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LOCAL STRUCTURE OF TURBULENCE IN AN INCOMPRESSIBLE VISCOUS FLUID AT VERY HIGH REYNOLDS NUMBERS*

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1. We shall denote by

$$U_\alpha(P) = U_\alpha(x_1, x_2, x_3, t), \quad \alpha = 1, 2, 3,$$

the velocity components at time t in a point with rectangular Cartesian coordinates x_1, x_2, x_3. When studying turbulence it is natural to consider the velocity components $U_\alpha(P($ in each point $P = (x_1, x_2, x_3, t)$ of a region G under consideration of the four-dimensional space (x_1, x_2, x_3, t) as **random variables** in the sense in which this is done in the theory of probability (see the paper by Millionshchikov[1] concerning such an approach).

Denoting by $\overline{A}$ the mathematical expectation of the random quantity A, we assume that

$$\overline{U_\alpha^2} \text{ and } \overline{\left(\frac{dU_\alpha}{dx_\beta}\right)^2}$$

are finite and bounded in each bounded subdomain of the region G.

We introduce in the four-dimensional space (x_1, x_2, x_3, t) new coordinates

$$y_\alpha = x_\alpha - x_\alpha^{(0)} - U_\alpha(P^{(0)})(t - t^{(0)}), \qquad s = t - t^{(0)}, \tag{1}$$

where $P^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, t^{(0)})$ is a fixed point in the region G. We note that the coordinates y_α in an arbitrary point P depend on the random quantities $U_\alpha(P^{(0)})$ and thus themselves are random variables. The velocity components are in the new coordinates equal to

$$W_\alpha(P) = U_\alpha(P) - U_\alpha(P^{(0)}). \tag{2}$$

Let the points $P^{(k)}$, k = 1, 2, ..., n, which in the system of coordinates (1) have the coordinates $y_\alpha^{(k)}$ and $s^{(k)}$, lie in the region G for some fixed values of the quantities $U_\alpha(P^{(0)})$. We can then define a 3n-dimensional conditional probability distribution law F_n for the quantities $W_\alpha^{(k)} = W_\alpha(P^{(k)})$, α = 1, 2, 3, k = 1, 2, ..., n, for given $U_\alpha^{(0)} = U_\alpha(P^{(0)})$.

Generally speaking the distribution law F_n depends on the parameters $x_\alpha^{(0)}$, $t^{(0)}$, $U_\alpha^{(0)}$, $y_\alpha^{(k)}$, $s^{(k)}$.

Definition 1. Turbulence is called locally homogeneous in the region G if for any fixed n, $y_\alpha^{(k)}$, and $s^{(k)}$ the distribution law F_n is independent of $x_\alpha^{(0)}$, t, and $U_\alpha^{(0)}$ so long as all points $P^{(k)}$ are located in G.

Definition 2. Turbulence is called locally isotropic in the region G if it is homogeneous and if moreover the distribution laws mentioned in definition 1 are invariant under rotation and mirror reflection of the original system of coordinate axes (x_1, x_2, x_3).

Compared with the concept of **isotropic turbulence** introduced by Taylor[2] our definition of locally isotropic turbulence is different first of all in that in our definition it is necessary that the distribution laws F_n are independent of $t^{(0)}$, i.e., that there is stationarity in time, and more broadly in that the limitation is imposed only upon the distribution laws for velocity differences and not for the velocities themselves.

2. The isotropy hypothesis in Taylor's sense is well verified experimentally in the case of turbulence caused by the passage of a current through a lattice (see[3]). In the majority of other cases of practical interest it can be considered only as a very rough approximation to reality even for small regions G and very large Reynolds numbers.

In contrast it seems to the author very plausible that in an arbitrary turbulent flow with sufficiently large Reynolds number* $R = LU/\nu$ in sufficiently small regions G of the four-dimensional space (x_1, x_2, x_3, t) which are not too close to the boundaries of the flow or to other singularities of it, to a good approximation the local isotropy hypothesis is realized. Under "small region" we understand a region the linear dimensions of which are small compared with L and the time dimensions small compared with T = U/L.

It is natural that in such a general and somewhat undefined formulation the assumption now put forward can not be proved rigorously.† To make possible its experimental verification for different specific cases we show in the following a few consequences of the hypothesis of local isotropy.

*L and U denote here typical scales for length and velocity for the flow as a whole.

† We give here only a few general considerations in aid of the proposed hypothesis. For very large R one can represent turbulent flow as follows: on the average flow (characterized by the mathematical expectations $\overline{U}_\alpha$) are superimposed "first order ripples" consisting in disordered shifts of different volumes of the fluid with diameters of order $l^{(1)} = l$ (where l is the Prandtl mixing length) with respect to one another; the order of magnitude of the velocities of these relative shifts are denoted by $v^{(1)}$; first order ripples for very large R turn out in turn to be unstable and superimposed upon them are "second order" ripples with a mixing length $l^{(2)} < l^{(1)}$ and relative velocities $v^{(2)} < v^{(1)}$; such a process of a consecutive reduction of turbulent ripples goes on until for ripples of some sufficiently large order n the Reynolds number

$$R^{(n)} = \frac{l^{(n)} v^{(n)}}{\nu}$$

turns out to be sufficiently small that the influence of the viscosity on the n-th order ripple be already appreciable and would prevent the formation on them of ripples of n + 1st order.

