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THEORETICAL FOUNDATIONS OF THE CPT THEOREM

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THE discrete symmetries play an important role in elementary particle physics, since one can generally connect them with additional conservation laws. These are first of all the symmetry with respect to reflections of space P, charge conjugation C and time reversal T. A little more than ten years ago there were hardly any doubts that the invariance of the theory under these transformations reflect strict laws of nature. Then we became witnesses to an experimental overthrow of this viewpoint: it turned out that parity P, charge-conjugation parity C and combined parity CP are not conserved in a series of processes. It is true that until now we do not know the profounder reasons for such an asymmetry of the interactions of elementary particles. Nevertheless such a turn of events has induced in some physicists an impatient desire to "finish off" with another symmetry of the discrete type, which has much stronger foundations. We have in mind CPT-invariance. In this connection we should like from the very beginning to point out the distinguished role of CPT-invariance. First of all this role consists in the fact that a strong reflection (or CPT-transformation) is not an ordinary unitary transformation. Indeed, the action of the CPT (also called Θ or R) operator on an in-state maps it into an out-state, so that one can speak about eigenvalues of the CPT-operator (or CPT-parity) only in a conventional way.* Such nonunitary operators were introduced by Wigner and later by Schwinger.[1, 2]

Another essential distinction is the relation of CPT-invariance to the locality properties of fields and the character of commutation relations among fields.† Consequently, a violation of CPT will affect to a larger or lesser extent problems of locality of the theory and the relation between spin and statistics. Therefore, before one subjects the CPT-symmetry to any doubts it is desirable once more to remember the basic assumptions lying at its foundation. This is also necessary because one often hears directly contradicting statements about CPT-invariance. On the one hand (e.g., Matthews[3]) it is stated that "in a Lorentz-invariant theory the invariance under strong reflections (CPT) does not lead to any additional restrictions or selection rules, if one assumes normal connection between spin and statistics." On the other hand, the opinion is widespread that a violation of CPT-invariance signifies violation of local commutativity or microcausality.

The development of the axiomatic method has allowed one to prove the CPT theorem from a very general position and to show[4, 5] that CPT-invariance is equivalent to weak local commutativity (WLC)* and therefore a theory may be CPT-invariant but not microcausal (cf. Example 1 at the end of this article). Thus the CPT-invariance observed in nature is not a strong support to the hypothesis of local commutativity.

It is interesting to compare this result of the axiomatic approach with the usual proof of CPT-invariance (the Pauli–Lüders theorem[6]) which is based on the existence of a local Lagrangian or Hamiltonian.

The purposes of the present report are:

First, to expose (without rigorous reasoning) the CPT theorem in the axiomatic approach.

Second, to compare this proof with the usual treatment of the CPT theorem in the Lagrangian method.

Third, to show how the CPT theorem leads to the relation between spin and statistics

Fourth, to list a series of examples and to discuss briefly the possibility of extending CPT and other symmetries to unstable particles.

This report is not intended as a review of work on the CPT theorem.

Among the numerous aspects of the CPT problem we are naturally able to throw light only on the most important ones, from our point of view. (Therefore this talk can be considered to a certain degree as a popular exposition of some facts which are well known in a narrow circle of specialists–axiomatists.)

I. THE CPT THEOREM IN THE AXIOMATIC APPROACH

For simplicity we first consider a neutral scalar field, and then we formulate the theorem in its general form.

We assume the following axioms of field theory to be valid.

The relativistic quantum postulate: Every state is described by a unit vector of the Hilbert space $\mathcal{H}$. The relativistic transformation law is given by the continuous unitary representation of the inhomogeneous Lorentz group (Poincaré group):† $\{a, \Lambda\} \to U(a, \Lambda)$. The unitary representation $U(a, 1)$ can be written in the form

$$= \exp(iP^{\mu}a_{\mu}),$$

where P is a Hermitean operator interpreted as the energy-momentum of the theory. $P^{\mu}P_{\mu} = m^2$ is the

* Thus, if $|\alpha_{in}, r>$ is the in-state of the particle α with spin r, $\Theta|\alpha_{in}, r> = <\alpha_{out}$, i.e., transforms into the out-state of the antiparticle with spin −r.

