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PHYSICAL PRINCIPLES OF THE THEORY OF BRITTLE FRACTURE CRACKS

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I. INTRODUCTION. CERTAIN ESSENTIAL STAGES OF THE DEVELOPMENT OF THE THEORY.

The theoretical strength of an ideal crystal lattice, corresponding to simultaneous breaking of all the intermolecular bonds, is very large, altogether one-tenth the value of Young's modulus. The strengths of real solids are smaller by several orders of magnitude, this being usually connected with the existence of lattice defects. From among the various types of defects, we shall consider here only cracks. Although actually there are not so many brittle materials (glass, quartz, etc.), the question of cracks in brittle bodies is of great practical importance because many seemingly ductile materials (metals) fail in "brittle" fashion. The problem of brittle failure has been paid much attention, particularly during the last two decades.

In real solids there are always a large number of cracks. Our everyday experience indicates that so long as the load applied to a body remains small, the cracks do not grow and the body retains its bearing strength even if the load is increased. However, as soon as the load reaches a certain sufficiently large value, different for each crack, the latter begins to expand. Cracks may develop in different fashions. In some cases the cracks grow very rapidly until the body is completely ruptured, even if the load is constant but has reached the critical value, i.e., the state is unstable. In other cases the cracks expand slowly with increasing load, the growth stops as soon as the increase of the load is interrupted, and the dimensions of such stable-equilibrium cracks are connected in some manner with the load applied to the body.

The theory of cracks has developed in natural fashion, as is customary in most fields of physics and mechanics, by expanding and improving existing concepts, by advancing new ideas, by raising new questions and problems, and by covering and explaining an ever increasing circle of details. Modern theory starts with the concept of the body as a continuous brittle medium that obeys Hooke's law (the linear connection between the stress and the strain) up to the breakdown stress, and which consequently is described by classical theory of elasticity. In earlier papers, Inglis (1913)[1] and Muskhelishvili (1919)[2] considered the problem of the theory of elasticity for a body under the influence of a tensile load, having a cavity of elliptic cross section, particularly a thin cut that can imitate a crack. An analysis of the elastic equilibrium of the body makes it possible to determine the stress and strain fields, including the profile of the elongated cut, but a solution is possible for a cut of any size. In addition, the profile of the tip of the crack is rounded (Fig. 1), and the stresses and strains of the body near the edge of the crack are infinite for all finite loads and cut dimensions. Since a real body can withstand only stresses that do not exceed a certain limit, it follows that a body weakened by a crack should fail no matter how small the load.

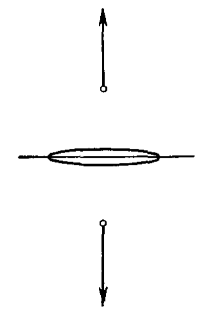

FIG. 1

A way out of this contradiction was pointed out by Griffith (1920)[3], who laid the groundwork of the theory of cracks. Griffith was the first to understand that it is impossible to develop an adequate theory of cracks with the aid of the concepts of elasticity theory only. It is necessary to introduce into consideration additional quantities characterizing the molecular cohesion forces and the resistance of the material to rupture. The quantity he used was an energy constant, namely the surface tension γ, i.e., the work necessary for the formation of a unit of new surface. Griffith considered unstable cracks in a homogeneous field of tensile stresses and found the critical value of the stress at which a crack of given length begins to grow catastrophically. Irwin[4] and Orowan[5] greatly extended the class of materials to which the theory of cracks could be applied, noting that certain substances that are plastic in ordinary tests behave like brittle ones in the case of crack formation. In such materials the plastic deformation is concentrated in a thin surface layer of the crack, and instead of the surface tension γ it is necessary simply to use a certain effective quantity γ_{eff}, which includes the energy consumed in plastic deformation. We note that the quantity γ_{eff}, which turns out to be larger by 2–3 orders of magnitude than γ, has not yet been derived theoretically.

The questions of crack stability, the shape of the tip of the crack, and the infinite stresses have interested many physicists and mechanics specialists, who have attempted to explain the appearance of the physically unlikely singularities of the tip of the crack, which follow from the theory, namely the rounding-off of the profile and the infinite stresses. This question was dealt with by Rebinder[6], Mott[7], Lifshitz[8], Frenkel'[9], Khristianovich (see[10]), and others.

Of very great importance for the mathematical theory of cracks was Irwin's 1957 paper[11], where he connected in general form the rate of release of elastic energy, which determines the critical conditions for the start of the crack growth, with the so-called coefficient of stress intensity at the tip of the crack.

The great importance of Irwin's formula lies in the fact that it serves as the basis for the formulation of a general method of solving problems involving arbitrary cracks within the framework of the so-called "energy" approach. Irwin himself considered principally unstable cracks, and some of his statements concerning cracks that grow slowly with increasing load indicate that he did not understand this process completely.

Starting in 1955, Khristianovich, Barenblatt[12,13], and their coworkers in the USSR developed a new trend in the theory of brittle-fracture cracks, based on the idea advanced by Khristianovich (see[10]) that the stresses at the tip of the crack are finite. During the course of these investigations, Barenblatt analyzed in general form the question of stable-equilibrium cracks as a problem in elasticity theory, in which the crack dimension is determined as a function of the applied load. In 1959, Barenblatt formulated in final form[14,15] a theory which, first, by virtue of the very formulation of the problem, encompassed all cases of both stable and unstable cracks and, second, contained a different "strength" (as we shall henceforth call it) approach to cracks. Instead of the energy parameter (surface tension) he introduces directly into the elasticity-theory equations the molecular cohesion forces acting on the tip of the crack and responsible for the resistance of the material to rupture. It should be noted that a condition equivalent to the finite-stress hypothesis was advanced by Lifshitz[8] (1948) in the related problem of the propagation of "twins"[9].

The introduction of cohesion forces has made it possible to explain the reason for the existence of infinite stresses on the tips of the crack in the solutions typical of the energy approach, to eliminate in a physically correct manner these nonrealistic infinite values, and to explain the detail of the structure (the profile) of the tips of the cracks, just where the process of the rupture of material develops.

In the macroscopic sense, both approaches, the force approach and the energy approach, turned out to be equivalent, as expected, since the new approach is more "microscopic" and consequently does not contradict the older one, but only supplements and refines it. The entire macroscopic effect of the cohesion forces is determined by a single constant of the material, which depends in integral fashion on the cohesion forces. The general formulation developed by Barenblatt for the problem of cracks has contributed, in particular, to a subsequent formulation of problems within the framework of the energy approach as applied to arbitrary (stable and unstable) cracks. Leonov and Panasyuk[16] proposed a similar crack model, in which account is taken of the cohesion forces, but not in as complete and general a form, although it was presented independently (very shortly afterwards).

