breaking of the waves. Therefore in the presence of even small dispersion, strong saturation with higher harmonics need not necessarily occur, and does not occur if the wave amplitude is not very large. Incidentally, even at a sufficiently large amplitude, the qualitative considerations of self-focusing remain in force if a definite almost-periodic wave in x is established and there is equilibrium of the harmonics or a "competition" between the dispersion and the nonlinear wave breaking. In particular, effects of self-focusing appear even in the limiting case of a single soliton[28]. Let us consider, for example, the soliton of Fig. 9a, the amplitude of which changes from y (the density of the shading in Fig. 9a corresponds to the height of the soliton). In the media with negative dispersion, the sections with larger amplitude move more rapidly and the soliton bends, as shown in Fig. 9b. As a result of the bending, focusing is produced and the amplitude of the central part begins to increase. This leads to a restoration of the shape of the soliton, but with the regions of larger amplitude moving along y. We see therefore that a soliton in a medium with negative dispersion does not experience self-focusing—it vibrates like a stretched string. But in the case of positive dispersion the situation changes: the sections with increased amplitude lag somewhat and the new batches of perturbation tend to them as a result of the bending; as a result, the soliton compresses into a compact formation also in the y direction.

4.2. Self-contraction of Wave Packets

The phenomenon of contraction of a nonlinear wave can occur not only in the transverse but also in the longitudinal direction relative to the direction of wave propagation. To reveal this effect, let us consider a plane wave packet with slowly-varying amplitude and phase. We assume that the amplitude of the wave is small, so that it does not differ strongly from sinusoidal, i.e., that the higher harmonics, which are in equilibrium with the fundamental, are small. Then the wave can be characterized by a wave number k and a frequency ω of the fundamental harmonic. In this case the main average nonlinear effect is the dependence of the phase velocity or of the frequency on the amplitude a; therefore at a low amplitude, when it suffices to confine oneself only to the first nonvanishing correction

$$\omega(k, a) = \omega_0(k) + \alpha a^2, \tag{4.1}$$

where $\omega_0(k)$ corresponds to the frequency of the linear wave, and the second term to the nonlinear correction. If k and a^2 vary with x, then the phase $\varphi(x, t)$, which for a monochromatic wave has the form $\varphi = kx - \omega t$, will now no longer be a linear function of x and t; it can, however, be assumed that, as before,

$$k = \frac{\partial \varphi}{\partial x}, \quad \omega = -\frac{\partial \varphi}{\partial t}. \tag{4.2}$$

We see from (4.2) that the wave number varies with time in accordance with the equation

$$\frac{\partial k}{\partial t} = -\frac{\partial \omega}{\partial x} = -v_g \frac{\partial k}{\partial x} - \alpha \frac{\partial a^2}{\partial x}, \tag{4.3}$$

where we have taken into account expression (4.1) for the frequency, and denoted the group velocity by $v_g = \partial\omega_0/\partial k$. We now recognize that the energy in the wave packet, which is a quantity quadratic in the amplitude, is transported (in the considered approximation in a) with the group velocity. Then for the energy conservation law we can assume the equation

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x}(v_g a^2) = 0. \tag{4.4}$$

From (4.3) and (4.4) it follows that under certain conditions a plane wave is unstable against breakdown into individual wave packets. In fact, let us apply to a monochromatic wave with wave number k_0 and amplitude a_0 a small perturbation:

$$k = k_0 + k' \exp(-i\nu t + i\varkappa x), \quad a = a_0 + a' \exp(-i\nu t + i\varkappa x),$$

where $\nu \ll \omega$, $\kappa \ll k_0$ are the frequency and wave number of the modulation. Then in the linear approximation we obtain from (4.3) and (4.4) the dispersion relation

$$\nu = v_g \varkappa \pm \sqrt{\alpha v_g' a_0^2}\,\varkappa, \tag{4.5}$$

where $v_g' = \partial v_g/\partial k$. We see that when $\alpha v_g' < 0$ we have instability of the type of breakdown of the wave into packets and self-contraction of the wave packets. This result was first obtained by Lighthill[29a].

The physics of the instability is explained in Fig. 10. Let us assume that $\alpha > 0$. Then at the points A and A′ the phase velocity of the wave is larger than at the point B, and in section a the wave number, which is proportional to the number of nodes per unit length, will increase, while in section b it will decrease with time. Consequently, when $v_g' < 0$, the wave packet in the region of a will lag and will amplify the amplitude at the point A, while in region b it will move ahead and amplify the wave at the point A′.

An interesting example of unstable nonlinear waves are the gravitational Stokes waves on the surface of a liquid whose depth is much larger than the wavelength. In this case the dispersion equation of the linear approximation is of the form $\omega = \sqrt{gk}$ (see, for example,[3,30]). For a stationary wave of finite (but small) amplitude, we obtain the following nonlinear dispersion equation[30]:

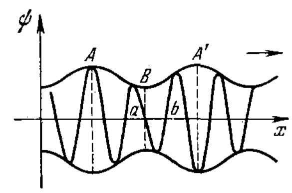


FIG. 10

$$\omega(k, a^2) = \sqrt{gk}\left(1 + \frac{a^2k^2}{2} + \ldots\right), \tag{4.6}$$

where a is the amplitude of the oscillations of the free surface of the liquid. It follows from (4.6) that

$$\alpha v_g' = -\frac{gk}{8} < 0. \tag{4.7}$$

Thus, the gravitational waves on the surface of "deep" water are unstable against longitudinal perturbations[29a]. From the foregoing formulas we see that the waves are stable against transverse perturbations (i.e., they do not become self-focused).

The conclusion that periodic gravitational waves on water are unstable created a sensation in hydrodynamics in its time and seemed unlikely. But subsequently it was derived again by different methods and was confirmed experimentally[31,32]. Now no one doubts the instability of gravitational waves in deep water, meaning that the superstition of the "tenth wave" has definite physical justification.

4.3. Parabolic Equation

Both self-focusing and self-contraction of wave packets, and furthermore with allowance for diffraction, can be described with the aid of a parabolic equation[33-36]. Let us imagine that we have an almost-plane wave with wave vector $\mathbf{k}_0$ and frequency $\omega_0 = \omega(\mathbf{k}_0)$. In the linear approximation, the system of equations for small deviations from the equilibrium position can be represented after expressing certain quantities in terms of others in the form of one linear equation for the complex amplitude ψ, namely $D(\omega, \mathbf{k})\psi_{\mathbf{k}\omega} = 0$. In particular, a solution of the dispersion equation $D(\omega, \mathbf{k}) = 0$ is given by the ω_0 and $\mathbf{k}_0$ chosen by us. Near these values, confining ourselves to terms linear in $\omega - \omega_0$ and $\mathbf{k} - \mathbf{k}_0$, the equation for the Fourier harmonic $\psi_{\mathbf{k}\omega}$ can be represented in the form $A(\mathbf{k}_0, \omega_0)\,[\omega - \omega(\mathbf{k})]\psi_{\mathbf{k}\omega} = 0$, where A = const can be omitted, and $\omega(\mathbf{k})$ is the solution of the dispersion equation. We expand $\omega(\mathbf{k})$ near $\mathbf{k} = \mathbf{k}_0$:

$$\omega(\mathbf{k}) \approx \omega(\mathbf{k}_0) + v_g\left(\sqrt{(k_0+\varkappa_x)^2 + \varkappa_\perp^2} - k_0\right) + \frac{1}{2} v_g' \varkappa_x^2 \approx$$
$$\approx \omega(k_0) + v_g\left(\varkappa_x + \frac{\varkappa_\perp^2}{2k_0}\right) + \frac{1}{2} v_g' \varkappa_x^2,$$

where $\boldsymbol{\kappa} = \mathbf{k} - \mathbf{k}_0$ is a small deviation of the wave vector $\mathbf{k}_0$ ($\mathbf{k}_0 = \mathbf{k}_{0x}$). If we substitute this expression in the equation $[\omega - \omega(\mathbf{k})]\psi_{\mathbf{k}\omega} = 0$ and change over from the Fourier harmonics to the variables $\mathbf{r}$ and t, then putting $\omega - \omega_0 = i(\partial/\partial t)$ and $\boldsymbol{\kappa} = -i\nabla$ we obtain a parabolic equation for ψ. When the nonlinear addition to the frequency (4.1) is included, this equation takes the form

$$i\left(\frac{\partial \psi}{\partial t} + v_g \frac{\partial \psi}{\partial x}\right) + \frac{1}{2} v_g' \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2}\frac{v_g}{k_0}\Delta_\perp \psi - \alpha |\psi|^2 \psi = 0, \tag{4.8}$$

where $v'_g = \partial v_g/\partial k_0$.

