

PROBLEMS OF THE ELECTRON THEORY OF METALS*
 IV. THERMODYNAMIC AND KINETIC PROPERTIES
 OF METALS IN A MAGNETIC FIELD

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THE second part of the present survey^[1] dealt with the thermodynamic and kinetic properties of conduction electrons in metals in stationary fields at quite high temperatures, and the third with their kinetic properties.^[2] However, a number of fundamentally important effects do not arise in this temperature range.

At extremely low temperatures, a metal which is neither a ferro- or antiferromagnet, because of the diamagnetic quantization of the electron orbits, is a strong dia- or paramagnet, with a magnetic susceptibility whose absolute value can be arbitrarily large (it can reach 10³ in magnetic fields of the order of 10 kOe). In the case of paramagnetism this leads to the appearance of a domain or periodic structure and the associated phenomenon of phase transitions.

In semimetals and semiconductors in very strong magnetic fields H, the "magnetic energy" μH (where μ is the Bohr magneton for a conduction electron) is the largest of the contributions to the electron energy ("ultraquantum-mechanical case"). As a result it becomes possible to have semimetal-intrinsic semiconductor and intrinsic semiconductor-metal transitions.

The kinetic behavior at low temperature exhibits quantum oscillations of the magnetoresistance (Shubnikov-de Haas effect), which were not treated in^[2]. In the classical domain in strong magnetic fields one can have a characteristic screening of a constant current, which is confined to a thin layer near the surface of the metal (static skin effect). At low temperatures this screening markedly affects all the kinetic characteristics of the metal and, in particular, leads to a new type of skin effect, which differs from the normal and anomalous skin effects in having a completely different kind of damping of the electric and magnetic fields.

The surface of a conductor plays a decisive role in the very special kinds of oscillations of surface impedance that occur in very weak magnetic fields,^[45] caused by magnetic surface levels.

This paper is devoted to the phenomena enumerated above, which were discovered during the past few years. The presentation is almost always such that a knowledge of the earlier parts of the survey is not required. In order not to make reading difficult, we have used a minimal number of references, so that the references do not give a complete bibliography.

*This paper is the fourth part of a survey; the preceding three parts were published earlier.^[1-3]

I. THERMODYNAMICS

1. Physical Nature of Quantum Oscillations and Strong Magnetism

It is well known (cf, for example, ^[3]) that an electron in a constant magnetic field H || z axis moves in a helix with its axis along the magnetic field, so that its motion in the plane perpendicular to the magnetic field is periodic with a frequency equal to the cyclotron frequency Ω. To a periodic motion in classical mechanics there corresponds discrete levels in quantum mechanics where, according to the correspondence principle,* in the quasiclassical case the level separation is related to the classical frequency by the equation:

$$\Delta\epsilon = \hbar\Omega \propto B, \tag{1.1}$$

where B is the magnetic induction.

The classical frequency Ω is (cf., for example, ^[3])

$$\Omega = \frac{eB}{m^*c}, \tag{1.2}$$

where e is the electron charge, c is the velocity of light, and m* is the effective mass of the electron:

$$m^* = \frac{1}{2\pi} \frac{\partial S}{\partial \epsilon}, \tag{1.3}$$

(S is the area of the section of the surface ε(p) = ε by the plane p_z = const, p is the quasimomentum and ε is the energy of the electron). From (1.1)-(1.3),

$$\Delta S = \frac{ehB}{c},$$

so that

$$S(\epsilon, p_z) = \frac{ehB}{c} n, \tag{1.4}$$

where n is an integer.† It is important to emphasize

*In the general case the correspondence principle follows from the following simple arguments. The resonance frequency is equal to Ω and to Δε/ħ in the classical and quantum cases respectively. But as ħ → 0 this must be one and the same frequency, so that in lowest approximation Δε/ħ ≈ Ω and Δε ≈ ħΩ. For an arbitrary dispersion law, such a relation, which implies the continuity of Δε/ħ as a function of ħ, is non-trivial (since ħ appears as a parameter in the highest derivative ∂/∂r) and can be regarded as a physical limitation on the possible form of the dispersion law. Even in the simplest case of a free electron in a periodic field the relations (1.1) are by no means simple (cf. ^[50]).

† Knowing only the classical Hamiltonian, one can obtain the energy levels not only in the lowest approximation but also in the next approximation, and also describe the wave functions. The point is that a hermitian operator is determined by the corresponding classical quantity to order ħ² inclusive, since different ways of symmetrizing give differences of order ħ² (cf. ^[20]). This is related to the fact that [a,b] = ih {a,b} where {...} is the classical Poisson bracket, and h occurs always accompanied by i.

that (1.4) was obtained from (1.1) and thus, according to the derivation, correctly determines the lowest approximation not only for S but also for ΔS . (Otherwise, in the quasiclassical case, where by definition $n \gg 1$, Eq. (1.4) as the lowest approximation would simply be a truism.)

According to (1.4) the energy levels are infinitely degenerate in the generalized quasimomentum P_x . This is natural physically: P_x determines the position of the center of the orbit in the crystal lattice (since $p_x = P_x + (eB/c)y$ and $y(t) = (p_x(t) - P_x)/(eB/c)$, t is the time of revolution in the orbit), which is unimportant, since the interatomic spacing a in the quasiclassical case is small compared to the Larmor radius r , and we are interested only in the lowest approximation in a/r ; P_x is an integral of the motion in this same approximation, since the vector potential was taken in the form $(-Hy, 0, 0)$.

Let us explain why the magnetic induction B , rather than the magnetic field intensity, appears in the basic formula (1.4). The problem of the behavior of the charged quasiparticles—the conduction electrons—in a magnetic field is a typical problem of field theory for the behavior of a system of free charges in vacuum (in the present case, the vacuum for the quasiparticles). Thus, strictly speaking, we should consider only the microscopic magnetic field produced at a given point by all the charges, moving in orbits whose radii are of the order of the Larmor radius r . If the distance a between the charges is small compared to r , as is usual in metals (where $a \sim 10^{-8}$, while r for $B \sim 10^4$ Oe is $\sim 10^{-3}$ cm), the microscopic magnetic field at a point is given by $4\pi(r/a)^2(l/a) \gg 1$ for the electrons, and thus is "compressed" (l is the mean free path of the electron).

By definition the magnetic induction B is such an averaged selfconsistent field; it alone is felt by each of the electrons, and it alone will determine the magnetic moment M . (This was first pointed out by D. Shoenberg^[7] and later proved by A. Pippard.^[8]) It is understood that the connection between M and B is nonlocal: M at a given point is expressed in terms of the values of B at all points at a distance $2r_{\max}$ from the given point. (The fact that it is B that appears in all the formulas is natural, since $p = P - eA/c$, while $\text{curl } A = B$, since $\text{div } B = 0$.)

Let us first consider the main features of the energy spectrum in a magnetic field for uniform B . The dependence $\epsilon_n(p_z)$ given by (1.4) is shown in Fig. 1; the different curves correspond to different n . From Fig. 1 we see how the quantization in the magnetic field changes the electron spectrum. Because of the presence of the continuous parameter p_z the spectrum remains continuous, but in general the limit of the spectrum shifts. (Thus, if we formally neglect the spin, for a free electron $\epsilon = (n + 1/2)\hbar\Omega + (p_z^2/2m)$, and ϵ_{\min} is equal to $\hbar\Omega/2$ and not to zero as for $B = 0$.) But, as the energy increases, there are abrupt changes in the degree of degeneracy, since new "branches" of the spectrum appear. (For example, in Fig. 1 there are four branches when $\epsilon = \epsilon_0^{(1)}$, while there are five when $\epsilon = \epsilon_0^{(2)}$.) This leads to singularities in the density of states $\nu(\epsilon)$. Near extremal values of the energy ϵ_i , $\nu(\epsilon) \approx |\epsilon - \epsilon_i|^{-1/2}$. (To obtain $\nu(\epsilon)$ we must write the density of states in

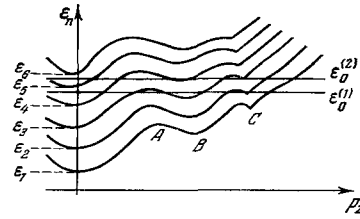


FIG. 1

the interval dp_z . Since the number of states is conserved when the magnetic field is switched on, even in the case of a Fermi liquid (see below), to each spin projection there corresponds a number of states $dp/h^3 = dp_x dp_y dp_z/h^3 = dS dp_z/h^3 \rightarrow \Delta S dp_z/h^3$. Using (1.4) this gives the Landau formula $dN = (eB/ch^2) dp_z$.)

Let us consider any effect that is determined at absolute zero (as is usually the case) only by electrons with energies equal to the limiting Fermi energy ϵ_0 (cf. Fig. 1). Since the total number of electrons in metals is conserved, and the spectrum changes in the magnetic field, ϵ_0 must itself depend on magnetic field. But the change in ϵ_0 is of the order $\Delta\epsilon$, while the changes of interest to us are, as we shall show in this section, proportional to $(\Delta\epsilon)^{1/2}$, so that because of the smallness of $\Delta\epsilon/\epsilon_0 \ll 1$, we can disregard the change in ϵ_0 .

With increasing magnetic field, since $\Delta\epsilon \sim B$, at some value of B one of the branches (or part of one of the branches, if there are points of type A or B in Fig. 1, where ϵ is an extremal) "quits the game," so that $\epsilon = \epsilon_0$ will correspond to a singularity of $\nu(\epsilon)$. It is clear that this also leads to a singularity in any physical quantity. Such a singularity is repeated every time another branch, or part of one, "disappears" (and also when the limiting Fermi energy passes through a singularity in the spectrum, for example, a point like C), i.e., when the number of branches having the energy ϵ_0 changes by one. We thus find from (1.4) the period of the oscillations (in the reciprocal of the magnetic field):

$$1 = \Delta n = \Delta \left(\frac{cS}{ehB} \right) = \frac{cS}{eh} \Delta \left(\frac{1}{B} \right),$$

$$\Delta \left(\frac{1}{B} \right) = \frac{cS}{eh}.$$

From our earlier remarks it is clear that S should be taken at the point $\epsilon = \epsilon_0$ and $p_z = p_0$, where p_0 corresponds to either an extremum of $\epsilon(p_z)$ or to points of the type of C. It is convenient to transform the equation $\epsilon'_n(p_z) = 0$ in the following way (using (1.4)):

$$\left(\frac{\partial \epsilon}{\partial p_z} \right)_n = \frac{\partial(\epsilon, n)}{\partial(p_z, n)} = \frac{\partial(\epsilon, S)}{\partial(p_z, S)} = \frac{\partial(\epsilon, S)}{\partial(\epsilon, p_z)} \bigg/ \frac{\partial(p_z, S)}{\partial(\epsilon, p_z)} = - \left(\frac{\partial S}{\partial p_z} \right)_\epsilon \bigg/ \left(\frac{\partial S}{\partial \epsilon} \right)_{p_z}.$$

From this we see that an extremum of $\epsilon_n(p_z)$ is reached at the point where $S'(p_z) = 0$ ($S = S_{\text{ext}}$), i.e.,

$$\Delta \left(\frac{1}{B} \right) = \frac{cS_{\text{ext}}(\epsilon_0)}{eh}. \tag{1.5a}$$

Formally $\epsilon'_n(p_z) = 0$ can also occur, according to (1.5), when

$$\frac{\partial S}{\partial \epsilon} = 2\pi m^* = \pm \infty,$$

which corresponds to singular sections that are self-intersecting, or to the "boundary" sections between open and closed sections. Such sections also "appear"

in the oscillations and contribute "their" area S_p :

$$\Delta \left(\frac{1}{B} \right) = \frac{cS_p(\epsilon_0)}{eh}. \quad (1.6)$$

Let us estimate the amplitude of these oscillations (for simplicity in the case of extremal sections). Since one branch of the spectrum "drops out," the change in the physical quantity is in general caused by the electrons in an interval Δp_z corresponding to $\Delta n = 1$ (for fixed B). Near the extremal section, we find from (1.4),

$$S''(p_0)(\Delta p_z)^2 = \frac{ehB}{e},$$

i.e.,

$$\Delta p_z \approx \left(\frac{ehB}{cS''(p_0)} \right)^{1/2} \approx p_0 \left(\frac{ehB}{cS} \right)^{1/2},$$

where p_0 is the characteristic size of the Fermi surface in the direction of p_z .

Thus the relative magnitude of the effect is proportional to $d = \Delta p_z/p_0$, i.e.,

$$d \approx \left(\frac{ehB}{cS} \right)^{1/2}. \quad (1.7)$$

In the case of a thermodynamic quantity, we should also consider that the magnetic moment has a purely relativistic origin (since the change of the quasimomentum in the magnetic field is determined solely by the relativistic term eA/c). Since the sign of the velocity v of the charges is irrelevant, the additional factor is of order

$$b = \left(\frac{v}{c} \right)^2. \quad (1.8)$$

So far we have been discussing the case of zero temperature $T = 0$ and infinite mean free path of the electrons $l = \infty$ (where the concept of energy levels of the electron has a rigorous meaning). Let us now turn to the case of finite l and T .

If $l \gg 2\pi r$, between collisions the electron can carry out a large number of revolutions and thus exhibit the periodicity of its motion; the natural width of the level \hbar/τ is small compared to the level spacing $\hbar\Omega$, and in the lowest approximation in $(\Omega\tau)^{-1}$ the amplitude of the quantum oscillations is independent of l . But if $l \ll 2\pi r$, most of the electrons do not succeed in completing even a single orbit, so they move aperiodically, the natural width of the level "covers" a large number of levels, and to say which of the branches passed through $\epsilon = \epsilon_0$ is meaningless. However, there are electrons that traverse paths of order $2\pi r$ and to which all the previous arguments are applicable. These are the electrons that make the oscillations possible; the amplitude of the oscillations is proportional to the number of such electrons, i.e.,

$$\exp \left(-\frac{2\pi r \alpha}{l} \right) \approx \exp \left(-\frac{2\pi \alpha}{\Omega\tau} \right) = d_1,$$

where the constant $\alpha \sim 1$ and cannot be determined from qualitative arguments. The exact calculation (cf. [4]) shows that $\alpha = \pi$ and

$$d_1 \approx \exp \left(-\frac{2\pi^2}{\Omega\tau} \right). \quad (1.8a)$$

Finite temperature has a fundamental effect on the oscillation amplitude; the effect is no longer determined only by $\epsilon = \epsilon_0$ if the energy smearing covers several levels:

$$T > \hbar\Omega. \quad (1.9)$$

We associate with the temperature an "effective lifetime" of the level (during which for finite T an electron can reach or leave the level):

$$\frac{\hbar}{\tau_{\text{eff}}} = T, \quad \tau_{\text{eff}} = \frac{\hbar}{T}. \quad (1.10)$$

Then making somewhat formal use of (1.8a), we obtain the correct result (found by direct calculation in [4,5]) for the dependence of oscillation amplitude on temperature:

$$d_2 \approx \exp \left(-\frac{2\pi^2 T}{\hbar\Omega} \right). \quad (1.11)$$

(Naturally, since the fundamental parameter is still $\hbar\Omega/\epsilon_0$, which tells us how quasiclassical the situation is, we should take Ω in (1.11) at $\epsilon = \epsilon_0$, $p_z = p_0$.)

Summarizing all the estimates and taking a "magnetic energy" of order $B^2/8\pi$, we find the order of magnitude of the magnetic correction $\delta\theta$ to the thermodynamic potential θ (singular sections are not included for simplicity; in any case as one can easily see from the estimate of the corresponding Δp_z , their contributions are much smaller):

$$\delta\theta = \sum_{\alpha} \frac{B^2}{8\pi} \left(\frac{v}{c} \right)^2 \left(\frac{\hbar\Omega}{\epsilon_0} \right)^{1/2} \exp \left(-\frac{2\pi^2}{\Omega\tau} - \frac{2\pi^2 T}{\hbar\Omega} \right) f_{\alpha} \left(\frac{cS_{\text{ext}}^{\alpha}}{ehB} \right) \ll \frac{B^2}{8\pi},$$

$$\frac{cS_{\text{ext}}^{\alpha}}{ehB} \sim \frac{\epsilon_0}{\hbar\Omega}, \quad |f_{\alpha}(x)| \approx 1, \quad (1.12)$$

where the sum is taken over all extremal sections of all zones, and $f_{\alpha}(x)$ is an oscillating function with a period of order 1. (This estimate can of course be gotten from the formulas of [4,5], if we consider that $e^2/\hbar \approx e^2 a/\hbar a \approx e^2/ap \approx \epsilon/p \approx v$.)

We note that the factor $(v/c)^2$ has a purely quantum mechanical origin, since $v \approx \hbar/ma$ (this follows immediately from the indeterminacy principle $\Delta p \cdot \Delta x \approx \hbar$). Thus $B^2(v/c)^2$ can be written in the more "usual" form $n\hbar\Omega \cdot \hbar\Omega/\epsilon_0$, where $n\hbar\Omega$ is the energy of the "completely oriented" diamagnetic moments (n is the total charge density), while $\hbar\Omega/\epsilon_0$ is the fraction of the moments that are oriented by the field B .

Before proceeding to further discussion, we also note that formula (1.4) determined only the kinetic energy of the charges. But the total energy of an electron in a magnetic field, including the spin paramagnetism, is

$$\epsilon_{\sigma}(p_z, n) = \epsilon_n(p_z) + \sigma \mu_0 B, \quad \mu_0 = \frac{e\hbar}{2m_0c}, \quad \sigma = \pm 1, \quad (1.13)$$

where $\epsilon_n(p_z)$ is given by (1.4) and m_0 is the mass of a free electron.

Let us discuss in a little more detail the question of why the charge, spin and paramagnetic moment of a quasiparticle—the conduction electron—turn out to be the same as for the free electron. The corresponding total quantities—rigorous integrals of the motion—for the whole ensemble of particles and excitations obviously coincide. It is also clear that the total number of states does not depend on the interaction, [6a] since the number of states is the number of linearly independent wave functions of the system, i.e., the number of basis vectors, which depends only on the dimensionality of the corresponding space, and is unchanged if the number of particles is conserved. (For the case of a finite number of states and Fermi statistics the con-

servation of the number of states is physically obvious: if all the states are occupied, a reduction in their number when the field is switched on or off would require a reduction in the number of particles.) If the corresponding interactions, for example, spin-orbit interaction, are small (in good metals with a number of free electrons per atom of order 1 this is natural: the only characteristic energy* is $e^2/a \approx \hbar^2/2ma^2 \approx \epsilon_0$, while the spin-orbit interaction contains the additional small parameter v/c), so that in lowest approximation the charge, spin and paramagnetic moment, for the branches of the spectrum of interest to us, do not depend on the dynamic characteristics of the quasiparticles and we get our result. (This argument is not applicable to semimetals: there still another small parameter appears—the number of electrons per atom. The effect of the strong spin-orbit interaction is that the spin and the momentum do not commute, σ itself cannot be taken as an integral of the motion, and formula (1.13) is meaningless. We note in passing that the coulomb interaction of quasiparticles, in contrast to the coulomb interaction of bare electrons, is small because of the smallness of the number of excitations and is proportional to T^2 .)

So far we have discussed effects related to the finite separation of levels with a given p_z , i.e., with $\Omega \neq 0$ in formula (1.1).

When $\Omega = 0$ the lowest quasiclassical approximation corresponds to a continuous spectrum. Naturally the transition from the discrete spectrum to a continuous one, and the associated singularities, are of interest.

The vanishing of Ω , i.e., the increase to infinity of the period of revolution (in the plane perpendicular to the magnetic field) can obviously have two origins. If the orbit is finite it can occur if in approaching some point on the orbit in the x, y plane, the velocity goes to zero sufficiently rapidly, i.e., is parallel to the magnetic field. Since the velocity is along the normal to the surface $\epsilon(\mathbf{p}) = \epsilon$, the corresponding point can be either of type A (Fig. 2a), in which case $\Omega \neq 0$, or it can correspond to the saddle point B and a self-intersection of the orbits. Then for a given p_z , $\Omega(\epsilon) = 0$ at an isolated point ϵ , where there is a transition from two sections with, correspondingly, two branches of Ω and the energy levels, to one section with double the number of levels (the transition is examined in detail in [6a, 19, 20]).

The second reason why motion in the orbit may not be periodic is that the orbit may be infinite in extent. It is clear that then $\Omega = 0$ fills the whole energy range, and the transition to $\Omega = 0$ corresponds to the transition from closed to open orbits, i.e., to self-intersecting trajectories (curve I in Fig. 2b). Since there is an exponentially small probability of tunneling from one closed orbit (α) to another (β), interaction of the orbits occurs. The discrete levels are smeared out over a band of relative width of order $\exp(-|\epsilon - \epsilon_S|/\hbar\Omega)$, where $\epsilon = \epsilon_S$ corresponds to self-intersection.

For motion in open orbits, in an exponentially narrow energy interval there is a resonance probability (also proportional to $\exp(-|\epsilon - \epsilon_S|/\hbar\Omega)$ of total internal reflection from the "low barrier." This results in exponentially narrow forbidden bands in the continuous spectrum.