Judging from the literature, Barenblatt's work has gained wide recognition; its results are used and are being further extended. This can be seen, for example, from the fact that Barenblatt's theory is described in detail and is named after him in some articles in the seven-volume encyclopedia of fracture[31]. The ideas of this theory, in particular, were applied by Kosevich and Pastur[17] to the theory of propagation of "twins"—an important phenomenon essential for the explanation of the mechanism of plastic deformation.

At the same time, Barenblatt's work has recently been subjected to vicious attacks on the part of some authors[25-28]. The resultant conclusion of the critical remarks is most concisely stated by the following citation: "... the exposition of the erroneous and formal concepts of G. I. Barenblatt have occupied in our literature a place not commensurate with their real significance"[27]—we see thus that the situation turned out to be rather strange.

Since we are dealing with a certain physical phenomenon, its understanding, and its theoretical description, there is undisputed interest in a general examination of the physical principles and ideas on which modern theory of cracks is constructed, and the present article has been written to this purpose. For a better acquaintance with the problem of cracks we can recommend the review[15] (see also the book[30]). A more up to date bibliography can be found in the review[18].

II. UNSTABLE CRACK (THE GRIFFITH PROBLEM)

Let us consider an infinite body situated in a homogeneous field of tensile stresses σ_{yy} produced by a uniform load $p = \sigma_{yy} = \text{const}$ applied at infinity. Assume that the body contains a linear crack of width $2l$, which stretches out in the direction of the c axis, perpendicular to the plane of Fig. 2. It is required to determine the critical stress p_0 at which the crack begins to expand without limit.

Assume that there is no crack, then the uniformly stretched body has an elastic energy $p^2/2E$ per unit volume (E is Young's modulus). Assume now that the crack appears. It opens up under the influence of the tensile stresses and the stresses in its vicinity weaken (the material "relaxes"). The characteristic dimension of this vicinity (shown shaded in Fig. 2) is obviously of the order of l. Consequently, an elastic energy W is released, of the order of $W \sim (p^2/2E)(2l)^2$ per unit length of the crack. On the other hand, when the crack opens up, work is performed to overcome the cohesion forces acting between the opposite edges of the cut, and this work, or the surface energy of the crack, is $\Pi = 2\gamma \cdot 2l$ per unit length of the body (we write 2γ, since two surfaces are produced).

If we disregard the changes in the temperature,

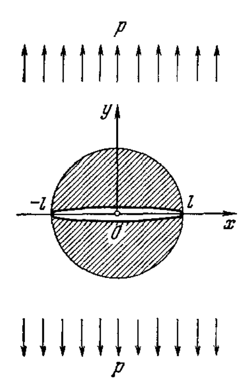

FIG. 2

which we assume to be constant, then, apart from a constant, the free energy F of the body is made up of the volume elastic energy U and the surface energy Π of the crack. The spontaneous occurrence of the crack, not accompanied by work done by external forces, leads to the following change in the free energy

$$\Delta F = \Delta(U + \Pi) = -W + \Pi \sim -(p^2/2E)4l^2 + 4\gamma l.$$

We see that ΔF (or F), as a function of l, has a maximum corresponding to $p = p_0 \sim (\gamma E/l)^{1/2}$. The value of p_0 at which a crack with half-width $l \sim \gamma E/p_0^2$ is at equilibrium is the critical stress. This equilibrium is unstable, because the free energy is not minimal but maximal. At $p > p_0$ we have $\partial F/\partial l < 0$ and an already existing crack of given width begins to expand without limit. At a load below critical we have $\partial F/\partial l > 0$ and the crack should contract*.

Thus, the criterion for unstable equilibrium and for the start of growth of the crack is as follows:

$$\frac{\partial F}{\partial l} = \frac{\partial}{\partial l}(-W + \Pi) = -\frac{\partial W}{\partial l} + 4\gamma \leqslant 0. \quad (1)$$

By means of a more accurate calculation, using the results of Inglis[1] for the quantity W, Griffith[3] found the exact value of the critical stress

$$p_0 = [2E\gamma/\pi(1-\nu^2)\,l]^{1/2}, \quad (2)$$

where ν is the Poisson coefficient. This formula pertains to the case of planar deformation (to which we shall henceforth confine ourselves), wherein the displacements of the medium in the z direction, perpendicular to the plane of the figure (see Fig. 2), are equal to zero, and the normal stresses σ_{zz} differ from zero. This and all the subsequent formulas can be derived also for the case of a planar loaded state, when the displacements in the z direction are such that there is no stress σ_{zz}.

Experiment confirms well the Griffith critical condition $p_0 l^{1/2}$ = const. In Griffith's own experiments[3], cracks of different widths were made on glass, and the load at which the glass broke was measured. It turned out that $p_0 l^{1/2} \approx 26$ kg/cm$^{3/2}$, corresponding, for example, to $p_0 = 58$ kg/cm^2 and $2l = 0.38$ cm.

III. COEFFICIENT OF STRESS INTENSITY AND CRITERION FOR CRACK GROWTH

The rate of release of elastic energy with increasing area of the crack, which essentially determines the critical conditions, can be readily calculated only in the simplest cases. In the case of non-uniform loads or in the case when the crack does not have a simple shape, the calculation of the elastic energy is a very complicated problem. This difficulty was overcome by Irwin[11] with the aid of the following device. As already noted above, the mathematical methods of elasticity theory, when applied to a loaded body with a thin cut, make it possible in principle to find the stress field in the body and the displacements of the points on the surfaces of the stretched cut, i.e., the profile of the crack. On the free surface of the crack itself, naturally, there is no stress (the cohesion forces between the edges of the cut have not yet been taken into account in the theory), so that the resultant solution of the equations for the elastic equilibrium, yields a rounded profile for the end (tip) of the crack (Figs. 1 and 3). For this reason, the strain of the medium, and consequently also the stress exerted by the medium at the point O, turns out to be infinite, and the stress has a discontinuity at this point.