From the energy point of view the process of turbulent shifts can naturally be represented as follows: first order ripples absorb energy from the average motion and transfer it successively to higher order ripples, while the energy of the smallest ripples is scattered into thermal motion thanks to viscosity.

Because of the random mechanism of transferring motion from lower order ripples to higher order ripples it is natural to postulate that within regions of space which are small compared to $l^{(1)}$ the small higher order ripples are approximately subject to a statistical regime which is spatially

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3. We shall denote by y the vector with components y_1, y_2, y_3 and we consider the random quantities

$$W_\alpha(y) = W_\alpha(y_1, y_2, y_3) = U_\alpha(x_1+y_1, x_2+y_2, x_3+y_3, t) - U_\alpha(x_1, x_2, x_3, t). \quad (3)$$

Because of the assumption of local isotropy their distribution laws are independent of x_1, x_2, x_3, and t. For the first moments of the quantities $W_\alpha(y)$ it follows from local isotropy that

$$\overline{W_\alpha(y)} = 0. \quad (4)$$

We turn therefore to a study of the second moments*

$$B_{\alpha\beta}(y^{(1)}, y^{(2)}) = \overline{W_\alpha(y^{(1)})(W_\beta(y^{(2)})}. \quad (5)$$

From local isotropy it follows that

$$B_{\alpha\beta}(y^{(1)}, y^{(2)}) = \frac{1}{2}[B_{\alpha\beta}(y^{(1)}, y^{(1)}) + B_{\alpha\beta}(y^{(2)}, y^{(2)}) - B_{\alpha\beta}(y^{(2)}-y^{(1)}, y^{(2)}-y^{(1)})]. \quad (6)$$

This equation allows us to restrict ourselves to second moments of the form $B_{\alpha\beta}(y, y)$. For them

$$B_{\alpha\beta}(y, y) = \overline{B}(r)\cos\theta_\alpha\cos\theta_\beta + \delta_{\alpha\beta}B_{nn}(r), \quad (7)$$

where

$$r^2 = y_1^2+y_2^2+y_3^2, \quad y_\alpha = r\cos\theta_\alpha, \quad \delta_{\alpha\beta}=0 \text{ when } \alpha\neq\beta,$$
$$\delta_{\alpha\beta} = 1 \text{ when } \alpha=\beta, \quad (8)$$
$$\overline{B}(r) = B_{dd}(r) - B_{nn}(r),$$

$$\left.\begin{aligned} B_{dd}(r) &= \overline{[w_1(r, 0, 0)]^2}, \\ B_{nn}(r) &= \overline{[w_2(r, 0, 0)]^2}. \end{aligned}\right\} \quad (9)$$

For r = 0 we have

$$B_{dd}(0) = B_{nn}(0) = \frac{\partial}{\partial r}B_{dd}(0) = \frac{\partial}{\partial r}B_{nn}(0) = 0, \quad (10)$$

$$\left.\begin{aligned} \frac{\partial^2}{\partial r^2}B_{dd}(0) &= 2\overline{\left(\frac{\partial w_1}{\partial y_1}\right)^2} = 2a^2, \\ \frac{\partial^2}{\partial r^2}B_{nn}0 \cdot 2\overline{\left(\frac{\partial w_2}{\partial y_1}\right)} &= 2a_n^2. \end{aligned}\right\} \quad (11)$$

We obtained Eqs. (6) to (11) without using the assumption that the fluid is incompressible. From that assumption follows the equation

$$r\frac{\partial B_{dd}}{\partial r} = -2\overline{B}, \quad (12)$$

which makes it possible to express B_{nn} in terms of B_{dd}. From (12) and (11) follows that

$$a_n^2 = 2a^2. \quad (13)$$

It is, moreover, easy to calculate that (assuming incompressibility) the average scattering of energy per unit time and unit mass is equal to

$$\overline{\varepsilon} = \nu\left\{2\overline{\left(\frac{dw_1}{dy_1}\right)^2} + \overline{\left(\frac{dw_2}{dy_2}\right)^2} + \overline{\left(\frac{dw_3}{dy_3}\right)^2} + \overline{\left(\frac{dw_2}{dy_1}+\frac{dw_1}{dy_2}\right)^2} + \overline{\left(\frac{dw_3}{dy_2}+\frac{dw_2}{dy_3}\right)^2} + \overline{\left(\frac{dw_1}{dy_3}+\frac{dw_3}{dy_1}\right)^2}\right\} = 15\nu a^2. \quad (14)$$

4. We consider a coordinate transformation

$$y'_\alpha = \frac{y_\alpha}{\eta}, \quad s' = \frac{s}{\sigma}. \quad (15)$$

The velocities, kinematic viscosity, and average energy scattering, are in the new set of coordinates expressed by the following equations

$$W'_\alpha = W_\alpha\frac{\sigma}{\eta}, \quad \nu' = \nu\frac{\sigma}{\eta^2}, \quad \overline{\varepsilon}' = \varepsilon\frac{\sigma^3}{\eta^2} \quad (16)$$

We make now the following hypothesis:

First similarity hypothesis. The distributions F_n for locally isotropic turbulence uniquely determine the quantities ν and $\overline{\epsilon}$.