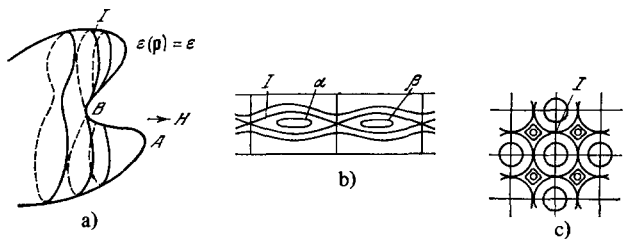


FIG. 2

Thus, as we approach the energy for self-intersection, ϵ_S , from the closed orbit side, the levels of the discrete spectrum are broadened into bands; then as we move away from ϵ_S , the "chinks" between the bands narrow and tend to zero. The parameter corresponding to the smearing of levels or of forbidden bands of the continuous spectrum is P_X ; it is a reflection of the non-equivalence (in the next approximation in a/r) of different positions of the orbit centers in the lattice. (For details on the nature of the spectrum, cf. [6b, c].)

If the xy plane is a symmetry plane, the orbits with self-intersection run through the whole lattice in this plane (Fig. 2c). In this case the inclusion of the interaction between orbits shows the importance of the commensurability of the field-dependent de Broglie wavelength and the period of the crystal lattice. Near the self-intersection, where the effect is nonexponentially small, the character of the spectrum is determined by the expansion of the reduced reciprocal magnetic field h^{-1} (for a square lattice, $h^{-1} = 2\pi c(eB\hbar a^2)^{-1}$, where a is the lattice period) in continued fractions: $h^{-1} = s + 1/[s_1 + 1/(s_2 + \dots)]$. The spectrum consists of s levels, each split into s_1 sublevels; each sublevel is in turn split into s_2 sublevels, etc. The separation of levels, sublevels, etc, oscillates with B^{-1} with a universal period. Each rational point of h^{-1} is singular for an infinite mean free path l ; finite l leads to a smearing out of this complex system of levels. It should be remembered that the total width of the split Landau level is of order $\exp(-|\epsilon - \epsilon_S|/\hbar\Omega)$. (The character of the spectrum is described in more detail in [19].)

The study of the spectrum in the region of "interaction" of orbits is extremely important for the study of a variety of quantum effects, for example for studying magnetic breakdown.

2. Physical Reason for Domain and Periodic Diamagnetic Structures. Character of Phase Transitions

In the preceding section we obtained the estimate (1.12) of the diamagnetic contribution to the thermodynamic potential. Considering that $\epsilon_0/\hbar\Omega \gg 1$, and that consequently $f_\alpha(B^{-1})$ is a rapidly oscillating function, we find for the magnetic moment M the formula

$$M = -\frac{\delta\theta}{\delta B} \approx \sum_{\alpha} B \left(\frac{v}{c}\right)^2 \left(\frac{\epsilon_0}{\hbar\Omega}\right)^{1/2} \exp\left(-\frac{2\pi^2}{\Omega\tau} - \frac{2\pi^2}{\hbar\Omega}\right) f'_\alpha\left(\frac{eS_{\text{ext}}^\alpha}{ehB}\right). \quad (2.1)$$

Then the magnetic susceptibility $\chi = -\delta M/\delta B$ is

$$\chi \approx \sum_{\alpha} \left(\frac{v}{c}\right)^2 \left(\frac{\epsilon_0}{\hbar\Omega}\right)^{3/2} \exp\left(-\frac{2\pi^2}{\Omega\tau} - \frac{2\pi^2}{\hbar\Omega}\right) f''_\alpha\left(\frac{eS_{\text{ext}}^\alpha}{ehB}\right). \quad (2.2)$$

For sufficiently low temperatures, in pure samples, when

$$\Omega \gg \frac{2\pi^2}{\tau}, \quad \frac{2\pi^2 T}{\hbar}, \quad (2.3)$$

*This argument may not be valid for the case of high Z nuclei.

χ in formula (2.2) can reach arbitrarily large values and correspond to strong magnetism. (This was first pointed out by Shoenberg.^[7]) In particular, for $\chi \gtrsim 1$, we need for (2.3)

$$\frac{\epsilon_0}{\hbar\Omega} \gtrsim \left(\frac{c}{v}\right)^{4/3} \sim 10^3. \quad (2.4)$$

A specific feature of electron diamagnetism is that, when $T = 0$, $\tau = \infty$, it is the stronger, the weaker the magnetic field: $M/B \propto B^{-1/2}$, $\chi \propto B^{-3/2}$. The term "diamagnetism" is used here and in the sequel only to mean that strong magnetism occurs, i.e., in the sense of the diamagnetic quantization of Landau. But the magnetic susceptibility, according to (2.2), in general consists of the sum of "diamagnetic" and "paramagnetic" components (where we name them according to their sign), and can give either strong diamagnetism ($-\chi \gtrsim 1$) or strong paramagnetism ($\chi \gtrsim 1$). The latter leads to an instability of the homogeneous state (cf. below).

The nonoscillatory contribution $\delta\theta_L$ to θ because of the Landau diamagnetism is due to all the branches in Fig. 1, and is therefore much greater than $\delta\theta$ (cf. (1.12)): $\delta\theta_L \approx \hbar\Omega(\hbar\Omega/\epsilon_0) \approx B^2(v/c)^2$. However, it changes smoothly, and at low temperatures (when (2.3) is satisfied) its contribution to the total χ and M is obviously small: $M_\alpha/B \approx (v/c)^2$, $\chi \sim (v/c)^2$.

Since for $T = 0$, $\tau = \infty$, and $B \rightarrow 0$, not only χ but also $M/B \sim B^{-1/2}$ has a singularity, it is possible to have $|M| \gtrsim |B|$. For this, according to (2.1) it is necessary that

$$2\pi^2 T \ll \hbar\Omega < \epsilon_0 \left(\frac{v}{c}\right)^4,$$

i.e.,

$$T \ll \frac{1}{2\pi^2} \epsilon_0 \left(\frac{v}{c}\right)^4, \quad l > 2\pi^2 a \left(\frac{c}{v}\right)^4$$

(a is the mean distance between electrons) or $T \lesssim 10^{-4} 10^{-5} \text{K}$, $l > 10^{-10} 10^{-3} \text{cm}$. Experiments at such temperatures and free paths are not realizable at present, so we shall assume from now on that

$$T \gg \frac{1}{2\pi^2} \epsilon_0 \left(\frac{v}{c}\right)^4, \quad \left|\frac{M}{B}\right| \ll 1. \quad (2.4a)$$

Even under these conditions formula (2.2) can give large positive values of χ . But we know that there is thermodynamic stability only for states corresponding to $\partial H/\partial B > 0$ (cf. [9]), i.e., $4\pi\chi < 1$. Thus an increase in χ when $4\pi\chi_{\max} > 1$ leads to a segregation into magnetic phases with different values of B (Fig. 3), i.e., to the appearance of magnetic domains. This was first noted by Condon^[10] (cf. also [66]); the shape of the walls between domains was found by Privorotskiĭ,^[11a] the sizes of domains were found in [11b].

A direct experimental proof of the existence of diamagnetic domains was given in [64]: a splitting of an NMR line because of the different values of the magnetic moment was observed (for a presentation of the results of this model, and its discussion, see [65]).

Physically this is related to the fact that the magnetic field changes the density of states and consequently, also, the internal energy of the electron gas, and when $4\pi\chi_{\max} > 1$ it is favorable to have the readjustment of the density of states associated with the change in B on segregation of phases.

Since, according to (2.2), the value of χ oscillates,

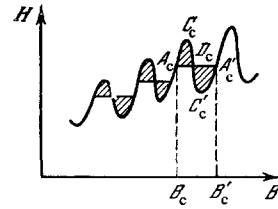


FIG. 3

the equilibrium curve has a series of periodically repeated flat regions of segregation (one is shown dashed in Fig. 3).

An unstable homogeneous state of the magnet can also occur when the splitting into domains is still not favored thermodynamically, if χ contains both paramag-

netic and diamagnetic components: $\chi = \sum_{\alpha=1}^{\alpha_0} \chi_\alpha$ (cf. (2.2))

with $4\pi\chi < 1$, but where, in the absence of interaction, instability against phase separation would already have appeared for the paramagnetic components: $4\pi\chi_0 = 4\pi\sum \chi_\nu > 1$ (where ν labels those α for which $\chi_\nu > 0$). The phase separation does not occur because of the self-consistent interaction with components for which the homogeneous state is stable. This means that the interaction leads to "mixing," i.e., it gives an effective attraction of the diamagnetic ($\chi_\alpha < 0$) and paramagnetic ($\chi_\alpha > 0$) components.* Thus the instability of the ν components produces a tendency to separation into phases, while the attraction between them (negative surface energy) causes a mixing. In such a case it is natural to expect an ordering of the phases, i.e., the appearance of a self-consistent periodic structure, whose characteristics are determined by the properties of the thermodynamic system. Thus the period of the structure is macroscopic and of the order of the interaction radius—the Larmor radius r . A necessary condition for the appearance of a magnetic structure is to have several de Haas–van Alphen periods ($\alpha_0 > 1$).

One can understand from general considerations how the transition from the homogeneous to the periodic structure will occur.† The requirement of thermodynamic stability leads to continuity in the transition of the corresponding thermodynamic potential (thus a pos-

*The separation into magnetic phases when $(\partial H/\partial B)_T^{\min} < 0$ is analogous to the separation into liquid-vapor in a multicomponent system when $(\partial p/\partial \rho)_T^{\min} < 0$; the pressure corresponds to a magnetic field intensity equal to the sum of the partial fields for the components corresponding to the extremal sections or bands, while the specific volume $V = \rho^{-1}$ corresponds to the magnetic induction. The interaction between components reduces formally to the statement that the "specific volumes" of the different components must coincide. (B obviously depends only on the coordinates.) As a result, although separation into phases with B and B' is favorable for the ν components, it is forbidden by the other bands, for which such "specific volumes" are unfavorable. But for an inhomogeneous induction B , the different χ_α change differently, the total susceptibility can increase, which is thermodynamically favorable (cf. below).

†The possible occurrence of a periodic structure was first pointed out in [12]. This paper also gave a theory of such structures. In [13] the occurrence of periodic structures was noted, but the solution found there corresponds to $4\pi\chi > 1$, which is thermodynamically absolutely unstable, and therefore can never be realized (cf. below).

itive specific heat c_V means that the free energy will be continuous and monotonic as a function of temperature T). This means that at the transition point either an infinitesimal amount appears of a new phase with essentially new properties, i.e., with a finite amplitude A of spatial oscillations, or that the new phase itself differs infinitesimally from the old one (i.e., the amplitude of the oscillations is infinitesimal).

In the first case the appearance of the new phase is associated with seed regions that appear through fluctuations, and which, in order to be thermodynamically favorable, must be quite large (since there is surface energy at the boundaries of the phases). It is thus possible to have superheating and supercooling, and a preservation of a metastable homogeneous phase. The transition point is thus a point of intersection of the thermodynamic potentials for the homogeneous phases, and is not really a singularity (cf. ^[14] Secs. 81 and 83)—this is a phase transition of the first kind.

In the second case the surface energy is absent, so that superheating and supercooling are not possible: the new state appears abruptly over the whole volume and the transition point is a singularity of the thermodynamic potential of the system (phase transition of the second kind). The abrupt change of state over a whole macroscopic volume must obviously be "prepared": the dimensions of the regions of unfavorable state (correlation radius r_c) arising from fluctuations must increase without limit as the transition point is approached. In the Landau theory this occurs according to the law (cf. ^[14] Sec. 119)

$$r_c \approx a \left| \frac{T - T_c}{T_c} \right|^{-1/2} = a\tau^{-1/2}$$

(a is the separation between particles, T_c is the transition temperature). So long as the correlation radius is small compared to the interaction radius ρ (the Larmor radius in our case), the fluctuating nonuniformity in the oscillation amplitude can be disregarded and the amplitude can be considered to be uniform and "real" (not a fluctuation) and determinable from the conditions of thermodynamic equilibrium. The inequality $r_c \ll r$ corresponds to $|T - T_c| \gg (a/r)^2 T_c$ ($a \sim 10^{-8}$ cm, $r \sim 10^{-3}$ cm in magnetic fields $H \sim 10^4$ Oe, $T_c \sim 1^\circ$ K; only for such temperatures are the quantum oscillations important, i.e., $|T - T_c| \gg 10^{-10}$ °K). A violation of these inequalities occurs only far beyond the limits of possible experiments.

If $(a/r)^2 \lesssim N^{-1/2}$ (N is the number of particles) the inequality $r_c \ll r$ is satisfied over the whole range of validity of thermodynamics, since the measurement of temperature is meaningful only to within the fluctuations: $\tau \gg N^{-1/2}$. (We note that $(a/r)^2 \sim n^{-4/3}$, where n is the enormous density of the Fermi particles.)

When we neglect fluctuations, the correction to any thermodynamic potential that is associated with the magnetic field depends on the "total" induction. Since the inhomogeneous contribution near the point of a transition of the second kind is small compared to the homogeneous contribution ($A \rightarrow 0$ for $\tau \rightarrow 0$), θ can be expanded in series in powers of this contribution. Regarding the oscillation period λ as given* (i.e., guaran-

teeing a minimum of the thermodynamic potential for a given amplitude A of the spatial oscillations), we get an expansion of θ in powers of A , which is analogous to the familiar Landau expansion

$$\theta = \theta_0 + \alpha A^2 + \beta A^4 + \gamma A^6 + \dots \quad (2.5)$$

For $\alpha = 0$, $\beta > 0$ we get phase transitions of the second kind of the Landau type (cf. ^[14] Secs. 137, 138) with a finite jump in the specific heat c_H (if the transition occurs for a fixed magnetic field and changing temperature) or in the magnetic susceptibility (if H changes and T is fixed). One can show that the relative size of the jump is of order $(\hbar\Omega/\epsilon_0)^{1/2}$.

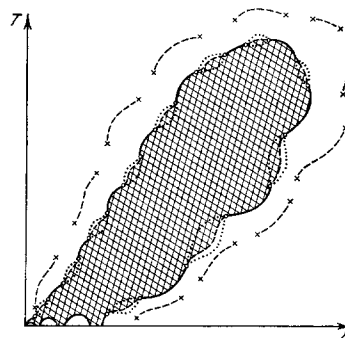


FIG. 4. Phase diagram. — line of phase transitions of the second kind; --- line of phase transitions of the first kind; ···· — line of absolute instability; O — critical point of second-kind phase transitions; X — critical point of first-kind phase transitions. The cross-hatching shows the region of existence of a spatially periodic structure.

The curve of phase transitions of the second kind may terminate, intersecting the curve of phase transitions of the first kind at the critical point for the second-kind transitions (cf. ^[14] Sec. 140), where, in formula (2.5), $\alpha(T_0, B_0) = \beta(T_0, B_0) = 0$. For $\alpha = a_1 T_1 + b_1 H_1$, $\beta = a_2 T_1 + b_2 H_1$ (where $T_1 = T - T_0$, $H_1 = H - H_0$, $|T_1/T| \ll 1$, $|H_1/H| \ll 1$, the investigation of the minimum in (2.5) is conveniently done in the (α, β) coordinate system, after which we go over to coordinates (T_1, H_1) . When $\gamma > 0$ it turns out that the direct extension of the curve of second kind phase transitions is a line of absolute instability of the homogeneous phase with $A = 0$ (concerning the investigation of the curve bounding the region of instability, see later in Sec. 6); tangent to it at the point (T_0, H_0) are the line of phase transitions of the first kind (corresponding to a separation into a homogeneous phase with $A = 0$ and a periodic phase with $A \neq 0$) and the line of absolute instability of the periodic phase with $A \neq 0$ (Fig. 4). On the latter the specific heat c_H or the corresponding susceptibility χ go to infinity like $|T_1|^{-1/2}$ or $|H_1|^{-1/2}$ (cf. Sec. 6).

It is of course, possible to have a phase transition not only from a homogeneous to a periodic structure, but also one where on the background of an inhomogeneous structure there appears with finite amplitude a contribution (finite or infinitesimal) with a new period that differs from the old period by a finite amount.

The periodic dependence of the quantum oscillations on B^{-1} in the uniform case leads to repeated transitions when we change B^{-1} for given T .

All the difficulties in constructing a theory of the second-kind transitions refer to the experimentally un-

*For $T = T_c$ it is finite; assuming the function $\lambda(T)$ to be regular, we get a correction to λ_c near T_c that is linear with temperature. The proof is given below.

attainable region $r_c \geq \rho$ (if such a region exists at all).

Phase transitions associated with the diamagnetic Landau quantization also occur in the appearance of a domain structure (cf. Fig. 3). In the external magnetic field H_0 parallel to the sample surface, when $H_t = H_0$, the independent parameter is H and the potential

$$\delta\theta_t = -\left(\frac{1}{4\pi}\right) B \delta H. \quad (2.6)$$

has a minimum. When $H = H_C$, "boiling" occurs—a phase transition of the first kind from B_C to B'_C . If the field H_0 is normal to the surface, it coincides with the average over the surface of the sample of the magnetic induction \bar{B} :

$$H_0 = \bar{B}. \quad (2.7)$$

(This follows from the continuity of $\int B_n dS$.) From (2.7), $H_C = cB_C + (1-c)B'_C$, where c is the concentration of the phase B_C . The points B_C and B'_C correspond to $4\pi\chi < 1$, and are found from the minimum of the thermodynamic potential. Since, according to (2.7), \bar{B} is given, the independent parameter is B , and we seek a

minimum of θ_t^* ($\delta\theta_t^* = \frac{1}{4\pi} H \delta B$) under the supplementary condition (2.7). This means that $\theta_t = \theta_t^* + A \int B dr$ has an unconditional minimum. Determining the constant A from $\delta\theta_t/\delta B = 0$, we find $A = -H/4\pi$, and

$$\delta\theta_t = -\frac{1}{4\pi} B \delta H = -M \delta B + 2\pi \delta M^2 + \frac{\delta H^2}{8\pi} \equiv \delta\theta + 2\pi \delta M^2 + \frac{\delta H^2}{8\pi}. \quad (2.6a)$$

Thus θ_t has a minimum with respect to the independent variable B , so that (cf. Fig. 3) the points B_C and B'_C are determined by the equality of the areas $A_C C'_C D_C$ and $D_C C'_C A'_C$. When H_0 changes in the range $B_C \leq H \leq B'_C$ there is an "evaporation" accompanied by a jump in the susceptibility, associated with a second-kind transition of the Landau type. Consequently the situation is analogous to the usual separation in the vapor-liquid system: at constant volume the independent parameter is the mean specific volume v , the nonuniqueness of $v(p)$ permits one to vary the pressure p for a given v and the thermodynamic potential Φ ($d\Phi = v dp$) has a minimum as a function of v ; the value of the potential Φ per particle, i.e., the chemical potential, is the same in the two phases.

Thus, phase transitions of the first and second kind can correspond to a transition to a periodic or to a domain structure.

In order to distinguish the phase transitions related to the appearance of periodic and domain structures, we must make use of the fact that the latter is dependent on the geometry of the sample, while the first is not. Thus, for example, if H_0 is rotated in a plane perpendicular to a fourfold axis (sample dimensions $L_x, L_y, \sqrt{r}L_x \gg L_y \gg r$), the nature of the phase transitions when $H_0 \parallel x$ and $H_0 \parallel y$ will be the same for periodic structures, and will differ markedly for domain structures (transitions of the first kind for $H_0 \parallel x$ and transitions of the second kind for $H_0 \parallel y$).

An important feature of such phase transitions is the macroscopic size of the interaction radius. The finite size of the sample leads to a relative width of the transition (both in temperature and magnetic field) of order $(r/D)^2$, where D is the smallest linear dimension of the sample. Within this region all thermodynamic quantities are rapidly varying, but analytic, functions.

Formula (2.2) shows that inhomogeneous structures arise for weak magnetic fields and low temperatures in pure samples (for $T = 0$ and $\tau = \infty$ the susceptibility $\chi \rightarrow \infty$ for $B \rightarrow 0$), i.e., for

$$2\pi^2 T < \hbar\Omega < \epsilon_0 (4\pi)^2 \left(\frac{r}{c}\right)^{4/3}, \quad \Omega\tau \gg 1, \quad \Omega \propto B. \quad (2.8)$$

Interference of different transitions leads to a reduction in χ and a smearing out of the transition temperature or even complete impossibility of an inhomogeneous structure, if χ is too small. This means that one needs extremely weak mosaic character of the crystal, high stability with time, and spatial uniformity of the magnetic field (its variation must be small compared to the period ΔB of the oscillations of χ , i.e., small compared to $B\hbar\Omega/e_0 \sim B^2$).