† More correctly, CPT-invariance reflects the propert y of weak local commutativity (cf. infra).

* *The most recent investigations, in particular, the abandonment of* so-called localizable theories destroy this connection: a theory may be CPT-invariant and nonlocalizable. In such a theory MLC loses its meaning, in general.

† $x \to x' = \Lambda x + a$, Λ is a Lorentz transformation, a is a four-vector.

mass operator. The eigenvalues of P_μ are in the forward light-cone. There exists a unique invariant state $|0\rangle$ or Φ_0:

$$U(a, \Lambda)|0\rangle = |0\rangle.$$

The postulate of existence of local operators $\varphi(x)$ defined on $\mathscr{H}^{*\prime}$—i.e., of renormalized Heisenberg operators.

The postulate of Poincaré invariance: under a transformation $\{a, \Lambda\}$ of the Poincaré group $\varphi(x)$ transforms according to the law

$$U(a, \Lambda)\varphi(x)U^{-1}(a, \Lambda) = \varphi(\Lambda x + a).$$

We do not assume the postulate of local commutativity.

If the theory under consideration exhibits symmetry (invariance) under discrete transformations these can be represented by means of unitary operators. Thus, if there is invariance under space reflection and charge conjugation, then there exist the unitary operators U(P) and U(C) such that

$$U(P)\varphi(x)U^{-1}(P) = \varphi(Px) = \varphi(-\mathbf{x}, x_0), \quad U(C)\varphi(x)U^{-1}(C) = \varphi^*(x) = \varphi(x$$

(for a neutral scalar field). These operators are defined up to an arbitrary phase $(U(P)|0\rangle = e^{i\alpha}|0\rangle)$. In the presence of CPT invariance there exists a "unitary" operator $U(CPT) \equiv \Theta$ such that†

$$(\Theta\varphi(x)\Theta^{-1})^T = \varphi(-x),$$
$$\Theta\Psi = \Psi'^*, \quad \Psi^*\Theta^{-1} = \Psi'. \qquad (1)$$

In axiomatic field theory all properties are formulated in terms of vacuum expectation values of simple products (Wightman functions) or of T- and R-products (Lehmann–Symanzik–Zimmermann,[7] Bogolyubov, Medvedev, Polivanov[8]).

If the theory is CPT invariant then in the language of Wightman functions $W(x_2, \ldots, x_n)$ this means that

$$W(x_1, \ldots, x_n) \equiv \langle 0|\varphi(x_1)\ldots\varphi(x_n)|0\rangle = \langle 0|\varphi(-x_n)\ldots\varphi(-x_1)|0\rangle, \qquad (2)$$

an immediate consequence of (1).

The requirements of P-, C-, Cr- etc. invariance can be written in similar form.

The CPT theorem states: Let $\varphi(x)$ be a Hermitian scalar field satisfying the listed axioms. If the condition (2) is satisfied for all x_i then at Jost points the condition of weak local commutativity is satisfied, i.e.,

$$\langle 0|\varphi(x_1)\ldots\varphi(x_n)|0\rangle = \langle 0|\varphi(x_n)\ldots\varphi(x_1)|0\rangle. \qquad (3)$$

Conversely, if the WLC condition (3) is satisfied in a (real) neighborhood of a Jost point, the CPT-invariance condition (2) is satisfied everywhere. In other words, CPT-invariance is a necessary and sufficient condition for WLC.

Since local commutativity implies WLC, any field theory of a local Hermitian field exhibits CPT-symmetry.

Definition. The set of vectors

$$\xi_1 = x_1 - x_2, \ldots, \xi_{n-1} = x_{n-1} - x_n$$

is called a Jost point if for arbitrary $\lambda_i \geq 0$

$$\left(\sum_{i=1}^{n-1} \lambda_i \xi_i\right)^2 < 0, \quad \sum_{i=1}^{n-1} \lambda_i \neq 0. \qquad (4)$$

It is obvious that the Jost points do not exhaust the whole region where $\xi_i^2 < 0$, i.e., where the vectors are spacelike, but at any Jost point all $\xi_i^2 < 0$. We do not reproduce here the proof of the theorem (cf. [4,5]). We only note that the proof is essentially based on the analytic properties of $W(x_1, \ldots, x_n)$ which are consequences of the positivity of the spectrum of P^0 and Poincaré invariance. (The scheme of the proof: if (2) is valid, it is also valid at Jost points, but the latter are points of holomorphy of W, so that the signs of all can be changed, leading to the WLC condition (3).)

