SOVIET PHYSICS USPEKHI

A Translation of Uspekhi Fizicheskikh Nauk

SOVIET PHYSICS USPEKHI (Russian Vol. 93, Nos. 1 and 2) MARCH-APRIL 1968

SELF-FOCUSING AND DIFFRACTION OF LIGHT IN A NONLINEAR MEDIUM

S. A. AKHMANOV, A. P. SUKHORUKOV, and R. V. KHOKHLOV

Moscow State University

Usp. Fiz. Nauk 93, 19–70 (September, 1967)

CONTENTS

1. INTRODUCTION

1.1. Effects of Self Interaction of Electromagnetic Waves in a Nonlinear Medium. Experimental Data

AMONG the nonlinear optical effects that have been intensely investigated in recent years, a special position is occupied by effects of self-action of powerful light waves. Self-action effects are connected with the dependence of the complex dielectric constant (complex refractive index) on the intensity of the propagating wave. The appearance of this dependence can be connected with various physical causes. Electrostriction in a light field leads to the appearance of a pressure $p = E^2/8\pi \cdot \rho\partial\epsilon/\partial\rho$ (E—intensity of light field, ρ—density, ϵ—dielectric constant), which changes the density in the region occupied by the light beam, and consequently also the refractive index of the medium. In a liquid, a strong light field leads to orientation of the anisotropically polarized molecules, owing to the interaction with the induced dipole moment; the medium then becomes anisotropic and the average refractive index for the orienting field increases. This effect is customarily called the high-frequency Kerr effect; as in the well known static Kerr effect, the variation of the refractive index is produced in this case, as a result of the "alignment" of the molecules in the field. The refractive-index increment, which depends on the intensity of the light wave, can also be connected with the nonlinearity of the electron polarization. Finally, the change of the density, and consequently also of the refractive index, can be connected with heating due to dissipation of energy by the powerful light wave. A more detailed discussions of these effects will be deferred to Sec. 1.2, and we shall turn now to an examination of the singularities of the wave phenomena in media with a refractive index that depends on the intensity of the light wave. It is possible to explain many features of the wave processes without resorting to a detailed consideration of the physical mechanism of the nonlinearity, confining oneself to a phenomenological description of the polarization of the medium. The phenomenonlogical effects of the self-action are described by the odd terms (with respect to the field E) in the expansion of the nonlinear part of the polarization of the nonlinear medium with respect to the field, that is, if we represent the i-th component of the vector of the nonlinear polarization $P^{(nl)}$ in the form

$$P_i^{(\mathrm{nl})}=\chi_{ijk}E_jE_k+\chi_{ijkl}E_jE_kE_l+\chi_{ijklm}E_jE_kE_lE_m+\chi_{ijklmn}E_jE_kE_lE_mE_n+\ldots \qquad (1.1)$$

then the effects of self action are described by the tensors χ_{ijkl}, χ_{ijklmn}, etc. The corresponding expansion of the complex dielectric constant in powers of the field is (we use a symbolic notation; A is the amplitude of the wave)

$$\varepsilon=\varepsilon_0+\varepsilon_2|A|^2+\varepsilon_4|A|^4+\ldots, \qquad (1.2)$$

where ϵ_0 is the linear dielectric constant (the asymptotic value of ϵ as $A\to 0$), and ϵ_2, ϵ_4, $\epsilon_6\ldots$ are the complex quantities whose real and positive parts can have, in general, arbitrary signs. (It is easy to see that ϵ_2 is determined by the tensor χ_{ijkl}, ϵ_4 by χ_{ijklmn}, etc., $\epsilon_0=\epsilon_0'+i\epsilon_0''$, $\epsilon_2=\epsilon_2'+i\epsilon_2''$, ...) Unlike such well known nonlinear optical effects as the generation of harmonics and parametric processes, where waves interact at several strongly differing frequencies, during the process of self-action the wave remains quasimonochromatic, and the effect itself becomes manifest in a change of its amplitude, polarization, form of the angular or frequency spectrum. The classification of the possible variants of self-action of light waves in nonlinear media can be obtained by using (1.2) and simple qualitative considerations. Assume that we are dealing with the propagation of an unmodulated (in time) harmonic wave $E=(1/2)A\exp(i\omega t-\mathbf{k}_0\cdot\mathbf{r})+\text{c.c.}$ in a medium with a refractive index in the form (1.2). Then the imaginary parts of the coefficients ϵ_2, ϵ_4, ... will be connected with effects of nonlinear absorption, and the total damping decrement of the wave in the medium is