Let us consider some section of the crack near the end and let the x′ axis lie in the plane of the crack and be perpendicular to its contour. The origin O is placed at the end point of the crack (see Fig. 3). As follows from the mathematical solution, the asymptotic profile of edge v(x′) and the tensile stress $\sigma_{yy}(x')$ on the continuation of the plane of the crack, behave at distances $|x'|$ small compared with the dimensions of the entire crack like

$$v = 4(1-\nu^2)E^{-1}N\,|x'|^{1/2} + O(|x'|^{3/2}), \quad (3)$$

$$\sigma_{yy} = N(x')^{-1/2} + O(1), \quad (4)$$

where N is a constant. This constant, which has the dimension dyne/cm$^{3/2}$, determines completely the behavior of the stress and strain field at the end of the crack and plays an important role in the theory. It is called the coefficient of stress intensity. In the solution of the equations of elasticity theory, N depends on the loads and on the dimensions of the crack. For example, in the case of uniform load p = const and a crack representing a strip of width $2l$, we have $N = p(l/2)^{1/2}$.

Let us assume that the crack has broadened and its end O has shifted to the right a small distance a, such that the asymptotic formulas (3) and (4) are still valid at $|x'| \sim a$. Assume that while this has occurred the boundaries of the body, to which the loads are applied, have remained immobile, so that the external forces perform no work whatever. Let us find the change of the elastic energy of the body, which in this case is connected only with the expansion of the crack. To this end, we draw a hypothetical cut along the plane of the crack from x′ = 0 to x′ = a and assume that fictitious "external" forces, equal and opposite to the true stresses $\sigma_{yy}(x')$, are applied to the surfaces of the cut in such a way that the surfaces are held together. Assume that these forces weaken gradually to zero. Then the surfaces move apart just as gradually to a final new profile $v_1(x') = v(x' - a)$ (see Fig. 3). The work performed by the fictitious for-

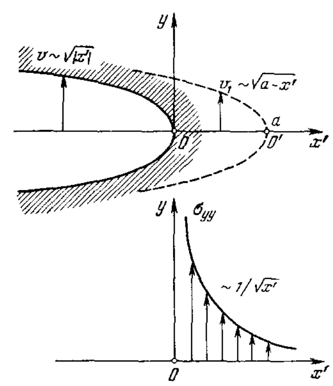

FIG. 3

*Real cracks do not close up at subcritical loads. It is possible that extraneous atoms become concentrated on the surface or else some transformations occur and prevent the surfaces from joining together.

ces when the cut opens up is just equal to the change of the elastic energy of the body. This work is negative, and consequently the elastic energy decreases. The energy released per unit volume of the crack in a direction perpendicular to the plane of the figure is obviously equal to

$$\Delta W = 2\cdot\frac{1}{2}\int_0^a \sigma_{yy}(x')\,v(x'-a)\,dx = \frac{2\pi(1-\nu^2)}{E}N^2a.$$

The coefficient 1/2 in front of the integral sign is the result of the linearity of the connection between the displacements of the surface and the forces as the latter decrease gradually. The region $x' < 0$ makes no contribution to the integral, since the surfaces are free in this case.

As is clear from the derivation itself, the formula is valid for any small variation of the crack area δS (in this case $\delta S = a\cdot 1$), and the release of elastic energy of the body connected with such a variation, $\delta W = -\delta U$, is equal to

$$\delta W = 2\pi(1-\nu^2)E^{-1}N^2\delta S. \qquad (5)$$

It is determined only by the local properties of the edge of the crack, calculated by the intensity coefficients N. Formula (5) can be derived also by calculating rigorously the elastic energy of the body for any concrete case, and replacing the combinations of loads and dimensions, which enter in the formula for the rate of energy release, by N. Thus, for an infinite body with a linear crack of width $2l$, situated in a uniform field of external stresses p, we have according to Inglis[1] $W = \pi(1-\nu^2)p^2l^2E^{-1}$, i.e.,

$$\frac{\partial W}{\partial l} = \frac{2\pi(1-\nu^2)\,p^2l}{E},$$

which, together with the expressions $N = p(l/2)^{1/2}$ and $\delta l = \delta S/2$, yields (5). The fact that p and l fall out from the formula for $\partial W/\partial l$ and are replaced by one quantity N confirms the decisive role of the edge of the crack.

Combining expression (5) with the inequality of the type (1), corresponding to arbitrary variation of δS, for the entire body with the crack

$$\delta F = -\delta W + \delta\Pi \leqslant 0, \quad \delta\Pi = 2\gamma_{\text{eff}}\delta S, \qquad (6)$$

Irwin obtained a criterion for the growth (for the "start of rapid propagation") of the crack

$$\pi(1-\nu^2)E^{-1}N^2 > \gamma_{\text{eff}}. \qquad (7)$$

IV. CASE WHEN THE FORCES ARE APPLIED FROM INSIDE THE CRACK

The condition that the free energy be stationary at equilibrium, $\delta F = 0$, does not hold in the case when forces are applied from the inside of the crack, since the entire surface of the crack moves when its area varies, and the forces perform work. Yet the processes in which the external forces are applied to the value not from the outer surface (or not only from the outer surface), but from the interior of the crack, are of great interest. These include, for example, the case of splitting of material or its rupture from the interior by a gas at high pressure*. As applied to these cases, the equilibrium condition must be written in the form corresponding to the variation of the crack not at constant displacements of the points where the forces are applied, but at constant forces.

In a reversible process at constant temperature, the increment of the free energy δF is equal to the work done by the external forces δR. The equality $\delta F = \delta R$ means that the internal state of the body is at equilibrium: if $\delta R = 0$, then $\delta F = 0$, i.e., the free energy is stationary. Assume that the forces are applied to the surface of the body; then

$$\delta R = \int \mathbf{p}\delta\mathbf{u}\,df,$$

where **p** is the force per unit area, **u** is the displacement of the point on the surface of the body from the free state, df is a surface element, and the integral is taken over all the surfaces.

When the elastic body is in mechanical equilibrium, the forces are linearly connected with the displacements. Therefore, in the case of an equilibrium deformation of the body from the free state ($\mathbf{p} = 0$, $\mathbf{u} = 0$) to a certain state corresponding to **p** and **u**, the forces should grow in proportion to the deformation. As a result, the elastic energy in this state turns out to equal half the work of the forces **p** on the displacements **u** (the Clapeyron theorem)

$$U = \frac{1}{2}A, \quad A = \int \mathbf{pu}\,df. \qquad (8)$$

If the crack grows slightly and its boundaries are not clamped, but the loads **p** are constant, then the boundaries shift and the forces applied to them perform work $(\delta R)_p = \delta\int \mathbf{p}\cdot\mathbf{u}\,df = 2\delta U$. The equilibrium condition $\delta F = \delta U + \delta\Pi = \delta R$ takes in this case the form of the condition for the stationarity of the thermodynamic potential $\Phi = F - 2U$:

$$(\delta\Phi)_p = \delta\Pi - (\delta U)_p = 0. \qquad (9)$$

Thus, the increment of the area of the crack releases material to such a degree, that the external forces deform the body still more and perform work δR, which not only replenishes the energy $\delta\Pi$ consumed in the formation of the new surface, but is also stored in the elastic energy of the body, with $\delta\Pi = (\delta U)_p = (\delta R)_p/2$.