For the case of fields with arbitrary spins the existence of the operator Θ or CPT-invariance is expressed in the form of the identities (for all Wightman functions)

$$\langle 0|\varphi_1(x_1)\ldots\varphi_n(x_n)|0\rangle = (-1)^J i^F \langle 0|\varphi_n(-x_n)\ldots\varphi_1(-x_1)|0\rangle, \qquad (5)$$

where F is the number of fields of half-integral spin, and J is the total number of undotted spinor indices. In a more familiar notation, for particles of spin $\frac{1}{2}$ the action of Θ is defined as:

$$(\Theta\psi_\alpha(x)\Theta^{-1})^T = i\gamma_5\psi_\alpha(-x). \qquad (6)$$

Then in the language of Wightman functions the CPT-invariance condition will be

$$\langle 0|\psi_{\alpha_1}(x_1)\ldots\psi_{\alpha_n}(x_n)|0\rangle = i^n(\gamma_5)_{\alpha_1\alpha_1'}\ldots(\gamma_5)_{\alpha_n\alpha_n'}\langle 0|\psi_{\alpha_n'}(-x_n)\ldots\psi_{\alpha_1'}(-x_1)|0\rangle \qquad (7)$$

etc.

We now discuss the action of Θ on the operators $\varphi_{\text{in, out}}$ and what properties of the observables are consequences of CPT.

If one assumes a linear relation between $\varphi(x)$ and $\varphi_{\text{in, out}}$ (the Yang–Feldman equation*), then

$$\varphi_{\text{in}} = \varphi(x) + \int \Delta^R(x - x', m)\, j(x')\, d^4x', \qquad (8)$$

$$\varphi_{\text{out}} = \varphi(x) + \int \Delta^A(x - x', m)\, j(x')\, d^4x',$$
$$(j(x) \equiv (\Box - m^2)\varphi(x)). \qquad (8')$$

Applying the operator Θ we have

$$(\Theta\varphi_{\text{in}}(x)\Theta^{-1})^T = \varphi(-x) + \int \Delta^A(-x - x', m)\, j(x')\, d^4x'. \qquad (9)$$

Comparing (8′) and (9) we find

$$(\Theta\varphi_{\text{in}}(x)\Theta^{-1})^T = \varphi_{\text{out}}(-x). \qquad (10)$$

On the other hand, since the fields $\varphi_{\text{in}}(x)$ are weakly local with respect to each other, there exists a "unitary" operator V, such that

*Strictly speaking, one should talk about smeared-out operators $\varphi_f = \int\varphi(x)f(x)dx$ where the thest function f is smooth, since the action of $\varphi(x)$ on Ψ in $\mathcal{H}$ leads to a nonnormalizable vector.

†We explain the properties of the operator Θ: if $\Psi = \int f(x_1) - f(x_n)\varphi(x_1) - \varphi(x_n)(d_x)|O\rangle$ then $\Theta\Psi = \Psi^{\prime x} < O|\int f(-x_1) - f(-x_n)\varphi(x_1) - \varphi(x_n)(d_x)$ consenquently $\Theta(\alpha\Psi + \beta\phi) = \alpha\Psi_{+\beta\psi}{}^{\prime x}$. The definition of Θ as an antiunitary operator acting in the same Hilbert space can be found in the book [5] Sec. 3.5.

*In the general case the asymptotic fields are introduced on the basis of the Haag-Ruelle scattering theory [4]. However, until now there is no rigorous proof of (10) without use of the locality axiom, for the case of a nonlinear relation between φ in and $\varphi(x)$.

$$(V\varphi_{\mathrm{in}}(x)V^{-1})^T=\varphi_{\mathrm{in}}(-x); \tag{11}$$

and similar for $\varphi_{\mathrm{out}}(x)$.