$$\delta=\delta_0+\delta_2|A|^2+\delta_4|A|^4+\ldots \qquad (1.2a)$$

$\delta_0=k_0\epsilon_0''/2\epsilon_0'$, $\delta_2=k_0\epsilon_2''/2\epsilon_0'$); on the other hand, the real parts of these coefficients are connected with the corrections to the phase velocity.

The total eikonal of a plane harmonic wave contains in this case the nonlinear additions (ϵ_0', $\epsilon_4'\ll\epsilon_0'$)

$$s^{(\mathrm{nl})}=\frac{\varepsilon_2'|A|^2k_0}{2\varepsilon_0'}z+\frac{\varepsilon_4'|A|^2k_0}{2\varepsilon_0'}z+\ldots \qquad (1.3)$$

the wave is accelerated or decelerated, and this can influence, generally speaking, the efficiency of the wave interactions in the nonlinear medium. However, particular interest is attached to effects connected with nonlinear additions to the real part of ϵ in the case of bounded beams. These effects give rise here to a new important physical effect, which can be called nonlinear refraction of the light rays. Indeed, in the field of a bounded light beam, a medium which is initially homogeneous becomes optically inhomogeneous by virtue of (1.2); the refractive index of the medium is now determined by the distribution of the intensity of the propagating wave. In order to reveal the main features of the nonlinear-refraction effect, we confine ourselves to allowance for the lowest-order (and consequently largest) nonlinear term in the expansion (1.2). It is obvious that the character of the nonlinear refraction is determined by the sign of ϵ_2'. In a medium with $\epsilon_2'>0$ (usually realized in the high-frequency Kerr effect, in electrostriction in the field of an intense light wave, and sometimes as a result of heating of the medium in the light field; for more details see Sec. 1.2) the regions of maximum intensity are simultaneously also the optically densest regions. In this case the nonlinear refraction should lead, obviously, to a concentration of the energy—the peripheral rays are deflected in a region where the field is maximal. This effect is called self-focusing of the light beam. An exceedingly important circumstance, which distinguishes the self-focusing effect on other nonlinear optical effects, is its "avalanche" character. Indeed, in a medium with $\epsilon_2'>0$ even a weak increase of the intensity in some section of the light beam leads to a concentration of the rays in this region, and consequently to an additional increase of the intensity; the latter increases the effect of nonlinear refraction, etc.

In linear optics, the growth of the field in a focal point of an optical system is hindered by diffraction. A similar role is played by diffraction also in self focusing; here, however, as we shall show later, the diffraction effects are far from always capable of compensating the nonlinear refraction.

An elementary analysis of the main laws connected with self-focusing can be performed by considering the behavior of the light rays on the boundary of a beam having a rectangular distribution of the amplitude (Fig. 1). Assume that in a nonlinear medium with $\epsilon_2'>0$ and $\epsilon_2''=0$ there propagates a cylindrical beam of radius a; then by virtue of (1.2) the refractive index outside the beam is $n=n_0=\epsilon_0^{1/2}$, and inside the beam $n=n_0+n_2|A|^2$ ($n_2/n_0=\epsilon_2/2\epsilon_0$). The rays incident on the boundary of the beam from the outside move from a medium which is optically denser to a medium with lower optical density; consequently, at sufficiently large angles φ, total internal reflection is possible. The critical angle corresponds to a ray whose inclination θ_0 to the beam axis is

$$\theta_0=\arccos\left(\frac{n_0}{n_0+n_2|A|^2}\right). \qquad (1.4)$$

Rays with $\theta>\theta_0$ emerge to the outside, and rays with $\theta<\theta_0$ return to the axis. In a beam whose phase front at the entrance to the nonlinear medium is plane, the angle θ is determined by diffraction:

$$\theta_d=\frac{0.61\lambda_0}{n_0 2a}. \qquad (1.5)$$

The relative contribution of the nonlinear refraction and diffraction to the behavior of such a beam can be estimated by comparing the angles θ_0 and θ_d.