We have calculated above the elastic energy δW released when the force-free area of the crack increases by δS, under conditions when the external forces do not perform work (when the boundaries are fixed). The value of δW is expressed in terms of the stress intensity coefficient by means of formula (5). Let us vary the same equilibrium crack by an amount δS at fixed loads; the elastic energy of the body increases in this case by an amount $(\delta U)_p$, which is exactly equal to δW. We shall show that this result is valid also if arbitrary forces are applied to the surface of the crack. In fact,

*It is interesting that the latter takes place in the fracture of transparent polymers such as Plexiglas under the influence of intense laser radiation. Experiments have shown [20] that bubbles of high-pressure gas are produced inside the specimens, and the resultant disk-like cracks develop from the bubble because of the bursting action of the gas pressure.

the increment of the elastic energy upon variation of the crack with the forces unchanged consists of the increment δU_1, which is connected with the opening up of a new section of the cut and the work δR done by all the forces on the displacements of all the surfaces.

But δU_1, as before, is determined only by the coefficient of stress intensity, $\delta U_1 = -\delta W$, while δR, in accordance with the constancy of the forces and Clapeyron's theorem, is equal to $\delta R = \delta A = 2\delta U$. Therefore $\delta U = \delta U_1 + \delta R = -\delta W + 2\delta U$, hence $\delta U = \delta W$. Thus, for any system of forces p applied to the body, including also to the surface of the crack, we have

$$(\delta U)_{p=\text{const}} = \delta W = 2\pi(1-\nu^2)E^{-1}N^2\delta S, \qquad (10)$$

where N corresponds to the given system of forces. In accordance with (9), Irwin's criterion (7) remains valid always, regardless of whether the forces are applied from the outside or from the inside of the crack.

V. STABLE EQUILIBRIUM CRACKS AND FRACTURE OF THE BODY

In the case of a homogeneous field of tensile stresses and an unloaded surface of the crack, the free energy of the body at equilibrium is maximal, and therefore the equilibrium is unstable. One of the attributes of instability is the fact that the equilibrium (critical) stress p_0 decreases with increasing dimension of the crack (the width in the planar case). It is clear that if the crack expands somewhat for some reason, then the applied stress turns out to be higher than the critical value corresponding to the new width, and the growth will accelerate. The crack turns out to be unstable in the case of the sufficiently strong load, if the distribution of the external forces is such that the stresses produced by them in the plane of the crack do not decrease (or decrease too slowly) with increasing outward distance from the end of the crack. In this case the total external tension force on the entire area of the crack increases in proportion to this area, i.e., the rupturing action of the load does not weaken sufficiently as the crack expands.

There exist, however, stable equilibrium states. In these states, the free energy has not a maximum but a minimum. It is physically clear that the equilibrium is stable if the linear dimension of the crack (width, radius) are not inversely but directly proportional to the applied loads and the stresses produced by the external forces decrease sufficiently rapidly in the plane of the crack when the outward distance from its ends increases. In this case, following a random increase of the area of the crack, the total tensile force applied to the entire area increases more slowly than the area itself, and the growth stops. We can say it differently: the applied load becomes smaller than the equilibrium value corresponding to the growing crack.

The process of development of cracks in brittle materials can be visualized qualitatively as follows. The material contains microcracks. With increasing tensile load applied to the body, there is reached at some point a stress that is critical with respect to a crack of definite dimension present at that point. The dimension of the microcrack, naturally, is small compared with the characteristic length over which the stresses due to the external loads change noticeably (Fig. 4). Therefore the microcrack turns out to be practically in a homogeneous stress field. After the critical stress is reached, the crack becomes unstable and begins to expand rapidly. However, at some instant its dimension increases to a value comparable with the characteristic scale over which stresses vary. If the load is concentrated to some degree, as a result of which the stresses on the whole decrease along the plane of the crack, as in Fig. 4, then the growth of the crack begins to slow down. Under definite conditions (the loads are concentrated, not too high, and are applied sufficiently far from the boundaries of the body), the crack alternately stops growing, becomes unstable, and the body can remain in such a state for a long time if the load remains unchanged. If these necessary conditions are not satisfied, then the growing crack reaches the limits of the body, and this can lead to its destruction. It is important that when the dimensions of the crack becomes comparable with the dimensions of the body, the crack becomes unstable even when it would be stable under the same conditions and at the same loads in an unbounded medium.

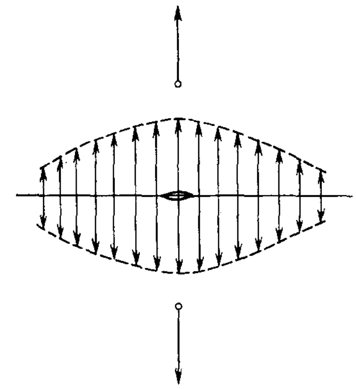

FIG. 4

The foregoing considerations, first clearly formulated by Barenblatt[14], are completely and fully confirmed also by the mathematical solutions obtained likewise by him for the first time in[14,15]. These solutions, which demonstrate the regularities of the equilibrium cracks, were found by Barenblatt with the aid of the "force" method which will be discussed in Ch. VI. Here we shall obtain these results on the basis of the energy approach, since we are already familiar with the energy method. The gist of the energy method is contained, in essence in formulas (5)—(7) or (9), but only formulas (6) and (7), disregarding the stability question, should be written in the form of equalities corresponding to the equilibrium state (in the case of stable equilibrium, the free energy or the thermodynamic potential are maximal and the sign of the inequality in (6) is reversed).

If we set up, on the basis of the solution of the problem of elasticity theory for a body with given loads and with a crack of given shape but of still unknown dimension, an equation for the coefficient of the stress intensity N, then we get from (7)

$$\pi(1-\nu^2)E^{-1}N^2 = \gamma \qquad (11)$$

the required connection between the dimension and the loads.

