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KINETIC PHENOMENA IN SUPERCONDUCTORS

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I. INTRODUCTION

IN connection with the transition of a substance from the normal to the superconducting state, its physical properties change very substantially. The electromagnetic, thermal, and kinetic properties of superconductors differ from the properties of normal metals, which are fairly well described by the ordinary quantum theory of solids. In the microscopic theory of superconductivity[1-3] it is shown that the basic characteristics of the superconducting state are due to the pair correlation of the electrons and the associated appearance of a gap in the energy spectrum of superconductors.

The present review is devoted to a description of kinetic phenomena in superconductors; in this connection we have confined the investigation to the phenomena of thermal conductivity and the attenuation of sound. Investigation of heat transfer processes in superconductors was started quite a long time ago. The specific properties of these phenomena (see, for example,[4]) led to the creation of the two-fluid model,[5] within the framework of which it was possible to describe qualitatively some of the observed relationships.

Transport phenomena in superconductors have already been briefly discussed in review articles on superconductivity[6-9] which appeared soon after the development of the microscopic theory. At the present time one is able to rather completely describe the phenomena of thermal conductivity and the absorption of sound in superconductors on the basis of the contemporary microscopic theory. The description of transport processes, as it appears to us, is at the present time a rather completed branch of the physics of the superconducting state. The obtained temperature dependences of the kinetic coefficients actually turn out to differ substantially from the well-known results for normal metals. The basic property of superconductors consists in the presence of a condensate of Cooper pairs, whose motion is not accompanied by the transport of entropy. The number of electronic excitations, which play a basic role in kinetic phenomena, depends on the temperature, decreases upon its reduction below T_c (T_c denotes the critical temperature), and tends to zero as $T \to 0$. This dependence of the number of carriers on temperature and the corresponding changes of the mean free path lead to an extremely distinctive picture of kinetic phenomena in superconductors.

II. THERMAL CONDUCTIVITY OF SUPERCONDUCTORS

1. Mechanisms of Thermal Conductivity

As is well-known, the heat flux in a metal is composed of two components: the electronic thermal conductivity κ_e and the thermal conductivity of the lattice κ_p, so that $\kappa = \kappa_e + \kappa_p$. The steady-state nature of the thermal conductivity process is guaranteed by the presence of several relaxation mechanisms. In connection with an investigation of the electronic contribution to the heat flux, such mechanisms are as follows: collisions of electrons with impurities (which determines κ_{ed}), with phonons (κ_{ep}), and interelectronic collisions (κ_{ee}). The lattice thermal conductivity κ_p is determined by the interaction of phonons with electrons (κ_{pe}), by phonon-phonon collisions (κ_{pp}), and by the scattering of phonons by impurities, boundaries, and defects of the crystal (κ_{pd}).

Thus, six mechanisms of thermal conductivity exist. The thermal conductivity corresponding to a given type of carrier is approximately determined by the sum of the thermal resistances. Therefore

$$\varkappa_e^{-1} \approx \varkappa_{ed}^{-1} + \varkappa_{ep}^{-1} + \varkappa_{ee}^{-1}, \qquad \varkappa_p^{-1} \approx \varkappa_{pe}^{-1} + \varkappa_{pp}^{-1} + \varkappa_{pd}^{-1}.$$

Depending on the conditions, the different mechanisms, of course, play unequal roles. In an ordinary metal the electronic component κ_e makes the major contribution to the heat flux, and under the most favorable conditions (see, for example,[10]) κ_p amounts to only a few per cent of the total thermal conductivity.

In superconductors the electronic thermal conductivity plays the main role in pure or slightly impure samples. As $T \to 0$ (κ_e falls exponentially as $T \to 0$; see below) and also in substances containing a large concentration of impurities, the role of κ_p increases appreciably. A detailed discussion of the role of the different mechanisms is given below in Sec. 6 after a calculation of the coefficients of thermal conductivity corresponding to the different mechanisms. We note, incidentally, that since the basic properties of the superconducting state are associated with a rearrangement of the electronic system, the quantities κ_{pp} and κ_{pd} are described by the same laws which also hold for normal metals.

2. Scattering of Electrons by Impurities

The mechanism[11] under consideration (see also[12]) plays a major role in superconducting samples containing small concentrations of impurities. In this connection, of course, the temperature region close to absolute zero, where the number of electronic excitations is exponentially small, is excluded.

The heat flux is determined by the relation

$$Q = -\varkappa \frac{\partial T}{\partial x} = 2 \int \varepsilon v_x f \frac{dp}{(2\pi)^3};$$

here f denotes the perturbed distribution function for a superconductor's electronic excitations, which are due to the effect of the temperature gradient and which interact with the impurity atoms. For its determination, let us write down the corresponding kinetic equation

$$\frac{\partial f}{\partial x}\frac{\partial \varepsilon}{\partial p_x}-\frac{\partial f}{\partial p_x}\frac{\partial \varepsilon}{\partial x}=-\frac{f-f_0}{\tau}; \quad (1)$$

here ϵ denotes the energy of an electronic excitation, given by $\epsilon = \sqrt{\xi^2 + \Delta^2(T)}$ (ξ denotes the energy of an ordinary electron measured from the Fermi surface, $\xi = (p^2 - p_0^2)/2m = v_F(p - p_0)$, v_F and p_0 denote the velocity and momentum, respectively, at the Fermi surface, and $\Delta(T)$ is the gap in the energy spectrum), and $f_0 = [\exp(\epsilon/T) + 1]^{-1}$ (the temperature is measured in energy units, $k = 1$). On the left-hand side one can substitute f_0 instead of f; then Eq. (1) reduces to the form

$$\frac{\partial f_0}{\partial \varepsilon}\frac{\varepsilon}{T}\frac{\partial \varepsilon}{\partial p_x}\frac{\partial T}{\partial x}=\frac{f-f_0}{\tau}. \quad (2)$$

We note that the temperature dependence of the excitation energy comes entirely from the kinetic equation.

The time τ for the relaxation of a superconductor's electronic excitations by impurities appears on the right-hand sides of Eqs. (1) and (2). In order to determine this quantity let us write down the Hamiltonian for the interaction of ordinary electrons with impurity atoms in the representation of second quantization:

$$\mathscr{H}'=\sum_{\mathbf{k}}(a^+_{\mathbf{k},1/2}a_{\mathbf{k}',1/2}+a^+_{\mathbf{k},-1/2}a_{\mathbf{k}',-1/2})V_{\mathbf{kk}'},$$

where $a^+_{\mathbf{k},1/2}$ and $a_{\mathbf{k},-1/2}$ denote the amplitudes of second quantization with momentum **k** and spin $1/2$ ($-1/2$) corresponding to the creation and annihilation of an electron, and $V_{\mathbf{kk}'}$ denotes the matrix element.

According to[2], the superconducting state is described by Fermi amplitudes $\alpha_{\mathbf{k}0}$ and $\alpha_{\mathbf{k}1}$, where

$$\alpha_{\mathbf{k}0}=u_{\mathbf{k}}a_{\mathbf{k},1/2}-v_{\mathbf{k}}a^+_{-\mathbf{k},-1/2},\quad \alpha_{\mathbf{k}1}=u_{\mathbf{k}}a_{-\mathbf{k},-1/2}+v_{\mathbf{k}}a^+_{\mathbf{k},1/2},$$
$$\left.\begin{matrix}u^2_{\mathbf{k}}\\ v^2_{\mathbf{k}}\end{matrix}\right\}=\frac{1}{2}\left(1\pm\frac{\xi}{\varepsilon}\right). \quad (3)$$

Introducing these amplitudes into $\mathscr{H}'$, we find the following result for elastic scattering ($|\mathbf{k}| = |\mathbf{k}'|$):

$$\mathscr{H}'=\sum\frac{\xi}{\varepsilon}V_{\mathbf{kk}'}(\alpha^+_{\mathbf{k}0}\alpha_{\mathbf{k}'0}+\alpha^+_{\mathbf{k}1}\alpha_{\mathbf{k}'1}),$$

where $\alpha^+_{\mathbf{k}i}\alpha_{\mathbf{k}i}$ denote the occupation numbers of the excitation.

Then expressing the probability for the scattering of an electronic excitation with the aid of the well-

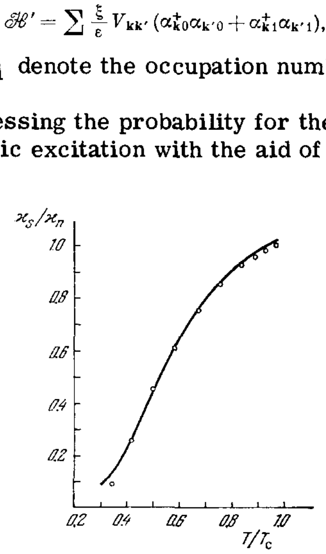

FIG. 1

known quantum-mechanical formula $W_{ab} = 2\pi|\mathscr{H}'|^2_{ab}\rho_E$ (the statistical weight $\rho_E = p^2 d\Omega dp/dE$), we find

$$\tau=\tau_0\frac{\varepsilon}{|\xi|}, \quad (4)$$

where τ_0 is the relaxation time of the ordinary electrons and does not depend on the energy.

Substituting the expression obtained for τ into formula (2), we find the desired distribution function, and then we calculate the corresponding coefficient of thermal conductivity

$$\left.\begin{aligned}\varkappa_{ed}&=\frac{2}{3}\frac{p_0^3\tau_0}{\pi^2 m}F(T),\\ F(T)&=T^{-1}\int_{-\infty}^{\Delta}\varepsilon^2\frac{\partial f_0}{\partial\varepsilon}d\varepsilon\\ &=\frac{\Delta^2(T)}{T}\left[\exp\left(\frac{\Delta}{T}\right)+1\right]^{-1}+2T\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{s^2}\exp\left(-\frac{s\Delta}{T}\right)\\ &\quad+2\Delta\ln\left[1+\exp\left(-\frac{\Delta}{T}\right)\right].\end{aligned}\right\} \quad (5)$$

Thus, we see that the electronic thermal conductivity is described by a universal dependence on the temperature and decreases with its lowering according to the law (5). A graph of this dependence is plotted in Fig. 1. In addition, experimental data obtained in an investigation of impure samples of Tl[13] are also shown. It is clear that the theory is in rather good agreement with experiment. If we set $\Delta = 0$ in formula (5), then we arrive at the linear law for the variation of the quantity κ_{ed} which is well-known in the ordinary theory of metals (see, for example,[10]).

The electronic thermal conductivity κ_{ed} is calculated in articles[14,15] by using an exact formula for the coefficient of thermal conductivity[16] (see below, Sec. 8) and by application of the method of Green's functions. As a result the authors arrive at the same formula (5) which was obtained in[11] with the aid of the kinetic equation.

3. Thermal Conductivity of Pure Superconductors

The thermal conductivity κ_{ep} associated with the scattering of electrons by phonons plays a major role in samples of high purity as $T \to T_c$.

In order to investigate the question of temperature dependence[17,18], let us write down the kinetic equation

$$\frac{\partial f}{\partial x}\frac{\partial \varepsilon}{\partial p_x}-\frac{\partial f}{\partial p_x}\frac{\partial \varepsilon}{\partial x}=\left(\frac{\partial f}{\partial t}\right)_{\text{coll}},$$

where f denotes the electronic distribution function which we shall seek in the form

$$f=f_0+\frac{\partial f_0}{\partial\varepsilon}\varphi(\varepsilon,\Omega)\frac{\partial T}{\partial x}.$$

In order to write down an expression for the collision integral in the case under consideration, we make the transformation (3) to new Fermi amplitudes in the electron-phonon interaction Hamiltonian

$$\mathscr{H}'=\sum_{\substack{\mathbf{k},s,\mathbf{q}\\ \mathbf{k}'=\mathbf{k}+\mathbf{q}}}V_{\mathbf{kk}'}a^+_{\mathbf{k}s}a_{\mathbf{k}'s}b^+_{\mathbf{q}}+\text{c.c.} \quad (6)$$

($b^+_{\mathbf{q}}$ denotes the amplitude for the creation of a phonon with momentum q, $s = 1/2$ or $-1/2$, and $V_{\mathbf{kk}'}$ denotes the matrix element), after which the initial Hamiltonian takes the form

$$\mathscr{H}'=\sum_{\mathbf{k},\mathbf{q}}V_{\mathbf{kk}'}[(u_{\mathbf{k}}u_{\mathbf{k}'}-v_{\mathbf{k}}v_{\mathbf{k}'})(\alpha^+_{\mathbf{k}0}\alpha_{\mathbf{k}'0}+\alpha^+_{\mathbf{k}1}\alpha_{\mathbf{k}'1})$$

$$+(u_{\mathbf{k}}v_{\mathbf{k}'}+u_{\mathbf{k}'}v_{\mathbf{k}'})(\alpha^+_{\mathbf{k}0}\alpha^+_{\mathbf{k}'1}+\alpha_{\mathbf{k}0}\alpha_{\mathbf{k}'1})]\,b^+_{\mathbf{q}}+\text{c.c.} \tag{6'}$$

With the expression obtained for $\mathscr{H}'$ taken into consideration, the kinetic equation is written in the form

$$\begin{aligned} f_0^2 e^{\varepsilon/T}\frac{\xi}{T}\frac{p_x}{m} &= \int |V|^2 N_0\left(1+\frac{\xi\xi'-\Delta^2}{\varepsilon\varepsilon'}\right)[\varphi(\varepsilon', \Omega')-\varphi(\varepsilon, \Omega)] \\ &\times e^{\varepsilon'/T} f_0(\varepsilon) f_0(\varepsilon')\,\delta(\varepsilon'-\varepsilon-\omega)\,d\mathbf{q} + \int |V|^2 N_0\left(1+\frac{\xi\xi'-\Delta^2}{\varepsilon\varepsilon'}\right) \\ &\times [\varphi(\varepsilon', \Omega')-\varphi(\varepsilon, \Omega)]\exp\left(\frac{\varepsilon}{T}\right) f_0(\varepsilon) f_0(\varepsilon')\,\delta(\varepsilon-\varepsilon'-\omega)\,d\mathbf{q} \\ &+ \int |V|^2 N_0\left(1-\frac{\xi\xi'-\Delta^2}{\varepsilon\varepsilon'}\right)[\varphi(\varepsilon', \Omega')-\varphi(\varepsilon, \Omega)] \\ &\qquad\times \exp\left(\frac{\omega}{T}\right) f_0(\varepsilon) f_0(\varepsilon')\,\delta(\varepsilon'+\varepsilon-\omega)\,d\mathbf{q}, \\ N_0 &= \left[\exp\left(\frac{\omega}{T}\right)-1\right]^{-1}. \end{aligned} \tag{7}$$

