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ELASTIC SCATTERING OF HIGH-ENERGY PARTICLES BY NUCLEI

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I. OPTICAL MODEL

UNTIL recently, the basis of all the calculations of elastic scattering of fast particles by nuclei was the semiphenomenological optical model (see, e.g., [1-3]). After the phase shifts are determined from the solution of the wave equation with an experimentally selected potential, the scattering cross section $\sigma(\theta)$ is obtained by simple summation. For example, for scattering of spinless particles

$$\sigma(\theta) = |A(\theta)|^2, \tag{1}$$

$$\mathcal{A}(\theta) = \lambda i \sum_{l=0}^{\infty} (2l+1)(1-e^{2i\eta_l}) P_l(\cos\theta)/2. \tag{2}$$

At high energies, when the de Broglie wavelength λ becomes many times smaller than the dimensions of the target nucleus, and an appreciable contribution is made to the sum over l by a very large number of terms, the sum in formula (2) is conveniently replaced by an integral, and it is best to use the asymptotic relation

$$P_l(\cos\theta) \approx J_0((2l+1)\sin\theta/2) \approx J_0(l\theta), \tag{3}$$

where J_0 is the well known Bessel function.

In the high-energy region it is possible to use also another approximation, namely, to express the scattering phase shifts in terms of a complex refractive index $K(T, r) = n(T, r) + ik(T, r)$, i.e., we can put

$$\eta_l(T) = \frac{1}{2} \int_{\lambda l}^{\infty} \frac{rK(T, r)\,dr}{[r^2+(\lambda l)^2]^{1/2}}, \tag{4}$$

where

$$n(T, r) = \alpha(T)k(T, r), \quad k(T, r) = A\sigma_t(T)d(r), \tag{5}$$

σ_t is a typical cross section for the interaction between the scattering particle and the nucleon of the nucleus, α is the ratio of the real and imaginary parts of the amplitude of elastic scattering of this particle by the intranuclear nucleon, and $d(r)$ is the nucleon density in a target nucleus with atomic number A; this density is chosen by comparison with experiment. Here and throughout T is the kinetic energy of the scattered particle in the laboratory frame.

Since the characteristics of the interaction of the particle with the bound intranuclear nucleon differs somewhat from the corresponding characteristics of free-particle interaction, it becomes necessary to introduce into (5) correction factors chosen from a comparison with experiment and dependent, generally speaking, on the energy. The large number of adjustment parameters that must be introduced into the formulas cannot fail to give rise to an unsatisfactory feeling: the optical model turns out to be too "attached" to experiment.

The degree of agreement between the optical curve and experiment is seen in Figs. 1—3. It is particularly difficult to fit the theoretical curves to the experimental data at large scattering angles, where higher diffraction minima and maxima make a contribution. As a rule, if the experimental data are sufficiently accurate, it is possible to obtain here only qualitative agreement, otherwise it is necessary to use too complicated optical potentials with a large number of parameters, which change quite irregularly with changing energy of the scattering particles and with changing type of target nucleus.

This is particularly noticeable for light nuclei such as deuterons, helium, etc. To obtain agreement with experiment in this case, it is necessary to choose optical potentials with a radial dependence that is difficult to reconcile with what we know from experiments on electron scattering. For example, in order for the optical curve to coincide with the experimental data of the Brookhaven Laboratory on elastic scattering of protons

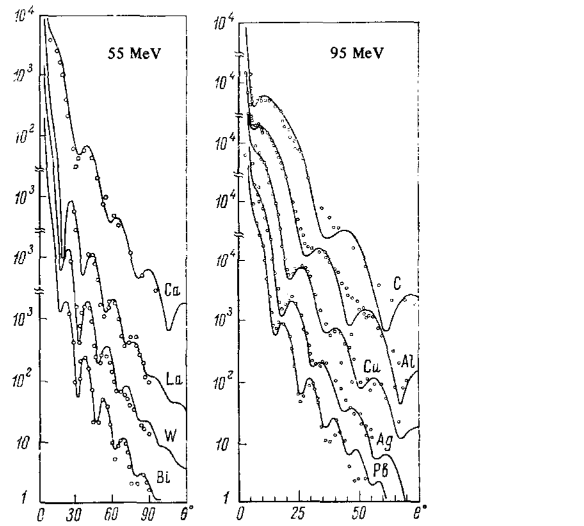

FIG. 1. Differential cross section of elastic scattering of protons by different nuclei (mb/sr, c.m.s.). Circles—experimental data [4-6], curves-optical-model calculation [7].

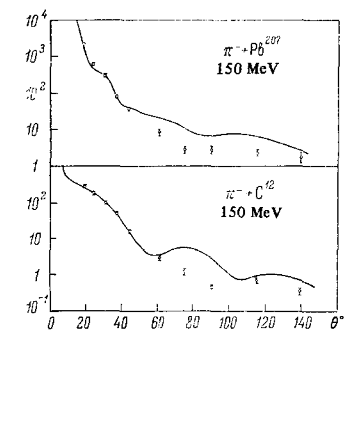

FIG. 2. Differential cross section of elastic scattering of pions (mb/sr, c.m.s.) [8]. Curves–optical-model calculation.

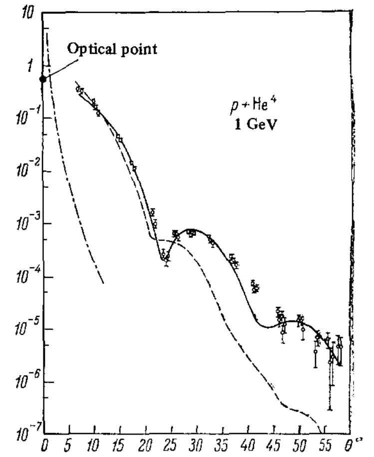

FIG. 3. Cross section of elastic scattering of protons by helium (b/sr, c.m.s.) [9]. Dashed line—optical calculation with Gaussian potential obtained from experiments with scattering of electrons by He^4 nuclei. An approximate agreement with experiment can be obtained in this case only for the first diffraction peak. Solid curve—optical calculation in the case when the potential is chosen in the form of a Woods-Saxon function that decreases rapidly in the boundary and has a radius $c = 1.6 \times 10^{-13}$ cm and a diffuseness parameter $\alpha = 3.1 \times 10^{-14}$ cm. Dash-dot curve—cross section of Coulomb scattering by the He^4 nucleus, calculated from the Mott formula with a Gaussian form factor obtained from experiments with electron scattering. At angles $\theta > 2$–$3°$, this cross section becomes negligibly small.

and He^4 at an energy T = 1 GeV[9], it is necessary to have a potential with a sharp boundary, whereas experiments with electron scattering indicate that the boundary of the He^4 nucleus is highly diffuse. Allowance for the dimensions of the scattered proton only increases the effective diffuseness parameter.

The use in optical calculations of a Gaussian potential taken from experiments on electron scattering does not reproduce the sharp minimum in the angular distribution at $\theta \approx 24°$, and yields at large scattering angles cross sections that are too high by two or three orders of magnitude compared with experiment (see Fig. 3). This is physically connected with the obvious fact that the diffraction effects and the refraction of the de Broglie wavelength of the scattered proton become all the more noticeable, the more inhomogeneous the scattering and refractive the media.

We see thus that the optical model does not take into account some very important physical details. To cope with the situation, let us consider in greater detail the passage of a particle through a medium made up of individual scattering and absorbing centers (nucleons).

II. THEORY OF MULTIPLE DIFFRACTION SCATTERING

In the ordinary optical model the microstructure of the medium is not considered and the substance is represented as some continuous medium. We can hope that a more accurate analysis will enable us to establish corrections that must be introduced in such an approximate optical picture and to determine more accurately the limits of its applicability.

We neglect for the time being the dependence of the interactions on the spins. This is fully justified at high

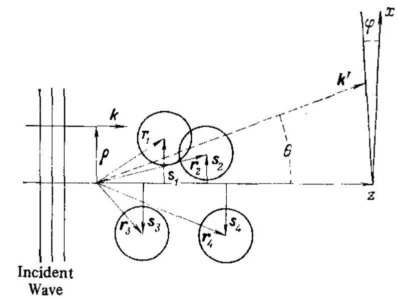

FIG. 4. Scattering of a particle with momentum k by a system consisting of several nucleons. The positions of the individual nucleons are determined by the three-dimensional vectors $\mathbf{r}_i$; $\mathbf{s}_i$ are two-dimensional vectors characterizing the positions of the nucleons in a plane perpendicular to the momentum $\mathbf{k}$ (the z axis is parallel to $\mathbf{k}$); ρ is the two-dimensional impact-parameter vector in the same plane; $\mathbf{k}$ is the momentum of the scattered particle; θ is the polar scattering angle; the azimuthal scattering angle φ, characterizing the rotation of the plane (k, z) about the z axis, is reckoned from the (x, z) plane.

energies, where the spin effects affect significantly only the details, and do not change the general picture of the interaction (see Ch. III below). The scattering of the incoming particle from an individual nucleus will be characterized by a scattering amplitude $\mathcal{A}$, but since the nucleons now need not necessarily be located at the origin ($\mathbf{r}_i \neq 0$); Fig. 4), the azimuthally-symmetrical expression (2) is no longer suitable for this amplitude.

