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PARTIAL CONSERVATION OF AXIAL CURRENT IN PROCESSES INVOLVING "SOFT" MESONS

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1. INTRODUCTION

1.1. Fundamental Hypotheses

1.1.1. Laws of conservation and of symmetry play a fundamental part in present-day hadron physics. Arguments based on symmetries and conservation laws enable us to make predictions about the mass spectrum of hadrons and the amplitudes for various processes without considering the dynamics of strong interactions (SI).

In the general case the existence of a symmetry of the Hamiltonian means that the levels of the system are degenerate. If, for example, a particle moves in a centrally symmetric field, so that the Hamiltonian is invariant under rotations around the center, the energies are degenerate for states with a fixed value of the orbital angular momentum and various values of the projection of the angular momentum along one of the coordinate axes. The transformations of the group intermingle these states with each other.

A well known example of a symmetry in elementary-particle physics is the isotopic invariance of SI. It turns out that mesons and baryons with different electric charges and the same values for the other quantum numbers are grouped in families of particles which are called isotopic multiplets. The strong interactions remain unchanged under isotopic transformations which intermingle the states within a multiplet. Examples of isotopic multiplets are the proton and neutron, the π^+, π^0, and π^- mesons, and so on.

Isotopic invariance is not rigorous. For example, the masses of the proton and neutron differ by 1.3 MeV, and those of charged and neutral pions by 4.6 MeV. The ratio of these mass differences to the characteristic masses for the strong interactions is of the order of 1/100 to 1/1000 and can serve as the parameter for the symmetry breaking. Ordinarily mass differences within isotopic multiplets are ascribed to the electromagnetic interaction.

Since the isotopic symmetry explains all experimentally known cases in which particle masses are equal to within a few million electron volts, it is clear that the strong interactions have no symmetry higher than the isotopic one. It is possible, however, that there exist additional approximate symmetries with breaking parameters of the order of 1/10, so that the mass splittings within the multiplets are of the order of 100 MeV. Much attention has been given to the search for such symmetries.

In particular, a number of results have been obtained in recent years, the best known being the Adler-Weissberg relation,[1,2] which indicate that the strong interactions are approximately invariant with respect to some group of transformations which includes along with the isotopic transformations transformations which mix states with different parities. The fundamental ideas whose development led to these results were put forward in papers by Nambu and his collaborators[3-5] and by Gell-Mann and his collaborators.[6-8]

1.1.2. Let us examine how one can construct the multiplets of a group which includes transformations with parity change. As an example we take the multiplet which includes the nucleon—a particle with mass m = 940 MeV and spin and parity $\frac{1}{2}^+$. The same multiplet must include a state with the opposite parity. For the usual realization of the symmetry this state is a particle with the quantum numbers $\frac{1}{2}^-$.

Nambu[3] pointed out a possibility for constructing multiplets which is quite different in principle. If there were to exist a massless pseudoscalar particle, then a state with the quantum numbers $\frac{1}{2}^-$ could be formed by a stationary nucleon and a pseudoscalar meson with zero energy. Since in this case the transformations which change the parity add to the state a meson with zero energy, the other terms of the multiplet are: nucleon plus two mesons with zero energy, nucleon plus three mesons, and so on. Another example of a multiplet is the succession of states: vacuum, one meson with zero energy, two mesons with zero energy, and so on. We remark that in the general case for a continuous group of transformations there exist two possibilities for realizing a symmetry: with one-particle states and with many-particle states. From the mathematical point of view the first possibility corresponds to linear representations of the group and the second to nonlinear representations.

Let us now see which of these possibilities gives the better correspondence with the experimental mass spectrum in the case of a group which includes transformations with parity change. The resonance with quantum numbers $\frac{1}{2}^-$ which has mass closest to that of the nucleon and could be its partner in a multiplet is the resonance N*(1480), so that the mass splitting is about 500 MeV. The situation is no better for other particles, say π mesons. The corresponding scalar meson, if it indeed exists, apparently has mass ≈ 700 MeV.

In the case of a nonlinear realization of the symmetry it is necessary, as we already noted, that there exist a massless pseudoscalar particle. No such particle has been found experimentally. The minimum mass is that of the π meson (μ = 140 MeV). If we regard this quantity as small in comparison with the characteristic mass m_{char} of the strong interactions, we can try to identify the π meson with the required particle. We adopt this hypothesis throughout what follows.

The hypothesis that the ratio μ/m_{char} is small may seem paradoxical at first sight, since it is often assumed that $\mu \sim m_{char}$. There are, however, some experimental facts which indicate that 140 MeV can in some sense be regarded as a small quantity. For example, as the square of the momentum transfer varies from zero to μ^2 the charge form-factor of the proton changes by about 4 percent. There are also theoretical predictions based on treating μ/m_{char} as small which agree well with experiment. We mention only the Kroll-Ruderman relation for the amplitude at threshold for photoproduction of π mesons. Therefore the assumption $\mu/m_{char} \ll 1$ is not obviously senseless, although in adopting this assumption we cannot expect accuracy better than say 10 percent. Furthermore it cannot be excluded that regarding μ/m_{char} as a small quantity may be permissible in some cases and not in others.

In view of these stipulations, let us return to the consideration of the symmetry in question and assume that $\mu = 0$. As is well known, in addition to the spectra the symmetry determines relations between the amplitudes for processes which differ by the replacement of particles by others from the same multiplet. For example, owing to isotopic invariance the difference of the amplitudes for elastic scattering of π^- and π^+ mesons by protons is equal to $2^{1/2}$ times the amplitude for charge exchange between π^- mesons and protons.

In our present case a single multiplet contains states with different numbers of π mesons with zero energy. Therefore there are relations between the amplitudes for processes with different numbers of "soft" π mesons. These predictions are analogous to the low-energy theorems for reactions involving soft γ-ray quanta, which also connect[10] the matrix elements for processes with different numbers of particles—radiative and nonradiative.

The low-energy theorems for processes involving photons or mesons correspond to the invariance of the theory with respect to nonlinear transformations of the fields (so that states with different numbers of particles are combined in a multiplet). In the case of the electromagnetic interaction these nonlinear transformations are gauge transformations of the fields with variable phase factors.

1.1.3. Let us go on to the description of the group structure of the symmetry in question. It is a remarkable fact[11] that this structure is almost uniquely determined by the spectrum of the particles: the existence of the isotopic multiplets and of the triplet of massless π mesons.

As is well known, the structure of a group is given by the commutation relations between the generators—the operators for infinitesimal transformations. In this connection let us recall that the invariance of the Hamiltonian $\mathscr{H}$ with respect to a group of transformations means that the commutators of the Hamiltonian $\mathscr{H}$ with the generators of the group are equal to zero. In other words, the generators are time-independent and are operators corresponding to conserved quantities.

Since the symmetry we are considering includes isotopic invariance, the list of generators includes the generators V^i (i = 1, 2, 3) of the isotopic group, for which the commutators are well known to be

$$[V^i, V^k] = i\varepsilon^{ikl}V^l. \tag{1.1}$$

A group whose generators obey these commutation relations is denoted by SU(2).

Our group also includes transformations with change of the parity of the state. As was noted earlier, these transformations in particular change the vacuum into a π meson, a particle with isospin unity. Therefore the corresponding generators form an isotopic vector A^i (i = 1, 2, 3). This means that

$$[V^i, A^k] = i\varepsilon^{ikl}A^l. \tag{1.2}$$

Let us now consider the commutators $[A^i, A^k]$. Owing to the antisymmetry with respect to the indices i, k, these commutators can be written in the form

$$[A^i, A^k] = i\varepsilon^{ikl}\tilde{V}^l, \tag{1.3}$$

where the $\tilde{V}^l$ are operators independent of the time. If the $\tilde{V}^l$ are not identically zero, they can be regarded as the generators of certain transformations which do not change the parity. There are two possibilities: Either these transformations are new ones, or else they coincide (to within a factor) with some already known, i.e., with the isotopic transformations.

We shall show that the assumption that there exist "new" operators $\tilde{V}^i$, which are not the same as the generators of isotopic rotations, is in contradiction with experimental data. In fact, in the case of linear representations this assumption contradicts the fact that there is no degeneracy in the mass spectrum of the particles which is not explained by the isotopic invariance. In the case of the nonlinear representations, in which the operators mix one-particle and many-particle states, there would have to be a triplet of massless scalar particles, which also contradicts experiment.

Accordingly we have shown that the operators $\tilde{V}^i$ are proportional to the generators V^i of the isogroup. If the proportionality coefficient is not equal to zero, the generators A^i can be normalized so that

$$[A^i, A^k] = i\varepsilon^{ikl}V^l. \tag{1.4}$$

This relation gives the final determination of the group structure. If the generators A^i commute with each other, we have a different group. The choice between the two is made on the basis of a more detailed consideration of the experimental consequences. In view of the results of such a consideration, we rule out the possibility that the commutator $[A^i, A^k]$ is equal to zero, and adopt Eq. (1.4).

The commutation relations (1.1), (1.2), and (1.4) can be rewritten in the form

$$\left.\begin{array}{l}[(V^i \pm A^i)/2,\ (V^k \pm A^k)/2] = i\varepsilon^{ikl}(V^l \pm A^l)/2,\\ [(V^i + A^i)/2,\ (V^k - A^k)/2] = 0.\end{array}\right\} \tag{1.5}$$

On comparing these with Eq. (1.1) we see that the operators $(V^i \pm A^i)/2$ are the generators of two independent groups SU(2). This is expressed by saying that the symmetry group is the direct product SU(2) ⊗ SU(2).

Naturally the group SI(2) ⊗ SU(2) also has representations with a finite number of particles in the multiplet. For example, this is the symmetry of a system of interacting massless nucleons[8] (see also Sec. 1 of Chapter 8), in which the operators $(V^i + A^i)/2$ and $(V^i - A^i)/2$, which correspond to the isospins of mass zero nucleons with left-handed and right-handed helicities, are separately conserved.

One could imagine that "originally" there were massless nucleons; therefore not only the commutation relations but also the group representations were simple. The interaction brought it about that the nucleon acquired a mass and a massless π meson appeared; there was a rearrangement of the multiplets, while there was no change in the commutation relations of the generators, since the interaction does not break the symmetry. Such a rearrangement is called spontaneous symmetry breaking,[4,12,13] although this does not mean that the symmetry is not exact.

1.1.4. Let us now consider the connection between the symmetry of the strong interactions and the properties of the weak and electromagnetic interactions of hadrons. We shall elucidate this connection with the example of isotopic symmetry.

The generators V^i (i = 1, 2, 3) of the isogroup can be represented in the form of space integrals of the zeroth components of vector currents v^i_μ:

$$V^i = \int v_0^i(t, \mathbf{x})\, d^3x. \tag{1.6}$$

The time independence of the operators V^i corresponds to conservation of the currents v^i_μ:

$$\partial_\mu v_\mu^i = 0. \tag{1.7}$$

On the other hand, in the electromagnetic and weak interactions (we here have in mind the weak interaction of leptons with hadrons without change of strangeness) also involve vertain vector currents, which in general have no relation at all to the currents v^i_μ introduced above. It is usually assumed,[8,14] however, that it is precisely these currents in Eq. (1.6) which appear in the electromagnetic interaction (more exactly, in its isovector part) and the weak interaction of hadrons, and this is verified experimentally.

When we include a group of transformations which change the parity, conserved axial currents a^i_μ (i = 1, 2, 3) appear. We shall assume[3,7,8] that these currents determine the axial part of the weak interaction between hadrons and leptons without change of strangeness.

We can now inquire about the selection rules obeyed by the matrix elements of the currents v^i_μ and a^i_μ. We assume that the currents are components of isotopic vectors. In other words, their commutators with the generators of isotopic rotations are of the form

$$[V^i,\, v_\mu^k] = i\varepsilon^{ikl} v_\mu^l,\quad [V^i,\, a_\mu^k] = i\varepsilon^{ikl} a_\mu^l. \tag{1.8}$$

The transformation properties of the operators v_μ and a_μ with respect to the group SU(2) ⊗ SU(2) are determined by the form of the commutators of the currents with the operators $(V^i \pm A^i)/2$. We shall assume[8] that these are analogous to the commutation relations for the corresponding generators, i.e.,

$$\left.\begin{aligned} &[(V^i \pm A^i)/2,\ (v_\mu^k + a_\mu^k)] = i\varepsilon^{ikl}(v_\mu^l \pm a_\mu^l),\\ &[(V^i \pm A^i)/2,\ (v_\mu^k \mp a_\mu^k)] = 0. \end{aligned}\right\} \tag{1.9}$$

We note that the weak current i_μ without change of strangeness is of the form $i_\mu = v_\mu + a_\mu$ and according to (1.9) satisfies the condition

$$[V^i - A^i,\ i_\mu] = 0. \tag{1.10}$$

It is easy to understand the relation (1.10) in the framework of simple models in which the group representations are linear. In such models the generators $(V^i + A^i)/2$ and $(V^i - A^i)/2$ are composed of left-handed fields $(1 + \gamma_5)\Psi/2$ and right-handed fields $(1 - \gamma_5)\Psi/2$. The hadronic weak current, in analogy with the leptonic current, is constructed from left-handed fields only, and this leads to the relation (1.10).

With this sort of approach there is an obvious extension of the commutation relations to the case of the current i^S_μ which causes leptonic interactions of hadrons with change of strangeness, $\Delta S = \pm 1$. Assuming that this current is also made up of left-handed fields, we get

$$[V^i - A^i,\ i_\mu^S] = 0. \tag{1.11}$$

This relation, together with a hypothesis about the isotopic properties of the operator i^S_μ (usually i^S_μ is regarded as a component of an isospinor), determines the transformation properties of i^S_μ with respect to the group.

1.1.5. In concluding this section we list once again the fundamental hypotheses whose consequences are considered in the present paper.

I. In the limit of vanishing pion mass the SI Hamiltonian is invariant with respect to transformations which mix states with different parities.[3-5] The generators A^i (i = 1, 2, 3) of these transformations, which we shall also call axial charges, are independent of the time (for $\mu^2 = 0$).

If the commutator $[A^i, A^k]$ is different from zero (see the discussion in 1.1.3) the operators A^i can be normalized so that

$$[A^i,\ A^k] = i\varepsilon^{ikl} V^l, \tag{1.12}$$

where V^i are the generators of the isotopic rotations.

The relation (1.12) closes the algebra of the vector and axial charges and means that the symmetry group of the strong interactions is SU(2) ⊗ SU(2).[8]

The generators A^i can be represented as space integrals of the zeroth components of conserved (for $\mu^2 = 0$) axial currents a^i_μ:

$$\left.\begin{aligned} &A^i(t) = \int d^3x\, a_0^i(t, \mathbf{x}),\\ &\frac{d}{dt} A^i(t) = 0,\quad \partial_\mu a_\mu(t, \mathbf{x}) = 0. \end{aligned}\right\} \tag{1.13}$$

The matrix elements of the currents a_μ acquire a direct physical meaning through the second main hypothesis.

II. It is assumed that the axial currents $a^\pm = a^1 \pm a^2$ associated with the SU(2) ⊗ SU(2) symmetry of the strong interactions are identical[3,7,8] with the axial currents of the weak interactions of hadrons with leptons without change of strangeness.

To determine the transformation properties of the currents with respect to the group it is necessary to prescribe their commutators with the generators of the group. We include assumptions about the form of these commutators in Hypothesis III.

III. Hypotheses about the transformation properties of the Hamiltonians of various interactions with respect to the group SU(2) ⊗ SU(2).[8,15] In the case of the weak interactions we shall assume that

$$[V^i - A^i,\ \mathcal{H}^W] = 0, \tag{1.14}$$

where $\mathcal{H}^W$ is the Hamiltonian for both nonleptonic and leptonic weak interactions (the operators $V^i - A^i$ do not act on the lepton fields).

For the commutator of the electromagnetic current j_μ with A^i we assume

$$[A^i, j_\mu] = i\varepsilon^{i3l}a^l_\mu \qquad (1.15)$$

(we recall that the current j_μ is of the form of an isoscalar current and the third component of an isovector current).

In the derivation of certain relations it is possible to include a "semistrong" interaction which breaks the SU(2) ⊗ SU(2) symmetry and in particular is responsible for the nonzero mass of the π meson. For the Hamiltonian $\mathscr{H}^{\text{br}}$ of this interaction we shall assume that

$$[A^i, [A^k, \mathscr{H}^{\text{br}}]] \sim \delta^{ik}. \qquad (1.16)$$

We note that if the symmetry is broken the operators A^i are time-dependent. It is assumed, however,[8] that the interaction that breaks the symmetry is such that the equal-time commutation relations are not changed.

Hypotheses I—III allow us to calculate the amplitudes for processes involving π mesons at a nonphysical point where the four-momentum of the π meson[1)] is equal to zero. In order to connect the amplitude at this point with experimentally measurable quantities one uses Hypothesis IV.

IV. Extrapolation formulas.[16] If there are no singularities of the amplitude in the range of π-meson energies $E_\pi \sim \mu$, or if the contribution of any such singularities to the amplitude is for some reason small, the hypothesis that the π-meson mass is small leads to a representation of the amplitude as a polynomial in the momentum of the π meson. If there are indeed singularities in the region in question, their contribution is taken into account separately. For example, in the amplitude for πN scattering one must deal separately with the contribution of the nucleon pole diagram.

1.2. Purpose of this Review

There have been many papers on the consequences of the hypotheses I—IV formulated above, and the results have aroused lively interest. One usually gets an idea of the correctness or incorrectness of physical theories from the answers to two questions: How simple and beautiful are the basic ideas? How broad a range of experimental facts does the theory describe, and how good is the agreement between theory and experiment?

Evidently a widespread opinion about the hypotheses in question (they are often called briefly the hypothesis of partial conservation of axial current) is that the foundations of the theory are not comprehensible and are of the nature of a recipe, but on the other hand the consequences receive excellent experimental support.

It seems to us that the situation is rather the reverse. The concept of a spontaneously broken symmetry is simple and beautiful (although it is possible that the foregoing exposition has not convinced the reader of this), while the number of verified predictions is small. However, the existing experimental confirmations (the Goldberger-Treiman and Adler-Weissberg relations) are rather impressive, and when the clarity of the basic assumptions is taken into account they give us hope that the theory will stand the test of time.

Of course further comparison of experimental relations with experimental data will be of decisive importance. Therefore our main attention in this review will be given to deriving on the basis of Hypotheses I—IV formulas for the amplitudes for specific processes and to comparing these relations with experiment. To keep the details of the calculations from obscuring the main idea, we would like to emphasize here that the treatment of all processes is essentially the same, and roughly speaking can be divided into two stages.

First, one calculates the theoretical value of the amplitude for π-meson momentum equal to zero. The answer is always unique if Hypotheses I—III are adopted. There is a simple recipe for the calculation, based on relations analogous to Ward's identity in electrodynamics. Use is made of a reduction formula for the amplitude. The derivation of the reduction formula can be found in various books, e.g., [17]. But a reader who accepts this formula as "natural" will have no difficulty with what follows.

The second step in the treatment of any process is to extract from the existing experimental data the value of the amplitude at the nonphysical point where the momentum of the π meson is zero. The amplitude can be continued to this point, provided that there exists a simple analytic expression which gives a good description of the behavior of the amplitude in the region of small pion energies. To find such formulas and compare them with experiment is a separate problem, in general unrelated with the SU(2) ⊗ SU(2) symmetry. The question of the extrapolation formulas for the amplitudes for specific processes is discussed in the sections dealing with them.

In the treatment of each process we have tried to expound the results obtained as fully as possible. It seemed to us that this could be done, since the method of derivation of the theoretical relations is a rather standard one. Here it should be emphasized that we are considering only the consequences of Hypotheses I—IV, and shall not discuss the hypothesis of vector-meson dominance, the dispersion sum rules, SU(3) ⊗ SU(3) symmetry, and so on, for which also there is an extensive literature.

To make the review really complete, we must of course not only discuss individual reactions in detail, but also look through all possible processes which involve "soft" pions. If we do not consider processes which are fantastic from the experimental point of view, there is a limited list of such processes.

Moreover, it turns out that for some processes Hypotheses I—IV do not lead to any new consequences. For example, the amplitude for the decay $\omega \to 3\pi$ must be equal to zero in the limit of zero momentum of any one of the π mesons. But this condition always holds, independently of the correctness of any hypotheses, owing to the kinematics of the decay. We shall not discuss such cases here.

The processes considered in this paper are listed in the next section, but we shall at once remark that in our opinion at least two further reactions could also be

[1)]Hereafter when we speak of momenta of particles we mean their four-momenta.

considered: the production of a π meson in a neutrino experiment, $\nu + N \to 1 + N + \pi$,[18,19] and the production of a π meson in πN collisions, $\pi N \to 2\pi N$.[20] The description of these processes is omitted mainly for lack of space, and we confine ourselves to referring to the original papers.

We shall now make some remarks about the nature of the exposition. The subject matter of the review is extensive, and in many places the exposition is rather brief. For the most part this review is a "working paper" rather than a popular introduction to this branch of physics. We have tried to derive and formulate the results with a degree of rigor not inferior to that of the original papers.

The parts written in most detail are Secs. 2.1 and 2.2, 3.1 and 3.2 in Chapters 2 and 3. Here essentially all necessary methods of calculation are expounded. In particular, the treatment of the amplitude for πN scattering (Sec. 3.2 in Chapter 3) can serve as a "model" for the description of any other process. An acquaintance with these sections (except Subsection 3.2.7) and with the introduction to Chapter 4 is in our opinion enough for the reader who is interested only in the way the results are derived. In the other parts we have avoided repetition, and all of the calculations which do not differ from ones already encountered are treated concisely.

We have tried to include in this review all necessary information from the phenomenological description of the reactions considered. As a rule these are given at the beginnings of the respective chapters, together with the notations. We hope that these sections will not deter the reader from reaching the subsequent sections which have more content.