It follows from (10) and (11) that

$$\varphi_{\mathrm{out}}(x)=(\Theta(V\varphi_{\mathrm{in}}(x)V^{-1})^T\Theta^{-1})^T. \tag{12}$$

The product of the two "unitary" operators $\Theta^{-1}V^T$ is the (genuinely) unitary operator

$$S=V^{-1}\Theta^T=(V^{-1})^T\Theta, \tag{13}$$

which coincides with the scattering matrix S, since

$$\varphi_{\mathrm{out}}=S^*\varphi_{\mathrm{in}}S,\quad S^*S=1. \tag{14}$$

The relation (13) is extremely useful for the sequel. It is easy to check that (13) implies

$$(\Theta S\Theta^{-1})^T=S. \tag{15}$$

The relation (15) allows one to derive the symmetry properties of the S-matrix implied by the CPT-invariance of the theory. For an arbitrary matrix element we have

$$(\Phi, S\Psi)=(\Psi', S\Phi'),\quad \Phi'=\Phi^*\Theta^{-1},\quad \Psi'^*=\Theta\Psi.$$

Considering that

$$\Theta|\alpha,\ \mathrm{in}\rangle=\langle\bar{\alpha},\ \mathrm{out}|, \tag{16}$$

it follows

$$\langle\alpha,\ \mathrm{in}|S|\beta,\ \mathrm{in}\rangle=\langle\bar{\beta},\ \mathrm{out}|S|\bar{\alpha},\ \mathrm{out}\rangle=\langle\bar{\beta},\ \mathrm{in}|S|\bar{\alpha},\ \mathrm{in}\rangle, \tag{17}$$

since $S|\mathrm{out}\rangle = |\mathrm{in}\rangle$. Here $|\bar{\alpha}\rangle$ denotes the corresponding antiparticle state with reversed spins.

It is interesting to consider the relations (17) in terms of energy variables. For neutral scalar particles the S-matrix can be expanded into normal-ordered products of in- or out-operators:

$$\begin{aligned} S&=\sum_n\frac{1}{n!}\int S_n(x_1,\ \ldots,\ x_n):\varphi_{\mathrm{in}}(x_1)\ldots\varphi_{\mathrm{in}}(x_n):(dx) \\ &=\sum_n\frac{1}{n!}\int S_n(x_1,\ \ldots,x_n):\varphi_{\mathrm{out}}(x_1)\ldots\varphi_{\mathrm{out}}(x_n):(dx).\end{aligned} \tag{18}$$

We assume normal statistics for the in- and out-fields. This does not constitute an additional assumption and follows from the definitions (8), (8′) and the relations (10).

From (10), (15), (18) we obtain in the momentum representation on the mass shell

$$\begin{gathered} S_n(p_1,\ \ldots,\ p_n)=S_n(-p_1,\ \ldots,\ -p_n), \\ p_i^2=m^2;\quad p_{i0}=+(\mathbf{p}_i^2+m^2)^{1/2},\quad p_j^2=m^2;\quad p_{j0}=-(\mathbf{p}_j^2+m^2)^{1/2}, \\ 0\leqslant i\leqslant\nu,\quad \nu<j\leqslant n. \end{gathered} \tag{19}$$

It is characteristic that even for a neutral scalar field this relation is not a simple consequence of Lorentz invariance (as might seem at a first glance). It also requires certain analytic properties of the S_n (cf. Example 2 at the end of this article).

A series of examples of interactions of the nonlocal type without WLC confirm the nontrivial character of (19), which reflects the CPT-invariance of the scalar theory (cf. Examples 1 and 2).