To obtain the correct expression, we recall that the asymptotic wave function after elastic scattering represents, in the general case, a sum of a plane incident wave e^{ikz} and a scattered spherical wave with amplitude $\mathcal{A}(\theta, \varphi)$:

$$\Psi(x) \backsim e^{ikz} + \mathcal{A}(\theta, \varphi)\, e^{ikz}/r, \tag{6}$$

where

$$\mathcal{A}(\theta, \varphi) = \frac{\lambda i}{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (2l+1)(\delta_{0m} - S_{lm})\, e^{im\varphi} P_l^m(\cos\theta), \tag{7}$$

$\lambda = \hbar/k$ is the de Broglie wavelength in the c.m.s., δ_{0m} is the known Kronecker symbol, and P_l^m is an associated Legendre polynomial of order m. The particular case of azimuthal symmetry (2) is obtained under the assumption

$$S_{lm} = e^{2i\eta_l}\delta_{0m}. \tag{8}$$

If we confine ourselves to sufficiently high energies, when the scattering by the nucleon occurs mainly in the region of small angles θ and the significant l are very large, we can use the asymptotic relation between the polynomial P_l^m and the Bessel function J_m [10,11]

$$P_l^m(\cos\theta) \approx (-1)^m [l + (1/2)]^m J_m[(2l+1)\sin(\theta/2)] \tag{9}$$

(formula (3) is a particular case of this relation).

Taking next into account the integral representation of the Bessel function

$$J_m(x) = \frac{(-i)^{-m}}{2\pi} \int_0^{2\pi} e^{ix\cos\xi + im\xi}\, d\xi$$

and replacing the summation over l by an integral, we write the amplitude (7) in the form

$$\mathcal{A}(\theta, \varphi) = \frac{\lambda i}{2\pi} \int_0^{2\pi} d\xi \sum_{l=0}^{\infty} \left(l + \frac{1}{2}\right) e^{i(2l+1)\sin(\theta/2)\cos\xi} \ldots$$

$$= \left\{1 - \sum_{m=-l}^{l} i^{-m}\left(l+\frac{1}{2}\right)^m S_{lm} e^{im(\varphi+\xi)}\right\}$$

$$= \frac{i}{2\pi\lambda}\int_0^{2\pi} d\xi \sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right) e^{i(2l+1)\sin(\theta/2)\cos\xi}\left[1-e^{i\eta(\rho,\,\psi)}\right]\rho\, d\rho,$$

where $\rho = \lambda l$ is the impact parameter, $\psi = \xi + \varphi$, and

$$\sum_{m=-l}^{l} i^{-m}\left(l+\frac{1}{2}\right) S_{lm} e^{im\psi} \equiv e^{i\eta(\rho,\,\psi)}.$$

The function $\eta(\rho, \psi)$ introduced in this manner will henceforth be called the "phase."

If the variable ξ is now given the physical meaning of the angle between the direction of the vector ρ and the x axis, then in the region of small angles θ it is possible to put approximately $\rho(\mathbf{k}-\mathbf{k}') \approx 2k\rho\cos(\xi - \varphi + \pi)\sin(\theta/2)$, where $\mathbf{k}$ and $\mathbf{k}'$ are respectively the momenta of the incoming and scattered particles. The scattering amplitude is then written in the compact form

$$\mathcal{A}(0, \varphi) \equiv \mathcal{A}(q) = \frac{i}{2\pi\lambda}\int e^{i(\mathbf{k}-\mathbf{k}')\boldsymbol{\rho}}\left(1-e^{i\eta(\boldsymbol{\rho})}\right)d^2\boldsymbol{\rho} = \frac{i}{2\pi\lambda}\int e^{i\mathbf{q}\boldsymbol{\rho}}\Gamma(\boldsymbol{\rho})\,d^2\boldsymbol{\rho}, \tag{10}$$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the momentum transfer

$$\Gamma(\boldsymbol{\rho}) = 1 - e^{i\eta(\boldsymbol{\rho})} \tag{11}$$

is the so-called "profiling function," and the integration is carried out over the entire plane of the impact parameter $\boldsymbol{\rho}$.

We note that expression (11) can be regarded as the Fourier transform of the function $\Gamma(\rho)$: if the amplitude $\mathcal{A}(\mathbf{q})$ is known, this function can be obtained by taking the inverse transformation

$$\Gamma(\boldsymbol{\rho}) = \frac{\lambda}{2\pi i}\int e^{-i\mathbf{q}\boldsymbol{\rho}}\mathcal{A}(\mathbf{q})\,d^2\mathbf{q}, \tag{12}$$

where the integration is over the entire plane perpendicular to the initial momentum $\mathbf{k}$.

Expressions (10) and (11) will serve as a basis for further calculations of the scattering of the fast particle by a nucleus*. We shall attempt to express the total amplitude of such a scattering in terms of the amplitudes for scattering by individual nucleons. The first to use this approach was Glauber[12-14].

A high-energy incident particle can collide with some single nucleon inside the target nucleus or else experience successive collisions with several such nucleons. In those cases when the scattering is through an angle $\theta > \pi/2$, repeated scattering by one of these nucleons is also possible. In addition, since the radius of the interaction of the incoming particle with the nucleon can be larger than the distance between the individual nucleons in the nucleus, it may happen that the particle interacts simultaneously immediately with several intranuclear nucleons.

In the general case, the mathematical analysis of such a picture is a very complicated problem, but it is possible to obtain significant simplifications by using the "diffraction approximation."

We shall assume that the scattering from any internuclear nucleon occurs in such a way as if this nucleon were to be a small piece of an absorbing and refracting medium (see Fig. 4). Then, in analogy with the situation in optics, to determine the scattering amplitude it is necessary to find only the phase shift. The phase due to a single nucleon is

$$\chi(\boldsymbol{\rho}) = \int K(\mathbf{r})\,dr, \tag{14}$$

where $K(\mathbf{r})$ is the local refractive index, and the integration is along the particle trajectory. The main point of Glauber's theory is the assumption that the particle trajectory deviates insignificantly from a straight line, and consequently the integration along the curvilinear path in (14) can be replaced by integration along the initial motion of the particle z. Thus, the phase shift due to scattering by any one nucleon turns out to be independent of the presence of other nucleons. When a particle moves inside a nucleus having A nucleons, the total refractive index is

$$K(\mathbf{r}, \mathbf{r}_1, \ldots, \mathbf{r}_A) = \sum_{i=1}^{A} K_i(\mathbf{r}-\mathbf{r}_i)$$

and accordingly the total phase shift is equal to the sum of the phase shifts due to the scattering by the individual nucleons:

$$\chi(\boldsymbol{\rho}, \mathbf{s}_1, \ldots, \mathbf{s}_A) = \sum_{i=1}^{A}\chi_i(\boldsymbol{\rho}-\mathbf{s}_i). \tag{15}$$

In this case there is no need at all for knowing the details of the individual interactions of the incoming particle with the nucleons of the nucleus: the phase shift $\chi_i(\rho - \mathbf{s}_i)$ is determined by the Fourier component (12) of the scattering amplitude of the incident particle on the corresponding nucleon. All the partial amplitudes for scattering by individual nucleons must in this case, of course, be transformed to some common coordinate frame, for example, the laboratory frame. In accordance with expression (10), the total change of the amplitude of the incident wave is

$$\delta\mathcal{A}_{\mathrm{inc}} = \frac{i}{2\pi\lambda}\int e^{i\mathbf{q}\boldsymbol{\rho}}\Phi_f^*(\mathbf{r}_1, \ldots, \mathbf{r}_A)\,\Gamma(\boldsymbol{\rho}, \mathbf{s}_1, \ldots, \mathbf{s}_A)\,\Phi_{\mathrm{in}}(\mathbf{r}_1, \ldots, \mathbf{r}_A) \times \prod_{i=1}^{A} d^3r_i\, d^2\rho, \tag{16}$$

where

$$\Gamma(\boldsymbol{\rho}, \mathbf{s}_1, \ldots, \mathbf{s}_A) = 1 - \prod_{i=1}^{A}\{1-\Gamma_i(\boldsymbol{\rho}-\mathbf{s}_i)\}$$
$$= \sum_{i=1}^{A}\Gamma_i(\boldsymbol{\rho}-\mathbf{s}_i) + \sum_{\substack{i,\,j=1\\ i\neq j}}^{A}\Gamma_i(\boldsymbol{\rho}-\mathbf{s}_i)\,\Gamma_j(\boldsymbol{\rho}-\mathbf{s}_j)$$
$$+ \sum_{\substack{i,\,j,\,k=1\\ i\neq j\neq k}}^{A}\Gamma_i(\boldsymbol{\rho}-\mathbf{s}_i)\,\Gamma_j(\boldsymbol{\rho}-\mathbf{s}_j)\,\Gamma_k(\boldsymbol{\rho}-\mathbf{s}_k) + \ldots \tag{17}$$

is the profiling function of the nucleus, Φ_{in} and Φ_f are the wave functions describing the ground state and the motion of the nucleus before and after scattering. The physical meaning of the introduction of the wave functions of the nucleus into the amplitude (16) lies in the fact that the product $\Phi_f^*\Phi_{in}$ determines the distribution of the scattering centers inside the target.

We see that the wave scattered by the nucleus is a

*In the particular case of azimuthal symmetry, when (8) is valid and consequently the phase $\eta(\mathrm{p}, \psi)$ does not depend on ψ, it is possible to integrate over the angle variable in (10), after which we obtain the well known expression

$$\mathcal{A}(\theta) = ik\int_0^{\infty} J_0(2k\rho\sin\theta/2)\left(1-e^{i\eta(\rho)}\right)\rho\, d\rho,$$

which coincides with (2) if we make in the latter the asymptotic substitution (3) and take into account the fact that the phase $\eta(\rho)$ is now twice as large as the phase used in expression (2).

sum of terms corresponding to the scattering by individual intranuclear nucleons, scattering by two nucleons, either in sequence or simultaneously (in our approximation there is no difference), scattering by three nucleons, etc.