Let us consider by way of an example a rectangular crack (a strip) of width $2l$, when the loads are symmetrical both with respect to the plane of the crack and with respect to the perpendicular plane passing through the center of the strip. We locate the origin $x = 0$ will be placed in the center of the cut (see Fig. 5). In accordance with Muskhelishvili's solution[2,19], for such a problem

$$N = \frac{(2l)^{1/2}}{\pi} \int_0^l \frac{p(x)\,dx}{(l^2 - x^2)^{1/2}}, \tag{12}$$

where $p(x)$ are the tensile normal stresses σ_{yy} produced in the plane of the cut by the external loads in the absence of the cut. Combining (12) with (11) we obtain an equation for the width of the equilibrium crack:

$$N = \frac{(2l)^{1/2}}{\pi} \int_0^l \frac{p(x)\,dx}{(l^2 - x^2)^{1/2}} = \frac{K}{\pi}, \tag{13}$$

where K denotes the quantity

$$K = [\pi E \gamma^9 / (1 - \nu^2)]^{1/2}. \tag{14}$$

The only material constant K entering in the dependence of the dimension of the crack on the load seems, at first glance, a random combination of the other constants E, ν, and γ. Actually, however, K is not simply a symbol: it has a definite physical meaning, which will become clear after we become acquainted with the force approach in the next chapter.

Let us assume that the stresses are uniform in a band of width l_0 and there are no stresses outside this band: $p(x) = p$ when $|x| < l_0$ and $p(x) = 0$ when $|x| > l_0$, with $l < l$ (Fig. 5). Integrating (13), we obtain an equation for $l(p, l_0)$

$$p(2l_0)^{1/2} = K(l_0/l)^{1/2}[\arcsin(l_0/l)]^{-1}. \tag{15}$$

It is easy to show that this equation relative to l has no real roots if $p < p_0 = K(2/l_0)^{1/2}/\pi$. This means that the homogeneous stress p applied to the band opens the cut and transforms it into a crack only in the case when p exceeds the quantity

$$p_0 = K(2/l_0)^{1/2}/\pi = [2E\gamma/\pi(1-\nu^2)\,l_0]^{1/2}. \tag{16}$$

In the limit as $p = p_0$, the equilibrium width $2l$ is precisely equal to $2l_0$, as is seen from formula (15). When $p > p_0$ and at a fixed width of the strip of applied stresses $2l_0$, the equilibrium width of the crack $2l$ increases monotonically with increasing p, starting with $2l_0$. In the limit when $p \gg p_0$ and $l \gg l_0$, which corresponds to the limit of equal and opposite concentrated forces $P = p \cdot 2l_0$ per unit length of the strip, applied in the center of the crack, we have

$$l = P^2/2K^2, \quad P = p \cdot 2l_0. \tag{17}$$

But the value of p_0 determined by formula (16) is none other than the critical Griffith's stress (2), and the obtained $l(p)$ dependence at l_0 = const and $0 < p < \infty$ can be interpreted in full accordance with the qualitative picture described above. So long as the stresses are smaller than the critical value for the largest cut still contained in the homogeneous field, the cut will not open up. (A cut of smaller width will not open up a fortiori.) As soon as the stress reaches p_0, the cut opens up immediately to a width $2l = 2l_0$ (instability). With further increase of the stresses, the crack becomes stable,

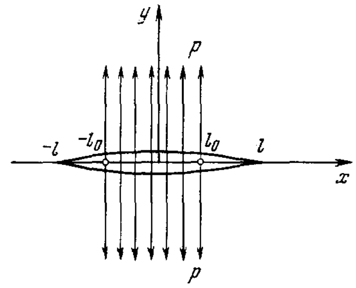

FIG. 5

since there are no external stresses outside its ends, and increases gradually with increasing load. A perfectly analogous solution is obtained also in the axially-symmetrical case of a disc-like crack[14].

An investigation of equilibrium cracks in a bounded body[13] (in the problem formulation shown in Fig. 6) has made it possible to explain important features of the influence of the boundaries on the crack. Let us consider two limiting cases. If the load is applied very far from the crack, then the stresses in the plane of the crack are practically uniform, and after reaching its critical stress a crack of any dimension becomes unstable, so that the body breaks in two. On the other hand, if the load is applied close to the crack, then it might seem that we would have a typical case of a stable crack. It turns out, however, that when the load reaches a value at which the width of the crack is approximately equal to half the width of the body, the crack becomes unstable. In the intermediate cases there are regions of stability and instability on the equilibrium plot of the crack dimension vs. the load.

The loss of stability of a large crack is obviously due to the fact that the limited length of the material separating the crack from the boundary of the body is in an overloaded state compared with the stresses that would be experienced by an infinite body.

VI. INTRODUCTION OF COHESION FORCES INTO THE EQUATION OF MECHANICAL EQUILIBRIUM

The fact that the stresses increase without limit in the material near the end of the crack (formula (4)), which follows from elasticity theory if the surface of the cut is assumed to be perfectly free, offers evidence that the solution of the problem in such a "macroscopic"

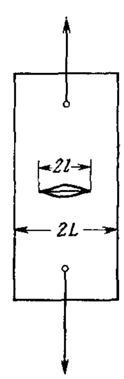

FIG. 6

formulation is incorrect in a certain vicinity of the end. Physically the reason for this circumstance is perfectly clear. When the substance becomes laminated, the molecular cohesion forces that are actually in effect between the opposite sides of the rupturing material are overcome gradually, i.e., the surface is in fact not free at the end. Owing to the action of the cohesion forces, the edges of the crack come close together, forming not a rounded profile, as follows from formula (3), but a continuous profile as shown in Fig. 7 (we recall how two glued pieces of a substance break away from each other). Therefore the strains and the stresses in the body near the end are finite.

Actually, the theory based on the energy approach contains a certain internal inconsistency. The strain and stress that increase without limit, as follows from the theory, are described by the equations of the linear theory of elasticity, which by very definition are valid only at small strains. At the same time, no account is taken of the strains and stresses produced in the medium by the cohesion forces, which really act on the opposite sides of the cut and are precisely the ones that prevent the occurrence of unrealistically infinite values. The work required to overcome the cohesion forces, which do not enter in the equations of the mechanical equilibrium of the body, is introduced in the form of a certain empirical constant in the condition for the stationarity of the free energy or of the thermodynamic potential, which must be used to supplement the equations of the theory of elasticity. This last operation, which goes beyond the framework of the concepts of elasticity theory, makes up for the indicated incompleteness of the initial mechanical-equilibrium equations.