We furthermore take into account the fact that the wave vector $\mathbf{q}$ of a thermal phonon is small in magnitude in comparison with the electron momentum. Therefore one can expand the function $\varphi(\Omega')$ in powers of $\mathbf{q}' = \mathbf{p} - \mathbf{p}^*$ ($\mathbf{p}^*$ is a vector directed along $\mathbf{p}$ and having a length equal to $|\mathbf{p}'|$). Such a method of investigating the kinetic equation was first developed in[19]. We seek a solution of the equation, which is obtained after integration over the angles and by the substitution $\epsilon\epsilon'[1 \pm (\xi\xi' - \Delta^2)/\epsilon\epsilon']/|\xi||\xi'| \approx 2$ (see below, Sec. 3), in the form $\varphi(\epsilon, \Omega) = \varphi_1(\Omega) + \varphi_2(\epsilon, \Omega)$. After simple but cumbersome calculations we arrive at the following expression for the quantity φ which determines the perturbed electronic distribution function:

$$\varphi = \frac{a(\Omega)}{T^4\Phi(T)}\int_b^\infty f_0^2 e^z\sqrt{z^2-b^2}\,dz; \tag{8}$$

$$\begin{aligned} b &= \frac{\Delta}{T}, \quad \Phi(T) \approx 96\zeta(4)\ln(1+e^{-b}) \\ &+ \sum_{s=1}^\infty s^{-5}e^{-2bs}(80b^4s^4+160b^3s^3+240b^2s^2+240bs+120) \\ &\qquad -\ln(e^b+1)\sum_{s=1}^\infty s^{-4}e^{-2bs}(64b^3s^3+96b^2s^2+96bs+48). \end{aligned}$$

Then the heat flux is calculated according to the formula $Q = 2\int \epsilon v_x f dp$, which leads to the following final expression:

$$\varkappa_{ep} = \frac{\text{const}}{\Phi(T)T^2}\left[b^2\sum_{s=1}^\infty(-1)^{s+1}K_2(bs)\right]^2 \tag{9}$$

($K_2(bs)$ denotes the Bessel function of imaginary argument).

In the temperature region close to T_c where the contribution of the electron-phonon interaction to the heat flux is most important, we find

$$\varkappa_{ep} = \varkappa^n_{ep}\frac{36}{\pi^4}\frac{\Phi(T_c)}{\Phi(T)}\left[2\sum_{s=1}^\infty\frac{(-1)^{s+1}}{s^2}e^{-bs}+2b\ln(1+e^{-b})-\frac{b^2}{2(e^b+1)^2}\right]^2 \tag{10}$$

(κ^n_{ep} denotes the thermal conductivity of the normal metal). The corresponding theoretical curve is plotted and experimental data for pure samples of tin[20] and indium[21] are given in Fig. 2.

In sufficiently pure superconductors, effects may appear which are associated with the presence of overlapping energy bands. In this connection the total coefficient of thermal conductivity, which is determined by the scattering of electrons by thermal phonons, turns out to be an additive function of the coefficients pertaining to different bands[22] (for more details about the conditions under which many-band effects arise, see below, Ch. III, Sec. 1d). We shall not consider processes in which the electrons pass from one band to another since such transitions in general are accompanied by a change of the electron momentum by an amount $\sim p_F$, which is impossible at the frequencies $\omega \sim T$ of the thermal phonons under consideration. The indicated additivity is violated upon the introduction of impurities whose presence leads to intraband transitions.[23] For path lengths l satisfying the condition $l < \xi_0$ (ξ_0 denotes the size of a Cooper pair), many-band effects cease to play a role,[24] and the usual single-band approximation becomes valid.

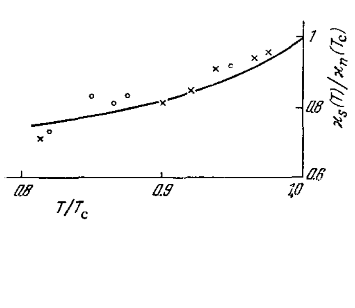

FIG. 2. The experimental points correspond to: × – tin, [20] ○ – indium; [21] the solid curve corresponds to the theoretical value. [17]

4. Lattice Thermal Conductivity

Now let us consider the question of the transport of heat by phonons, associated with their scattering by the electronic excitations of a superconductor.[25,26] A temperature range ($T \lesssim 0.3$ to $0.5\ T_c$) exists where this mechanism plays a decisive role.

Let us write down the kinetic equation for the distribution function of N phonons

$$-\frac{\partial N}{\partial T}\frac{\partial T}{\partial x}u_0\frac{q_x}{|\mathbf{q}|} = \left(\frac{\partial N}{\partial t}\right)_{\text{coll}}, \tag{11}$$

where u_0 denotes the speed of sound, and $\mathbf{q}$ is the momentum of a phonon.

Using expression (6) for the electron-phonon interaction Hamiltonian $\mathscr{H}'$, in which a transformation has been made to the new Fermi amplitudes which describe the superconducting state, we arrive at the following expression for the collision integral:

$$\begin{aligned} \left(\frac{\partial N}{\partial t}\right)_{\text{coll}} &= \int |V_{\mathbf{kk}'}|^2\left\{\left(1+\frac{\xi\xi'-\Delta^2}{\varepsilon\varepsilon'}\right)[f'(1-f)(N+1)-Nf(1-f')]\right. \\ &\times\delta(\varepsilon'-\varepsilon-\omega)+\frac{1}{2}\left(1+\frac{\xi\xi'-\Delta^2}{\varepsilon\varepsilon'}\right)[(N+1)ff'-N(1-f)(1-f')] \\ &\left.\times\delta(\varepsilon'+\varepsilon-\omega)\right\}\frac{p^2\,dp\,do}{4\pi^2}. \end{aligned} \tag{12}$$

The collision integral (12) differs from the usual expression by the factors $\{1 \pm [(\xi\xi' - \Delta^2)/\epsilon\epsilon']\}$, whose appearance is associated with the presence of the factors $(u_{\mathbf{k}}\cdot u_{\mathbf{k}'} - v_{\mathbf{k}}\cdot v_{\mathbf{k}'})$ and $(u_{\mathbf{k}}\cdot v_{\mathbf{k}'} + u_{\mathbf{k}'}\cdot v_{\mathbf{k}})$ in the Hamiltonian $\mathscr{H}'$ given by Eq. (6'). We seek the desired phonon distribution function in the form

$$N = N_0 - r(x)\frac{q_x}{T}\frac{\partial N_0}{\partial x},$$

where $x \equiv \omega/T$. According to Eqs. (2) and (4) the electron distribution function f is given by

$$f = f_0 + \frac{p_x}{m}\frac{\partial f_0}{\partial\varepsilon}\frac{\varepsilon}{T}\tau_0\frac{\xi}{|\xi|}\frac{\partial T}{\partial x}, \quad f_0 = \left[\exp\left(\frac{\varepsilon}{T}\right)+1\right]^{-1}.$$

Then we substitute the expressions for N and f into relation (12) and integrate over the angles. Assuming approximately $L\epsilon\epsilon'/|\xi||\xi'| \approx 2$, $L = 1 \pm (\xi\xi' - \Delta^2)/\epsilon\epsilon'$ (this gives an error $\sim\Delta^2\omega^2/4(\epsilon\epsilon' \mp \Delta^2)\epsilon\epsilon'$, i.e., an error ~ 0.1 for Q when $\Delta/T = 1.5$) and then integrating over the energy ϵ, we obtain the following expression for the desired function $r(x)$:

$$r \approx \frac{\text{const}}{T^2}[2x - 2\ln(e^{b+x}+1)(e^b+1)^{-1} + D(x)(2b - x + 2\ln(e^{x-b}+1)(e^b+1)^{-1}]^{-1}\frac{\partial T}{\partial x};$$

here

$$b = \frac{\Delta}{T}, \quad x = \frac{\omega}{T},$$

$$D(x) = \begin{cases} 1, & x \gg 2b, \\ 0, & x < 2b. \end{cases}$$

We have taken into account the fact that the creation of a pair of excitations by a phonon is possible only in the case when $\omega \geq 2\Delta$.

Now let us consider the thermal flux in the lattice

$$Q = \sum_{\mathbf{q}} N_{\mathbf{q}} u_0 \frac{q_x}{|\mathbf{q}|} u_0 \mathbf{q} = \frac{T^4}{6\pi u_0^3}\int_0^\infty \frac{x^4 r(x) e^x}{(e^x - 1)^2}dx.$$

Evaluation of the integral leads to the following final expression for the coefficient of lattice thermal conductivity:

$$\varkappa_{pe} = BT^2F(T),$$

$$F(T) = -8b^4(e^b-1)^{-1} - 8b^3(e^b-1)^{-1} + 6\zeta(3)(e^b+1) - 3(e^b+1)\sum_s s^{-3}e^{-2bs}(4b^2s^2+4bs+2) + 6\zeta(4)(e^b-1) - (e^b-1)\sum_s s^{-4}e^{-2bs}(8b^3s^3+12b^2s^2+12bs+6) + 32b^3(e^{2b}-1)^{-1} + a^4\sum_s\{se^{-2bs} - Ei[-s(2b-a)]\} + 6\sum_s s^{-3}e^{-2bs},$$

$$a \approx 2b - 0.16, \quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{13}$$

The theoretical curve plotted according to Eq. (13) and experimental data for an In-Tl alloy[27] are shown in Fig. 3.

When $\Delta = 0$, formula (13) goes over into the well-known formula of the quantum theory of metals (see, for example,[10]), describing the contribution of the lattice thermal conductivity which decreases as the temperature is lowered. As is well-known, this decrease is associated with a corresponding decrease in the number of thermal phonons.

In superconductors, in contrast to normal metals, the value of κ_{pe} turns out to increase as the temperature is lowered, as is evident from Eq. (13). This property is related to the increase in the mean free path of the phonons due to the exponential decrease in the number of electronic excitations (the mean free path of the phonons, which is determined by their scattering by electronic excitations, is equal to $l_{pe} \approx (n_e\sigma)^{-1}$, where n_e is the number of electronic excitations and σ is the cross-section).

At a sufficiently low temperature the phonons begin to be primarily scattered by lattice defects and the boundaries of the crystal (as is well-known,[28] the phonon-phonon interaction does not play a role at temperatures $T \ll \Theta$). Here $\kappa_p \approx \kappa_{pd} \sim T^3$, and the lattice thermal conductivity decreases as $T \to 0$. A maximum of the lattice thermal conductivity should be observed for the temperature at which $\kappa_{pe} \approx \kappa_{pd}$.

5. Comparison with Experiment

The temperature dependences (5), (9), and (13) obtained above enable one to explain to a considerable extent the totality of experimental data existing at the present time, which has been obtained during investigations of the thermal conductivity of superconductors.

As already mentioned above in Sec. 1, six mechanisms of thermal conductivity exist whose relative contributions depend very significantly on the conditions of the experiment.

For pure superconductors and also for samples containing small concentrations of impurities $\kappa \approx \kappa_e$ (where for very pure superconductors $\kappa \sim \kappa_{ep}$, but for slightly impure ones $\kappa \sim \kappa_{ed}$) except for the temperature region very close to absolute zero. Here the thermal conductivity falls with a decrease of the temperature in agreement with formulas (5) and (9) (see Fig. 5 below, curve 1). At first this drop is slow, but then it becomes exponential, which is related to the corresponding decrease in the number of electronic excitations.

Actually for qualitative estimates we shall use the simple formula of kinetic theory, $\kappa = c_V l v/3$ (c_V is the heat capacity, l is the mean free path, and $v = v_F$). In the case of interest to us $c_V \sim n \sim \exp(-\Delta/T)$, and l varies according to a power law.

The indicated dependence rather well describes the experimental data obtained upon investigation of the thermal conductivity of Al and Zn in[29] and later in[31], Tl,[13] Sn,[13,20] and In[21] (for a review of the experimental data, see also[4,7,30]).

At very low temperatures $\kappa_e \ll \kappa_p$; then $\kappa \approx \kappa_{pd}$. In this connection the thermal conductivity is basically determined by the scattering of phonons by impurities, defects, and by their reflection from the crystal boundaries. As $T \to 0$ it tends to zero according to the law $\kappa \sim T^3$. The abrupt increase in the role of the lattice thermal conductivity is an intrinsic anomaly of the superconducting state (in ordinary metals $\kappa \approx \kappa_e$ at all temperatures) and is associated with a decrease of the normal component of the electronic fluid. The condensate of Cooper pairs has zero entropy, and therefore does not give any contribution to the thermal flux.

The thermal conductivity of Pb, Sn, In, and Ta in

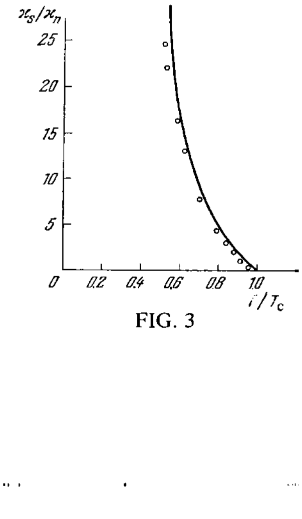

FIG. 3

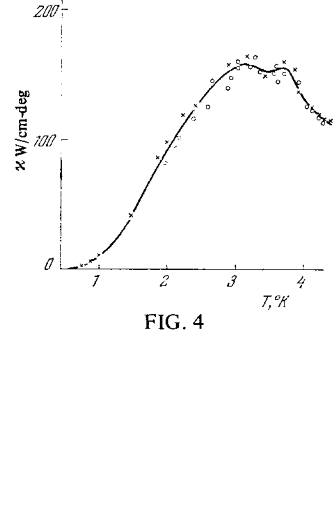

FIG. 4

the low-temperature region was investigated in article[32]; the properties of Sn were studied in[37], and the properties of Tl, Sn, Al, Ga, Zn, Cd, and Pb were studied in[13,36,42,43]. In fact, it was established that as $T \to 0$ (for Tl, for example, for $T \lesssim 0.3$ to $0.4°K$[13]) a superconductor, in regard to the nature of its heat transfer properties, becomes no different from a dielectric. It is seen in Fig. 4 that as $T \to 0$ the nature of the temperature dependence $\kappa(T)$ of tin[13] changes: the thermal conductivity begins to vary according to the law $\kappa_{pd} \sim T^3$.