a) When $\theta_0<\theta_d$, the beam spreads, but the rate of this spreading is smaller than in a linear medium.

b) When $\theta_0=\theta_d$ (nonlinear refraction compensates for the diffraction spreading completely), the dimensions and the form of the beam unchanged when the beam propagates in the nonlinear medium. The beam produces for itself a unique optical waveguide, in which its propagates without divergence. This mode is customarily called self-trapping of the wave beam.

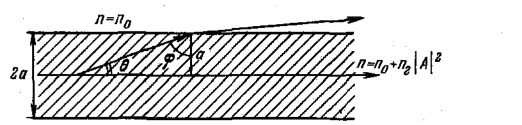

FIG. 1. Illustration of the derivation of a formula for the critical power of a beam, transported without divergence (self-trapping) in a nonlinear medium. The shaded area is occupied by the beam.

Using (1.4) and (1.5), it is easy to verify that the condition $\theta_0 = \theta_d$ imposes a requirement only on the total power of the beam and on the nonlinearity of the medium. Indeed, from (1.4) we get

$$\frac{\theta_d^2}{2} = \frac{n_2 |A|_{cr}^2}{n_0} \text{ and } P_{cr} = \frac{\lambda_0^2 c\,(1,2,2)^2}{256 n_2} \quad (1.6)$$

is the critical power of the self-trapping beam.

c) When $\theta_0 > \theta_d$ (and consequently $P > P_{cr}$) the rays are deflected towards the beam axis—self-focusing takes place. In this case the nonlinear medium acts like a positive lens. Its equivalent focal distance can be easily estimated by using formula (1.6).

Introducing the diffraction length $R_d = k_0 a^2/2 \cong a/\theta_d$, we find from (1.6) that the condition $\theta_0 = \theta_d$ is equivalent to the condition

$$R_d = \frac{a}{2}\sqrt{\frac{n_0}{n_2 |A|^2}} \equiv R_{nl}. \quad (1.7)$$

The quantity r_{nl}, which has the dimension of length, can be called the effective self-focusing length. Indeed, the diffraction divergence can be treated as a result of the action of a defocusing lens with focal distance R_d; then the equality $R_{nl} = R_d$ corresponds to the focal distance of a system of two lenses becoming infinite ($Z_f^{-1} = R_{nl}^{-1} - R_d^{-1}$). In beams of large "supercritical" power we have, with sufficient degree of accuracy, $Z_f \simeq R_{nl}$ ($R_{nl} \ll R_d$) (see Fig. 2)—the behavior of the beam is well described by the geometrical-optics approximation, and the diffraction effects hardly come into play.

The important role of the effects of nonlinear refraction in the propagation of intense electromagnetic radiation in a material medium, and the possibility of the occurrence of the effect of self focusing, were first indicated by Askar'yan in [1], which was published in 1962. In 1964, Talanov [2] calculated the profile of the beam that becomes self-trapped ($P = P_{cr}$) in a medium with $n = n_0 + n_2 |A|^2$. The conditions for self-focusing of electromagnetic waves in a nonlinear were discussed by Keldysh [3], and finally, Chiao, Garmire, and Townes [4], considered the problem of self-trapping of a wave beam in a nonlinear medium as applied to the optical range and to laser technology. Townes and his co-workers considered the nonlinear polarization mechanisms capable of leading to self-trapping, investigated the profiles of the self-trapping beams, discussed the factors determining the diameter of the self-trapping beam, etc. Estimates of the value of P_{cr}, given in [4], have shown that self-trapping effects should be observed even at moderate laser powers, but