Similar reasoning applied to a spinor field of spin ½, with the assumption that

$$\left.\begin{aligned}(\Theta\Psi_{\mathrm{in}}(x)\Theta^{-1})^T&=i\gamma_5\psi_{\mathrm{out}}(-x), \\ (\Theta\overline{\Psi}_{\mathrm{in}}(x)\Theta^{-1})^T&=i\overline{\psi}_{\mathrm{out}}(-x)\gamma_5,\end{aligned}\right\} \tag{20}$$

lead, for instance, in the case of the matrix element (with state vectors on both sides omitted) of the interaction of scalar fields with a fermion to a relation of the form

$$S_n(p_1,\ p_2;\ k_1\ldots k_n)=\gamma_5S_n(-p_1,\ -p_2;\ -k_1,\ \ldots,\ -k_n)\gamma_5; \tag{21}$$

where p_1 and p_2 are the four-momenta of the incident and outgoing fermions, and k_i are the boson four-momenta ($i = 1, \ldots, n$).

This relation is clearly satisfied in a theory with local commutativity and normal commutation relations for the in-fields. If WLC or other postulates are violated it does not in general hold.

We stress the fact that in the derivation of (19) and (21) we have nowhere made use of the postulate of local commutativity or any concrete form of the S-matrix in the Lagrangian method. This was a purely axiomatic reasoning.

II. THE USUAL PROOF OF THE CPT THEOREM

We briefly recall the usual reasoning used in proving the CPT theorem in the Lagrangian (or Hamiltonian) approach, and make some critical remarks in this connection.

For example in Pauli's article[6] the fundamental assumptions are formulated as follows:

a) there exists a relation between spin and statistics which follows from CPT-invariance;

b) the Lagrangian is invariant under proper Lorentz transformations;

c) the equations have a local character, i.e., all quantities are spinors or tensors of finite rank and involve only derivatives of finite order;

d) kinematically independent spinor fields anticommute; then, assuming a definite form of the transformation law of spinors under CPT and antisymmetrizing all products with respect to all permutations of spinors and symmetrizing with respect to boson fields one can show that $\mathscr{L}$ remains invariant under a strong reflection.

The principal assumption which restricts the usualness of this proof is the assumption that a local Lagrangian exists in the Heisenberg picture.

It follows from the preceding section that the locality of the interaction is not necessary for the validity of the CPT theorem.

In addition, it is impossible to define in a mathematically rigorous manner the concept of an $\mathscr{L}(x)$ in the Heisenberg picture.

Insofar as the Schrödinger equation in the interaction picture is concerned, Haag's theorem shows us that in relativistic quantum field theory such a picture exists only in the case when there is no interaction between the particles!

A more acceptable proof of the CPT theorem can be obtained starting from a representation of the renormalized S-matrix in the form of a T-ordered exponential:

$$S=T\exp\left(i\int\mathscr{L}_{\mathrm{int}}^{\mathrm{in}}(x)\,d^4x\right)=T\exp\left(i\int\mathscr{L}_{\mathrm{int}}^{\mathrm{out}}(x)\,d^4x\right),$$

where $\mathscr{L}_{\mathrm{int}}^{\mathrm{in}}$ is the renormalized interaction Lagrangian.

Then, assuming as usual the action of the operator on the fields according to Eqs. (10), (20) and normal commutation relations for the in-fields, it is easy to show that

$$(\Theta S\Theta^{-1})^T = S, \quad \text{since} \quad (\Theta \mathscr{L}_{\text{int}}^{\text{in}}(x)\,\Theta^{-1})^T = \mathscr{L}_{\text{int}}^{\text{out}}(-x), \qquad (15)$$

and consequently relations of the type (17) hold for the matrix elements of the S-matrix. But, in distinction from the axiomatic proof, here again it was necessary to assume local commutativity of $\mathscr{L}_{\text{int}}^{\text{in}}(x)$. From what was said before it is clear that the condition (15) remains valid even under weaker assumptions about $\mathscr{L}_{\text{int}}^{\text{in}}(x)$. Because of lack of time we do not dwell on this subject in more detail.

Now, regarding the quotation (from Matthews) in the introduction. It is clear that Lorentz invariance and normal spin-statistics for the in(out) fields do not suffice for a proof of the CPT theorem. In Matthews' lectures[3] the matrix elements are represented à la Lehmann–Symanzik–Zimmermann.[7] Requiring Lorentz invariance of this representation on the mass shell only, it will be valid also in a nonlocal theory, where, in general, CPT is violated. If one assumes that this expression is valid and Lorentz invariant also off the mass shell, we are led to the local commutativity of the Heisenberg operators and thus automatically to CPT-invariance.