Equation (4.8) can be transformed into another frequently-used form by substituting in (4.8) $\psi = ae^{i\varphi}$ (a—amplitude and φ is an increment to the phase of the wave, both being assumed real), and then separating the real and imaginary parts. We then obtain the following systems:

$$\frac{\partial\varphi}{\partial t}+v_g\frac{\partial\varphi}{\partial x}+\frac{1}{2}v'_g\left(\frac{\partial\varphi}{\partial x}\right)^2+\frac{v_g}{k_0}(\nabla_\perp\varphi)^2+\alpha a^2-\frac{v'_g}{2a}\frac{\partial^2 a}{\partial x^2}-\frac{v_g}{2k_0 a}\Delta_\perp a=0,$$

$$\frac{\partial a^2}{\partial t}+v_g\frac{\partial a^2}{\partial x}+v'_g\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}a^2\right)+\frac{v_g}{k_0}\nabla_\perp(a^2\nabla_\perp\varphi)=0. \quad (4.9)$$

It is important that the coefficients of (4.8) (and also of the system (4.9)) are determined completely by the nonlinear dispersion equation (4.1). Therefore waves of different nature can be regarded from a unified point of view. In particular, for an electromagnetic field in a nonlinear medium with dielectric constant $\epsilon = \epsilon_0(\omega)(1 + \beta|E|^2)$ the equation for the complex amplitude of the electric field $E(r, t)$ takes the form

$$2i(E_t+v_gE_x)-k_{\omega\omega}v_g^3E_{xx}+\frac{v_g}{k_0}\Delta_\perp E+2v_gk_0\beta|E|^2E=0. \quad (4.10)$$

At $E_t = 0$ *this equation describes the diffraction of a* stationary light beam with allowance for the nonlinear effects. If we neglect the nonlinear term and the term with E_{xx} (which is of the order of the ratio of the wavelength to the width of the beam), then we obtain the so-called parabolic equation of the approximate theory of diffraction[33]. In this connection, Eq. (4.8) is customarily called the nonlinear parabolic equation. Equation (4.10) was investigated in a number of papers in connection with the phenomenon of the self-action of light (see, for example, the reviews[35,36]). The results obtained thereby, as seen from the foregoing, can be extended also to other types of waves, which are described by the more general equation (4.8).

Let us consider again the problem of the instability of a plane wave: $\psi_0 = a_0\exp(-i\nu_0 t)$, where a_0 is the initial amplitude and ν_0 is the nonlinear increment to the frequency, which according to (4.8) is given by $\nu_0 = \alpha a_0^2$. Let us assume that the wave ψ_0 is perturbed somewhat, and that its amplitude a and phase φ vary little in space and in time: $a = a_0 + a'$, $\varphi = \nu_0 + \varphi'$. The small quantities a' and φ' should be regarded as real. After substituting this expression in (4.9) and discarding terms quadratic in a' and φ', we obtain a system of equations for a' and φ':

$$a_0\left(\frac{\partial\varphi'}{\partial t}+v_g\frac{\partial\varphi'}{\partial x}\right)+\hat{\mathscr{L}}a'+2\alpha a_0^2a'=0, \quad (4.11)$$

$$\frac{\partial a'}{\partial t}+v_g\frac{\partial a'}{\partial x}-a_0\hat{\mathscr{L}}\varphi'=0, \quad (4.12)$$

where $\hat{\mathscr{L}}$ is an operator equal to

$$\hat{\mathscr{L}}=-\frac{v_g}{2k_0}\Delta_\perp-\frac{1}{2}v'_g\frac{\partial^2}{\partial x^2}.$$

Equations (4.11) and (4.12) are linear, and their solution can therefore be sought in a form proportional to $\exp(-i\nu t + i\kappa r)$. $\hat{\mathscr{L}}$ can be regarded here as equal to the number

$$L=\frac{v_g}{2k_0}\varkappa_\perp^2+\frac{1}{2}v'_g\varkappa_x^2,$$

and, as a condition for the solvability of (4.11) and (4.12), we obtain the following expression for the frequency:

$$\nu=v_g\varkappa_x\pm\sqrt{L(2\alpha a_0^2+L)}. \quad (4.13)$$

If $\kappa_\perp = 0$ and κ_x^2 is small, then we obtain from this the previous result (4.5), i.e., instability with respect to self-contraction at $\alpha v'_g < 0$. As seen from (4.13), the conclusion of self-contraction when $\alpha v'_g < 0$ is valid only at sufficiently small κ_x^2. If $\kappa_x^2 > 4\alpha a_0^2/v'_g$, then the instability with respect to self-contraction is stabilized by the diffraction spreading of the wave packet. In the other limiting case $\kappa_x = 0$, (4.13) leads to self-focusing if $\alpha < 0$. Self-focusing also begins only with sufficiently small $\kappa_\perp$, namely $\kappa_\perp^2 < 4\alpha a_0^2 k_0/v_g$. Since the minimum possible value of $\kappa_\perp$ is of the order of the reciprocal radius R of the cylindrical beam, it follows therefore that in the approximation under consideration, where only the correction quadratic in the amplitude is included for the frequency (4.1), self-focusing begins only at sufficiently large power of the wave beam, proportional to $a_0^2R^0$.

The stabilizing role of diffraction makes it possible for stationary focused beams or self-contracted wave packets to exist. In fact, let us consider, for example, the case $\alpha v'_g < 0$ and let us assume that there is no dependence on y or z. We consider a solution of the type of a traveling packet $\psi = e^{-i\nu_0 t}u(x - v_g t)$. For the function u we obtain from (4.8)

$$\frac{\partial^2 u}{\partial x^2}=\frac{2\alpha}{v'_g}u^3-\frac{2\nu_0}{v'_g}u. \quad (4.14)$$

We have already encountered an equation of this type in the analysis of periodic solutions of the Korteweg—de Vries equations. We can again regard (4.14) as an equation for a nonlinear oscillator with potential energy

$$W(u)=-\frac{\alpha}{2v'_g}u^4+\frac{\nu_0}{v'_g}u^2.$$

When $\alpha v'_g < 0$ and $\nu_0 v'_g < 0$, the potential W(u) has a well, so that both periodic envelope waves and localized packets of the soliton type are possible.

The equation for the latter is of the form[34]

$$a=A\,\mathrm{sech}\left[A\left(-\frac{\alpha}{v'_g}\right)^{1/2}(x-u_0t)\right],\quad \nu_0=\alpha A^2/2. \quad (4.15)$$

Similarly, for the case of self-focusing we can find solutions localized in the transverse direction if $\alpha < 0$. These solutions correspond to beams that produce a waveguide by themselves and propagate in the form of narrow filaments.

4.4. Nonlinear Geometrical Optics

The results of the preceding section describe only the initial stage of the processes of modulation and focusing. An investigation of these processes at sufficiently large t becomes greatly complicated by the complicated form of the fundamental equations. We shall therefore consider first a simplified system of equations, which is obtained from (4.9) if one neglects in the first equation of (4.9) the terms containing the second derivatives in the amplitude:

$$\frac{\partial\varphi}{\partial t}+v_g\frac{\partial\varphi}{\partial x}+\frac{1}{2}v'_g\left(\frac{\partial\varphi}{\partial x}\right)^2+\frac{v_g}{2k_0}(\nabla_\perp\varphi)^2+\alpha a^2=0, \quad (4.16)$$

$$\frac{\partial a^2}{\partial t}+v_g\frac{\partial a^2}{\partial x}+v'_g\frac{\partial}{\partial x}\left(a^2\frac{\partial\varphi}{\partial x}\right)+\frac{v_g}{2k_0}\Delta_\perp(a^2\nabla_\perp\varphi)=0. \quad (4.17)$$

We note that these equations can also be obtained from a three-dimensional analysis of Eqs. (4.2) and (4.3):

$$\frac{\partial \mathbf{k}}{\partial t} = -\frac{\partial \omega}{\partial \mathbf{r}}, \quad \frac{\partial a^2}{\partial t} + \frac{\partial}{\partial \mathbf{r}} (\mathbf{v}_g a^2) = 0, \tag{4.18}$$

by putting in them

$$\mathbf{k} = \mathbf{k}_0 + \frac{\partial \varphi(\mathbf{r}, t)}{\partial \mathbf{r}}, \quad \omega = \omega_0 - \frac{\partial \varphi(\mathbf{r}, t)}{\partial t} \tag{4.19}$$