We remark that the phenomenological information is here given in summary style. Therefore in each case we give references to textbooks and papers in which the details can be found. As a rule, it will suffice to know the phenomenology of strong and EM processes to the extent of Nishijima's book,[21] and that of weak processes, from Okun's book.[22]

1.3. Plan of the Review. Literature

As already stated, our main attention in this article is given to the specific consequences of our particular hypotheses. Discussions of broader questions, beyond that in the introduction, are found in Sec. 1.2 of Chapter 1, on the breaking of SU(2) ⊗ SU(2) symmetry, and in Chapter 8, which gives a brief description of a way of deriving the consequences of Hypotheses I—IV which is quite different from that used in the main part of this article.

The main basis for the division into chapters is the ascription of the various processes to particular interactions. The type of interaction is important in principle, since in the comparison of the predictions of the theory with experiment one is testing assumptions about the structure of the Hamiltonians of the various interactions in relation to the group SU(2) ⊗ SU(2).

In Chapter 2 we consider the weak processes: β decay of the neutron, μ capture by protons, and neutrino reactions; in Chapter 3, strong processes: πN and $\pi\pi$ scattering, and the decay $\chi \to 2\pi\eta$; in Chapter 4, EM processes: photoproduction and electroproduction of π mesons; in Chapter 5, leptonic decays with change of strangeness: K_{l3} and K_{e4} decays; in Chapter 6, weak nonleptonic interactions: the decays $K \to 3\pi$ and decays of hyperons; in Chapter 7, the decay $\eta \to 3\pi$ (it is assumed that this process is due to a virtual electromagnetic interaction).

The bibliography does not pretend to be a complete one. Only those papers are listed whose results are given in the review. There are no references at all to papers which consider the same processes but start from different, even if very similar, hypotheses (for example, hypotheses of partial conservation of axial current with change of strangeness).

During the writing of this article a number of monographs and reviews has appeared,[23-28] which deal with similar ideas and can be recommended to the reader. A distinguishing feature of these articles in comparison with the present one is that they discuss also "adjacent" branches of physics, such as dispersion sum rules and the symmetry SU(3) ⊗ SU(3).

In conclusion we give some references to papers on questions touched on in our review but not presented in sufficient detail. The quark model and the derivation of the commutation relations between currents and charges are well treated by Adler and Dashen in an authors' commentary on a collection of papers.[24] For the study of these questions we can also recommend the clearly written papers of Gell-Mann.[8] Phenomenological Lagrangians satisfying SU(2) ⊗ SU(2) symmetry are not considered in adequate detail in our review; this lack can be supplied by a review by Gasiorowicz and Geffen.[28] We also may mention a detailed article by Gell-Mann and Lévy.[7] Finally, it should be kept in mind that not all authors base their treatments on the concept of spontaneous symmetry breaking. Subtractionless dispersion relations are often used to derive the same results. That approach is essentially equivalent to the one adopted in the present review, but in our view it is of a more formal nature, and therefore is not used here. It can be found in other reviews, for example in the book [24], or in the original papers.[29,7]

In this paper we use the following notations:

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, \quad \Box = -\partial_\mu\partial_\mu = \partial_0^2 - \partial^2,$$
$$\hat{a} = a_0\gamma_0 - \mathbf{a}\boldsymbol{\gamma}, \quad \boldsymbol{\tau} = \frac{\boldsymbol{\sigma}}{2}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma_5 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. PARTIAL CONSERVATION OF AXIAL CURRENT

In this chapter we consider the consequences of the conservation of axial current in the limit of zero pion mass for various weak processes. In Sec. 2.1 we shall derive the Goldberger-Treiman relation, which connects the axial β-decay constant with the lifetime of the charged π meson; in Sec. 2.3 we obtain some relations for the amplitudes of neutrino-induced reactions; and in Sec. 2.4, a relation for the effective pseudoscalar constant in the process of μ capture by protons.

Using the example of the matrix element of the current a_μ between nucleon states we shall show that when the pion mass is taken into account the divergence of the axial current is different from zero and satisfies the relation[29,7]

$$\partial_\mu a_\mu^i = (\mu^2/c)\,\varphi^i \qquad (i = 1, 2, 3), \tag{2.1}$$

where c is a constant. This equation is commonly spoken of as the hypothesis of partial conservation of axial current. The general question of the breaking of SU(2) ⊗ SU(2) symmetry is discussed in Sec. 2.2.

2.1. The Goldberger-Treiman Relation

2.1.1. Let us consider the matrix element between nucleon states of the weak current without change of strangeness, $i_\mu = v_\mu + a_\mu$, which occasions some weak processes:[30] β decay of the neutron, $n \to pe^-\bar{\nu}$, μ capture by protons, $\mu^- + p \to n + \nu$, and the interaction between neutrino and proton, $\nu + p \to n + \mu$. The amplitudes for these processes can be written in the form

$$M = (G/\sqrt{2})\, l_\mu \langle p | v_\mu^+ + a_\mu^+ | n \rangle, \quad l_\mu = \bar{u}_l \gamma_\mu (1 + \gamma_5) u_\nu, \tag{2.2}$$

where G is the weak interaction constant, $G = 1.01 \times 10^{-5} m_p^{-2}$, and u_ν and u_l are lepton wave functions.

We shall be interested in the matrix element of the axial current $a^+ = a_\mu^1 + ia_\mu^2$, which is described by two independent form-factors $g(k^2)$ and $h(k^2)$:

$$M_\mu = \langle p | a_\mu^+ | n \rangle = g(k^2)\, \bar{u}_2 \gamma_\mu \gamma_5 u_1 - h(k^2)\, \bar{u}_2 \hat{k} \gamma_5 u_1 k_\mu,$$

where u_1, u_2 are the wave functions of the initial and final nucleons, which have momenta p_1, p_2 ($k = p_1 - p_2$). There is no term $u_2 \gamma_5 \sigma_{\mu\nu} k_\nu u_1$ in (2.2); this term has positive G parity, opposite to the G parity of the terms retained. We assume that for the operator a_μ the quantum number G is equal to -1.

Hypotheses I and II, as formulated in Chapter 1, lead to a relation between the form-factors $g(k^2)$ and $h(k^2)$. According to these hypotheses the axial current is conserved in the limit of zero pion mass ($\mu^2 = 0$). In this limit the longitudinal part of the matrix element M_μ must be zero, and this gives the very important equation[3]

$$g(k^2) = k^2 h(k^2). \tag{2.3}$$

By using this equation we can write the matrix element M_μ in the form

$$M_\mu = g(k^2)\, [g_{\mu\nu} - (k_\mu k_\nu / k^2)]\, \bar{u}_2 \gamma_\nu \gamma_5 u_1. \tag{2.4}$$

The pole at $k^2 = 0$ in this expression is due to the fact that conservation of the axial current is possible only if a massless pseudoscalar particle exists.[12,13] This is not surprising, since, as discussed in Chapter 1, the SU(2) ⊗ SU(2) symmetry can be exact only in the limit $\mu^2 = 0$.

It is clear that a nonvanishing pion mass causes the pole to be shifted to the point $k^2 = \mu^2$. If we regard the pion mass as small, we can neglect the change of the residue at the pole and write M_μ in the form[3]

$$M_\mu \approx g(k^2)\, \{g_{\mu\nu} - [k_\mu k_\nu / (k^2 - \mu^2)]\}\, \bar{u}_2 \gamma_\nu \gamma_5 u_1. \tag{2.5}$$

The pole contribution to the amplitude is calculated directly, starting from the diagram of Fig. 1. This diagram gives for $g(k^2 = \mu^2)$

$$g(\mu^2) = f_\pi g_r / m \sqrt{2}, \tag{2.6}$$

where g_r is the constant of the πNN interaction, $g_r^2/4\pi = 14.6$, m is the mass of the nucleon, and f_π is the constant of the $\pi \to \mu\nu$ decay, defined as

$$\langle 0 | a_\mu^+(0) | \pi^- \rangle = i f_\pi k_\mu. \tag{2.7}$$

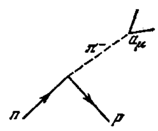

FIG. 1

Neglecting the difference between $g(\mu^2)$ and g(0), we get for the axial constant g_A the result

$$g_A \equiv g(k^2 = 0) \approx f_\pi g_r / m \sqrt{2}. \tag{2.8}$$

This equation was first derived in [31] and is called the Goldberger-Treiman relation. In the derivation of this relation we have used the conservation of axial current for $\mu^2 = 0$ and the assumption that μ^2 is small.

2.1.2. Let us now compare the Goldberger-Treiman relation with the experimental data. The experimental values of g_A and g_r are $g_A = 1.18$ and $g_r^2/4\pi = 14.6$, and the constant f_π is connected with the probability of the decay $\pi \to \mu\nu$ in the following way:

$$w(\pi \to \mu\nu) = (G^2 f_\pi^2 \mu / 8\pi)\, m_\mu^2\, [1 - (m_\mu^2/\mu^2)^2], \tag{2.9}$$

where G is the weak interaction constant, $G = 1.01 \times 10^{-5} m_p^{-2}$. The experimental value of $w(\pi \to \mu\nu)$ is 3.85×10^7 sec^{-1},[2)] and this gives

$$f_\pi = 0.93\mu. \tag{2.10}$$

The resulting values of the right and left sides of (2.8) are 1.35 and 1.28, respectively. Accordingly, the error in the Goldberger-Treiman relation is about 10 percent. In deriving it we have neglected terms of order μ^2/m_{char}^2 in comparison with unity. From the comparison with the experimental data it follows that in this case m_{char} is relatively small: $m_{\text{char}} \sim 3\mu$.

2.2. Breaking of the SU(2) ⊗ SU(2) Symmetry

2.2.1. It is easy to see that inclusion of the mass of the π meson in the pole contribution to the matrix element for β decay has the result that the matrix element of the current a_μ is no longer transverse, as follows from Eq. (2.5):

$$k_\mu M_\mu = -i \langle p | \partial_\mu a_\mu^+(0) | n \rangle = -(\mu^2 g(k^2)/(k^2 - \mu^2)) \cdot 2m \bar{u}_2 \gamma_5 u_1. \tag{2.11}$$

We shall show that this nonconservation of the axial current can be written in the form of the operator equation

$$\partial_\mu a_\mu^+ = (\mu^2/c)\, \varphi^+, \tag{2.12}$$

where φ^+ is the renormalized operator of the charged π-meson field, $\langle \pi^+ | \varphi^+ | 0 \rangle = 1$, and the constant c has the value $c = 1/f_\pi \approx g_r / 2^{1/2} m g_A$. In fact, the matrix element of φ^+ between nucleon states is

$$\langle p | \varphi^+ | n \rangle = -(k^2 - \mu^2)^{-1} \langle p | j_\pi^+ | n \rangle = -[i \sqrt{2} g_r(k^2)/(k^2 - \mu^2)]\, \bar{u}_2 \gamma_5 u_1, \tag{2.13}$$

where $f_\pi^+ = -(\Box - \mu^2)\varphi^+$, and $g_r(k^2)$ is equal to g_r for $k^2 = \mu^2$. If, as in the derivation of the Goldberger-Treiman relation, we neglect the dependence of the functions $g_r(k^2)$ and $g(k^2)$ on their argument in the region $k^2 \sim \mu^2$, Eqs. (2.11) and (2.12) are equivalent.

The reason for the operator equation (2.12) can also

[2)] If no special reference is given, experimental data are taken from the tables of [32].

be demonstrated intuitively in a different way. Let us separate out from the axial current a_μ^+ the term responsible for the decay of the π meson:

$$a_\mu^+ = \tilde{a}_\mu^+ - \partial_\mu \varphi^+ c^{-1}, \tag{2.14}$$

where $\langle 0 | \tilde{a}^+ | \pi^- \rangle = 0$. For zero pion mass the axial current is conserved, which gives

$$\Box \varphi = -c \partial_\mu \tilde{a}_\mu. \tag{2.15}$$

If we assume that as the π meson's mass is changed its source remains the same as for $\mu^2 = 0$, then the equation for the operator φ takes the form

$$(\Box - \mu^2)\varphi = -c\partial_\mu \tilde{a}_\mu \tag{2.16}$$

and when we use the definition (2.14) we arrive at the relation (2.12).

It is very important to emphasize that Eq. (2.12) does not contain any additional information beyond the hypothesis of the conservation of the axial current in the limit $\mu^2 = 0$. It expresses only the trivial fact that the pole corresponding to the π-meson intermediate state is at $k^2 = \mu^2$ and not at $k^2 = 0$.

In fact, if we look at an arbitrary matrix element of the divergence $\partial_\mu a_\mu$ of the axial current, then for $k^2 \to \mu^2$ the only contribution to it is that of the π-meson pole, and Eq. (2.12) holds independently of any assumptions.[33] For $k^2 \neq \mu^2$ the matrix elements of the pion field operator φ have no physical meaning and Eq. (2.12) can be regarded as the definition of φ.

Therefore in itself the assertion that the divergence of the axial current is proportional to the field does not lead to any consequences at all. The intrinsic meaning of Eq. (2.12) is that the axial current is conserved for $\mu^2 = 0$. Along with the assumption that μ^2 is small this allows us to make predictions for various quantities. In what follows we shall use Eq. (2.12), not Eq. (1.13), in order immediately to take into account the nonzero pion mass in the pole denominators.

2.2.2. Equation (2.12) is a convenient starting point for the discussion of the general question of the breaking of $SU(2) \otimes SU(2)$ symmetry, since the amount by which the divergence of the axial current differs from zero can serve as a measure of the symmetry breaking.

By means of (2.12) the matrix element of the operator $\partial_\mu a_\mu^+$ between the hadron states A and B can be represented in the form

$$\langle B | \partial_\mu a_\mu^+(0) | A \rangle = -[i\mu^2/c(k^2-\mu^2)]\, T(\pi A \to B) + O(\mu^2), \tag{2.17}$$

where $k = p_B - p_A$, $T(\pi A \to B)$ is the amplitude for the strong process $\pi + A \to B$, and $0(\mu^2)$ is a term of order μ^2 which does not contain the pion pole. The first term in the right member of (2.17) is separated off because in it the parameter μ^2 is "made dimensionless" by the quantity $(k^2 - \mu^2)$, which can also be small if $k^2 \sim \mu^2$. Therefore in the region $k^2 \lesssim \mu^2$ the symmetry is strongly broken. This breaking can be taken into account exactly, however,

As for the terms $O(\mu^2)$ in (2.17), nothing is known about them in the general case. The magnitude of these terms goes to zero for $\mu^2 \to 0$, but can be rather considerable for the actual value of μ^2. In particular, the terms $O(\mu^2)$ led to a violation of the Goldberger-Treiman relation by about 10 percent.

If we do not consider detailed dynamical models, the only way of getting some specific idea of the contribution $O(\mu^2)$ is to prescribe the transformation properties of the operator φ with respect to the group $SU(2) \otimes SU(2)$. Starting from model arguments[7] and considerations of simplicity, we shall assume that

$$[A^i, \partial_\mu a_\mu^k] \sim \delta^{ik}. \tag{2.18}$$

We emphasize that a nonvanishing $\partial_\mu a_\mu$ means that the strong interactions contain an admixture of a "semistrong" interaction which breaks the symmetry and in particular is responsible for the appearance of the pion mass. If we use the relations

$$\frac{dA^i(t)}{dt} = \int d^3x \partial_\mu a_\mu^i(t, \mathbf{x}), \quad \frac{dA^i}{dt} \sim [A^i, \mathscr{H}], \tag{2.19}$$

the assumption (2.18) can be formulated as a hypothesis about the properties of the Hamiltonian $\mathscr{H}^{br}$ of the "semistrong" interaction,

$$[A^i, [A^k, \mathscr{H}^{br}]] \sim \delta^{ik}. \tag{2.20}$$

2.2.3. Accordingly, the corrections owing to the breaking of the $SU(2) \otimes SU(2)$ symmetry can be roughly divided into two categories.

First we must take into account the explicit dependence on μ in the phase volumes and the contributions of the nearest singularities, located at distances of the order of μ. These corrections can make a contribution of the order of unity.

Second, by using the group properties of the Hamiltonian $\mathscr{H}^{br}$, one can sometimes manage to find the relatively small corrections to quantities which depend weakly on μ^2.

The special care which must be taken in dealing with the nearest singularities is of course not a specific feature of the $SU(2) \otimes SU(2)$ symmetry. For example, the isotopic relations for the small-angle scattering amplitudes are strongly violated owing to exchange of a photon. The difference is that the $SU(2) \otimes SU(2)$ symmetry is much more strongly broken than the isotopic symmetry, and singularities located at distances of several hundred million electron-volts can turn out to be "nearest" ones.

2.3. The Adler Relation for Neutrino Reactions

2.3.1. A beautiful possibility for testing the relation (2.17) in inelastic neutrino reactions $\nu + A \to l + B$ was pointed out by Adler.[34] He showed that if the lepton that is produced moves in the same direction as the neutrino the amplitude for the reaction is proportional to the matrix element of the divergence of the axial current, and thus, according to Eq, (2.17), to the amplitude for the strong process $\pi A \to B$.

The proof of this statement is based on the fact that when the lepton mass is neglected the lepton factor

$$l_\mu = \bar{u}_l \gamma_\mu (1 + \gamma_5) u_\nu \tag{2.21}$$

is proportional to the momentum transfer $k = (p^\nu - p^l)$. Therefore the only part of the hadron factor that appears in the answer is the matrix element of the divergence of the weak current, $\partial_\mu(v_\mu + a_\mu)$. The vector current is conserved, and we have left only the divergence of the axial current. If the neutrino and lepton move in the same direction, then $k^2 \approx 0$ and we can use only the first term in the expression (2.17) for the divergence of the axial current. The final expression for the cross section of the neutrino reaction is as follows:

$$\frac{d^2\sigma}{d\Omega_l dW}=\left(\frac{Wk_{20}}{k_0 m_A}\right)^2(k_0^2-\mu^2)^{1/2}\frac{g_r^2 g_A^2}{16m^2\pi^3}[1-{}^1/_2 m_l^2(\mu^2 k_{20}^2+m_l k_0)^{-1}k_0]^2\sigma(\pi A\to B), \quad (2.22)$$

where $k_0 = (W^2 - m_A^2 + \mu^2)/2W$ and $k_{20} = (m_A^2 + 2m_A E_\nu - W^2)/2W$; m_l, m_A, and W are the lepton mass and the invariant masses of the states A and B.

2.3.2. Adler's formula (2.22) is far from exhausting all the consequences of Hypotheses I–IV for the neutrino experiment. For example, in the reaction $\nu + N \to 1 + N'$, where N and N′ are nucleons, one can check Eq. (2.3), $k^2h(k^2) = g(k^2)$, which follows from the conservation of axial current for $k^2 \gg \mu^2$. In the region $k^2 \sim \mu^2$ this relation must be replaced by $g(k^2) = (k^2 - \mu^2)h(k^2)$.

Moreover, in inelastic neutrino reactions at small values of the momentum transferred to leptons one can use a generalization of the Goldberger-Treiman relation, Eq. (2.23), which enables us to express the matrix element $\langle B|a_\mu|A\rangle$ of the axial current in terms of the amplitude for the strong process $\pi + A \to B$:[35]

$$\langle B|a_\mu|A\rangle=(i/c)\{g_{\mu\nu}-[k_\mu k_\nu/(k^2-\mu^2)]\}T_\nu, \quad (2.23)$$

where $k_\nu T_\nu$ is the amplitude for the reaction $\pi + A \to B$.

Additional relations arise in cases in which the states A, B contain "soft" π mesons,[8] but we shall not discuss them in detail. The method for derivation of such relations is expounded in the following chapters.

2.4. The Effective Pseudoscalar Constant in μ Capture

Besides the β decay of the neutron and the scattering of neutrinos by nucleons, there is another physical process described by the matrix element (2.2) of the axial current; this is μ capture by a proton, $\mu + p \to n + \nu$. Unlike the β decay of the neutrino, this process enables us to measure the form-factor $h(k^2)$. The experimental determination of this form-factor for $k^2 \lesssim \mu^2$ is of great interest for comparison with theoretical predictions.

To derive these predictions we write the function $h(k^2)$ in the form

$$h(k^2)=(f_\pi g_r/m\sqrt{2})(k^2-\mu^2)^{-1}+r(k^2), \quad (2.24)$$

where we have separated out the contribution of the π-meson pole diagram (see Fig. 1) and denoted the remainder by $r(k^2)$.

For $k^2 \sim \mu^2$ the pole term is of order μ^{-2} and must give the main contribution owing to our assumption that μ^2 is effectively small. This assertion has already been used in the derivation of the Goldberger-Treiman relation. Keeping only the pole contribution, we get for the effective pseudoscalar constant for the capture of a slow μ meson by a proton[36] (the value of k^2 in this process is $k^2 = -k_0^2 = -m_\mu^2[1 + (m_\mu/m)]^{-1}$)

$$g_P\equiv 2mm_\mu h(k^2=-m_\mu^2/[1+(m_\mu/m)])\approx -f_\pi g_r\sqrt{2}m_\mu/(\mu^2-k_0^2)\approx 8.85. \quad (2.25)$$

We emphasize that the prediction (2.25) is based only on the hypothesis of the smallness of the pion mass. The expected accuracy of this prediction, as in the case of the Goldberger-Treiman relation, is of the order of 10 percent.

Using the conservation of axial current (for $\mu^2 = 0$), we can get a more precise prediction for the constant g_P.[37,38] The entire uncertainty in the calculation of g_P is due to the contribution $r(k^2)$, since the pole term is known exactly. Since the quantity $r(k^2)$ is relatively small, it suffices to calculate it with the "usual" 10 percent accuracy in order to predict the value of g_P to one-percent accuracy.