III. THE CONNECTION OF SPIN AND STATISTICS

The experimental facts indicate that integral spin particles are subject to Bose–Einstein and those of half-odd-integral spin are subject to Fermi–Dirac statistics. So far no systems obeying parastatistics have been observed.

The spin-statistics theorem asserts that in quantum field theory a nontrivial field of integral spin cannot anticommute at spacelike separated points, and that a nontrivial field of half-integral spin cannot commute, at such points.

In passing on to commutation relations between different fields the picture becomes more complicated. It turns out that "anomalous" commutation relations can be realized, where two integral spin fields, or one integral and one half-integral spin field, anticommute, and two half-integral spin fields commute. At the same time the theory exhibits symmetries of a special type. Owing to these symmetries one can always select in such theories the fields in such a manner that they obey normal commutation relations and are related to the original fields by means of so-called Klein transformations. In this sense a theory with anomalous commutation relations is equivalent to one with normal commutation relations.

We shall not prove all these assertions since the proof can be found in the book by Streater and Wightman.[5] We illustrate the facts on the case of a scalar field $\varphi(x)$ (and its Hermitian adjoint $\varphi^*(x)$, to be considered a different field!). Assume that

$$[\varphi(x), \varphi^*(y)]_+ = 0, \quad \text{if} \quad (x-y)^2 < 0. \qquad (22)$$

Then $\varphi(x)\Psi_0 = 0$, $\varphi^*(x)\Psi_0 = 0$. In a field theory where φ and φ^* commute or anticommute with all other fields this implies $\varphi = \varphi^* = 0$. The "simplest" proof follows from the Källén–Lehmann representation

$$\langle 0 \,|\, [\varphi(x), \varphi^*(y)]_- \,|\, 0\rangle = -i\int_0^\infty d\varkappa^2 \rho(\varkappa^2)\,\Delta(x-y, \varkappa^2) = 0, \qquad (x-y)^2 < 0. \qquad (23)$$

Comparing (22) and (23) it follows (using analyticity)

$$\langle 0 \,|\, \varphi(x)\,\varphi^*(y) \,|\, 0\rangle = 0,$$

or

$$\varphi(x)\,\Psi_0 = 0$$

(this means that $\|\varphi(f)\Psi_0\| = \|\varphi^*(f)\Psi_0\| = 0$ where $\varphi(f) = \int \varphi(x) f(x) d^4x$, $f(x)$ being a smooth function from the test function space, and the norm).

The second part of the theorem ($\varphi = \varphi^* = 0$) is proved on the basis of analyticity of Wightman functions.

IV. SOME APPLICATIONS OF CPT INVARIANCE

1. We first consider stable particles. We show that CPT invariance implies the equality of particle and antiparticle masses. Let Ψ_+^{in} denote a state of a particle of mass m, i.e.,

$$(P^\mu P_\mu)\,\Psi_+^{\text{in}} = m^2 \Psi_+^{\text{in}}. \qquad (24)$$

Considering that

$$(\Theta P^\mu \Theta^{-1})^T = P^\mu. \qquad (25)$$

we obtain from (24) and (25)

$$\Theta P^\mu P_\mu \Theta^{-1} \Theta \Psi_+^{\text{in}} = \Psi_-^{\text{out}*} (\Theta P^\mu P_\mu \Theta^{-1})^T = m^2 \Psi_-^{\text{out}*},$$

where Ψ^{out} is an antiparticle state of opposite spin, or

$$(P^\mu P_\mu)\,\Psi_-^{\text{in}} = m^2 \Psi_-^{\text{in}},$$

as required.