The fact that the phonon thermal conductivity begins to play a major role as the temperature is lowered is experimentally confirmed in the following way. The mean free path of the long-wavelength thermal phonons changes significantly upon plastic deformation of the sample because in this connection they begin to be primarily scattered by the dislocations which arise during such a deformation. However, for the electrons, which have a small wavelength, the appearance of dislocations does not play an important role, and, as before, they are primarily scattered by impurity atoms. Therefore the thermal conductivity of the sample must decrease following plastic deformation if the phonons make the major contribution to the heat flow, and it should not change in practice if the electronic component plays the major role.

Measurements carried out in[33] showed a decrease by a factor of six in the thermal conductivity of superconducting lead upon plastic deformation at $T = 1°K$, which also confirms the arguments given above about the increase in the role of κ_p as $T \to 0$.

Strong contamination of the sample decreases the mean free path of the electrons, and at the same time the electronic contribution to the thermal conductivity also decreases in accordance with the formula $\kappa = l c_v v/3$. In this connection $\kappa_p \gg \kappa_e$ and $\kappa \approx \kappa_p$ at all temperatures. In the intermediate temperature range ($T = 0.3$ to $0.5\ T_c$) and as $T \to T_c$ the scattering of phonons by electrons plays the main role, and the thermal conductivity is described by formula (13) (Fig. 5, curve 2). In this connection the function κ_{pe} increases as the temperature is lowered, which explains the experimentally observed anomalous behavior of the thermal conductivity of superconducting alloys. Experimental data for an In-Tl alloy[27] are shown in Fig. 3. An increase of κ_p is observed in[34,35] for Nb at $T = 0.4\ T_c$.

The temperature dependence of the thermal conductivity for an alloy of Pb + 10% Bi[36] is shown in Fig. 6. At first the function $\kappa(T)$ increases with reduction of the temperature, in agreement with Eq. (13), and then after reaching a maximum it begins to decrease as $T \to 0$. In the temperature region corresponding to the maximum, $\kappa_{pe} \approx \kappa_{pd}$ (see Sec. 4). Upon a further reduction of the temperature, the phonons are mainly scattered by impurities, which also corresponds to a decrease of the function $\kappa(T)$.

In the intermediate case of not very pure superconductors κ_e plays the dominant role in the temperature region close to T_c, and in view of this κ falls according to the law (5) as the temperature decreases (see Fig. 5). At a sufficiently low temperature κ_p turns out to be larger than κ_e and the subsequent behavior of

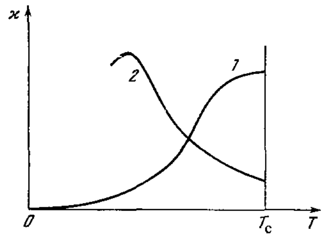

FIG. 5

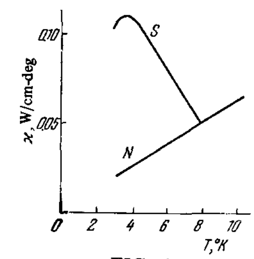

FIG. 6

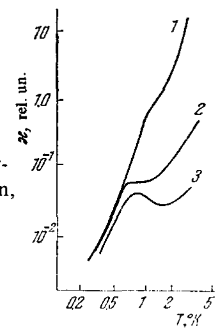

FIG. 7. The curves correspond to the following compositions: 1 – pure Sn, 2 – Sn + 0.3% In, 3 – Sn + 3% In.

the thermal conductivity will be determined by curve 2 in Fig. 5, which corresponds to formula (13). Such a dependence was observed experimentally for Sn,[37] Hg,[38] and Pb.[40] In Fig. 7[37] it is shown how a gradual increase in the impurity concentration leads to an increase in the role of the lattice thermal conductivity and the appearance of a corresponding maximum.

The temperature dependence of the thermal conductivity of superconducting contaminated lead, obtained in[40], is shown in Fig. 8. The decrease of $\kappa(T)$ for $T \sim T_c$ is associated with the corresponding behavior of the quantity κ_e, which plays the major role for $T \sim T_c$. Then, as is evident from the figure, the thermal conductivity of Pb starts to increase, and later a maximum of the thermal conductivity is observed, associated with the maximum of κ_p.

6. Basic Types of Thermal Conductivity

Theoretical and experimental investigations of the processes of heat transport in superconductors thus point to the existence of three types of temperature dependence of the thermal conductivity.

The first type (Fig. 9) is exhibited by pure or slightly impure superconductors. As already mentioned above, the electronic thermal conductivity κ_e plays a dominant role in them as $T \to T_c$ and in the

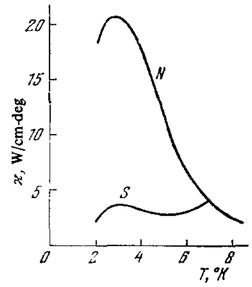

FIG. 8

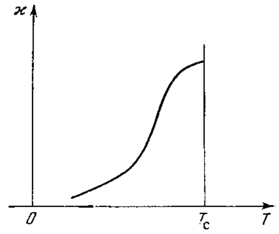

FIG. 9

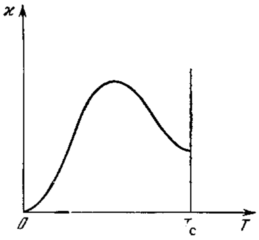

FIG. 10

intermediate temperature range. Upon a reduction of the temperature, κ here basically falls according to the exponential law described by formulas (5) and (9), the first corresponding to the case of a slightly impure superconductor, and the second corresponding to a pure superconductor. The temperature dependence of κ for thallium[13], which belongs to the first type under consideration, was shown in Fig. 1. As $T \rightarrow 0$ the lattice thermal conductivity begins to play the major role: $\kappa_{pd} \sim T^3$ and κ begins to tend to zero according to a power law.

The second type of thermal conductivity involves superconductors with a large impurity concentration (Fig. 10). In contrast to substances of the first type, the thermal conductivity of these strongly contaminated superconductors increases for $T \sim T_c$ as the temperature is lowered. In this connection the lattice thermal conductivity κ_{pe}, described by formula (13), plays a major role. A characteristic property of the case under consideration is the presence of a maximum in the thermal conductivity; after passing through its maximum $\kappa(T)$ tends to zero according to the law determined by the function $\kappa_{pd} \sim T^3$. An alloy of Pb + 10% Bi (see Fig. 6) is an example of a superconductor of the second type.

Finally, the third type of thermal conductivity (Fig. 11), which describes the behavior of not very pure superconductors, is a combination of the first type (for $T \sim T_c$) and of the second type (in the intermediate temperature range). The data on Pb and Sn (see Fig. 8) cited in[37,13,40] enable us to regard the corresponding samples as examples of superconductors that pertain to the intermediate case under consideration.

In these samples κ_{ed} plays the major role at $T \sim T_c$. Therefore the thermal conductivity first drops with a lowering of the temperature according to the law (5). Then, however, κ_p becomes larger than κ_e, and in what follows the superconductor behaves like a substance belonging to the second type considered above.

7. Effect of Anisotropy on the Thermal Conductivity of Superconductors

An isotropic model was used above in connection with the investigation of different mechanisms of thermal conductivity. The dispersion law of the electrons was assumed to be quadratic. The electron-phonon interaction was described by a constant quantity, which was independent of direction.

Real metals are characterized by an intrinsically anisotropic Fermi surface. The energy gap $\Delta(p)$ is, in general, a function of direction. However, the anisotropy of the gap is not large; in connection with this the isotropic BCS model, as is well known, describes the experimental data quite well.

Thermal conductivity is an integral effect. Therefore the averaging carried out during its calculation generally does not lead to any appreciable anisotropy in the coefficient of thermal conductivity. Therefore, the formulas cited in Secs. 2–4, obtained in an isotropic model, describe the experimental data quite well.

Uniaxial crystals (for example, Ga and Zn) constitute an exception. In these superconductors a noticeable difference in the form of the function $\kappa(T)$ is observed experimentally[42,43] for different orientations of the crystal.

Let us calculate the electronic thermal conductivity of a uniaxial superconductor.[41] The kinetic equation, describing the behavior of the electronic excitations which are interacting with impurities in the anisotropic case under consideration, has the form

$$\frac{f}{T^2}\xi_k\left(\frac{\partial \xi_k}{\partial k}\nabla T\right) = I_{coll}, \qquad (14)$$

where I_{coll} denotes the collision integral; in this connection $\epsilon_k = \sqrt{\xi_k^2 + \Delta^2(k)}$.

The energy gap is small at temperatures close to the critical temperature, and therefore effects associated with its anisotropy do not play an important role. The situation is different at sufficiently low temperatures which satisfy the condition $T \ll \Delta_{min}$. Let us consider a region of such low temperatures, for which the gap actually ceases to depend on T and the distribution function of the electronic excitations is determined by the exponential $\exp(-\Delta/T)$. In the anisotropic case the gap $\Delta(k)$ will take a minimal value with respect to certain extremal directions. In a uniaxial crystal with an extremal direction which coincides with a principal axis of its symmetry, a large fraction of the electronic excitations will move along this direction.

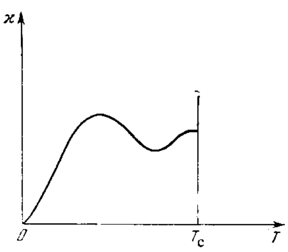

FIG. 11

The collision integral which appears in Eq. (14) may be written in the form (see Sec. 1)

$$I_{coll} = -2\pi \int |V_{kk'}|^2 (u_k u_{k'} - v_k v_{k'})^2 \delta(\varepsilon_k - \varepsilon_{k'}) (f_k - f_{k'})\, d\tau_k.$$

$$\left.\begin{matrix} u_{\mathbf{k}}^2 \\ v_{\mathbf{k}}^2 \end{matrix}\right\} = \frac{1}{2}\left(1 \pm \frac{\xi_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}}\right). \tag{15}$$

It has been taken into account that in the anisotropic case only the energy of an excitation is conserved during elastic scattering by impurities. In contrast to the isotropic case, the absolute magnitude of the momentum is not conserved.

We seek the desired distribution function $f_{\mathbf{k}}$ in the form $f_{\mathbf{k}} = f_{\mathbf{k}}^0(1 + \chi_{\mathbf{k}})$. After integration over $\xi_{\mathbf{k}}$ the kinetic equation takes the form

$$-\frac{\varepsilon_{\mathbf{k}}}{T^2}\left(\frac{\partial\xi_{\mathbf{k}}}{\partial\mathbf{k}}\nabla T\right) = \frac{\pi}{2}\int |V_{\mathbf{kk}'}|^2 \frac{\xi_{\mathbf{k}}+\xi_{\mathbf{k}'}}{\xi_{\mathbf{k}}|\xi_{\mathbf{k}'}|}\left(\frac{\partial^2\tau}{\partial\xi_{\mathbf{k}}\,\partial o'}\right)(\chi_{\mathbf{k}}-\chi_{\mathbf{k}'})\,do'. \tag{15'}$$

Equation (14′) is solved for the extremal directions. Let us denote the coefficients of thermal conductivity along the extremal direction and along a direction perpendicular to it by $\kappa_\parallel$ and $\kappa_\perp$, respectively. We introduce the quantities V_0 and V_π to denote the matrix elements for scattering through angles of zero degrees and π radians, respectively, for excitations which were initially moving in the extremal direction. The solution of Eq. (14′) leads to the following expressions for the quantities $\kappa_\parallel$ and $\kappa_\perp$:

$$\varkappa_\parallel \approx \frac{8\Delta^2}{5\pi T}\left(\frac{\partial\xi}{\partial k}\right)_F \frac{1}{1.8V_\pi^2+0.2V_0^2}\exp\left(-\frac{\Delta}{T}\right), \tag{16}$$

$$\varkappa_\perp \approx \frac{10\Delta^2}{3\pi\Delta''}\left(\frac{\partial\xi}{\partial k}\right)_F \frac{1}{V_0^2+V_\pi^2}\exp\left(-\frac{\Delta}{T}\right) \tag{16'}$$

(Δ'' denotes the second derivative of the gap with respect to the angle θ between ∇T and the vector $\mathbf{k}$ near an extremum). The exponential part of the temperature dependence of κ is the same in all directions, as is clear from Eqs. (16) and (16′). We see that the ratio $\kappa_\perp/\kappa_\parallel$ turns out to be given by

$$\frac{\varkappa_\perp}{\varkappa_\parallel} \approx \frac{2T}{\Delta''}\,\frac{1.8V_\pi^2+0.2V_0^2}{V_\pi^2+V_0^2}, \tag{17}$$

i.e., the thermal conductivity in a direction perpendicular to the extremal direction turns out to be approximately T/Δ times smaller than the extremal value. The dependence of the thermal conductivity on the angle θ (the case $\kappa_\perp/\kappa_\parallel = 1/5$ is considered) is shown in Fig. 12, the angle being measured from the extremal direction and the thermal conductivity having the form $\kappa = \kappa_\parallel \cos^2\theta + \kappa_\perp \sin^2\theta$. Anisotropy of the thermal conductivity has been experimentally observed in crystals of Ga,[42] Zn, and Gd.[43] A detailed comparison of the results obtained by investigation of Zn samples with the theory[41] is carried out in[43]. As is evident from Fig. 13, the agreement is quite good. The anisotropy of the gap in Zn turns out to be characterized by the following data: $\Delta_{\min} = 1.2\,T_c$ or $\sim 1.0°K$ and $\Delta_{\max} - \Delta_{\min} \sim 0.55\,T_c$.

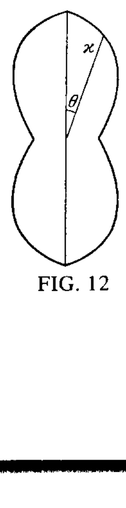

FIG. 12

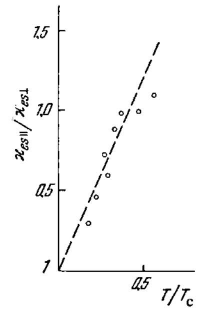

FIG. 13. $\kappa_{es\parallel}$ and $\kappa_{es\perp}$ are the thermal conductivities of Zn samples along and perpendicular to the hexagonal axis [43] (the dashed line represents the theory [41]).