In the approximation adopted

$$r(-k_0^2)\approx r(k^2=0,\ \mu^2=0), \quad (2.26)$$

and the quantity r ($k^2 = 0$, $\mu^2 = 0$, can be expressed in terms of the radius of the axial form-factor $dg(k^2)/dk^2$ at $k^2 = 0$. For this purpose we must expand $g(k^2)$ and $h(k^2)$ in (2.3) in powers of k^2. The result found for g_P is

$$g_P=-8.85+2mm_\mu[dg(k^2)/dk^2]|_{k^2=0}. \quad (2.27)$$

An analysis of the data on the neutrino experiment leads to the estimates[39]

$$dg(k^2)/dk^2\,(\text{at } k^2=0)=(0.4-1.1\ \text{GeV}^{-2}. \quad (2.28)$$

The corresponding correction to the pole value of g_P ranges from 1.5 to 14 percent. We note that the sign of the correction is evidently already definitely fixed.

Testing the relation (2.27) is very difficult. At the present time the constant g_P has been measured to about 40 percent accuracy.[40] It must be kept in mind, however, that no other predictions based on Hypotheses I–IV that could pretend to one-percent accuracy have as yet been found.

3. SOME STRONG PROCESSES INVOLVING π MESONS

3.1. The Adler Selfconsistency Condition

A direct consequence of the partial conservation of axial current is that the amplitude for a strong process is zero for zero pion momentum if there are no contributions to the process from pole diagrams corresponding to the emission of a π meson from an external line. We shall give a proof of this assertion, which is called the Adler selfconsistency condition,[41] and shall examine[42] for the example of the decay $X(960) \to \eta 2\pi$ to what sort of consequences this condition leads.

3.1.1. In what follows we shall make frequent use of a reduction formula.[17] According to this formula, in particular, the amplitude for a process $A \to B + \pi$, where A and B are arbitrary states of hadrons, can be represented in the form

$$M(2\pi)^4\delta^4(p_2+q-p_1)=-\int dxe^{iqx}(\Box-\mu^2)\langle B|\varphi(x)|A\rangle. \quad (3.1)$$

Here p_1, p_2 are the momenta of the states A and B, and q is the momentum of the π meson. The reduction formula (3.1) defines the amplitude off the mass shell as the corresponding Green's function multiplied by the reciprocal $(\Box - \mu^2)$ of the free-particle propagator. It is clear that the quantity defined in this way coincides for $q^2 = \mu^2$ with the amplitude for the process $A \to B\pi$, since in this limit we have left in the right member of (3.1) only the part of the Green's function which is proportional to $(q^2 - \mu^2)^{-1}$.

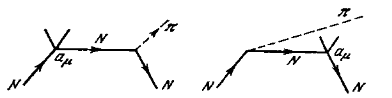

FIG. 2

Replacing the π-meson field in (3.1) by its expression (2.12) in terms of the divergence of the axial current, and integrating by parts, we get

$$M(2\pi)^4\delta^4(p_2+q-p_1) = -\frac{c}{\mu^2}\int dx e^{iqx}(\Box-\mu^2)\langle B|\partial_\mu a_\mu(x)|A\rangle = \\ = \frac{ic}{\mu^2}(q^2-\mu^2)q_\mu\int dx e^{iqx}\langle B|a_\mu(x)|A\rangle. \quad (3.2)$$

In the right member of Eq. (3.2) we have dropped the so-called surface terms, i.e., expressions of the type of

$$\int dx e^{iqx}\frac{\partial}{\partial x_0}\varphi(x)\Big|_{x_0=\pm\infty}.$$

For $x_0 \to \pm\infty$ the operator $\varphi(x)$ is the same as the free-field operator with mass μ. Therefore such an expression can be different from zero only for $q^2 = \mu^2$. In the general case the surface terms contribute to the amplitude only for definite values of q^2 that correspond to the masses of particles which have the quantum numbers of the operators in question.

Since the amplitude is a continuous function of q^2 we can take as the amplitude on the mass shell the limit of the expression (3.2) for $q^2 = \mu^2$. Then the question of surface terms does not arise, and we omit them everywhere in what follows.

We note that the explicit dependence of the right member of (3.2) on q^2 and μ^2 is essentially fictitious. To verify this, we must separate off the contribution of the diagram that contains the π meson pole (Fig. 2). Formally this can be done simply, by using the form (2.15) for expressing the partial conservation of axial current. Substituting (2.15) in (3.1), we get

$$M(2\pi)^4\delta^4(p_2+q-p_1) = -icq_\mu\int dx e^{iqx}\langle B|\tilde{a}_\mu|A\rangle, \quad (3.3)$$

where the matrix element $\langle B|\tilde{a}_\mu|A\rangle$, unlike $\langle B|\tilde{a}_\mu|A\rangle$, does not contain the π-meson pole.

Equation (3.3) allows us to find the amplitude M for zero momentum of the π meson. For $q \to 0$ in the right member of (3.3) the only contributions are from pole parts of the matrix element $\langle B|\tilde{a}_\mu|A\rangle$ which correspond to diagrams in which the axial current is connected with an external line. We shall analyze the case when there are such diagrams in Sec. 3.2, using the example of πN scattering. If there are no such diagrams, the amplitude must go to zero for $q = 0$:

$$M(A \to B\pi) \underset{q\to 0}{\longrightarrow} 0. \quad (3.4)$$

Equation (3.4) expresses the content of Adler's selfconsistency condition.

3.1.2. As an example let us see to what experimental consequences this condition leads in the case of the decay $X \to \eta 2\pi$. It is assumed that X(960) is a pseudoscalar meson with isospin zero.

If we assume that the matrix element for this decay is linear in the energies, it can be represented in the form

$$M(X \to \eta 2\pi) = f + g(q_1+q_2)^2. \quad (3.5)$$

where we have taken into account the Bose statistics for the π mesons, which means that M is symmetric in the pion momenta q_1 and q_2.

The vanishing of the amplitude when the momentum of one of the π mesons goes to zero leads to the relation

$$f + g\mu^2 = 0, \quad (3.6)$$

and the matrix element in the physical region of the decay can be written in the form

$$M(X \to \eta 2\pi) = \text{const}(1+\alpha Y). \quad (3.7)$$

where $Y = (2T/T_{max}) - 1$, T is the kinetic energy of the η meson, and

$$\alpha = -[(m_X-m_\eta)^2-4\mu^2]/[(m_X-m_\eta)^2+2\mu^2] = -0.43. \quad (3.8)$$

The sign of α is apparently confirmed by the existing experimental data,[43] but the statistics are insufficient for the determination of the absolute value of α.

The theoretical accuracy of the prediction (3.8) for the quantity α is evidently not high. In fact, we have assumed that it is legitimate to treat the amplitude as linear in the energies of the π mesons; i.e., we have assumed that the characteristic mass of the strong interactions is much larger than E_π. But this can scarcely be satisfied with good accuracy, since in the case in question E_π varies over a rather wide range—from zero to 340 MeV (when the momentum of "the other" π meson is zero). In this sense the most favorable case is that of threshold πN scattering, for which the physical region of variation of the π meson's energy is closest to the point $E_\pi = 0$.

3.2. πN Scattering at Low Energies

Hypotheses I–IV (see Chapter 1) allow us to derive a number of consequences for the amplitude for πN scattering at low energies, which are discussed in the present section. The plan of the exposition is as follows: in Subsection 3.2.1 we present the necessary information from the phenomenological description of πN scattering; in 3.2.2 we discuss the consequences of the Adler selfconsistency condition; in 3.3.3 the isotopically odd part of the amplitude is calculated; the results obtained are compared with experiment in 3.2.4 and 3.2.5; and the extrapolation formulas and their experimental testing are discussed in 3.2.6.

3.2.1. Phenomenology of πN scattering (cf., e.g., [21,34]). The isotopic structure of πN scattering is described by two independent amplitudes T^+ and T^-:

$$T = \Psi_2^+\Psi_1(\varphi_1\varphi_2)T^+ + 2\Psi_2^+[(\varphi_2\tau),(\varphi_1\tau)]\Psi_1 T^-, \quad (3.9)$$

where $\Psi_{1,2}$ and $\varphi_{1,2}$ are isotopic functions for nucleons and π mesons, the index 1 referring to the initial and 2 to the final state; $\tau = \sigma/2$ are the isospin matrices. The quantities $T^\pm$ are connected with the amplitudes for the processes $\pi^\pm p \to \pi^\pm p$ and the scattering amplitudes $T_{3/2}$ and $T_{1/2}$ in states with total isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$ by the relations

$$\left.\begin{aligned} T^+ &= (T_{\pi^-p}+T_{\pi^+p})/2 = (T_{1/2}+2T_{3/2})/3, \\ T^- &= (T_{\pi^-p}-T_{\pi^+p})/2 = (T_{1/2}-T_{3/2})/3. \end{aligned}\right\} \quad (3.10)$$

We write the spatial structure of $T^\pm$ in the form

$$T^\pm = \bar{u}_2[C^\pm + {}^1/_{2m}\sigma_{\mu\nu}k_\mu q_\nu B^\pm]u_1, \quad (3.11)$$

where u_1, u_2 are the wave functions of the initial and final nucleons with momenta p_1, p_2; k and q are the momenta of the initial and final mesons. The amplitudes $C^{\pm}$ and $B^{\pm}$ are functions of the variables ν and t:

$$\nu = (k+q)(p_1+p_2)/4m = k(p_1+p_2)/2m = q(p_1+p_2)/2m, \quad t=(k-q)^2. \tag{3.12}$$

The crossing symmetry conditions can be written in the following way:

$$C^{\pm}(\nu, t) = \pm C^{\pm}(-\nu, t), \quad B^{\pm}(\nu, t) = \mp B^{\pm}(-\nu, t). \tag{3.13}$$

In what follows we shall consider the threshold values of the amplitudes $C^{\pm}$ and $B^{\pm}$ and of the first derivatives of $C^{\pm}$ with respect to ν and t. These quantities (or, more exactly, their real parts) are expressed in terms of the phases of the s and p waves by the equations

$$\left.\begin{aligned} &C = 4\pi[1+(\mu/m)]a, \quad B = 8\pi m[a_1 - a_3 + (1/4m^2)a], \\ &\frac{\partial C}{\partial \nu} = \frac{8\pi m}{m+\mu}\left(2a_3 + a_1 + b - \frac{2}{3}a^3 + \frac{a}{2m\mu}\right), \\ &\frac{\partial C}{\partial t} = \frac{2\pi}{m+\mu}\left[2\left(m+3\mu+\frac{3}{2}\frac{\mu^2}{m}\right)a_3 + ma_1 - \frac{m-\mu}{4m^2}a + \frac{2}{3}\mu a^3\right], \end{aligned}\right\} \tag{3.14}$$

where a is the scattering length and b is the radius of the s waves, and a_1, a_3 are the scattering lengths of the p waves in the states with total angular momentum $\frac{1}{2}$ and $\frac{3}{2}$ (the isotopic indices are omitted). The scattering lengths and radii are connected with the scattering phase shifts of the s and p waves by the relations

$$\left.\begin{aligned} &\delta^s = a|\mathbf{q}| + b|\mathbf{q}|^3 + O(|\mathbf{q}|^5), \\ &\delta^p_{2J} = a_{2J}|\mathbf{q}|^3 + O(|\mathbf{q}|^5), \end{aligned}\right\} \tag{3.15}$$

where q is the three-dimensional momentum in the c.m.s.

We need to separate the amplitude for πN scattering into pole and nonpole parts T_p and $\tilde{T}$:

$$T = T_p + \tilde{T}, \tag{3.16}$$

where by the pole part we mean the contribution of nucleon pole diagrams, with the vertex for the πNN interaction taken in the form $-f\bar{\Psi}\gamma_\mu\gamma_5\tau\Psi\partial_\mu\varphi$, $f = g_r/2m = (1.01 \pm 0.01)\mu^{-1}$. If we take the vertex in a different form (for example, $ig_r\bar{\Psi}\gamma_5\tau\Psi\varphi$), the resulting pole diagrams will differ from those with the pseudovector coupling by terms which contain no pole, and the question reduces to a redefinition of $\tilde{T}$.

We give the expressions for the pole contributions to the amplitudes $C^{\pm}$ and $B^{\pm}$, and also the values of these contributions at the threshold of the scattering:

$$C_p^+ = 2mf^2(t-k^2-q^2)\mathcal{L}^-, \quad C_p^+|_{\nu=\mu} = -f^2\frac{\mu^2}{m} = -0.15\mu^{-1}, \tag{3.17}$$

$$C_p^- = -2f^2\nu - mf^2(t-k^2-q^2)\mathcal{L}^+, \quad C_p^-|_{\nu=\mu} = 2\mu f^2(\mu/2m)^2 = 0.011\mu^{-1}, \tag{3.18}$$

$$B_p^+ = -4m^2f\mathcal{L}^+, \quad B_p^+|_{\nu=\mu} = -4f^2\frac{m}{\mu} = -27.4\mu^{-2}, \tag{3.19}$$

$$B_p^- = -2f^2 + 4m^2f^2\mathcal{L}^-, \quad B_p^-|_{\nu=\mu} = 4f^2(\mu/2m) = 0.022\mu^{-2}, \tag{3.20}$$

where

$$\mathcal{L}^{\pm} = \left[2m\nu + \frac{1}{2}(t-k^2-q^2)\right]^{-1} \pm \left[2m\nu - \frac{1}{2}(t-k^2-q^2)\right]^{-1}$$

and in the calculation of the threshold values $(\mu/2m)^2$ is neglected in comparison with unity.

3.2.2. After these preliminary remarks let us now examine to what consequences the hypothesis of conservation of axial current leads. For this purpose we let the momentum of one of the mesons go to zero (with the other particles on the mass shell), and to calculate the limit of the amplitude we use the relation (3.3):

$$T \xrightarrow[q\to 0]{} -icq_\mu\langle p_2|a_\mu(0)|p_1, k\rangle. \tag{3.21}$$

For $q \to 0$ we need consider in the right member only the pole part of the matrix element of the axial current, corresponding to the diagrams of Fig. 2, which is singular for $q \to 0$. Equation (3.21) expresses the Adler selfconsistency condition for the πN scattering amplitude. For small q the axial vertex which occurs in the diagrams of Fig. 2 is of the form $g_A\bar{u}\gamma_\mu\gamma_5 u$ [see Eq. (2.5)]. After being multiplied by q_μ this vertex is identical with the pseudovector vertex of the πNN interaction, since $c = 2^{1/2}f/g_A$. Therefore the right member of Eq. (3.21) coincides, for $q \to 0$, with the pole part T_p of the amplitude, as defined by Eq. (3.16), and for the nonpole part $\tilde{T}$ we get

$$\tilde{T}|_{q=0} = 0. \tag{3.22}$$

We shall discuss the experimental consequences of this relation in Subsection 3.2.5.

3.2.3. Equation (3.22) states that the constant term in the expansion of the nonpole part of the amplitude for πN scattering in a power series in the momenta of the π mesons is equal to zero. We shall now find the terms of the expansion which are linear in k or q.

To calculate the linear terms we must consider the limit of the amplitude as the momenta of the two π mesons go to zero. To do so we write a reduction formula with respect to the two π meson fields, replace them with the divergences of axial currents, and integrate by parts:

$$\begin{aligned} &T_{\pi+p}(2\pi)^4\delta^4(p_1+k-p_2-q) = \\ &= i\int dx\,dy\exp(-ikx+iqy)(\Box_x-\mu^2)(\Box_y-\mu^2)\langle p_2|T\{\varphi^-(y)\varphi^+(x)\}|p_1\rangle \xrightarrow[k,\,q\to 0]{} \\ &\to ic^2\int dx\,dy\exp(-ikx+iqy)\langle p_2|[\partial_\nu a_\nu^-(y), a_0^+(x)]\delta(x^0-y^0) \\ &\quad + ik_\mu[a_\mu^+(x), a_0^-(y)]\delta(x^0-y^0) + k_\mu q_\nu T\{a_\mu^+(x), a_\nu^-(y)\}|p_1\rangle. \end{aligned} \tag{3.23}$$

In the integration by parts in (3.23) we have taken into account the fact that the operations of time ordering and differentiation with respect to time do not commute; it follows from the definition of the T product that

$$T\{\varphi(y)\partial_\mu a_\mu(x)\} = \theta(x^0-y^0)\partial_\mu a_\mu(x)\varphi(y) + \theta(y^0-x^0)\varphi(y)\partial_\mu a_\mu(x), \tag{3.24}$$

where

$$\theta(x^0) = \begin{cases} 1, & x^0 > 0, \\ 0, & x^0 < 0. \end{cases}$$

Taking the derivative outside the sign of the T product and using the fact that $(\partial/\partial x^0)\theta(x^0) = \delta(x^0)$, we get

$$T\{\partial_\mu a_\mu(x)\varphi(y)\} = \partial_\mu T\{a_\mu(x)\varphi(y)\} + \delta(x^0-y^0)[\varphi(y), a_0(x)]. \tag{3.25}$$

We have used this equation in the derivation of (3.24).

It is clear that in letting k and q go to zero we must give separate attention to the singular contributions of one-particle intermediate states. There are such terms in the last term in the right member of (3.23). It is easily verified that for the case of pseudovector πNN coupling they are identical with the pole part of the amplitude for πN scattering. The proof of this is completely analogous to the proof given in 3.2.2.

As for the contributions of many-particle states to $k_\mu q_\nu\langle p_2|T\{a_\mu^+(x)a_\nu^-(y)\}|p_1\rangle$, they are obviously quantities of second and higher orders in k and q. Therefore the equal-time commutators of the zeroth components of the axial current with each other and with the divergence of the current, which arise in the right member of (3.23), give the nonpole part of the amplitude to and including terms linear in k and q.

If we set $\mathbf{k} = \mathbf{q} = 0$ (with also $k_0 = q_0$), the exponentials in the relation (3.23) are equal to unity, and the nonpole part of the amplitude can be expressed in terms of commutators containing the axial charge:

$$(2\pi)^4 \delta^4 (p_1 + k - p_2 - q)\, \widetilde{T}_{\pi^+p} \xrightarrow[k,\, q\to 0]{} ic^2 \int dt \{\langle p_2 | [\dot{A}^-(t), A^+(t)] + ik_0 [A^+(t), A^-(t)] | p_1\rangle\}. \quad (3.26)$$

We shall show that the first term in Eq. (3.26) is associated with the isotopic amplitude $\widetilde{T}^+$. To do this we differentiate the commutator of the axial charges

$$[A^+(t), A^-(t)] = 2V^3 \quad (3.27)$$

with respect to the time:

$$[\dot{A}^+(t), A^-(t)] + [A^+(t), \dot{A}^-(t)] = 0. \quad (3.28)$$

The expression for the amplitude for π^-p scattering can be obtained from (3.26) by interchanging the indices "+" and "−". It follows from (3.28) that the first term in (3.26) remains unchanged by this interchange and consequently belongs to the isotopically even part of the amplitude. This term, by the way, is relatively small, since it is proportional to μ^2 (for $\mu^2 = 0$ the axial current is conserved and $\dot{A} = 0$).

The second term in (3.26) is proportional to the commutator $[A^+, A^-]$, which obviously changes sign on interchange of the isotopic indices, and therefore gives the value of the isotopically odd part of the amplitude when the meson momenta go to zero:

$$\widetilde{T}^- (2\pi)^4 \delta^4 (0) \xrightarrow[k,\, q\to 0]{} c^2 k_0 \int \langle p_2 | 2V^3 | p_1\rangle dt = c^2 k_0 \int \langle p_2 | p_1 \rangle dt = c^2 \frac{k_0 p_0}{m} \bar{u}_2 u_1 (2\pi)^4 \delta^4 (0), \quad (3.29)$$

where we have used covariant normalization of the states, $\langle p_2 | p_1 \rangle = (p_{10}/m)\bar{u}_2 u_1 (2\pi)^3 \delta^3(p_2 - p_1)$. We thus get for the amplitude C^-:

$$C^- \xrightarrow[k,\, q\to 0]{} c^2\nu + C_p^-. \quad (3.30)$$

3.2.4. Two ways of comparing this relation with experiment have been proposed. One can assume[45,16] that the expression (3.30) is valid right down to the threshold of πN scattering and thus predicts the scattering length. We shall consider this in detail in the next section.

Another possibility[1,2] is to use the dispersion relation for the amplitude C^- at t = 0. As is well known, this relation is

$$\frac{C^-(\nu)}{\nu} = \frac{g_r^2 \mu^2}{2m^2} \frac{1}{\nu^2 - (\mu^2/2m)^2} + \frac{1}{\pi} \int_\mu^\infty d\nu' \frac{k' [\sigma_-(\nu') - \sigma_+(\nu')]}{\nu'^2 - \nu^2}, \quad (3.31)$$

where $\sigma_\pm$ are the total cross sections of the $\pi^\pm p$ interaction and k is the three-dimensional momentum of the pion in the laboratory reference system. Substituting the expression (3.30) for C^- in the left member of (3.31) for $\nu \to 0$ and using the explicit form (3.18) for the pole term C_p^-, we arrive at a relation first obtained by Adler[1] and by Weissberg:[2]

$$1 - \frac{1}{g_A^2} = \frac{2m^2}{\pi g_r^2} \int_\mu^\infty \frac{d\nu}{\nu^2} k [\sigma_+(\nu) - \sigma_-(\nu)], \quad (3.32)$$

where we have used $c = g_r / 2^{1/2} m g_A$. This relation agrees excellently with the experimental value $g_A = 1.18$.

We emphasize that Eq. (3.32) is approximate, since the dispersion relation holds for the physical amplitude, i.e., for $k^2 = q^2 = \mu^2$, and the expression (3.30) for C^- is derived for k, q → 0. Accordingly, the dispersion relation allows us to take into account the dependence of C^-/ν on ν^2, but not that on k^2 and q^2, which we have neglected.