One of the main conditions for the validity of such an expansion is the requirement that the wave propagating inside the nucleus be little distorted by diffraction effects; only in this case is it possible to apply formulas (10) and (11) to each interaction of the wave inside the nucleus, and to speak of summation of the phases. We recall that the formulas (10) and (11) pertain to the scattering of the plane wave and, in addition, in the derivation of relation (15) it was assumed that the trajectory of the scattered particle remains almost a straight line.

In scattering by a nucleon with dimensions r_N, the diffraction effects begin to come strongly into play only at distances

$$d \gtrsim r_N^2/\lambda_L$$

(λ_L is the de Broglie wavelength in the laboratory frame), and therefore the condition for the applicability of the expansion (17) is the requirement that the "shadow region" d greatly exceed the dimensions of the nucleus R:

$$r_N^2/\lambda_L \gg R.$$

Since $R/r_N \sim A^{1/3}$, this is equivalent to the requirement

$$\lambda_L \ll r_N A^{1/3} \approx A^{1/3} 10^{-13}\,\text{cm}, \tag{18}$$

which is satisfied already at energies of several hundred MeV.

The assumption that the total refractive index $K(\mathbf{r}, \mathbf{r}_1, \ldots, \mathbf{r}_A)$ is a simple sum of the refractive indices for the scattering by the individual nucleons of the nucleus presupposes that the particle moves quite freely in the interval between two acts of scattering inside the nucleus. This, of course, is also only an approximation of the real process, which is apparently perfectly valid for the deuterium nucleus, but may lead to additional errors in the case of heavier nuclei. If we speak in the language of field theory, then in this case only the δ function is retained in the total propagator describing the motion of the particle between two interactions, and the entire remaining part is discarded.

Another limitation of the theory is the need for considering only the region of not too large scattering angles, since relations (10)–(12) have been derived precisely in such an approximation. As a result, repeated scattering by the same nucleon is excluded and there are no identical factors in the sums of formula (17).

We see therefore that the picture considered by us still remains quite approximate in many respects. There is wide scope here for various refinements and extensions. Nevertheless, as will be shown below, even at this stage it is possible to explain many new details of nuclear interactions and to make an important step forward compared with the usual optical model.

We now transform expression (16) into a form more convenient for numerical calculations. To this end we recognize first that the nuclear wave functions are

$$\Phi_{\text{in}}(\mathbf{r}_1, \ldots, \mathbf{r}_A) = e^{i\mathbf{P}'\mathcal{R}}\varphi(\mathbf{r}_1, \ldots, \mathbf{r}_A), \quad \Phi_{\text{f}}(\mathbf{r}_1, \ldots, \mathbf{r}_A) = e^{i\mathbf{P}\mathcal{R}}\varphi(\mathbf{r}_1, \ldots, \mathbf{r}_A),$$

where $\mathbf{P}$ and $\mathbf{P}'$ are the momenta of the target nucleus before and after scattering,

$$\mathcal{R} = \sum_{i=1}^{A} \mathbf{r}_i/A$$

is a vector determining the position of the mass center of this nucleus, and φ is the wave function describing the internal state of the nucleus. We introduce, further, the relative coordinates $\mathbf{r}_i' = \mathbf{r}_i - \mathcal{R}$ of the nucleons in the c.m.s. and denote $\mathbf{s}_i' = \mathbf{s}_i - \mathbf{s}$ and $\boldsymbol{\rho}' = \boldsymbol{\rho} - \mathbf{s}$, where $\mathbf{s}$ is a projection of the vector $\mathcal{R}$ on the plane perpendicular to the momentum $\mathbf{k}$ of the primary particle. Formula (16) is then written in the form

$$\delta\mathcal{A}_{\text{inc}} = \frac{i}{2\pi\lambda} \int e^{i\mathbf{q}\boldsymbol{\rho}} d^2\rho \int e^{i(\mathbf{P}-\mathbf{P}')\mathcal{R}} \varphi^*(\mathbf{r}_1, \ldots, \mathbf{r}_A)\, \Gamma(\boldsymbol{\rho}, \mathbf{s}_1, \ldots, \mathbf{s}_A)$$
$$\times \varphi(\mathbf{r}_1, \ldots, \mathbf{r}_A)\, \delta\left(R - A^{-1}\sum_{i=1}^{A} \mathbf{r}_i\right) d^3\mathcal{R} \prod_{i=1}^{A} d^3 r_i = A(\mathbf{q}) \int e^{i\mathbf{q}\mathbf{s} + i(\mathbf{P}-\mathbf{P}')R} d^3\mathcal{R},$$

where

$$\mathcal{A}(\mathbf{q}) = \frac{i}{2\pi\lambda} \int e^{i\mathbf{q}\boldsymbol{\rho}'} d^2\rho' \int \varphi^*(\mathbf{r}_1', \ldots, \mathbf{r}_A')\, \Gamma(\boldsymbol{\rho}', \mathbf{s}_1, \ldots, \mathbf{s}_A)$$
$$\times \varphi(\mathbf{r}_1', \ldots, \mathbf{r}_A')\, \delta\left(A^{-1}\sum_{k=1}^{A} \mathbf{r}_k'\right) \prod_{k=1}^{A} d^3\mathbf{r}_k$$

(the function Γ, as seen from (17), depends only on the differences $\boldsymbol{\rho} - \mathbf{s}_i = \boldsymbol{\rho}' - \mathbf{s}_i'$).

Integrating in this expression over the coordinates of the mass center, we obtain

$$\delta\mathcal{A}_{\text{inc}} = (2\pi)^3\, \delta_\perp(\mathbf{k} - \mathbf{k}' + \mathbf{P} - \mathbf{P}')\, \delta[\mathbf{k}(\mathbf{P} - \mathbf{P}')/k]\, \mathcal{A}(\mathbf{q}); \tag{19}$$

here the δ functions express the momentum conservation law, and the function $\mathcal{A}(\mathbf{q})$ is the amplitude for the scattering of the particle by the nucleus in the standard normalization.

With the aid of relations (12) and (17), we can express the profiling function Γ in terms of the amplitudes of inelastic scattering of the particle by the intranuclear nucleons, with relative c.m.s. momenta $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_A$:

$$\mathcal{A}(\mathbf{q}) = \frac{i}{2\pi\lambda} \int e^{i\mathbf{q}\boldsymbol{\rho}} d^2\rho \int \varphi^*(\mathbf{r}_1, \ldots, \mathbf{r}_A) \left\{1 - \prod_{k=1}^{A} \left[1 - \frac{\lambda}{2\pi i}\right.\right.$$
$$\left.\left. \times \int e^{-i\mathbf{q}_f(\boldsymbol{\rho} - \mathbf{s}_f)} \mathcal{A}_k(\mathbf{q}_k)\, d^2 q_k\right]\right\} \varphi(\mathbf{r}_1, \ldots, \mathbf{r}_A)\, \delta\left(\sum_{k=1}^{A} \mathbf{r}_k/A\right) \prod_{k=1}^{A} d^3\mathbf{r}_k. \tag{20}$$

In this expression we have omitted the prime signs of the space vectors, since it will be implied henceforth that all the spatial coordinates in the formulas for the nuclear scattering amplitudes pertain to the c.m.s. of the nucleus.

In accordance with our conclusion, all the nucleon amplitudes in (20) pertain in our case to the laboratory frame. This is not quite convenient in practice, since in most experimental and theoretical investigations it is customary to specify the amplitudes for scattering by a nucleon in the c.m.s. of the incoming particle and of the nucleon. As applied to our case, this means that a separate c.m.s. should be chosen for each amplitude $\mathcal{A}_i$, since the momenta of all the internal nucleons, generally speaking, are different. It is easy to show, however, that for a region of not too large scattering angles, when the recoil effects are insignificant, the scattering amplitude in the laboratory frame and the scattering amplitude in the c.m.s. of the incoming particle plus the nucleon differ only in the fact that the momentum transfer $\mathbf{q}_i$ must be chosen respectively in the laboratory frame or in the c.m.s.[13]:

$$A_i^{(L)}(\mathbf{q}_i^{(L)}) = A_i^{(C)}(\mathbf{q}_i^{(C)}). \quad (21)$$

Under all the transformations, however, it is necessary to retain the relation, which is satisfied in the c.m.s., between the square of the three-dimensional momentum transfer q^2 and the corresponding value of the 4-momentum transfer t, viz., $q^2 = -t$; therefore in all the final formulas the quantity q^2 should be interpreted as $|t|$.

It is furthermore convenient to represent the nucleon amplitude in the c.m.s. in the form

$$\mathcal{A}_i(\mathbf{q}) = ZA^{-1}\mathcal{A}_{pi}(\mathbf{q}) + (1 - ZA^{-1})\mathcal{A}_{ni}(\mathbf{q}), \quad (22)$$

where

$$\mathcal{A}_{Ni}(q) = \{\mathrm{Re}\,\mathcal{A}_{Ni}(0) + i\,\mathrm{Im}\,\mathcal{A}_{Ni}(0)\}\,e^{-\beta_N q^2/4} = (\sigma_{Nt}/4\pi\lambda)(i + \alpha_N)\,e^{-\beta_N q^2/4}; \quad (23)$$

here i is the imaginary unit, α_N is the ratio of the real and imaginary parts of the amplitude for the scattering through an angle $\theta = 0$, which does not depend on the choice of the coordinate system, and β_N is a constant known from experiment. This expression is a perfectly good approximation in the entire considered region of not too large scattering angles.

The approach to the calculation of the nuclear cross sections, which is based on formulas (20)–(30), is now customarily called the theory of multiple diffraction scattering, or simply Glauber's theory.