(where the "unperturbed" wave vector $\mathbf{k}_0$ is assumed to be directed along the x axis), and by confining ourselves to terms quadratic in $\mathbf{k} - \mathbf{k}_0 = \nabla\varphi$). Since Eqs. (4.18) correspond to the geometrical-optics approximation, the system (4.16) and (4.17) can be called the fundamental equations of the nonlinear "geometrical optics" for small-amplitude waves, when the local profile of the wave is determined by one harmonic. This system of equations describes processes of nonlinear self-action of "rapidly oscillating" waves, characterized by the fact that the changes of the amplitude, wavelength, and other parameters are sufficiently small over distances on the order of the wavelength and in a time comparable with the period of the oscillations. The most general approach to the nonlinear wave processes in this limiting case was developed by Whitham[37,38], who devised a general method that makes it possible to obtain equations for slowly varying parameters in an approximation that can now be called "adiabatic." The approximation considered here corresponds in the Whitham formalism to waves of small amplitude.

Let us now consider the nonlinear evolution of the envelope of a wave having an amplitude a_0 prior to the application of the perturbation. In this case it is convenient to redefine the phase, $\varphi \to \varphi - \alpha a_0^2 t$, so that the last nonlinear term in (4.16) is replaced by $\alpha(a^2 - a_0^2)$. Let us consider the one-dimensional case, when all the quantities depend only on x and t, and change over to new variables*

$$\xi = x - v_g t, \quad \tau = v_g' t. \tag{4.20}$$

We see that ξ has the meaning of the coordinate in a reference frame moving with the group velocity. In terms of these variables, Eqs. (4.16) and (4.17), with the redefined phase, take the form

$$\frac{\partial \varphi}{\partial \tau} + \frac{1}{2}\left(\frac{\partial \varphi}{\partial \xi}\right)^2 + \frac{\alpha}{v_g'}(a^2 - a_0^2) = 0, \tag{4.21}$$

$$\frac{\partial a^2}{\partial \tau} + \frac{\partial}{\partial \xi}\left(a^2 \frac{\partial \varphi}{\partial \xi}\right) = 0, \tag{4.22}$$

which have the same form as the hydrodynamic equations. Here $\varphi(\xi, t)$ plays the role of the velocity potential and a^2 plays the role of density (the "adiabatic exponent" is $\gamma = 2$). Accordingly, the square of the speed of "sound" is

$$c_0^2 = \frac{\alpha a_0^2}{v_g'}. \tag{4.23}$$

We see that an unstable wave corresponds to $c_0^2 < 0$, i.e., to "negative compressibility."

Let us consider first the case when the nonlinear stationary wave is stable, i.e., the square of the propagation velocity of the modulation oscillations (4.23) is positive. In this case the simplest nonlinear solutions of Eqs. (4.21) and (4.22) are "simple waves" of the modulation perturbations, propagating against the background of an infinite stationary wave with parameters a_0, k_0, $\omega_0 = \omega(k_0, a_0^2)$. Proceeding in the same manner as in ordinary gasdynamics, we can easily obtain the following equation for a simple wave propagating in the positive direction*:

$$\frac{\partial a}{\partial t} + c_0\left(1 + 3\frac{a - a_0}{a_0}\right)\frac{\partial a}{\partial \xi} = 0, \tag{4.24}$$

where c_0 is defined in (4.23). It follows from (4.24) that in a reference system where the medium is at rest, the point where the amplitude of the wave is equal to some fixed value a moves with a velocity

$$\left(\frac{dx}{dt}\right)_a = v_g + v_g' c_0\left(1 + 3\frac{a - a_0}{a_0}\right) \tag{4.25}$$

(we have returned to the initial variables x and t in accordance with (4.20)). It is seen from (4.25) that the larger a, the greater the velocity of the point of the perturbation profile corresponding to this value of the amplitude. As a result, the perturbation profile will become steeper until a break in the amplitude occurs, with a corresponding break in the wave number $k = k_0 + (\partial\varphi/\partial\xi)$. Of course, the approximation of nonlinear geometrical optics becomes meaningless here, since the gradients of the amplitude and of the wave number become large near the breaking point. Therefore further analysis of the evolution of the perturbations can be based on the more exact theory that goes outside the limits of geometrical optics (see below).

More general nonlinear solutions of Eqs. (4.21) and (4.22), which do not reduce to simple waves, can be obtained, just as in hydrodynamics, with the aid of the "hodograph" transformation, choosing as the independent variables a and $\kappa \equiv (\partial\varphi/\partial\xi)$; the quantities ξ and τ should be defined in this case as functions of a and κ. The corresponding transformation was first carried out by Lighthill[29]. Introducing in place of the phase $\varphi(\xi, \tau)$, which enters in Eqs. (4.21) and (4.22), a new function

$$\Phi = \varkappa\xi - \left[\frac{\varkappa^2}{2} + \frac{\alpha}{v_g'}(a^2 - a_0^2)\right]\tau - \varphi(\xi, \tau) \tag{4.26}$$

(where it is assumed that the quantities ξ and τ are certain functions of a and κ), we can obtain

$$\tau = -\frac{v_g'}{2\alpha a}\left(\frac{\partial\Phi}{\partial a}\right)_\varkappa, \quad \xi = \left(\frac{\partial\Phi}{\partial\varkappa}\right)_a - \frac{\varkappa v_g'}{2\alpha a}\frac{\partial\Phi}{\partial a}, \tag{4.27}$$

with the function $\Phi(a, \kappa)$ satisfying the following linear equation:

$$\frac{\partial^2\Phi}{\partial a^2} + \frac{1}{a}\frac{\partial\Phi}{\partial a} - \frac{4\alpha}{v_g'}\frac{\partial^2\Phi}{\partial\varkappa^2} = 0. \tag{4.28}$$

When $\alpha/v_g' > 0$ (i.e., when $c_0^2 > 0$), Eq. (4.28) has the form of an axially-symmetrical wave equation in cylindrical coordinates (the role of the "radius" is played by a and that of the "time" by κ).

If the initial conditions

$$a = a_0(\xi), \quad \varkappa = \varkappa_0(\xi) \qquad (\tau = 0), \tag{4.29}$$

are specified, then the solution of (4.28) can easily be

*We note that in terms of the variables (4.20), Eq. (4.8) takes the form of the Ginzburg-Landau equation in superconductivity theory [39], and the cases of unstable and stable plane waves correspond here to states above and below the transition temperature.

*We recall that this equation pertains to a system moving with the group velocity v_g relative to the medium.

obtained by using, for example, the general Riemann method. This solution was obtained and investigated in detail in[29]. Like the fundamental equations of geometrical optics, it becomes meaningless when shock-wave formation begins, owing to the nonlinear steepening of the envelope profile.

At this stage, a significant role is assumed by the last two terms in the first equation of (4.9), which we have neglected in the geometrical-optics approximation. Taking into account terms with higher derivatives, we obtain in the one-dimensional case in place of (4.21) the equation

$$\frac{\partial\varphi}{\partial\tau}+\frac{1}{2}\left(\frac{\partial\varphi}{\partial\xi}\right)^2+\frac{\alpha}{v_g'}(a^2-a_0^2)-\frac{1}{2a}\frac{\partial^2 a}{\partial\xi^2}=0. \qquad (4.30)$$

To clarify effects determined by the last term, let us consider first a simple case, when the quantity $a - a_0$ in the initial perturbation is small compared with a_0. For stable waves ($\alpha v_g' > 0$) this quantity will likewise not increase subsequently, so that in the first approximation the last term of (4.30) can be written in the form $(1/2a_0)(\partial^2 a/\partial\xi^2)$. Then the system (4.30) and (4.22) reduces to the Korteweg–de Vries equation, which in this case takes the form (for perturbations propagating to the right)[40]

$$\frac{\partial\varkappa}{\partial\tau}+\left(c_0+\frac{3}{2}\varkappa\right)\frac{\partial\varkappa}{\partial\xi}+\beta\frac{\partial^3\varkappa}{\partial\xi^3}=0, \qquad (4.31)$$

where

$$\varkappa=\frac{\partial\varphi}{\partial\xi},\quad \beta=-\frac{1}{8c_0},\quad c_0^2=\frac{\alpha}{v_g'}a_0^2. \qquad (4.32)$$

Since the dispersion parameter β is negative in this case, the perturbation breaks up into a series of negative solitons and a wave packet propagating to the right. The relation between the packet and the solitons is determined by the initial perturbation in accordance with the results of Sec. 3.8.