2. Unstable particles. The one-particle state is, generally speaking, unstable, i.e.,

$$S\,|\alpha, \text{in}\rangle \neq |\alpha, \text{in}\rangle.$$

If one assumes that, as before (taking into account the decay interaction), the theory is CPT invariant, one can establish the equality of lifetimes of particle and antiparticle. The decay probability of the particle is defined by

$$\sum_\beta \delta^4(p_\beta - p_\alpha)\,|\langle\beta, \text{in}\,|\,\tilde{S}\,|\,\alpha, \text{in}\rangle|^2, \qquad (26)$$

where S is the S-matrix from which a four-dimensional delta function has been removed and $|\alpha \text{ in}\rangle$ is the in-state of the unstable particle. CPT-invariance implies (cf. (17))

$$\langle\beta, \text{in}\,|\,\tilde{S}\,|\,\alpha, \text{in}\rangle = \langle\bar{\alpha}, \text{in}\,|\,\tilde{S}\,|\,\bar{\beta}, \text{in}\rangle. \qquad (27)$$

The decay probability of the antiparticle is proportional to

$$\sum_\beta \delta^4(p_\beta - p_\alpha)\,|\langle\bar{\beta}, \text{in}\,|\,\tilde{S}\,|\,\bar{\alpha}, \text{in}\rangle|^2. \qquad (28)$$

Using the relation (27) and Lorentz invariance, one can reduce the expression (28) to (26).

Other examples of applications of CPT-invariance to K-meson decays can be found in the reviews [9-11].

3. Let us briefly discuss the extension of the concepts of P, C, and other symmetries to unstable particles. The most consistent approach to introducing unstable particles into the theory is the determination of the corresponding singularities on unphysical sheets of

the matrix elements of stable particle interactions. Thus, consider the process*

$$\pi^+ + p \rightarrow \pi^+ + \pi^+ + n.$$

Assume that the matrix element of this process, as a function of the invariant mass q^2 of the two final pions has a pole in the unphysical sheet at the point

$$q^2 = M^2 - i\lambda \qquad (\lambda > 0).$$

We denote the residue at this pole by

$$T(s, t;\ M^2 - i\lambda).$$

Then if $\lambda/M^2 \ll 1$, one can use the value of $T(s, t, M^2)$ as a definition of the transition matrix element for $\pi^+ + p$ with the formation of an unstable particle (a ρ meson, say; we do not consider other quantum numbers here).

It is obvious that if the original theory has a certain symmetry (e.g., CPT) then the quantum numbers corresponding to this symmetry can be attributed to the unstable particle. Up to terms of the order λ/M^2 one can introduce creation–annihilation operators for the unstable particles, etc. If the particles have a large width there appear complicated unsolved problems.

In a completely analogous manner one can define the matrix elements for the formation of two or more unstable particles.

4. We give two examples of theories satisfying the spectral condition and Lorentz and translation invariance, but where local commutativity does not hold. In the first example the theory is CPT invariant, and in the second example this invariance is violated. For simplicity we consider neutral, zero spin particles.

Example 1. We write an S-matrix of the form

$$S = \exp(i\Lambda), \qquad \Lambda = \Lambda^*$$

and select Λ of the form

$$\Lambda = g \int :\varphi^3_{\text{in}}(x): d^4x.$$

The matrix element for the scattering of two particles $(p, q \rightarrow p', q')$ in the g^2-approximation will have the form $\delta^4(p + q - p' - q') \times \{\delta[(p+q)^2 - m^2] + \delta[(p-q')^2 - m^2] + \delta[(p-p')^2 - m^2]\}$. It satisfies the symmetry requirement (19) which follows from CPT-invariance, but violates the analyticity properties which are implied by local commutativity.

Example 2.

$$\Lambda = g \int \{\Phi^{(+)}_{\text{in}}(x)\, \Phi^{(+)}_{\text{in}}(x)\, \varphi^{(-)}_{\text{in}}(x):\ + \text{h. c.}\}\, d^4x.$$

The field Φ_{in} describes a particle of mass m, and φ_{in} describes a particle of mass μ. In the g^2-approximation the matrix element is

$$(p,\ q \rightarrow p',\ q') \sim \delta^4(p + q - p' - q')\ \Theta(p_0 + q_0)\ \delta((p+q)^2 - \mu^2).$$

It does not exhibit CPT-invariance, but satisfies all the postulates, except local commutativity.

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*The neutron is considered stable.

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DISCUSSION

V. B. Berestetskiĭ:

Aren't the requirements imposed on the S-matrix by the CPT theorem in its rigorous form tautological? In fact they contain the definition of antiparticles.