Better and internally more consistent is the other, "force" approach to the problem of equilibrium between the body and the crack. In this approach, the cohesion forces are introduced into the equations of the mechanical equilibrium of the elastic body from the very outset, as external forces with respect to a continuous medium bounded from the inside by the surfaces of the crack. In this formulation, the entire problem can be solved completely on the basis of the static equations of elasticity theory, such as the equations of equilibrium of forces in the volume and on the surfaces, with a condition that is natural for any closed problem, namely that there be no unphysical singularities in the solution, i.e., that the stresses be finite. No additional energy conditions have to be introduced in such an approach, since it is not necessary to introduce the concept of surface energy, which is foreign to the theory of elasticity. The cohesion forces, which are external with respect to the elastic medium, do not differ in any way from any other external forces. They produce strains in the medium, perform positive or negative work upon suitable displacement of the points of their application, and this work is transformed (in the algebraic sense) into elastic energy of the body. When account is taken of the forces in the equilibrium equation, the surface energy, as work consumed in overcoming the cohesion forces when the cut is opened, is separated from the elastic medium. It is stored by the source providing the cohesion forces, namely the molecular bonds, which are external with respect to the surfaces of the crack. The energy balance in the system "elastic medium plus source of cohesion forces," describing the real body in accordance with the model of an "ideally brittle body," is automatically satisfied, just as in any other "elastic body plus source of external loads" system.

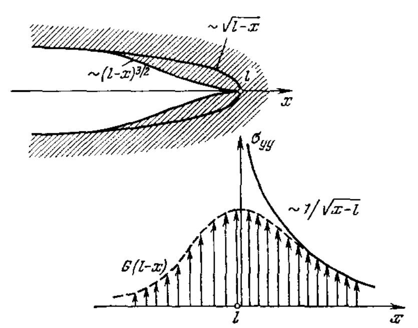

FIG. 7

The "force" approach to the problem of the crack affords greater opportunities for its investigation, since it is less "macroscopic" than the energy approach. It makes it possible to take into account the real additional deformation of the body, due to the cohesion forces, which is insignificant over the extent of the greater part of the surface of the crack, but which is very appreciable near its end, where the cohesion forces change radically the profile and the distribution of the stresses. Since the cohesion forces decrease very rapidly with increasing distance between the surfaces, they act practically only in a small region $\sim$ d near the end of the crack, where the separation is very small, so that in the entire remaining "macroscopic" part of the crack the surface remains free and the profile of the crack does not differ here from that dictated by the theory that does not take the cohesion forces into account. All this is illustrated by Fig. 7, which shows the true crack profile and stress distribution as well as those which correspond to complete freedom of the surface and are extrapolated to the end of the crack. The true stresses near the end are equal approximately to the maximum surface density of the cohesion forces, i.e., to the theoretical strength of an ideally brittle body. Of course, we go here beyond the accuracy of the theory, since the linear scheme of elasticity theory is extended up to very large stresses, which already rupture the material, but this is the model assumed in elasticity theory for an ideally brittle body.

The "force" approach to the problem of a body with a crack, described above, was developed in the papers of Barenblatt[14-15], which were stimulated by the ideas of Khristianovich[10] concerning the smoothness of the joining together of the edges of the crack and the finite nature of the stresses as a condition with the aid of which it is possible to determine the dimension of the equilibrium crack if the loads are given.*

The cohesion forces are effective only in a very small vicinity of the end of the crack. On the other hand, in the region where they are actually effective,

*These ideas were used by Khristianovich in analyzing certain problems concerning the expansion of vertical fissures in mountain rocks by liquids, when the cohesion forces are negligible compared with the lateral pressure of the mountain and the pressure of the liquid.

the surface density of the forces is larger by several orders of magnitude than the stresses produced in this place by the external loads in the absence of a crack, which obviously are of the order of the real strength (the material is ruptured as a result of concentration of the stresses near the end of the crack before the theoretical strength is reached). Therefore rupturing loads have little influence on the profile of the end of the crack, which is established under the influence of the much stronger cohesion forces and consequently has a form that is universal for a given material, independent of the applied load and of the dimensions of the crack (the autonomy of the end of the crack). This circumstance greatly simplifies the mathematical formulation of the described ideas.

Let us consider, just as in Ch. V, a rectangular crack—a strip of width $2l$ under symmetrical load (Fig. 5). The asymptotic expansion of the solution of the equations of inelastic equilibrium near the end of the cut $x = l$ is expressed, as before, by formulas (3) and (4), in which $x' = x - l$, and the constant N in the higher-order terms of the expansion is given by formula (12). The stress p(x) in formula (12) is, by definition, the total stress produced in the plane of the cut by all the external forces when there is no cut. In this case, the external forces are both the loads and the cohesion forces. Accordingly, we mark with the index t the total or summary quantities (N_t, $p_t(x)$), and retain the designation p(x) for stresses produced only by the loads. We denote the cohesion force per unit area per cut by G ($G > 0$)*.

The density of the cohesion forces G depends on the distance between the edges of the crack, but owing to the proposed universality of the profile of the end we can regard G as a function of the distance $l - x$ from the end of the crack. The stresses p(x) and $G(l - x)$ act in opposite directions (the loads tend to open the cut and the cohesion forces hinder the opening), so that the total stress is equal to

$$p_t(x) = p(x) - G(l - x). \quad (18)$$

From among all the solutions admitted by the equations of elastic equilibrium and characterized by arbitrary values of the constant N_t, the only ones realized are those which do not contain physically meaningless infinite stresses, i.e., the one in which $N_t = 0$ and the first terms of the expansions (3) and (4) vanish. In this case the profile of the end is given by $v \sim |x'|^{3/2} = (l - x)^{3/2}$ (see Fig. 7). At a short distance away from the end, the profile gradually is transformed into the one obtained at the same crack width but without allowance for the cohesion forces. The condition that the stresses be finite, $N_t = 0$, yields an equation that determines the dimension of the crack. In fact, let us substitute (18) in (12), separate the terms corresponding to the loads and to the cohesion forces, and equate the integral to zero:

$$N_t = N + N_G = 0; \quad (19)$$

here N is the quantity defined by (12), i.e., the coefficient of the stress intensity calculated with allowance for the cohesion forces, and

$$N_G = -\frac{(2l)^{1/2}}{\pi} \int_0^l \frac{G(l-x)\,dx}{(l^2 - x^2)^{1/2}}.$$

Because of the rapid decrease of the function $G(l - x)$ at distances $l - x \sim d \ll l$, this integral simplifies and the quantity N_G turns out to be independent of l, namely, $N_G = -K/\pi$, where K is a constant characteristic of the given material and having an integral dependence on the cohesion forces:

$$K = \int_0^\infty \frac{G(t)\,dt}{t^{1/2}}, \qquad t = l - x. \quad (20)$$

Equation (19) takes the form

$$N = \frac{(2l)^{1/2}}{\pi} \int_0^l \frac{p(x)\,dx}{(l^2 - x^2)^{1/2}} = \frac{K}{\pi}. \quad (21)$$

It yields the dimension of the equilibrium crack. Comparing the fundamental equations (21) and (13) obtained with the force and the energy approaches, we see that these equations are identical if the material constants K defined by formulas (20) and (14) are identical. It is obvious that the proof of the identity of these quantities is a proof of the complete equivalence of both approaches when it comes to determining the dimensions of the cracks for given loads.