The general qualitative picture of the evolution outlined above remains in force, of course, also in the case when $a - a_0 \sim a_0$.

In particular, for all $a - a_0$, Eqs. (4.30) and (4.22) are satisfied by solutions describing stationary envelope waves (solitary and periodic). Such solutions can be easily obtained by substituting in (4.30) and (4.22) expressions such as

$$a^2=\rho(\xi-V\tau),\quad \varphi=\varphi_1(\xi-V\tau)+\mathrm{const}\cdot\tau. \qquad (4.33)$$

The expressions describing the soliton are

$$\rho=\rho_0-(\rho_0-\rho_{min})\,\mathrm{sech}^2\left(\frac{\xi-V\tau}{l}\right),\quad \frac{d\varphi}{d\xi}=\frac{V(\rho-\rho_0)}{\rho},$$
$$l=\frac{1}{c_0}\left(\frac{\rho_0}{\rho_0-\rho_{min}}\right)^{1/2},\quad V=c_0\left(\frac{\rho_{min}}{\rho_0}\right)^{1/2}, \qquad (4.34)$$

where ρ_{min} is the minimum value of ρ in the soliton. We see, in accord with the foregoing, that the soliton represents here a "well" moving with velocity V against the background of a stationary wave with amplitude $a_0 = \rho_0^{1/2}$. The phase difference between the points on both sides of the soliton (formally between $\xi = \infty$ and $\xi = -\infty$) is

$$\Delta\varphi=\int_{-\infty}^{\infty}\varphi'(\xi)\,d\xi=-2\,\mathrm{arctg}\left(\frac{\rho_0-\rho_{min}}{\rho_{min}}\right)^{1/2}. \qquad (4.35)$$

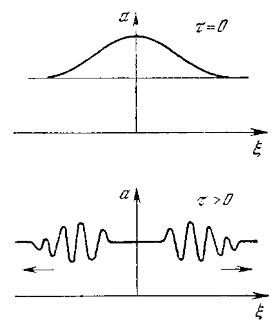


FIG. 11

A special case is a soliton with $\rho_{min} = 0$. In this case

$$\rho=\rho_0[1-\mathrm{sech}^2(\xi/l)],\quad V=0, \qquad (4.36)$$

$$l=\frac{1}{c_0}=\left(\frac{v_g}{\alpha\rho_0}\right)^{1/2}, \qquad (4.37)$$

$$\Delta\varphi=-\pi. \qquad (4.38)$$

We see that such a soliton is at rest relative to the wave in question (we recall that all this pertains to a reference frame moving with the group velocity of the stationary wave v_g).

All these results make it possible to advance the following qualitative considerations with respect to modulation processes in the case of stable waves ($\alpha v_g' > 0$).

If at the initial instant of time there is a certain space-limited perturbation against the background of a plane wave, then at large values of t it is transformed into two trains of waves traveling in opposite directions away from the region of the initial perturbation*, as shown in Fig. 11. The oscillations with the largest scale can have profiles close to those of solitons (in the region of the minima).

4.5. Qualitative Features of the Evolution of Unstable Waves

Let us stop to discuss now certain qualitative features of the evolution of modulated waves at $\alpha v_g' < 0$ (when the plane wave is "unstable" relative to small perturbations).

In this case the fundamental equations in the geometrical-optics approximation become elliptical (the "sound" velocity is imaginary). Putting in (12.28)

$$r=\left(-\frac{4\alpha}{v_g'}\right)^{1/2}a, \qquad (4.39)$$

we obtain an axially symmetrical Laplace equation

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{\partial^2\Phi}{\partial\varkappa^2}=0. \qquad (4.40)$$

The analytic solution of the Cauchy problem for this equation can be obtained by the same Riemann method as in the case (4.28) (introducing complex characteristics), as was indeed done by Lighthill. Referring the reader to his papers[29] for details, we present here only some results (see also[41]). Assume that at t = 0 there is a symmetrical wave packet with wave number k_0, i.e.,

$$a=a_0 f(\xi^2/L^2),\quad \varkappa=0 \qquad (t=0),$$
$$f(0)=1,\quad f(z)\to 0 \qquad (z\to\infty), \qquad (4.41)$$

*In a reference frame moving with velocity v_g relative to the medium.

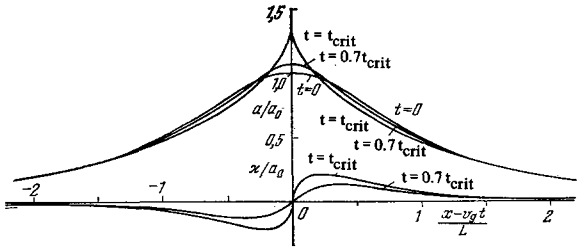


FIG. 12

where f(z) is an analytic function, a_0 is the maximum amplitude, and L is the length of the packet. Then with increasing t the amplitude remains a continuous even function of ξ, and the increment to the wave number $\kappa(\xi, t)$ is odd, the value of the amplitude at the center of the packet a(0, t) increasing with increasing t. At a certain value $t = t_0$, the derivative $(\partial a/\partial \xi)$ becomes infinite:

$$\left(\frac{\partial a}{\partial \xi}\right)_{\xi=0, t=t_0} = \infty \qquad (4.42)$$

(the function $a(\xi, t)$ itself remains continuous in general). The time t_0 is connected with the parameters of the packet at the initial instant of time t = 0 in the following manner:

$$t_0 = \frac{\text{const}\cdot L}{2\,|\,\alpha v_g'\,|^{1/2} a_0}, \qquad (4.43)$$

where const is a dimensionless constant that depends neither on a_0 nor on L. In particular, for the concrete initial wave-packet profile

$$a = \frac{a_0}{1+(\xi/L)^2}. \qquad (4.44)$$

Lighthill obtained[29] const = 0.69. Figure 12 shows Lighthill's results, characterizing the evolution of the packet with increasing t in the case (4.44) ($t_0 = t_{crit}$).

We note also that for elliptic equations the Cauchy problem is incorrect: small changes of the initial conditions lead (generally speaking, quite rapidly) to an appreciable change of the solution (as is manifest by the "instability" effect referred to above).

Near $t = t_0$, the foregoing theory based on geometrical optics becomes inapplicable, and it is necessary to use Eq. (4.30) in lieu of (4.21). The term with the second derivative in (4.30) then causes the sharp peak at the center of the packet to be replaced by oscillations of the envelope profile. A qualitative investigation of the development of these oscillations, carried out in[42], and a numerical solution of the parabolic equation (4.8), obtained in[43] for the case when the initial profile of the envelope has the form of a local perturbation against the background of a plane wave, leads to the picture shown in Fig. 13 (in a reference frame moving with the group velocity v_g). Oscillations that propagate in both directions away from the initial perturbation are produced in the central region, so that the width of the region increases and, as a result, the wave breaks up into wave packets whose amplitudes are larger by 1.5–2 times than the amplitude a_0 of the initial wave, and whose form is sufficiently close to stationary wave packets of the soliton type, described by formulas (4.15). The self-modulation process described here is in qualitative agreement with experiments carried out for gravitational waves on deep water[31,32], as can be seen from Fig. 14, which is taken from[32].

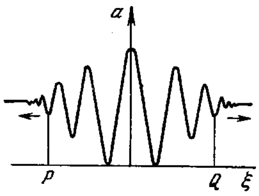


FIG. 13

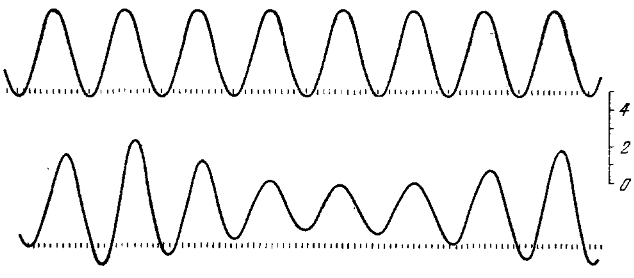


FIG. 14

5. ELECTROACOUSTIC WAVES

In the propagation of electromagnetic waves in a continuous medium an important role is frequently played by the already mentioned electrostriction effects, which are manifest in the fact that the pressure of the high-frequency field of the electromagnetic wave changes the density of the medium ρ, and with it also the dielectric constant $\epsilon(\omega, \rho)$.