V. Ya. Faĭnberg:

No, since the particle–antiparticle concepts are introduced in the language of in- (or out-) operators, i.e., for noninteracting particles. In any theory where, for instance, weak local commutativity is violated (hence also locality), one can introduce particles and antiparticles, but the interaction (S-matrix) will not be CPT invariant.

D. A. Kirzhnits:

Can one make any statement on whether, in principle, the condition of weak local commutativity reduces to the first three postulates, or whether this condition represents an independent requirement?

V. Ya. Faĭnberg:

Dyson has shown that if fields possess the property of weak local commutativity (WLC) the Wightman functions will be analytic and single-valued in a real neighborhood of spacelike points. Thus, in general WLC extends the domain of analyticity and is thus an additional requirement.

M. K. Polivanov:

Since weak local commutativity is hierarchically a weaker requirement than local commutativity, a violation of weak local commutativity will without doubt lead to a violation of locality in the strong sense.

V. Ya. Faĭnberg:

This is a correct statement.

S. Matinyan:

Recently the assertion has frequently been made that the proof of the CPT theorem does not require the whole apparatus of quantum field theory, but can be "proved" in an S-matrix theory (Chew, Stapp). On the other hand, Jost has criticized Stapp's attempt in this direction, showing that in fact Stapp introduces into his proof those analytic properties of the S-matrix elements which are required for the CPT theorem. Can you say something in this regard? How seriously should one take the S-matrix approach?

V. Ya. Faĭnberg:

It is now clear that Jost's criticism of Stapp seems to be valid only within the so-called localizable theories (Meĭman, Jaffe). In the case of nonlocalizable theories, when the matrix elements can grow exponentially or faster in momentum space, the analytic properties of the S-matrix which are equivalent to CPT invariance do not imply the existence of local operators: the concept of weak local commutativity somehow loses its meaning and the S-matrix approach turns out to be more general (cf. footnote [3)]).

A. A. Komar:

You have asserted that in the axiomatic approach CPT invariance is equivalent to weak local commutativity. However, if one goes over to observable (to the S-matrix) it was necessary to assume additionally the existence of the Yang–Feldman equations. Isn't this a rather stringent new hypothesis, which brings the axiomatic proof of the CPT theorem closer to the proof in the Lagrangian formalism?

V. Ya. Faĭnberg:

In the transition to observables in the theory of asymptotic fields and particles (the Haag–Ruelle scattering theory) one also assumes the completeness of the in-states:

$$\mathcal{H} = \mathcal{H}_{\text{in}},$$

and local commutativity. However, the latter assumption does not seem to be necessary: it is only necessary that the commutator of two fields decrease sufficiently rapidly in spacelike directions. In the case of a linear relation (â la Yang–Feldman) between the in-operators and the Heisenberg operators, the requirement of local commutativity is not necessary for proving the CPT-properties of the S-matrix.

L. B. Okun':

What happens if one writes a Lagrangian which is not symmetrized in boson fields, or antisymmetrized in fermion fields? Can one use such a Lagrangian for computing matrix elements, etc., or will some problems arise? The symmetrization or antisymmetrization are needed not only for the proof of the CPT-theorem.

V. Ya. Faĭnberg:

If the fields which enter into $\mathcal{L}(x)$ are subject to such commutation relations that $\mathcal{L}(x)$ satisfies local (or weakly local) commutativity, then there certainly exists an operator Θ such that

$$(\Theta \mathcal{L}(x) \Theta^{-1})^T = \mathcal{L}(-x),$$

and the theory will be CPT-invariant. In this case the symmetrization–antisymmetrization procedure for the boson and fermion operators which occur in $\mathcal{L}(x)$ reduces to eliminating so-called contact infinities. In the interaction picture this procedure is equivalent to a transition from ordinary products to normal-ordered products. If the fields are nonlocal, the theory will certainly nc be CPT-invariant. Thus, roughly speaking, in a local theory the symmetrization is equivalent to some kind of renormalization of $\mathcal{L}(x)$

Translated by M. E. Mayer