8. Investigation of the Intermediate State

An appreciable decrease in the thermal conductivity of a superconductor is observed experimentally upon its transition to the intermediate state. This phenomenon was first observed in[44], where samples containing Pb + 0.1% Bi were investigated. This effect is also mentioned in a number of other articles, for example, in[45–47] where the effect of a decrease in κ was observed for monocrystals of Hg, Sn, and In. This effect was investigated in[48,49] over a broad range of temperatures. A decrease in the thermal conductivity is noted both at low temperatures, where the phonon thermal conductivity is the main term, and also in the region of higher temperatures where the electronic contribution is the major term. In the first case, which is considered in[50,51], the decrease in κ is associated with a decrease of the phonon mean free path associated with their transition from the superconducting phase to the normal phase (in the experiments mentioned above, the region of the normal phase was arranged to be perpendicular to the sample axis along which the heat flux is being propagated; therefore the phonons crossed the boundary between the normal and superconducting phases). The average mean free path of the phonons was decreased due to the presence of the normal regions, which also led to a reduction in the observed heat flow.

Let us consider further the region of higher temperatures where the electronic contribution is the major contribution (the sample is assumed to be sufficiently pure). In[52] it is shown that at the boundary between the normal and superconducting phases, a unique reflection of the electronic excitations takes place in connection with transmission of heat across the layers. Namely, an investigation of the equations

$$i\frac{\partial f}{\partial t} = -\left(\frac{\nabla^2}{2m}+\mu\right)f + i\Delta(\mathbf{r})\varphi,$$

$$i\frac{\partial\varphi}{\partial t} = \left(\frac{\nabla^2}{2m}+\mu\right)\varphi - i\Delta(\mathbf{r})f,$$

$$f(\mathbf{r},t) = \langle\Phi_0|\psi(\mathbf{r},t)|\Phi_1\rangle,\quad \varphi = (\mathbf{r},t) = \langle\Phi_0|\psi^+(\mathbf{r},t)|\Phi_1\rangle,$$

(where ψ^+ and ψ are Heisenberg operators, Φ_0 and Φ_1 are the wave functions in the space of occupation numbers which describe the ground and excited states of the system, respectively, and μ denotes the chemical potential; the set of quantities f and φ has the meaning of the wave function of the quasiparticles) leads to the result that the momentum actually does not change upon reflection, but the quantity ξ and the velocity $\partial\epsilon/\partial\mathbf{p}$ change sign. In other words, upon reflection on "electron" changes into a "hole" and vice versa.

Let us consider the temperature interval $T_0 \ll T \ll T_c$ (T_0 is the temperature at which the phonon thermal conductivity begins to play a major role). Then excitations with an energy ϵ close to Δ (the number of other excitations is exponentially small) play a major role. Here the coefficient for above-barrier reflection turns out to be given by

$$W = 1 - f(\mathbf{n}_z) \sqrt{\frac{\varepsilon - \Delta}{\Delta'}} \tag{18}$$

($f \sim 1$, n_z is the unit normal vector).

As we see, the coefficient W turns out to be a quantity of the order of unity, and consequently heat exchange through the boundary between the superconducting and normal phases turns out to be intrinsically difficult. A situation arises which is analogous to heat exchange between solid and liquid helium.[55,56] In both cases there is a temperature jump which, in the case of the intermediate state, is related to the heat flow Q by the following equation:

$$Q = \frac{\sqrt{\pi}}{2} f_0 \left(\frac{p_0}{\pi}\right)^2 \Delta \sqrt{\frac{\Delta}{T}} \exp\left(-\frac{\Delta}{T}\right) \delta T,$$

where $Q = (\partial W/\partial T)\delta T$, W is calculated according to the formula

$$W = \int\limits_{v_z > 0} 2n_0(\varepsilon)\, \varepsilon\, (1 - W)\, v_z \frac{d^3\mathbf{p}}{(2\pi)^3}, \quad f_0 \sim 1, \; n_0(\varepsilon) = \exp(-\varepsilon/T),$$

and W is defined by Eq. (18). The additional thermal resistance is defined by the formula

$$\Delta R = f_0^{-1} \left(\frac{2\pi}{p_0}\right)^2 \sqrt{\frac{\varphi(\eta) T}{\pi \alpha\, d \Delta^3}} \exp\left(\frac{\Delta}{T}\right) \tag{19}$$

($a = \sqrt{\alpha a/\varphi(\eta)}$ denotes the period of the structure in the intermediate state[57]). The temperature dependence of ΔR (in which the electron mean free path does not appear, as is evident from Eq. (19)) is in quite good agreement with the experimental data.

In[54] it is shown that in the case when $a_s \ll a_n$ (a_s and a_n denote, respectively, the thicknesses of the superconducting and normal layers) the effect of tunneling of the thermal excitations through the s-layer turns out to be important for $T \ll T_c$, said effect leading to a power-law dependence of the thermal resistance on temperature.

In[53], the problem of heat transmission along layers of normal and superconducting phases is investigated by the method of the kinetic equation. For $T \ll T_c$ the electrons in the normal regions play the major role. In this connection the thermal conductivity ζ_{ik} (i, k denote x, y; the z axis is normal to the boundary of the region) of the layered structure turns out to be given by

$$\varkappa_{ik} = \frac{7\zeta(3)}{12\pi^3} T \frac{a_n^2}{a} \int \frac{d(\cos\theta)\, d\varphi}{K(\theta, \varphi)} n_i n_k \delta(\cos\theta), \tag{20}$$

where $n = \partial E/\partial \mathbf{p}/|\partial E/\partial \mathbf{p}|$, $K(\theta, \varphi)$ denotes the Gaussian curvature of the Fermi surface, $a = a_n + a_s$ is the period of the structure, and the angles θ and φ determine the mutual orientation of n and the z axis. The excitations moving parallel to the boundary separating the phases give the major contribution to the thermal conductivity. It is essential that in the case under consideration of a complex structure, κ does not depend on the mean free path. In this connection, as is evident from Eq. (20), the thermal conductivity of the pure normal phase turns out to be $la/a_n^2 \gg 1$ times larger than the thermal conductivity of the intermediate state.

In the case of a "filamentary structure," which is thermodynamically more favorable near the boundary of the intermediate state, the thermal conductivity turns out to depend on the mean free path. In this connection

$$\varkappa = \eta \frac{T}{24} \frac{a^2 \ln(l/a)}{K(0)\, l_0}, \tag{21}$$

where η denotes the concentration of the normal phase, and l is the mean free path, $l_0 = l(0)$; it is assumed that the mean free path of the electrons is much larger than the diameter of the normal filaments. In this case excitations whose velocity of motion makes a small angle with the axis of the filament give the major contribution to the thermal conductivity.

9. Thermal Conductivity of Superconductors with a Strong Electron-phonon Interaction

Experimental investigation of the thermal conductivity of pure, superconducting samples of Pb and Hg[38,39] showed that for $T \sim T_c$ $\kappa(T)$ falls off with decreasing temperature more abruptly than for other metals (see Fig. 14). The thermal conductivity of these superconductors is therefore poorly described by formula (9). This property of lead and mercury is associated with their membership in the group of so-called anomalous superconductors. In addition to Pb and Hg, the following also belong to this group: Nb, Ga, In, NbN, Bi films, and others. In these substances the electron-phonon interaction which determines the superconductivity is not weak, and therefore their properties (see the review[7]) are poorly described by the usual theory, which was developed in the weak-coupling approximation. Anomalous superconductors are characterized by a value of the ratio T_c/θ which is significantly larger than that associated with other metals. Thus, for example, for lead $T_c/\theta \approx 0.1$ (for comparison we note that for Al, for example, $T_c/\theta \approx 1/400$) which, according to the formula $T_c \approx \theta \exp(-1/g)$, leads to a value of the coupling constant $g_{Pb} \approx 0.4$.

From formulas (5), (9), and (13) it is seen that the thermal conductivity of superconductors is determined by a universal function of the parameter $b = \Delta/T$. According to the usual theory[1,6]

$$\left.\frac{\Delta}{T}\right|_{T \to T_c} = 3.06 \sqrt{1 - \frac{T}{T_c}}.$$

This relation, which was obtained in the weak coupling approximation, turns out to be incorrect for anomalous superconductors; the experimentally-noted discrepancy between theory and experiment is also related to this.

Articles[58-61,62-63] are devoted to theoretical investigations of the properties of superconductors possessing strong coupling. The specific characteristics of the thermal conductivity of anomalous superconductors are mentioned in[64].

The temperature dependence of the gap as $T \to T_c$, which is essential in order to solve the problem of thermal conductivity of interest to us, is found in[60,61].

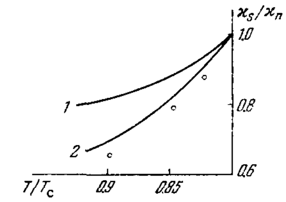

FIG. 14

It turned out that for Pb

$$\left.\frac{\Delta}{T}\right|_{\substack{\mathrm{Pb},\\ T\to T_c}} \approx 4\sqrt{1-\frac{T}{T_c}}.$$

Since the electron-phonon interaction is not weak, in order to calculate the electronic thermal conductivity of anomalous superconductors,[60,61] we shall start from a general definition of the coefficient of thermal conductivity[16]

$$\varkappa(T,\omega)=(2\omega T)^{-1}\int d^4x\,\langle[\hat{q}(x),\hat{q}(0)]\rangle, \tag{22}$$

where $\hat{q}$ denotes the operator of heat flow, an expression for which can be written in the form[65]

$$\hat{q}=\left(\frac{\nabla\nabla'}{2m}-\mu+V\right)\frac{\nabla-\nabla'}{2mi}\psi^+(x')\psi(x)\Big|_{x=x'},$$

where V is the potential energy. Expression (22) may be reduced to the form

$$\varkappa=(2\omega)^{-1}\sum_{\omega_{n'}}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\omega_{n'}\omega'_-}{m^2}p^2\{G^n(\mathbf{p},\omega_{n'})G^n(-\mathbf{p},-i\omega'_-)$$
$$+F^{+n}(\mathbf{p},\omega_{n'})F(-\bar{\mathbf{p}},-\omega'_-)\},$$
$$\omega'_-=\omega_{n'}-\omega_1,\quad \omega_1=-i\omega,\quad \omega_{n'}=(2n'+1)\pi T; \tag{23}$$

here G^n and F^n are the temperature Green's functions of the electronic excitations which interact with the impurities, and according to[66] G^n and F^n may be written in the form

$$G^n=\frac{i\tilde{\omega}+\xi}{\tilde{\omega}^2+\xi^2+\tilde{\Sigma}^2(\omega)},\qquad F^{+n}=\frac{\tilde{\Sigma}^*(\omega)}{\tilde{\omega}^2+\xi^2+\tilde{\Sigma}^2(\omega)},$$
$$\tilde{\omega}=\eta\omega,\quad \tilde{\Sigma}(\omega,T)=\eta\Sigma(\omega,T),\quad \eta=1+\frac{i}{2\tau\sqrt{\omega^2-\Sigma^2(\omega)}},$$

where τ is the relaxation time.

The function $\Sigma(\omega, T)$ represents the self-energy part, describing the Cooper pairing of the electrons. In the weak coupling approximation its frequency dependence can be neglected. In this connection $\Sigma(\omega, T) \equiv \Delta(T)$ ($\Delta(T)$ denotes the energy gap). For anomalous superconductors the function $\Sigma(\omega, T)$ is determined by solving an integral equation[67]

$$\Sigma(\omega_n,T)=\frac{T}{(2\pi)^3}g^2\sum_{\omega_{n'}}\int\frac{\omega^2(\mathbf{p}-\mathbf{k})}{\omega^2(\mathbf{p}-\mathbf{k})+(\omega_n-\omega_{n'})^2}\,\frac{\Sigma(\omega_{n'},T)}{\omega_{n'}^2+\xi^2+\Sigma^2(\omega_{n'},T)}\,d\mathbf{k},$$

where it turns out to be very essential to take the "retardation" terms, described by the expression $(\omega_n - \omega_n')^2$ in the denominator of the phonon Green's function, into account.

Let us substitute the expressions for the Green's functions into Eq. (23), and then integrate with the aid of the theory of residues. We replace the summation (see[14] for a similar calculation) by an integration in the complex variable plane. It is essential that here the singularities are points which satisfy the equation $\omega = i\Sigma(\omega)$, i.e., points which are a solution of the equation for determination of the energy gap Δ. As a result it turns out that the usual static coefficient of thermal conductivity is described by expressions (5)[11] in which, however, the gap $\Delta(T)$ corresponding to a given anomalous superconductor appears; in lead, for example, as we already noted above, $\Delta/T \approx 4\sqrt{1-(T/T_c)}$ as $T \to T_c$.

Expression (9) with the appropriate function $\Delta(T)$ also describes the thermal conductivity of pure anomalous superconductors, which is associated with the scattering of electrons by phonons. The corresponding theoretical curve, which describes the experimental data quite well, is shown in Fig. 14.

The large value of the coefficient α in the formula

$$\left.\frac{\Delta}{T}\right|_{T\to T_c}=\alpha\sqrt{1-\frac{T}{T_c}}$$

determines a much sharper than usual decrease in the coefficient of thermal conductivity of anomalous superconductors as the temperature is lowered.

10. Thermal Effects

a) Thermoelectric effect. The anisotropy of a crystal leads to the possibility of observing a distinctive thermoelectric effect in superconductors. If a temperature gradient is created in the sample, then its existence leads to the appearance of a normal current $j^{(n)}$. However, in the isotropic case this current is completely cancelled by the counter, superconducting current $j^{(s)}$, and thus both the total current and the field created by it are absent. Thus, the presence of a temperature gradient only leads to the usual thermal conductivity and to the small additional effect of convective thermal conductivity.[68,69,11]

This effect is analogous to the circulation of the normal and superfluid components of non-uniformly heated He II. In[11] it is shown that the ratio Q_{conv}/Q for the convective heat flow $Q_{conv} = TSv_n$ (S denotes the entropy, v_n denotes the velocity of the normal component) is a quantity of the order of $\sim k(T/T_c)^{1/2}/(p_0^2/m)$, which even for $T \sim T_c$ amounts to a quantity of the order of $\sim 10^{-5}$ to 10^{-4}. However, in the anisotropic case when there is not one direction associated with ∇T but several preferred directions, the current $j^{(n)}$ in general is not completely cancelled by the current $j^{(s)}$. The first questions about the thermoelectric effect in superconductors were considered in[70,68], where it was shown on the basis of an appropriate generalization of the London theory that taking anisotropy into account makes the usual conclusion about the absence of thermoelectric phenomena in superconductors untrue.