3.2.5. Since near threshold $\nu^2 \sim k^2 \sim q^2 \sim \mu^2$, we can assume that in this region the quantity $\widetilde{C}/\nu$ depends not only on q^2 and k^2, but also on ν^2. Then we get for the isotopically odd scattering length[45,16]

$$a^- = \left\{4\pi\left[1 + \left(\frac{\mu}{m}\right)\right]\right\}^{-1} C^-|_{\nu=\mu} = c^2\mu/4\pi \left[1 + \left(\frac{\mu}{m}\right)\right] = 0.10\mu^{-1}, \quad (3.33)$$

where the numerical value of the constant c is taken equal to

$$c = g_r/\sqrt{2}\, m g_A = 1{,}2\mu^{-1}$$

and the small contribution of the pole diagram ($\sim 10^{-3}\mu^{-1}$) has been dropped. We note that here and in what follows we have taken the kinematic factors into account exactly, without neglecting terms of order μ/m.

The theoretical value (3.33) is to be compared with the experimental data on the scattering lengths,

$$a^- = (0.086 \pm 0.005)\, \mu^{-1}\ {}^{44}, \quad a^- = (0.093 \pm 0.005)\, \mu^{-1}\ {}^{46},$$

and we see that there is very good agreement between theory and experiment.

Let us now proceed to the consideration of the isotopically even amplitude. As was already pointed out in 3.2.4, the constant term in the expansion of $\widetilde{C}^+$ in powers of k and q is equal to zero (more exactly, is $\sim \mu^2$) owing to Eq. (3.22), and the expansion of $\widetilde{C}^+$ begins with terms quadratic in k and q. From the explicit expression (3.17) for the pole term in C^+ we see that it is also quadratic in μ near threshold. It follows that the isotopically even scattering length must be small in comparison with the isotopically odd length, which is linear in μ [cf. Eq. (3.33)]:

$$a^+/a^- \ll 1. \quad (3.34)$$

The experimental data confirm this prediction of the theory:

$$a^+ = -(0.002 \pm 0.006)\, \mu^{-1}\ {}^{44}, \quad a^+ = -(0.011 \pm 0.005)\, \mu^{-1}\ {}^{46}.$$

3.2.6. We have derived the predictions (3.33) and (3.34) essentially by using extrapolation formulas for the amplitude for πN scattering. Namely, we have assumed that the amplitude is the sum of a pole term, which is treated exactly, and nonpole terms, which we expanded in a series, keeping only the first term of the expansion. It is clear that an independent check[47] of the extrapolation formulas is of interest, and we devote the present subsection to this.

We consider the amplitude for πN scattering in the region $\nu \sim t^{1/2} \sim \mu$. If the amplitude had no singularities in this region the assumption that the pion mass can be treated as small would lead to a representation of the amplitude as a polynomial in ν and t. The singularities in the given region are due to nucleon pole diagrams, to scattering by way of the isobar N*(1236), and also to two-particle intermediate states (threshold singularities). In the calculation of the scattering lengths no account was taken of the contribution of the isobar (resonance in the p wave), since we were considering the amplitude for zero three-dimensional momenta of the particles. We shall now derive the sum rules for the amplitudes of the p waves, and in this case the contribution of the isobar must be dealt with separately. As for the contribution of the threshold singularities, it is proportional to the square of the amplitude near threshold and is small, since the πN scattering lengths are small (see Eqs. (3.33), (3.34).

Keeping these remarks in mind, and taking into account the requirements of crossing symmetry, we finally write the extrapolation formulas for the amplitudes $C^{\pm}$, $B^{\pm}$ in the following form:

$$C^+ = C_p^+ + C_{33}^+ + c_1^+ + c_2^+(kq) + c_3^+\nu^2 + O(\{k, q\}^3), \tag{3.35}$$

$$C^- = C_p^- + C_{33}^- + c^-\nu + O(\{k, q\}^3), \tag{3.36}$$

$$B^+ = B_p^+ + B_{33}^+ + O(\{k, q\}^1), \tag{3.37}$$

$$B^- = B_p^- + B_{33}^- + b^- + O(\{k, q\}^1), \tag{3.38}$$

where the indices p and 33 refer to the contributions of the nucleon and the isobar, and c_i^+, c^-, and b^- are the coefficients of the expansion; $O(\{k, q\}^n)$ denotes terms of order n in k, q. Besides the higher terms of the expansion, the quantities $O(\{k, q\}^n)$ include imaginary parts of the amplitudes and nonanalytic terms associated with the threshold singularities.

Just as in the case of the contribution of the nucleon pole, we must define more precisely what is meant by the contribution of the isobar in Eqs. (3.35)–(3.38). To describe the particle with spin $\frac{3}{2}$ we shall use the Rarita-Schwinger formalism, i.e., describe it with a quantity Ψ_μ (μ = 0, 1, 2, 3). In this formalism the propagator is of the form

$$(P^2 - M^2)^{-1}\{(\hat{P} + M)[-g_{\mu\nu} + (\gamma_\mu\gamma_\nu/3) + ((\gamma_\mu P_\nu - \gamma_\nu P_\mu)/3M) + (2P_\mu P_\nu/3M^2)] - 2(P^2 - M^2)/3M^2[\gamma_\mu P_\nu - \gamma_\nu P_\mu + (\hat{P} + M)\gamma_\mu\gamma_\nu]\}, \tag{3.39}$$

where M is the mass of the isobar and P its momentum.

Finally, we take the vertex of the πNN* interaction in the form

$$\lambda\bar{\Psi}_\mu\Psi\,\partial_\mu\varphi, \tag{3.40}$$

where the constant $\lambda_{N^{*++}\to p\pi^+}$ is equal to $2.16\mu^{-1}$, which corresponds to the isobar's width Γ = 120 MeV.

We emphasize that the contribution of the isobar to the πN scattering amplitude is calculated ambiguously, since, as is well known, a particle with spin $\frac{3}{2}$ off the mass shell contains an admixture of states with spin $\frac{1}{2}$. These states give a contribution not containing a resonance denominator, which overdetermines the coefficients of the expansion in (3.35)–(3.38). The choice of the propagator in the form (3.39) has a simple physical meaning: the scattering through the isobar does not contribute to the s wave amplitude at the threshold.

It is particularly important to keep in mind the possible uncertainty in the calculation of the contribution of the pole terms, owing to the fact that in what follows we shall derive some relations by dropping all the terms of the series except the pole terms. It is clear that this is permissible only if the result actually does not depend on the choice of the nonpole parts of the isobar propagator and the interaction vertices.

Let us now examine the relations between observable quantities to which the extrapolation formulas (3.35)–(3.38) lead without any additional hypotheses and assumptions. We shall confine ourselves to listing the predictions for the amplitudes of the s and p waves, regarding which there are more or less reliable data from phase-shift analyses. As can be seen from (3.14), this restriction means that we can examine the threshold values of the following quantities: $C^{\pm}$, $B^{\pm}$, $\partial C^{\pm}/\partial\nu$, $\partial C^{\pm}/\partial t$.

The most interesting test is that of the expansion (3.36) for the amplitude C^-, since it has been used for calculating the isotopically odd scattering length. It follows from (3.36) that in the approximation adopted the single coefficient c^-, together with the pole terms, determines both the threshold value of C^- and the threshold value of the derivative $\partial C^-/\partial\nu$. Eliminating c^-, we get

$$\left(\frac{\partial C^-}{\partial\nu} - \frac{C^-}{\nu}\right)\Big|_{\nu=\mu} = \left(\frac{\partial C^-}{\partial\nu} - \frac{C^-}{\nu}\right)_p\Big|_{\nu=\mu} + \left(\frac{\partial C^-}{\partial\nu} - \frac{C^-}{\nu}\right)_{33}\Big|_{\nu=\mu}. \tag{3.41}$$

We note that although the contribution of the isobar to the amplitude C^- depends on the nonresonance part of the propagator of the particle with spin $\frac{3}{2}$ and therefore is not uniquely determined, this ambiguity drops out of the difference of $\partial C^-/\partial\nu$ and C^-/ν.

Generally speaking the relation (3.41) should have about the same accuracy as the prediction (3.33). In fact, the terms retained in (3.41), i.e., the pole terms, are of the order of μ^0, and those dropped owing to crossing symmetry are of order μ^2. If we were considering not the invariant amplitudes, but the actual phases of the s and p waves, i.e., quantities having no definite crossing symmetry, the accuracy of the predictions would be poorer, $\sim\mu$. A comparison of the relation (3.41) with experiment is given in Table I, and shows that the agreement is good.

The expansion (3.35)–(3.38) also allows us to find the threshold values of B^+ [48,49] and $\partial C^-/\partial t$, which to the approximation considered are expressed in terms of the pole terms. These relations also agree well with experiment. The predictions for the quantities $\partial C^-/\partial t$ and B^+ also satisfy the requirement that they be independent of the choice of the nonpole part of the propagator. We get practically the same result whether they are calculated by means of the propagator (3.39) or by the dispersion method with the isobar width neglected.

In the case of the amplitudes B^- and $\partial C^+/\partial t$ the pole contributions are not uniquely determined and one cannot get predictions for them. In Table I the contribution of the isobar is calculated by means of the propagator (3.39) and is given for the sake of completeness. Finally, the quantity $\partial C^+/\partial\nu$, as can be seen from Table I, is mainly given by the contribution of the isobar, which in this case is unambiguous. However, it is hard to estimate the relative sizes of the isobar contribution and the neglected nonpole terms to $\partial C^+/\partial\nu$ theoretically, and therefore it is not clear what error to expect owing to the omission of the nonpole terms.

3.2.7. Accordingly, we have examined the amplitudes for πN scattering and found confirmation for all three main hypotheses (I, II, and IV in the introduction) used in the soft-pion method: a) that it is possible to expand the amplitudes in powers of the momenta of the π mesons; b) the conservation of axial current for μ^2 = 0;

Table I.

Quantity	Nucleon contribution ($\nu=\mu$)	Contribution of isobar ($\nu=\mu$)	Theoretical prediction	Experimental value	
				from [44]	from [46]
C^+	$-f^2\mu^2/m=-0,15$ *)	0	≈ 0	$-0,03$	$-0,13$
C^-	$f^2\mu^3/2m^2=0,01$	0	1,44	1,24	1,34
$B^-/2m$	$f^2\mu^2/2m^3=0,002$	$0,22\lambda^2=1,02$	—	0,75	0,84
$B^+/2m$	$-2f^2/\mu=-2,04$	$-0,10\lambda^2=-0,49$	$-2,53$	$-2,42$	$-2,41$
$\frac{\partial C^+}{\partial\nu}$	$2f^2\mu/m=0,30$	$0,63\lambda^2=2,92$	3,22	3,65	3,72
$\frac{\partial C^-}{\partial\nu}-\frac{C^-}{\nu}$	$-4f^2=-4,09$	$-0,14\lambda^2=-0,67$	$-4,76$	$-4,89$	$-4,75$
$\frac{\partial C^+}{\partial t}$	$\frac{f^2}{m}=0,15$	$0,17\lambda^2=0,78$	—	1,88	1,91
$\frac{\partial C^-}{\partial t}$	$-f^2/\mu\,(1+\mu/2m)=-1,10$	$-0,05\lambda^2=-0,23$	$-1,33$	$-1,42$	$-1,37$

*All quantities are given in the system of units $\mu=\hbar=c=1$.

c) the assumption (3.27) about the form of the commutator of the axial charges.

3.3. $\pi\pi$ Scattering

In the present section we examine the consequences of Hypotheses I—IV for the $\pi\pi$ scattering amplitude. Our procedure is approximately the same as in the case of πN scattering. A special feature here is the necessity of including the "semistrong" interaction, which breaks the SU(2) ⊗ SU(2) symmetry. The main results were obtained in a paper by Weinberg,[16] and his review in a report at the Vienna Conference[50] can be recommended for more detailed information on the matter and for the bibliography.

3.3.1. In this introductory subsection we shall present the necessary information about the phenomenological properties of $\pi\pi$ scattering (a more detailed exposition can be found, for example, in the book [51]).

The invariant amplitude for the process $\pi(q_1)+\pi(q_2)\to\pi(-q_3)+\pi(-q_4)$ which satisfies the requirements of crossing symmetry and isotopic invariance is of the form

$$T=(\varphi_1\varphi_2)(\varphi_3\varphi_4)A(q_1,q_2;q_3,q_4)+(\varphi_1\varphi_3)(\varphi_2\varphi_4)A(q_1,q_3;q_2,q_4)+(\varphi_1\varphi_4)(\varphi_2\varphi_3)A(q_1,q_4;q_2,q_3), \tag{3.42}$$

where φ_i (i = 1, 2, 3, 4) are the isotopic wave functions of the π mesons, and $A(q_1,q_2;q_3,q_4)$ is an invariant function of the momenta of the particles which is symmetric with respect to the interchanges $q_1\longleftrightarrow q_2$; $q_3\longleftrightarrow q_4$; $q_1\longleftrightarrow q_3$; $q_2\longleftrightarrow q_4$.

We assume for $A(q_1,q_2;q_3,q_4)$ an expansion in powers of the momenta, and confine ourselves to the quadratic terms in this expansion (for a discussion of this hypothesis see 3.3.6 and 3.3.7). Then

$$A(q_1,q_2;q_3,q_4)=\alpha+\beta\,[(q_1+q_2)^2-\mu^2]+\gamma\,[q_1^2+q_2^2+q_3^2+q_4^2-3\mu^2] \tag{3.43}$$

where α, β, γ are coefficients in the expansion.

In this approximation the nonvanishing phase shifts are $\delta_{0,2}$ for s waves in states with isotopic spins T = 0, 2 and δ_1 for the p wave; these are expressed in terms of the invariant amplitudes for the process in the following way:

$$E\delta_0/q=(1/32\pi)\,T_0=(1/32\pi)\,[3T(\pi^+\pi^-\to\pi^0\pi^0)+T(\pi^+\pi^+\to\pi^+\pi^+)]=(\mu^2/32\pi)\,[7\beta+5g+8\beta\,(q^2/\mu^2)], \tag{3.44}$$

$$E\delta_2/q=(1/32\pi)\,T_2=(1/32\pi)\,T(\pi^+\pi^+\to\pi^+\pi^+)=(\mu^2/16\pi)\,[g-\beta-2\beta\,(q^2/\mu^2)], \tag{3.45}$$

$$E\delta_1/q^3=T_1/96\pi q^2\cos\theta=(1/96\pi q^2\cos\theta)\,[2T(\pi^+\pi^0\to\pi^+\pi^0)-T(\pi^+\pi^+\to\pi^+\pi^+)]=\beta/24\pi, \tag{3.46}$$

where E, q, θ are the energy, the absolute value of the three-dimensional momentum, and the scattering angle in the c.m.s.; $g=\alpha+\gamma\mu^2$.

We introduce the scattering lengths $a_{0,2}$ and a_1 of the s and p waves, and parameters b characterizing the dependence of the s waves on the energy

$$\left.\begin{aligned} &E\delta_{0,2}/q\approx\mu a_{0,2}+q^2b_{0,2},\\ &E\delta_1/q^3\approx a_1. \end{aligned}\right\} \tag{3.47}$$

3.3.2. From the assumptions of a quadratic expansion of the amplitude and of crossing symmetry three relations follow for the parameters we have introduced,

$$\beta/4\pi=b_0=-2b_2=6a_1=(1/3\mu)\,(2a_0-5a_2). \tag{3.48}$$

None of these relations can at present be checked experimentally.

3.3.3. The Adler selfconsistency condition (see Sec. 3.1) has the consequence that the amplitude is zero when the momentum of one of the mesons is zero and the other mesons are on the mass shell. This means that the coefficient α in (3.43) must be equal to zero:

$$\alpha=0. \tag{3.49}$$

The condition (3.49) does not in itself lead to any relations between observable quantities, in contrast with the case of πN scattering. This is due to the fact that we are taking the dependence of the amplitude on the masses into account, and in the physical region the amplitude involves only the combination $\alpha+\gamma\mu^2$. We must take the dependence on the masses into account because, as follows from Eqs. (3.43) and (3.49), the total amplitude near threshold is a quantity of the order of μ^2.

3.3.4. To derive the consequences of Hypothesis II it is convenient to consider the concrete process $\pi^+(q_1)+\pi^0(q_2)\to\pi^0(-q_3)+\pi^+(-q_4)$; the amplitude for it is

$$T(\pi^+\pi^0\to\pi^0\pi^+)=\beta\,[(q_1+q_2)^2-\mu^2]+\gamma\,[q_1^2+q_2^2+q_3^2+q_4^2-3\mu^2]. \tag{3.50}$$

We take $q_1=-q_3=q$, $q_2=-q_4=p$, $p^2=\mu^2$ and let q go to zero. As in the case of πN scattering, the partial conservation of axial current gives for $q_0\to 0$ ($\mathbf{q}=0$)

$$(2\pi)^4 \delta^4(0)\, T(\pi^+\pi^0 \to \pi^0\pi^+) \underset{q\to 0}{\longrightarrow}$$
$$\underset{q\to 0}{\longrightarrow} ic^2\sqrt{2}\int dt\, \langle \pi^+ | [A^3(t), \dot{A}^+(t)] + ik_0 [A^3(t), A^+(t)] | \pi^0\rangle + (Ok^3) \quad (3.51)$$

Using for the commutator of the axial charges Eq. (1.4), $[A^3(t), A^+(t)] = V^3$, and the relation $\langle \pi^+ | \pi^+\rangle = 2p_0(2\pi)^3\delta^3(0)$, we get from a comparison of Eqs. (3.51) and (3.50)

$$\beta = 2c^2, \quad (3.52)$$

$$(2\pi)^4\delta^4(0)\,\gamma\mu^2 = ic^2\sqrt{2}\int dt\, \langle \pi^+ | [\dot{A}^3(t), A^+(t)] | \pi^0\rangle. \quad (3.53)$$

Equation (3.52) determines the absolute magnitude of the parameters appearing in Eq. (3.48).

3.3.5. Accordingly, after using the same hypotheses as in the consideration of the amplitude for πN scattering, we have one unknown coefficient γ left in the expansion (3.43). For massless π mesons this coefficient does not occur in the expression for the amplitude on the mass shell. For $\mu^2 \neq 0$ this is not true, owing to the breaking of the SU(2) ⊗ SU(2) symmetry. As has already been pointed out, at energies of the order of the pion mass the term proportional to γ in Eq. (3.43) in general makes an important contribution.

The coefficient γ is determined from the additional assumption (2.18) about the properties of the interaction that breaks the SU(2) ⊗ SU(2) symmetry: $[\dot{A}^i(t), A^k(t)] \sim \delta^{ik}$. Then $[\dot{A}^3(t), A^+(t)] = 0$, and

$$\gamma = 0, \quad (3.54)$$

which together with the condition (3.49) leads to the following relation:

$$7a_2 = -2a_0. \quad (3.55)$$

For the scattering lengths $a_{0,2}$ we have finally

$$a_0 = 0.2\mu^{-1}, \quad a_2 = -0.06\mu^{-1}. \quad (3.56)$$

3.3.6. We note that although the scattering lengths are small, according to Eqs. (3.44), (3.45) the s-wave phase shifts increase rapidly with the energy. For example, for total energy 500 MeV δ_0 is about 35°. It must be kept in mind, however, that we have everywhere neglected the imaginary part of the $\pi\pi$ scattering amplitude, which is not small for $\delta_0 = 35°$. Therefore this result can be regarded only as an estimate of the quantity δ_0.

The existing experimental data evidently lead to large $\pi\pi$ scattering lengths ($a_0 \sim 1\mu^{-1}$), which contradict the prediction (3.56). Values of the scattering phase shifts have been obtained from analyses of the reactions $\pi N \to 2\pi N$,[52] and decays $K \to 3\pi$,[53] and $K \to 2\pi e\nu$.[54] It is not clear, however, how reliable data on the scattering lengths are which are obtained by these indirect methods.

3.3.7. In conclusion we shall make some remarks concerning the hypotheses used in the derivation of the relations (3.56). The main one is the assumption that the amplitude can be written in the form of the polynomial (3.43). In particular this means that the contribution of threshold singularities is neglected, which is permissible only for small scattering lengths. Therefore we can judge the correctness of Hypotheses I–III from comparing the theoretical predictions derived in this section with experiment only in cases in which the experimental scattering lengths are actually small. We recall that in the case of πN scattering the condition that the scattering lengths be small was satisfied.

The expansion (3.43) can also fail if the amplitude has a pole at a distance $\sim\mu$ from the threshold. The width of the corresponding resonance could also be of the order of μ. We note that a strong s-wave interaction of π mesons would not affect the predictions we already know for the amplitude for πN scattering. In fact it is essential to include this interaction only in calculating the quantity $\partial C^+/\partial t$, for which we cannot obtain any predictions at all.

4. PHOTOPRODUCTION AND ELECTROPRODUCTION OF π MESONS

In this chapter we come to the consideration of processes caused by electromagnetic or weak interactions. The main difference from strong processes is that these interactions break the SU(2) ⊗ SU(2) symmetry even in the limit $\mu^2 = 0$.

In what follows we shall often use the reduction formula for the electromagnetic or weak process $A \to B\pi$

$$M \equiv \langle B\pi | \mathscr{H}(0) | A\rangle = -i\int dx e^{iqx} (\Box - \mu^2)\langle B | T\{\mathscr{H}(0)\varphi(x)\} | A\rangle, \quad (4.1)$$

where $\mathscr{H}$ is the Hamiltonian of the interaction. Carrying out the same operations as in the derivation of Eq. (3.23) we get for the value of the amplitude M for zero pion momentum the result

$$M \underset{q\to 0}{\longrightarrow} ic\int dx e^{iqx}\langle B | T\{\partial_\mu a_\mu(x)\mathscr{H}(0)\} | A\rangle =$$
$$= ic\langle B | [\mathscr{H}(0), A(0)] | A\rangle + cq_\mu\int dx e^{iqx}\langle B | T\{a_\mu(x)\mathscr{H}(0)\} | A\rangle, \quad (4.2)$$

i.e., the amplitude at this point is expressed in terms of the commutator of the axial charge A with the Hamiltonian (in the second term only the pole contributions need be taken into account). Using definite assumptions about the form of these commutators and the extrapolation formulas for the amplitude, we can obtain from (4.2) predictions for experimentally measurable quantities.