In this case the modulation processes in electromagnetic waves are accompanied by acoustic oscillations that are coupled to them (electroacoustic waves). The propagation of electroacoustic waves is described by a system of Maxwell's equations together with the hydrodynamic equations (for liquids and gases)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{f}_E, \quad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \qquad (5.1)$$

where f_E is the density of the electromagnetic-field pressure forces, which can be written in accordance with[44] in the form

$$\mathbf{f}_E = \frac{1}{16\pi}\left\{\nabla\left(|E|^2 \rho \frac{\partial \varepsilon}{\partial \rho}\right) - |E|^2 \nabla \varepsilon\right\}. \qquad (5.2)$$

The striction effects, in particular, can play a predominant role in an isotropic sufficiently rarefied plasma[45-52]*. Substituting in (5.2) the expression for the dielectric constant of the plasma $\epsilon = 1 - 4\pi n e^2/m_e\omega^2$, where $n = n\{|E|^2\}$ is the density of the particles (which depends on the field), we arrive at an expression first derived in[45]:

$$\mathbf{f}_E = -\frac{ne^2}{4m_e\omega^2}\nabla |E|^2. \qquad (5.3)$$

It is seen from (5.3) that the gradient of the electromagnetic-field pressure forces is directed so as to decrease the field-energy density, i.e., the plasma-particle density is smaller where the field is larger, and vice versa. As a result it turns out that a sufficiently strong electromagnetic field can propagate in a non-

*Concerning other media, see, for example, [53,35,36,50].

isothermal plasma ($T_e \gg T_i$), where ion-acoustic oscillations attenuate weakly, even in the case when the equilibrium value of the dielectric constant $\epsilon(n_0) = 1 - \omega_0^2/\omega^2$ is negative, i.e., the field frequency is lower than critical $\omega^2 < \omega_0^2 = 4\pi n_0 e^2/m_e$[50,51]. In this case the field propagates in the form of electroacoustic solitons described by the relations (for $\omega_0 - \omega \ll \omega_0$)

$$E(x, t) = a \operatorname{sech}[\mu(x - Vt)], \tag{5.4}$$

$$\mu^2 = \frac{\omega_0^2 - \omega^2}{c^2}, \quad V = c_s\left(1 - \frac{a^2}{32\pi n_0 T_e |\varepsilon_0(\omega)|}\right)^{1/2}; \tag{5.5}$$

here E(x, t) is the slowly-varying amplitude of the electric field, and $c_s = (T_e/m_i)^{1/2}$ is the velocity of the ion sound. The variation of the density inside the soliton is determined in this case by the expression

$$n(x, t) = n_0 \left\{1 - \frac{2(\omega_0^2 - \omega^2)}{\omega_0^2} \operatorname{sech}[\mu(x - Vt)]\right\}, \tag{5.6}$$

i.e., the plasma density inside the soliton is lower than that of the unperturbed plasma, as a result of which the field is "trapped" in the soliton.

It follows from (5.5) that the propagation velocity of an electroacoustic soliton is always smaller than the velocity of sound c_s and approaches the latter if the maximum amplitude a decreases. It is also seen from (5.5) that the largest possible value of the amplitude a is

$$E_s = (32\pi n_0 T_e |\varepsilon_0(\omega)|)^{1/2}. \tag{5.7}$$

When $a \to E_s$ the velocity of the soliton tends to zero. Writing formula (5.4) for this case in the form

$$E(x) = (E_s \operatorname{sech}[\mu(x - x_0)], \tag{5.8}$$

where x_0 is the coordinate of the vertex of the "limiting" soliton, we obtain an expression for the field in a stationary nonlinear skin layer for normal incidence of a plane electromagnetic wave on a collisionless plasma[49]. The quantity x_0 is determined in this case by the value of the field amplitude on the plasma boundary: $E_0 = E_s \operatorname{sech}(\mu x_0)$ (Fig. 15). When $E_0 > E_s$ the plasma cannot "withstand" the pressure of the wave field and a stationary state of the plasma is impossible. On the other hand, if a wave with variable amplitude is incident on the plasma, i.e., $E_0 = E_0(t)$, then under certain conditions, electroacoustic waves* "split away" from the skin layer, and are transformed, as they propagate in the interior of the plasma, into solitons described by formulas (5.4)–(5.6)[51].

6. DYNAMIC AND STOCHASTIC INTERACTION OF WAVES

6.1. Three-wave Processes

Let us now consider the general case of three-dimensional nonlinear oscillations. Of course, in the most general case one can hardly advance far enough, since there is still no complete mathematical theory for the solution of partial nonlinear differential equations. However, if it is assumed that the amplitude of the oscillations is not very large, then one can use the perturbation-theory method[54-60]. At first approximation, we simply have a linear theory with the superposition

*The generation of electroacoustic waves is most effective if the characteristic amplitude modulation time T is of the order of $1/c_s\mu$, i.e., of the time required by the sound to traverse the skin layer.

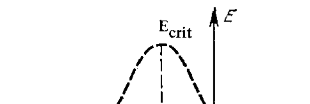


FIG. 15

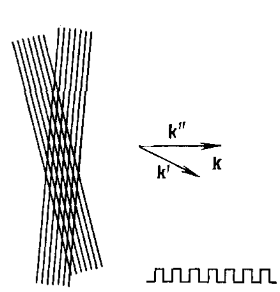


FIG. 16

principle, so that an arbitrary perturbation can be represented in the form of an aggregate of natural oscillations:

$$\Psi = \sum_{\mathbf{k}} a_{\mathbf{k}} \Psi_{\mathbf{k}}^0 \exp(i\mathbf{k}\mathbf{r} - i\omega_{\mathbf{k}} t), \tag{6.1}$$

where the quantity Ψ has, generally speaking, several components, $\Psi_{\mathbf{k}}^0$ is the polarization "vector" (suitably normalized), $a_{\mathbf{k}}$ is the amplitude (complex), k is the wave number, and $\omega_{\mathbf{k}}$ is the natural frequency. If we now take a weak nonlinearity into account, then the presence of quadratic terms will give rise in the equation of motion to terms of the form $a_{\mathbf{k}'} \times a_{\mathbf{k}''} \exp[i(\mathbf{k}' + \mathbf{k}'')\mathbf{r} - i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''})t]$. Such a term plays the role of a driving force with frequency $\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''}$ and wave vector $\mathbf{k}' + \mathbf{k}''$. We note that since Ψ and $\Psi_{\mathbf{k}}^0$ are real, the terms of the sum (6.1) must include pairwise conjugate terms, so that $a_{-\mathbf{k}} = a_{\mathbf{k}}^*$, $\omega_{\mathbf{k}} = -\omega_{-\mathbf{k}}$. Therefore, in addition to the force indicated above, there should be present also a driving force at the frequency difference $\omega_{\mathbf{k}'} - \omega_{\mathbf{k}''}$ and wave vectors $\mathbf{k}' - \mathbf{k}''$. The appearance of beats at combination frequencies $\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''}$ and wave vectors $\mathbf{k}' + \mathbf{k}''$ is graphically demonstrated in Fig. 16, which shows the superposition of two plane waves. Each of the shaded strips in Fig. 16 can be regarded as a wave of stepwise form, in which the amplitude changes jumpwise from zero to unity—zero where there is a black line and unity between them (see the lower part of Fig. 16). Upon superposition (intersection) of such waves, their products are produced—black strips, where at least one of the amplitudes is "black," and a white field between them. We see that a moire pattern is produced upon superposition of the waves, namely a wave with wave vector $\mathbf{k}' - \mathbf{k}''$ (in addition, a more frequent ripple is produced at the sum $\mathbf{k}' + \mathbf{k}''$). When the waves $\mathbf{k}'$ and $\mathbf{k}''$ propagate, the moire pattern also propagates, and if it coincides with one of the natural waves, i.e., if $\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''}$ turns out to be equal to the natural frequency $\omega_{\mathbf{k}}$ corresponding to the wave vector of the beats $\mathbf{k} = \mathbf{k}'' + \mathbf{k}''$, then the nonlinear driving force leads to a buildup of the wave k. The amplitude of the wave varies slowly with time, so that

$$\frac{\partial a_{\mathbf{k}}}{\partial t}=\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}',\mathbf{k}''} a_{\mathbf{k}'} a_{\mathbf{k}''} e^{-i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}})t}. \quad (6.2)$$

Here $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$, since Eq. (6.2) is the Fourier harmonic, with wave vector $\mathbf{k}$, of the initial nonlinear equation of motion, and $V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$ is the matrix element of the interaction, which is determined by the concrete form of the equation of motion. If the detuning $\Delta = \omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} - \omega_{\mathbf{k}}$ vanishes and resonance sets in, then even in the case of a small nonlinearity a strong change of the oscillation amplitudes can occur after the lapse of a sufficiently large time—one wave can break up into two others and two waves can coalesce into one. Such processes, in which three waves interact, are called three-wave processes, and the spectrum (the dependence of the frequency $\omega_{\mathbf{k}}$ on k) for which the condition $\omega_{\mathbf{k}} = \omega_{\mathbf{k}'} + \omega_{\mathbf{k}''}$ can be satisfied is called a decay spectrum.