The surface tension γ, which enters in formula (14), is determined by the work that must be performed against the cohesion forces if the body is cut by a plane and the two halves are moved away to infinity without producing any strains in the volume. In other words, if y is the normal distance between the parallel surfaces of both halves, and the cohesion force per unit area is G(y), then we have for the two surfaces*

$$2\gamma = \int_0^\infty G(y)\,dy. \quad (22)$$

By virtue of the universality of the profile of the crack, the distance between the edges is a universal function of the distance $t = l - x$ from the end: $v = v(t)$. Consequently

$$\gamma = \int_0^\infty G[v(t)]\frac{dv}{dt}\,dt. \quad (22')$$

If we substitute here the profile of the end of the crack $v = \text{const} \cdot t^{3/2}$ with the actual value of the constant and perform certain mathematical transformations, then we can see that the integrals (20) and (22) are actually connected with each other by relation (14) (this calculation can be found in the paper by Willis[21].

The dependence of the dimension of the crack on the load contains a single material constant K, determined by the cohesion forces in accordance with formula (20). It characterizes the resistance of the material to rupture in brittle cracking (the larger K, the larger the load necessary to produce a crack of given dimension), and it is natural to call it the cohesion modulus. For an

*We recall that in the formulation of the boundary conditions, in accord with the general scheme of the linear theory of elasticity, the external forces are specified on the undeformed boundary, in this case on the plane of the still unopened cut.

*Strictly speaking, the lower limit of the integral is not zero but a distance on the order of atomic dimensions, the same which conditionally distinguishes between the concept of an elastically-stretched solid medium and that of a ruptured material.

experimental determination of K it is possible to use any formula that expresses the dimension of the crack in terms of the load, for example formula (17).

VII. FINITE CHARACTER OF THE STRESSES AND STATIONARITY OF THE ELASTIC POTENTIAL

The static equations of elasticity theory, which constitute equations for the equilibrium of the forces (internal stresses and those applied from the outside) in the volume and on the surfaces of the body, constitute a completely closed system, and if the problem is correctly formulated they require no supplementation. The problem of a body with a crack, with allowance for cohesion forces in the equations for the equilibrium of the forces, is perfectly correctly formulated, and the absence of divergences in the solution is also natural, just as for any other physical problem in which there are no reasons for encountering infinities. The finiteness condition is not a hypothesis and requires no proof, just as in dozens of other physical problems, where out of all the possible particular solutions of equations one discards those containing unphysical infinities, or, equivalently, the arbitrary constants of these particular solutions are set equal to zero. Since the equations of the theory of elasticity are internally closed, the condition that the stresses be finite, being a condition that selects the true equilibrium crack from among the set of solutions admitted by the equations for the equilibrium of the forces, should follow also from the energy principles of the equilibrium of the elastic body, from which follow the force equilibrium equations themselves. Let us verify this directly.

In this case the condition for the equilibrium of the body is the condition of the stationarity of the thermodynamic potential at constant forces, since the varied surface of the crack is not free. However, as is especially emphasized above, when account is taken of the cohesion forces in the equilibrium equation, regarded as external forces applied to part of the surface of the body, the surface energy is assigned to a source of forces and does not belong to the elastic medium itself. Therefore the free energy F or the thermodynamic potential Φ of the elastic medium does not contain the surface energy Π. Apart from constants that depend on the temperature, we have $F = U$, $\Phi = -2U$, and the stationarity condition for the thermodynamic potential upon variation of the area of the crack with fixed forces (9) assumes the form of stationarity of the elastic potential U

$$(\delta U)_{p=\mathrm{const}} = 0. \tag{23}$$

The general formula (10), in which N should be taken to mean the total quantity N_t corresponding to all the forces, loads, and cohesion forces, yields immediately the condition $N_t = 0$ for the finiteness of expressions (19).

We emphasize that the concept of variation of the area of the crack at fixed forces has a perfectly exact meaning, which admits of no leeway in its interpretation. Thus, if the tip of the crack is shifted and the half-width of a rectangular crack l increases virtually by δl, then the cohesion forces acting prior to the variation, say on the section from $x = l - d$ to $x = l$, remain the same as before the shift, and in the newly opened section of the cut, from $x = l$ to $x = l + l$, there are no cohesion forces after the variation, just as there were none before.

The condition of the stationarity of the elastic potential $(\delta U)_p = 0$ denotes physically that in the case of infinitesimally small virtual expansion of the equilibrium crack by δl, the work performed by the loads, accurate to small quantities of higher order than δl, is consumed in overcoming the cohesion forces that prevent the displacement of the surface of the crack, and the body neither acquires nor releases elastic energy in this case (again accurate to small quantities of higher order).

It is precisely this aspect of the theory, the question of the stationarity of the elastic potential, which somehow was not understood and which took the brunt of the criticism. This aspect was subsequently extended also to the condition of finiteness of the stresses[25-28] (see also[29] concerning this question).

Contributing to a clear understanding of the internal properties and the macroscopic equivalence of the theories based on the energy and on the force approaches is an article by Ishlinskiĭ[22], in which, using a very simple example of a linear crack with constant loads applied from the inside to a narrower band, and with constant cohesion forces G acting on a small section d at the end of the crack, all the calculations are carried out in explicit form and to full conclusion. Complete formulas are given for the profile of the crack and for the distribution of the stresses, the elastic energy of the body is calculated, and everything is done without using, as it were, any "theorems" such as the Irwin formula for the rate of release of energy, or the condition of stationarity of the elastic potential in the force approach. The results are obtained by direct calculation of the elastic energy of the body before and after the variation of the crack and by letting the variation go to zero.

VIII. REAL MATERIALS. QUASIBRITTLE DAMAGE

The theory based on introducing cohesion forces in the equations of elastic equilibrium of a body is in any case more perfect and more physical than the theory in which one introduces simply a surface tension, since the force approach leads to the same macroscopic results as the energy approach but, in addition, describes qualitatively correctly the vicinity of the tip of the crack and makes it possible to eliminate the unrealistic singularities of the energy approach. A certain skepticism with respect to the literal understanding of the results of the force theory might concern the validity of including in the gross scheme of the linear theory of elasticity of short-range forces of molecular nature. However, the significance of the theory is much greater than indicated, and its results, concerning the structure of the tip of the crack and its vicinity, have a perfectly realistic physical meaning for an entire class of materials of practical importance. As shown by numerical estimates, in materials such as silicate and organic glasses, the dimension of the end region of the crack d, where cohesion forces act, turn out to be large compared with the interatomic distances, and this indeed justifies the use of the macroscopic analysis for this region.