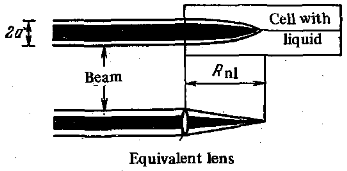

FIG. 2. Picture of self-focusing in a nonlinear medium, of a light beam with a power in excess of critical. The bounded beam (of radius a) of "supercritical" power with a plane phase front becomes self-focused in a medium with $n_2 > 0$ at a distance $R_{nl} = (a/2)\sqrt{n_0/n_2 |A|^2}$; the nonlinear medium is equivalent to a certain degree to a gathering lens with a focal distance R_{nl}, but the paths of the rays are different, see also Fig. 9.

in any rate in liquids with a large Kerr constant (for CS_2, $n_2 = 9 \times 10^{-12}$ cgs esu and, according to (1.6), $P_{cr} = 10$ kW).

The foregoing investigations stimulated new theoretical and experimental research in the field of self-focusing of powerful electromagnetic waves, and the overwhelming number of them were connected with the problem of self-focusing and self-trapping of intense laser beams in liquids and solids.*

In theoretical papers published in 1965 and in the first half of 1966, they investigated the dynamics of self focusing of a beam with $P > P_{cr}$ (see [5,6,14]—this question was not considered in [1-4]); in [6,14], particular attention was paid to factors determining the self-focusing length and to the rate of growth of the field intensity. In [6,16,74] they discussed the behavior of complicated beams in a nonlinear medium; in [7,8,40,41,62,65]—the nonstationary processes occurring in self-focusing, etc. In approximately the same time, the self-focusing effect was reliably recorded experimentally. The first experimental observations of self focusing were reported by Pilipetskiĭ and Rustamov [20], who photographed in their experiments narrow glowing filaments excited in organic liquids by a focused ruby-laser beam. At the present time there is no longer any doubt that the effect of self focusing becomes manifest in most experiments on the propagation in liquids of powerful light beams generated by Q-switched lasers. These included, first of all, experiments on the observation of stimulated Raman scattering (SRS) and stimulated Mandel'shtam-Brillouin scattering (SMBS), where the self-focusing effects leads to a sharp lowering of the stimulated scattering threshold (see [10,12,15,17-19]), to a change in the rate of growth of the stimulated-emission components with distance [18,42], to a strong deformation of the angular spectrum of the stimulated-scattering components [13], etc. Although these experiments do yield a certain amount of information on the self-focusing effect as such (in [18], in particular, they estimated experimentally the critical power P_{cr} in this manner) and on the structure of the beam at the exit from the cell with the self-focusing liquid (see, for example, [10], they cannot, of course, replace a detailed study of the dynamics of self-focusing. From among the investigations of the self-focusing phenomenon proper, the most substantial is that of Chiao, Garmire and Townes [11], who traced with the aid of a specially developed microscopic procedure the dynamics of the behavior of an intense ruby-laser beam in carbon disulfide (CS_2).

It was established in these experiments that the initially unfocused light beam with sufficiently homogeneous amplitude and phase fronts becomes compressed in a nonlinear liquid into a thin filament of practically constant radius, ~15 μ. The experimental setup is shown in Fig. 3. The diffraction-bounded

*At the same time it must be mentioned that in the first investigations on self-focusing, considerable attention was paid to the features of this effect in a plasma. (In a plasma self-focusing is the result of redistribution of the density.) In the present review we are unable to dwell in detail on this interesting question; we note only that a thorough analysis of different variants of self focusing in a plasma was published recently by Litvak [58-61].