Decay conditions are particularly easy to satisfy in a plasma, in view of the abundance of different oscillation modes[24 61]. For example, in a compressible plasma ($\beta = H^2/8\pi nT \neq \infty$), the Alfven and the magnetosonic waves, while not decay waves themselves, can decay into a pair of Alfven and magnetosonic waves. Helicons (whistlers) are decay waves in themselves. Many examples of decay processes can also be cited from the field of nonlinear optics[62-63].

Three-wave processes make possible the transformation of certain waves into others and by the same token give rise to a complicated process of energy transfer in the phase space of the wave numbers.

6.2. Interaction of Three Waves

Let us consider the simplest case when there are only three waves $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$, related by the resonance condition. We denote their amplitudes by a_i: $a_1 = a_{\mathbf{k}}$, $a_2 = a_{\mathbf{k}'}$, $a_3 = a_{\mathbf{k}''}$ and their respective frequencies by ω_1, ω_2, and ω_3. We assume that all the frequencies are positive and furthermore $\omega_1 = \omega_2 + \omega_3 > \omega_2 > \omega_3$.

From (6.2) we have for a_1

$$\frac{\partial a_1}{\partial t} = V_1 a_2 a_3, \quad (6.3)$$

where $V_1 = V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$.

We can similarly obtain equations for a_2 and a_3. To this end, in the former case we interchange $\mathbf{k}$ and $\mathbf{k}'$ in (6.3):

$$\frac{\partial a_2}{\partial t} = V_2 a_1 a_3^*, \quad (6.4)$$

and in the latter case we substitute $\mathbf{k}''$ for $\mathbf{k}$ and $\mathbf{k}$ for $\mathbf{k}'$:

$$\frac{\partial a_3}{\partial t} = V_3 a_1 a_2^*, \quad (6.5)$$

where $V_2 = V_{\mathbf{k}',\mathbf{k},-\mathbf{k}''}$, $V_3 = V_{\mathbf{k}'',\mathbf{k},-\mathbf{k}'}$.

In the case when the nonlinear equations for the amplitudes are the result of the Fourier transformation of the initial partial differential equations of motion, the coefficients $V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}$ have the same phase—they are either pure imaginary or real. For simplicity we shall assume them to be real (imaginarity can always be eliminated by introducing an additional factor i in the amplitudes $a_{\mathbf{k}}$). In Eqs. (6.2)—(6.5), the dynamic variables are the amplitudes $a_{\mathbf{k}}$. Changing the normalization of the polarization vectors $\Psi_{\mathbf{k}}^0$ in (6.1), we can introduce additional factors in $a_{\mathbf{k}}$ for the purpose of simplifying the form of the equations. Using this, we normalize the amplitudes $a_{\mathbf{k}}$ in such a way that the energy of the k-th wave $\mathscr{E}_{\mathbf{k}}$ is equal to $\mathscr{E}_{\mathbf{k}} = \omega_{\mathbf{k}} |a_{\mathbf{k}}|^2$. Its momentum $\mathbf{P}_{\mathbf{k}}$ is then $\mathbf{P}_{\mathbf{k}} = \mathbf{k} |a_{\mathbf{k}}|^2$. In analogy with quantum mechanics, the quantity $N_{\mathbf{k}} = |a_{\mathbf{k}}|^2$ can be interpreted as the number of quanta in the state $\mathbf{k}$. In the case of classical processes, this quantity is usually called the number of waves.

By virtue of the energy and momentum conservation laws for the interacting waves, the matrix elements in (6.3)—(6.5) cannot be purely arbitrary. In fact, multiplying (6.3)—(6.5) respectively by $\omega_i a_i^*$ and $\mathbf{k}_i a_i^*$ ($\mathbf{k}_1 = \mathbf{k}$, $\mathbf{k}_2 = \mathbf{k}'$, $\mathbf{k}_3 = \mathbf{k}''$), adding them to their complex conjugates, *and taking into account the energy and momentum conservation laws*

$$\omega_1 N_1 + \omega_2 N_2 + \omega_3 N_3 = \text{const}, \quad \mathbf{k}_1 N_1 + \mathbf{k}_2 N_2 + \mathbf{k}_3 N_3 = \text{const}, \quad (6.6)$$

we obtain

$$\omega_2 (V_1 + V_2) + \omega_3 (V_1 + V_3) = 0, \quad \mathbf{k}_2 (V_1 + V_2) + \mathbf{k}_3 (V_1 + V_3) = 0, \quad (6.7)$$

where we have taken into account the fact that $\omega_1 = \omega_2 + \omega_3$, $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$. Equations (6.7) constitute a system of four scalar equations for two quantities, $V_1 + V_2$ and $V_1 + V_3$. It follows therefore that $V_1 = -V_2 = -V_3$. In addition, since reversal of the sign of all the $\mathbf{k}$ denotes a changeover to the complex-conjugate quantities, and V_i is real by definition, it follows that $V_{\mathbf{k},\mathbf{k}',\mathbf{k}''} = V_{-\mathbf{k},-\mathbf{k}',-\mathbf{k}''}$. In addition, as seen from (6.2), it can be assumed that $V_{\mathbf{k},\mathbf{k}',\mathbf{k}''} = V_{\mathbf{k},\mathbf{k}'',\mathbf{k}'}$. Thus, the symmetry conditions can be represented in the form[59,63,64]

$$V_{\mathbf{k},\mathbf{k}',\mathbf{k}''} = V_{\mathbf{k},\mathbf{k}'',\mathbf{k}'} = V_{-\mathbf{k},-\mathbf{k}',-\mathbf{k}''} = -V_{\mathbf{k}',\mathbf{k},-\mathbf{k}''} = -V_{\mathbf{k}'',\mathbf{k},-\mathbf{k}'}. \quad (6.8)$$

Besides (6.6), Eqs. (6.3)—(6.5) have one more integral of motion. In fact, if we multiply (6.3) by a_1^* and (6.4) by a_2^* and add them together with their complex conjugate relations, then by virtue of the symmetry condition we obtain zero in the right-hand side, i.e.,

$$N_1 + N_2 = \text{const}. \quad (6.9)$$

Analogously, from (6.4) and (6.5) we can obtain $N_1 + N_3 = \text{const}$, but this relation is a consequence of (6.6) and (6.9) if it is recognized that $\omega_1 = \omega_2 + \omega_3$.

If we substitute for N_i in (6.6) and (6.9) the squares of the amplitudes $N_i = a_i^2$, then we obtain the equations for an ellipsoid (6.6) and a cylinder (6.9) with axis along a_3. The amplitudes a_i can change upon interaction only in such a way that the point (a_1, a_2, a_3) moves along the line of intersection of these surfaces. If it is recognized that $\omega_1 > \omega_2 > \omega_3$, then the intersection line has the form shown in Fig. 17.

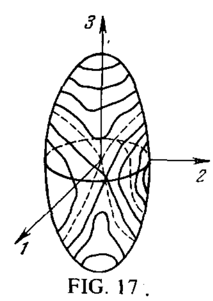


FIG. 17.