The occurrence of an inhomogeneous magnetic moment leads to the appearance of an electrostatic potential φ . According to the conditions for thermodynamic equilibrium, the chemical potential ζ is constant over the system, and in order to assure constancy of the total electron density (which follows from the condition of electrical neutrality, which is satisfied up to terms of order $(a/r)^2$ (cf., for example, [15]), we must introduce $\varphi(r)$. Since $e\varphi(r) \ll \hbar\Omega$, φ need not be considered in

the quantization, so that, if $N = N_0 + \Delta N$ ($\Delta N = -\frac{\delta\Delta\Omega(r)}{\delta\zeta}$)

is the oscillatory part of N for constant ζ , $\Omega(r)$ is the density of the potential Ω , N_0 is the part of the electron density that in lowest approximation is independent of magnetic field and monotonic), we have

$$N_0(\zeta) = N_0(\zeta + e\varphi) + \Delta N\{B(r)\}, \quad (2.9)$$

$$e\varphi(r) = -\frac{1}{v\epsilon_0} \Delta N\{B(r)\} \approx \hbar\Omega \left(\frac{\hbar\Omega}{\epsilon_0}\right)^{-2} \exp\left(-\frac{2\pi^2 T}{\hbar\Omega} - \frac{2\pi^2}{\Omega\tau}\right). \quad (2.9a)$$

The quantity ζ is determined from the identity of the total numbers of electrons in the magnetic field and in its absence. Knowledge of ζ and $\varphi(r)$ allows one to determine the density of electrons of each zone at the point r .

Using this formula and the estimate made earlier for the correction to the thermodynamic potential, it is easy to see that, in the region of interest to us, the potential φ has no effect on either the quantization in the magnetic field or on the magnetic susceptibility. Thus all calculations can be done without $\varphi(r)$, and by substituting the computational results in (2.9a) we get $\varphi(r)$.

The spatially periodic structure also affects other properties of the magnet. The propagation of electromagnetic and ultrasonic waves is changed because of the periodic superstructure; in particular, for sufficient amplitude of the waves, one gets peculiarities associated with the periodicity in time of the phase transitions. New types of resonances appear because of new branches of the normal vibrations in the superstructure. A spatial modulation of the specific volume occurs (magnetostriction in a periodic field).

It is clear that periodic structures, diamagnetic domains and all the related effects are also possible in a ferromagnet, where even in the absence of an external field we have $B = 4\pi M_0$ ($M_0(T)$ is the spontaneous magnetic moment).

3. Thermodynamics of Domain and Periodic Structures

In order to simplify our treatment, we restrict ourselves to the case where the magnetic field is along one of the principal crystal axes, so that $\mathbf{M} \parallel \mathbf{B} \parallel \mathbf{H} \parallel z$. We shall consider only the lowest approximation in the mean free path l : $l = \infty$. Then there is no characteristic length along the z direction, and so nothing depends on z . Since $\text{curl } \mathbf{H} = 0$ this means that

$$H = H_z = \text{const}, \quad M = M_z(x, y), \quad B = B_z(x, y). \quad (3.1)$$

To determine the equation of state, i.e., the dependence of M on B , which, as was shown in Sec. 1 (cf. Fig. 1) has a nonlocal functional character:

$$M = M\{B\}, \quad (3.2)$$

we must calculate the correction $\delta\theta$, because of diamagnetic and spin quantization, to the corresponding thermodynamic potential (cf. (2.6a)). (Since this correction is always small, according to (1.12), the contributions to all the thermodynamics potentials in the corresponding variables coincides; cf. ^[14] Sec. 24). M is then given by:

$$M = -\frac{\delta\theta}{\delta B}. \quad (3.3)$$

It is most convenient to calculate the potential Ω :

$$\Omega = -T \sum_k \ln(e^{\frac{\xi - \epsilon_k}{T}} + 1) \quad (3.4)$$

(where the sum is over all states).

In the uniform case the energy levels have already been given in formula (1.13). According to (1.13) the inhomogeneous contribution depends only on M , which is always small compared to B (cf. (2.4a) and the discussion of this question in Sec. 2). This enables us to use perturbation theory on M in finding the energy levels in a nonuniform field B : the degeneracy in P_x is then lifted, since in the nonuniform field the position of the center of the orbit, which is determined by P_x becomes significant.

In the one-dimensional case, where B depends on only the one coordinate and P_x is again an integral of the motion, the calculation is elementary (and was first done in ^[16]); in the two-dimensional case it is somewhat more complicated, since P_x is no longer conserved and one must determine the "correct" zero order functions in the case of an infinite degeneracy (the computation is given in ^[17]).

The order of magnitude of the relative contribution to the level separation is obvious: $M/H \approx M/B \approx \chi \hbar \Omega / \epsilon_0$ (compare with (2.1) and (2.2)). This small addition has no effect on the estimates made in Sec. 2 and affects mainly the argument of the oscillating function; the number of levels $\epsilon = \epsilon_0$ is no longer $\epsilon_0 / \hbar \Omega$, but because of the change $\Delta \epsilon$, is $(\epsilon_0 / \hbar \Omega) [1 + (aM/H)]^{-1}$ ($a \sim 1$). In the linear approximation in M , the small contribution to the phase is thus of order χ (cf. (2.2)). The next approximation gives a very small correction $(M/B)^2 (\epsilon_0 / \hbar \Omega) \lesssim (v/c)^2 (\hbar \Omega / \epsilon_0)^{1/2}$ (cf. (2.1)). This means that even in the argument of the periodic function we can remain with the linear approximation in M . But this means that one can only get the first order of perturbation

theory in the quantization rules and wave functions.* In the general case the correct zero-order wave functions and the energy levels correct to first order have the form

$$\left. \begin{aligned} \psi^{(0)} &= \int c(P_x) \psi_{\hbar P_x}^{(0)} dP_x, \\ c(P_x) &= \exp \left[\frac{i}{\hbar} \int^x x(P'_x) dP'_x \right], \\ \epsilon &= \epsilon_{\hbar P_x}^{(0)} + \epsilon', \end{aligned} \right\} \quad (3.5)$$

where ϵ' is a continuous quantum number, and $x(P_x)$ is gotten from the equation

$$\epsilon' = \frac{1}{H} \frac{\partial S}{\partial \epsilon} \oint P_y(P_x) dP_x B_1 \left(x, \frac{c(P_x - P_x)}{eH} \right), \quad B_1 = 4\pi M = B - H, \quad (3.6)$$

while $\epsilon_{\hbar P_x}^{(0)}$, $\psi_{\hbar P_x}^{(0)}$ correspond to the zeroth approximation ($B = H$) and are given, for example, in ^[11].

The calculation of the thermodynamic potential for known energy levels can be done similarly to the procedure in the uniform case (cf. ^[11]). The result is simplest in the one-dimensional case:^[17]

$$\begin{aligned} \Omega &= \sum_\alpha \int dP_x \Omega_{LK}^{(\alpha)} \left[\frac{cS_\alpha}{ehH} - \frac{2c}{ehH^2} \int_{P_{x\alpha}^{\min}}^{P_{x\alpha}^{\max}} P_y(P_x) M \left(\frac{P_x - P_x}{eH/c} \right) dP_x \right], \\ M &= \frac{B - H}{4\pi}, \end{aligned} \quad (3.7)$$

where $\sum_\alpha \Omega_{LK}^{(\alpha)}$ is the potential found by Lifshitz and

Kosevich^[5] in a uniform field H . Just as in the uniform case, to obtain the final formula one can replace the chemical potential by the limiting Fermi energy.

The formulas found apply only to an ideal electron gas and do not include the Fermi liquid interaction; but the latter has no effect on the period oscillations and the order of magnitude of their amplitude (cf. ^[1, 18]). But the specific form of $M(\mathbf{r})$ is not very important, and the character of this function can be gotten from very general arguments.

As was shown at the beginning of this section, in the general case all quantities are periodic functions whose argument is a linear function of the magnetic moment. This means that the density of any thermodynamic potential $\theta_1(\mathbf{r})$ for the magnet must, when we include translational symmetry, have the form (where B is again along a crystal axis)

$$\begin{aligned} \theta(\mathbf{r}) &= \sum_\alpha f_\alpha \left\{ \int K_\alpha(\mathbf{r} - \mathbf{r}') M(\mathbf{r}') d\mathbf{r}' \right\}, \\ \theta &= \int \theta_1(\mathbf{r}) d\mathbf{r}, \end{aligned} \quad (3.8)$$

where the central symmetry of the crystal guarantees that $K_\alpha(\mathbf{r})$ is an even function. Knowing the thermodynamic potential enables us to determine the moment M

*It is for just this reason that we can disregard the spin-orbit force in the homogeneous field, associated with the force $\mu_0 \nabla(\sigma B)$. This also allows us to use the correspondence principle (1.1), with Ω the classical frequency in the nonuniform field, to get the quantization rules. It can be found using perturbation with respect to M in the classical Hamilton-Jacobi equation; it is more convenient than Newton's equations when one varies the period.

using formula (3.3) and to write the fundamental equation for determining $B(x, y)$:

$$B - 4\pi M \{B\} = H = \text{const.} \quad (3.9)$$

It should however be remembered that this equation alone can give not only equilibrium solutions, but also metastable and even absolutely unstable solutions. In fact the equation of state (3.3) guarantees only an extremum of the thermodynamic potential and not a minimum value, and is only a necessary (but not sufficient condition) for equilibrium. To get rid of the absolutely unstable solutions, one should write the second variation of the total thermodynamic potential, the magnetic contribution to which is related not only to the quantization but also to the variation of field in the magnet. In Sec. 2 it was shown that this potential is always the potential $\delta\theta = -B\delta H/4\pi$ (formulas (2.6), (2.6a)); the arguments obviously do not depend on whether one is discussing domain or periodic structures. Since H is not varied ($H = \text{const}$) and $-M\delta B = \delta\theta$ (cf. (3.3)),

$$\theta_t = \theta_0 + 2\pi \int M^2 dr, \quad M = \frac{B-H}{4\pi} = -\frac{\delta\theta}{\delta B}. \quad (3.10)$$

The formulas (3.10), (3.8), (3.3) and (3.9) enable us in principle to construct a thermodynamics of magnets in both the equilibrium and metastable states. Since the thermodynamic potentials in these states differ by a finite amount and have a minimum in both cases, the investigation of the variational derivatives gives nothing more. To the equilibrium state will correspond the smallest value of θ_t , which can be selected by comparing the values of θ_t for the different solutions (substituted in (3.10)).

Because of the long range interaction, metastable solutions in magnets may prove to be extremely stable, since one always needs macroscopic inclusions of the new phase (of the order of r , i.e., 10^{-3} - 10^{-4} cm). During boiling we know that even for slight superheating the critical size of the centers is the size of an atom. It is true that the heating must occur very rapidly since the characteristic relaxation times have an atomic origin, and are extremely small (even in extremely pure metals at helium temperatures they are $\sim 10^{-9}$ sec). We now turn to a more interesting region—the appearance of an inhomogeneous structure. We first study the case where such a structure appears during a transition of the second kind.

4. Theory of Diamagnetic Phase Transitions of the Second Kind

The fundamental equation of the theory of periodic structures is

$$B - 4\pi M \{B\} = H = \text{const}, \quad B = B(\mathbf{R}), \quad \mathbf{R} \equiv (x, y). \quad (4.1)$$

Equation (4.1) always has the uniform solution B_0 :

$$B_0 - 4\pi M(B_0) = H, \quad (4.2)$$

which may not be unique. Let us examine the conditions for the existence of an inhomogeneous solution infinitely close to the homogeneous one:

$$B = B_0 + B_1. \quad (4.3)$$

We have

$$B_1 = 4\pi \hat{\chi} B_1, \quad (4.4)$$

where $\hat{\chi}$ is a linear integral operator. From the homogeneity of space (with respect to shifts by a period of the crystal lattice, which in the lowest approximation is infinitesimal compared to the distance r) it follows that this operator must have a difference form, while the invariance of the crystal under inversion ($\mathbf{R} \rightarrow -\mathbf{R}$) assures its evenness, so that (4.4) can be written in the form

$$B_1(\mathbf{r}) = \int \chi(\mathbf{R}-\mathbf{R}') B_1(\mathbf{R}') d\mathbf{R}', \quad \chi(-\mathbf{R}) = \chi(\mathbf{R}). \quad (4.5)$$

Setting

$$B_1 = \text{Re} [A \exp(i\mathbf{kR})], \quad (4.6)$$

we arrive at the equation for the period of spatial oscillations

$$4\pi \tilde{\chi}(\mathbf{k}_0) = 1, \quad \tilde{\chi}(\mathbf{k}) = \int \chi(\mathbf{R}) \cos(\mathbf{kR}) d\mathbf{R}. \quad (4.7)$$

For $4\pi\chi < 1$ the condition for solvability is not only necessary, but also a sufficient condition, for the appearance of periodic structure. To prove this we expand $\theta\{B\}$ in terms of B_1 . Considering that (3.10) should then coincide with (4.5), and taking account of the symmetry of the crystal, we find

$$\begin{aligned} \theta_t = \theta_0 + \frac{1}{8\pi} \int B_1^2(\mathbf{R}) d\mathbf{R} - \frac{1}{2} \int \chi(\mathbf{R}-\mathbf{R}') B_1(\mathbf{R}) B_1(\mathbf{R}') d\mathbf{R} d\mathbf{R}' \\ + \frac{1}{12\pi} \int \int j(\mathbf{R}-\mathbf{R}', \mathbf{R}-\mathbf{R}'') B_1(\mathbf{R}) B_1(\mathbf{R}') B_1(\mathbf{R}'') d\mathbf{R} d\mathbf{R}' d\mathbf{R}'' \\ + \frac{1}{16\pi} \int \int g(\mathbf{R}-\mathbf{R}', \mathbf{R}-\mathbf{R}'', \mathbf{R}-\mathbf{R}''') B_1(\mathbf{R}) B_1(\mathbf{R}') B_1(\mathbf{R}'') \\ \times B_1(\mathbf{R}''') d\mathbf{R} d\mathbf{R}' d\mathbf{R}'' d\mathbf{R}''' + \dots \end{aligned} \quad (4.8)$$

The term linear in B_1 is absent from (4.8) because it must have (from translational symmetry) the form, $K_0(B_0) \int B(\mathbf{R}) d\mathbf{R}$, but stability with respect to uniform perturbations ($B_1 = \text{const}$) means that

$$\left. \frac{\partial \theta_t}{\partial B_1} \right|_{B_1=0} = K_0(B_0) = 0 \quad \left(\text{a } \left. \frac{\partial^2 \theta}{\partial B_1^2} \right|_{B_1=0} = 1 - 4\pi\chi > 0 \right)$$

by assumption).

We write (4.8) in the form

$$\theta_t = \theta_0 + \frac{\pi}{2} \int (1 - 4\pi \tilde{\chi}(\mathbf{k})) |\tilde{B}_1(\mathbf{k})|^2 d\mathbf{k} + \dots \quad (4.9)$$

(where the tilde denotes Fourier components of the corresponding quantity).

If (4.7) has a solution, then $4\pi \tilde{\chi}_{\text{max}} \geq 4\pi \tilde{\chi}(\mathbf{k}_0) = 1$, and according to (4.9) a small inhomogeneity is very favorable, and the homogeneous state is unstable. As already mentioned in Sec. 1, instability of the homogeneous state may be related to either the appearance of a periodic structure or to separation into diamagnetic domains. The latter already occurs (cf. Fig. 3) when $4\pi\chi < 1$ (i.e., $\frac{\partial H}{\partial B} > 0$); only such values of χ are realized. Thus, for the existence of a periodic structure it is necessary* to have $4\pi \tilde{\chi}_{\text{max}} \geq 1$ (maximum is taken over \mathbf{k}) when $4\pi\chi = 4\pi \tilde{\chi}(0) < 1$.

*One can show that for a given band and a given section, $\chi_\mu(\mathbf{R})$ has a fixed sign, and

$$\text{sign } \chi_\mu(\mathbf{R}) = \text{sign } \chi_\mu, \quad 4\pi |\tilde{\chi}_\mu| \leq 4\pi |\tilde{\chi}_\mu(0)| = 4\pi |\chi_\mu|.$$

Thus for the realization of a periodic structure one needs several extremal sections and $4\pi\chi_0 > 1$ with $4\pi\chi < 1$. (The definition of χ_0 is given in Sec. 1).

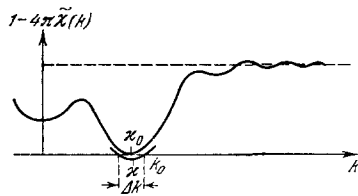


FIG. 5

We shall assume that this condition is satisfied. The second order phase transition point corresponds to the first appearance of a root of (3.7), i.e., $4\pi\chi_{\max} = 1$, so that we get a multiple root of (3.7) (Fig. 5). Near this point, the interval of k in which $1 - 4\pi\chi < 0$ is small: $|\Delta k/k_0| \ll 1$. This means (cf. Fig. 5) that when $|\Delta k| \gtrsim k_0$, where $1 - 4\pi\chi(k) \gtrsim 1$, the value of B_1 must be small compared to its value in the interval where $1 - 4\pi\chi < 0$, since the term written in (4.9) is controlling for small \tilde{B}_1 (as assumed initially), and a state with a small spread Δk is very unfavorable. This means that even when we include the higher terms in the expansion near the transition point, B_1 can be represented in the form

$$B_1(\mathbf{R}) = A(\mathbf{R}) \exp(i\mathbf{x}\mathbf{R}) + A^*(\mathbf{R}) \exp(-i\mathbf{x}\mathbf{R}) + C(\mathbf{R}), \quad |C| \ll |A|, \quad (4.10)$$

$$\tilde{\chi}(\mathbf{x}) = \tilde{\chi}_{\max}, \quad \nabla \tilde{\chi}(\mathbf{x}) = 0, \quad (4.11)$$

where $A(\mathbf{R})$ is a slowly varying function (over distances large compared to the radius of interaction in the kernels in (4.8)—the Larmor radius r), the asterisk denotes the complex conjugate, and $C(\mathbf{R})$ may correspond to any k , changes arbitrarily, but is small compared to $A(\mathbf{R})$ (strictly speaking, its rapidly varying part need not have a small amplitude).

The form (4.10) enables one to solve by successive approximations* the nonlinear equation for $B_1(\mathbf{R})$ which follows from $\delta\theta_t/\delta B_1 = 0$:

$$B_1(\mathbf{R}) = 4\pi \int \chi(\mathbf{R}-\mathbf{R}') B_1(\mathbf{R}') d\mathbf{R}' + \int f(\mathbf{R}-\mathbf{R}', \mathbf{R}-\mathbf{R}'') B_1(\mathbf{R}') \times B_1(\mathbf{R}'') d\mathbf{R}' d\mathbf{R}'' + \int g(\mathbf{R}-\mathbf{R}', \mathbf{R}-\mathbf{R}'', \mathbf{R}-\mathbf{R}''') B_1(\mathbf{R}') B_1(\mathbf{R}'') \times B_1(\mathbf{R}''') d\mathbf{R}' d\mathbf{R}'' d\mathbf{R}''' + \dots \quad (4.12)$$

Substituting (4.10) in (4.12), we find (to second order in A) the value of C :

$$C = E \exp(2i\mathbf{x}\mathbf{R}) + E^* \exp(-2i\mathbf{x}\mathbf{R}) + 2D, \quad (4.13)$$

$$E = \frac{\tilde{f}(\mathbf{x}, -\mathbf{x})}{1 - 4\pi\tilde{\chi}(2\mathbf{x})} A^2, \quad D = \frac{\tilde{f}(\mathbf{x}, -\mathbf{x})}{1 - 4\pi\tilde{\chi}(0)} |A|^2. \quad (4.14)$$

In the next approximation, in addition to the third harmonics, first harmonics appear from the terms of third order in B_1 . The corresponding equation (which, of course, coincides with the condition for orthogonality of the "perturbing" inhomogeneous terms in (4.12) to the solution of the homogeneous equation) has the form

*It is clear that the choice of the zero order approximation in the form (4.10) already completely determines the behavior of the higher approximations, provided we take account of the condition for solvability of an inhomogeneous equation when the homogeneous equation has a nontrivial solution. Then the solution of the equation $f(\mathbf{R}-\mathbf{R}') \Phi(\mathbf{R}') d\mathbf{R}' = F(\mathbf{R}) \exp(i\mathbf{s}\mathbf{R})$ with $F(\mathbf{R})$ a slowly varying function has the form

$$\Phi(\mathbf{R}) = \exp(i\mathbf{s}\mathbf{R}) H^{-1} \left(i \frac{d}{d\mathbf{R}} F(\mathbf{R}) \right), \quad H(\xi) = \int \exp[i\mathbf{R}(\xi - \mathbf{s})] h(\mathbf{R}) d\mathbf{R}$$

$$-2\pi \sum_{i=1}^2 \frac{\partial^2 A}{\partial R_i^2} \frac{\partial^2 \tilde{\chi}}{\partial \alpha_i^2} + iA - 4\beta A |A|^2 = 0, \quad (4.15)$$

$$\tau = 4\pi\tilde{\chi}(\mathbf{x}) - 1, \quad 2\beta = 3\tilde{g}(\mathbf{x}, \mathbf{x}, -\mathbf{x}) + \frac{2\tilde{f}(\mathbf{x}, 2\mathbf{x})\tilde{f}(\mathbf{x}, \mathbf{x})}{1 - 4\pi\tilde{\chi}(2\mathbf{x})}, \quad (4.16)$$

$$\alpha_i^{-1} = -2\pi \frac{\partial^2 \tilde{\chi}}{\partial \alpha_i^2} > 0;$$

here $\alpha_i > 0$, since $\tilde{\chi}$ has a maximum at the point \mathbf{x} ; the directions of the x, y , axes are chosen so that $\frac{\partial^2 \tilde{\chi}}{\partial \kappa_x \partial \kappa_y} = 0$.