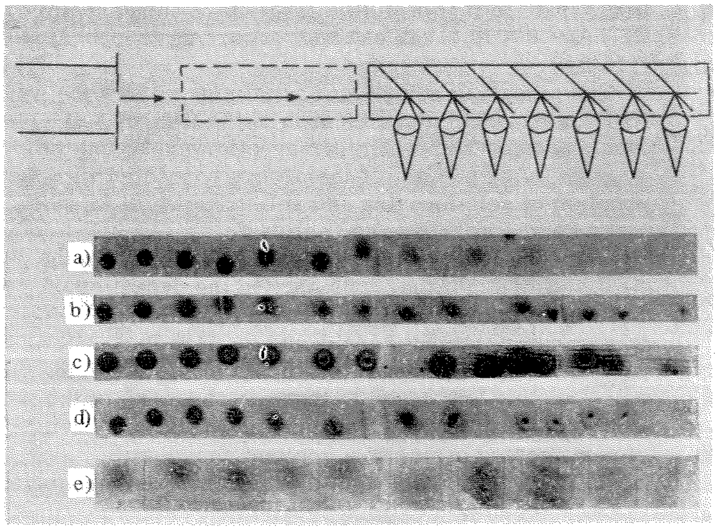

FIG. 3. Diagram of the experiment of Townes et al., in which self-focusing of light was experimentally observed. (The illustration is taken from [11]). The diffraction-bound beam is separated with the aid of a diaphragm from the Q-switch ruby-laser beam and and is guided to a cell with CS_2. The structure of the beam in different cross sections is viewed in a microscope. Photographs, b, c, d, and e show the evaluation of the beam as it propagates along the cell (different photographs correspond to different boundary conditions, photograph e corresponds to a complicated beam that becomes stratified by self-focusing.

beam of 0.5 mm diameter and 10—100 kW power was guided to a cell with CS_2; the evolution of the beam was observed with the aid of a thin plate that disturbed the beam little, and a microscope. It was established experimentally that the compression of the beam into a narrow filament occurs at a distance $z \simeq R_{nl}$ (in the experiment, at $P = 90$ kW, $2a = 0.5$ mm, and $n_2 = 9 \times 10^{-12}$ cgs esu they obtained $R_{nl} = 12$ cm; the calculated value of R_{nl} was 11 cm). As expected, decrease of a by one-half decreased R_{nl} by one-half. The critical power of the beam at the input, at which the waveguide propagation mode set in, was $P_{cr} = 25$ kW. This is approximately double the value of P_{cr} determined from (1.6). It is interesting that the presence of beam compression is not always monotonic; deviations from monotonicity are particularly noticeable at $P \gg P_{cr}$ (Fig. 3c).

The authors of[11] investigated the profile of a self-trapping beam and found that it is close to that calculated in[2,4]. Many authors observed narrow light channels in self-focusing liquids by photographing the ray of a ruby laser in the cell from the side and from the end[20,21,15,18,108]. The data obtained in this manner also contained valuable information. It must only be borne in mind that the observation of the filament in itself cannot always be ascribed to self-focusing; in the medium, a unique space-time selection of filamentary laser radiation takes place in the medium in the presence of optical breakdown. (This circumstance was pointed out by the authors of[21].) An important experimental result, which was reliably established in the indicated investigations, is that self focusing of the beam as a whole is usually not observed. Quite typical is the breakdown of the beam in a self-focusing medium into a multiplicity of "hot" filaments (Fig. 4). Moreover, even self-focused filaments which are at first glance homogeneous (with diameter ~30--50 μ) have, as shown by a thorough investigation (see[48,105]) a fine structure and break up into filaments of 3--5 μ diameter. It is interesting that the appearance of such "hyperfine" filaments is accompanied by a number of new phenomena (a sharp increase of the intensity of scattering at an angle 90°, breakup of a laser pulse, etc.).

An exceedingly interesting question is that of the influence of self-focusing effects on the propagation of laser beams in solids. Although much less has been done to date in this region than in self-focusing in liquids, there are data that offer evidence of a possible contribution of self-focusing to the mechanism of damage to optically transparent glasses and crystals. A characteristic photograph (from[42]) of the picture of damage to optical glass in the field of a ruby laser is shown in Fig. 5; analogous observations were reported also by the authors of[47]. Data which apparently also pertain to self-focusing are contained in[63], where the

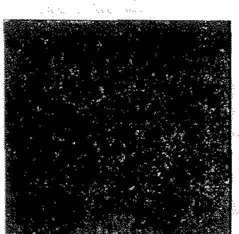

FIG. 4. Thin self-focusing filaments of Stokes SRS radiation excited by a ruby laser in nitrobenzene, photographed from the end of the cell. The upper and lower photographs are made through a polarizer in two orthogonal positions. The polarization of the laser radiation was circular, that of the filaments was linear and arbitrary.