The coordinate transformation (15) with

$$\eta = \lambda = \sqrt{\frac{\nu}{a}} = \frac{\nu^{\frac{3}{4}}}{\overline{\varepsilon}^{\frac{1}{4}}}, \quad (17)$$

and

$$\sigma = \frac{1}{a}\sqrt{\frac{\nu}{\overline{\varepsilon}}} \quad (18)$$

leads to the quantities $\nu' = 1$, $\overline{\epsilon}' = 1$. Therefore by virtue of the similarity hypothesis which we have assumed, the corresponding function

$$B'_{dd}(r') = \beta_{dd}(r') \quad (19)$$

must be the same for all cases of locally isotropic turbulence. The equation

$$B_{dd}(r) = \sqrt{\nu\overline{\varepsilon}}\,\beta_{dd}\left(\frac{r}{\lambda}\right) \quad (20)$$

shows in connection with what we have done earlier that the second moments $B_{\alpha\beta}(y^{(1)}, y^{(2)})$ in the case of locally isotropic turbulence are uniquely expressed in terms of ν, $\overline{\epsilon}$, and the universal function β_{dd}.

5. To determine the behavior of the function $\beta_{dd}(r')$ for large r′ we introduce one more hypothesis:

Second similarity hypothesis.* If the absolute magnitudes of the vectors $y^{(k)}$ and their differences $y^{(k)}-y^{(k')}$ (where $k \neq k'$) are large compared to λ then the distribution laws F_n will uniquely determine the quantities $\overline{\epsilon}$ and will be independent of ν.

We put

$$y''_\alpha = \frac{y'_\alpha}{k^3}, \quad s'' = \frac{s'}{k^2}, \quad (21)$$

where y'_α and s′ are defined in accordance with Eqs. (15), (17), and (18). Since for any k

$$\overline{\varepsilon}' = \overline{\varepsilon}'' = 1,$$

for r′ large compared to $\lambda' = 1$ we have by virtue of the hypothesis assumed here

isotropic. Within small time intervals it is natural to assume this regime to be approximately stationary even in the case when the flow as a whole is not stationary.

Since for very large R the differences $W_\alpha(P) = U_\alpha(P) - U_\alpha(P^{(0)})$ of the velocity components in neighboring points P and $P^{(0)}$ of the four-dimensional space (x_1, x_2, x_3, t) are determined almost exclusively by the higher order ripples, the scheme proposed here leads us also to the local isotropy hypothesis in small regions G in the sense of the definitions of Secs. 1 and 2.

*All results of Sec. 3 and completely analogous to those obtained in [1],[2], and [4] for the case of isotropic turbulence in Taylor's sense.

*In terms of the schematic exposition of turbulence given in footnote *, λ is the scale of the smallest ripples the energy of which is directly scattered into thermal motion due to viscosity. The meaning of the second similarity hypothesis lies in the fact that the mechanism of energy transfer from larger to smaller ripples is for ripples of intermediate order, for which $l^{(k)}$ is larger than λ, independent of the viscosity.

$$B''_{dd}(r'') \sim B'_{dd}(r'') = \beta_{dd}\left(\frac{r'}{k^3}\right).$$

On the other hand, it follows from Eq. (21) that

$$B''_{dd}(r'') = \frac{1}{k^2} B'_{dd}(r') = \frac{1}{k^2}\beta_{dd}(r').$$

Thus, for large r′

$$\beta_{dd}\left(\frac{r'}{k^3}\right) \sim \frac{1}{k^2}\beta_{dd}(r),$$

whence

$$\beta_{dd}(r') \sim C(r')^{\frac{2}{3}}, \tag{22}$$

where C is an absolute constant. By virtue of (17), (20), and (20) for r large compared to λ,

$$B_{dd}(r) \sim C\varepsilon^{-\frac{2}{3}} r^{\frac{2}{3}}. \tag{23}$$

From (23) and (12) we conclude easily that for r large compared to λ

$$B_{nn}(r) \sim \frac{4}{3} B_{dd}(r). \tag{24}$$

We note in connection with this last formula that for r small compared to λ because of (13) the following relation holds

$$B_{nn}(r) \sim 2B_{dd}(r). \tag{25}$$

[1] M. Millionshchikov, Dokl. Akad. Nauk SSSR **22**, 236 (1939).
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[4] T. V. Karman, J. Aeronaut. Sc. **4**, 131 (1937).

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