We introduce κ_0 , the point at which (cf. Fig. 5)

$$4\pi\tilde{\chi}(\kappa_0) = 1, \quad \nabla \tilde{\chi}(\kappa_0) = 0. \quad (4.17)$$

In addition to determining κ_0 , the two equations (4.17) also give the relation between T and H , i.e., the curve of phase transitions from the homogeneous to the periodic structure in the T, H plane. If the transition is observed at fixed H , then $\tau \sim T - T_0(H)$ (where $T_0(H)$ is the transition point); if T is fixed, then $\tau \sim H - H_0(T)$.

Setting $A = |A| \exp(i\psi)$, we find that if $\psi \neq 0$ then $\nabla \psi \sim |A|^{-2}$, and is a rapidly oscillating function of \mathbf{R} (since $|A|$ is small by assumption), which was shown above to be false. Therefore $\psi = 0$ and A is real. Substitution of (4.10), (4.13) and (4.14) with real A in (4.8) leads to the expression

$$\theta_t - \theta_0 = \int \left[-U(A) + \sum_{i=1}^2 \frac{1}{2\alpha_i} \left(\frac{\partial A}{\partial R_i} \right)^2 \right] d\mathbf{R} \equiv \int \theta_t d\mathbf{R}, \quad (4.18)$$

$$U(A) = \frac{1}{4} \tau A^2 - \frac{1}{2} \beta A^4.$$

From the fact that, as one sees from the general arguments above, a periodic structure arises at the transition point with period $2\pi\kappa_1^{-1} \approx r$, the form (4.18) for the expansion of θ_t in terms of the slowly varying (compared to r) small correction A can be obtained directly. It is sufficient to remember that: 1) from the requirement of a minimum of θ_t (already averaged over distances of order r) as a function of A at the transition point, we get $\delta\theta_t/\delta A = 0$; 2) $\delta H/\delta A \sim \tau$ (since $4\pi\tilde{\chi}(\kappa_0) = 1$), but this gives $\delta^2\theta_t/\delta A^2 \sim \tau$; 3) the presence of a minimum of $\theta_t(A)$ for $\tau = 0$ requires $\delta^3\theta/\delta A^3 = 0$, $\delta^4\theta/\delta A^4 = \beta > 0$; 4) the expansion in the small quantity ∇A (small because of the slow variation of $A(\mathbf{R})$) can contain only even powers of ∇A because of the invariance under the substitution $\mathbf{R} \rightarrow -\mathbf{R}$.

Let us explain the meaning of the requirement $\beta > 0$. If $\beta < 0$, it follows from (4.18) that the point $\tau = 0$ is in general not singular—there is already a periodic structure with finite amplitude, the transition has occurred earlier at finite A (since the transition when $A \rightarrow 0$ has just been studied), i.e., we have a phase transition of the first kind.

Since $\alpha_i > 0$ (cf. (4.16)), $\theta_t \geq -U(A) \geq -U_{\max}$, where the equality $\theta_t = -U_{\max}$ is attained for uniform A . The term containing derivatives in (4.18) vanishes, θ_t takes a form characteristic for second-kind transitions of the Landau type (cf. (2.5)), and leads to a transition from $A = 0$ when $\tau < 0$ to $\pm A_0$, $A_0 = \frac{1}{2} \sqrt{\tau/2\beta}$, when $\tau > 0$. The states with $\pm A_0$ differ only by a phase shift. This difference can be important in a finite sample, where such states are analogous to domains. From the definitions of κ and κ_0 it is easy to find the dependence of the spatial periods $2\pi\kappa_1^{-1}$ on τ : $\kappa - \kappa_0 \sim \tau$. Thus, the

period of oscillation changes linearly with τ near the transition point, but the amplitude is proportional to $\sqrt{\tau}$.

All the above discussion is particularly lucid in the one-dimensional case ($A = A(y)$), when the functional (4.18) can be interpreted formally as the action for the one-dimensional motion of a particle with mass α in time y over the coordinate A , where emergence of the particle to infinity is forbidden (since B must be finite), $U(A)$ is the "potential energy." The transition between the states with $\pm A_0$ in this case has domain character, and occurs (when $\tau > 0$) according to the law

$$A = A_0 \operatorname{th} \left[\frac{1}{2} \sqrt{\tau \alpha} (y - y_0) \right]. \quad (4.19)$$

From the definition (4.7) it is clear that when $k \rightarrow \infty$ the function $\tilde{\chi}(k)$ oscillates, tending to zero. This means that when the temperature (or the magnetic field) is changed, new roots of Eqs. (4.11) can appear, and correspondingly there may be new phase transitions. If any extremum $\tilde{\chi}(k)$ for $k > 0$ is degenerate, one can have a separation into phases with different periods.

In all the preceding discussion, it was assumed that there is a unique solution κ of (4.11). It is understood that this is so only for a plane rectangular lattice: the presence of a symmetry axis of order higher than two guarantees a corresponding number of solutions of (4.11) — the "star" of the vectors κ . If no three of them add to zero, the whole treatment remains valid, but is somewhat more involved. But if some of the solutions of (4.11) sum to zero, the third order term in the expansion of $\theta_t(A)$ does not vanish, a phase transition of the second kind is impossible, and the periodic structure appears as the result of a phase transition of the first kind.

5. Critical Point and Domain Structure

In the preceding section it was assumed that the instability of the uniform state appears when $\kappa \neq 0$. Now let us consider the case when $\tilde{\chi}_{\max} = \tilde{\chi}(0) = \chi$. (The null necessarily corresponds to an extremum, because of the central symmetry of the crystal: $\tilde{\chi}(-k) = \tilde{\chi}(k)$, so that $\nabla \tilde{\chi}(-k) = -\nabla \tilde{\chi}(k)$, from which $\nabla \tilde{\chi} = 0$, i.e., there is an extremum which, because of the initial stability of the uniform state, is a maximum). The point of transition to the nonuniform structure when $\chi < 0$ corresponds to $4\pi\tilde{\chi} = 1$ according to (4.7). In this case we can use the theory developed in the previous section, but it is simpler to remark immediately that when $\kappa_0 = 0$ the quantity B varies slowly, so that $\theta_1\{B\}$ can be expanded in powers of ∇B . In lowest approximation θ coincides with the "local" uniform density $\theta_1^0(B)$. In the next approximation,

$$\theta_1\{B\} = \theta_1^0(B) + \sum_{i=1}^2 \frac{1}{2\alpha_i} \left(\frac{\partial B}{\partial R_i} \right)^2. \quad (5.1)$$

Using arguments similar to those in Sec. 4, we see that this point corresponds to a separation into diamagnetic domains. Outside the domain wall the derivatives in $\theta_1\{B\}$ should be dropped.

Since $4\pi\chi = 1$, and consequently in the uniform case $\frac{\partial H}{\partial B} = 0$, from the stability of the uniform state on the one side of the transition point it follows that $\frac{\partial^2 H}{\partial B^2} = 0$,

$\frac{\partial^3 H}{\partial B^3} > 0$. Setting $B = B_0 + B_1$, $T = T_0 + T_1$, $H = H_0 + H_1$, and noting that

$$\delta\theta_t = -B\delta H, \quad (5.2)$$

we find

$$\theta_t = \theta_0 + \left(a_2 T_1 - \frac{H_1}{4\pi} \right) B_1 + \frac{1}{3} a_3 T_1 B_1^2 + \frac{1}{4} a_4 B_1^4. \quad (5.3)$$

This expansion (whose validity was justified in Sec. 2) is analogous to the Gibbs expansion near an ordinary critical point of the liquid-vapor system (cf. for example, ^[14] Sec. 83), except that H plays the role of the pressure and B that of the specific volume v . (It is understood that, near an "ordinary" critical point, experimental investigation for a given relation of p and v , corresponding to a shifted H_0 , is extremely difficult.) Thus all the results of the Gibbs treatment of critical points easily carry over to the present case. In analogy to formula (84.10) of Sec. 137 in ^[14], we have

$$c_H \propto (a_3 T_1 + 3a_4 B_1^2)^{-1}. \quad (5.3a)$$

In particular, on the equilibrium curve, where $B_1 \propto |T_1|^{1/2}$, the specific heat $c_H \propto T^{-1}$, while in a magnetic field equal to the "critical" one ($H_1 = 0$), when we find

from the minimum of θ_t (i.e., from $\frac{\partial \theta_t}{\partial B_1} = 0$) that $B_1 \propto T^{-1/3}$, formula (5.3a) gives $c_H \propto T_1$

The understanding of the sizes and shapes of diamagnetic domains has an intrinsic interest. The most interesting and important case is the one where the sample dimensions L are large compared to the Larmor radius r , so that the size g of the domains is much greater than the thickness d of the domain walls. In this case the solution of the problem breaks up into a "microscopic" and a "macroscopic" part.

The "microscopic" part consists in explaining the shape of the domain wall, i.e., the distribution of the induction in a layer of thickness $\sim d$. Since $d \ll g$, in lowest approximation in d/g the wall may be treated as a plane, but the problem is one-dimensional so that according to Maxwell's equations $B_n = \text{const}$, $H_t = \text{const}$ (n is the normal, whose direction we shall take as the y axis; t is the tangent to the wall). Then according to (4.1)

$$H_t\{B_t(y); B_n\} = H_t, \quad H_n\{B_t(y); B_n\} = H_n(y). \quad (5.4)$$

Equations (5.4) are completely analogous to the usual equations of state near the boundary between coexisting phases, except that B_n should be regarded as a given parameter (since in the region near the sample surface it does not in general coincide with the "bulk" equilibrium value). Therefore the conditions for approach of B_t and B_n to saturation as $y \rightarrow \pm \infty$ (i.e., for $|y| \gg d$, but of course $|y| \ll g$) are identical with the usual condition $H_t = H_c$, which corresponds to equality of areas on the curve of $H_t = H_t(B_t)$ for given B_n , analogous to the curve in Fig. 3, where H is replaced by H_t and B by B_t . As a result we get a definite connection at infinity ($y \rightarrow \pm \infty$) between H_t and B_n :

$$H_t = H_\kappa(B_n). \quad (5.5)$$

This condition is particularly clear in the isotropic case, when H_t is independent of B_n .

A specific form for $B_t(y)$ and $H_n(y)$ can be gotten in

the vicinity of the critical point, where (5.1) holds. The corresponding formula for the isotropic case was first found in [11a]. It is naturally analogous to (4.19) (the thickness of the domain wall, in particular, is of order $r_{\max}\mu^{-1/2}$, where r_{\max} is the maximum Larmor radius) and enables us to find the surface energy Δ at the domain boundary:

$$\Delta = \frac{r_{\max}\mu^{1/2}(B-B')^2}{24\sqrt{2\pi}}, \quad \mu = 1 - 4\pi\chi \ll 1. \quad (5.6)$$

The "macroscopic" problem consists in explaining the shapes and dimensions of domains in the lowest approximation in $d/g \ll 1$, i.e., when $d = 0$. After assigning the period D of the domain structure, one should solve the magnetostatic problem; then Eq. (5.5), which serves as an auxiliary boundary condition on the domain boundaries, determines the shapes of the boundaries (they were found in various limiting cases in [11b]).

After this, using (2.7) we can determine the sizes of the domain regions; in the case of a flat plate

$$g_1 B + (D - g_1) B' = H_0 D, \quad (5.7)$$

$$g_1 = \frac{(H_0 - B')D}{B - B'}, \quad g_2 = D - g_1 = \frac{(B - H_0)D}{B - B'}.$$

Finally, calculating the total thermodynamic potential for a known structure of the field, and including the surface energy at the domain boundaries and minimizing with respect to D , we determine D . The value of D is of order [11b]

$$D \approx \sqrt{\mu r} L, \quad \mu = 1 - 4\pi\chi \ll 1, \quad (5.8)$$

where L is the plate thickness.

From (5.6)–(5.8) the validity of the approximations is clear:

$$\frac{d}{g} \sim \frac{1}{\mu} \sqrt{\frac{r}{L}} \ll 1 \quad \text{и} \quad \frac{g}{L} \sim \sqrt{\frac{r}{L}} \mu \ll 1.$$

Far from the critical point, when $\mu \gg 1$ and correspondingly $\chi_{\max} \gg 1/4\pi$ (cf. Fig. 3), the computations become considerably more complicated. [11b] The domain wall has a very complicated form. In going from A_C to A'_C , $B(y)$ oscillates and has a series of gradually damped, narrow and high maxima, and correspondingly deep minima. The surface energy and domain size are given in order of magnitude by (5.6), (5.7); but now, when $\mu \gg 1$, we must replace $\mu^{1/2}$ by $\mu^{1/3}$.

6. Curve of Absolute Instability for a Uniform and Periodic Structure

The validity of the expansion of the thermodynamic potential (analogous to the Landau expansion in the theory of phase transitions) which results from the long range interaction at distances $r \gg a$ (where a is the mean separation of charges) enables us to study the singularity on the curve of absolute instability (where in the uniform case $\frac{\partial H}{\partial B} = 0$). Suppose that for given H and $T = T_0$ the equation for B , which we write symbolically in the form

$$\hat{L}\{T, H; B\} = 0, \quad (6.1)$$

has the particular solution $B_0(\mathbf{R})$, so that

$$\hat{L}\{T_0, H; B_0(\mathbf{R})\} = 0. \quad (6.2)$$

Let us examine the character and stability of the solution when $T = T + T'$. We set $B = B_0 + \psi$. Then

$$\hat{L}_1\psi + T'\hat{L}_2\{B_0(\mathbf{R})\} + \frac{1}{2}\hat{L}_3\psi^2 + \dots = 0, \quad (6.3)$$

where

$$\hat{L}_1 = \frac{\delta \hat{L}}{\delta B_0}, \quad \hat{L}_2\{B_0(\mathbf{R})\} = \frac{\partial}{\partial T} \hat{L}\{T, H; B_0(\mathbf{R})\}, \quad \hat{L}_3 = \frac{\delta^2 \hat{L}}{\delta B_0^2}. \quad (6.4)$$

The variation of T with H fixed has been taken only as an example. The treatment is completely similar for fixed T and varied H . The solution and examination of (6.3) is done the same as for (4.12), and its character is determined by the presence or absence of a solution of the homogeneous equation $\hat{L}_1\psi = 0$. For a given H , such a solution first appears at the point $T = T_0(H)$. The set of such points for different H gives a curve $T_0 = T_0(H)$. The only difference is that the terms in the thermodynamic potential that are cubic in the amplitude can either vanish (because of symmetry or at isolated values of H), or not be present, since, in contrast to (4.12), the kernel in (6.3) need not be a difference kernel, since the nonuniformity of $B_0(\mathbf{R})$ violates the translational symmetry of the system. In the latter case the expansion of the thermodynamic potential in $A_1 = A - A_0$ (where A_0 is the amplitude of the periodic structure on the absolute instability curve) and $T_1 = T - T_0$ has the form

$$\theta_t = \theta_0 + \alpha T_1 A_1 + \frac{1}{6} \beta A_1^3. \quad (6.5)$$

When $\alpha T_1/\beta < 0$ there is no minimum of θ_t , while for $\alpha T_1/\beta > 0$ there is a relative minimum for $A_1 = (-2\alpha T_1/\beta)^{1/2}$ and $\theta_t - \theta_0 \propto |T_1|^{3/2}$. This means that as we approach the absolute instability curve for a given external field H_0 , the specific heat tends to infinity like $|T_1|^{-1/2}$. For an isothermal approach to the curve ($H_0 = H + H_1$) the susceptibility tends to infinity like $|H_1|^{-1/2}$. There is such a singularity, in particular, at the critical point for a phase transition of the second kind (cf. Sec. 2).

7. Phase Diagram

Let us now examine the phase diagram in a magnetic field, taking account of the appearance of periodic and domain structures. At high temperatures $T \gg \hbar\Omega/2\pi^2$ the susceptibility associated with the Landau diamagnetism and the Pauli paramagnetism is small for all temperatures and a nonuniform structure is impossible even as a metastable state. (This is easily shown using perturbation theory.)

As the temperature is lowered, at some temperature there first appears a local minimum for the nonuniform field. Then several different cases are possible in principle.

The local minimum may first appear for a finite amplitude of the inhomogeneity, so that the minimum value of θ_t will again guarantee a uniform induction B . The function $\theta_t(A)$ has at least two minima, between which there must be a relative maximum. Thus (in addition to the minimum for uniform induction B , i.e., when $A = 0$), $\theta_t(A)$ suddenly acquires a relative maximum and minimum; the point $T = T_C$ where the minimum is "born" corresponds to a threefold degenerate solution:

$$\frac{\partial \theta_t}{\partial A} = \frac{\partial^2 \theta_t}{\partial A^2} = 0,$$

which determines the temperature T_C and the amplitude at the point (Fig. 6a). The appearance of the local minimum shows the possibility of existence of the corresponding phase, though perhaps as a metastable phase, while its vanishing means absolute instability of such a phase. The curve of appearance of minima in the T-H plane thus bounds the region of absolute instability of the given state (analogous to the curve $(\frac{\partial p}{\partial v})_T = 0$ for the separation of the liquid-vapor system).

If then the minimum of θ_t , as it drops, reaches the same value at $T = T_C$ as it had in the preceding absolute minimum when $A = 0$, there is a separation into a uniform ($A = 0$) and a nonuniform ($A \neq 0$) phase, i.e., a phase transition of the first kind.* When $T < T'_C$ the nonuniform state becomes stable, while the uniform state is metastable when $T'_C < T < T''_C$ and absolutely unstable for $T < T''_C$ (in the last case the singularity near $T = T''_C$ is the same as above; it should be understood that we are working with fixed H and not T only for concreteness).

It may, however, happen that even before the separation into a uniform and a nonuniform phase (if it occurs at all—since the minimum for $A \neq 0$ may begin to move upward after reaching some temperature), instability of the uniform state for $A = 0$ appears. Since, as already pointed out and demonstrated in Sec. 4, the expansion of $\theta_t(A)$ contains only even terms in A, this will indicate a second-kind transition as studied earlier (Fig. 6b); the minima at $\pm A$ correspond to phases with the same periods and different “starting points” (cf. Sec. 4). But if the minima of $\theta_t(A)$ for different $A \neq 0$ coincide, there is a separation into a structure with different periods.

Since, when we change the external conditions, the minima on the $\theta_t(A)$ curves may shift up or down (though their motion for fixed T and varying magnetic field is periodic) it follows from (3.2) and the fact that θ_t is a superposition of functions periodic in B^{-1} , that for given $T < \hbar\Omega/2\pi^2$ the number of transitions of a given type is $\sim \frac{1}{4}(\epsilon_0/T)(v/c)^{4/3}$ and a variety of different combinations of the cases cited can occur. In particular the new structure may appear at only a single point (H_{00}, T_0) (Fig. 7 we show the curve of $\theta_t(A)$ for two structures I and II).

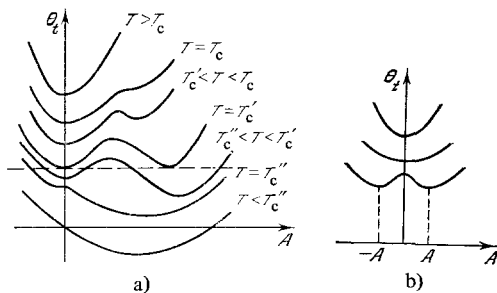


FIG. 6

*It is clear from this that the periodic solution with finite amplitude always first appears as a metastable state, unfavored compared to the uniform solution.

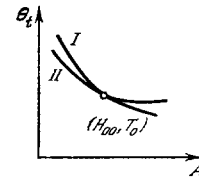


FIG. 7

Considering all this, we turn to consideration of Fig. 4, where we show the state diagram of the magnet in the T-H diagram; the transition to the variables T and H_0 is clear from the relation between H_0 and H given above. In Fig. 4, we have taken into account that the curve of phase transitions cannot break off at the periodic structure and may end at the critical point for a domain structure; the character of the transition to the domain structure is the same all along the curve and depends only on the orientation of H (for concrete-ness H_0 was taken perpendicular to the sample surface in Fig. 4). We also took into account that (in addition to the periodicity of χ_α in B^{-1}) according to (2.8) the transition curve is bounded from above as well as below in B, and is bounded above in temperature. The form of the state diagram would be much more complicated in the next approximation in a_0/r^2 (a_0 is the interatomic spacing), where the commensurability of a_0^2 and ehH/c is important (cf. also [19]).

The singularities on the state diagram were given in Sec. 2 and demonstrated in Sec. 5.