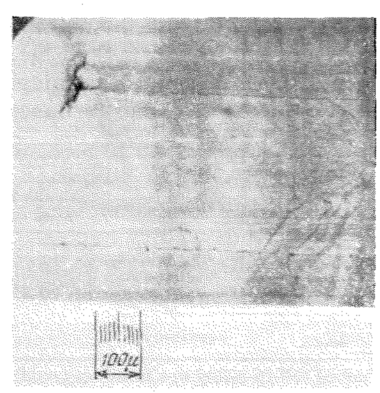

FIG. 5. Picture showing the destruction of optical glass in the field of a ruby laser. The long channel of diameter ~ 4 μ, which is clearly seen on the photograph, is difficult to explain without resorting to the self-focusing concepts (from [42]).

propagation of a giant pulse of a ruby laser in quartz was investigated at helium temperature. The effects of self focusing in a NaCl crystal are discussed in[97].

Self-focusing is undoubtedly the most interesting but not the only consequence of the presence of a nonlinear correction of the real part of the dielectric constant. We note first of all that in media with $n_2 < 0$ nonlinear refraction leads to a defocusing of the powerful wave beam. The causes of appearance of this effect are quite varied, but the most important of them is heating of the medium, connected with dissipation of the energy of light beam. Although thermal effects can apparently play a definite role also in the propagation of short laser pulses (see, for example, the theoretical estimates made in[40,41,112]), their action is strongest on a light beam in the steady state, when a stationary temperature gradient is established (the time of establishment of the stationary state is $\tau_T \simeq a^2/\kappa$, where κ is the coefficient of thermal conductivity of the medium for typical cases and exceeds 10^{-1} sec). The stationary thermal defocusing of the beam of a He-Ne laser was observed in[44], and in an argon laser in[144]. The results of the experiment described in[44] are shown in Fig. 6; we see that even at relatively high radiation intensities and lower losses (the hexane investigated in[44] has $\delta_0 \simeq 10^{-2}\ \text{cm}^{-1}$ at $\lambda \simeq 0.6\ \mu$) the effect of thermal defocusing can be appreciable.

Finally, we must point out also a large group of problems connected with self action of time-modulated waves in a nonlinear medium. Apparently, the first to call attention to the possible appearance of effects of this type was Ostrovskiĭ, who demonstrated in a paper published in 1963[23] that the envelope can become noticeably deformed in a medium with $n = n_0 + n_2 |A|^2$. A more detailed analysis performed recently (see[24,29]) with account of the dispersion spreading of pulses, has shown that it is possible to trace a very interesting space-time analogy in problems of self-action of modulated waves: the character of the variations of the spatial modulation (angular spectrum) and temporal modulation (frequency spectrum) due to self-action is in many respects the same, and is expressed in terms of comparable parameters.* The temporal analog of the spatial self-focusing is the effect of self-compression of wave packets, which was calculated in the cited papers.

A large number of problems is connected with the self-action of electromagnetic waves in absorbing media. Besides the already mentioned problems involving the propagation of waves in media with nonlinear absorption, special mention should be made of effects of self-action in active media, particularly in saturating laser amplifiers (see, for example,[25-28,98,99]). Theoretical and experimental investigations made in this field by Basov and co-workers[25-27] have shown that the self-action connected with the dependence of the refractive index on the intensity of the wave can

*Precisely the same analogy exists in the theory of nonlinear interactions of modulated waves; the influence of the spatial and temporal modulation of the waves on such processes as generation of optical harmonics, parametric amplification, is expressed in terms of comparable parameters and obeys common laws; see [30,101].