We have considered only one of the possible derivations of the relations of this theory, in our opinion the simplest one, which makes it possible to explain in the most lucid form the relation between Glauber's theory and the usual optical model. There are other detailed published methods of deriving Glauber's formula: the diagram method[29,30], the method based on the use of Watson's multiple scattering theory[33-35], and a few others (see, e.g.,[36]). In these approaches, the main relations of Glauber's theory are considered from somewhat different points of view, making it possible to explain more clearly which premises are the most important for the picture in question and in which directions this picture can be generalized. The analysis and comparison of different approaches to Glauber's theory would be the subject of a separate large review. The purpose of our review is to describe only the main ideas of Glauber's theory and to ascertain the extent to which this theory agrees with experiment. The interested reader can become acquainted with the more profound details in the already cited literature.

III. NUCLEON-DEUTERON SCATTERING

In order to be able to calculate the nuclear cross section from (20)–(23), it is necessary to determine first, with the aid of some model, the density of the distribution of the nucleons in the nucleus

$$\rho(\mathbf{r}_1, \ldots, \mathbf{r}_A) = \varphi^*(\mathbf{r}_1, \ldots, \mathbf{r}_A)\,\varphi(\mathbf{r}_1, \ldots, \mathbf{r}_A). \quad (24)$$

This can be done relatively easy in the case of the deuterium nucleus, for which there exists a sufficiently well developed theory. The cross sections calculated from Glauber's theory turn out in this case quite sensitive both to the choice of the deuteron wave function and to the magnitude of the real part of the nucleon amplitude $\mathrm{Re}\,\mathcal{A}_N \sim \alpha_N$ (Figs. 5–7). This circumstance can

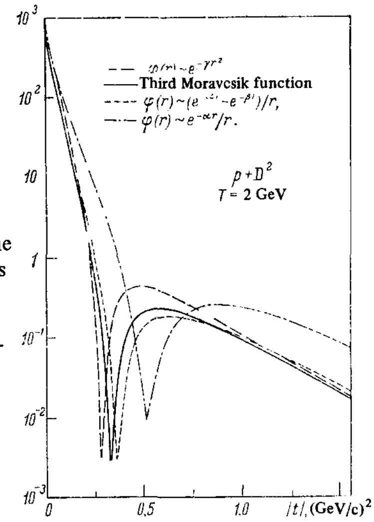

FIG. 5. Change of the cross section of elastic scattering of the proton by the deuterium nucleus as a function of the different choices of the wave function of the deuteron $\varphi(r)$ (mb/sr; laboratory frame) [20].

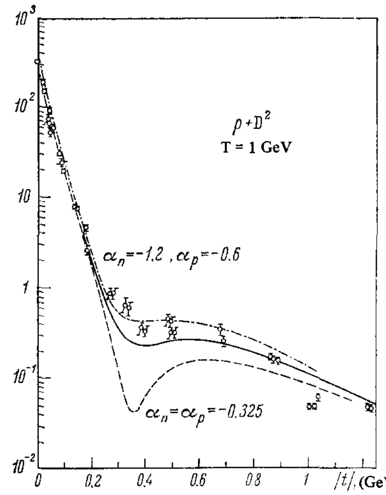

FIG. 6. Cross section for the elastic scattering of a proton by deuteron at different values of the parameter α (mb/sr; laboratory frame). Proton energy 1 GeV [9,21,38]. Solid curve–cross section calculated for the experimental values α_p and α_n with allowance for the contribution of the S and D waves to the φ function of the deuteron.

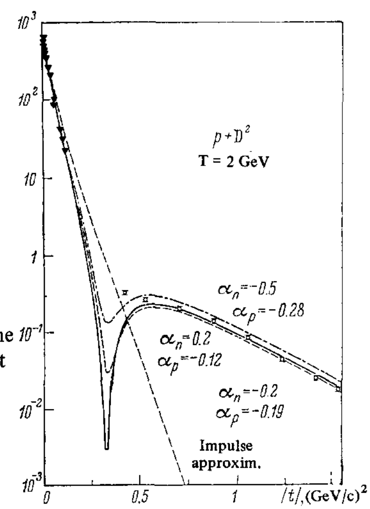

FIG. 7. The same as Fig. 6. Proton energy 2 GeV [20]. The cross section calculated in the impulse approximation is shown separately (only the term with h = 1 in the product $\prod_{k=1}^{A}[\ldots]$ in formula (20)) is retained. The contribution of the D-wave to the φ function of the deuteron is not taken into account.

be used for an experimental determination of all these quantities. In particular, recently many experiments with deuterium have been devoted to the measurement of α_N – a parameter that plays a very important role in the physics of elementary particles[15]; Glauber's theory gives a sufficiently reliable basis for the analysis of these experiments.

As applied to deuterium, Glauber's theory deserves special attention also from a different point of view. As is well known, the main source of our information concerning the cross sections for the interactions of elementary particles with the neutron are difference experiments with hydrogen and deuterium. It is very important here to be able to calculate with sufficient accuracy the correction connected with effects of screening of the nucleons in the deuterium nucleus[16]. The accuracy of calculation of this correction frequently greatly exceeds the direct cross section measurement errors. Glauber's theory makes it possible to take sufficiently correct account of the screening effects[13].

From the data on proton-deuteron scattering shown in Figs. 6 and 7 we can see that the theory of multiple diffraction scattering is in fair agreement with the known experimental data in the regions of the first and second maxima, but for $|t| \approx 0.35$ (GeV/c)^2 the theory predicts a sharp minimum, whereas experiment reveals only a change in the slope of the curve.

The situation is even worse with pion scattering (Fig. 8). In this case the experimental data also reveal no deep diffraction minimum at $|t| \approx 0.35$ (GeV/c)^2, and furthermore exceed noticeably the theoretical values.

The discrepancies between the calculations and experiment still remain in force if attempts are made to use for the deuteron different forms of spherically-symmetrical wave functions (see Fig. 5) or, more accurately, to write out the elastic πN and NN scattering amplitudes with allowance for the term with q^4 in the argument of the exponential of formula (23), or finally to vary the cross sections σ_p and σ_n but leave the deuteron cross sections σ_{pd} and $\sigma_{\pi d}$ unchanged. An attempt to take into account the dependence of the interactions on the spins likewise ended in failure: the depth of the diffraction minimum at $|t| \approx 0.35$ (GeV/c)^2 decreased in this case, but it was impossible to obtain agreement with the experimental points[19,37,39,40].

It is seen from Figs. 6 and 7 that the depth of the minimum is a very sensitive function of α_p and α_n. Calculations have shown that agreement with experiments on proton-deuteron scattering can be obtained by proposing such a dependence of $\alpha_p(t)$ and $\alpha_n(t)$, that at t = 0 the values of α_p and α_n coincide with the values determined from experiments on elastic pp scattering and from the dispersion relations, and at $|t| \approx 0.35$ (GeV/c)^2 the values $\alpha_p = -0.6$ and $\alpha_n = -1.2$ acquire much larger (absolute) values. In this case, however, the agreement between the experimental and theoretical data is lost in the case of scattering of protons by helium (see the next chapter).

Some progress towards improving the agreement between calculations and experiment was attained by taking into account the intranuclear motion of the nucleons, which turns out to be particularly important in the case of pion-deuteron scattering, owing to the presence of closely-lying resonances in the πN scattering cross sections[32]. Apparently, however, the simplest and most convincing explanation of the discrepancy between the theoretical and experimental data was recently found by Michael and Wilkin[37]. They called attention to the fact that even a small admixture of the D state in the wave function of the deuteron leads to the possibility of highly intense transitions between the S and D states precisely in the momentum-transfer region $|t| \sim 0.35$ (GeV/c)^2, as a result of which the diffraction minimum of the theoretical curve becomes smoothed out.

An analogous phenomenon was established long ago in the case of electron-deuteron scattering; if the D state of the deuteron is not taken into account, then a sharp minimum is formed in this case also in the momentum dependence of the form factor F(t).

Since the spin dependence turns out to be insufficient for the elimination of the minimum in the theoretical curve, it can be neglected for simplicity. In this case, the amplitude $\mathcal{A}_{ss'}$ of the elastic scattering of a high-energy particle by the deuteron nucleus with transition of the nucleus from a state with spin projection s into a state with spin projection s′ can be written in the form

$$\mathcal{A}_{11} = [\mathcal{A}_p(\mathbf{q}) + \mathcal{A}_n(\mathbf{q})[\{F_1(\mathbf{q}/2) - [F_2(\mathbf{q}/2)/\sqrt{2}]\} + \frac{i}{2\pi\lambda}\int \mathcal{A}_p[(\mathbf{q}/2)+\mathbf{q}']\,\mathcal{A}_n[(\mathbf{q}/2)-\mathbf{q}']\{F_1(\mathbf{q}'/2) - [F_2(\mathbf{q}'/2)\,P_2(\cos\theta)/\sqrt{2}]\}\,\mathbf{q}'\,d\mathbf{q}'\,d\theta,$$

$$\mathcal{A}_{10} = [\mathcal{A}_p(\mathbf{q}) + \mathcal{A}_n(\mathbf{q})]\,[F_1(\mathbf{q}/2) + \sqrt{2}\,F_2(\mathbf{q}/2)] + \frac{i}{2\pi\lambda}\int \mathcal{A}_p[(\mathbf{q}/2)+\mathbf{q}']\,\mathcal{A}_n[(\mathbf{q}/2)-\mathbf{q}']\,[F_1(\mathbf{q}') + \sqrt{2}\,F_2(\mathbf{q}')\,P_2(\cos\theta)]\,\mathbf{q}'d\mathbf{q}'d\theta,$$

$$\mathcal{A}_{1,-1} = -\frac{i}{2\pi\lambda}\int \mathcal{A}_p[(\mathbf{q}/2)+\mathbf{q}']\,\mathcal{A}_n[(\mathbf{q}/2)-\mathbf{q}']\,[(3/2\sqrt{2})\,F_2(q')\sin^2\theta]\,\mathbf{q}'d\mathbf{q}'d\theta,$$

$$\mathcal{A}_{00} = 0;$$

here

$$F_1(\mathbf{q}) = \int_0^\infty (u^2 + w^2)\,J_0(\mathbf{q}r)\,dr, \qquad F_2(\mathbf{q}) = \int_0^\infty w\left(2u - \frac{1}{\sqrt{2}}w\right)\mathcal{J}_2(\mathbf{q}r)\,dr$$

are the spherical and quadrupole form factors of the deuteron, u(r) and w(r) are the wave functions of the S and D states of the deuteron, determined, for example, in[41], and J_0 and J_2 are known Bessel functions. Details of the derivation of all the relations can be found in[13,37,40].