8. Ultraquantum Case and the Metal-Dielectric Transition

So far we have been discussing magnetic fields that are weak compared to the Fermi energy: $\hbar\Omega \ll \epsilon_0$; this is the quasiclassical case. Then a knowledge of the classical dispersion law $\epsilon = \epsilon(p)$ was sufficient for finding the main quantum corrections to all physical quantities. As a result the experimental study of quantum effects in this range of magnetic fields was a convenient means for studying the classical dispersion law. (Thus, the periods of quantum oscillations were directly determined by the areas of the extremal and singular sections of the classical Fermi surface.)

As the magnetic field increases, the associated energy may become sizable. Such strong magnetic fields are actually reached in semimetals like bismuth and in semiconductors with a Fermi energy that is small (because of the small number of carriers) and usually with a low effective mass. Generally speaking, when $\hbar\Omega \gg \epsilon_0$, when magnetostriction essentially changes the crystal lattice, the energy spectrum for the quantum case cannot be gotten from the classical spectrum, and only the systematics of the states is preserved from the quasiclassical case. Since in all attainable magnetic fields both the classical radius of the orbit and the Bohr radius $\sqrt{ch/eH}$ are large compared to the lattice constant, in lowest approximation the degeneracy in P_x is also preserved (cf. Par. 1). As a result $\epsilon = \epsilon_s(p_z; H)$, where $s = 0, 1, \dots$ is a discrete quantum number. Since we can say nothing beforehand about the distances between the different discrete levels for fixed p_z (in particular the spin-orbit coupling is not weak), it is meaningless to

keep the three quantum numbers used earlier—the band number q , the spin projection σ and the number n of the diamagnetic Landau level.

It is meaningful to keep only the division into electrons and "holes," for which, respectively, the energy $\epsilon_S(p_Z)$ has a minimum or a maximum in the energy interval of interest to us. The investigation of the region of such strong magnetic fields in the "quantum" and "ultraquantum" cases (when the level separation becomes the largest parameter of the problem) is of particular interest since it enables us, by shifting the band edge (cf. below) to change the Fermi energy itself, to affect the number of carriers (when there are several bands), and to observe anomalies when the Fermi energy passes through singular values of the energy.

On the other hand the theoretical treatment of such a case is interesting in principle. The main idea in introducing quasiparticles is to study a small number (and therefore a weakly interacting system) of excitations, appearing on the background of a given vacuum. External fields, and in particular a constant magnetic field, change this vacuum (just as they change the character of the quasiparticle spectrum, leading to the appearance of singularities in the spectrum (cf. Sec. 1)). However, so long as the external fields are small (and this condition coincides, in the case of constant field, with the condition for the system to be quasiclassical), the change in the vacuum can be treated as a small perturbation and we may talk about a change of the dispersion law for the quasiparticles with "the same vacuum." But if the vacuum is unchanged the only characteristic of the quasiparticles is their dispersion law, and it is natural that in lowest approximation the "new" law of dispersion in external fields can be obtained from the "old" dispersion law. This is precisely the procedure in both classical and quasiclassical theory. Thus the basic classical equation of motion of the electron in external fields, $\dot{\mathbf{p}} = e\mathbf{E} + (e/c)\mathbf{v} \times \mathbf{H}$ can be obtained in a way completely analogous to the procedure in field theory (cf., for example, [20]), if we take into account that: 1) the Lagrangian must be determined uniquely to within a total time derivative, and the quasimomentum to within a reciprocal lattice vector; 2) the charge on a conduction electron is the same as the charge on a free electron (cf. Sec. 1).

When the level separation is considerably greater than the Fermi energy, i.e., when the energy associated with the magnetic field is of the order of the energy associated with the vacuum, and the external field is not a small perturbation, finding the new dispersion law which appears on the background of the new vacuum, from the old, classical dispersion law is in general not possible in principle. This means that the problem must be formulated differently (cf. [20, 22]). In particular one must assume that the spectrum of the quasiparticles in the constant magnetic field is given. One must also assume that the wave functions (or the matrix elements) are given.

The problem now consists of, first, understanding which physical results are independent of the dispersion law, and, secondly, finding methods for experimentally reconstructing the quantum Hamiltonian, in particular, the energy levels—a problem analogous to that which arose earlier when we were given the dispersion law $\epsilon = \epsilon(\mathbf{p})$ in the classical case.

It is natural to expect, as is the situation in all cases known at present, that the distance between the discrete levels will grow with increasing H . (Saturation is improbable, but trivial, since it means a disappearance of field dependence in all physical quantities. For the systematic oscillations with H there is not even a characteristic parameter that would determine the oscillation period.) Then, since the number of states is conserved (cf. Sec. 1), a larger and larger number of classical states will be assigned to each level, so that the density of states $\nu_S(p_Z; B)$ also will grow. If the states belonging to different energies are not shifted about by the interaction, $\nu_S(p_Z; B)$ is connected with $\epsilon_S(p_Z; B)$ by the formula which follows from the conservation law for the number of states:

$$\nu = \nu(\epsilon_S; p_Z), \quad \epsilon = \epsilon_S(p_Z; B). \quad (8.1)$$

At absolute zero ($T = 0$) the shift of the boundaries of the s branch may "destroy" one of the branches when passing through the Fermi energy. Then all the electronic properties of the metal will have singularities that are smeared out at finite temperatures (cf. [22]).

Let us illustrate this situation first in the simplest case, when there are equal numbers of electrons (e) and holes (h) with the dispersion laws

$$\epsilon_e = \epsilon_1 + \frac{p^2}{2m_e}, \quad \epsilon_h = \epsilon_2 - \frac{(p - p_2)^2}{2m_h} \quad (8.2)$$

and spin masses $m_{e,h}^S$, which do not change in the magnetic field. Then, in a magnetic field along the z axis,

$$\left. \begin{aligned} \epsilon_e &= \epsilon_1 + \left(k + \frac{1}{2}\right) \mu_e B \pm \mu_e^S B + \frac{p_z^2}{2m_e}, \\ \epsilon_h &= \epsilon_2 - \left(k + \frac{1}{2}\right) \mu_h B \pm \mu_h^S B - \frac{(p_z - p_2)^2}{2m_h}, \\ k &= 0, \dots, \mu_{e,h} = \frac{|e|\hbar}{m_{e,h}}, \quad \mu_{e,h}^S = \frac{|e|\hbar}{2m_{e,h}^S c}. \end{aligned} \right\} \quad (8.3)$$

The origins of the electron and hole spectra are therefore shifted; the distance between the "floor" of the electron band (ϵ_e^{\min}) and the "ceiling" of the hole band (ϵ_h^{\max}) changes:

$$A = \epsilon_h^{\max} - \epsilon_e^{\min} = \epsilon_2 - \epsilon_1 + bB, \quad (8.4)$$

and if $b < 0$, when $B = B_c$, for which $A = 0$, the bands diverge, and for $T = 0$ the metal changes to a dielectric (an effect that was first predicted by Davydov and Pomerenchuk [21]). If B is produced by a current through the sample, we can have the phenomenon of an "intermediate" or "mixed" state—the separation into metallic and semiconductor phases.

Let us consider this process in the general case. It is clear that for B near B_c the number of current carriers is always small, so that we are interested only in the narrow regions near extrema, i.e., $\epsilon_e = \epsilon_e^0(p_Z) \approx \epsilon_{\min} + \alpha(p_Z - p_1)^2$, $\epsilon_h = \epsilon_h^0(p_Z) \approx \epsilon_{\max} - \beta(p_Z - p_2)^2$ and $\nu_{e,h}(p_Z; B) \approx \nu_{e,h}(p_{1,2}; B_c)$. As a result, when $B = B_c - B'$, $0 < B' \ll B_c$ we find for the numbers of electrons n_e and holes n_h and the chemical potential ζ , the values

$$n_e = n_h \propto (\zeta - \zeta(B_c))^{1/2}, \quad \zeta(B_c + B') = \zeta(B_c) - \gamma B'. \quad (8.5)$$

For the thermodynamic potentials we have

$$0 \propto n_{e,h} (\zeta - \zeta(B_c)) \sim |B'|^{3/2}. \quad (8.6)$$

(this is evident for the energy, but the small corrections to θ associated with $n_{e,h}$ are the same for all thermody-

dynamic quantities in the corresponding variables^[14]. For $B' > 0$, $n_e = n_h = 0$ and $\theta = 0$. Thus, when $T = 0$, at the point $B = B_c$ there is a phase transition of the "two-and-a-half kind": the magnetic moment $M = -\frac{\partial \theta}{\partial B}$ is continuous, the susceptibility goes to plus or minus infinity when $B = B_c - 0$ according to the law $(-B')^{-1/2}$, and the conductivity goes to zero proportionally to $n_e \sim (-B)^{1/2}$ ($\chi \rightarrow \infty$ means of course, that separation occurs; cf. above).

It may, however, happen that the transition from metal to dielectric proceeds via a phase transition of the first kind because of annihilation of electron-hole pairs through their interaction, as is the case under pressure in the absence of a magnetic field (cf. ^[24]).

If the number of electrons is not the same as the number of holes, but one of the spectrum branches goes out beyond the Fermi boundary, the number of particles in the disappearing branch goes to zero, nevertheless, so we can make the same computation in this case, with the same result—a phase transition of the "2 1/2 kind."

At finite temperatures, low compared with the Fermi energy, the phase transition is smeared out and replaced by an anomaly: near B_c the gas of charges in the disappearing branch becomes nondegenerate and then, as the energy gap increases, the number of charges drops exponentially. The features of the electron characteristics near B_c were treated in ^[22,23] and observed experimentally in bismuth, antimony and their alloys in ^[59-62].

The sharp drop in the relative resistance with increase of magnetic field (dielectric-metal transition) is clearly shown in Fig. 8 (taken from ^[62]), and the change in the character of the magnetoresistance in Fig. 9 (from ^[60]).

If the electron and hole levels overlap more and more as the magnetic field increases, (for the spectrum (8.2) this means that in (8.4), $b > 0$), the numbers of electrons and holes increase without limit with increasing B . This case permits a constructive computation only when (8.1) holds. The calculation is of greatest interest for the case when the main contribution to the magnetic moment comes from one branch of one of the zones (electron or hole). Then from the $M(B, T)$ dependence we can reconstruct the $\epsilon(p_z, B)$ corresponding to that branch. The computation is rather tiresome, so there is no point in giving it here.

Finally a case is possible where in sufficiently strong field only electron or only hole branches remain. Then, since the number N of charges is conserved, for $B \rightarrow \infty$ their density of states $\nu \rightarrow \infty$, only the states in a small region of p_z near the ground state $\epsilon_0^{\min}(B)$ are occupied, so that*

$$\epsilon_0(p_z, B) = \epsilon_0^{\min}(B) + \frac{1}{2} \beta (p_z - p_0)^2, \quad \nu_0(p_z, B) \approx \nu(p_0(B), B) \equiv \nu_0(B) \quad (8.7)$$

*It is understood that the dependence on B is connected in particular with magnetostriction (which may lead to an isomorphous phase transition of the first kind). The magnetostriction must be taken into account in determining ϵ_0 since $n = N/V = \nu_0(p_z(\nu_0) - p_0)$ where V is the volume of the conductor, is not conserved. But this has no effect on the order of magnitude of ϵ_0 , since the compression is not large, so long as the conductor supports the deformation. In order to disregard the magnetostriction, we shall regard ν as the density of states over the whole volume (and not per unit volume).

($\beta > 0$ if the branch is an electron branch, and $\beta < 0$ if it is a hole branch). The Fermi energy $\epsilon_0 - \epsilon_0^{\min}$ is proportional to $(n/\nu_0)^2 \rightarrow 0$, so that the gas of charges is a Boltzmann gas at any finite temperature. Its free energy is

$$F = -NT \ln \int_0^\infty \exp\left(-\frac{\epsilon_{\min} + 0.5\beta p_z^2}{T}\right) \nu(p_z) dp_z,$$

and the magnetic moment at constant volume (found in ^[26]) is

$$M = -\left(\frac{\partial E}{\partial B}\right)_V = -N \frac{\partial \epsilon_0}{\partial B} + NT \frac{\partial}{\partial B} \ln \nu_0. \quad (8.8)$$

Neglecting the change in volume ($p = -\frac{\partial E}{\partial V} = N \frac{\partial \epsilon_0}{\partial V}$, so that $v = v(H, p)$), we can, by measuring the magnetic moment, find the dependence on the magnetic field of the energy ϵ_0 of the ground state and the density of states ν_0 in it. These are of course only the beginnings of information about the quantum Hamiltonian. The development of the theory ought to give, in principle, the possibility of reconstructing the Hamiltonian from the experimental data, just as we have reconstructed the classical Hamiltonian (the dispersion law) for a large number of metals. Formula (8.8) for $T = 0$ obviously means that the magnetization is nominal.* We note that Fermi liquid effects are negligibly small in semimetals and semiconductors, where the number of charge carriers is small (cf., for example, ^[40]), so that this treatment is consistent.

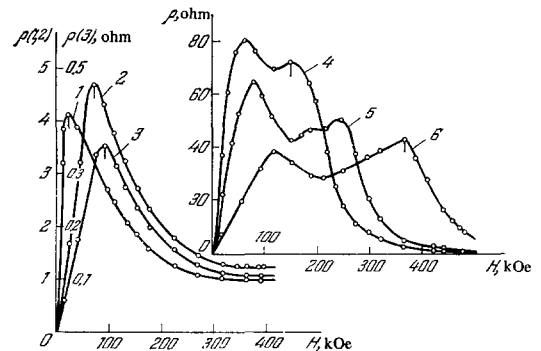


FIG. 8. Dependence of change of resistance ρ on H for orientation $H \parallel C_3$ (current $i \parallel C_2$). Bi-Sb alloys; Sb concentration in at%: 1 - 8.8; 2 - 8.9; 3 - 9.1; 4 - 10.5; 5 - 12; 6 - 15.8.

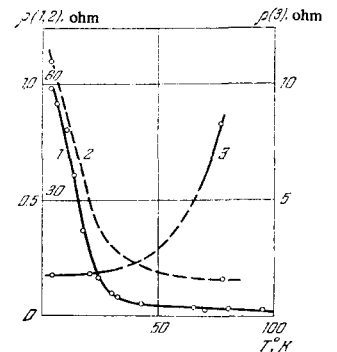


FIG. 9. Temperature dependence of ρ for $Bi_{89.5}Sb_{10.5}$ sample, for: 1 - $H = 0$; 2 - $H < H_c$; 3 - $H > H_c$.^[60]

*The case of fixed bottom of the band is special. It corresponds to the absence of the first term in (8.8), and was considered, for quadratic dispersion, by Rumer.^[25]

9. Variable Fields. Nonlinear Quantum Pseudoresonance

So far we have considered the static case, where electrostatics and magnetostatics are separate, and we can speak of quantum oscillations of the magnetic moment and of the electrical resistance. In a variable electromagnetic field one measures one quantity, the tensor of the total surface impedance, relating the field intensity on the conductor surface and the total current passing through it. In good conductors with a high charge density, retaining the displacement current in Maxwell's equations is usually an improvement in accuracy (cf. also Part II, Sec. 10 and 11). Thus the total current is determined by the intensity on the surface of the varying magnetic field. In the one-dimensional case of a conductor filling a halfspace* the impedance \hat{Z} is given by the expression

$$E_t^{(S)} = \frac{c}{4\pi} \hat{Z} [Hn]^{(S)} \quad (9.1) \dagger$$

(the subscript t denotes the tangential component, (S) the surface, n the normal to the surface, E and H the intensities of the varying fields.

Knowing the tensor \hat{Z} enables us to find the solution to the purely external problem of the field in vacuum with the boundary conditions (9.1) and to express the complex coefficient of reflection of the wave in terms of \hat{Z} . (The standard electromagnetic boundary conditions, continuity of $E_t^{(S)}$ and $H_t^{(S)}$, together with (9.1) give six equations, expressing $E_t^{(S)}$, $H_t^{(S)}$, E_t^{refl} in terms of E_t^{inc}).

The quantum oscillations of \hat{Z} are naturally determined by the quantum oscillations of both the magnetic moment M and the conduction current density j. Strictly speaking, in the nonstationary case these two quantities cannot even be separated. Just as in Sec. 1, we are considering a typical case of field theory—free charges in fields E and $B_0 + B$ (where B is determined by the constant magnetic field and B by the varying field). Then the Maxwell equations are:

$$\text{rot } E = -\frac{i\omega}{c} B, \quad \text{rot } H = \frac{4\pi}{c} j_{\text{tot}}, \quad (9.2)$$

but j_{tot} is related to the density matrix \hat{n} in the external fields E and B (cf. [27]):

$$j(r) = \frac{1}{2} e \text{Sp} \{ \hat{v} \delta(r - \hat{r}) + \delta(r - \hat{r}) \hat{v} \} \hat{n}. \quad (9.3)$$

(The physical meaning of this formula is obvious: in the classical case $e v \delta(r - r(t))$ is the density of current produced at the point r by a charge located at the point r(t). The formula can be obtained directly in the quasiclassical case from the definition $j = -c \delta \hat{\mathcal{H}} / \delta A$, where $\hat{\mathcal{H}} = \text{Sp} (\hat{\mathcal{H}} \hat{n})$ is the mean value of the Hamiltonian operator.)

At low frequencies ($\omega \ll \nu$, where ν is the characteristic collision frequency) the system almost succeeds in "following" the instantaneous values of the varying fields, and we can, in first approximation, take for M and j in these formulas their values at the particular time.

*This case is the one of most practical importance. High conductivity means short wavelength in the conductor, compared to which all the characteristic dimensions of the conductor can be taken to be infinite.

† $[Hn] \equiv H \times n$.

In the case of frequencies so low that we have the normal skin effect, the depth δ of the skin layer in an isotropic conductor has the form

$$\delta = \sqrt{\frac{c^2}{2\pi\omega\sigma(1+4\pi\chi)}}. \quad (9.2a)$$

The order of magnitude of χ was estimated earlier (formula (2.2)). From the same arguments as in Sec. 2, one easily estimates the relative magnitude of the quantum oscillations of the conductivity, $\Delta\sigma/\sigma$. Its dependence on T and τ is the same as that of χ ; the contribution to $\Delta\sigma$, as for χ , comes from the part $(\hbar\Omega/\epsilon_0)^{1/2}$ of the electrons near the extremal sections, but the factor $(v/c)^2$ is absent, since $\Delta\sigma$, unlike χ , is not relativistic in origin. As a result

$$\frac{\Delta\sigma}{\sigma} \approx \left(\frac{\hbar\Omega}{e_0} \right)^{1/2} \exp\left(-\frac{2\pi^2 T}{\hbar\Omega} - \frac{2\pi^2}{\Omega t}\right) \ll 1. \quad (9.3a)$$

Actually the magnitude of $\Delta\sigma/\sigma$ may be still smaller, since σ is determined by all the electron groups, while the usually observed oscillations of $\Delta\sigma$ are associated with anomalously small bands.

Comparing (9.3a) and (2.2), we see that

$$s = \frac{4\pi\chi}{\Delta\sigma/\sigma} \approx \left(\frac{v}{c} \frac{e_0}{\hbar\Omega} \right)^2. \quad (9.4)$$

(The estimate (2.2) does not hold when $4\pi\chi \gtrsim 1$, but in that case the contribution $\Delta\sigma/\sigma \ll 1$ is very small compared to the contribution when $4\pi\chi \approx 1$.)

Usually one reaches magnetic fields $\hbar\Omega \ll \epsilon_0 v/c$, and the resistance oscillations are determined by the oscillations of the magnetic moment. [28] The situation changes only when we go to sufficiently high frequencies or to the low-frequency limit. In the low-frequency range, when $\delta > dS^{-1} \gg d$ (d is the thickness of the conductor), the quantity appearing in (9.1) is the difference of the values of H_t on the sample surfaces (it determines the total current) and the estimate shows that the determining factor is the σ oscillations (Shubnikov-de Haas effect).

At sufficiently high frequencies when the anomalous skin effect appears, and $\delta < r$, the magnetic moment is determined by the varying field only over a small part of the orbit and χ decreases by a factor δ_{eff}/r (δ_{eff} is the effective damping distance for B). When $\omega\tau \gg 1$, there is an additional reduction of χ by a factor $\omega\tau$, since the magnetic moment is unable to "follow" the change in B.* As a result, the oscillations of \hat{Z} at sufficiently high frequencies are determined by the oscillations of j (cf. [27]).

A characteristic feature of the quantum oscillations in varying fields is the nonlinearity that appears already at relatively low values of the incident field. When $\omega\tau \ll 1$, according to our remarks the total induction $B = B_0 + B$ appears in (2.2). Field intensities in the incident wave of the order of the period of the quantum oscillations, i.e.,

$$B \approx B_0 \frac{\hbar\Omega}{e_0} \approx \Delta B, \quad (9.5)$$

are sufficient to make the nonlinear effects important. The case of $B \gg \Delta B$ is especially interesting, since it leads to a unique pseudoresonance. [34]

*It is understood that both of these also reduce the quantum oscillations Δj but they also reduce the classical value of j in the same way, so that the relative values $\Delta j/j$ is unchanged.