An estimate of the dimension of the tip region is obtained from expression (20) for the cohesion modulus K. Obviously, in order of magnitude we have

$$K = \int_0^\infty \frac{G(t)\,dt}{t^{1/2}} \sim G_m d^{1/2},$$

where G_m is the characteristic scale of the cohesion forces, which is of the order of the maximum value of these forces and certainly does not exceed the breakdown stress for an ideal (defect-free) material. For silicate glass, for example, the ideal strength is approximately 0.05 of Young's modulus $E = 6.7 \times 10^{11}$ dyne/cm^2. On the basis of the experimental data obtained by measuring the dimensions of the cracks at given loads, the surface tension is $\gamma = 2.1 \times 10^3$ erg/cm^2 and $\nu = 0.24$. Using formula (14), which connects K and γ, we get

$$d \sim \pi E\gamma / G_m^2 (1-\nu^2) \sim \pi\gamma/\alpha^2 E (1-\nu^2) \sim 4 \cdot 10^{-6} \text{ cm}$$

which exceeds the interatomic distances by two orders of magnitude.

The dimensions of the end region are even larger in materials such as organic glass. Thus, in polymethylmethacrylate, according to measurements of the crack dimensions—bands of width $2l$—under the influence of concentrated loads Q per unit length applied at the center of the band, the cohesion modulus turns out to be $K = Q(2l)^{1/2} = 1.1 \times 10^8$ dyne/cm$^{3/2}$ in the range $l = 16$–49 cm, with $Q \approx (1.7$–$2.2) \times 10^8$ dyne/cm$^{3/2}$[23]. For this material, $E = 2.45 \times 10^{11}$ dyne/cm^2 and, in accordance with formula (14), $\gamma_{eff} = 1.5 \times 10^5$ erg/cm^2. G_m can be assumed to be of the order of the yield point, $G_m \sim 7 \times 10^8$ dyne/cm^2. Hence $d \sim K^2/G_m^2 \sim 10^{-2}$ cm. Indeed, macrophotography of the cracks shows that the edges of the crack converge smoothly and the dimensions of the end region amount to several dozen microns.

Experience shows that the structure of the end region of cracks in organic glass has a very interesting character: the edges of the crack are drawn together by some thin filaments, fibers of the material, the number of which per unit area increases with increasing distance from the end, the cohesion of the edges being realized, as it were, by tension of these filaments. Since the nature of the cohesion forces is not discussed at all in the force approach, the theory can be applied directly also to the case of cohesion realized by "stretched filaments." The use of this model has made it possible to consider the kinetics of the growth of the cracks in similar material[24]. We note that the nature of the filaments has not yet been explained.

In general, since the rupture of the material occurs precisely in the end region of the crack, for any analysis of the mechanism and for details of the destruction process it is necessary to have correct ideas concerning the structure of the end region, the profile of the crack, and the distribution of the stresses, something that cannot be obtained without taking into account the cohesion forces. Ideas concerning the structure of the end region are essential for the study of the mechanism of propagation of cracks under the influence of the load (growth kinetics of the cracks), and for consideration of the fatigue strength of the materials (it turns out that, theoretically, stable cracks grow very slowly even at a constant load, and the material fails ultimately). All this attaches special significance to the results obtained concerning the structure of the ends of the cracks by the force method.

The possibilities of using the theory of brittle fracture go far beyond the limits of this comparatively narrow class of materials that are actually brittle (silicate glass, fused quartz, and a few others). Experimental investigations have shown that upon formation of cracks, individual materials which are perfectly plastic in ordinary tension tests are fractured in such a way that the plastic deformations are concentrated in a thin layer near the surface of the crack. The results of experiments on such materials confirm Griffith's formula for the critical stress $p_0 l^{1/2} = \text{const}$. However, the value of the effective energy γ_{eff}, determined in these measurements from the values of the constant in this formula, turns out to be much larger than the surface tension, which, of course, is not known, but which can be estimated from data on the cohesion forces. Thus, for example, for U8 carbon steel ($E = 2.06 \times 10^{12}$ dyne/cm^2), experiment yields $\gamma_{eff} = 7.5 \times 10^5$ erg/cm^2, which is several hundred times larger than the surface tension.

Thus, all the formulas describing cracks in ideally brittle bodies can be extended also to "quasibrittle" bodies, if the constants γ and K entering in these formulas are regarded as generalized constants that take into account the effect of plastic deformations in the surface layer (the ideas of "quasibrittle" damage, as already mentioned in Ch. I, where were advanced in the papers of Irwin and Orowan[4,5]). Measurements of the equilibrium cracks in metals gave the following values of the cohesion moduli, on the basis of which we can estimate the dimensions of the end region. For steel No. 4330 we have $K = 2.5 \times 10^{10}$ dyne/cm$^{3/2}$, and for the 2218-T87 aluminum alloy $K = 10^{10}$ dyne/cm$^{3/2}$[23]. For the first of these two materials, $E = 2 \times 10^{12}$ dyne/cm^2 and the ultimate strength is $\sigma_t = 1.5 \times 10^{10}$ dyne/cm^2, and for the second $E = 0.8 \times 10^{12}$ cm^2 and $\sigma_t = 0.4 \times 10^{10}$ dyne/cm^2, respectively. An estimate based on the formula $d \sim K^2/\sigma_t^2$ shows that the dimension of the end region is of the order of several millimeters, so that the theory of quasibrittle damage is applicable to cracks much larger than this quantity.

It should be stated that there is still no theory that takes plastic deformation into consideration and is capable of explaining the value of the effective energy loss σ_{eff}. We emphasize that the order of magnitude of σ_{eff} cannot be determined from dimensionality considerations, unlike, for example, the approximate order of magnitude of the surface tension: $\gamma \sim Ea$, where a is the atomic dimension (more accurately, $\gamma \approx 0.1Ea$). Besides E, a, or γ, the material is characterized by other quantities having the same dimensionality as E, namely the shear modulus $\mu \approx E$ and the ultimate strength σ_t. Consequently, the effective energy γ_{eff} can be represented in the theory only in the form $\gamma_{eff} = \gamma f(E/\sigma_t)$, where f is some function of the dimensionless ratio E/σ_t or μ/σ_t, and it is obviously impossible to predict beforehand that f has the experimental value 10^2–10^3.

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