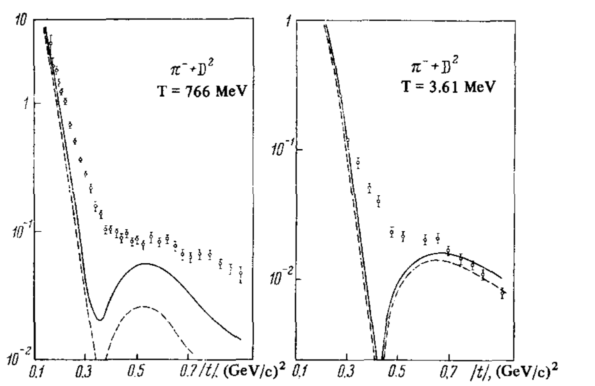

FIG. 8. Cross section for the scattering of π^- mesons by a deuterium nucleus (mb/GeV/c)^2) [27,28,31]. Dashed curve–cross section without allowance for spin flip solid curve–total cross section, with allowance for spin effects. The contribution of the D-wave to the φ-function of the deuteron is not taken into account. The calculations are performed using the "covariant generalization of the Glauber theory" [29,30].

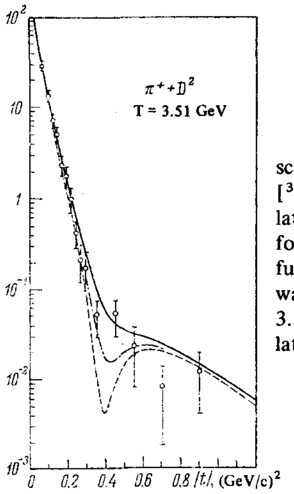

FIG. 9. Cross section of elastic π^+D scattering at T = 3.51 GeV (mb/(GeV/c)^2) [37,38]. Solid and dash-dot curves—calculation by Glauber's theory with allowance for the S and D waves in the deuteron φ-function, when the contribution of the D wave is assumed to be equal to 7 and 3.5%, respectively. Dashed curve—calculation without allowance for the D wave.

The results of the calculations are given in Figs. 6 and 9; the improvement of the agreement between experiment and theory is particularly appreciable in the case of pion-deuteron scattering. The depth of the minimum in the theoretical curve turns out to be practically proportional to the probability of the D states. This uncovers interesting possibilities for an experimental study of the contribution of the D state in the deuterium nucleus.

IV. NUCLEON SCATTERING BY THE HELIUM NUCLEUS

For more complicated nuclei we do not have at present a good theory, and it becomes necessary to use different, frequently quite crude, model representations.

If we assume approximately that the nucleons in the nucleus can be regarded independently of one another (we shall see later that in many cases this is not so poor an approximation), then the density ρ factors out:

$$\rho(\mathbf{r}_1, \ldots, \mathbf{r}_A) = \prod_{k=1}^{A} \rho(\mathbf{r}_k). \qquad (25)$$

The amplitude (20) would in this case also have the form of a product of independent integrals with respect to the nucleon coordinates $\mathbf{r}_i$, were it for the δ-function in the amplitude. This function can be eliminated with the aid of the Gartenhaus-Schwartz transformation[17]. Unfortunately, the resultant expression has, generally speaking, a complicated form and it is frequently preferable to work directly with an expression containing a δ function. However, if it is assumed that the nucleons inside the nucleus are acted upon by a harmonic potential, then the transformed expression turns out to be quite simple. In particular, if all A nucleons of the nucleus with radius R are in the 1S state, then

$$\rho(\mathbf{r}) \equiv \varphi^*(\mathbf{r})\,\varphi(\mathbf{r}) = (\pi R^2)^{-3/2} e^{-r^2/R^2} \qquad (26)$$

and the amplitude for scattering by the nucleus is

$$\mathcal{A}(\mathbf{q}) = \frac{i}{2\pi\lambda\!\!\!^{-}} e^{q^2R^2/4A} \int e^{i\mathbf{q}\boldsymbol{\rho}} d^2\rho \left\{1 - \prod_{k=1}^{A}\left[1 - \frac{\lambda\!\!\!^{-}}{2\pi i}\int d^2\delta\, e^{-i\boldsymbol{\rho}\boldsymbol{\delta}-(R\delta/2)^2}\mathcal{A}_{Nk}(\delta)\right]\right\}$$
$$= \frac{i}{2\pi\lambda\!\!\!^{-}} e^{q^2R^2/4A} \int e^{i\mathbf{q}\boldsymbol{\rho}} d^2\rho \left\{1 - \prod_{k=1}^{A}\left[1 - \frac{\sigma_t(1-i\alpha)}{2\pi(R^2+\beta^2)} e^{-\rho^2/(R^2+\beta^2)}\right]\right\}, \qquad (27)$$

where σ_t, α, and β are quantities averaged over the numbers of protons and neutrons (see expression (22)).

Since the expression in the square brackets does not depend on k, we can rewrite the product in the form of a sum

$$\prod_{k=1}^{A}[1-\chi] = [1-\chi]^A = \sum_{k=0}^{A}\frac{(-1)^k A!}{k!\,(A-k)!}\chi^k,$$

after which all the integrals in (27) can be evaluated analytically and the scattering amplitude becomes[18]

$$\mathcal{A}(\mathbf{q}) = \frac{i}{2\lambda\!\!\!^{-}}(R^2+\beta^2)\, e^{q^2R^2/4A} \times \sum_{k=1}^{A}\frac{(-1)^{k+1} A!}{k\cdot k!\,(A-k)!} \times \left[\frac{\sigma_t(1-i\alpha)}{2\pi(R^2+\beta^2/R^2)}\right]^k e^{-q^2(R^2+\beta^2)/4k}. \qquad (28)$$

The N-th term in (28) corresponds to scattering with N-fold collision inside the nucleus (see formula (17)). From experiments with electron scattering it is known that the functions (25) and (26) are a good approximation for the He^4 nucleus. In this case all four nucleons (A = 4) are actually in the 1S state, forming a closed shell.

The results of calculations by formula (28) are compared in Fig. 10 with the experimental data. Good agreement occurs only in the region of the principal diffraction peak, and at larger values of q the theoretical curve passes much lower than the experimental points. The discrepancy cannot be eliminated by selecting the parameters, although the depth of the first minimum is quite sensitive to the value of α.

Let us attempt to improve the agreement between calculation and experiment by choosing a more complicated expression for the density ρ. It might be assumed that just as four spheres attracting each other assume a stable position such that a hole is produced in the center of their arrangement, so are the nucleons in the

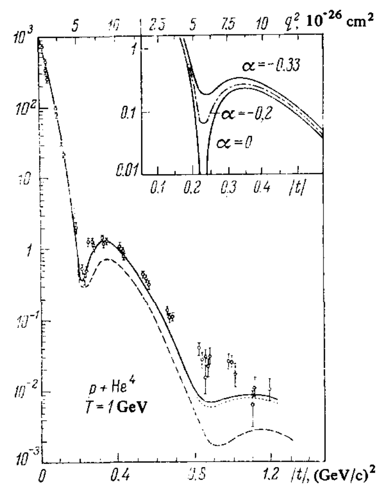

FIG. 10. Cross section of elastic scattering of protons by the helium nucleus (mb/sr; c.m.s.) [9,19]. Solid curve—results of calculation for the density (29), with account taken of the repulsion at the center of the nucleus. Dashed curve—calculation in accordance with the simpler formula (26). Dotted curve—results of calculation for the density (30), with account taken of the repulsion at the center of the nucleus and the nucleon correlation. The influence of the change of the parameter α on the depth of the first diffraction minimum is shown separately (for simplicity it is assumed that $\alpha = \alpha_n = \alpha_p$; this part of the calculations was performed with the density (26) [18]).

Values of the parameters determining the density $\rho(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$ obtained from the condition of best agreement between the experimental and the theoretical cross sections $\sigma(\theta)$ [19]

Expression for density	R, 10^{-13} cm	γ	S, 10^{-13} cm	C	D
(26)	1,37	—	—	—	—
(29)	1,32	0,555	—	0,858	—
(30)	1,25	0,436	0,41	1	1

He^4 so arranged that ρ is minimal at the center of the nucleus. This can be regarded phenomenologically as the result of the joint action of the harmonic potential considered above and a certain repulsion potential at the center of the nucleus. The corresponding expression for the nucleon density is chosen in the form

$$\rho(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \prod_{k=1}^{4} e^{-r_k^2/R^2}(1 - Ce^{-\gamma^2 r_k^2/R^2}), \tag{29}$$

where the constants C and γ are chosen from the condition that the theoretical cross section agree in the best manner with the experimental one (see the table; an expression for the amplitude $\mathcal{A}(q)$ is obtained in this case, as can be readily visualized, by a simple modification of formula (28)).