It is seen from this figure that only a wave with maximum frequency ω_1 can "decay," i.e., can decrease strongly in amplitude even under conditions when the initial amplitudes of the two other waves are small: $a_1 \gg a_2, a_3$. As to a_2 and a_3, in the case of slight excitation of the waves resonantly coupled with them, they change their amplitude little, executing small oscillations about the initial value. From this figure, by virtue of its symmetry, we can conclude also that the process of wave decay has the character of periodic conversion of the wave a_1 into the two other waves a_2 and a_3, and a return to the initial state.

The entire picture is similar to the free motion of a solid. In particular, the process of the decay of the wave a_1 into a_2 and a_3 is similar to the free motion of an asymmetrical top[65] caused to rotate about an axis passing near the principal inertial axis, such that the moment of inertia I_2 lies between the two others, $I_3 > I_2 > I_1$ (see Fig. 51 of[65]). This is not surprising, since the equations for the amplitudes (6.3)—(6.5) are very similar to Euler's equations for the free motion of an asymmetrical solid. Accordingly, the analytic expressions for the motion of a solid[65] are similar to those for the time variation of the amplitudes of three resonantly-interacting waves with the same phase[63].

6.3. Interaction of High-frequency Waves with Low-frequency Waves

Let us consider another relatively simple case, where the number of interacting waves is arbitrary, but then one of the waves has a very small wave number and low frequency[66]. This case corresponds to Fig. 15, when the moire pattern from the high-frequency waves can fall into resonance with a natural low-frequency wave.

We denote the amplitude of the low-frequency wave by b, and its wave vector and frequency respectively by κ and ν. By assumption, $\kappa \ll k_0$, where k_0 is the wave number of the fundamental wave with amplitude a_0. As seen from (6.2), waves with wave vectors $\mathbf{k}_0 + n\kappa$ enter into the interaction; we shall denote their amplitudes by a_n. We assume that the detuning is equal to zero, i.e., $\omega_{\mathbf{k}} - \omega_{\mathbf{k}-\kappa} = \nu$. Allowing for the smallness of κ, we obtain

$$\frac{\partial \omega}{\partial \mathbf{k}} \varkappa = v_g \varkappa = \nu. \tag{6.10}$$

We see therefore that in the considered case of small κ the resonance condition corresponds to equality of the group velocity of the high-frequency packet to the phase velocity of the low-frequency waves. Since κ is small, the matrix element in (6.2) can be regarded as independent of n, so that the equation for a_n can be written in the form

$$\frac{\partial a_n}{\partial t} = iV\left(a_{n+1}b^* + a_{n-1}b\right), \tag{6.11}$$

where V is simply a certain number assumed to be real. (As will be shown later, it is convenient to normalize the amplitudes in such a way that the matrix element of the interaction iV is pure imaginary.) Analogously, we have for b

$$\frac{\partial b}{\partial t} = iV \sum_n a_n a^*_{n-1}. \tag{6.12}$$

We assume that the matrix elements of the interaction are the same in (6.11) and (6.12), as is the case if the amplitudes a_n and b are normalized in a definite manner.

We assume that at the initial instant of time there is excited only one wave with amplitude a_0^0 and there is a small admixture of a low-frequency wave b. We assume for simplicity that b is a real quantity (the phase of b depends on the choice of the coordinate origin). Then, using the well-known recurrence relation for Bessel functions $2[dJ_n(x)/dx] = J_{n-1}(x) - J_{n+1}(x)$ we can easily verify that a solution of (6.11) is

$$a_n = a_0^0 i^n J_n\left(2V \int b\,dt\right). \tag{6.13}$$

We see therefore that owing to the interaction with the low-frequency wave, there appear, with increasing time, higher and higher harmonics a_n, and as $t \to \infty$ they become equalized in the mean. The appearance of the harmonics a_n corresponds simply to modulation of the initial wave, as is seen from the relation

$$\Psi = \sum_n a_n \Psi_n^0 e^{-i(\omega_n t - \mathbf{k}_n \mathbf{r})} = e^{-i(\omega_0 t - \mathbf{k}_0 \mathbf{r})} \sum_n a_n \Psi_n^0 e^{i(n\varkappa \mathbf{r} - n\nu t)}. \tag{6.14}$$

If, in addition to the assumptions made above, we can neglect the dependence of Ψ_n^0 on n, then (6.14) takes the form

$$\Psi = a_0^0 \Psi_0 e^{i(\mathbf{k}_0 \mathbf{r} - \omega_0 t)} \exp\{i[\mathbf{k}_0 \mathbf{r} - \omega_0 t] - i\lambda \cos(\nu t - \varkappa \mathbf{r})\},$$

where $\lambda = 2V \int b\,dt$. We see therefore that in the case considered here we have a pure phase modulation.

Substituting the obtained expressions for a_n from (6.13) into Eq. (6.12) for b, we obtain

$$\frac{\partial b}{\partial t} = -V \sum_n J_n(\lambda) J_{n-1}(\lambda). \tag{6.15}$$

But the expression in the right-hand side, by virtue of the well-known addition theorem for Bessel functions, vanishes. Thus, modulation of the high-frequency wave occurs at constant amplitude of the low-frequency wave. If we take into account the slight asymmetry of the matrix elements of the interaction, i.e., their dependence on n, then the amplitude of the low-frequency wave will also vary in time—it will either increase or experience low-frequency oscillations. Everything depends on whether the predominant transformation of the high-frequency waves is in the Stokes or in the anti-Stokes direction.

6.4. Weak Turbulence

The effect of the growth of the number of harmonics and the tendency to distribution of the energy over a very large number of waves are also retained in the general case of arbitrary three-wave processes. If the number of excited waves is very large, then as a result of interaction with a large number of waves the phase of each individual wave varies in a complicated irregular manner with time, so that the entire process of exchange of energy acquires an irregular character. But it is precisely because of this irregularity that we can describe the process in a different language, by using the statistical approach. In this limiting case it is assumed that the waves constitute random quantities, which can be regarded as practically uncorrelated—such an approximation is well known in physics as the random-phase approximation.

In the random-phase approximation, the problem of describing the dynamics of nonlinear waves reduces to a determination of the time dependence of the mean squares of the amplitudes, i.e., $N_{\mathbf{k}}(t)$. The corresponding equations for $N_{\mathbf{k}}$ can be obtained by averaging Eqs. (6.2) over the phases. To this end, we first multiply (6.2) by $a^*_{\mathbf{k}}$, add the result to its complex conjugate, and average over the phases:

$$\frac{\partial N_{\mathbf{k}}}{\partial t}=\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}\langle a^*_{\mathbf{k}}a_{\mathbf{k}'}a_{\mathbf{k}''}\rangle e^{-i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}})t}+\text{c.c.} \quad (6.16)$$

where the angle brackets denote averaging over the phases, $N_{\mathbf{k}} = |a_{\mathbf{k}}|^2$, and c.c. denotes the complex conjugate. If we assume the phases to be random and the amplitudes $a_{\mathbf{k}}$ to be completely uncorrelated, then we obtain in the right-hand side of (6.16) simply zero. In fact, a slight correlation between the amplitudes $a_{\mathbf{k}}$ does nevertheless arise, by virtue of the amplitude equation (6.2) itself, owing to its nonlinearity. At small values of $a_{\mathbf{k}}$, this correlation is naturally very weak. To take it into account, we represent $a_{\mathbf{k}}$ in the form $\overset{0}{a}_{\mathbf{k}} + \delta a_{\mathbf{k}}$, where $\overset{0}{a}_{\mathbf{k}}$ is the fundamental part of the amplitude with the random phase (this part of the amplitude can be regarded as independent of the time), and $\delta a_{\mathbf{k}}$ is a small increment that takes into account the correlation of the amplitudes. In the first approximation in the correlation, Eq. (6.2) takes the form

$$\frac{\partial}{\partial t}(\delta a_{\mathbf{k}})=\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}\overset{0}{a}_{\mathbf{k}'}\overset{0}{a}_{\mathbf{k}''}e^{-i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}})t}. \quad (6.17)$$

This equation can be integrated with respect to time from $-\infty$ to $+\infty$, and assuming that the correlation weakens as $t \to -\infty$, i.e., the exponential of (6.17) contains a small increment νt ($\nu > 0$), we obtain

$$\delta a_{\mathbf{k}}=\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}',\mathbf{k}''}a_{\mathbf{k}'}a_{\mathbf{k}''}\pi\delta(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}}). \quad (6.18)$$

We have retained here only the real part of the integral of the exponential, which is the only part that will enter in the final result, and in addition we have omitted the zero index for $a_{\mathbf{k}'}$ and $a_{\mathbf{k}''}$ in the right-hand side.