4.1. The Photoproduction of π Mesons

In this section we consider the amplitude for photoproduction of π mesons from nucleons near threshold. In Subsection 4.1.1 we prove the Kroll-Ruderman theorem,[9] according to which the photoproduction amplitude is described, up to and including terms of zero order in the photon momentum k, by pole and contact diagrams. This statement is a particular case of a theorem of Low,[10] in proving which we shall follow.[18] In 4.1.2 it is shown that it follows from conservation of axial current that the terms in the nonpole part of the amplitude which are linear in the photon or pion momentum are equal to zero.[55,18] In 4.1.3 we make a comparison of the results with the available experimental data.

4.1.1. For definiteness we consider the production of a π^+ meson from a proton

$$\gamma + p \to n + \pi^+. \quad (4.3)$$

The amplitude for this process can be written in the form

$$T(\gamma p \to n\pi^+) = e\varepsilon_\mu M_\mu = -e\varepsilon_\mu\langle n\pi^+ | j_\mu(0) | p\rangle, \quad (4.4)$$

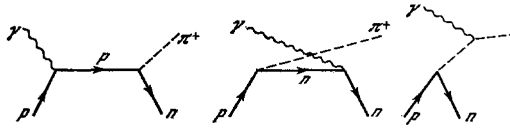

FIG. 3

where ϵ_μ is the polarization vector of the photon, $j_\mu(0)$ is the operator of the electromagnetic current of the hadrons, and $e^2/4\pi = 1/137$.

In the matrix element M_μ it is convenient to separate out the contributions of the pole diagrams (Fig. 3): the nucleon terms M_μ^p and M_μ^n, the π-meson term M_μ^π, and also the contact term M_μ^c, which is obtained from the pseudovector πNN vertex by the substitution $\partial_\mu \to \partial_\mu + ieA$:

$$M_\mu = M_\mu^p + M_\mu^n + M_\mu^\pi + M_\mu^c + \widetilde{M}_\mu. \tag{4.5}$$

The explicit expressions for these contributions are as follows:

$$M_\mu^p = if\sqrt{2}\,\bar{u}_2\hat{q}\gamma_5(\hat{p}_2 + \hat{q} - m)^{-1}[\gamma_\mu - (k^p/2m)\sigma_{\mu\nu}k_\nu]u_1, \tag{4.6}$$

$$M_\mu^n = -if\sqrt{2}\,\bar{u}_2[-(k^n/2m)\sigma_{\mu\nu}k_\nu](\hat{p}_1 - \hat{q} - m)^{-1}\hat{q}\gamma_5 u_1, \tag{4.7}$$

$$M_\mu^\pi = -if\sqrt{2}[(q-k)^2 - \mu^2]^{-1}(2q-k)_\mu\bar{u}_2(\hat{q} - \hat{k})\gamma_5 u_1, \tag{4.8}$$

$$M_\mu^c = if\sqrt{2}\,\bar{u}_2\gamma_\mu\gamma_5 u_1, \tag{4.9}$$

where p_1, p_2 are the momenta of the initial and final nucleons, q is the momentum of the π meson, and $k^{p,n}$ are the anomalous magnetic moments of the proton and neutron.

The Kroll-Ruderman theorem essentially states that the matrix element $\widetilde{M}_\mu$ is linear in the momentum of the photon, and if we regard this momentum as small at threshold we can neglect the term $\widetilde{M}_\mu$ in (4.5). The proof is based only on the requirement that the vertex for photon emission be transverse. It is easily verified that the sum of the pole and contact terms satisfies the transversality condition separately, so that the condition $k_\mu\widetilde{M}_\mu = 0$ must be satisfied. Let us expand $\widetilde{M}_\mu$ in a power series in the photon momentum k. Since all of the contributions to the amplitude that are singular at $k \to 0$ have been removed, the quantity $\widetilde{M}_\mu(0)$ must be finite. Then, because the constant vector $\widetilde{M}_\mu(0)$ and the arbitrary vector k_μ are orthogonal, we have $\widetilde{M}_\mu(0) = 0$. In other words, the expansion of $\widetilde{M}_\mu$ begins with terms linear in k.

We can calculate the cross section for photoproduction with some accuracy by keeping only the first four terms in (4.5). The accuracy of the prediction can be improved if, as in the case of πN scattering, we make use of the properties of the amplitude with respect to the crossing transformation. To insure that the form-factors have a definite parity under this transformation, we must examine the sum M_μ^+ of the amplitudes for photoproduction of π^+ mesons from protons and of π^- mesons from neutrons. The crossing properties of the amplituees for production of neutral mesons are the same as for the sum of the amplitudes for production of charged mesons.

The expansion of M in terms of invariant amplitudes is given by four independent form-factors V_i (i = 1, 2, 3, 4):

$$\widetilde{M}_\mu = \sum_{i=1}^{4} V_i\bar{u}_2 O_\mu^i u_1,$$

where

$$\left.\begin{aligned} O_\mu^1 &= \gamma_5\sigma_{\mu\nu}k_\nu, & \eta_1 &= +1,\\ O_\mu^2 &= \gamma_5[(p_1+p_2)_\mu(qk) - q_\mu(k, p_1+p_2)], & \eta_2 &= +1,\\ O_\mu^3 &= \gamma_5[\gamma_\mu(qk) - q_\mu\hat{k}], & \eta_3 &= -1,\\ O_\mu^4 &= -i\varepsilon_{\mu\nu\rho\sigma}\gamma_\nu k_\rho q_\sigma, & \eta_4 &= +1. \end{aligned}\right\} \tag{4.10}$$

The form-factors V_i depend on the invariant variables $t = (k-q)^2$ and $\nu = (k+q)(p_1+p_2)/4m$. The numbers η_i give the parities of the corresponding form-factors with respect to the crossing transformation:

$$V_i^\pm(\nu, t) = \pm\eta_i V_i^\pm(-\nu, t).$$

It follows from (4.10) that in the approximation which is linear in k and q we need take into account only the contribution of the form-factor V_1^+ to the photoproduction amplitude, and this contribution is the same for the cases of production of π^+ and π^- mesons. The zeroth-order terms are of opposite signs, and therefore the coefficient $V_1^+(0,0)$ drops out of the expression for the sum of the cross sections for production of charged mesons:[49,9]

$$\left\{\frac{k}{|q|}\left[\frac{d\sigma}{d\Omega}(\gamma p \to n\pi^+) + \frac{d\sigma}{d\Omega}(\gamma n \to p\pi^-)\right]\right\}_{q\to 0} = \frac{\alpha}{4\pi}\frac{f^2m^2}{(m+\mu)^2}, \tag{4.11}$$

where the rapidly varying factor $k/|q|$ in the left member is due to the phase volume (k and $|q|$ are the absolute values of the three-dimensional momenta of the photon and the π meson in the c.m.s.).

4.1.2. So far we have used only gauge invariance. We shall show that it follows from conservation of axial current that $V_1^+(0,0) = 0$.

To do so we use Eq. (4.2), where we must substitute $j_\mu(0)$ for $\mathscr{H}(0)$ in the present case. For the commutator of the current with the axial current we assume (see Chapter 1)

$$[A^-(0), j_\mu(0)] = a_\mu^-(0). \tag{4.12}$$

For the limiting value of the amplitude for $q \to 0$ we get from (4.2), (4.5), (4.12), and (2.5)

$$M_\mu^\pi + M_\mu^c + \widetilde{M}_\mu \underset{q\to 0}{\longrightarrow} if\sqrt{2}\,\bar{u}_2\{\gamma_\mu\gamma_5 - [k_\mu\hat{k}/(k^2-\mu^2)]\gamma_5\}u_1, \tag{4.13}$$

where we have used the fact that the nucleon pole contributions to the photoproduction amplitude and to the second term in (4.2) are equal.

From the explicit forms of M_μ^π and M_μ^c, Eqs. (4.8) and (4.9), it follows that

$$\widetilde{M}_\mu \underset{q\to 0}{\longrightarrow} 0, \quad V_1(q=0) = 0. \tag{4.14}$$

This result can be formulated in another way, as the prediction that the quantity V_1^+ is small compared with V_1^-,

$$V_1^+/V_1^- \ll 1, \tag{4.15}$$

since according to (4.14) the expansion of V_1^+ begins with terms quadratic in the momenta, and V_1^- is only linear in k and q. The statement (4.15) is analogous to the prediction (3.34) about the ratio of the isotopically even and isotopically odd scattering lengths.

Using the relation (4.14), we can calculate up to quadratic terms not only the sum of the cross sections for production of charged mesons, but also the separate cross sections, and also the amplitude for production of neutral mesons, which according to the Kroll-Ruderman theorem contains only terms of first and higher orders in k.

We note that the relation (4.2) for the commutator of the axial charge and the electromagnetic current is equivalent to the assumption of the minimum electromagnetic interaction.[41] To verify this, we write the amplitude for photoproduction somewhat differently from Eq. (4.2):

$$(2\pi)^4\delta^4(p_1+k-p_2-q)\,T(\gamma p\to n\pi^+)$$
$$=-\int dx e^{iqx}(\Box-\mu^2)\langle n|\varphi^-(x)|\gamma p\rangle, \quad (4.16)$$

where we have used the reduction formula with respect to the π-meson field. Since a photon is involved in the process, we must write the hypothesis of partial conservation of axial current, Eq. (2.12), with the electromagnetic interaction included to first order in the charge. The assumption of minimum electromagnetic interaction means that in (2.12) we replace $\partial_\mu a_\mu^\pm$ by $\partial_\mu a_\mu^\pm \pm eA_\mu a_\mu^\pm$; we then get

$$(2\pi)^4\delta^4(0)\,T(\gamma p\to n\pi^+)\underset{q\to 0}{\to}-iec\int dx e^{-iqx}\langle n|A_\mu(x)a_\mu^-(x)|\gamma p\rangle$$
$$-icq_\mu\int dx e^{-iqx}\langle n|a_\mu^-(x)|\gamma p\rangle, \quad (4.17)$$

which, as is easily verified, is identical with the previous result (4.13).

4.1.3. Let us now compare our results with experiment. The theoretical and experimental values of the cross section for production of π^+ mesons at threshold are

$$\left\{\frac{k}{|\mathbf{q}|}\frac{d\sigma}{d\Omega}(\gamma p\to n\pi^+)\right\}_{\rm theor}=15.5\ \mu\text{b/sr}^{56}, \quad (4.18)$$

$$\left\{\frac{k}{|\mathbf{q}|}\frac{d\sigma}{d\Omega}(\gamma p\to n\pi^+)\right\}_{\rm exp}=(15.6\pm 0.5)\ \mu\text{b/sr}^{57} \quad (4.19)$$

and are in excellent agreement.

The prediction about the ratio of the threshold cross sections for production of π^+ and π^- mesons is also well confirmed:

$$\left\{\frac{d\sigma}{d\Omega}(\gamma n\to p\pi^-)\Big/\frac{d\sigma}{d\Omega}(\gamma p\to n\pi^+)\right\}_{\rm theor}=1.3^{56}, \quad (4.20)$$

$$\left\{\frac{d\sigma}{d\Omega}(\gamma n\to p\pi^-)\Big/\frac{d\sigma}{d\Omega}(\gamma p\to n\pi^+)\right\}_{\rm exp}=1.265\pm 0.075^{57}. \quad (4.21)$$

It must be admitted, however, that owing to Eq. (4.11) only one of the predictions (4.18), (4.20) can be regarded as a consequence of current algebra, since the sum of the cross sections for production of π^+ and π^- mesons can be determined from only gauge invariance and the hypothesis that the amplitude can be expanded in a power series in the momenta.

It is very interesting to compare with experiment the predictions about the size of the cross section for production of neutral pions, which is determined by the linear terms and cannot be calculated without using the result (4.14). According to existing experimental estimates[58]

$$\left\{\lim_{q\to 0}\left[\frac{k}{|\mathbf{q}|}\frac{d\sigma}{d\Omega}(\gamma p\to p\pi^0)\right]\right\}_{\rm exp}=\begin{pmatrix}0.07\pm 0.02\\0.06\pm 0.04\end{pmatrix}\mu\text{b/sr} \quad (4.22)$$

whereas the theoretical value is

$$\left\{\lim_{q\to 0}\left[\frac{k}{|\mathbf{q}|}\frac{d\sigma}{d\Omega}(\gamma p\to p\pi^0)\right]\right\}_{\rm theor}=0.18\ \mu\text{b/sr} \quad (4.23)$$

It must be kept in mind, however, that in the experimental determination of the cross section one uses an extrapolation of data obtained at energies at which the contribution of the p wave is at least an order of magnitude larger than the contribution of the s wave, with which we are concerned here.

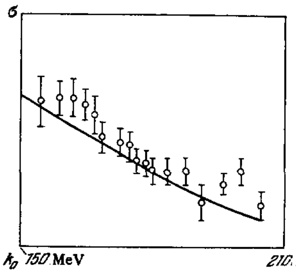

FIG. 4

If we keep the zeroth-order terms in the amplitude, then to this accuracy we can find not only the threshold value of the cross section, but also the energy-dependence of the cross section near the threshold. In Fig. 4, taken from [56], the theoretical curve (solid line) is compared with the experimental data. Since the zeroth-order terms are determined from the requirement of gauge invariance, the comparison of this prediction with experiment tests only the possibility of expanding the amplitude in a series. We cannot take the linear terms into account in calculating the energy dependence of the cross section, because we have not considered the contribution of the isobar, which is of the same order as the linear terms. In the calculation of the threshold value of the cross section—the amplitude of the s wave—it is not essential to take the isobar into account.

Accordingly, the comparison with experiment shows that the main features of photoproduction at low energies are satisfactorily described by the theory. But the relation (4.14), which is of more interest to us, cannot be tested on the basis of the existing experimental data.

4.2. Electroproduction of π Mesons

In this section we obtain theoretical predictions for the amplitude for electroproduction of π mesons for small values of the relative three-dimensional momentum of the final nucleon and meson. In Subsections 4.1.1 and 4.1.2 we consider the case of small momentum transferred to the leptons, and in 4.2.3 the case in which this momentum is relatively large, of the order of a BeV/c.

4.2.1. In the one-photon approximation the amplitude for electroproduction can be written in the form

$$\left.\begin{aligned}T(e^-p\to e^-n\pi^+)&=(4\pi\alpha/k^2)\,v_2\gamma_\mu v_1 M_\mu,\\ M_\mu&=-\langle n\pi^+|j_\mu(0)|p\rangle,\end{aligned}\right\} \quad (4.24)$$

where $v_{1,2}$ are the wave functions of the initial and final electrons and k is the momentum transferred to the electron (the momentum of the virtual photon).

If the momentum transfer is small, of the order of μ, the amplitude for electroproduction can be expanded in a series in k and q. It follows from the results obtained in the preceding section that up to terms linear in q the matrix element M_μ is given by the sum of the pole and contact diagrams. Explicit expressions for the contributions of these diagrams are given in Eqs. (4.6)—(4.9). In this approximation there is no difference between the amplitudes for photoproduction and for electroproduction.

4.2.2. It is clear that in order to obtain predictions specific to the case of electroproduction, we must consider the dependence of the amplitude on k^2. According to (4.2) and (4.12) the terms proportional to k^2 in the expansion of the nonpole part $\widetilde{M}_\mu$ can be expressed in terms of the derivative of the axial form-factor $g(k^2)$ of the nucleon with respect to k^2. The dependence of the pole diagrams on k^2 is due to the electric radii of the particles.

Therefore in second order in k and q the only remaining theoretically uncertain terms are those proportional to both k and q. But these terms are the same, to our approximation, for the cases of photoproduction and of electroproduction. Therefore in studying the process of electroproduction we can take the terms proportional to both k and q into account phenomenologically, expressing them in terms of the quantities $(V_3, V_4, \partial V_1/\partial\nu)_{q=0}$

Keeping these things in mind, we can easily derive an expression for the matrix element M_μ which is correct to and including terms of second order in q and k:[38]

$$\begin{aligned} M_\mu = & -if\sqrt{2}\,\bar{u}_2\{\hat{q}\gamma_5(\hat{p}_2+\hat{q}-m)^{-1}[\gamma_\mu(1+F'_p(0)k^2)-(k^\nu/2m)\sigma_{\mu\nu}k_\nu] \\ & +[\gamma_\mu F'_n(0)k^2-(k^n/2m)\sigma_{\mu\nu}k_\nu](\hat{p}_1-\hat{q}-m)^{-1}\hat{q}\gamma_5 \\ & +(\hat{q}-\hat{k})\gamma_5[(q-k)^2-\mu^2]^{-1}[(2q-k)_\mu+2F'_\pi(0)(q_\mu k^2-k_\mu(kq))] \\ & -\gamma_\mu\gamma_5-[F'_p(0)-F'_n(0)]k_\mu\hat{q}\gamma_5-(g'(0)/g_A)\gamma_5(k_\mu\hat{k}-\gamma_\mu\hat{k})\}u_1 \\ & -i\bar{u}_2\Big[V_4(0)\varepsilon_{\mu\nu\sigma\rho}\gamma_\nu k_\rho q_\sigma+iV_3(0)\gamma_5(\gamma_\mu(qk)-q_\mu\hat{k})+i\nu\frac{\partial V_1}{\partial\nu}\gamma_5\sigma_{\mu\xi}k_\xi\Big]u_1, \end{aligned} \tag{4.25}$$

where $F'_\pi(0)$, $F'_p(0)$, and $F'_n(0)$ are the values of the derivatives of the respective form-factors of the pion, proton, and neutron with respect to k^2 at $k^2 = 0$. We note that Eq. (4.25) applies only to the production of a π meson in the s wave, since we have not considered the contribution of the isobar. In this case the contribution of the form-factor V_4 can be neglected.

4.2.3. For small momenta of the virtual photon tests of the consequences of the conservation of axial current are made difficult by the fact that the main part of the matrix element is determined simply by the requirement of gauge invariance. In the case of electroproduction, however, one can select events in which the π meson produced is at rest relative to the final nucleon, and k is large. Then the conservation of the electromagnetic current does not (sic) allow us to find the amplitude with any accuracy, and Eq. (4.2) determines the matrix element M_μ with accuracy up to terms linear in the small momentum q:[60]

$$\begin{aligned} M_\mu = & \, i[g(k^2)/g_A]f\sqrt{2}\,\bar{u}_2[\gamma_\mu-(k_\mu\hat{k}/k^2)]\gamma_5 u_1 - \\ & -if\sqrt{2}\,\bar{u}_2\{\hat{q}\gamma_5(\hat{p}_2+\hat{q}-m)^{-1}[\gamma_\mu F_1^p(k^2)-{}^1/{}_2m\sigma_{\mu\nu}k_\nu F_2^p(k^2)] \\ & +[\gamma_\mu F_1^n(k^2)-{}^1/{}_2m\sigma_{\mu\nu}k_\nu F_2^n(k^2)](\hat{p}_1-\hat{q}-m)^{-1}\hat{q}\gamma_5\}u_1, \end{aligned} \tag{4.26}$$

where $F_{1,2}^{p,n}(k^2)$ are the charge and magnetic form-factors of the proton and neutron, and $g(k^2)$ is the axial form-factor. We note that the expression (4.26) is transverse only with accuracy up to the neglected terms, which are linear in q. In the calculation of M_μ one should, in general, also include separately the contribution of the isobar N*(1236). But this contribution is small at the threshold for meson production, of the order of 10 percent.

Generally speaking the relation (4.26) can be applied for arbitrary k^2. In the region of asymptotically large k^2, however, when the form-factors $F_{1,2}^{p,n}(k^2)$ become small, it can happen that the terms linear in q fall off more slowly with increasing k^2, so that it is essential to include them and (4.26) does not hold.

A complete test of (4.26) is obviously difficult, primarily because of the necessity of first determining the axial form-factor $g(k^2)$. One can therefore try to reverse the problem and regard (4.26) as the basis for determining $g(k^2)$ from experiments on electroproduction. The legitimacy of expanding the amplitude in a series in the momentum q could be tested by comparing with experiment the predictions for the cross section for production of neutral π mesons, which in the approximation considered is expressed solely in terms of the contribution of pole diagrams and does not contain any quantities not now known. Besides this, Eq. (4.26) imposes serious restrictions on the spin structure of the amplitude for production of charged π mesons, and this also offers possibilities for testing (4.26).

5. LEPTONIC DECAYS OF K MESONS

In this chapter we consider the following processes: $K \to l\nu$ (K_{l2} decays), $K \to \pi l\nu$ (K_{l3} decays), $K \to \pi\pi e\nu$ (K_{e4} decays), where l denotes a muon or electron and ν a neutrino.

Section 5.1 contains the necessary information from the phenomenology of these decays, which can be found expounded in detail in [22,30,61] Sections 5.2 and 5.3 give derivations of the theoretical predictions for the form-factors describing the K_{l3} and K_{e4} decays.[62,63] In Sec. 5.4 we discuss the extrapolation formulas for the form-factors, and in Sec. 5.5 we compare the results obtained with experiment.

5.1. Phenomenology of Leptonic Decays of K Mesons

Leptonic decays of K mesons are caused by the weak hadron current with change of strangeness, i_μ^S. It is usually assumed that i_μ^S is a component of an isotopic spinor.