The thermoelectric coefficients, whose determination also permits us to determine the feasibility of observing the effect, may be calculated on the basis of the microscopic theory. It is impossible to determine them from experiments with normal metals since they essentially depend on the parameters of the superconducting state (see below).

The thermoelectric effect in superconductors is considered in[71] on the basis of the contemporary microscopic theory. The question of the feasibility of its experimental observation is investigated. The initial equations for solution of the problem are Maxwell's equation and also the expressions which determine the superconducting and normal currents:

$$\operatorname{rot}\mathbf{H}=\frac{4\pi}{c}(\mathbf{j}^{(n)}+\mathbf{j}^{(s)}), \tag{24}$$

$$j_\alpha^{(n)}=b_{\alpha\beta}\nabla_\beta T, \tag{24'}$$

$$j_\alpha^{(s)}(\mathbf{q})=-\frac{c}{4\pi}K_{\alpha\beta}(\mathbf{q})A_\beta(\mathbf{q}), \tag{24''}$$

where $b_{\alpha\beta}$ are the thermoelectric coefficients, and $K_{\alpha\beta}$ is the so-called Pippard tensor which relates the current to the vector potential; $K_{\alpha\beta}$ is calculated on the basis of the microscopic theory.[72]

Let us consider an infinite superconducting plate (see Fig. 15; s is the axis of the crystal, all quantities depend on y). In the case of a half-space, the non-vanishing component of the current j_x is given by

$$j_x(y)=\frac{cH}{4\pi\delta}\exp(-y/\delta),$$

as is evident from Eqs. (24)–(24″), where $\delta = K_{xx}(0)$ denotes the penetration depth, and H is the field in the interior of the sample (see below). For a plate[70] one finds

$$j_x=\frac{cH}{4\pi\delta}\frac{\operatorname{sh}(y/2\delta)}{\operatorname{ch}(a/2\delta)}.$$

The relations obtained in the planar case under consideration are, of course, gauge invariant. Thus, anisotropy of the crystal leads to the existence of a circulating current, which does not vanish inside a surface layer of thickness $\sim\delta$. The magnetic field created by this current is determined by the relation

$$H_z(y)=\frac{4\pi}{c}\frac{d}{dy}[K^{-1}(0,T)\,j^{(n)}(y)],$$

$$j^{(n)}=j_x^{(n)}-\frac{K_{xy}}{K_{yy}}j_y^{(n)},\qquad K(q)=K_{xx}(q)-\frac{K_{xx}^2(q)}{K_{yy}(q)}. \tag{25}$$

It is important to note that even for Pippard superconductors the components of the Pippard tensor corresponding to $q\to 0$ enter into expression (25) for the field, i.e., for an investigation of thermoelectricity all superconductors are London-type. This is associated with the fact that for the possible values of the temperature gradient, $\nabla T \lesssim 0.1$ deg-cm^{-1}, the normal current associated with ∇T varies slightly over distances of order $\sim\xi_0$. In connection with this, values of $q \ll \xi_0^{-1}$ play a major role in the formula

$$A_x(y)=\int j_n(\mathbf{q})\,e^{iqy}[q^2+K(\mathbf{q})]^{-1}d\mathbf{q},$$

which determines the vector potential. In addition, of course, $q\ll\delta^{-1}$ with the exception of a very small temperature interval near T_c.

The normal current, determining the magnetic field according to Eq. (25), and at the same time also the thermoelectric coefficients are found by solving the kinetic equation (see Eqs. (2) and (15))

$$-\frac{\varepsilon}{T}\frac{\partial f_0}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial\mathbf{p}}\nabla T=-2\pi\frac{|\xi|}{\varepsilon}\int|V|^2(f'_{\mathbf{p}}-f'_{\mathbf{p}'})\frac{\partial\sigma}{v_F}$$

(where $d\sigma$ denotes an element of the Fermi surface), whose solution may be written in the form[73]

$$f_1=-\frac{\partial f_0}{\partial\varepsilon}\frac{\varepsilon}{T}\Lambda\nabla T$$

($\Lambda=\tau(\mathbf{p})\mathbf{v}$ denotes the vector mean free path).

Finally the following expression is obtained for the field created by the thermoelectric current:[71]

$$H_z(y)=\frac{2\pi ck}{e}(\nabla T)^2\sin 2\theta\,(\varphi_{\parallel}-\varphi_{\perp})\frac{d}{dT}\left[F\left(\frac{\Delta}{T}\right)\frac{\delta_L^2(T)}{\delta_L^2(0)}\right],$$

$$\varphi_{\parallel,\perp}=\int\frac{d\sigma}{v_F}v_{\parallel,\perp}^2(\mathbf{n})\Big/\int\frac{d\sigma}{v_F}v_{\parallel,\perp}^2, \tag{26}$$

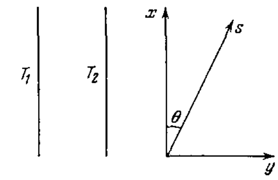

FIG. 15

where θ denotes the angle between s and the 0x axis (see Fig. 15), τ is the relaxation time, k is Boltzmann's constant, $v_{\parallel,\perp}$ denote, respectively, the components of the Fermi velocity parallel to and perpendicular to the axis of the crystal s, and $F(x) = x(e^x+1)^{-1}-\ln(1+e^{-x})$.

We see that the field is expressed in terms of a universal function of T/T_c and quantities which characterize the normal metal. As $T\to T_c$ we find

$$H_z=\frac{ck}{e}(\nabla T)^2\cdot\frac{1,4\pi}{T_c}\sin 2\theta\,(\varphi_{\parallel}-\varphi_{\perp})\left(1-\frac{T}{T_c}\right)^{-2}. \tag{27}$$

The effect increases as $T\to T_c$ (with the exception of a small interval near T_c where the circulating current vanishes).

Let us estimate the possible magnitude of the field:

$$H\sim\frac{ck}{e}(\nabla T)^2\frac{\Lambda}{v_F T_c}\left(1-\frac{T}{T_c}\right)^{-2}.$$

If, for example, $\nabla T\approx 0.1$ deg/cm and $\Lambda\approx 0.1$ cm (see, for example,[74,30]; the samples therefore must be sufficiently pure), then fields $H\sim 10^{-3}$ Gauss are possible for $T/T_c\approx 0.99$, and fields $H\sim 10^{-5}$ Gauss are possible for $T/T_c\approx 0.9$.

As the temperature is lowered or upon contamination of the sample, the thermoelectric field, and together with it the magnetic moment, decrease. In this connection an investigation of the thermoelectric effect in superconductors may be used[75] as a simple method for experimental observation of the quantization of magnetic flux.

The thermoelectric effect appears most strongly upon fulfilment of the following conditions: 1) the sample must be sufficiently pure; 2) one should choose a uniaxial crystal; 3) the temperature must be close to T_c.

The values of the field H given above are quite accessible to experimental observation, which therefore makes it of interest to set-up the appropriate experiments.

b) The thermomagnetic effect. As is well known, thermomagnetic effects arise due to the action of a magnetic field on the thermal flux (see, for example,[76]). In superconductors it turns out to be possible to observe the Leduc-Righi effect, consisting in the appearance of a temperature gradient perpendicular to the direction of the resultant heat flow.

In order to investigate this effect[77] let us write down the kinetic equation for electronic excitations in the presence of a temperature gradient along the x axis and a magnetic field which is perpendicular to the direction of heat flow:

$$-\frac{\partial f}{\partial\varepsilon}\frac{\varepsilon}{T}v_x\frac{\partial T}{\partial x}+\frac{eH}{c}\left(v_y\frac{\partial f}{\partial p_x}-v_x\frac{\partial f}{\partial p_y}\right)\frac{\xi}{|\xi|}=-\frac{f-f_0}{\tau}. \tag{28}$$

The relaxation time τ is given by[11] $\tau=\tau_0\epsilon/|\xi|$ (see above, Sec. 2). Naturally it is assumed that the size of the sample is smaller than the penetration depth of the field. Solving Eq. (28) by the method of successive approximations ($f = f_0 + f_1 + f_2$), we find that

$$f_1 = \frac{p_x}{m}\tau_0 \frac{\partial f_0}{\partial \varepsilon}\frac{\varepsilon}{T}\frac{\partial T}{\partial x}\frac{\xi}{|\xi|}, \quad f_2 = \tau_0^2 v_y \frac{1}{T}\frac{eH}{mc}\frac{\partial f_0}{\partial \varepsilon}\frac{\partial T}{\partial x}\frac{\varepsilon^2}{|\xi|}. \tag{29}$$

The magnitude of the Leduc-Righi effect is determined by the coefficient

$$L = \frac{\partial T}{\partial y'} \Big/ \frac{\partial T}{\partial x'} H$$

(the x′ axis coincides with the direction of the resultant heat flow). It is easy to see that $L = Q_y/Q_x H$, where

$$Q_x = 2\int \varepsilon v_x f_1\, dp, \qquad Q_y = 2\int \varepsilon v_y f_2\, dp.$$

With the aid of Eq. (29) we find

$$L = \frac{\tau_0 e}{mc}. \tag{30}$$

The Nernst-Ettingshausen effect, consisting in the appearance of an electric field perpendicular to the direction of the resultant heat flow, obviously is not present in the case of superconductors.

III. ABSORPTION OF SOUND IN SUPERCONDUCTORS

The attenuation of longitudinal sound waves in superconductors was first investigated just before the appearance of the theory of superconductivity in[78,79]. Investigation of this effect in Pb and Sn, respectively, led to the conclusion about the monotonic decrease of the attenuation as the temperature is lowered below T_c. The temperature dependence turns out to be much more abrupt in comparison with the relation $\gamma \sim (T/T_c)^4$, which was obtained upon investigation of the two-fluid model of Gorter and Casimir.

The question of the attenuation of sound was one of the first questions to be investigated in the theory of superconductivity. This circumstance is not accidental. The point is that the corresponding measurements enable us to obtain abundant information about the properties of the electronic spectrum of the excitations in a superconductor. The temperature and angular dependences of the energy gap are very accurately determined with the aid of ultrasonic measurements.

In connection with an examination of the question of the attenuation of sound, one should distinguish two limiting cases. The first of these corresponds to frequencies ω which satisfy the condition $\omega \gg \tau^{-1}$ (ultrasound) where τ is the relaxation time, and the second case corresponds to frequencies which satisfy the condition $\omega \ll \tau^{-1}$ (long-wavelength sound). In the first case one can generally neglect relaxation processes and regard the absorption of sound simply as a process involving the absorption of sound quanta by free electronic excitations. In this case the attenuation is completely analogous to the well-known Landau damping of plasma waves,[80] and actually corresponds to absorption of the wave by resonant electrons.

In the second case the relaxation processes naturally play a decisive role, and the dissipation of the sound wave energy is determined by investigating the appropriate kinetic equation.

1. Ultrasonic Attenuation

a) **Isotropic case.** In the case under consideration the period of the sound wave is much smaller than the relaxation time. In this connection, as we have already noted, the interaction of the wave with the electronic system may be regarded as the emission and absorption of sound quanta by the electronic excitations.

Writing the probability for the absorption of a sound quantum and the probability for the inverse process with the aid of the u, v transformation given by Eq. (3), we obtain the following expression for the absorption coefficient:[26,17]

$$\gamma \sim \int |V|^2 \Big[\Big(1 + \frac{\xi\xi' - \Delta^2}{\varepsilon\varepsilon'}\Big)(f - f')\,\delta(\varepsilon' - \varepsilon - \omega) + \Big(1 - \frac{\xi\xi' - \Delta^2}{\varepsilon\varepsilon'}\Big)(1 - f - f')\,\delta(\varepsilon' + \varepsilon - \omega)\Big]\, dp.$$

Carrying out the corresponding integration, we arrive at the following formula for the ratio of the coefficients for ultrasonic absorption in the normal and superconducting states (for $\omega \lesssim T$):

$$\frac{\gamma_s}{\gamma_n} = \frac{x - \ln[(e^{b+x}+1)(e^b+1)^{-1}] + D(x)(2b - x + 2\ln[(e^{x-b}+1)(e^b-1)^{-1}])}{\ln[(e^x+1)/2]}, \tag{31}$$

where $x = \omega/T$ and $b = \Delta/T$; the function $D(x)$ (see above is introduced in connection with the fact that the creation of a pair of excitations is possible only in that case when $\omega \geq 2\Delta$.

For $x = \omega/T \ll 1$ we arrive at the following formula which was previously obtained in[1]:

$$\frac{\gamma_s}{\gamma_n} = \frac{2}{e^b + 1}. \tag{32}$$

The experimentally-observed abrupt decrease in the absorption as the temperature is reduced below T_c is described by a "Fermi" function, as is evident from Eq. (32). It is determined by the sharp increase in the function $\Delta(T)$ as the temperature moves away from T_c and is associated with a decrease in the number of electronic excitations.

Formula (32) directly relates the energy gap $\Delta(T)$ to the experimentally measured ratio γ_s/γ_n. Its simplicity enables one to use very effectively ultrasonic measurements to determine the shape of the function $\Delta(T)$. Experimental data[81] obtained in connection with an investigation of ultrasonic attenuation in monocrystals of Sn are shown in Fig. 16. The very good agreement between theory and experimental data should be noted. It is clear from this example that ultrasonic

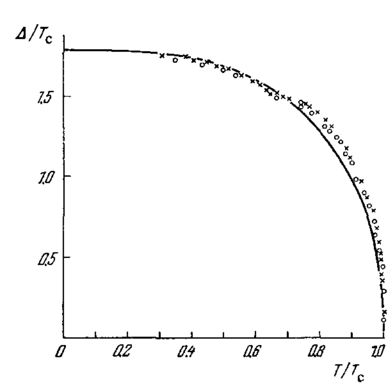

FIG. 16

measurements actually enable one to establish the nature of the function $\Delta(T)$, which is the fundamental characteristic of the superconducting state, with a high degree of accuracy.

b) **Influence of the gap anisotropy on ultrasonic attenuation.** 1) Absorption coefficient. Reconstruction of the function $\Delta(n)$. The absorption of ultrasound is extremely sensitive to the structure of the energy spectrum of the substance being investigated. Therefore the corresponding investigations, as already noted above, enable one to obtain rather complete information about the temperature and angular dependence of the gap $\Delta(T, n)$ in the excitation spectrum.