We now substitute in the right-hand side of (6.16) $a_{\mathbf{k}} = \overset{0}{a}_{\mathbf{k}} + \delta a_{\mathbf{k}}$ for each of the amplitudes, and retain only terms of fourth order in the amplitudes. In the obtained expression, all the phases can already be regarded as uncorrelated, so that the mean values of the quadruple products are transformed into products of paired correlation functions, i.e., they are expressed in terms of products of $N_{\mathbf{k}}$. Since there are many such terms, this entire procedure has a somewhat cumbersome appearance, but the final result, after taking into account the conditions for the symmetry of the matrix elements (6.8), is very simple:

$$\frac{\partial N_{\mathbf{k}}}{\partial t}=\int W_{\mathbf{k}\mathbf{k}'}(N_{\mathbf{k}'}N_{\mathbf{k}''}-N_{\mathbf{k}}N_{\mathbf{k}'}-N_{\mathbf{k}}N_{\mathbf{k}''})\,\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})\frac{d\mathbf{k}'}{(2\pi)^3}, \quad (6.19)$$

where $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$.*

In place of the sum over $\mathbf{k}'$, we have written here an integral, as is usually done on changing over to a continuous variable: $\sum_{\mathbf{k}'} = \int d\mathbf{k}'/(2\pi)^3$. In Eq. (6.19), the transition probability $W_{\mathbf{k}\mathbf{k}'}$ can be very easily expressed in terms of the matrix element of the interaction (which by assumption is real);

$$W_{\mathbf{k}\mathbf{k}'}=W_{\mathbf{k}'\mathbf{k}}=W_{\mathbf{k}\mathbf{k}''}=2\pi V^2_{\mathbf{k}\mathbf{k}'\mathbf{k}''}. \quad (6.20)$$

It is easy to verify, and this is perfectly natural, that the right-hand side of (6.19), which can be called the term representing the collisions between waves, conserves the energy and the momentum of the waves. It follows therefore, in particular, that Eq. (6.19) has the Rayleigh-Jeans distribution $N_{\mathbf{k}} = \text{const}/\omega_{\mathbf{k}}$ as a stationary solution. This can easily be verified directly by making the substitution $N_{\mathbf{k}} = \mathscr{E}_{\mathbf{k}}/\omega_{\mathbf{k}} = \text{const}/\omega_{\mathbf{k}}$ in the right-hand side of (6.19) and taking into account the presence of a δ function under the integral sign.

The aggregate of a large number of waves that interact weakly with one another is customarily called weak turbulence. Thus, Eq. (6.20) can be regarded as a kinetic equation for weak turbulence with account taken of only some three-wave interaction processes.

It turns out that weak turbulence can have properties similar to strong turbulence. As shown in [68,69], in many cases the transition probabilities $W_{\mathbf{k}\mathbf{k}'}$ in (6.19) have the property that the probabilities of transitions with strong change in the wave vector turn out to be much lower than the probabilities of transitions with change of k by an amount on the order of k. Consequently, the process of relay-like energy transfer over the spectrum first proposed for ordinary turbulence by Kolmogorov[70] and Obukhov[71] can also take place in weak turbulence. When account is taken of the fact that the interaction term of (6.19) is quadratic in $N_{\mathbf{k}}$, the condition for the transfer of energy over the spectrum determines the spectrum of the oscillations $\mathscr{E}_{\mathbf{k}}$ without the need for solving in detail the kinetic equation for the waves.

Let us consider, for example, the case of capillary waves[69] having a decay oscillation spectrum $\omega_{\mathbf{k}} = \sqrt{\sigma k^3/\rho}$. Let $\mathscr{E}_{\mathbf{k}}$ be the spectral energy function, i.e., $\mathscr{E}_{\mathbf{k}}dk$ is the oscillation energy in the interval dk (we regard k here as a scalar—the absolute value of the wave vector). We denote by $\dot{\mathscr{E}}$ the flow of energy over the spectrum. The quantity $\dot{\mathscr{E}}$ is obviously proportional to $k\,\partial\mathscr{E}/\partial t$, which according to the kinetic equation for the waves should be proportional to the square $\mathscr{E}^2_{\mathbf{k}}$, i.e., with dimensionality taken into account we have $\dot{\mathscr{E}} = A\omega_{\mathbf{k}}(\mathscr{E}_{\mathbf{k}}k)^2k^2/\sigma$. Here σ/k^2 is the only quantity at our disposal with the dimension of energy, and was added for the purpose of making A a dimensionless constant. From this we get

$$\mathscr{E}_k=\text{const}\cdot(\dot{\mathscr{E}})^{1/2}(\sigma\rho)^{1/4}k^{-11/4},\quad N_k\sim k^{-17/4}. \quad (6.21)$$

This spectrum is an exact solution of the kinetic equation for the wave[69]. It corresponds to the energy of inertial nonlinear energy transfer over the spectrum in the direction of large wave numbers, where dissipation by viscosity takes place.

6.5. Four-wave Processes

If the spectrum is of the non-decay type, i.e., the dependence of $\omega_{\mathbf{k}}$ on k is such that it is impossible to

*Equation (6.19) coincides in form with the well-known kinetic equation for phonons in a solid (see, for example [67]) in the classical limit ($\hbar \to 0$).

satisfy the condition $\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} = \omega_{\mathbf{k}-\mathbf{k}'}$, then it becomes necessary to take into account the next higher (cubic in the amplitude) term of the wave interaction. In this case the resonant transfer of the wave energy over the spectrum is due to four-wave processes, when the following conditions are satisfied:

$$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4, \quad \omega_1+\omega_2=\omega_3+\omega_4 \tag{6.22}$$

for the conversion of two waves, $\mathbf{k}_1$ and $\mathbf{k}_2$, into two others, $\mathbf{k}_3$ and $\mathbf{k}_4$, or analogous conditions for the decay of one wave into three or for the coalescence of three waves into one. In this case the interaction is a small quantity of third order in the amplitude, and in the case of random phases it is of third order in the number of waves $N_{\mathbf{k}}$. Of course, four-wave processes are fundamental also in the case when the nonlinearity is cubic from the very outset, as is the case, for example, in the equation (4.8) considered above, which describes self-focusing and self-contraction of wave packets. These processes themselves can also be interpreted in the language of four-wave interactions of waves with close wave vectors $\mathbf{k}$ and $\mathbf{k} \pm \kappa$[72,31,32]. More detailed information on four-wave processes can be found in[56,60,62,63,73].

7. CONCLUSION

As seen from this review, there has been a considerable shift of emphasis during the last few years in the theory of nonlinear waves, from investigations of individual nonlinear waves, such as simple and shock waves or Stokes waves on the surface of a liquid, to the study of entire classes of nonlinear wave processes. One such class constitutes waves in weakly dispersive media. For a very large number of objects, such waves are described by the Korteweg—de Vries equation. The use of numerical calculations and profound analytic investigations has brought about great clarity not only for stationary solutions of this equation, but also for time-dependent ones.

Another class of phenomena attracting more and more attention in different branches of physics is that of slow nonlinear processes in almost-periodic waves—self-focusing and self-contraction of wave packets. These processes are being intensely investigated in nonlinear optics, where they have already become one of the traditional trends. They turned out to be more unexpected for hydrodynamics—the theoretical and experimental proof of instability of periodic Stokes waves over deep water, the determination of whose profiles and the proof of whose existence had consumed so many efforts, has made it possible to examine the wave motion of liquids in an entirely new light.

Self-focusing and self-contraction of wave processes constitute a particular case of a more extensive class of phenomena—stimulated scattering of waves by waves. In optics, an example of this process may be stimulated scattering of light by phonons, corresponding to the Mandel'shtam-Brillouin effect. Stimulated scattering of waves by waves plays an important role in plasma, where complicated nonlinear processes with excitation of a very large number of waves can frequently come into play. Analogous phenomena also take place in excitation of a broad spectrum of waves over water. These phenomena have the character of stochastic interaction between waves; they are called weak turbulence. Weak turbulence has much in common with the interaction of phonons in solids, constituting thus a rather extensive circle of physical phenomena. By now its theory, based on expansion in terms of the small interaction of the waves with one another, has been developed sufficiently fully, and it can apparently serve as a starting point for the investigation of more complicated stochastic processes with large amplitudes of moderate and strong turbulence.

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