It is seen from Fig. 10 that at T = 1 GeV it is possible to obtain in this way sufficiently good agreement with the experimental points. Noticeable differences are observed only in the region of the second minimum. The reason for this is not yet clear. It may be that this is caused by the fact that the Glauber theory is not valid at large values of q, or it may be due to spin effects that are not accounted for in the theory.

Certain authors attempted to attribute the discrepancy at large q to intranuclear correlations of the nucleons. In this case the density ρ can be written in the form

$$\rho(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \prod_{k=1}^{4} e^{-r_k^2/R^2}(1 - Ce^{-\gamma^2 r_k^2/R^2}) \prod_{n<m}^{4} (1 - De^{-r_{nm}^2/S^2}), \tag{30}$$

where r_{nm} is the distance between two nucleons, and the value of the parameters is again chosen to obtain agreement between the theoretical cross section and experiment.

The expression for the amplitude $\mathcal{A}(q)$ now becomes much more complicated. In particular, the Gartenhaus-Schwartz transformation[17] is now of little use and it is simpler here to deal with an expression containing the δ function $\delta\left(\sum_{i=1}^{4} \mathbf{r}_i/A\right)$.

Substituting the density (30) in the amplitude (20) and calculating the integrals over all the variable with the exception of the coordinates of the four intranuclear nucleons, we obtain

$$\mathcal{A}(q) = \int \rho(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)(G_1 + G_2 + G_3 + G_4)\,\delta(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4)\,d^3(\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4), \tag{31}$$

where the functions G_1, G_2, G_3, and G_4 pertain respectively to single, double, triple, and quadruple collisions inside the nucleus. These functions are given by[19]

$$G_1 = (i/2\pi\lambda)(f_p + f_n)\,e^{i\mathbf{q}\mathbf{s}_1} e^{-\beta^2 q^2/4}, \tag{32a}$$

$$G_2 = -(3i/8\pi^2\beta^2\lambda) f_p f_n e^{i\mathbf{q}(\mathbf{s}_1+\mathbf{s}_2)/2} e^{-\beta^2 q^2/8} e^{-(\mathbf{s}_1-\mathbf{s}_2)^2/2\beta^2}, \tag{32b}$$

$$G_3 = (i/48\pi^3\beta^4\lambda) f_p f_n (4f_p f_n - f_p^2 - f_n^2)\, e^{i\mathbf{q}(\mathbf{s}_1+\mathbf{s}_2+\mathbf{s}_3)/3} e^{-\beta^2 q^2/12} e^{-[(s_1^2+s_2^2+s_3^2)-(s_1+s_2+s_3)^2/3]/\beta^2}, \tag{32c}$$

$$G_4 = -(i/128\pi^4\beta^6\lambda)[f_p^2 f_n^2 - f_p f_n (f_p - f_n)^2]\, e^{-\beta^2 q^2/16} e^{-[(s_1+s_2+s_3)^2+(s_1^2+s_2^2+s_3^2)]/\beta^2}; \tag{32d}$$

here $\sigma_{pt}(1 - i\alpha_p) = f_p$, $\sigma_{nt}(1 - i\alpha_n) = f_n$, and

$$\beta^2 = (\beta_p^2 + \beta_n^2)/2.$$

The integration of the δ function in (31) eliminates the coordinate $\mathbf{r}_4$ (this has already been taken into account in the expression for G_4). Then the remaining integrand acquires the form of a sum of terms proportional to the exponentials $\exp\left(-\sum_{n,m}^{3} a_{nm} r_n r_m\right)$. Such an integral can be calculated analytically (see[19]) or numerically. We have presented formulas (31) and (32) in order to show how to perform the calculations when the density ρ has a more complicated form than (26).

The results of the numerical calculations with the density (30) are shown in Fig. 10. Allowance for the intranuclear correlations merely shifts the theoretical curve slightly, and is perfectly negligible within the limits of the experimental errors. However, if account is taken of only the correlation and the drop of the density of the center of the nucleus is disregarded (i.e., if we put C = 1 in (30)), then the theoretical curve turns out to be much lower than the experimental points (for details see[19]).

It should be borne in mind that since the expression for the density ρ is fitted only in accordance with one experiment at a fixed energy T, it is necessary for the time being to approach with caution the values of the parameters listed in the table above as well as to the

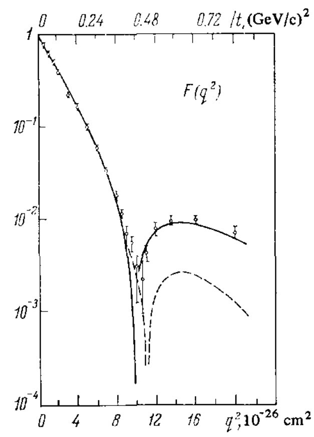

FIG. 11. Comparison of the "electron" form factor of the He^4 nucleus $F(q^2) = [\sigma(\theta)_{\exp}/\sigma(\theta)_{\text{Mott}}]^{1/2}$, determined from experiments with scattering of electrons (experimental points taken from [26]) with the "proton" form factor determined from experiments with proton scattering. Solid curve—values of the "proton" form factor calculated for the density (30) and the values of the parameters from the table. The internal structure of the scattering proton is taken into account with the aid of formulas (33)–(36). Dashed curves—corresponding values of the "proton" form factor calculated under the assumption that the nucleon distribution in the helium nucleus is determined by formula (30) with the parameter C = 1 (i.e., without allowance for the decrease of the density at the center of the nucleus). We see that in this case the "electron" and "proton" form factors differ quite noticeably [19].

very conclusion that the theory agrees with experiment. For more reliable conclusions it is necessary first of all to compare the obtained values of the parameters with the experimental data at several energies T. At the present time, unfortunately, there are still no such data. An additional argument favoring the correctness of the analysis is the fact that the form factor determined from the experiments with scattering of protons by the He^4 nucleus also explains well the experiments with electron-helium scattering (Fig. 11).

The fact that practically the same form factor is used to describe the scattering of electrons and protons in Glauber's theory is an important feature distinguishing this theory from the optical model.

Generally speaking, one can expect some differences between the "electron" and "proton" form factors because the densities of the protons and the neutrons in the nucleus differ somewhat from each other; however, these differences are small, particularly if exclusive account is taken of the distribution of the density inside the scattering proton itself; the dimensions and the structure of the electron can then be neglected*.

V. SCATTERING BY CARBON AND OXYGEN NUCLEI

In light nuclei with atomic numbers A > 4, as shown by experiments with electron scattering, the distribution of the nucleon density is also well described by an expression corresponding to a harmonic potential.

If it is recognized that only four nucleons can be in the S state, and the remainder occupies states with larger values of l, then

$$\rho(\mathbf{r}_1, \ldots, \mathbf{r}_A) = \prod_{k=1}^{4} \rho_S(\mathbf{r}_k) \prod_{k=5}^{A} \rho_P(\mathbf{r}_k); \tag{37}$$

here the first product describes the S shell, and the densities $\rho_S(\mathbf{r}_k)$ must be chosen in the form (27); the second product pertains to the remaining A − 4 nucleons (A < 16) in the P-state with densities

$$\rho_P(\mathbf{r}_k) = (2/3\pi^{1/2}R^2)\, \mathbf{r}_k^2 e^{-\mathbf{r}_k^2/R^2}. \tag{38}$$

After substituting these expressions in (20) and eliminating the δ function with the aid of the Gartenhaus-Schwartz transformation, the nuclear scattering amplitude is written in the form

$$\mathcal{A}(\mathbf{q}) = \frac{i}{2\pi\lambda\!\!\!^{-}} e^{\mathbf{q}^2R^2/4A} \int e^{i\mathbf{q}\boldsymbol{\rho}}\, d^2\rho \times$$
$$\times \int \prod_{k=1}^{4} \rho_S(\bar{\mathbf{r}}_k) \prod_{k=5}^{A} \rho_P(\mathbf{r}_k) \left\{1 - \prod_{j=1}^{A} \left[1 - \frac{\sigma_t(1-i\alpha)}{2\pi\beta} e^{-(\boldsymbol{\rho}-\mathbf{s}_j)^2/\beta^2}\right]\right\} \prod_{n=1}^{A} d^3r_n$$

$$= \frac{i}{2\pi\lambda\!\!\!^{-}} e^{\mathbf{q}^2R^2/4A} \int e^{i\mathbf{q}\boldsymbol{\rho}}\, d^3\rho \left\{1 - \left[1 - \frac{\sigma_t(1-i\alpha)}{2\pi(R^2+\beta^2)} e^{-\boldsymbol{\rho}^2/(R^2+\beta^2)}\right]^4\right.$$
$$\left.\times \left[1 - \frac{\sigma_t(1-i\alpha)}{2\pi}\left(\frac{1}{R^2+\beta^2} - \frac{2R^2}{3(R^2+\beta^2)^2} + \frac{2R^2\beta^2}{3(1+\beta^2)^3}\right) e^{-\boldsymbol{\rho}^2/(R^2+\beta^2)}\right]^{A-4}\right\}; \tag{39}$$

for simplicity the nucleon amplitude (23) is assumed in this expression to be averaged over the pp and pn interactions.

The integrand in (39) can be expanded in a power series (using the Newton binomial formula), after which all the integrals can be calculated analytically[19], however, the obtained expression is quite complicated.

The calculations simplify greatly if it is assumed that the nucleus is spherically symmetrical; in this case (see formulas (10) and (14)) we have

$$\mathcal{A}(\mathbf{q}) = \frac{i}{\lambda\!\!\!^{-}} \int_0^\infty J_0(\mathbf{q}\rho)\{\ldots\}\rho\, d\rho, \tag{40}$$

where q = 2 sin (θ/2)/λ, and the symbol {...} denotes the expression in the curly brackets in (39). The integration in (4) can be readily carried out by numerical methods.