A strikingly expressed anisotropy of the ultrasonic absorption in superconductors is observed in a number of experimental articles (see below). Formulas (31) and (32), which were derived in the isotropic approximation, are not applicable for an analysis of the corresponding experimental data. Articles[82-84] are devoted to a theoretical consideration of the effect of anisotropy on ultrasonic attenuation.

At first we confine the investigation to a pure superconductor.[82]

The attenuation is described by the imaginary part of the polarization operator $\pi(\mathbf{q}, i\omega_n)$ which is given by

$$\Pi(\mathbf{q}, i\omega_n) = \frac{T}{(2\pi)^3}\sum_{\omega_{n'}}\int d^3p'\,[G(\mathbf{p})\,G(\mathbf{p}-\mathbf{q}) - F(\mathbf{p})\,F(\mathbf{p}-\mathbf{q})],$$

where $\omega_n = (2n+1)\pi T$,

$$G(\mathbf{p}, \omega_n) = \left(\frac{u_{\mathbf{p}}^2}{\varepsilon_{\mathbf{p}}-\omega_n} - \frac{v_{\mathbf{p}}^2}{\varepsilon_{\mathbf{p}}+\omega_n}\right) g(\mathbf{p}),$$

$$F(\mathbf{p}, \omega_n) = \frac{\Delta(\mathbf{p})}{2\varepsilon_{\mathbf{p}}}\left(\frac{1}{\varepsilon_{\mathbf{p}}-\omega_n} + \frac{1}{\varepsilon_{\mathbf{p}}+\omega_n}\right) g(\mathbf{p}).$$

$$u_{\mathbf{p}}^2, v_{\mathbf{p}}^2 = \frac{1}{2}\left(1 \pm \frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\right);$$

the dimensionless function $g(\mathbf{p})$ depends on direction in an essential way.

Let us change from a summation over ω_n to an integration with the aid of the replacement

$$T\sum_{\omega_n} = \frac{i}{4\pi}\int_{\Gamma} \mathrm{th}\left(\frac{\beta\omega}{2}\right) d\omega$$

where Γ denotes a contour consisting of two straight lines parallel to the imaginary axis.

Evaluation leads to the following expression (for $\omega < 2\Delta$) for $\mathrm{Im}\,\pi$:

$$\mathrm{Im}\,\Pi = \frac{1}{(2\pi)^2}\int d^3p\,g^2(\mathbf{p})\,\frac{\xi^2}{\varepsilon^2}\,[f(\beta\varepsilon-\beta\omega) - f(\beta\varepsilon)]\,\delta(\varepsilon-\varepsilon'-\omega),\ \beta = T^{-1}.$$

The desired ratio γ_s/γ_n is described by the formula

$$\frac{\gamma_s}{\gamma_n} = 2\,\langle f(\beta\Delta)\rangle_n,$$

$$\langle\varphi\rangle_n = \int \frac{d\sigma'}{v_F'}\,g^2(\mathbf{n}')\,\delta(\cos x)\,\varphi(\mathbf{n}') \Big/ \int \frac{d\sigma'}{v_F'}\,g^2(\mathbf{n}')\,\delta(\cos x) \qquad (33)$$

($d\sigma$ denotes an element of the Fermi surface, x denotes the angle between the direction of the Fermi velocity and the direction of sound propagation), from which it is clear that the absorption of sound is actually characterized by considerable anisotropy.

A formula which is very convenient for analyzing experimental data is obtained by considering the ultrasonic absorption in the region of low temperatures which satisfy the condition

$$1 < \frac{T_c}{T} < \frac{2}{\alpha}\ln\left(\frac{v_F}{u}\right)$$

(u denotes the velocity of sound, α is the coefficient of anisotropy, which is equal to the ratio of the change in $\Delta(\mathbf{n})$ on the Fermi surface to the minimum value Δ_0). It has the form

$$\gamma_s(\mathbf{q}) \sim \omega(\mathbf{q})\exp\left(-\frac{\Delta_n^{\min}}{T}\right), \qquad (34)$$

where $\omega(\mathbf{q})$ is the phonon frequency, and $\Delta_n^{\min}$ denotes the minimum value of the energy gap on the circumference of a stereographic projection of the Fermi surface perpendicular to the vector $\mathbf{n} = \mathbf{q}/q$.

In the region of lower temperatures, at which measurements have apparently not yet been carried out, the absorption is determined by the absolute minimum of the gap, Δ_0. In this connection

$$\gamma \sim \exp(-\beta\Delta_0)\sum \cos^3\chi_0^{-1},$$

where χ_0 denotes the angle between the velocities at the absolute minimum point and the direction of $\mathbf{q}$.

From formula (34) it is immediately evident that the absorption coefficient γ_s decreases exponentially with temperature, and also the argument of the exponential depends on direction. According to this formula, the absorption is determined by the minimum value of the gap along the line $\mathbf{q}\cdot\mathbf{v} = 0$. This result has a clear physical meaning. In fact, in the case under consideration of the absorption of high frequency sound, which (see Ch. I above) may be regarded as a direct quantum process involving the absorption of a phonon by an electron, the energy conservation law $\epsilon_{\mathbf{p}+\mathbf{q}} - \epsilon_{\mathbf{p}} = \omega_{\mathbf{q}}$, $\omega_{\mathbf{q}} = \mathbf{u}\cdot\mathbf{q}$, must be satisfied. Taking the inequality $u \ll p$ into account, the cited condition may be written in the form

$$\mathbf{v}\mathbf{q} = \omega. \qquad (35)$$

Thus, as one would expect, electrons moving in a plane of equal phase with the sound wave introduce the major contribution to the resonance absorption under consideration. Since the speed of sound satisfies the inequality $u \ll v_F$, one can approximately write condition (35) in the form $\mathbf{v}\cdot\mathbf{q} = 0$, i.e., in fact electronic excitations whose velocities are perpendicular to the direction of sound propagation introduce the major contribution to the absorption. At low temperatures the Boltzmann distribution of the electronic excitations in a superconductor also leads to formula (34), in which the minimum value of the gap along the line $\mathbf{v}\cdot\mathbf{q} = 0$ appears.

The question of the feasibility of reconstructing the energy gap $\Delta(\mathbf{n})$ from measurements of ultrasonic attenuation in the low-temperature region is discussed in article[83]. A method is proposed which, in the presence of a simply-connected Fermi surface, permits one to reconstruct the function $\Delta(\mathbf{n})$ in many cases. This method consists in the following. The function $f(\mathbf{n}) = \Delta_n^{\min}$ is determined with the aid of ultrasonic measurements. Then for each point of the level line γ_a of this function, a large circle $c(\mathbf{n})$ is constructed, perpendicular to the direction $\mathbf{n}$. The envelope of the obtained family of circles represents the level line of the function $\Delta(\mathbf{n})$. Unfortunately, in the case of a multiply-connected Fermi surface the cited procedure does not enable one to uniquely reconstruct the Fermi surface since it is impossible to determine to which of its unconnected parts the found level lines belong.

2) Experimental data. Anisotropy of the gap has been observed in many superconducting elements with the aid of ultrasonic methods. To begin with this refers to superconducting tin, which is investigated in[85-88] (see in[89] for a detailed review of the experimental data). The form of the dependence of the absorption[86] on the orientation of the sound wave is immediately evident from Fig. 17.

In[90] it is mentioned that the difference $\Delta_{max} - \Delta_{min}$ decreases as the sample becomes more impure. According to[24], the introduction of impurities actually causes the investigated superconductor to all the more closely approach the isotropic model. The effects of the gap anisotropy are important as long as $l > v/\Delta_{av}$.

According to[87-89] the orientation of the vector q relative to the crystallographic axes given by $q \perp [101]$ corresponds to a minimum gap $2\Delta = 3.9\ T_c$ referred to the region $q \cdot v = 0$ on the Fermi surface; however if, for example, $q \perp [112]$, then $(2\Delta/T_c)_{min} = 4.4$, and the orientation $q \perp [111]$ corresponds to $(2\Delta/T_c)_{min} = 4.8$. Agreement with Pokrovskiĭ's theorem[91] about the proportional change with temperature of the gaps pertaining to different crystallographic directions is also noted.

With the aid of ultrasonic measurements anisotropy is also observed in Ga,[92] Va and Zn,[93] Ta,[94] Re,[95] In,[96] and Nb.[97] In Zn, for example, $\Delta_{min} = 3.4\ T_c$, and $\Delta_{max} = 3.8\ T_c$. In Nb the orientation $q_{min} \parallel [100]$ corresponds to $\Delta = 3.6\ T_c$, and $q \parallel [111]$ corresponds to $\Delta = 3.4\ T_c$.

The energy spectrum of Nb is investigated in article[98] by measuring its heat capacity in the low-temperature region. According to the data given in this article, pure superconducting Nb is characterized by two gaps, $2\Delta_1 = 3.5\ T_c$ and $2\Delta_2 = 0.3\ T_c$. Thus, an investigation of the properties of Nb by different methods leads to different values of Δ_{min} and Δ_{max}. This is associated with the fact that the many-band structure of a superconductor (see below) basically appears in connection with a measurement of the heat capacity, but the anisotropy of the gap plays the major role in connection with an investigation of the directional dependence of the ultrasonic absorption.

3) Absorption of sound in an anisotropic superconductor in the presence of impurities. Ultrasonic absorption in a pure anisotropic superconductor was

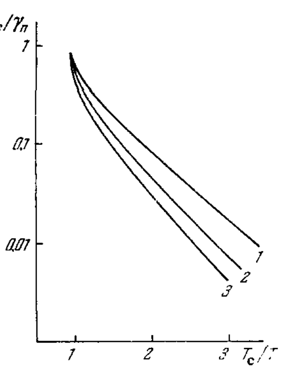

FIG. 17. The curves correspond to the following orientations of q: 1 – q ∥ [001], 2 – q ∥ [110], and 3 – q ⊥ [001].

considered above. Now let us calculate[84] the absorption coefficient with the scattering of electronic excitations by impurities taken into account. The problem is solved for both a bulk superconductor and for a lamina whose thickness $d < l$ (l denotes the mean free path of the excitations).

We shall regard the sound wave as a factor which deforms the crystal lattice.[99,100,106] The absorption coefficient is determined by the formula $\gamma_s = T\dot{S}/W$ (S denotes the entropy density of the gas of elementary excitations, which is related to the distribution function $n(p, r, t)$ in the usual manner, and W denotes the energy density of the sound wave). Writing down the collision integral in the relaxation time approximation, we find

$$\gamma_s = -\frac{2}{W}\int \nu' |\chi|^2 \frac{\partial n_0}{\partial \varepsilon} \frac{d\varepsilon\, ds}{\left|\frac{d\varepsilon}{dp}\right|}, \qquad (36)$$

where ν' denotes the frequency of collisions between excitations and impurities, and moreover $\nu' = \nu|\xi|/\epsilon$[11] (see above, Ch. II, Sec. 1),

$$n = n_0 - \frac{\partial n_0}{\partial \varepsilon}\chi, \quad n_0 = [\exp(\varepsilon/T) + 1]^{-1}, \qquad \varepsilon(p, r, t) = \varepsilon(p) + \varepsilon_1(p, r, t)$$

and the appearance of the correction $\epsilon_1(p, r, t)$ is due to the displacement in the lattice which the sound wave describes.

The function χ which determines the perturbed distribution function is found by solving a kinetic equation analogous to the equation used in[100] in order to solve the problem of sound absorption in a normal metal. The following two cases are considered separately: 1) a bulk metal or lamina with specularly reflecting boundaries; 2) a lamina with boundaries which scatter diffusely.

In the first case in the region of most interest, which turns out to be the low-temperature region $T < \delta\Delta = \Delta_{max} - \Delta_{min}$, the ratio γ_s/γ_n turns out to be given by (for $ql \ll 1$)

$$\frac{\gamma_s}{\gamma_n} \sim \frac{T}{\delta\Delta}\exp\left(-\frac{\Delta_{min}}{T}\right). \qquad (37)$$

In the opposite limiting case, $ql \gg 1$, the ratio of the quantities $(ql)^{-1}$ and $(T/\delta\Delta)^{1/2}$ turns out to be essential. Thus, for example, if $(ql)^{-1} < (T/\delta\Delta)^{1/2}$, then

$$\gamma_s^{max} \sim \exp\left(-\frac{\Delta_{min}}{T}\right)\left(\frac{T}{\delta\Delta}\right)^{1/2}\gamma_n;$$

however, if $(ql)^{-1} > (T/\delta\Delta)^{1/2}$, then

$$\gamma_s^{max} \sim \exp\left(-\frac{\Delta_{min}}{T}\right) ql \left(\frac{T}{\delta\Delta}\right)\gamma_n.$$

The most complicated angular dependence turns out to occur in the temperature interval $(ql)^{-1} < (T/\delta\Delta)^{1/2}$, $\exp(\delta\Delta/T) > ql\,(\delta\Delta/T)^{1/2}$. In this connection, the anisotropy of the pre-exponential factor turns out to be important.

The absorption in a lamina, γ_s^{lam}, having diffusely-scattering boundaries is described in the temperature range $\delta\Delta \lesssim T$ (it is precisely upon fulfilment of this condition that the temperature change of the gap turns out to be most noticeable) by the formula $\gamma_s^{lam} = 2n_0(\Delta_0)\gamma_n^{lam}$, where $\gamma_n^{lam} = \gamma_n^{bulk} d/l$, and Δ_0 denotes the value of the gap at the point $v_x = v_z = 0$ on the Fermi surface. This formula may be used in order to reconstruct the energy gap as a function of direction and temperature.

In the isotropic case, the formulas obtained in[84] go over into the usual relation (32). This is not accidental, since elastic scattering on impurities cannot be the relaxation mechanism for a system of electronic excitations which are changing their energy in the field of the sound wave. Therefore, in the isotropic case absorption in the presence of impurities turns out to be the same as in a free electron gas (see Sec. 1). In the anisotropic case, however, just as in the presence of overlapping energy bands, the picture becomes substantially more complicated since upon scattering by impurities the electrons may go into another band or into another part of the Fermi surface. An intermingling of the corresponding ψ-functions occurs. The change of the electronic states affects the picture of sound wave absorption, which also leads to relationships which differ from formula (32) of the usual BCS theory.

c) **Ultrasonic absorption in superconductors with overlapping energy bands.** In the presence of overlapping energy bands the absorption of ultrasound is described by the Hamiltonian $\mathscr{H} = \sum_{i,k,q} \lambda_i a_{k'i}^{+} a_{ki} b_q$, where $\mathbf{k}$, $\mathbf{k}'$ denote the electron momenta, and $\mathbf{q}$ is the phonon momentum.