To simplify the calculation we assume that the conditions for a normal skin effect are satisfied: $\delta \gg l$, so that $\delta \gg r$ (where δ is the skin depth). In this case the field at distances r, l can be taken to be uniform, and all the relations are local. In lowest approximation we can use the formulas obtained for the static case.

We shall assume that the magnetic susceptibility is small (i.e., according to (2.2), $(v/c)^2(\epsilon_0/\hbar\Omega)^{3/2} \ll 1$), and treat the magnetic moment as a perturbation in the Maxwell equations. Then, in lowest approximation the varying electromagnetic field in the metal has the usual form: $\mathbf{E} = \mathbf{E}_0 \exp(-z/\delta) \cos(\omega\tau - z/\delta)$, $H_1 = H_{1,0} \exp(-z/\delta) \cos(\omega\tau - z/\delta)$, while in the next approximation the zero-order magnetic moment appears in the Maxwell equations, i.e., (since we showed earlier that we can use the static formulas) when $(\epsilon_0/\hbar\Omega)(H_1/H)^2 \ll 1$,

$$M = M_0 \cos(kH_1 + \beta),$$

where $\beta = cS/e\hbar H$, $k = (cS/e\hbar H^2) \sin \alpha$ (α is the angle between the direction of H and the normal to the metal surface). In order for nonlinear effects to be important it is necessary that $kH_{1,0} \gg 1$. In this case the value of M oscillates rapidly with depth at distances $\delta/kH_{1,0}$ small compared to the skin depth δ . It is therefore natural that the field intensity at the surface will be determined only by the value of the magnetic moment on the surface:

$$M = M_0 \cos(kH_{1,0} \cos \omega t + \beta).$$

In particular, the reflection coefficient for the n -th harmonic is related to the n -th Fourier coefficient of the expansion of the function $\cos(kH_{1,0} \cos \omega t + \beta)$, i.e.,

$$c_n = \frac{1}{\pi} \int_0^{2\pi} \exp[i(kH_{1,0} \cos x - nx)] dx = \exp(-i\pi n/2) J_n(kH_{1,0}).$$

It is easily seen that when $kH_{1,0} \gg 1$ several cases are possible. If $n > kH_{1,0}$, c_n decreases exponentially with increasing $n/kH_{1,0}$; if $n < kH_{1,0}$, then $c_n \approx n(kH_{1,0})^{-3/2} \times [\text{nkH}_{1,0} - (n + 1/2)\pi/2]$. Finally, for $n = kH_{1,0}$, there is a peculiar pseudoresonance: $c_n \sim n^{1/3}$, the relative half-width being proportional to $n^{-2/3}$. As is clear from our remarks, this pseudoresonance has an essentially quantum mechanical nonlinear character.

Let us estimate the amplitude of the resonance harmonic. First, it is proportional to the amplitude of the magnetic moment, which, for $\hbar\Omega > 2\pi^2 T$, is of order $M \approx H(v/c)^2(\epsilon_0/\hbar\Omega)^{1/2}$ (cf. (2.1)). Secondly there is a factor $\sqrt{\omega/\sigma}$ because of the fact that the effect is nonlinear.

The conductivity σ appearing in the equations has a different order of magnitude from the conductivity σ_0 in the absence of the magnetic field (cf. [2]). If the sections of the Fermi surface are closed curves, the calculations show that $\sigma \approx (r/l)\sigma_0$, if the number of electrons is not equal to the number of holes, and $\sigma \approx (r/l)^2\sigma_0$ if $n_e = n_h$. (It is assumed that H is not parallel to the surface of the sample.)

II. KINETICS

10. Static Skin Effect

Starting from simple physical arguments, we obtain the formula for the current density \mathbf{j} and the uncompen-

sated charge density ρ' in the conductor. If n is the distribution function and n_0 is the equilibrium distribution function in the absence of the electric field,

$$\mathbf{j} = \frac{2e}{\hbar^3} \int v n dp, \quad \mathbf{v} = \frac{\partial \epsilon}{\partial \mathbf{p}}, \quad (10.1)$$

$$\rho' = \frac{2e}{\hbar^3} \int (n - n_0) dp. \quad (10.2)$$

(In formula (10.2) we have taken $\rho' = 0$ in the equilibrium state.)

The external field causes a quasiparticle having energy $\epsilon = \epsilon(\mathbf{p})$ to acquire the energy $\Delta\epsilon(\mathbf{p})$. It is clear that the number of quasiparticles now having energy ϵ is equal to the number of particles that previously had energy $\epsilon - \Delta\epsilon(\mathbf{p})$, i.e., $n(\mathbf{p}) = n_0(\epsilon(\mathbf{p}) - \Delta\epsilon(\mathbf{p}))$, or, in the linear approximation in the field (which is altogether sufficient in practice and which is all we shall consider)

$$n(\mathbf{p}) = n_0(\epsilon) - \frac{\partial n_0}{\partial \epsilon} \Delta\epsilon(\mathbf{p}).$$

Since in the equilibrium state (in the absence of the field) there are no currents or uncompensated charges in normal metals,

$$\mathbf{j} = -\frac{2e}{\hbar^3} \int v \frac{\partial n_0}{\partial \epsilon} \Delta\epsilon(\mathbf{p}) dp, \quad (10.3)$$

$$\rho' = -\frac{2e}{\hbar^3} \int \frac{\partial n_0}{\partial \epsilon} \Delta\epsilon(\mathbf{p}) dp. \quad (10.4)$$

(Mathematically, the term with $n_0(\epsilon)$ in the expression for \mathbf{j} vanishes because of the central symmetry of the function $\epsilon(\mathbf{p})$: $\epsilon(-\mathbf{p}) = \epsilon(\mathbf{p})$.)

Considering that $-\frac{\partial n_0}{\partial \epsilon} \approx \delta(\epsilon - \epsilon_0)$, since $\epsilon_0 \gg T$,

while $dp = |eH/c| d\epsilon dp_z dt$, we find

$$\mathbf{j} = \frac{2e}{\hbar^3} \left| \frac{eH}{c} \right| \sum_{\epsilon(\mathbf{p})=\epsilon_0} \int v \Delta\epsilon dp_z dt \equiv \frac{1}{e} \sum \langle v \Delta\epsilon \rangle, \quad (10.3a)$$

$$\rho' = \frac{2e}{\hbar^3} \left| \frac{eH}{c} \right| \sum_{\epsilon(\mathbf{p})=\epsilon_0} \int \Delta\epsilon dp_z dt \equiv \frac{1}{e} \sum \langle \Delta\epsilon \rangle. \quad (10.4a)$$

When there are several bands present the sum goes over all bands. In the absence of collisions

$$\dot{\mathbf{p}} = e\mathbf{E} + \frac{e}{c} [\mathbf{v}\mathbf{B}], \quad (10.5)$$

and so

$$\dot{\epsilon} = \frac{\partial \epsilon}{\partial \mathbf{p}} \dot{\mathbf{p}} = \mathbf{v}\mathbf{p} = e\mathbf{E}\mathbf{v}. \quad (10.6)$$

If the distance which the quasiparticle traverses between collisions is Δr (this is only an order of magnitude estimate)

$$\Delta\epsilon \approx e\mathbf{E}\Delta r. \quad (10.7)$$

Thus

$$\sigma_{ik} \approx \sum \langle v_i \Delta r_k \rangle, \quad (10.8)$$

where the angular brackets denote an average over the Fermi surface in the sense of (10.3a) and (10.4a).

From the formulas we see that the change of the conductivity tensor in a constant magnetic field is related to the change in the path traversed by the electron in a given direction without collision. While the length of this path along the magnetic field $\mathbf{B} \parallel z$, just as in the absence of the field, is of the order of the mean free path l (so that $\sigma_{zz}(\mathbf{B}) \approx \sigma_{zz}(0) \approx \sigma_0$ —the conductivity for $\mathbf{B} = 0$), perpendicular to the field $\Delta r_{\perp} \approx r \approx l\gamma$, and

the corresponding components σ_k in strong field ($r \ll l$) are markedly reduced. There is a marked anisotropy: if either i or k does not coincide with z , $\sigma_{ik} \lesssim \sigma_0 \gamma$, $\gamma \sim r/l \sim B^{-1}$, $\gamma \ll 1$.

From the Onsager symmetry principle we have for the kinetic coefficients: $\sigma_{ik}(H) = \sigma_{ki}(-H)$. Then $\sigma_{ii}(-H) = \sigma_{ii}(H)$ is an even function of γ and consequently the linear term in the expansion of $\sigma_{ii}(\gamma)$ vanishes, $\sigma_{xx} \approx \sigma_{yy} \approx \sigma_0 \gamma^2$.

The components σ_{xy} , σ_{yx} are also "peculiar." They are obviously determined by the drift in crossed electric and magnetic fields, and this velocity is equal to cE/B and is independent of the dispersion law.

Therefore*

$$\sigma_{xy} = \sum (\pm en) \frac{c}{H}, \quad (10.9)$$

where n , the number of carriers in a given band, is taken with a plus sign for electrons and a minus sign for holes, since the motion of a hole produces a field in the opposite direction to a current of electrons. If $n_e = n_h$, in the absence of collisions the drift current (10.9) is absent, the term linear in H^{-1} in σ_{xy} vanishes, and

$$\sigma_{xy} \approx \sigma_0 \gamma^2. \quad (10.10)$$

As a result, using the Onsager principle,

$$\hat{\sigma} = \sigma_0 \begin{pmatrix} a_{11} \gamma^2 & & & \\ -a_{12} \gamma \left(1 - \frac{n_e}{n_h}\right) + \tilde{a}_{12} \gamma^2 & a_{22} \gamma^2 & & \\ -a_{13} \gamma & & -a_{23} \gamma & \\ & & & a_{33} \end{pmatrix}, \quad (10.11)$$

$$\gamma = \frac{r}{l} \propto B^{-1} \quad (a_{ik} \approx 1)$$

$$\hat{\rho} = \hat{\sigma}^{-1} = \begin{pmatrix} b_{11} \left[\left(1 - \frac{n_e}{n_h}\right) + \gamma \alpha_{11} \right]^{-1} & b_{12} \gamma^{-1} \left[\left(1 - \frac{n_e}{n_h}\right) + \gamma \alpha_{12} \right]^{-1} & b_{13} \gamma^{-1} \\ -b_{21} \left[\left(1 - \frac{n_e}{n_h}\right) - \gamma \alpha_{21} \right]^{-1} \gamma^{-1} & b_{22} \left[\left(1 - \frac{n_e}{n_h}\right) + \gamma^2 \alpha_{22} \right]^{-1} & b_{23} \gamma^{-1} \\ -b_{31} \gamma^{-1} & & b_{33} \end{pmatrix}, \quad (10.12)$$

$$(b_{ik} \approx \alpha_{ik} \approx 1),$$

which coincides with the results of the exact calculation (cf. [2]). (It would not be difficult to give the analogous arguments and obtain the asymptotic form for $\bar{\sigma}$ and $\bar{\rho}$ for open surfaces, and the Kapitza law for polycrystals. In the latter case one would have to take into account that when the magnetic field is oriented at an angle $(\pi/2) - \varphi$ to the axis of the crimped cylinder, $\epsilon(\mathbf{p}) = \epsilon$ (Fig. 10) the path $\Delta y \approx r/(\varphi + \gamma)$.

From the formulas we see the difference in principle between the behavior of a conductor with and without the magnetic field. In the latter case, all (nonvanishing in the anisotropic case) components of the conductivity tensor go to infinity with increasing free path (since $\sigma_{ik} \sim l$). In the magnetic field, when $l \rightarrow \infty$ (so that the magnetic field automatically becomes strong, $r \ll l \rightarrow \infty$), the components σ_{xx} , σ_{yy} , and for $n_e = n_h$, the components σ_{xy} , σ_{yx} also, tend to zero, since they are proportional to l^{-1} (according to the formulas given above $\sigma_0 \gamma^2 \sim l$ (r/l) $^2 \sim l^{-1}$). There is an especially clear distinction between the behavior of a bulky conductor when $H \neq 0$ and $H = 0$ in the case of $n_e = n_h$: for $H = 0$ the sample with an infinite free path is an ideal conductor (resistance $\rho = 0$), while for $H \neq 0$ it is an ideal dielectric ($\rho = \infty$).

*We are here discussing closed Fermi surfaces. The case of open Fermi surfaces is treated similarly (cf. [3^b]).

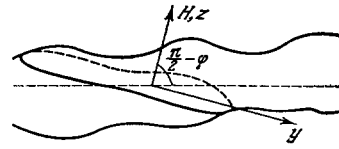


FIG. 10

Curiously this leads to a minimum on the curve of $\rho_H(T)$, where T is the temperature. In fact, so long as $l < r$, $\rho \sim l^{-1}$ along with l^{-1} decreases with decreasing temperature. But when we reach $l \approx r$ in a given magnetic field, further increase of l with decrease in temperature leads to an increase of $\rho \sim l$.

In a nonuniform field the drift velocity depends on the local characteristics of the orbit and the electron and "hole" currents no longer compensate. As a result

$$\sigma_{xy}(\xi) \approx \sigma_0 \left\{ \gamma f\left(\frac{\xi}{r}\right) + \gamma^2 \right\}, \quad (10.13)$$

where $f(1) = 1$, $f(x)$ damps to zero over distances of order unity (ξ is along the normal to the conductor surface). Near the surface the relation $\sigma_{ii}(-H) = \sigma_{ii}(H)$ is not valid, and so $\sigma_{xx} \approx \sigma_{yy} \approx \gamma \sigma_0$.

If the sample is thin (thickness $d \ll l$) the main contribution, even in the total current, comes, according to (10.13) from a layer of order r near the surface, and we have a static skin effect, predicted in [3⁰] and calculated in detail in [15]. The total current is independent of the plate thickness.

We now proceed to construct a consistent theory of the static skin effect. First we point out that even the basic system of equations in the microscopic treatment differs from the equations in the macroscopic case. [9]

In the treatment of geometric effects and their influence on the current and field distributions in the weakly inhomogeneous case, the relation between the current density and electric field $\mathbf{E} = -\nabla\varphi$ (since $\text{curl } \mathbf{E} = 0$) obviously remains the same as in the infinitely extended metal:

$$\mathbf{j} = \hat{\sigma} \mathbf{E} = -\hat{\sigma} \nabla \varphi, \quad (10.14)$$

where σ_{ik} is independent of the coordinates.

The fundamental equation for determining the field distribution in the sample is the second order equation for φ , the equation of continuity of charge in the static case:

$$\text{div } \mathbf{j} = 0, \quad (10.15)$$

i.e.,

$$(\nabla, \hat{\sigma} \nabla) \varphi = 0. \quad (10.16)$$

The boundary conditions for this equation guarantee the equation of continuity of the charge on the surface. Charges do not pass through the surface of the conductor (their emission can be neglected to high accuracy), outside the conductor $\mathbf{j} = 0$, and so

$$\mathbf{j}_n = 0 \quad (10.17)$$

(n is the internal normal to the conductor surface), i.e.,

$$(\hat{\sigma} \nabla \varphi)_n = 0. \quad (10.18)$$

(In the general case where there are contacts present, $\mathbf{j}_n = \mathbf{i}_n$, where \mathbf{i}_n is the current brought to the particular point through the contacts.)

For known φ the charge density is determined by the equation

$$\rho' = \frac{1}{4\pi} \operatorname{div} \mathbf{E} = \frac{1}{4\pi} \Delta\varphi. \quad (10.19)$$

Because of the high charge density of the electrons in a conductor, in lowest approximation we may take (since $\rho' \ll \rho_0$)

$$\rho' = 0. \quad (10.20)$$

In a thin plate ($d \ll l$), and for constructing a microscopic theory even in the general case, the connection between \mathbf{j} and φ must be found from microscopic theory, where, because of the inhomogeneity of the field $\hat{\sigma}$ is an operator in the coordinates. The conservation of charge is guaranteed automatically in a correct microscopic theory, and whatever $\varphi(\mathbf{r})$, the operator $\hat{\sigma}$ must be such that (10.15) is satisfied automatically as a result of the form of $\hat{\sigma}$.

In the microscopic theory the condition for reflection of charges from the conductor surfaces must also be such as to automatically conserve charge on the surface, i.e., so that (10.17) is automatically satisfied.

As for Eq. (10.20), the microscopic theory naturally gives not only a relation of \mathbf{j} to φ , but also an integral relation of ρ' to φ . As a result (10.20) is an integral equation for determining φ (and so requires no auxiliary boundary conditions).

In connection with the fact that (10.20) is the fundamental equation for φ , we give the justification for the substitution of (10.20) for (10.19) in more detail. From the definition (10.2) for ρ' it is clear that $\rho' = e^2 \gamma \hat{L}\nu$, where $\nu \approx n/\epsilon_0$ is a characteristic charge density, and \hat{L} is a linear integral operator of order unity*: $\hat{L}\varphi \approx \varphi$ (compare (10.2) with (10.1)). If φ changes over distances of order λ , it follows from (10.19) that $\hat{L}\varphi \approx (\lambda_0/\lambda)^2 \varphi$, where $\lambda_0 = (e\sqrt{4\pi\nu})^{-1}$. In good metals with a number of electrons of the order of one per atom, λ_0 is of the order of the interatomic spacing 10^{-8} cm, in bismuth type semimetals $\lambda_0 \approx 10^{-7}-10^{-8}$ cm; in extrinsic semiconductors, even at such low temperature that degeneracy becomes important, for $n \approx 10^{12}$ cm $^{-3}$ and an effective mass $m \approx 10^{-29}$ gm, $\lambda_0 \approx 10^{-6}$ cm (since $\lambda_0 \propto m^{-1/2} n^{-1/6}$, it depends very weakly on n).

This means that keeping the right side in (10.19) is only an improvement of accuracy even in the region where the boundary condition for the electrons scattered from the surface is "developing," so that, in fact, (10.19) reduces to $\hat{L}\varphi = 0$, which is formally equivalent to (10.20). It is clear that the replacement of \mathbf{E} by \mathbf{D} in (10.19) cannot change the character of the estimates made. In varying fields the replacement of (10.19) by (10.20) is equivalent to neglecting the displacement current. It should be emphasized that ρ' differs from the extremely small $\hat{L}\varphi$ by a dimensional factor, so to determine the value of ρ' we must, after getting φ from (10.20), use Eq. (10.19). (This corresponds to the next approximation to the solution of (10.19).)

The replacement of (10.19) by (10.20) also has a fundamental meaning. The point is that, as is clear from (10.19), λ_0 determines the depth for attenuation of external fields. This means that it is only to an accuracy determined by the ratio λ_0/λ that we can speak of the resistance as an internal characteristic of the conductor,

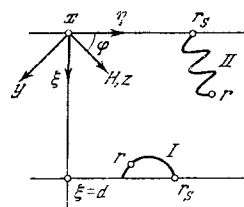


FIG. 11

independent of external electric fields. (For example, by bringing a current-carrying conductor between the plates of a condenser, and changing the potentials on the condenser plates, one could to this accuracy change the output of Joule heat in the conductor for a given current strength and given contacts.)

Mathematically, having determined ρ' inside the sample as a function of φ and the external currents on the surface, we arrive, according to (10.19), at a Poisson-type equation, for whose single-valued solution we should give the potential on the surface. Thus the complete solution requires either the simultaneous assignment on the conductor of the current density and potential, or giving the "outputs" beyond the conductor and a treatment of the interior problem.*

We now proceed to the solution of this problem in the lowest approximation in d/l (i.e., for $l = \infty$) for a plate $0 \leq \xi \leq d$ (Fig. 11). To find $\Delta\epsilon$ appearing in formulas (10.3a), (10.4a), we must determine the character of the scattering of electrons at the surface. We shall assume that the scattering is diffuse, i.e., after a collision the particle "forgets" its prehistory and establishes a Fermi distribution. (If the magnetic field is not too weak (cf. Sec. 13), so that the de Broglie wave length is small compared to the characteristic sizes of surface defects, this assumption is valid.) But the corresponding distribution is not in general an equilibrium distribution, but has a different chemical potential. Its value will be determined by the density of reflected particles, given by the law of conservation of charge on the surface. But this means identity of the fluxes of incident and reflected particles, and not of their densities.

For diffuse reflection the energy $\delta\epsilon$ "in addition" to $\epsilon = \epsilon_0$ with which the electron emerges from a point of the surface r_s , is isotropic over angles and (since $\epsilon = \epsilon_0$) can depend only on r_s . The energy acquired along the path from r_s to r (cf. Fig. 11) in the absence of collisions ($l = \infty$), is, according to (10.6), $-e[\varphi(r) - \varphi(r_s)]$, so that

$$\Delta\epsilon = \delta\epsilon(r_s) + e[\varphi(r_s) - \varphi(r)] - [h(r_s) - \varphi(r)]e. \quad (10.21)$$

$h(r_s)$ is an unknown function which must be determined from the conditions for conservation of charge on the surface, i.e., from the boundary condition guaranteeing that (10.17) is satisfied.