It is possible to obtain a more accurate expression for the amplitude $\mathcal{A}(\mathbf{q})$ if the coordinates of the nucleons are antisymmetrized in accordance with the Pauli principle during the course of the determination of the density $\rho(\mathbf{r}_1, \ldots, \mathbf{r}_A)$. For example, in the case of scattering by C^{12} and O^{16}, the configuration of the protons and accordingly of the neutrons has the form $(1S_{1/2})^2(1P_{3/2})^4$ for C^{12} and $(1S_{1/2})^2(1P_{3/2})^4(1P_{1/2})^2$ for O^{16}. The harmonic wave functions of the nucleon in the S and P states are given by

$$\left.\begin{aligned}
&\varphi_S(\mathbf{r}; J=1/2, J_z=\pm 1/2) = \varphi_S(\mathbf{r})\chi_\pm,\\
&\varphi_P(\mathbf{r}; J=1/2, J_z=1/2) = \varphi_P(\mathbf{r})(-Y_{10}\chi_+ + \sqrt{2}Y_{11}\chi_-)/\sqrt{3},\\
&\varphi_P(\mathbf{r}; J=1/2, J_z=-1/2) = \varphi_P(\mathbf{r})(-\sqrt{2}Y_{1-1}\chi_+ + Y_{10}\chi_-)/\sqrt{3},\\
&\varphi_P(\mathbf{r}; J=3/2, J_z=1/2) = \varphi_P(\mathbf{r})(\sqrt{2}Y_{10}\chi_+ + Y_{11}\chi_-)/\sqrt{3},\\
&\varphi_P(\mathbf{r}; J=3/2, J_z=-1/2) = \varphi_P(\mathbf{r})(-Y_{1-1}\chi_+ + \sqrt{2}Y_{10}\chi_-)/\sqrt{3},\\
&\varphi_P(\mathbf{r}; J=3/2, J_z=\pm 3/2) = \varphi_P(\mathbf{r})Y_{11}\chi_\pm;
\end{aligned}\right\} \tag{41}$$

here

$$\varphi_S(\mathbf{r}) = (1/\pi^{1/2}R)^{1/2} e^{-\mathbf{r}^2/2R^2}, \quad \varphi_P(\mathbf{r}) = (2/3\pi^{1/2}R^2)^{1/2} r e^{-\mathbf{r}^2/2R^2} \tag{42}$$

are the corresponding radial wave functions, Y_{lm} are spherical functions, $\chi_\pm$ are the spin functions of the nucleons (with the spin directed up or down; the direction of motion of the primary particles is chosen to be the z axis). The complete antisymmetrical wave functions describing the ground state of the target nucleus are expressed in terms of the Slater determinant

$$\varphi(\mathbf{r}_1, \ldots, \mathbf{r}_A) = (A!)^{-1/2} \|\varphi_n(\mathbf{r}_k)\|, \tag{43}$$

where $\varphi_n(\mathbf{r}_k)$ are single-particle wave functions (41).

If we disregard the charge-exchange process, the influence of which in the high-energy region is small, then the protons and the neutrons can be symmetrized separately:

$$\varphi(\mathbf{r}_1, \ldots, \mathbf{r}_A) = \varphi^{(p)}(\mathbf{r}_1, \ldots, \mathbf{r}_Z)\varphi^{(n)}(\mathbf{r}_{Z+1}, \ldots, \mathbf{r}_A), \tag{44}$$

$$\left.\begin{aligned}
&\varphi^{(p)}(\mathbf{r}_1, \ldots, \mathbf{r}_Z) = (Z!)^{-1/2} \|\varphi_n(\mathbf{r}_k)\|_{k=1,\ldots,Z},\\
&\varphi^{(n)}(\mathbf{r}_{Z+1}, \ldots, \mathbf{r}_A) = [(A-Z)!]^{-1/2} \|\varphi_n(\mathbf{r}_k)\|_{k=Z+1,\ldots,A}.
\end{aligned}\right\} \tag{45}$$

This simplifies the calculations.

Figure 12 shows the results of calculations for the

*The nuclear form factor appearing in experiments with scattering of extended protons is equal to

$$F(\mathbf{q}) = \int \rho(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}} d^3r = F_{\text{Я}}(\mathbf{q}) F_P(\mathbf{q}), \tag{33}$$

where

$$\rho(\mathbf{r}) = \int \rho_{\text{Я}}(\mathbf{r}+\mathbf{r}')\rho_p(\mathbf{r}')\, d^3\mathbf{r}, \tag{34}$$

$$\rho_{\text{Я}}(\mathbf{r}) = \int \rho(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)\, d^3(\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4) \tag{35}$$

(the nuclear density $\rho(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$ is symmetrical with respect to permutations of $\mathbf{r}_1$). To describe the proton form factor one can use the pole expression

$$F_P(\mathbf{q}) = [1 + (\mathbf{q}^2/a^2)]^{-2} \tag{36}$$

with parameter a = 0.71 GeV/c. We shall return to the question of scattering of extended particles in connection with scattering of nuclei by nuclei (see formulas (51)–(53)).

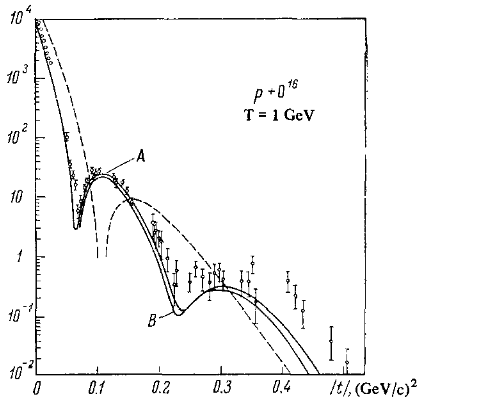

FIG. 12. Cross section for the elastic scattering of protons by the oxygen nucleus (mb/sr) [9,19]. Curve A–calculation by Glauber's theory with antisymmetrical wave function (43) B–corresponding curve for the case when the density ρ is chosen in the form (37). Dashed line–cross section calculated in the impulse approximation. t–square of the transferred 4-momentum.

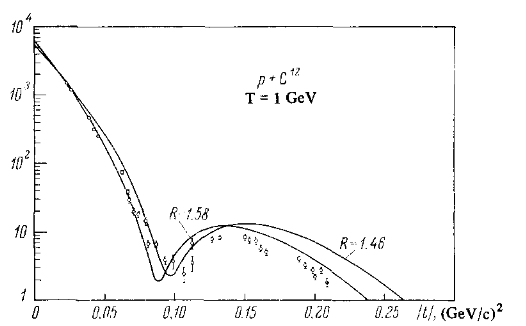

FIG. 13. Cross section for the elastic scattering of protons by the carbon nucleus (mb/sr) [9,19]. Curves–calculation by Glauber's theory for two values of the nuclear radius R (with allowance for the antisymmetrization of the nucleons (43)).

case of scattering of protons by oxygen ($R \approx 1.71 \times 10^{-13}$ cm, with allowance for the proton form factor). We see that theory and experiment are in good agreement; the antisymmetrization of the wave functions turns out to be relatively unimportant in this case. The calculations were performed for $|\alpha| = 0.3$ (the sign of α is immaterial); a change by more than 0.1 greatly deteriorates the agreement between the calculations and experiment.

There is much worse agreement in the case of scattering by carbon (Fig. 13). If we use for the nuclear radius the value $R = 1.58 \times 10^{-13}$ cm, obtained from experiments with electron scattering, then the minimum in the cross section occurs at too small values of the momentum transfer, and the calculated value of the cross section in the region of the second maximum turns out to be overestimated by approximately 50% compared with experiment. An attempt to decrease the radius R so as to obtain agreement between the positions of the experimental and theoretical minima leads to worse agreement both at small and at large t. Apparently the discrepancies are due to the fact that the C^{12} nucleus is strongly deformed, so that the expressions used in the calculation for the density $\rho(\mathbf{r}_1, \ldots, \mathbf{r}_A)$ are too inaccurate.

In the case of electron scattering, the main contribution is made by the impulse approximation, when the scattering cross section is expressed in terms of the densities of the individual nuclear nucleons. These densities are spherically symmetrical, and therefore the results of the calculations depend little on the degree of deformation of the nucleus. In the scattering of nucleons, an appreciable contribution is made by double scattering, for example scattering with a transition of the nucleus to the first excited state and subsequent scattering to the ground state. Deviations from sphericity become in this case very important and should become more strongly manifest in the region of the second diffraction maximum, where, as we shall show below, the principal role is played by double scattering.

Calculations performed by Drozdov on the basis of an approximate model of black ellipsoid[22] have shown that an increase of the deformation actually shifts the theoretical curve in the required direction.

An investigation of elastic scattering in the region of the second diffraction maximum can serve as an important source of information concerning the degree of nuclear deformation (for more details see[19]).