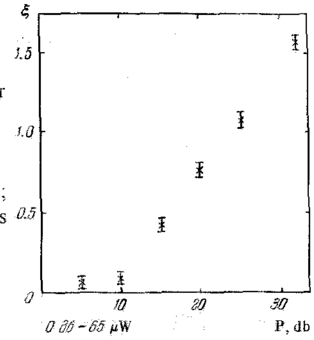

FIG. 6. Relative broadening of a helium-neon laser beam (maximal power $P \simeq 60$ MW, $\lambda = 0.63\ \mu$) after passing through a hexane-filled cell 98 cm long. Ordinates—$\zeta = (a - a_0)/a$ (2a — beam diameter; for power $P \simeq 65$ MW); abscissas—laser power and relative units (decibels). The plots are taken from [44].

become manifest here more strongly than in passive media.

Thus, the results of the investigations performed to date show that problems connected with self-action of waves occupy one of the central places in modern nonlinear optics. There is every reason for assuming it is just the self-action effects which determine the main features of the behavior of powerful light beams in the majority of material media, including active media of lasers themselves. It is interesting that in many cases, the self-action connected with broadening of the angular and frequency spectra can be observed simultaneously[116-118]. The purpose of the present review is to describe the modern status of theoretical and experimental investigations in this important field of laser physics. Principal attention will be paid in the review to effects of variation of the angular spectrum of a bounded beam in a passive medium with $n_2 > 0$ (self-focusing); these effects become manifest in experiments with powerful pulsed lasers most strongly, and as a rule they dominate. The theoretical part of the review (Secs. 2 and 3) is aimed at describing the mathematical formalism used in the theory of self-action of light beams. It must be emphasized that although crude and semiquantitative notions of the mechanism and dynamics of self-action have already been formulated, many problems remain unsolved to this very day; these are discussed in detail in Secs. 2 and 3. The main exposition is preceded by a small section in which we summarize the information on the physical processes that lead to self action (mechanisms of nonlinearity of the refractive index).

1.2. Mechanisms of Nonlinearity of the Refractive Index of a Material Medium

In this section we consider briefly the physical causes of the appearance of terms with ϵ_2, ϵ_4, etc. in the expansion (1.1). Before we go over to the concrete examination of different mechanisms of nonlinear polarizability, let us discuss the consequences resulting from a phenomenological analysis of nonlinear polarization. Using the expansion of the macroscopic nonlinear polarization in powers of the field (similar to (1.1)), the total steady-state response of a nonlinear medium to an arbitrary field $\mathbf{E}(\mathbf{r}, t)$ is

$$\mathbf{D}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) + 4\pi \mathbf{P}(\mathbf{r}, t) = \hat{\varepsilon}_0 \mathbf{E} + 4\pi \mathbf{P}^{(nl)} \qquad (1.8)$$

where $\mathbf{P}^{(nl)}$ is the nonlinear polarization of the medium) and can be written in the form (we go over to expression of the electric induction in terms of the components, and take into account only terms that lead to self-action)

$$D_i(r,\ t) = \int_0^\infty dt_1 \int d\mathbf{r}_1 \varepsilon_{ij}(t_1,\ \mathbf{r}_1)\, E_j(t-t_1,\ \mathbf{r}-\mathbf{r}_1)$$
$$+ 4\pi \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3 \int d\mathbf{r}_1 \int d\mathbf{r}_2 \int d\mathbf{r}_3 \chi_{ijkl}(t_1,\ t_2,\ t_3,\ \mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3)$$
$$\times E_j(t-t_1,\ \mathbf{r}-\mathbf{r}_1)\, E_k(t-t_1-t_2;\ \mathbf{r}-\mathbf{r}_1-\mathbf{r}_2)$$
$$\times E_l(t-t_1-t_2-t_3;\ \mathbf{r}-\mathbf{r}_1-\mathbf{r}_2-\mathbf{r}_3) + \ldots \quad (1.9)$$

Formula (1.9) takes into account the fact that both the linear and the nonlinear response of the medium at the "point" (r, t) are determined not only by the field at this point, but also by the fields in neighboring "points," that is, (1.9) takes into account the temporal and spatial dispersion of the linear and nonlinear susceptibility of the medium. Here and throughout we shall be interested henceforth in the response of the medium to an action of a special type—a quasimonochromatic wave

$$\mathbf{E} = \frac{1}{2}\mathbf{A}(r,\ t)\exp[i(\omega_0 t - \mathbf{k}_0\mathbf{r})] + \text{c.c.} \quad (1.10)$$

For a quasimonochromatic wave, the relative widths of the frequency and angular spectrum of which are small, the complex amplitude $A(r, t)$ is slowly varying function (compared with $\exp[i(\omega_0 t - k_0\cdot r)]$) of the coordinates and of the time.