5.1.1. K_{l2} **Decay.** The matrix element for this decay is determined by a single constant f_K:

$$M = \frac{G}{\sqrt{2}}\langle 0|i_\mu^S|K\rangle\bar{u}_\nu\gamma_\mu(1+\gamma_5)u_l, \quad \langle 0|i_\mu^S|K\rangle = if_K p_\mu, \tag{5.1}$$

where u_ν, u_l are the wave functions of the leptons, p_μ is the momentum of the K meson, and $G = (1.01 \pm 0.01) \times 10^{-5} m_p^{-2}$. The probability of $K_{\mu 2}$ decay is connected with f_K in the following way:

$$w(K^+ \to \mu^+\nu) = (G^2 f_K^2/8\pi)\, m_\mu^2 m_K[1-(m_\mu^2/m_K^2)]^2. \tag{5.2}$$

5.1.2. K_{l3} **Decays.** The matrix elements for K_{l3} decays depend on two independent functions $f_\pm(k^2)$, where k^2 is the square of the momentum transferred to the leptons:

$$\begin{gathered} M = G/\sqrt{2}\,\langle\pi|i_\mu^S|K\rangle\bar{u}_\nu\gamma_\mu(1+\gamma_5)u_l, \\ \langle\pi^-|i_\mu^S|K^0\rangle = \sqrt{2}\langle\pi^0|i_\mu^S|K^+\rangle = -[f_+(k^2)(p+q)_\mu + f_-(k^2)(p-q)_\mu], \end{gathered} \tag{5.3}$$

where p, q are the momenta of the K and π mesons. The contribution of the form-factor f_- to the amplitude is proportional to the mass of the lepton, and in the case of K_{l3} decay it can be neglected. If we further assume that the function $f_+(k^2)$ does not depend on its argument in the region considered $[0 < k^2 < (m_K - \mu)^2]$, then a measurement of the total probability of K_{l3} decay allows us to find the quantity f_+:

$$w(K^+ \to \pi^0 e^+ \nu) = (G^2 f_+^2 m_K^2 / 2 \cdot 768\pi^3)\, 0.58. \tag{5.4}$$

The ratio f_-/f_+, which is usually denoted by ξ, can be found from the probability of $K_{\mu 3}$ decay (two solutions), or from polarization experiments.

5.1.3. K_{e4} **Decays.** The matrix element can be written in the form

$$\left.\begin{aligned} M &= (G/\sqrt{2})\,\langle \pi\pi \,|\, i_\mu^S \rangle \,|\, K \rangle\, \bar{u}_\nu \gamma_\mu (1+\gamma_5)\, u_l, \\ \langle \pi^+\pi^- \,|\, i_\mu^S \,|\, K^+ \rangle &= -i\,\{(q_+ + q_-)_\mu f_1 + (q_+ - q_-)_\mu f_2 \\ &\quad + (p - q_+ - q_-)_\mu f_3 + i f_4 \varepsilon_{\mu\nu\rho\sigma} p_\nu q_\rho^+ q_\sigma^- \}, \end{aligned}\right\} \tag{5.5}$$

where p, q_+, q_- are the respective momenta of the K^+, π^+, and π^- mesons and the arguments of the functions $f_1, \ldots, f_4$ are omitted. The contribution of the form-factors f_3, f_4 to the decay probability is numerically suppressed, and if we regard f_1 and f_2 as constants we can write for the decay probability

$$w(K^+ \to \pi^+\pi^- e^+ \nu) = (G^2 m_K^7 / 2^{10} \cdot 360\pi^5)\,(f_1^2 \cdot 0{,}0296 + f_2^2 \cdot 0{,}0029). \tag{5.6}$$

The ratio f_1/f_2 can be determined from analysis of angular distributions.

5.2. The values of the amplitudes for K_{l3} and K_{e4} decays for zero pion momentum are given by Eq. (4.2) where for $\mathscr{H}$ we must substitute the operator i_μ^S. Since the π meson cannot be emitted from an external line, in this case a K meson line, the second term in the right member of (4.2) is equal to zero. For the commutators of the current i_μ^S with the generators of the group SU(2) ⊗ SU(2) we assume (see Introduction)

$$[V^i - A^i,\ i_\mu^S] = 0. \tag{5.7}$$

When we use this relation Eq. (4.2) leads to the following limiting value for the matrix element of $K_{\mu 3}^+$ decay:

$$\begin{aligned} \lim_{q\to 0} \langle \pi^0 \,|\, i_\mu^S \,|\, K^+ \rangle &= ic\sqrt{2}\,\langle 0 \,|\, [i_\mu^S,\ V^3] \,|\, K^+ \rangle \\ &= (ic/\sqrt{2})\,\langle 0 \,|\, i_\mu^S \,|\, K^+ \rangle = -c f_K p_\mu / \sqrt{2}, \end{aligned} \tag{5.8}$$

where we have operated on the product of wave functions with the generator V^3 of the isotopic group[3)] and have used the definition (5.1). It follows from (5.8) that

$$f_+(m_K^2) + f_-(m_K^2) = c f_K. \tag{5.9}$$

Analogously, we get for the form-factors of K_{e4} decay

$$f_3|_{q_+=0} = 0, \tag{5.10}$$

$$(f_1 - f_2)_{q_+=0} = 0, \tag{5.11}$$

$$(f_1 + f_2)_{q_-=0} = 2cf_+, \tag{5.12}$$

$$f_3|_{q_-=0} = c(f_+ + f_-). \tag{5.13}$$

3)The phase factors are chosen in such a way that $V^+ \mid K^+ > = V^- \mid K^0 > = 0$, $V^\pm \mid K^{0+} > = \mid K^{0+} >$, $V^3 \mid K^\pm > = \pm \mid K^\pm > /2$; $V^\pm \mid \pi^\pm > = 0$, $V^\pm \mid \pi^0 > = \mp 2^{1/2} \mid \pi^\pm >$, $V^\pm \mid \pi^\mp > = \pm 2^{1/2} \mid \pi^0 >$, $V^3 \mid \pi^\pm > = 0$, $V^3 \mid \pi^\pm > = \pm \mid \pi^\pm >$. We note that use is often made (in particular in the book [22]) of a choice of phases which gives the opposite sign to the state $\mid \pi^+ >$.

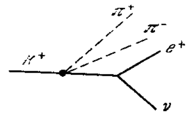

FIG. 5

5.3. In order for the conditions (5.9)–(5.13) to yield predictions for the amplitudes in the physical regions of the decays it is necessary to use some sort of extrapolation formulas for the form-factors. The simplest possibility is as follows: The quantities $f_\pm$, f_1, f_2 do not depend on their arguments, and the form-factor f_3 equals a constant plus the contribution of a diagram with the πK scattering block, as shown in Fig. 5 (this diagram contributes only to f_3). We now explain the last assumption. The isotopically odd part of the amplitude for πK scattering can be calculated in precisely the same way as the isotopically odd part of the amplitude for πN scattering was calculated in Sec. 3.2. It is not hard to verify that the contribution of the diagram of Fig. 5 to f_3 is given by

$$-(1/2)\, c^2 f_K [(q_+ - q_-)(2p - q_+ - q_-)] / [(p - q_+ - q_-)^2 - m_K^2]. \tag{5.14}$$

It can be seen from this expression that, depending on whether the momentum that goes to zero is q_+ or q_-, this contribution varies by the amount $c^2 f_K$, which according to (5.13), is comparable with f_3. Moreover, if we determine the constant part of f_3 from (5.13), then the relation (5.10) at once follows from the assumption that the entire dependence of f_3 on the momentum is due to the contribution (5.14), so that the hypotheses about the form of the extrapolation formula and about the commutators of the current i_μ^S with the vector and axial charges are selfconsistent.

Finally, the simplest solution for the form-factors f is

$$1 + \xi = c f_K / f_+,\quad f_1 = f_2 = c f_+,\quad f_3 = (c^2 f_K / 2) - \{(q_+ - q_-)\, p / 2\, [(p - q_+ - q_-)^2 - m_K^2]\}; \tag{5.15}$$

the quantity f_+ remains undetermined and must be taken from experiment.

5.4. We emphasize that the question of the form of the extrapolation formulas cannot be solved theoretically in the framework of the hypotheses we are considering. Therefore the assumption that the form-factors $f_\pm$, f_1, and f_2 are constants is not at all necessary and is rather due to the lack of experimental data and the desire to obtain predictions for even a rough comparison with experiment. At present there are only experimental estimates of the dependence of f_+ on k^2 [64]

$$f_+(k^2) \approx f_+(0)\,[1 + 0{,}023\,(k^2/\mu^2)]. \tag{5.16}$$

We see that as k^2 varies from 0 to m_K^2 the value of f_+ changes by about 30 percent. It is clear that effects of this order ought to be taken into account.

5.5. Let us now compare the solution (5.15) for the form-factors of the K_{l3} and K_{e4} decays with experiment. If we take the probability of the decay $K^+ \to \mu^+\nu$ to be $4.0 \times 10^6\ \mathrm{sec}^{-1}$, then we get for the probability of the decay $K^+ \to \pi^+\pi^+ e^+$ the value

$$w(K^+ \to \pi^+\pi^- e^+ \nu)_{\text{theor}} = 1{,}6 \cdot 10^3\ \mathrm{sec}^{-1}, \tag{5.17}$$

which is to be compared with the experimental value

$$w(K^+ \to \pi^+\pi^- e^+\nu)_{\exp} = (2.6 \pm 0.3)\cdot 10^3\ \mathrm{sec}^{-1} \qquad (5.18)$$

The discrepancy between theory and experiment is about 30 percent in the amplitude.

For the ratio of the form-factors f_1 and f_2 we have

$$(f_1/f_2)_{\text{theor}} = 1, \quad (f_1/f_2)_{\exp} = 0.8 \pm 0.3^{65}. \qquad (5.19)$$

We can also predict the value of the ratio ξ of the form-factors f_- and f_+ in $K_{\mu 3}$ decay:

$$1 + \xi = 1.3. \qquad (5.20)$$

There have been many papers on the experimental determination of the quantity ξ, but the situation is still rather unclear and we find it difficult to suggest any final value for ξ. Nevertheless it is important to point out that the majority of the results contradicts the prediction (5.19) and gives $\xi = -(0.5-1)$. A detailed review of the experimental papers can be found in a report by Rubbia (CERN, 1969).[66]

Improvement of the experimental data on the quantity ξ and the dependences of $f_\pm$ on k^2 is a matter of great interest. We point out that in comparing theoretical predictions with experiment it may be necessary to use a more realistic parametrization of the form-factors.

6. NONLEPTONIC DECAYS OF K MESONS AND HYPERONS

6.1. The Rule $\Delta T = \frac{1}{2}$

6.1.1. The hypothesis of the existence of SU(2) $\otimes$ SU(2) symmetry of the strong interactions is also useful in the discussion of weak nonleptonic interactions only in case the Hamiltonian of these interactions has definite transformation properties with respect to the group SU(2) $\otimes$ SU(2). As has already been said, these properties are characterized by the form of the commutation relations of the Hamiltonian $\mathscr{H}$ with the generators of the group.

The commutators of $\mathscr{H}$ with the vector generators determines its isospin structure, about which we make no hypotheses. We shall only assume that the bare weak interaction involves particles of left-handed helicity; this means that (see Chapter 1)

$$[V^i - A^i, \mathscr{H}] = 0. \qquad (6.1)$$

An example of a theory in which Eq. (6.1) holds is the well known model of the weak interactions[14] according to which the Hamiltonian $\mathscr{H}$ is equal to a product of charged currents.

6.1.2. Suppose the bare Hamiltonian does not satisfy the rule $\Delta T = \frac{1}{2}$, as, for example, in the model of charged currents. It will be shown[15,67–69] that nevertheless the relation (6.1) enables us to explain the suppression of some transitions with $\Delta T > \frac{1}{2}$, if we confine ourselves to the first terms in the expansions of the amplitudes for weak processes in powers of the momenta.

The proof of this assertion is based on the calculation of the limiting values of the amplitudes by means of Eq. (4.2). Let us first consider the s-wave amplitudes in hyperon decays. The pole diagrams (Fig. 6, a, b) give a contribution, as is easily verified, onto the

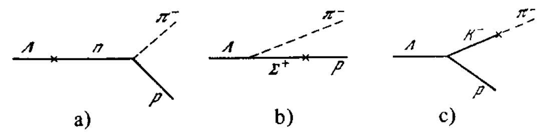

FIG. 6

p waves, and therefore in calculating the amplitudes of the s waves there is no second term in (4.2). Accordingly, we get, for example for the decay $\Lambda \to p\pi^-$,

$$M_S(\Lambda \to p\pi^-)|_{q=0} = ic\langle p|[\mathscr{H}V^+]|\Lambda\rangle = -ic\langle n|\mathscr{H}|\Lambda\rangle, \qquad (6.2)$$

where we have used the fact that $V^+|\Lambda\rangle = 0$, $V^-|p\rangle = |n\rangle$.

An analogous formula holds for the amplitude M_S $(\Lambda \to n\pi^0)$. The matrix element $\langle N|\mathscr{H}|\Lambda\rangle$ comes solely from the part of the Hamiltonian that changes the isospin by $\frac{1}{2}$. Therefore, if we neglect the change of the amplitude as we go from q = 0 to the physical value of q, the amplitudes for the decays $\Lambda \to p\pi^-$ and $\Lambda \to n\pi^0$ must satisfy the rule $\Delta T = \frac{1}{2}$ even if the Hamiltonian $\mathscr{H}$ contains terms with $\Delta T = \frac{3}{2}$. The same argument can also be applied to the s-wave part of the decays $\Xi \to \Lambda\pi$.

In the case of the Σ hyperons we get for q = 0

$$\left.\begin{aligned} M_S(\Sigma^+ \to n\pi^+) &= -ic[\sqrt{2}\langle n|\mathscr{H}|\Sigma^0\rangle + \langle p|\mathscr{H}|\Sigma^+\rangle], \\ M_S(\Sigma^+ \to p\pi^0) &= ic/\sqrt{2}\langle p|\mathscr{H}|\Sigma^+\rangle, \\ M_S(\Sigma^- \to n\pi^-) &= ic\sqrt{2}\langle n|\mathscr{H}|\Sigma^0\rangle, \end{aligned}\right\} \qquad (6.3)$$

which leads to the relation[67]

$$M_S(\Sigma^+ \to n\pi^+) + M_S(\Sigma^- \to n\pi^-) + \sqrt{2}M_S(\Sigma^+ \to p\pi^0) = 0, \qquad (6.4)$$

which differs from the prediction of the $\Delta T = \frac{1}{2}$ rule by the sign of the amplitude M_S $(\Sigma^+ \to n\pi^+)$. Therefore if the Hamiltonian satisfies the rule $\Delta T = \frac{1}{2}$ the amplitude M_S $(\Sigma^+ \to n\pi^+)$ must be equal to zero. But if $\mathscr{H}$ contains transitions with $\Delta T > \frac{1}{2}$ there is in general no reason to expect that this amplitude will be small. Experimentally M_S $(\Sigma^+ \to n\pi^+) \approx 0$, which is hard to explain in the framework of our hypotheses if $\Delta T \neq \frac{1}{2}$.

6.1.3. The amplitude of the p wave in hyperon decays is experimentally of the same order of magnitude as that of the s wave. At first glance it may seem that this contradicts the assumption that the pion mass is small on the scale of masses of the strong interactions, since the amplitude of the p wave contains a kinematic factor **q** and in actual cases $|\mathbf{q}| \sim \mu$. This is actually not so, since there exist diagrams in which the "comparison" mass is small. For example, the contribution of the pole diagrams, shown in Fig. 6, a, b, is proportional to $1/\Delta m$, where Δm is the difference of the baryon masses and $\Delta m \sim \mu$.

A quantitative treatment shows that the p-wave amplitude cannot be satisfactorily described if we confine ourselves to baryon intermediate states. This discrepancy may be due to the necessity of including the contribution of the K-meson pole diagram (Fig. 6, c). We shall assume that this is so and show that in the framework of the pole model the p-wave amplitudes must satisfy the rule $\Delta T = \frac{1}{2}$.

The isotopic selection rules for the amplitudes corresponding to the pole diagrams are determined by the properties of the weak $K\pi$ and BB′ transitions (B, B′ = Λ, Σ, Ξ, N). Terms with $\Delta T = \frac{3}{2}$ could contribute only to the amplitudes for $K\pi$ and πN transitions. From (6.3) and the experimental fact that the amplitude M_S ($\Sigma^+ \to n\pi^+$) is zero it follows that the $\Delta T = \frac{1}{2}$ rule holds for ΣN transitions. For the $K\pi$ vertex this rule follows from (4.2), which says that in the limit of zero pion momentum the amplitudes for the transitions $K \to \pi$ and K meson to vacuum are proportional. It is obvious that only terms with $\Delta T = \frac{1}{2}$ contribute to the latter amplitude.

Accordingly, in the framework of the pole model the $\Delta T = \frac{1}{2}$ rule must be satisfied for the p-wave amplitudes of hyperon decays. Unfortunately, this model cannot be tested in a reliable way, since the constants for $K\pi$ transitions and K-meson interactions with hyperons are not known.

6.1.4. Let us now proceed to the discussion of nonleptonic decays of K mesons. The scheme of the proof[15] of the $\Delta T = \frac{1}{2}$ rule for the decays $K \to 2\pi$, 3π is analogous to the case of hyperon decays: using the relation (4.2), we can connect the amplitudes for these decays, in the limit of vanishing momenta of the π mesons, with the matrix element for the transition of a K meson to vacuum, to which any part of the Hamiltonian with $\Delta T > \frac{1}{2}$ does not contribute.

There is a complication owing to the fact that one must consider separately the contribution of the pole diagrams (Fig. 7), which depends strongly on the momenta of the mesons and cannot be expanded in a series in these momenta. The change of the isotopic spin in the pole diagrams is determined by the vertices for the transitions $K \to \pi$ and K meson to vacuum, which, as stated above, satisfy the rule $\Delta T = \frac{1}{2}$. Therefore inclusion of the pole diagrams does not change our conclusions about the isotopic structure of the amplitudes.

6.1.5. Accordingly, using Hypotheses I–IV of Chapter 1, we can establish the $\Delta T = \frac{1}{2}$ rule for decays of K mesons and for the s waves in decays of Λ and Ξ hyperons. In the framework of the pole model the $\Delta T = \frac{1}{2}$ rule for the amplitudes of the p waves follows from the $\Delta T = \frac{1}{2}$ rule for the amplitudes of the s waves.

The assumptions we have used, however, apparently do not allow us to explain the great accuracy with which the rule $\Delta T = \frac{1}{2}$ is obeyed. Moreover, as we discussed earlier, the zero value of the amplitude M_S ($\Sigma^+ \to n\pi^+$) is more likely evidence that the $\Delta T = \frac{1}{2}$ rule for the decays is not due to the involvement of π mesons in these processes. Further information about the nature of this rule can be obtained in the study of the decays $K \to 3\pi$.

6.2. The Decays $K \to 3\pi$

6.2.1. In the preceding section we used a successive reduction of all the π meson fields to connect the amplitude for $K \to 3\pi$ decays with the amplitude for transition of a K meson to the vacuum, and in this way explained the suppression of transitions with $\Delta T > \frac{1}{2}$. However, interesting results appear already at the first step: Eq. (4.2) enables us to connect the amplitudes for $K \to 3\pi$ and $K \to 2\pi$ decays, which can be measured experimentally. Accordingly, one can calculate the probabilities and slopes of the π-meson spectra in $K \to 3\pi$ decays. We can not only explain the approximate validity of the rule $\Delta T = \frac{1}{2}$. but also predict the degree of its violation in decays $K \to 3\pi$, by expressing it in terms of the amount of the transitions with $\Delta T > \frac{1}{2}$ in the decays $K \to 2\pi$. In the present section we derive these predictions for the amplitudes of $K \to 3\pi$ decays[15,71-76] and discuss the assumptions used in this argument.

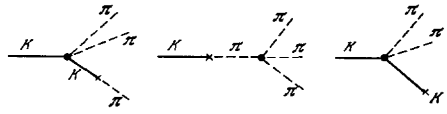

FIG. 7

6.2.2. Theoretically, as always, we can find the value of the amplitude for zero momentum of one of the π mesons. Let us examine, for example, the decay $K^+ \to \pi^+\pi^+\pi^-$. Equations (4.2) and (6.1) give for the amplitude for this decay

$$\langle\pi^+\pi^+\pi^-|\mathscr{H}|K^+\rangle_{q_3=0} = ic\langle\pi^+\pi^+|[\mathscr{H}, A^+]|K^+\rangle = ic\sqrt{2}\langle\pi^+\pi^0|\mathscr{H}|K^+\rangle, \tag{6.5}$$

$$\begin{aligned}\langle\pi^+\pi^+\pi^-|\mathscr{H}|K^+\rangle_{q_1=0} &= ic\langle\pi^+\pi^-|[\mathscr{H}, A^-]|K^+\rangle \\ &= ic\{\langle\pi^+\pi^-|\mathscr{H}|K^0\rangle - \sqrt{2}\langle\pi^+\pi^0|\mathscr{H}|K^+\rangle\},\end{aligned} \tag{6.6}$$

where q_3 is the momentum of the π^- meson and q_1 is that of one of the π^+ mesons. In deriving Eqs. (6.5) and (6.6) we have applied the operator V^- to the product of wave functions.

Let us assume, further, an expression for the amplitude linear in the energies of the π mesons. When the identity of the π^+ mesons is taken into account this is

$$\langle\pi^+\pi^+\pi^-|\mathscr{H}|K^+\rangle = a + b(pq_3), \tag{6.7}$$

where p is the momentum of the K meson.

Using Eqs. (6.5) and (6.6), we can easily find the coefficients a and b, and thus completely determine the amplitude. We must note from the very beginning, however, that the assumption about the form of the expansion is a very strong one, not checked experimentally to the needed accuracy, and we shall therefore return to a discussion of it in 6.2.4.