If at a given "orbital" time t the electron is at the point r and has a given p_z (in the zeroth approximation in \mathbf{E} , t and p_z , together with $\epsilon = \epsilon_0$, uniquely determine the state of the electron, i.e., its quasimomentum \mathbf{p}), then $r_s = r_s(r, t, p_z)$ and is found by identifying the path

*In particular, in the absence of external fields (except for emfs) for $\mathbf{D} = \mathbf{E}$, because of the continuity of φ and $\partial\varphi/\partial n$ that follows from the Maxwell equations, in determining φ we must require the vanishing of φ at infinity. Then $\lim_{\lambda_0 \rightarrow 0} (\lambda_0 (\partial\varphi/\partial n))$ is finite.

to the surface and the distance covered by the electron:

$$\mathbf{r} - \mathbf{r}_s = \mathbf{r}(t) - \mathbf{r}(\lambda) = \int_{\lambda}^t \mathbf{v}(t') dt', \quad \lambda < t, \quad (10.21a)$$

where λ is the moment of collision with the surface; since the orbit when extended formally beyond the surface may intersect the surface at several points, λ is the root of (10.21a) closest to t (less than t); it is clear that the collision occurs with the surface on which $v_n > 0$ (where \mathbf{n} is the internal normal to the surface.*). In the case of a plate, Eq. (10.21a) reduces to

$$\xi(\lambda) = \xi(t) - \xi + \xi_s, \quad \lambda < t, \quad \lambda = \lambda_{\text{min}}, \quad \left. \begin{array}{l} \xi_s = \begin{cases} 0 & v_{\xi}(\lambda) > 0, \\ d & v_{\xi}(\lambda) < 0. \end{cases} \end{array} \right\} \quad (10.22)$$

The function $h(\mathbf{r}_S)$ uniquely determines the current density. Substitution of (10.21) in (10.3a) gives

$$\mathbf{j} = \sum \langle \mathbf{v} h(\mathbf{r}_S) \rangle \quad (10.23)$$

and the potential φ ; from (10.21), (10.20) and (10.4a) it follows that

$$\varphi(\mathbf{r}) = \frac{\sum \langle h(\mathbf{r}_S) \rangle}{\sum \langle 1 \rangle}. \quad (10.24)$$

Since $\text{curl } \mathbf{E} = 0$, \mathbf{E}_x and \mathbf{E}_η (the choice of axes is shown in Fig. 11) do not depend on ξ , so that $\varphi(\mathbf{r}) = -\mathbf{E}_x x - \mathbf{E}_\eta \eta - \varphi_1(\xi)$.

Thus, using (10.24), it is convenient to look for

$$h(\mathbf{r}_S) = -E_x x - E_\eta \eta + h(\xi_s). \quad (10.25)$$

We now determine $h(\xi_S)$. According to (10.17)

$$j_n = \sum \langle v_n h \rangle^S = \sum \langle v_n h \rangle_+^S + \sum \langle v_n h \rangle_-^S = 0,$$

$$\langle f \rangle = \frac{2e^2}{\hbar^3} \left| \frac{eH}{c} \right| \int f dp_x dt = \overline{\langle f \rangle}, \quad \overline{\varphi} = \int_0^T \varphi dt, \quad \langle v_n \rangle = 0, \quad (10.26)$$

where the superscript S means that the point is taken on the surface, while the plus and minus refer, respectively, to electrons reflected from the same surface (type I in Fig. 11) or from the opposite surface (type II). Substituting (10.25) in (10.26), we find

$$h(0) - h(d) = \frac{1}{\sum \langle v_{\xi} \rangle_+} \left\{ E_x \sum \langle v_{\xi}(t)(x(t) - x(\lambda_0)) \rangle + E_\eta \sum \langle v_{\xi}(t)(\eta(t) - \eta(\lambda_0)) \rangle \right\}, \quad (10.27)$$

where λ_0 is given, for electrons colliding with the surface $\xi = 0$, by

$$\xi(\lambda_0) = \xi(t), \quad \lambda < t, \quad (10.22a)$$

and for electrons colliding with the surface $\xi = d$, by

$$\xi(\lambda_0) = \xi(t) + d, \quad \lambda < t. \quad (10.22b)$$

From (10.22) it is clear that

$$v_{\xi}(t) dt = v_{\xi}(\lambda) d\lambda, \quad (10.28)$$

which enables us to simplify (10.27) considerably:

$$h(0) - h(d) = d E_\eta \text{ctg } \varphi \quad (10.29)$$

*In the general case, the three equations (10.21a) together with the equation of the surface $G(\mathbf{r}_S) = 0$ determine four quantities: \mathbf{r}_S (three components) and λ .

(for the definition of φ see Fig. 11).

This is sufficient to determine the difference of potential from (10.24) and the current density from (10.23); change of h by a constant does not affect them. The function $h(\mathbf{r})$ (which is needed only when $\xi = 0, d$) can be written in the form

$$h = -E_x x - E_\eta \eta + E_\eta \xi \text{ctg } \varphi = -E_x x - \frac{E_\eta \eta}{\sin \varphi}. \quad (10.30)$$

Formulas (10.23), (10.24) (10.27) give the exact solution of our problem in any magnetic field (arbitrary magnitude and direction). An exception is $\varphi = 0$, when the result becomes meaningless. This is natural: when $\varphi = 0$ there are always electrons whose orbits are "shifted" in the plate, and which in this case move without collisions and guarantee infinite conductivity.

We substitute (10.30) and (10.22) in (10.23):

$$j = \sum \langle v(t)(K(t) - K(\lambda)) \rangle, \quad (10.31)$$

$$K(t) = -E_x x(t) - \frac{E_\eta \eta(t)}{\sin \varphi}, \quad (10.32)$$

where $K(t)$ is a function of t with period T . We note that it follows automatically from (10.31) and (10.28) that $j_{\xi}(\xi) = 0$.

From (10.31)

$$j_{\alpha}(\xi) = \sigma_{\alpha\beta}(\xi) E_{\beta} \quad (10.33)$$

(α, β take on the values x, η) where $\sigma_{\alpha, \beta}(\xi) \sim r$.

To calculate the current density from (10.31) it is convenient to use the fact that $K(\lambda)$ is periodic in λ with period T , while λ changes by an amount T when ξ changes by \bar{v}_{ξ} . We expand $K(\lambda^{\pm})$ in Fourier series in ξ :

$$K(\lambda^{\pm}) = \sum_{n=-\infty}^{\infty} \kappa_n^{\pm} \exp\left(\frac{2\pi i n \xi}{\bar{v}_{\xi}}\right); \quad (10.34)$$

then from (10.31)

$$j_{\alpha} = \sum_{m=-\infty}^{\infty} j_m^{\alpha}, \quad j_0^{\alpha} = \overline{v_{\alpha} K} - \frac{\bar{v}_{\alpha} \bar{v}_{\xi} K}{\bar{v}_{\xi}}, \quad (10.35)$$

$$j_m^{\alpha} = \frac{1}{2\pi i m} \left\{ T \exp\left[\frac{2\pi i n}{|\bar{v}_{\xi}|} \left(\frac{d}{2} + \text{sign } \bar{v}_{\xi} \left(\frac{d}{2} + \xi\right)\right)\right] \times \overline{v_{\alpha}(t) \exp\left(-\frac{2\pi i m \xi(t)}{\bar{v}_{\xi}}\right) \dot{K}(t') \exp\left(\frac{2\pi i m \xi(t')}{\bar{v}_{\xi}}\right)} \right\}. \quad (10.36)$$

Let us examine in more detail the case of strong magnetic fields: $r \ll d$.

Noting that $\overline{v_x x} = 0 = \overline{v_y y}$, $\sum \langle v_x y \rangle = -\sum \langle v_y x \rangle \sim n_e - n_h$, we find, for $d \gg r$,

$$j_0^{\alpha} = \mp \frac{(n_e - n_h) e c}{H \sin \varphi} E_{\eta, x}. \quad (10.37)$$

Let us explain the physical meaning of this result. If the number of holes in a metal is equal to the number of electrons, then $j_0^{\alpha} = 0$, and the current at any depth is related to terms of the type of (10.36). But because of the p_z integration, these terms oscillate and fall off rapidly as we move away from the plate surface (because $d \gg r$). The main contribution to the total current when $r \ll d$ comes only from a surface layer with thickness of order r near the surface (to see this one need only compute the total current in a region far from the surface of the plate). In lowest approximation in r/d (i.e., as $d \rightarrow \infty$) the current density on the surface is zero, at a depth of order r it rises to a value of order

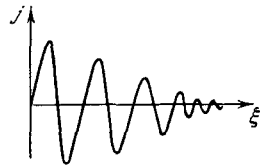


FIG. 12.

$n_e c E/H$, and at great depths $\xi \gg r$, it damps out, while oscillating, in proportion to $(r_0/\xi) \cos(\xi/r_0)$; $r_0 = (\bar{v}\xi)_{\max}$ (Fig. 12).

For just this reason the total current when $n_e = n_h$ is independent of the plate thickness and proportional to r^2 . The current in "deep" layers of the metal leads only to small Sondheimer oscillations,^[32] treated in detail in^[33], for which we can get the exact formula from (10.36). (When $d \gg r$ they are not difficult to calculate: in (10.36) the main contribution comes from a region near the limits of the p_z integration, i.e., near the turning points p_0 .)

The damping of the current in the bulk of the metal, shown in Fig. 12, is also a static skin effect. It occurs only when $n_e = n_h$ and does not change the dependence of the resistance on magnetic field. In fact, let us find the resistance of the plate. To do this we must remember that even though the plate is long in the direction μ (even compared to d) it is still finite, so the total current can only flow in the direction $\xi \perp \mu$, ξ , and $J_\mu = \int j_\mu d\mu = 0$ (the equation of continuity in integral form, guaranteeing absence of a flux of charge through the boundary of the plate in the μ direction). Since

$$J_{x, \eta} = \mp \frac{(n_e - n_h) e c}{B \sin \psi} dE_{\eta, x} + \frac{n_e e^2}{m v} r^2 a_p^{x, \eta} E_\beta, \quad a_p^{x, \eta} \approx 1,$$

$R = E_\xi / J_\xi$ has the form

$$R(B, d) \approx \begin{cases} R(0, d) \propto d^{-2} (n_e \neq n_h), \\ R(0, d) \left(\frac{d}{r}\right)^2 \propto r^{-2} (n_e = n_h) \end{cases}$$

and is independent of the plate thickness when $n_e = n_h$.

So far we have been discussing only the one-dimensional case. From the presentation it is, however, clear that the method described is applicable^[31a] to a sample of arbitrary shape with arbitrary arrangement and form of current lead contacts. The last question is of particular interest, since it is connected with the focusing of a current in a strong magnetic field. Then the sample resistance also changes significantly. In particular, for contact dimensions small compared to r , it is inversely proportional to the area of the contacts.^[31a]

The scattering of electrons from surfaces is always assumed to be diffuse. In semimetals it may be close to specular (because of the large deBroglie wavelength*). This leads^[31b] to the Kapitza law—a linear dependence of the resistance on magnetic field.

In good metals the reflection is, for this same rea-

*In semimetals there may be still another reason for specular reflection — the curving of the energy bands near the surface, at a distance of the order of the deBroglie wave length from the surface: the resulting potential barrier does not let charges through to it (cf. [69]). But if the bending of the band forms a potential well, the charges do not penetrate from the surface into the bulk of the semimetal. In this case surface levels appear, which can be detected from the resonance they produce.

son, almost specular for electrons colliding with the surface at small angles (cf. Sec. 13). As a result there is a possibility of determining the angular dependence of the reflection coefficient from the dependence of the resistance on magnetic field.

We have so far discussed only closed Fermi surfaces. Open surfaces also lead to the appearance of a static skin effect; the theory for this case was developed in^[31b]. Experimentally the static skin effect has been studied for cadmium,^[52] tungsten,^[53] indium and aluminum^[54] and bismuth.^[54,55]

11. Alternating Fields (Classical Case). Low Frequencies

It was shown in the preceding section that even a weak nonuniformity can significantly change the resistance of a conductor in a strong magnetic field ($\gamma \ll 1$). Since the term with the nonuniformity is proportional to γ , while the homogeneous term is proportional to γ^2 , it is sufficient for the characteristic distance associated with the inhomogeneity to be of order $r\gamma^{-1} \ll l$. The inhomogeneity may occur for various reasons. One of the simplest ways of getting it is the skin effect. In this case (as contrasted with the "ordinary" skin-effect, which is determined by the conductivity in a uniform field) it is natural to expect a different dependence of resistance on magnetic field even at very low frequencies, where the "usual" skin depth is of the order of the free path:

$$\delta \approx \sqrt{\frac{c^2}{2\pi\omega\sigma(B)}} \ll l. \quad (11.1)$$

In good, highly pure, metals at low temperatures, when $l \approx 1-3$ mm, this corresponds to $\omega_1 \approx 1-100$ sec.

It is clear from the start that the damping of the field and current when $\omega \gtrsim \omega_1$ will have a very peculiar character under the influence of the static skin effect. The almost uniform electric field, just as in Sec. 10, produces a current which drops mainly over distances of the order of the Larmor radius r , as in Fig. 12. Because of the difference in the damping of the field and the charge density, at not too high frequencies one can solve the problem by successive approximations. One must first determine the strength of the electric field. So long as it changes over distances large compared to r (this determines what we mean by "not too high frequencies"), the character of the scattering of the electrons from the surface is unimportant, and the presence of the surface does not affect the conductivity tensor. This permits one to continue the electric field into the halfspace outside the conductor $z \geq 0$ and write

$$j(z) = \int_{-\infty}^{\infty} \hat{\sigma}(|z-z'|) E(z') dz'. \quad (11.2)$$

As a result, the Maxwell equations in the one-dimensional case

$$E_\alpha'' = \frac{4\pi i \omega}{c^2} j_\alpha(\alpha = x, y), \quad j_z = 0 \quad (11.3)$$

are quickly solved by going over to Fourier components in the coordinates (where we of course take account of the jump in E'_α at $z = 0$, associated with the continuation of E_α as an even function):

$$-k^2 E_\alpha - 2E'_\alpha(0) = \frac{4\pi i \omega}{c^2} \tilde{\sigma}_{\alpha j}(k) E_j(k),$$

$$\tilde{\sigma}_{z j}(k) E_j(k) = 0 \tag{11.4}$$

and

$$\tilde{\sigma}_{i j}(k) = 2 \int_0^\infty \cos(kz) \sigma_{i j}(z) dz, \quad E(k) = 2 \int_0^\infty \cos(kz) E(z) dz, \tag{11.4a}$$

where i and j take the values x, y, z . Having determined $E(z)$ from (11.4), we must write the next approximation—a formula for rapidly falling j (which depends on the character of the reflection from the surface) analogous to our procedure in Sec. 10. In the case of interest, when E changes over distances large compared to r , the term to take account of the surface reflection will be the same as in the static case. (Since the characteristic distance over which $\hat{\sigma}$ changes is r , we can assume E to be homogeneous.) Thus the total current J_α is (cf. Sec. 10, $n_e = n_h$):

$$J_\alpha = J_\alpha^{(0)} + J_\alpha^{(stat.)} = (\hat{Z}_0^{-1})_{\alpha\beta} E_\beta(0) + (1-q) \frac{n_e e^2 r^2}{p} a_{\alpha\beta} E_\beta(0), \tag{11.5}$$

$a_{\alpha\beta} \sim 1,$

where \hat{Z}_0 is given by (11.4), J_α^{stat} is independent of ω, l , and q is the coefficient of reflection of charges from the surface, which in strong magnetic fields at large skin depths for good metals is probably close to zero (since the angles at which the charges strike the surface are not small).

Using the definition of impedance $E_\alpha(0) = Z_{\alpha\beta} J_\beta$, we find

$$\hat{Z}^{-1} = \hat{Z}_0^{-1} + \frac{n_e e^2 r^2}{p} \hat{a} (1-q). \tag{11.6}$$

The order of $\tilde{\sigma}_{\alpha\beta}(k)$ is easily determined in the same way as at the start of Sec. 10, if we remember that $\sigma_{\alpha\beta}(k) = \sigma_{\alpha\beta}(-k)$. In lowest approximation, over a wide range of frequencies, the impedance when $q \neq 1$ is independent of frequency and coincides with the static resistance of the plate with $l = \infty$ when $d \rightarrow \infty$.

Without giving the results of the computation for the whole range of frequencies and magnetic fields, we point out the peculiar resonance at superlow frequencies,^[35] which occurs for $n_e = n_h$ in a magnetic field normal to the surface (x, y) of the metal, when the surface coincides with a crystal plane. In this case, because of the symmetry of the problem, an electric field in the x, y plane produces a current only in that plane, so that $\sigma_{\alpha z} = \sigma_{z\alpha} = 0$, and in (11.4) $E_z = 0$, while $\sigma_{\alpha\beta}$ has the form

$$\hat{\sigma} = \sigma_0 \begin{pmatrix} \gamma^2 a_{11}, & \gamma^2 a_{12} + \gamma (kr)^2 b_{12} \\ \gamma^2 a_{12} - \gamma (kr)^2 b_{12}, & \gamma^2 a_{22} \end{pmatrix}. \tag{11.7}$$

When

$$\frac{4\pi\omega\sigma_0\gamma r^2}{c^2} = 1 \tag{11.8}$$

the determinant of Eqs. (11.4) and (11.7) vanishes, and the equations no longer have a solution. The reason is a resonance of one of the principal values of \hat{Z}_0 . To calculate \hat{Z}_0 near resonance, we must continue the expansion of $\sigma_{xy}(\gamma, k) = \sigma_{yx}(-\gamma, k) = \sigma_{xy}(\gamma, -k)$:

$$\sigma_{xy} = \sigma_0 (\gamma^2 a_{12} + \gamma (kr)^2 b_{12} + \gamma (kr)^4 c_{12}). \tag{11.9}$$

(we point out the meaning of this expansion: it says for example, that

$$j_x = \sigma_0 (\gamma^2 a_{11} E_x + \gamma^2 a_{12} E_y + \gamma r^2 b_{12} E'_y + \gamma r^4 c_{12} E''_y), \tag{11.10}$$

i.e., for $l = \infty$ the current density depends strongly on the fourth derivative of the electric field with respect to the coordinates!) The half-width of the resonance is of order $\gamma^{1/2}$, the effective attenuation depth of the field at resonance is $\sim r\gamma^{-1/4}$. Both the real and imaginary parts of the impedance \hat{Z}_0 (which also means, for specular reflection of the charges from the surface, Z) increase by the factor $\gamma^{-1/4}$ at resonance. For reflection other than specular, the weak spatial dispersion of E results, as in the static case, in a strong spatial dispersion of the current density, which damps over distances of order r . As a result, in lowest approximation in γ the impedance is independent of frequency and coincides with the static resistance of the plate with $l = \infty$ for $d \rightarrow \infty$ and fixed q .^[67,68] In the next approximation a dependence on ω and a resonant contribution to the impedance appears; at resonance it is of order $\gamma^{3/4}$; the corresponding correction to $Z'(B)$ is of order $\gamma^{1/4}$. The resonance is extremely sensitive to the geometry of the problem (as can be seen by considering the general case).

The characteristic feature of the resonance is the peculiar dependence of the resonance frequency on magnetic field

$$\omega \approx \Omega \left(\frac{c\Omega}{v\omega_0} \right)^2 \propto H^3 \tag{11.11}$$

(where ω_0 is the plasma frequency). The condition $\Omega\tau > 1$ means that frequencies

$$\omega > \frac{1}{\tau} \left(\frac{v\omega_0\tau}{c} \right)^{-2}, \tag{11.12}$$

are sufficient, which agrees with (11.1).

The resonance is most apparent in the lattices of cubic symmetry, when $a_{12} = 0$ in (11.7) and σ_{xy} is completely determined by the inhomogeneity of the varying field. Then the impedance is diagonal for a circularly polarized wave:

$$E_\pm = E_x \pm iE_y. \tag{11.13}$$

Equations (11.4) give

$$E'_\pm = \frac{4\pi i \omega}{c^2} j_\pm = \frac{4\pi i \omega}{c^2} (\gamma^2 \sigma_0 a_{11} E_\pm \pm i \sigma_0 \gamma r^2 b_{12} E'_\pm), \tag{11.14}$$

and at resonance E''_\pm in (11.14) vanishes, so one must retain E^{IV}_\pm .

Of particular interest in the nonresonant region is the appearance of a skin effect of a new type, different both from the normal and the anomalous skin effects: the varying electric field on the one hand, and the varying magnetic field on the other, fall off over distances of different orders of magnitude. This skin effect occurs over a wide range of frequencies (or constant magnetic fields) corresponding to $\delta_{an} \gg r \gg \delta_n$ (where δ_n and δ_{an} are the skin depths for the normal and anomalous effects, respectively). Since it is of the same kind as the static skin effect (cf. Sec. 10), in which the electric field is uniform, but the current attenuates into the depth of the metal, it is natural to call it the surface skin effect. The theory of the surface effect is developed in ^[67] (for a half-space) and ^[68] (case of a thin plate).