For $\omega \lesssim \Delta_i(T)$ and for pure samples, the absorption coefficient turns out to be equal to[101-103]

$$\frac{\gamma_s}{\gamma_n} = \sum_i \frac{f_i}{\exp\left(\frac{\Delta_i}{T}\right)+1}\,. \tag{38}$$

The summation in Eq. (38) goes over all bands; the quantities f_i do not depend on the temperature. Processes in which the electron, absorbing a quantum of sound, goes from one band to another are not considered since such transitions are generally accompanied by a change of the electron momentum by an amount $\sim p_0$, which is impossible for the sound wave frequencies $\omega \lesssim \Delta$ under consideration. As $T \to 0$, as is evident from Eq. (38), the band characterized by the smaller energy gap introduces the main contribution to the absorption, which leads to a deviation from the simple exponential dependence and to a certain slowing down of the decrease of the function $\gamma_s(T)$. This effect of a deviation in the absorption-coefficient temperature dependence from an exponential dependence as $T \to 0$ is noted in a number of experimental articles. Thus, in[93] this phenomenon was observed in superconducting vanadium. An analogous slowing down of the decrease is noted in superconducting tin.[90] In accordance with[24], the introduction of impurities gradually leads to the disappearance of many-band effects, which is evident from Fig. 18, taken from[90].

It should be noted that the effect of anisotropy leads to smaller changes in the gap than the effect due to overlapping bands. In this connection, we note that in the same article[90] data is presented for the gaps Δ_1 and Δ_2, which apparently pertain to different energy bands: $2\Delta_1/T_c = 2.8$ and $2\Delta_2/T_c = 8$ (compare with the data for Nb and see Item b2 above). It is mentioned in[89] that a deviation from exponential dependence is observed for one of the orientations of the

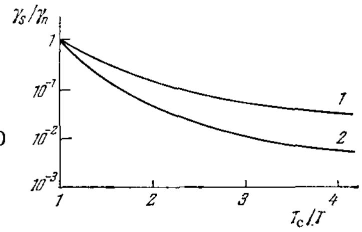

FIG. 18. The curves correspond to the following compositions: 1 – pure Sn, 2 – Sn + 0 0.1 at. % In.

sound wave with respect to the axes of An. In this connection we note that the effect of the many-band structure in Sn on tunneling experiments is mentioned in[104].

The absorption of longitudinal ultrasound in a two-band superconductor is investigated in[103] by using the method of correlation functions. In this connection the coefficients f_i appearing in Eq. (38) turn out to be given by

$$f_i = \frac{m_i^2}{\sum_{k=1,2} m_k^2} \qquad (i = 1, 2)$$

($m_{1,2}$ denote the effective electron masses for the first and second bands, respectively). In[105] a general formula is derived for the coefficient of ultrasonic absorption in superconducting alloys using a many-band anisotropic model. In this connection, it turns out that even in the presence of isotropic gaps in each of the bands, the absorption may turn out to be different depending on the direction of sound propagation relative to the crystallographic axes, provided that in connection with equality of the gaps the effective masses and the Fermi momenta are unequal in different bands.

d) **Ultrasonic absorption in strong-coupling superconductors.** Ultrasonic absorption in superconducting lead was experimentally investigated in[107]. It is noted that below T_c the attenuation falls off much more rapidly than is predicted by the usual theory of superconductivity. It is clear that this property is associated with the anomalous character of lead, which is a superconductor possessing a strong electron-phonon interaction. Similar effects were observed during investigations of Nb[108] and Hg[109], which are also anomalous superconductors. The noted property can be understood[110] on the basis of the theory of superconductors with strong coupling.[60-61]

The calculation is carried out on the basis of the method developed in[82] (see above, Sec. 1b). It is taken into account that the self-energy part $\Sigma(\omega_n, T)$, which appears in the Green's function and which describes the Cooper pairing, significantly depends on the "frequencies" ω_n ($\omega_n = (2n+1)\pi T$). This dependence has the form[60,61]

$$\Sigma(\omega_n, T) = C(T)\frac{\omega^2}{\omega^2 + \omega_n^2} + \Sigma' \qquad (\Sigma' \ll \Sigma),$$

where ω denotes the phonon frequency.

It turns out that even in the presence of strong coupling, the ultrasonic absorption is described by the formulas (31) and (32) of the usual theory (this fact is also mentioned in[111]). However, now the value of the gap corresponding to a given anomalous superconductor appears in these formulas. If the coefficient α in the formula

$$\frac{\Delta}{T}\Big|_{T\to T_c} = \alpha\left(1-\frac{T}{T_c}\right)^{1/2}$$

is equal to 3.06 in the usual case, then for Pb[60,61] $\alpha_{Pb} = 4$, which determines a more rapid decrease of the ultrasonic attenuation as the temperature is lowered below T_c. Experimental data[107] corresponding to measurements near T_c and a theoretical curve[110] constructed with the above considerations taken into account are shown in Fig. 19. It is clear that formula (32), with the dependence of Δ/T as $T \to T_c$ appropriate for lead, describes the experimental data cited in[107] quite well.

e) **Threshold phenomena.** The presence of a gap in the energy spectrum leads to the possibility of observing so-called threshold phenomena in superconductors.[112-114] If the absorption of a phonon of arbitrary frequency is possible in a normal metal, then in a superconductor at T = 0°K ultrasound will be attenuated only in the case when $\omega \geq 2\Delta(0)$. At finite temperatures the corresponding process of the production of a pair of excitations by a phonon turns out to be possible only if the phonon energy is not smaller than the threshold value, which is given by $\omega = 2\Delta(T)$. The presence of threshold effects in hypersonic absorption in Al was observed experimentally in[115].

According to[113], the threshold absorption at T = 0°K is described by the following general formula:

$$\frac{\gamma_s}{\omega} = \frac{\pi\tilde{g}}{(2\pi)^2 qv}\int_{-\infty}^{\infty} d\xi_+ \int_{\xi_+-vq}^{\xi_++vq} d\xi_- \frac{\varepsilon_+\varepsilon_- - \tilde{\xi}_+\tilde{\xi}_- + \Delta^2}{\varepsilon_+\varepsilon_-}\,\delta(\varepsilon_+ + \varepsilon_- - \omega),$$
$$\tilde{\xi}_\pm = \xi_{\mathbf{p}\pm\mathbf{q}/2}. \qquad (39)$$

Near threshold, when $\omega = 2\Delta[1 + (\eta/2)]$ $(\eta \ll 1)$ the *attenuation increases by a jump from zero to a finite* value $\tilde{g}u/8v$. Smearing of this jump of the attenuation takes place in an extremely small region of frequencies.

In the case when $\omega \gg 2\Delta$, one can neglect the quantity 2Δ in Eq. (39). Then, of course, the formula[116] describing the production of an electron-hole pair by a phonon in a normal metal is obtained.

In the anisotropic case, the threshold frequency ω_0, as one would expect, is determined by the minimum value of the gap on the line $\mathbf{q}\cdot\mathbf{v} = 0$. Here it turns out that near threshold the absorption $\gamma_s(\mathbf{n})$ is described by the law $\gamma_s(\mathbf{n}) \sim (\omega_0/v)(\omega - \omega_0)^{1/2}$.

Threshold effects are important at finite temperatures,[114] as already noted above, for frequencies satisfying the condition $\omega \geq 2\Delta(T)$. Thus, even sound quanta of small energy may decay into a pair of excitations in the temperature region close to the critical temperature. However, the contribution of these effects turns out to be small in comparison with the contribution due to processes involving the direct absorption of phonons by quasiparticles.[116] Threshold phenomena play an essential role in connection with the evaluation of the derivative $d\gamma_s/dT$ at $T = T_c$. According to Eq. (32), this derivative tends to infinity at the critical point. Taking account of the threshold absorption[114] leads to a finite value for it, which is observed experimentally.[85] For example, for $\omega < (u/v_F)T_c$

$$\frac{d\gamma_s}{dT}\Big|_{T=T_c} = \frac{1}{3}\gamma_n\frac{c^2}{\omega T_c}\left(\frac{s}{v_F}\right)^3 \quad (c = 3.05T_c^{1/2}).$$

In a normal metal, as is well known, the coefficient for the absorption of ultrasound does not depend on the temperature,[100] i.e., here the derivative under consideration turns out to be equal to zero. Therefore, a jump in the value of the derivative $d\gamma_s/dT$ takes place upon transition of the metal from the normal to the superconducting state. However, even at frequencies equal, for example, to 10^8 sec^{-1}, this jump turns out to be extremely small: $(T_c/\gamma_n)(d\gamma_s/dT) \sim 10^{-4}$. It is curious that in the case of high frequencies the derivative $d\gamma_s/dT$ turns out to be a negative quantity, which indicates an increase of the absorption coefficient associated with a reduction of the temperature near T_c.

With the threshold effects taken into account, the temperature dependence of the coefficient of ultrasonic absorption[114] turns out to differ from formulas (32) and (31), which were obtained under the assumption that $\omega \ll T$. For example, as $T \to 0$

$$\frac{\gamma_s}{\gamma_n} = \exp\left(-\frac{\Delta}{T}\right)\sqrt{\frac{2\pi\Delta T}{\omega(\omega-2\Delta)}} \quad (\omega \gg T).$$

The coefficient for the absorption of sound of rather high frequency may increase as the temperature is decreased to a value $T_{thresh.}$, which is determined by the equation $\omega = 2\Delta(T_{thresh})$. Such an increase in the hypersonic absorption at $\omega \sim 3\times10^{10}$ Hz is noted in superconducting indium in[117].

In[118] it is shown that at frequencies $\omega \sim 2\Delta$ and for sound propagation velocities close to v_F (the latter condition may be fulfilled in superconducting semiconductors), one should expect a noticeable anomaly in the sound dispersion law. Here the wavelength λ of the sound becomes comparable with the size ξ_0 of a Cooper pair. The problem of the dispersion of sound in superconducting metals is discussed in[119-121]. In[122] it is shown experimentally, in agreement with theory,[119] that no essential changes in the dependence on $\omega(\mathbf{q})$ were observed for $\mathbf{q}\cdot\mathbf{v}_F \ll \Delta$ ($\mathbf{q}$ denotes the wave number).

2. Absorption of Long Wavelength Sound

Above we have talked about two limiting cases which should be distinguished upon considering the problem of sound absorption. Up to now we have basically investigated the case $\omega \gg \tau^{-1}$ (ultrasound).

The absorption of long wavelength sound,[17,18] whose frequency satisfies the condition $\omega \ll \tau^{-1}$, is determined in the isotropic case by the scattering of electronic excitations by thermal phonons. We shall regard the sound field as a factor which deforms the lattice. The irreversibility of the deformation process

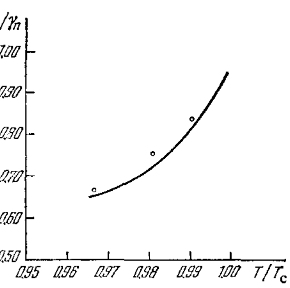

FIG. 19

leads to the absorption of sound energy. The problem thus reduces to the solution of the appropriate kinetic equation and the subsequent calculation of the dissipative function which determines the absorption of the sound wave.

The kinetic equation for the distribution function f of the electronic excitations in a superconductor, situated in a sound field and interacting with phonons, has the form

$$-\left(\frac{\partial f}{\partial t}\right)_{\mathrm{ac}}=\sum_{\mathbf{q}}|V|^2(u_{\mathbf{k}}u_{\mathbf{k}'}-v_{\mathbf{k}}v_{\mathbf{k}'})^2\{[f'(1-f)(N+1)-f(1-f')N]\,\delta(\varepsilon'-\varepsilon-\omega)+[f'(1-f)N-f(1-f')(N+1)]\,\delta(\varepsilon-\varepsilon'-\omega)\}+|V|^2(u_{\mathbf{k}}v_{\mathbf{k}'}+u_{\mathbf{k}'}v_{\mathbf{k}})^2[N(1-f)(1-f')-(N+1)ff']\,\delta(\varepsilon'+\varepsilon-\omega),$$

where N denotes the number of phonons of frequency ω.

Upon switching on the sound field, the electron turns out to be in a lattice with a somewhat changed constant, which leads to a dependence of its momentum on the deformation tensor.[99] Therefore

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{ac}}=\frac{\partial f}{\partial \varepsilon}\frac{\xi}{\varepsilon}\varepsilon_{ik}(\mathbf{k})u_{ik} \qquad (40)$$

(u_{ik} is the strain tensor, and ϵ_{ik} is a tensor depending on the direction of **k**). The collision integral, of course, has the same form as in the problem of the electronic thermal conductivity of a pure superconductor (see above, Ch. II, Sec. 3). A spherical Fermi surface is considered.

As usual, we seek the distribution function in the form

$$f=f_0+g(\varepsilon,\,\Omega),$$

where $f_0=[\exp(\epsilon/T)+1]^{-1}$, and $g(\epsilon,\Omega)$ is a function which depends on the energy of the electronic excitations and on the angles determining the direction of their motion. Let us represent it in the form of an expansion in terms of Legendre polynomials and, by integrating over the angles, we find the correction to the distribution function. Then let us evaluate the dissipative function $T\dot{S}$ (S denotes the entropy of the gas of electronic excitations). The absorption coefficient turns out to be a quantity which decreases with reduction of the temperature below T_c according to the law

$$\gamma_s=\gamma_n\frac{4\Phi(T_c)}{(e^b+1)^2\,\Phi(T)}, \qquad (41)$$

where $\gamma_n=\mathrm{const}\cdot T^{-5}$ is the sound absorption coefficient in the normal metal;[99] $\Phi(T)$ is determined by formula (8). The temperature dependence of the attenuation (41) is due to a decrease in the number of electronic excitations as $T\to 0$. One can show[99,17] that impurities do not play a role in the process under consideration. This is natural since, as we already mentioned above, elastic scattering in the present case cannot lead to the establishment of equilibrium.

Sound energy may also be absorbed by phonons interacting with the electronic excitations in a superconductor. In this case the relaxation mechanism will be the same as that used in connection with an investigation of the lattice thermal conductivity of superconductors (see above, Ch. II, Sec. 4). The solution of the corresponding kinetic equation for phonons located in a sound wave field, and the subsequent calculation of the dissipative function enable one to determine the temperature dependence of the absorption coefficient γ_{pe}.[17] The absorption of sound by phonons turns out to increase as the temperature is lowered, which is associated with an increase in the mean free path of the phonons as a consequence of a decrease in the number of electronic excitations in the superconductor. This mechanism plays a role at temperatures not too close to T_c.