Accordingly, in the framework of the assumption (6.7) the matrix element for the decay $K^+ \to \pi^+\pi^+\pi^-$ is completely determined. We can treat the other decays in an entirely analogous way. The final result can conveniently be put in the form

$$\langle 3\pi|\mathscr{H}|K\rangle = \gamma(1+\sigma y), \tag{6.8}$$

where $y = (2E_3 - E_1 - E_2)/m_K$, E_3 being the energy of the "odd" π meson and E_1 and E_2 the energies of the "even" mesons;

$$\left.\begin{aligned}
&\langle\pi^+\pi^+\pi^-|\mathscr{H}|K^+\rangle = (ic\sqrt{2}/3)\langle\pi^+\pi^-|\mathscr{H}|K_1^0\rangle[1+(1-6\delta)y],\\
&\langle\pi^0\pi^0\pi^+|\mathscr{H}|K^+\rangle =\\
&\quad = (ic\sqrt{2}/6)\langle\pi^+\pi^-|\mathscr{H}|K_1^0\rangle(1+\theta)\{1-2[1+(3\delta/[1+\theta])]\},\\
&\langle\pi^+\pi^-\pi^0|\mathscr{H}|K_2^0\rangle =\\
&\quad = -(ic\sqrt{2}/6)\langle\pi^+\pi^-|\mathscr{H}|K_1^0\rangle(1-2\delta)\{1+2[1\\
&\qquad + (3\delta/[1-2\delta])]y\},\\
&\langle\pi^0\pi^0\pi^0|\mathscr{H}|K_2^0\rangle = -(ic\sqrt{2}/2)\langle\pi^+\pi^-|\mathscr{H}|K_1^0\rangle(1+\theta-2\delta),\\
&\langle\pi^+\pi^-\pi^0|\mathscr{H}|K_1^0\rangle =\\
&\quad = (ic\sqrt{2}/2)\langle\pi^+\pi^-|\mathscr{H}|K_1^0\rangle(E_{\pi+}-E_{\pi-})(2\sqrt{2}\theta-3\delta).
\end{aligned}\right\} \tag{6.9}$$

The parameters θ and δ in (6.9) characterize the deviation from the rule $\Delta T = \frac{1}{2}$ in the decays $K \to 2\pi$ and are defined as follows:

$$\delta = \langle \pi^+\pi^0 | \mathscr{H} | K^+\rangle / \langle \pi^+\pi^- | \mathscr{H} | K_1^0\rangle,$$
$$\theta = [\langle \pi^0\pi^0 | \mathscr{H} | K_1^0\rangle - \langle \pi^+\pi^- | \mathscr{H} | K_1^0\rangle + 2\langle \pi^+\pi^0 | \mathscr{H} | K^+\rangle] / \langle \pi^+\pi^- | \mathscr{H} | K_1^0\rangle.$$

The quantity θ is equal to zero if the Hamiltonian $\mathscr{H}$ does not contain any transitions $\Delta T = \frac{5}{2}$. The absolute value of the parameter δ is known from experiment: $|\delta| = 1/22$. The sign of δ can be found from the ratio of the probabilities of the decays $K_1^0 \to 2\pi^0$ and $K_1^0 \to \pi^+\pi^-$, if we make the additional assumption that $\theta = 0$ [see also the discussion below, after Eq. (6.11)]. Then

$$1 - 2\delta = \langle \pi^0\pi^0 | \mathscr{H} | K_1^0\rangle / \langle \pi^+\pi^- | \mathscr{H} | K_1^0\rangle,$$

and the experimental data favor $\delta > 0$.[77] In the numerical calculations we take $\delta = 1/22$.

If we neglect electromagnetic corrections and identify the matrix elements of the Hamiltonian of nonleptonic interactions with the amplitudes of the physical processes, then the formulas (6.9) allow us to connect transitions with a given change ΔT of the isotopic spin in $K \to 3\pi$ and $K \to 2\pi$ decays. (The isotopic analysis of the $K \to 3\pi$ decays from the phenomenological point of view is given, in particular, in the papers.[78])

First, it is clear that since transitions with $\Delta T = \frac{7}{2}$ do not contribute to the amplitudes for $K \to 2\pi$ decays, the amplitude from such transitions must be zero also in the case of $K \to 3\pi$ decays. This condition leads to the relation

$$\frac{2\gamma(K^+ \to \pi^+\pi^0\pi^0)}{\gamma(K^+ \to \pi^+\pi^+\pi^-)} - \frac{\gamma(K_2^0 \to 3\pi^0)}{3\gamma(K_2^0 \to \pi^+\pi^-\pi^0)} = 0. \qquad (6.10)$$

The transitions with $\Delta T = \frac{5}{2}$ in $K \to 3\pi$ decays can be connected with the ratio of the amplitudes for different modes of decay of a given K meson (K^+ or K^0). According to (6.9) we have

$$\frac{\gamma(K_2^0 \to 3\pi^0)}{3\gamma(K_2^0 \to \pi^+\pi^-\pi^0)} - 1 = \frac{2\gamma(K^+ \to \pi^+\pi^0\pi^0)}{\gamma(K^+ \to \pi^+\pi^+\pi^-)} - 1 = \theta. \qquad (6.11)$$

This relation allows us to obtain from the existing data on $K \to 3\pi$ decays a limit on the possible value of θ, $\theta < 0.05-0.1$, from which it follows, in particular, that the contribution of the interaction with $\Delta T = \frac{5}{2}$ to the amplitude for the decay $K^+ \to \pi^+\pi^0$ does not exceed 10 to 20 percent.

The contribution of transitions with $\Delta T = \frac{3}{2}$ has its strongest effect on the magnitude of the ratio of the slopes of the spectra in the decays $K^+ \to \pi^+\pi^+\pi^-$ and $K^+ \to \pi^+\pi^0\pi^0$ ($K_2^0 \to \pi^+\pi^-\pi^0$):

$$-\frac{\sigma(K^+ \to \pi^+\pi^0\pi^0)}{2\sigma(K^+ \to \pi^+\pi^+\pi^-)} = -\frac{(K\sigma_2^0 \to \pi^+\pi^-\pi^0)}{2\sigma(K^+ \to \pi^+\pi^+\pi^-)} \approx \frac{1+3\delta}{1-6\delta} = 1.56. \qquad (6.12)$$

The $\Delta T = \frac{1}{2}$ rule makes these ratios unity.

Finally, the amplitude with $\Delta T = \frac{1}{2}$ gives the main contribution to the absolute value of the amplitude of any of these processes.

6.2.3. In comparing predictions about violations of the $\Delta T = \frac{1}{2}$ rule with experiment we must keep in mind the uncertainty associated with the inclusion of the electromagnetic mass differences of the π and K mesons.[79] The reason for the uncertainty is that instead of the parameter y in (6.8) one can introduce a different quantity, for example

$$y' = -(1/2m_K)\,[2(p-q_3)^2 - (p-q_1)^2 - (p-q_2)^2]. \qquad (6.13)$$

In the limit of isotopic symmetry the masses of $\pi^\pm$ and π^0 mesons are equal and $y = y'$. If, on the other hand, we take account of the mass splitting, then when y' is reduced to y terms appear in the constant part of the amplitude proportional to the electromagnetic mass differences of pions and kaons. Therefore different forms for writing the matrix elements lead to different predictions for the probabilities. This means that the predictions for the ratio of the quantities γ cannot be tested to an accuracy better than 5 percent. In this connection we note that an uncertainty in testing predictions of the value of the product γ_σ, which is associated with the electromagnetic mass differences of the particles, arises only when terms quadratic in y are included, which have been assumed small in the derivation of (6.9) [see also the discussion after Eq. (6.14)].

In connection with the above-mentioned difficulty in testing the rule $\Delta T = \frac{1}{2}$ with data on decay probabilities, it is of particular interest to compare the ratios (6.12) with experiment, since in this case the predicted effect of a violation of the $\Delta T = \frac{1}{2}$ rule is large. It can be seen from Table II, which gives the experimental data, that at present we cannot exclude the possibility of a 50 percent violation of the rule $\Delta T = \frac{1}{2}$ in the ratios of the slopes of the pion spectra in the different decays.

We emphasize that the prediction (6.12) is based on the assumption that the weak-interaction Hamiltonian contains transitions with $\Delta T = \frac{3}{2}$, which in particular are responsible for the decay $K^+ \to \pi^+\pi^0$. If it turns out that the ratio

$$-\sigma(K^+ \to \pi^+\pi^0\pi^0) / 2\sigma(K^+ \to \pi^+\pi^+\pi^-)$$

is close to unity, this will be a serious argument in favor of an electromagnetic origin of the decay $K^+ \to \pi^+\pi^0$.

Table II also shows comparisons with experiment for the absolute values of γ and σ for one of the decays ($K^+ \to \pi^+\pi^+\pi^-$), where they are most accurately known experimentally. It is seen that there is good agreement between the theoretical predictions and experiment.

6.2.4. This agreement with experiment is in the final analysis the only argument in favor of the validity of the expansion (6.7), which was fundamental to the derivation of all of these results. Regarding this expansion one must keep in mind the following remarks:

a) We have neglected terms quadratic in the momenta of the π mesons. First there are the terms proportional to q^2, which in principle cannot be found from experiment. Since the value of the main terms is $\sim m_K E_\pi$, the error caused by neglecting the dependence on q^2 is in general of the order of $\mu^2/m_K E_\pi \sim \frac{1}{4}$. Second, we have not included terms of second order in the energies of the π mesons. Since we are using the

Table II.

Quantity	Theory		Experiment [80] (January, 1969)
	$\Delta T = 1/2$	$\Delta T > 1/2$	
$10^6\gamma(K^+ \to \pi^+\pi^+\pi^-)$	1,6	1,6	1,92±0,01
$\sigma(K^+ \to \pi^+\pi^+\pi^-)$	1	0,73	0,85±0,04
$\frac{\sigma(K_2^0 \to \pi^+\pi^-\pi^0)}{\sigma(K^+ \to \pi^+\pi^+\pi^+)}$	1	1	0,77±0,12
$-\frac{\sigma(K_2^0 \to \pi^+\pi^-\pi^0)}{\sigma(K^+ \to \pi^+\pi^+\pi^-)}$	1	1,56	1,30±0,12

expansion (6.7) in a region of pion energies ranging from zero to $m_K/2$, it is necessary that the "comparison" mass for the additional power of the energy be much larger than $m_K/2$. The terms quadratic in the energy can in principle be measured experimentally. If we write the matrix element in the form

$$M \sim 1 + \sigma y + \tau y^2, \tag{6.14}$$

then in order that there be no great change in the predictions it is necessary that τ satisfy the relation $\tau \lesssim 0.1$. The available experimental data[80] give only the upper limit $\tau \lesssim 0.5$.

b) In the treatment given in Subsection 6.2.1 it was assumed that the amplitudes for $K \to 2\pi$ and $K \to 3\pi$ decays are real. In particular this means that we neglect the phases $\delta_{0,2}$ of the scattering in states with total isotopic spins 0 and 2 at a total energy equal to the mass of the K meson. Indirect experimental data give $\delta_0 \approx 35°$.[52] If we regard this value of δ_0 as correct, the neglect of the imaginary part of the amplitude is unjustified. We therefore point out that the assumption that δ_0 is small is used only in calculating the amplitudes for transitions with $\Delta T = \frac{1}{2}$ in the decays $K \to 3\pi$. The predictions about the slopes of the spectra follow from the fact that the part of the amplitude with $\Delta T = \frac{1}{2}$ is zero when the momentum of a particular π meson is zero, and their proof does not depend on the assumption that δ_0 is small. In the calculation of the amplitudes with $\Delta T > \frac{1}{2}$ it is assumed that δ_2 is small, not δ_0. Therefore if the expansion (6.7) is valid only for the part of the amplitude with $\Delta T > \frac{1}{2}$ the predictions about the degree of violation of the $\Delta T = \frac{1}{2}$ rule are practically unchanged.

7. THE DECAY $\eta \to 3\pi$

7.1. In the present chapter we consider the amplitude for the decay $\eta \to 3\pi$. It will be shown[81] that in the limit of exact $SU(2) \otimes SU(2)$ symmetry of the strong interactions this decay is forbidden and its matrix element is equal to zero. If we take into account the "semistrong" interaction which breaks the symmetry it is possible to calculate[42,82] the slope of the π-meson spectrum in the decay $\eta \to \pi^+\pi^-\pi^0$, which turns out to be in excellent agreement with experiment.

Conservation of G parity is violated in the decay $\eta \to 3\pi$, and therefore it is assumed that this process goes in second order in the electromagnetic interaction. The matrix element for the decay can then be written in the form

$$M = -\frac{e^{2'}}{2} \langle 3\pi | \int dx D_{\mu\nu}(x) T\{j_\mu'(x) j_\nu(0)\} | \eta\rangle, \tag{7.1}$$

where $D_{\mu\nu}(x)$ is the propagation function of the photon and j_μ is the EM current of the hadrons.

We assume that the amplitude M for the decay $\eta \to \pi^+\pi^-\pi^0$ is a quadratic function of the momenta:

$$M = a + b(pq_0) + d(q_+^2 + q_-^2) + fq_0^2, \tag{7.2}$$

where p, q_+, q_-, q are the respective momenta of the η, π^+, π^-, and π^0 mesons. In the physical region of the decay the expansion (7.2) assumes a linear dependence of the amplitude on the energies of the π mesons. As in the case of the K-meson decays, the assumption about the form of the expansion of the amplitude is extremely important and needs further experimental testing. The difference from the $K \to 3\pi$ decays is that we explicitly take into account a dependence of the amplitude on q^2. At the same time the neglect of the terms quadratic in the energies can be justified (or refuted) experimentally.

We shall show that the amplitude M given by (7.1) is zero in the limit of zero momentum of one of the π mesons (the other particles are on the mass shell). Using a reduction formula, for example for the π^+ meson, and integrating by parts, we get for the limiting value of the amplitude

$$\begin{aligned} M = (ie^2/2) \quad & \langle \pi^-\pi^0 | \int dx\, dy e^{iq+y} D_{\mu\nu}(x) (\Box_y - \mu^2) \\ & \times T\{j_\mu(x) j_\nu(0) \varphi^-(y)\} | \eta\rangle \underset{q_+\to 0}{\longrightarrow} -(ie^2c/2) \langle \pi^-\pi^0 | \\ & \times \int dx D_{\mu\nu}(x) T\{j_\mu(x) [j_\nu(0), A^-(0)] + [j_\mu(x) A^-(x^0)] j_\nu(0)\} | \eta\rangle. \end{aligned} \tag{7.3}$$

The commutator of the axial charge with the electric current, which appears in (7.3), is equal to the axial current [see Eq. (4.12)]. It follows from considerations of G parity that only the product of the commutator and the isoscalar part of the electromagnetic current contributes to the expression (7.3). Therefore the total isospin of the π^- and π^0 mesons must be unity, and this is forbidden by the requirement of Bose statistics, since the orbital angular momentum of the mesons is equal to zero. Therefore the entire amplitude is zero. If the reduction is applied to the π^0 meson the commutator $[j_\nu(0), A^3(0)]$ is itself equal to zero. Accordingly, two relations are obtained for the parameters of the expansion in Eq. (7.2),

$$a + 2d\mu^2 = 0, \quad a + (bm_\eta^2/2) + d\mu^2 + f\mu^2 = 0. \tag{7.4}$$

One further condition on the constants of the expansion can be obtained by considering the amplitude M for zero momentum of the π^0 meson when one of the charged π mesons (for definiteness, the π^-) is off the mass shell. Using the reduction formula for the π^- and π^0 mesons

$$\begin{aligned} M = (e^2/2) \langle \pi^+ | & \int dx\, dy\, dz\, e^{i(q_-y + q_0z)} D_{\mu\nu}(x) \\ & \times (\Box_z - \mu^2)(\Box_y - \mu^2) T\{j_\mu(x) j_\nu(0) \varphi^+(y) \varphi^3(z)\} | \eta\rangle, \end{aligned} \tag{7.5}$$

we find for $q_0 = 0$

$$\begin{aligned} M = -(e^2c^2/\sqrt{2}) \langle \pi^+ | & \int e^{iq_-y} (q_-^2 - \mu^2) D_{\mu\nu}(x) \\ & \times T\{j_\mu(x) j_\nu(0) [\varphi^+(y), A^3(y^0)]\} | \eta\rangle \\ & + \text{terms with the commutator } [j_\mu, A^3] = 0. \end{aligned} \tag{7.6}$$

We now use the assumption that the equal-time commutator of the axial charge with the pion field, $[A^i, \varphi^k]$, is proportional to δ^{ik} (i, k = 1, 2, 3) (see discussion in Sec. 2.2 of Chapter 2). Then $[\varphi^+(y), A^3(y^0)] = 0$, and M = 0 for $q^0 = 0$, $q_+^2 = \mu^2$ and arbitrary q_-^2. It follows from Eq. (7.2) that

$$a = d = 0. \tag{7.7}$$

The relations (7.4) and (7.7) determine the amplitude M to within a numerical factor, and in the physical region for the decay we get

$$M(\eta \to \pi^+\pi^-\pi^0) = f\mu^2 \left(1 - \frac{2pq_0}{m_\eta^2}\right) = \frac{f\mu^2}{3}\left[1 - \frac{2Q}{m_\eta}\left(\frac{3T - Q}{Q}\right)\right], \tag{7.8}$$

where T is the kinetic energy of the π meson and Q is the energy released.

Accordingly, the theoretical value of the slope of the pion spectrum in the decay $\eta \to \pi^+\pi^-\pi^0$ is $\alpha_{theor} = -2Q/m_\eta = -0.49$. The experimental value is $\alpha_{exp} = -0.478 \pm 0.038$.[83]

We note that the expansion does not apply to the pole diagram with an η meson in the intermediate state, which must be treated separately. It can be shown, however, that including it does not change the result. This is because the amplitude for $\pi\eta$ scattering, which is involved in this diagram, is zero for zero pion momentum owing to the Adler selfconsistency condition. Therefore the contribution of the pole term in the physical region is described up to a constant factor by Eq. (7.8) and does not change the prediction as to the slope of the spectrum.

7.2. It can be seen from (7.8) that the amplitude for $\eta \to 3\pi$ decay goes to zero for $\mu^2 = 0$. In this respect the situation in the decays $\eta \to 3\pi$ is different from that in the weak transitions $K \to 3\pi$ (see Chapter 6), where the mass of the π meson was neglected. The decays $\eta \to 3\pi$ are rather to be compared with the strong process $\pi^0\pi^0 \to \pi^0\pi^0$, which also has an amplitude proportional to μ^2.

The decay $\eta \to 3\pi$ is due to symmetry breaking, and therefore provides a possibility of testing the assumption (2.18) about the properties of the symmetry breaking interaction. At present the prediction about the slope of the pion spectrum in the decay $\eta \to \pi^+\pi^-\pi^0$ is the only experimentally tested consequence of this assumption.

8. THE CONSTRUCTION OF INVARIANT AMPLITUDES

This chapter gives a brief exposition of the way consequences of the SU(2)⊗ SU(2) symmetry can be derived by constructing so-called phenomenological, or effective, Lagrangians.[84,85] The word "phenomenological" as applied to a Lagrangian means that in calculating amplitudes on the basis of the Lagrangian one is to be guided by certain simple rules, which in the main reduce to the dropping of diagrams with closed loops. It will be shown that in this way we can derive the predictions for the amplitude of πN scattering which we have previously given (see Sec. 3.2 of Chapter 3).

Although effective Lagrangians do not enable us to obtain new results, an acquaintance with them can be useful for understanding the SU(2)⊗ SU(2) symmetry, and in some cases it enables us to find the answer for an amplitude quickly. The point is that the method for deriving results adopted in the main part of this review is rather formal. In the discussion of effective Lagrangians we get a closer acquaintance with the nonlinear representations and with the ideas which were expounded in Chapter 1.

The first section (Sec. 8.1) of this chapter is partly of a supplementary nature. Here we also give a very simple example of the realization of the symmetry for interacting massless nucleons.[8] In the second section we shall construct and write out explicitly the Lagrangian for interacting π mesons and nucleons which satisfies the requirements of SU(2)⊗ SU(2) symmetry.[86,7,84,85]

8.1. The Case of Massless Nucleons

We shall show that massless nucleons can form a linear representation of the group SU(2) ⊗ SU(2). We take the Lagrangian in the form

$$\mathscr{L} = -i\bar{\psi}\,\hat{\partial}\psi + \varkappa\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma_\mu\psi + \gamma\bar{\psi}\gamma_\mu\gamma_5\psi\bar{\psi}\gamma_5\psi, \tag{8.1}$$

where ψ is a spinor in ordinary and isotopic spaces and κ and γ are constants. The first term in (8.1) corresponds to the free fields, and the second describes the interaction.

The Lagrangian (8.1) is invariant under isotopic transformations

$$\delta\psi = i\tau\delta u\psi, \tag{8.2}$$

where δu is the parameter of an infinitesimal rotation in isotopic space.

We define the increment $\delta_C\psi$ (bracket operation) as follows:

$$\delta_C = \delta_2\delta_1 - \delta_1\delta_2, \tag{8.3}$$

where $\delta_{1,2}$ correspond to transformations with parameters $\delta u_{1,2}$. The fact that we have a group means that $\delta_C\psi$ is some transformation of the group, and we denote its parameter by δu_C.