12. Quantum Oscillations of the Resistance of a Metal (Shubnikov-de Haas Effect)

As was shown in Sec. 1 of part I, in sufficiently strong magnetic fields, when quasistationary diamagnetic levels can be established,

$$\Omega\tau > 2\pi^2, \quad (12.1)$$

and when the number of levels that play a significant role is rigorously determined: the smearing out of the Fermi energy is small compared to the separation between diamagnetic levels,

$$\hbar\Omega > 2\pi^2T, \quad (12.2)$$

quantization in the magnetic field leads to oscillations in both thermodynamic quantities (de Haas-van Alphen effect; cf. Sec. 1) and in kinetic quantities, such as the resistance (Shubnikov-de Haas effect). Let us consider the latter.

In order to picture the situation arising in this case, we introduce (following [38a]) (entirely formally, merely for simplicity of calculation) a mean free time τ . Then the quantum kinetic equation for the density matrix \hat{n}

$$\dot{\hat{n}} = \hat{\mathcal{H}}_0 \hat{n} + \hat{n}' \quad (12.3)$$

($n = n_0(\hat{\mathcal{H}}_0)$ is the equilibrium density matrix, $\hat{\mathcal{H}}_0$ is the Hamiltonian operator in the absence of the electric field) takes the form

$$\frac{d\hat{n}}{dt} = \frac{i}{\hbar} [\hat{\mathcal{H}}_0 \hat{n}] + \frac{\hat{n} - \hat{n}_0}{\tau} = 0, \quad (12.4)$$

where

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 - eE\hat{r}. \quad (12.5)$$

In the linear approximation in the electric field, substituting (12.5) and (12.3) in (12.4), we get

$$\frac{i}{\hbar} [\hat{\mathcal{H}}_0, \hat{n}'] + \frac{\hat{n}'}{\tau} = \frac{i}{\hbar} [eE\hat{r}, \hat{n}_0]. \quad (12.6)$$

Going over to matrix elements for a known spectrum and using the fact that $\hat{\mathcal{H}}_0$ and \hat{n}_0 are diagonal, and that

$$\begin{aligned} \frac{i}{\hbar} [r, n_0]_{nn'} &= \frac{i}{\hbar} (n_0(\epsilon_{n'}) - n_0(\epsilon_n)) r_{nn'} \\ &= \frac{i}{\hbar} \frac{n_0(\epsilon_{n'}) - n_0(\epsilon_n)}{\epsilon_{n'} - \epsilon_n} [r, \hat{\mathcal{H}}_0]_{nn'} = -\frac{n_0(\epsilon_{n'}) - n_0(\epsilon_n)}{\epsilon_{n'} - \epsilon_n} v_{nn'}, \end{aligned} \quad (12.7)$$

we find from (12.6)

$$\left[\frac{i}{\hbar} (\epsilon_{n'} - \epsilon_n) + \frac{1}{\tau} \right] n'_{nn'} = -\frac{n_0(\epsilon_{n'}) - n_0(\epsilon_n)}{\epsilon_{n'} - \epsilon_n} eEv_{nn'}, \quad (12.8)$$

so that

$$n'_{nn'} = -\frac{n_0(\epsilon_{n'}) - n_0(\epsilon_n)}{\epsilon_{n'} - \epsilon_n} \frac{eEv_{nn'}}{\frac{i}{\hbar} (\epsilon_{n'} - \epsilon_n) + \frac{1}{\tau}}. \quad (12.9)$$

Since $\epsilon_{n'} - \epsilon_n \approx (n' - n)\hbar\Omega$, it is clear from (12.1) and (12.2) that both the first and second factors in (12.9) have a sharp maximum at $n' = n$.

We know that the diagonal matrix elements in the quasiclassical case are equal to the time average of the classical quantity. [38b] But, as was shown in Chap. I, Sec. 1, the quantum oscillations are determined by the sections of the Fermi surface that have an extremal area, where the mean velocity of the electron vanishes (since $\bar{v}_x = \bar{v}_y = 0$, while

$$\bar{v}_z = \left(\frac{\partial \epsilon}{\partial n} \right)_{p_z} \approx \oint \frac{\partial \epsilon}{\partial p_z} \frac{dl}{v_{\perp}} \approx \oint \frac{\partial p_{\perp}}{\partial p_z} dl \approx \frac{\partial S}{\partial p_z}.$$

Thus, it appears that the term which would give the main contribution to the quantum oscillations drops out, and that this is related to the uniform averaging over the orbit, i.e., to the uniformity over space.

Consequently, the terms in (12.8) that are spatially inhomogeneous, even though they appear small in the equations, could be decisive for the resistance oscillations. (This was first pointed out in [37].)

It is easy to understand that, strictly speaking, there should be such terms in the kinetic equation, and then it is clear from our remarks that it may be necessary to retain them.

To be specific, we consider elastic scattering on impurities. In the classical case, conservation of energy in the collisions is guaranteed by the delta function $\delta(\epsilon - \epsilon')$, where in lowest approximation we may take the energy in the absence of the electric field. In the quantum case where, in crossed electric and magnetic fields ($\mathbf{E} \perp \mathbf{H}$) stationary states appear, that correspond to discrete levels for a given p_z ($z \parallel \mathbf{H}$), we must write $\delta(\epsilon - eEyy - \epsilon' + eEyy')$.

Then terms appear (inhomogeneous in the classical case) whose contribution should be estimated. (The fact that including the electric field in the collision integral reduces to these terms was shown by Kosevich and Andreev [38] using Bogolyubov's method. [39])

It is clear that the corresponding terms on the right side of (12.8) are of order $(eEr/\hbar\Omega)(n_0/\tau)$ (since $y \approx r$). On the other hand, the right side of (12.8) is of order $eEv(n_0/\hbar\Omega)(\Delta n/n_0) \approx eEv(n_0/\hbar\Omega)(\hbar\Omega/\epsilon_0)$. Since in the homogeneous field r plays the role of l in the nondiagonal terms in the conductivity, i.e., only the next approximation in $(\Omega\tau)^{-1}$ gives a contribution to the current, the contribution from the usual right side of (12.8) to the current is a term of order $eEv(n_0/\hbar\Omega)(\hbar\Omega/\epsilon_0)(1/\Omega\tau)$. Overall, the role of the "additional" inhomogeneous terms is $\epsilon_0/\hbar\Omega \gg 1$ times as important as that of the "usual" terms.

We emphasize that, as is clear from the above remarks, this is entirely due to the homogeneity of the field. It is easy to see that in a significantly inhomogeneous field, in particular for the anomalous skin effect, one should not include the "additional" dependence of the collision integral on the electric field (as is confirmed by direct computations).

The expression for the quantum integral for collisions with impurities is easily transformed. The summation in the collision integral runs only over differences of quantum numbers: the shift of the energy by the potential energy in the electric field has to be taken into account, in the linear approximation in E , only in quantities corresponding to the zeroth approximation. As a result we get (cf. [38])

$$\Delta\sigma_{\text{osc}}^{\alpha\beta} \approx -B^2 u_m^{\alpha\beta}(\epsilon_0) \frac{d \ln(S_{\text{exp}}(\epsilon_0))}{d\epsilon_0} \frac{\partial M_z}{\partial B}. \quad (12.10)$$

The oscillations of the chemical potential give a small contribution to the conductivity oscillations (cf. Sec. 1 of Part I). The other elements of the tensor $\Delta\sigma^{\text{ik}}$ can be calculated similarly.

Thus the oscillations of the conductivity are expressed in terms of the oscillations of the magnetic moment M_z , where the oscillation amplitude is determined in order of magnitude by the classical "mobility tensor"

$$u_{\alpha\beta}|_{S=S_{ext}, \epsilon=\epsilon_0} \equiv u_m^{\alpha\beta}.$$

A consistent calculation of the conductivity tensor in a magnetic field in the quasiclassical, as well as in the quantum and ultraquantum cases, using diagrammatic techniques, has been given by Abrikosov.^[63]

The construction of a quantum theory of thermomagnetic phenomena can be carried through in a similar quantum theory of galvanomagnetic phenomena, but requires the inclusion in the quantum case of the magnetic moment that develops and affects the kinetic coefficients (in the same way as in the case of a varying field; cf. Sec. 9, Part I).

13. Alternating Fields (Quantum Case). Surface Levels and Determination of Coefficient of Reflection of Electrons from Surfaces

Our whole discussion has been related to the quantization of the electron energy spectrum in a constant magnetic field in an infinite sample. But it is of considerable interest to discuss the character of surface levels, since they can, in particular, give information about the nature of the scattering of electrons from the surface of the conductor. It is clear beforehand that the character of the scattering is determined by the ratio of the de Broglie wavelength λ_D of the incident charges and the characteristic size of defects of the surface. In good metals λ_D is of the order of the interatomic spacing, so that for small angles of incidence, the reflection of electrons from the surface can be regarded as diffuse. This means that the natural width of a level is surely of the same order as their spacing (since it is determined by the same frequency) and we cannot speak of a quantization of the spectrum for such electrons. As the angle of incidence φ decreases, the de Broglie wavelength in the direction normal to the surface increases proportionally to φ^{-1} , so that at sufficiently small angles of incidence the reflection of electrons from the surface can be regarded as almost specular.

Such a situation arises if the "glancing" electrons are important. This is just what occurs in the anomalous skin effect, when the effective skin depth $\delta_{\text{eff}} \approx (\delta^2 l)^{1/3}$ is small compared to the mean free path l and, in the absence of a magnetic field, angles $\varphi \approx (\delta/l)^{2/3}$ are important,^[41] which may correspond to $\varphi \approx 10^{-3}$.

In a weak magnetic field ($r \gg l$) at high frequencies, angles $\varphi \approx (\delta_{\text{eff}}/r)^{2/3}$ (Fig. 13) are important. Thus the study of the quantization of the orbits shown in Fig. 13 is not of purely academic interest.

In the quasiclassical case the quantization of the levels of electrons colliding with the surface in a magnetic field parallel to it (for simplicity) is found from the same arguments as is the quantization in the infinite case (cf. Sec. 1, Part I), and leads to an analogous result (cf. for example^[31])

$$S(\epsilon, p_z, P_z) = \frac{nehH}{c} \quad (13.1)$$

(in weak fields the difference between H and B is unimportant), where S is the area of the orbit cut out by the sample surface in momentum space. It naturally depends on the conserved quantity P_x , which determines the position of the orbit ($y = (c/eH)(p_x - P_x)$), and is determined by the area of the section ($y \geq 0$):

$$\epsilon(p_x, p_y, p_z) = \epsilon, \quad p_z = \text{const}, \quad p_x \leq P_x. \quad (13.2)$$

The area of the orbit in momentum space differs from its area in coordinate space only by the factor $(eH/c)^2$ (since it follows from $\dot{\mathbf{p}} = e/c\mathbf{v} \times \mathbf{H}$ that $y = (cp_x/eH) + y_0$, $x = -(cp_y/eH) + x_0$). For small φ , as is clear from Fig. 13;

$$y = y_{\text{max}} - \alpha x^2, \quad y_{\text{max}} = x_{\text{max}}\varphi, \quad \alpha = R^{-1} \quad (13.3)$$

(where R is the radius of curvature at the point $= y_{\text{max}}$), so that

$$y_{\text{max}} = R\varphi^2, \quad x_{\text{max}} = R\varphi.$$

Quantization of the area of the orbit gives a quantization of the angles φ :

$$\frac{2}{3} R^2 \varphi^3 = \frac{neh}{eH}, \quad \varphi_n = \left(\frac{3}{2} \frac{neh}{eHR^2} \right)^{1/3}. \quad (13.4)$$

Since R is of the order of the Larmor radius ($\epsilon \approx \epsilon_0$),

$$\varphi_n \approx \left(\frac{eHh}{cp_0^2} n \right)^{1/3} \approx \left(\frac{h\Omega}{\epsilon} \right)^{1/3} n^{1/3}. \quad (13.5)$$

Since $x = v_x^0 t \approx v_0 t$, the period of revolution $T_\varphi = 2\Delta x_{\text{max}}/v_x^0 = 2R\varphi_n(v_x^0)^{-1}$, and the frequencies Ω_n of revolution in such orbits are

$$\Omega_n(\epsilon_0, p_z) = \frac{\pi v_x^0}{R\varphi_n} \approx \Omega \left(\frac{\epsilon_0}{h\Omega} \right)^{1/3} n^{-1/3}. \quad (13.6)$$

The spacing of the corresponding levels is

$$\Delta\epsilon_n = \hbar\Omega_n. \quad (13.7)$$

The natural width of the level $\delta\epsilon$ is determined by the time for a free path $(\delta\epsilon)_\tau \approx \hbar/\tau$ and the deviation of the scattering from purely specular reflection. If after each collision the probability for diffuse scattering is q ($q \ll 1$) then the number of orbits to lead to practically complete scattering is $(1-q)^n \approx 1/2$, $n \sim 1/q$ and $(\delta\epsilon)_q \approx \hbar\Omega_n/q$. Thus the necessary conditions for the existence of quasistationary levels (13.7) are

$$q(\varphi_n) \ll 1, \quad \Omega\tau \left(\frac{\epsilon_0}{h\Omega} \right)^{1/3} n^{-1/3} \gg 1, \quad (13.8)$$

which can be satisfied in weak fields H if $q[(\omega_0\tau)^{-1/2}] \ll 1$. For $n \approx 1$ Eq. (13.1) gives a crude approximation; since such n 's may be important one can obtain a more

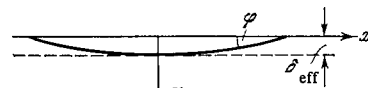


FIG. 13

accurate formula for the levels in the case of interest here (first done in^[42]) using the expansion of $\epsilon = \epsilon(p_x, p_y)$ near the point $\epsilon(p_x^0, p_y^0) = \epsilon_0$, $v_y(p_x^0, p_y^0) = 0$, $v_x^0 < 0$, or Eq. (13.3), with the substitution

$$x = i\hbar \frac{\partial}{\partial p_x} = \frac{ich}{eH} \frac{\partial}{\partial y}.$$

The result is

$$\epsilon_n = \hbar v_x^0(p_z) \left(\frac{eH}{h} \right)^{2/3} [2R(p_z)]^{-1/3} a_n, \quad \text{Ai}(-a_n) = 0, \quad (13.9)$$

where $\text{Ai}(x)$ is the Airy function ($\text{Ai}'' - x \text{Ai} = 0$).

For the case where the Fermi surface is a cylinder, (13.7), or the corresponding exact formula (13.9) for

$\epsilon_n - \epsilon_m$, directly determines the frequencies Ω of possible resonances:

$$\omega = k\Omega_n, \quad \Omega \propto \omega^{3/2},$$

which are very small, do not depend on the skin depth, and are proportional to $\omega^{3/2}$ ($\hbar\omega = \epsilon_n - \epsilon_m$ corresponds to the harmonic number of ω relative to the classical revolution frequency Ω : $\omega = (n - m)\Omega$). The corresponding magnetic fields are limited both from below (second condition in (13.8)) and from above, since with increasing H , φ increases $\propto H^{1/3}$ while for sufficiently large φ the reflection from the surface becomes essentially diffuse.

In the general case of an arbitrary Fermi surface, when ϵ_n and $\hbar\Omega_{nm} = \epsilon_n - \epsilon_m$ depend on p_z , resonance occurs, as in the classical case of cyclotron resonance, when $p_z = p_0$, when the frequency Ω deviates most slowly from resonance: $\Omega'(p_0) = 0$. It is understood that specular reflection may, in sufficiently weak magnetic fields, also affect classical effects, in particular, cyclotron resonance, which disappears for $q = 1$.^[43] For this it is naturally necessary that $q[(\delta/l)^{1/2}] \ll 1$.

The half-width of the resonance is determined by the natural width of the level (cf. above)

$$\frac{\Delta\omega}{\omega_n} \approx q(\varphi_n) + (\omega\tau)^{-1}. \quad (13.10)$$

The relative height of the resonance is essentially different for cylindrical and noncylindrical Fermi surfaces. Since in the latter case only the electrons in an interval $\Delta p_z/p_0 \approx (\Delta\omega/\omega)^{1/2}$ participate in the resonance (since $\Omega \propto (p_z - p_0)^2$), the relative height of the resonance will be correspondingly smaller.

Another important point is whether or not the resonance orbit shifts within the skin layer, since this (i.e., the quantity $(\hbar\Omega/\epsilon_0)^{2/3} r/\delta$) affects the effective path in the field and the relative number of electrons accelerated in the field. In the quasiclassical range ($n \gg 1$) an exact formula can be gotten for the impedance. When $\hbar\omega \ll T$ it looks especially simple, since we can then use the correspondence principle, replacing, for arbitrary q in formulas (9) and (10) of ^[44], $1/\tau$ by $1/\tau + i\omega$, α by ∞ , the "classical" integration over φ by a "quantum-mechanical" summation over φ_n , and taking $E(\mu)$ under the integral sign. For $\hbar\omega \gtrsim T$, $\frac{\partial n_0}{\partial \epsilon}$ is replaced by $[n_0^2(\epsilon_{k+k'}) - n_0(\epsilon_k)]/(\epsilon_{k+k'} - \epsilon_k)$, where k is the set of quantum numbers (n, p_z, P_x).

It is of particular interest to get the shape of the resonance curve for arbitrary $q(\varphi)$ (in the quasiclassical case, this is not difficult, using our remarks above). Comparison of experiment with theory would enable one to get information about the form of $q(\varphi)$.

Actually the transition from a resonance orbit shifting in the skin layer to a nonresonant orbit, which affects the shape of the curve, allows one to determine R from the relation $y_{\max} = R\varphi_n^2 \propto \delta_{\text{eff}}$ (where δ_{eff} is expressed in terms of the impedance), since $R\varphi_n \sim R^{1/3}$; cf. (13.4). According to (13.6) the resonance frequencies give v_x^0 . Knowing R and v_x^0 , we can from the dependence of the resonance width on n when $\omega\tau \gg q^{-1}$, find $q(\varphi)$ (cf. (13.10)).

A tilting of the magnetic field weakens the resonance for $p_z = 0$ (the electrons will penetrate into the depth of the metal, carrying out nonperiodic collisions with the

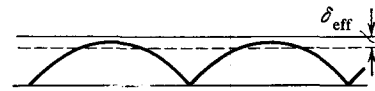


FIG. 14

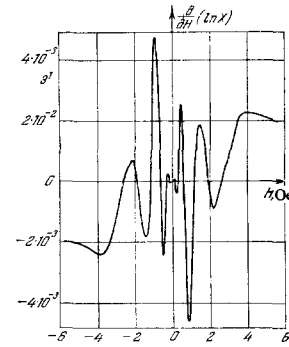


FIG. 15

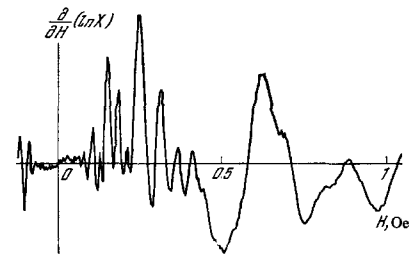


FIG. 16

surface of the metal). However, the presence of a new parameter—the inclination of the field, increases the information obtainable (especially for resonance when $p_z = 0$, when there is no drift in the bulk of the metal).

Analogous effects are also possible in superconductors.^[42, 71] V. G. Peschanskiy^[51] called attention to the existence of a classical resonance in weak fields in a thin (thickness $d \ll l$) rigorously plane-parallel plate at frequencies corresponding to $\sqrt{Rd}/v_0 = 2\pi n/\omega$, when $H \propto (\omega/n)^2$. The nature of the resonance is clear from Fig. 14. This effect can also be used for determining $q(\varphi)$, since the resonance has a half-width $[\gamma + q(\frac{1}{2}\sqrt{R/d})]$. For finding $q(\varphi)$ there are other useful effects, sensitive to the angle of collision of the electrons with the surface. The search for and study of these effects, which are of great interest, is essentially only just beginning.

The resonant oscillations described in this section were discovered by Khaikin^[45] and then observed by Koch and Kip^[46] and Koch and Kuo.^[47] The first classical explanation, relating the oscillations with electrons that do not collide with the surface, was proposed in ^[46]. Khaikin^[48] pointed out the decisive role of electrons reflected specularly from the metal surface; van Gelder^[49] noted the purely quantum character of the oscillations and obtained the correct frequency dependence of their period; in ^[42] the resonance period was discovered and a qualitative theory was given; in ^[56] the quantitative theory was given; in ^[57] there is a

quasiclassical computation of the surface levels, an experimental study of the oscillations in bismuth and a comparison with the theory. A detailed survey of experimental and theoretical work on diamagnetic surface levels was given in [58].

Figure 15 shows the results of the first experimental observation^[45] of surface oscillations on tin at $T = 3.8^\circ\text{K}$ at a frequency $\omega = 9.4$ MHz ($H \parallel C_4 \perp N \parallel C_2$, $J \parallel C_2$); Fig. 16 shows the detailed picture of the oscillations observed^[48] in bismuth ($T = 1.7^\circ\text{K}$, $\omega = 9.7$ MHz, $H \parallel C_1 \perp N \parallel C_3$). In both experiments, $\frac{\partial}{\partial H}(\ln X)$ was determined as a function of H (X is the imaginary part of the surface impedance).

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