VI. INTERACTION WITH HEAVY NUCLEI

For scattering by heavy nuclei, it is convenient to represent the amplitude (39) in a somewhat different form. For simplicity we again assume that the density $\rho(\mathbf{r}_1, \ldots, \mathbf{r}_A)$ is given by (25), i.e., it can be factored, and in addition it is expressed in terms of Gaussian exponentials. Then relation (20) can be rewritten with the aid of the Gartenhaus-Schwartz transformation in the form

$$\mathcal{A}(\mathbf{q}) = \frac{i}{2\pi\lambda} e^{q^2R^2/4A} \int e^{i\mathbf{q}\boldsymbol{\rho}}\, d^2\rho \left\{1 - \prod_{k=1}^{A}\left[1 - \frac{\lambda}{2\pi i}\int e^{-i\mathbf{q}_k\boldsymbol{\rho}}\mathcal{A}_k(\mathbf{q}_k)\, F(\mathbf{q}_k)\, d^2q_k\right]\right\}$$
$$= \frac{i}{2\pi\lambda} e^{q^2R^2/4A} \int e^{i\mathbf{q}\boldsymbol{\rho}}\, d^2\rho \left\{1 - \left[1 - \frac{\lambda}{2\pi i}\int e^{-i\delta\rho}\mathcal{A}(\delta)\, F(\delta)\, d^2\delta\right]^A\right\}, \quad (46)$$

where the form factor is

$$F(\delta) = \int e^{i\mathbf{r}_k\delta}\rho(\mathbf{r}_k)\, d^3\mathbf{r}_k,$$

and the density functions $\rho(\mathbf{r}_k)$ satisfy the normalization condition

$$\int \rho(\mathbf{r}_k)\, d^3\mathbf{r}_k = 1.$$

We note further that in nuclei with large values of A the density functions change with increasing A approximately like $1/A$—this is well known from experiments with electron scattering. Therefore, by determining the function

$$\eta(\rho) = \frac{\lambda A}{2\pi}\int e^{-i\delta\rho}\mathcal{A}(\delta)\, F(\delta)\, d^2\delta, \quad (47)$$

it is possible to replace the expression in the square brackets in (46)

$$\left[1 + \frac{i\eta}{A}\right]^A = \sum_{k=0}^{A}\left(\frac{A-k+1}{A}\,\frac{A-k+2}{A}\cdots\frac{A-1}{A}\right)\frac{(i\eta)^k}{k!} \quad (48)$$

approximately by an infinite series—the expansion of the exponential

$$e^{i\eta} = \sum_{k=0}^{\infty}\frac{(i\eta)^k}{k!}. \quad (49)$$

The expression for the nuclear scattering amplitude then assumes the following form, which is convenient for calculations

$$\mathcal{A}(\mathbf{q}) = \frac{i}{2\pi\bar{\lambda}} e^{q^2R^2/4A} \int e^{i\mathbf{q}\rho}[1 - e^{i\eta(\rho)}]\, d^2\rho. \qquad (50)$$

VII. APPLICATION OF GLAUBER'S THEORY TO THE INTERACTION OF TWO NUCLEI

Glauber's theory can be generalized also to the case of interactions of two nuclei. The initial relations of the theory (16) and (17) will have in this case the form

$$\mathcal{A}_{\text{inc}} = \frac{i}{2\pi\bar{\lambda}} \int e^{i\mathbf{q}\rho}\Phi_{\text{f}}^{(A)*}(\mathbf{r}_1, \ldots, \mathbf{r}_A)\, \Phi_{\text{f}}^{(B)*}(\mathbf{r}_{A+1}, \ldots, \mathbf{r}_{A+B}) \times$$
$$\times \Gamma(\rho, \mathbf{s}_1, \ldots, \mathbf{s}_{A+B})\, \Phi_{\text{in}}^{(A)}(\mathbf{r}_1, \ldots, \mathbf{r}_A)\, \Phi_{\text{in}}^{(B)}(\mathbf{r}_{A+1}, \ldots, \mathbf{r}_{A+B}) \prod_{i=1}^{A+B} d^3r_i\, d^2\rho, \qquad (51)$$

$$\Gamma(\rho_1, \mathbf{s}_1, \ldots, \mathbf{s}_{A+B}) = 1 - \prod_{i=1}^{A}\prod_{j=1}^{B}\{1 - \Gamma_{ij}(\rho - \mathbf{s}_i^{(A)} - \mathbf{s}_j^{(B)})\}, \qquad (52)$$

where the function Γ_{ij} is expressed with the aid of relation (12) in terms of the NN-scattering amplitude $\mathcal{A}_N(\mathbf{q})$.

All the succeeding relations are likewise generalized in obvious fashion. In particular, if we use for the nuclear densities $\rho^{(A)}(\mathbf{r}_1, \ldots, \mathbf{r}_A)$ and $\rho^{(B)}(\mathbf{r}_1, \ldots, \mathbf{r}_B)$ factored expressions that depend on Gaussian exponentials, then the nuclear amplitude becomes rid of the δ functions and the following factor

$$e^{q^2R_A^2/4A} e^{q^2R_B^2/4B}, \qquad (53)$$

where A and B are the mass numbers of the colliding nuclei, appears in front of the integral.

The application of Glauber's theory to the simplest case of the interaction of two nuclei (to the deuteron + nucleus interaction) was considered in[23,25].

If we now again proceed to the case of the N + nucleus scattering, then one of the exponentials in (53) vanishes, and the form factor of the nucleon automatically appears in the expression for the nuclear amplitude.

VIII. TRANSITION TO THE OPTICAL APPROXIMATION

In conclusion, it is useful to examine in greater details the manner in which the ordinary optical diffraction picture appears in Glauber's theory.

Figure 14 shows, with He^4 as an example, the contribution made to the scattering amplitude by the terms due to one, two, three, etc. intranuclear collisions. In a semi-log scale, the curves corresponding to the Gaussian exponentials assume the form of inclined straight lines. Summing these curves, we see that the first, principal peak in the angular distribution of the scattered particles is due practically entirely to single interactions within the nucleus. The mutual cancellation of the amplitudes resulting from single and double collisions leads to a minimum in the cross section near $q^2 = 0.2\ (\text{GeV}/c)^2$. For somewhat larger values of q, in the region of the second diffraction maximum, the angular distribution is determined practically entirely by double scattering processes. The mutual cancellation of the amplitudes $\mathcal{A}_2$ and $\mathcal{A}_3$ gives a minimum near $q^2 = 0.8\ (\text{GeV}/c)^2$, etc.

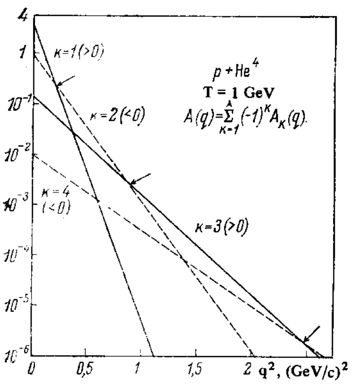

FIG. 14. Contribution made to the amplitude for elastic scattering of protons by helium at T = 1 GeV by the terms due to k intranuclear collisions (in relative units) [14]. The calculations were performed in accordance with formula (28), where we put for simplicity α = 0. The arrows denote the points at which the nuclear amplitude has a minimum as the result of the fact that two terms that make the largest contribution cancel each other.

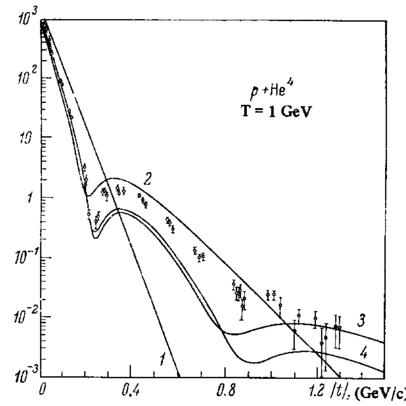

FIG. 15. The numbers 1–4 designate the scattering cross sections calculated with allowance for one, two, three, and four terms respectively in the scattering amplitude

$$\mathcal{A} = \sum_{k=1}^{n}(-1)^k\mathcal{A}_k \quad (n = 1, 2, 3, 4).$$

The calculations were performed with account taken of the real part of the amplitude of NN scattering α [19].

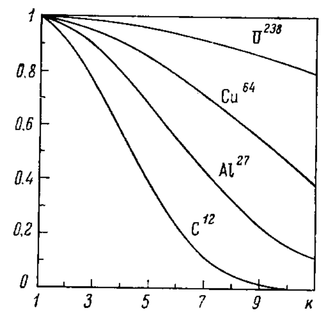

FIG. 16. Relative values of the coefficients in the expansions (48) and (49) as functions of the mass number of the target nucleus and the number k of the term of the expansion.

Thus, the diffraction picture in Glauber's theory results from interference of amplitudes due to different numbers of intranuclear collisions. This is clearly seen also in Fig. 15.

On going over to heavy nuclei, we obtain the usual optical expression for the nuclear amplitude, if we discard in formula (50) the factor $\exp(q^2R^2/4A)$ (we recall that A is sufficiently large), and the amplitude $\mathcal{A}(\delta)$ is independent of δ. (The latter is perfectly justified, since in heavy nuclei, whose radius greatly exceeds the radius of the NN interaction, the form factor F(δ) has a much larger peak at $\delta \approx 0$ than the amplitude $\mathcal{A}(\delta)$, the value of which at $\delta = 0$ can be taken outside the integral sign in formula (47).

Since formula (50) was obtained under the assumption that the finite series (48) can be replaced by the infinite expansion (49), the deviation of the ratio of the corresponding coefficients in (48) and (49)

$$\Delta(A, k) = \frac{A-k+1}{A}\,\frac{A-k+2}{A}\,\ldots\,\frac{A-1}{A}$$

from unity is a characteristic of the extent to which the Glauber theory of multiple diffraction scattering differs from the optical theory.

It is seen from Fig. 16 that the ratio $\Delta(A, k)$ at $k = 1$ and 2 is close to unity for all nuclei. This explains why the first diffraction maximum at small scattering angles is well approximated in all cases by the optical theory. At large values of k, the ratio $\Delta(A, k)$ is close to unity only for heavy nuclei; only in this case is the optical approximation sufficiently good.

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