In particular, for the isotopic group the parameter of the bracket operation of two transformations with parameters δu_1 and δu_2 is

$$\delta u_C = [\delta u_2 \delta u_1], \tag{8.4}$$

which corresponds to the commutation relations for the generators of the group,

$$[V^i, V^k] = ie^{ikl}V^l. \tag{8.5}$$

The Lagrangian (8.1) is also invariant under axial, i.e., parity-changing, transformations with parameter δv:

$$\delta\psi = i\tau\delta v\gamma_5\psi. \tag{8.6}$$

If we introduce left and right helical nucleons ψ_L, ψ_R,

$$\psi_L = (1+\gamma_5)\,\psi/2, \quad \psi_R = (1-\gamma_5)\,\psi/2,$$

then the states $\psi_{L,R}$ transform among themselves:

$$\delta\psi_L = \tau\,[(\delta u + \delta v)/2]\,\psi_L, \quad \delta\psi_R = \tau\,[(\delta u - \delta v)/2]\,\psi_R. \tag{8.7}$$

From a comparison of these relations with (8.2) it is clear that we have obtained two independent groups of left and right isotopic spins defined by the parameters $(\delta u + \delta v)/2$ and $(\delta u - \delta v)/2$. This formulation also expresses the fact that the symmetry group of the Lagrangian is the direct product SU(2) ⊗ SU(2). The states ψ_L and ψ_R form the representations $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ of this group (the numbers in the parentheses denote the magnitudes of the left and right isotopic spins).

8.2. The Effective Lagrangian of the πN and $\pi\pi$ Interactions

Let us now consider the case of real, not massless, nucleons. The transformations (8.6) take a nucleon into a state with a different parity. In the case of a nucleon with mass $m \neq 0$ we cannot construct such a state if there are no other particles, and the symmetry SU(2) ⊗ SU(2) must be violated for free nucleons. It is also easy to verify this directly by calculating the variation of the mass term $m\bar{\psi}\psi$ in the transformations (8.6); this variation is not equal to zero.

We shall assume, as in the main part of the review, that there exist massless mesons and that the axial transformations take a nucleon into a state nucleon plus meson. Let us find out how the nucleon and meson fields must transform in order for a SU(2) ⊗ SU(2) symmetry of the strong interactions to exist.

For an infinitesimal rotation in isotopic space defined by the parameter δu the increments of the fields are given by

$$\delta\psi = i\tau\delta\mathbf{u}\psi, \quad \delta\varphi = -[\delta\mathbf{u}\varphi]. \tag{8.8}$$

For an axial transformation with parameter δv we assume that the increment of the nucleon field is

$$\delta\psi = if_0\tau[\varphi\delta\mathbf{v}]\psi, \tag{8.9}$$

where $\tau[\varphi \times \delta\mathbf{v}]$ is the triple scalar product of the vectors τ, φ, and δv, and f_0 is a constant. We shall determine the transformation law of the pion field φ later from the condition that the group exist.

The action of the commutator of two axial transformations with parameters $\delta\mathbf{v}_1$ and $\delta\mathbf{v}_2$ on a nucleon field must reduce to a vector transformation with the parameter $\delta\mathbf{u} = [\delta\mathbf{v}_1 \times \delta\mathbf{v}_2]$. This condition leads to the following equation for the variation of the meson field:

$$\delta_C\psi = i\tau[\delta\mathbf{v}_2\delta\mathbf{v}_1]\psi = if_0\{[(\delta_2\varphi)\delta\mathbf{v}_1] - [(\delta_1\varphi)\delta\mathbf{v}_2]\}\tau\psi - if_0^2[[\varphi\delta\mathbf{v}_1][\varphi\delta\mathbf{v}_2]]\tau\psi, \tag{8.10}$$

where $(\delta_{1,2}\varphi)$ is the increment of the meson field. It is clear from (8.10) that $\delta\varphi$ must contain a constant part corresponding to a displacement of the field, and terms quadratic in φ:

$$\delta\varphi = \alpha\delta\mathbf{v} + \beta\varphi(\varphi\delta\mathbf{v}) + \gamma\delta\mathbf{v}\varphi^2. \tag{8.11}$$

The constants α, β, γ can be determined from (8.1), and we finally get for $\delta\varphi$ the result

$$\delta\varphi = (\delta\mathbf{v}/2f_0) + f_0[\varphi(\delta\mathbf{v}\varphi) - (\delta\mathbf{v}\varphi^2/2)]. \tag{8.12}$$

By a direct calculation one can verify that the transformations on φ defined by (8.12) themselves satisfy the group property. We note that the choice of the form of the transformations (8.9) and (8.12) is not unique. One can list all of the possibilities by starting from a linear representation of the group, the so-called σ model.[7] However, the various forms of the transformation reduce to each other by canonical transformations and are physically equivalent.[87]

A distinguishing feature of the transformations (8.9) and (8.12) is that they are nonlinear. For example, in the expression in (8.12) for the increment of the π-meson field the displacement $f_0\delta\mathbf{v}/2$ corresponds to a mixing of the pion field with the vacuum under axial transformations, and the terms quadratic in φ correspond to mixing with two-pion states. Owing to the nonlinearity of the transformations derivatives of the fields transform differently from the fields themselves, and therefore the various invariants of the group SU(2) ⊗ SU(2) are characterized by the numbers of derivatives they involve.

Differentiating (8.12), we get the transformation law for the derivative of the meson field

$$\delta(\partial_\mu\varphi) = f_0[\partial_\mu\varphi(\delta\mathbf{v}\varphi) + \varphi(\partial_\mu\varphi\delta\mathbf{v}) - \delta\mathbf{v}(\partial_\mu\varphi\varphi)]. \tag{8.13}$$

Instead of $\partial_\mu\varphi$ it is convenient to introduce the quantity

$$\varphi_\mu = \partial_\mu\varphi/(1 + f_0^2\varphi^2), \tag{8.14}$$

whose increment for axial transformations is given by

$$\delta\varphi_\mu = f_0[[\delta\mathbf{v}\varphi]\varphi_\mu]. \tag{8.15}$$

This transformation, like the transformation (8.9), is analogous to an isotopic rotation with the parameter $\delta\mathbf{u} = f_0[\varphi\delta\mathbf{v}]$. Therefore it is clear that if we do not include higher derivatives in our treatment invariant combinations of ψ and φ_μ can be constructed in precisely the same way as in the case of isotopic symmetry.

In particular, the product $\varphi_\mu\varphi_\mu$ is an invariant of the group SU(2) ⊗ SU(2), and the phenomenological Lagrangian for processes involving π mesons only can be represented in the form

$$\mathscr{L}^\pi = \varphi_\mu\varphi_\mu/2 = \frac{(\partial_\mu\varphi)^2}{2(1+f_0^2\varphi^2)^2} = (\partial_\mu\varphi)^2[1 - 2f_0^2\varphi^2 + 3f_0^4\varphi^4 + \ldots]/2. \tag{8.16}$$

The first term of the series corresponds to the kinetic energy of the π meson, the second to ππ scattering, the third to the process 2π → 4π, and so on.

In the approximation for the amplitudes which is quadratic in the momenta of the particles the Lagrangian $\mathscr{L}^\pi$ is uniquely determined. On the other hand, if for example we allow fourth-order terms in the pion momenta, then we can construct additional invariants $(\varphi_\mu\varphi_\mu)(\varphi_\nu\varphi_\nu)$ and $(\varphi_\mu\varphi_\nu)(\varphi_\mu\varphi_\nu)$, which describe processes with four or more mesons.

In order to derive relations for the amplitudes of various processes, starting from the Lagrangian (8.16), we must take into account all perturbation-theory diagrams in lowest order in the constant f_0. It is easily verified that these are contact diagrams, in which all the mesons are emitted from a single point, and diagrams with π-meson poles. Diagrams with a larger number of π mesons in the intermediate state are quantities of higher order in f_0. Since the relations obtained follow from SU(2) ⊗ SU(2) symmetry alone, the fact that lowest-order perturbation theory is used is unimportant. The predictions for the ratios of amplitudes of various processes remain valid in arbitrary order in f_0.

At first glance such a recipe for calculating amplitudes may seem unusual and unconvincing. Therefore it is helpful to point out the extremely close analogy with the calculation of the amplitudes for radiative processes. In order to find the amplitudes for radiative processes in the zeroth-order approximation in the photon momentum, we can proceed in the following way. We write out the Lagrangian of the strong interactions, which is the sum of the free terms, the vertex parts, and the amplitudes of all possible processes involving physical constants. We get the effective Lagrangian for radiative processes if we replace ∂_μ by $\partial_\mu \pm ieA_\mu$ in this expression. To calculate the amplitude for a radiative process it is necessary to take the sum of contact and pole diagrams. It is clear that it is not necessary to iterate the strong-interaction Lagrangian, and the entire procedure has no relation to the question of constructing a renormalized theory of the strong interactions.

The justification of this sort of recipe for calculating radiative processes is the conservation of the electromagnetic current, or the Ward identity for these processes. The proof is given in Sec. 4.1 of Chapter 4 for the case of photoproduction of π mesons. The recipe described above for calculating the amplitudes of processes involving soft pions can be justified by means of relations analogous to Ward's identity, as has been done in the main part of this review. Accordingly, the two methods are equivalent, and the choice between them is determined by considerations of convenience.

As an illustration of the method of invariants we shall reproduce the results derived in Chapter 2 for the amplitude for πN scattering. For this purpose let us find out what invariant of the group describes the isotopically odd part of the amplitude for πN scattering. In the approximation linear in the pion momentum the nonpole part of the amplitude is of the form

$$c^{-}\bar{\psi}\gamma_\mu \boldsymbol{\tau}[\partial_\mu \boldsymbol{\varphi}\boldsymbol{\varphi}]\psi, \tag{8.17}$$

where c^- is the constant introduced in (3.36).

We calculate the variation of (8.17) under the transformations (8.9), (8.12), (8.13) and confine ourselves to the lowest order in f_0:

$$\delta\{c^{-}\bar{\psi}\gamma_\mu \boldsymbol{\tau}[\partial_\mu \boldsymbol{\varphi}\boldsymbol{\varphi}]\Psi\} = -(c^{-}/2f_0)\bar{\psi}\gamma_\mu \boldsymbol{\tau}[\delta \mathbf{v}\,\partial_\mu \boldsymbol{\varphi}]\psi. \tag{8.18}$$

This change of the amplitude can be compensated by the variation of the kinetic term in the free Lagrangian

$$\delta[i\bar{\psi}\gamma_\mu\partial_\mu\psi] = f_0\bar{\psi}\gamma_\mu \boldsymbol{\tau}[\delta \mathbf{v}\,\partial_\mu \boldsymbol{\varphi}]\psi. \tag{8.19}$$

By comparing (8.18) and (8.19) we find that c^- and f_0 must obey the condition

$$c^- = 2f_0^2, \tag{8.20}$$

which is the same as the result (3.29) if

$$f_0 = c/\sqrt{2}. \tag{8.21}$$

Equation (8.21) follows from the explicit form of the axial current corresponding to the Lagrangian $\mathscr{L}^{\pi}$. We shall not go into this in more detail. The vanishing of the nonpole part of the isotopically odd part of the amplitude for πN scattering follows from the fact that the quantity $\bar{\psi}\psi\varphi^2$ cannot be made into an invariant.

There are also pole diagrams that contribute to the amplitude for πN scattering. To calculate then we must write the vertex of the πNN interaction in invariant form. It is clear that the combination

$$-f\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\boldsymbol{\varphi}_\mu\psi \tag{8.22}$$

does not change under the transformations of the group. This can be seen from the fact already mentioned that the axial transformations of the quantities ψ and φ_μ are formally the same as the isotopic transformations if we take $\delta u = f_0[\varphi\delta v]$. The first term of the expansion of (8.22) in a series in f_0 gives the pseudovector πNN coupling. The next term of the expansion corresponds to the contact diagram for the process $\pi N \rightarrow 2\pi N$.

We give the final result for the Lagrangian of pions and nucleons, which is invariant with respect to the group SU(2) ⊗ SU(2) (everywhere except in $\mathscr{L}^{\pi}$ we confine ourselves to the first-order terms in the pion momenta):

$$\mathscr{L} = \mathscr{L}^{\pi} + \mathscr{L}^{N} + \mathscr{L}^{\pi N} = (\boldsymbol{\varphi}_\mu^2/2) - \bar{\psi}(-i\hat{\partial} + m)\psi$$
$$- f\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\boldsymbol{\varphi}_\mu\psi - f_0^2\bar{\psi}\gamma_\mu\boldsymbol{\tau}[\boldsymbol{\varphi}\boldsymbol{\varphi}_\mu]\psi.$$

9. CONCLUSION

In Chapter 1 we formulated four hypotheses whose consequences are considered in the present review. In conclusion we shall list the main predictions whose agreement with experiment sustains our belief in the correctness of these hypotheses.

As for the extrapolation formulas (Hypothesis IV), they are used in the derivation of all the predictions, and we shall not discuss this. In any case it is desirable (and in principle possible) to test the extrapolation formulas independently of the other hypotheses. At present such a test can be made in the case of πN scattering (see the sum rules for the p-wave amplitudes in Sec. 3.2 of Chapter 3), and partially in the case of photoproduction of π mesons at threshold (see the predictions for the zeroth-order terms in the photon momentum in Sec. 4.1 of Chapter 4).

Now, the main results are the following:

1) the Goldberger-Treiman relation (see Sec. 2.1 of Chapter 2); this is based on Hypotheses I and II.

2) The smallness of the isotopically even πN scattering length confirms the Adler selfconsistency condition, which follows from Hypothesis I (see Secs. 3.1–3.3 of Chapter 3).

3) The prediction (3.33) for the value of the isotopically odd πN scattering length, which is equivalent to the Adler-Weissberg relation (see Sec. 3.2 of Chapter 3); this is based on Hypotheses I and II.

One can also describe, with various degrees of agreement with experiment, the decays K_{l3}, K_{e4} (Chapter 5), $K \rightarrow 3\pi$ (Sec. 6.2 of Chapter 6), and $\eta \rightarrow 3\pi$ (Chapter 7). Here definite assumptions (Hypothesis III) are used about the transformation properties of the interaction Hamiltonians responsible for the decays with respect to the group SU(2) ⊗ SU(2). For a final elucidation of the degree of agreement between theory and experiment in the case of these decays it is very important to get more accurate forms of the matrix elements in the physical region in order to test the extrapolation formulas.

We see that the number of verified results is small. Let us also list the predictions that have been derived in this review and which cannot be well tested at present because the experimental data are inadequate:

1. The relation (2.3) between the axial form-factors of the nucleon, which can be tested in the reaction $\nu + N \rightarrow N' + l$ at large values of the momentum transferred to the lepton.

2. The Adler relation (2.22) for inelastic neutrino reactions.

3. The generalized Goldberger-Treiman relation for inelastic neutrino reactions, Eq. (2.23), which holds for small momentum transfer to the leptons.

4. The relation (2.27) between the size of the effective pseudoscalar constant in μ capture by protons and the radius of the axial form-factor of the nucleon.

Agreement of these four predictions with experiment would allow us to be finally convinced of the existence of axial currents which are conserved (in the limit $\mu^2 = 0$) (Hypotheses I and II). The assumption (1.4) about the form of the commutator of the axial charges and the hypothesis (1.16) about the properties of the "semistrong" interaction could be tested in the study of $\pi\pi$ scattering.

5. The predictions for the low-energy parameters of scattering—the scattering lengths of the s and p waves and the radii of the s waves, calculated in Sec. 3.3 of Chapter 3. In the derivation of these relations the further assumption was made that the scattering lengths are small.

The transformation properties of the electromagnetic current of the hadrons and the weak interaction Hamiltonian (leptonic and nonleptonic) can be elucidated by comparing with experiment the predictions listed as points 6.—10.

6. The relation (4.15) between the form-factors V_1^+ and V_1^-, which describe the production of charged π mesons from nucleons, The definition of the quantities $V_1^\pm$ is given in Eq. (4.10) and the text preceding it.

7. The relation (4.25) between the amplitudes for photoproduction and electroproduction of π mesons at threshold, the electric radii of the π meson and of nucleons, and the radius of the axial form-factor.

8. The relation (4.26) for the amplitude for electroproduction of π mesons for large values, of the order of GeV/c, of the momentum transferred to the electron and small relative momentum of the nucleon and the meson. The relation connects the amplitude for electroproduction with the electromagnetic and axial form-factors of the nucleon.

9. The relation (5.9) for the form-factors which describe the decay $K \to \pi\mu\nu$. The definition of the form-factors is given in Eqs. (5.1) and (5.3).

10. The prediction (6.12) for the amount of deviation from the $\Delta T = \frac{1}{2}$ rule in decays $K \to 3\pi$ (see also Table II). In the derivation of these predictions the additional assumption was made that the decay $K^+ \to \pi^+\pi^0$ is due to a bare weak nonleptonic interaction with $\Delta T = \frac{3}{2}$.

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Note added in proof (December 26, 1969). A consistent use of the requirements of crossing symmetry allows us to derive low-energy theorems on the cross sections for photoproduction of π mesons from nucleons with better accuracy than appears from the exposition of Chapter 4. We shall list these theorems (all of these results are derived in the limit of zero three-momentum of the π meson):

1) The ratio of the cross sections for production of π^- and π^+ mesons is given by

$$\frac{d\sigma}{d\Omega}(\gamma n \to p\pi^-) \Big/ \frac{d\sigma}{d\Omega}(\gamma p \to n\pi^+) = 1 + \frac{2\mu}{m} + \frac{\mu^2}{m^2}(1 - k^p - k^n) + O(\mu^3) = 1{,}32.$$

Since the neglected terms, $0(\mu^3)$, are small quantities of high order, this relation should hold with very good accuracy, about 1 percent.

2) The cross section for production of π^0 mesons from protons is given by

$$\frac{k}{|\mathbf{q}|}\frac{d\sigma}{d\Omega}(\gamma p \to p\pi^0) = \frac{\alpha}{4\pi}\frac{f^2\mu^2}{(m+\mu)^2}\left[1 - \frac{\mu}{m}(k^p + 1) + O(\mu^2)\right] = 0{,}11\ \frac{\text{мк барн}}{\text{стер}}.$$

Comparison of this with Eq. (4.23) shows that inclusion of the terms linear in μ is important.

The cross section for production of π^0 mesons from neutrons is

$$\frac{k}{|\mathbf{q}|}\frac{d\sigma}{d\Omega}(\gamma n \to n\pi^0) = \frac{\alpha}{4\pi}\frac{f^2\mu^2}{(m+\mu)^2}\left(k^n\ \frac{\mu}{2m}\right)^2(1 + O(\mu)) = 0{,}003\ \frac{\text{мк барн}}{\text{стер}}.$$

The low-energy theorem for the electroproduction process can be formulated for the cross section, and not for the amplitude only, as was done in Subsection 4.2.2. Namely, we get the following predictions for the squares of the matrix elements of the interaction of transverse and longitudinal quanta; for $k^2 \sim \mu^2$ they should hold to about 1 percent accuracy.

$$\lim_{\mathbf{q}\to 0} |M_T|^2 = |M_\gamma|^2\left[1 + 2k^2\left(\frac{g'}{g_A} - \frac{k^n}{2m^2} - \frac{1}{8m^2}\right)\right],$$

$$\lim_{\mathbf{q}\to 0} |M_L|^2 = 8m^2 f^2\left[\frac{k_0(2\mu - k_0)}{2\mu k_0 - k^2} - \frac{\mu}{2m} + \right.$$
$$\left. + \frac{\mu^2}{4m^2}(1 + k^p - k^n) + \frac{\mu^2 - k^2}{2\mu^2 - k^2}\left(2\mu^2 F'_\pi - \frac{\mu^2 - k^2}{8m^2}\right) + \mu^2\left(\frac{g'}{g_A} + \gamma\right) + \frac{\mu^2 - k^2}{8m^2}\right]^2,$$

where $k_0 = \mu - (\mu^2 - k^2)/2(m + \mu)$, $g' = dg(k^2)/dk^2|_{k^2 = 0}$, and the quantities $|M_\gamma|^2$ and γ can be expressed in terms of the experimental value of the threshold cross section for photoproduction of π^+ mesons:

$$\frac{k}{|\mathbf{q}|}\frac{d\sigma}{d\Omega}(\gamma p \to n\pi^+) = \frac{\alpha}{32\pi}\frac{1}{(m+\mu)^2}|M_\gamma|^2,$$

$$|M_\gamma|^2 = 16 f^2 m^2\left[1 - \frac{\mu}{m} + \frac{\mu^2}{2m^2}(k^n + k^p + 2) + 2\mu^2\gamma\right].$$

The quantities $|M_T|^2$ and $|M_L|^2$ written out above determine the cross sections for the reactions $ep \to e'n\pi^+$ and $\pi^- p \to ne^+e^-$. For example, for the cross section for the electroproduction reaction we have

$$\frac{(-k^2)}{|\mathbf{q}|}\frac{d\sigma}{d\Omega_2\, d\varepsilon_2}(ep \to e'n\pi^+)$$
$$= \frac{\alpha}{32\pi^2}\frac{\varepsilon_2}{\varepsilon_1}\frac{1}{m\sqrt{m^2 + 2m(\varepsilon_2 - \varepsilon_1) + k^2}}\left[|M_T|^2 f_1(\varepsilon_1, \varepsilon_2, \theta) + |M_L|^2 f_2(\varepsilon_1, \varepsilon_2, \theta)\right],$$

where

$$f_1 = \frac{4\varepsilon_1\varepsilon_2\cos^2\left(\frac{\theta}{2}\right)}{(\varepsilon_1 - \varepsilon_2)^2 - k^2} + 2,$$

$$f_2 = \frac{4\varepsilon_1\varepsilon_2\cos^2\left(\frac{\theta}{2}\right)}{(\varepsilon_1 - \varepsilon_2)^2 - k^2}\,\frac{(-2k^2)}{(\varepsilon_1 - \varepsilon_2)^2}, \quad -k^2 = 4\varepsilon_1\varepsilon_2\sin^2\left(\frac{\theta}{2}\right)$$

(ϵ_1, ϵ_2 are the energies of the initial and final electrons, θ is the angle of scattering of the electron, and **q** is the momentum of the π meson). All quantities referring to electrons are given in the laboratory coordinate system, and those characterizing hadrons are in the c.m.s. of the final nucleon and π meson.

A more detailed exposition of these questions can be found in a paper by the present writers which has been sent to press under the title of "Low-energy Theorems for the Amplitudes for μ Capture and the Electromagnetic Production of Pions."

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