3. Absorption of Sound in the Intermediate State

The absorption of sound in the intermediate state is characterized by a number of essential features. First of all one should note the additional mechanism for the absorption of a sound wave, investigated in[123] and associated with the presence of alternating layers of the normal and superconducting phases, which is characteristic of the intermediate state of a system.

As is well known, the critical magnetic field depends on the pressure and temperature of the sample. Therefore its value changes in the presence of a sound wave. The magnetic field in the normal layers is equal to the critical field, and therefore a change of the latter leads to a movement of the boundary between the phases. In this connection, a variable magnetic field and the eddy currents associated with it appear in the normal layers. The Joule heat given off in this connection represents an additional mechanism for dissipation of the sound-wave energy.

In the case under consideration, one can write the critical field H_c in the form

$$H_c=H_{c0}(1+\alpha u_{ii})+\frac{\partial H_c}{\partial T}T';$$

here H_{c0} denotes the critical field in the absence of sound, $\alpha\approx 1$, $u_{ii}=\mathrm{div}\,\mathbf{u}$, $\mathbf{u}$ denotes the displacement of the lattice in the sound wave, $\mathbf{u}=\mathbf{u}_0\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$, T' denotes the change in the temperature which arises because of the generation or absorption of heat associated with the motion of the boundaries between the phases.

The mechanism described turns out to be essential in connection with an investigation of the attenuation of low-frequency sound, whose wavelength satisfies the conditions $\lambda\gg\delta$ and $\lambda\gg a_n$ (δ denotes the depth of the skin-layer and a_n denotes the thickness of the normal layers).

A general formula for the absorption coefficient is obtained in[123]. In the limiting case of very small frequencies, when $a_n\ll\delta$, the absorption for $T\ll T_c$ is described by the formula

$$\gamma=\frac{a_n^3[\alpha+1-(\mathbf{mn})^2]^2}{24\,(a_s+a_n)\,\rho\sigma u_e^3\delta^4}\left(\frac{cH_c}{2\pi}\right)^2, \qquad (42)$$

where **m** and **n** denote unit vectors in the direction of H_{c0} and **k**, ρ is the density of the metal, a_s is the thickness of the superconducting layers, u_l is the velocity of longitudinal sound, and σ denotes the static conductivity; it is assumed that the mean free path of the electrons is much smaller than a_n or δ, which also permits one to use σ. It is clear that in this case the absorption is proportional to the square of the sound wave's frequency ω^2 and is practically independent of temperature. This dependence becomes important when T is raised.

In the opposite limiting case, $a_n \gg \delta$, the absorption

$$\gamma = \frac{4\,[\alpha+1-(mn)^2]^2}{\rho\sigma\,(a_n+a_s)\,u_e^3\delta}\left(\frac{cH_K}{8\pi}\right)^2 \tag{43}$$

turns out to be proportional to $\omega^{1/2}$.

The ratio of the coefficient for the absorption of sound, associated with the mechanism under consideration, to the usual electronic absorption turns out to be a quantity of the order of $(e^2/c)(v_F/c)(\alpha/\xi_0)^2$ ($a = a_n + a_s$). This ratio may be larger than unity thanks to the factor a/ξ_0. Then the mechanism under consideration for the absorption of low-frequency sound, which is associated with the motion of the phase boundaries, begins to play a major role. In this connection the frequency dependence of the absorption turns out to be extremely distinctive. In the region of very small frequencies, the attenuation is proportional to ω^2. Then with an increase of the frequency a region appears (for $a_n \gg \delta$) where the absorption is proportional to $\omega^{1/2}$. At large frequencies, where the absorption is determined by the usual mechanism,[100] a dependence $\gamma \sim \omega^2$ again appears. The good agreement of the obtained results with formula (43) is mentioned in article[124] where the absorption of sound whose frequency varied in the interval from 21 to 90 MHz was experimentally investigated. The sound wave was absorbed in pure superconducting lead, existing in the intermediate state.

Above we have discussed the absorption of low-frequency sound. The attenuation of high-frequency sound vibrations by superconductors existing in the intermediate state is studied in[125]. In this connection the wavelength of the sound wave turns out to be of the same order of magnitude as the thickness of the layers. Also the conditions $a_n \ll l$ and $a_n \ll R$ (R denotes the Larmor radius in the critical magnetic field) are assumed to be satisfied. In[125] it is shown that in this case it is possible, with the aid of ultrasonic measurements, to determine the period of the laminar structure of the superconductor under investigation. It is shown that both for $R \gg l$ as well as for the opposite limiting case $R \ll l$ the monotonic part of the absorption is described by a function directly relating the period of the structure with the value of the critical magnetic field. For $R \ll l$, in addition, there is an oscillating correction to the absorption (the analogous effect in normal metals is mentioned in[126]), where the period of the oscillations also turns out to be related to the desired quantity $a = a_n + a_s$.

4. Absorption of Transverse Sound Waves

The attenuation of transverse waves in superconductors has been studied to much less extent than the absorption in them of longitudinal sound waves. The distinctive features of the attenuation of transverse sound are mentioned in articles[127,128], where Sn and In are investigated, and in[129-131] where the corresponding properties of Al are examined. The presence of two regions, differing sharply in their temperature dependences, is noted. In the first of these regions, existing immediately close to T_c, an extremely sharp drop in the absorption is observed (Fig. 20). Thus, for example, in[127] upon a total change in the temperature below T_c of only 0.01° the absorption of transverse sound of frequency $\omega \sim 30$ MHz decreases by a factor of two. After this abrupt drop, the temperature dependence is well described by the usual formula (32) of the BCS theory.

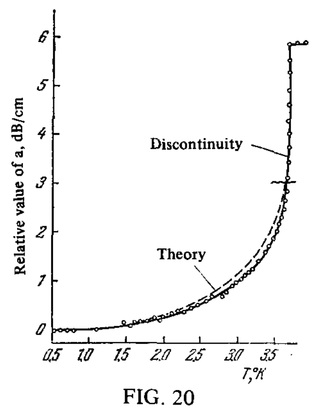

FIG. 20

In[7,132-134], the abrupt drop in the absorption is attributed to screening by superconducting currents associated with the field of the transverse sound wave.

Subsequent investigations have shown, however, that the mentioned effect is not universal. Thus, in[135] it is noted that the attenuation of transverse sound in Nb and Ta is not characterized by anomalies, and its temperature dependence over the entire temperature range is completely analogous to the dependence which describes the absorption of longitudinal sound. Investigation of pure Ga ($ql > 100$)[136] showed that a region of sharp decrease in the absorption is observed for a number of directions of sound propagation, but for other directions such an anomalous drop does not occur.

In[137] it is shown that in the region of high sound frequencies the absorption coefficient and the dispersion of the velocity of propagation of transverse sound to a considerable extent are determined by the contribution of the macroscopic electric fields. At frequencies $\omega \lesssim 10^8$ Hz these fields generally do not play a role (see also[138]), and the absorption, as usual, is related to a direct deformation interaction.

A calculation of the coefficient for the absorption of transverse sound[139] indicated that in a number of cases essential features of the absorption under consideration should be observed. For example, for values of the parameters satisfying the inequalities $\omega \ll T \ll \Delta \ll qv$, the temperature dependence of the absorption in Pippard superconductors is described by the formula

$$\gamma = \frac{8Nmv_F\omega}{3\pi^2\rho_m v_s}\,\frac{\Delta}{T}\exp\left(-\frac{\Delta}{T}\right)\left(\frac{\omega^2 mc^2 v_F}{3\pi^2 Ne^2 v_s^3}+\frac{\pi\Delta}{\omega}\right)^{-2}, \tag{44}$$

where ρ_m denotes the density of the metal and v_s denotes the velocity of the superfluid component. This is of interest in connection with organization of the corresponding experiments.

5. Characteristic Features of Kinetic Phenomena in Type II Superconductors

As is well-known,[140] type II superconductors existing in a mixed state are characterized by the presence of a system of vortex lines. The superconducting order parameter $\Delta(\mathbf{r})$ vanishes on the axis of a vortex line. The vortex structure and the associated appearance of a normal component are additional factors which must be taken into consideration in connection with an investigation of kinetic phenomena in type II superconductors. The experimentally noted properties of heat transfer and sound absorption processes in the mixed state are also associated with the effect of these factors.

The behavior of the electronic thermal conductivity was theoretically investigated in[141,142]: κ_e varies near H_{c2} according to a linear law, $\kappa_e \sim (H - H_{c2})$. An investigation of impure samples of lead and indium is carried out in[143]; it is noted, in agreement with theory, that under conditions when the electronic contribution to the heat flow is the major contribution, the ratio of the derivatives $d\kappa/dH$ and dM/dH (M denotes the magnetization) is a universal function of the temperature. A monotonic increase of the thermal conductivity as a function of the field, varying in the interval between H_{c1} and H_{c2}, is noted in[144] where monocrystalline samples of disordered alloys of tantalum and niobium were investigated. The measurements were carried out at a constant temperature close to T_c. Here the electronic component gives the major contribution to the heat flow. Similar behavior is noted upon investigation of pure Nb.[145,146] An intrinsic anisotropy of κ_e, depending on the angle between the direction of heat flow and the direction of the vortex lines, is observed in[147]. The behavior of κ is utilized to determine the values of H_{c1} and H_{c2}.

The most significant anomalies are observed upon investigation of the phonon thermal conductivity of type-II superconductors. The presence of large-scale structures, which correspond to Abrikosov lines, leads to a noticeable decrease in the mean free path of the thermal phonons, and associated with this there is an abrupt drop in the value of κ_p. This drop in the thermal conductivity upon reaching the field value H_{c1} was observed in[144] in the low-temperature region where, as already noted above (see Ch. II, Sec. 5), phonon thermal conductivity plays a major role. Its decrease with increasing field also leads to the result that in the investigated superconductor the electronic thermal conductivity κ_e again begins to play a major role; however, the value of κ_e increases with increase of H. In virtue of the arguments presented here, it is clear that the thermal conductivity $\kappa(H)$ of type II superconductors, investigated as a function of the field which varies in the interval $H_{c1} < H < H_{c2}$, is characterized in the low-temperature region by the presence of a minimum. This minimum has actually been observed in[144-146] (Fig. 21).

The ultrasonic absorption coefficient in a "dirty" superconducting alloy ($\tau T_c \ll 1$, τ denotes the relaxation time) is calculated in[148]. It is assumed that the field is not too large ($H_{c1} < H \ll H_{c2}$). The work done by the sound wave on the electron gas is determined, where the wavelength of the sound is assumed to satisfy the conditions $ql \ll q\delta_0 \ll 1$, $qd \ll 1$, where d denotes the distance between vortex lines. In the hydrodynamical approximation the dissipation of energy is described by the following formula:

$$\overline{\langle \dot{H} \rangle^t} = \int dV\, [\overline{\eta_{iklm} \dot{u}_{ik} \dot{u}^t_{lm}} + \overline{\sigma_{ik} E_i E^t_k}], \tag{45}$$

where E_i is the effective electric field in the coordinate system associated with the lattice, σ_{ik} is the conductivity tensor, η_{iklm} is the tensor of the coefficients of viscosity, and u_{ik} is the deformation tensor; the brackets $\langle \ldots \rangle$ indicate thermodynamic averaging, and the bar indicates averaging with respect to time.

The first term in (45) represents the ordinary absorption of a sound wave by an electron system. One can represent the damping coefficient associated with it in the form

$$\frac{\gamma_s}{\gamma_n} = \frac{\gamma_s}{\gamma_n}\bigg|_{\mathrm{BCS}} + K(T)\frac{B}{H_{c2}}; \tag{46}$$

here B denotes the magnetic induction in the superconductor; the function $K(T)$ is expressed in a definite way in terms of the quantities κ and Δ/T; κ denotes the parameter appearing in the Ginzburg-Landau theory. At $T = 0°$ the value of $K(0) \approx 1$.

The basic characteristics of sound absorption in a mixed state are associated with the second term which appears in Eq. (45). It describes the Ohmic losses associated with the appearance of induced electric fields. Upon a displacement of the crystal lattice due to the passage of the sound wave, the vortex lines are dragged by the ionic system of the metal. However, the elastic interaction between the vortex lines leads to the result that this dragging is not complete. Induced electric fields also appear due to the motion of the system of vortex lines relative to the lattice ions. In this connection, an extremely strong anisotropy is observed in the absorption of both longitudinal and transverse sound. Thus, for example, the coefficient for the absorption of longitudinal sound turns out to be proportional to $\sim \sin^6\theta$, where θ is the angle between the vectors $\mathbf{q}$ and H_0. In the case when the sound is propagated in the direction of the field, deformation of the vortex lattice does not occur, and the corresponding absorption coefficient turns out to be equal to zero. Waves propagating in a direction perpendicular to the field are absorbed most strongly.

Ultrasonic absorption in the mixed state in the presence of paramagnetic impurities is considered in[149]. Articles[151-155] are devoted to theoretical and experimental investigation of the question of the attenuation of sound in pure type II superconductors. In[153] a linear dependence of $[1 - (\gamma^l_s/\gamma^l_n)]$ on $[H_{c2}(T) - H]^{1/2}$ is experimentally obtained, which is in agreement with

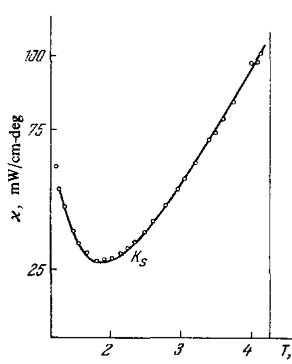

FIG. 21

theory,[151] where H denotes the external field, and γ^l denotes the coefficient for the absorption of longitudinal sound in pure Nb.

The presence of a vortex structure leads to the existence in type II superconductors of a type of phenomena similar to the effect of ordinary magnetoresistance in a normal metal. Also thermomagnetic and galvanomagnetic effects appear. The Ettingshausen effect[154-157] (the appearance of a temperature gradient perpendicular to the flowing current, where $\Delta T \sim (V_l/\kappa)(T/T_c)^2$;[155] here V_l denotes the longitudinal electric field intensity and κ is the coefficient of thermal conductivity), the longitudinal Nernst effect,[154,156,158] and the Hall effect[159-160] belong to this type. The generation of Peltier heat[155,157] is observed at the boundary separating the mixed ($H > H_{c1}$) and ordinary superconducting states.

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