For an analysis of the effects of self action of the wave (1.10) it is sufficient to consider only those components of the induction D, whose "fast" dependence on r and t is in the form $\exp[i(\omega_0 t - k_0 r)]$. In many problems on self-action it is possible to confine oneself in the allowance for the dispersion properties of the medium to a quasistatic approximation (the complex amplitudes in (1.9) can be regarded as functions that vary slowly compared with $\epsilon(t_1, r_1)$ and $\chi(t_1, t_2, t_3, r_1, r_2, r_3)$, and can be taken outside the integral sign). Then

$$D_i^{(l)} = \varepsilon_{ij}(\omega_0,\ k_0)\, A_j(r,\ t)\exp i(\omega_0 t - \mathbf{k}_0\mathbf{r}) \quad (1.11)$$

and

$$D_i^{(\mathrm{nl})} = 4\pi\chi_{ijkl}(\omega_0,\ k_0;\ \omega_0;\ k_0;\ \omega_0,\ k_0;\ -\omega_0;\ -k_0) \times A_j A_k A_l^* \exp[i(\omega_0 t - \mathbf{k}_0\mathbf{r})] \quad (1.12)$$

(we take into account only the nonlinear term of lowest order, and therefore of largest magnitude); a medium for which only χ_{ijkl} is significant is called cubic). If we are dealing with linear induction, then the quasistatic approximation in the allowance for the spatial dispersion is usually perfectly satisfactory. The same cannot be said, however, with respect to the temporal dispersion; the linear inertial properties of the medium become strongly manifest in many problems involving self-action of light pulses. It is frequently necessary therefore to use in lieu of (1.11) a more general expression, which is obtained from (1.11) by expanding $A(t - t_1, r)$ in a Taylor series:

$$D_i^{(l)} = \left\{\varepsilon_{ij}(\omega_0,\ \mathbf{k}_0)\, A_j + \sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}\frac{\partial^n \varepsilon_{ij}}{\partial\omega^n}\bigg|_{\omega=\omega_0}\frac{\partial^n A_j}{\partial t^n}\right\}\exp[i(\omega_0 t - \mathbf{k}_0\mathbf{r})]. \quad (1.13)$$

It is possible to take into account in similar fashion also "nonquasistatic" effects, due to spatial dispersion; in this case the expansion of the type (1.13) contains corresponding spatial derivatives.

The quasistatic approximation, as a rule, is perfectly satisfactory for a nonlinear susceptibility. In an overwhelming majority of cases of spatial dispersion, we can neglect in (1.12) (the spatial dispersion (perhaps the only important exception is striction nonlinearity), and assume that the effects of self-action are described by the spectral component of the tensor χ in the form

$$\chi(\omega;\ \omega;\ \omega;\ -\omega) \equiv \chi(\omega = \omega + \omega - \omega) \quad (1.14)$$

(we use here a general notation, which takes into account all the frequency arguments (see [35,36]) and later on we shall sometimes write, for brevity, simply $\chi(\omega)$ in lieu of (1.14)).

Allowance for the temporal dispersion of the nonlinear susceptibility in the band of the signal (1.10) can be made by going over to an expansion of the type (1.13).

In the general case, an arbitrary component of the nonlinear-susceptibility tensor contains both real and imaginary parts:

$$\chi_{ijkl}(\omega) = \chi'_{ijkl}(\omega) + i\chi''_{ijkl}(\omega). \quad (1.15)$$

The real part $\chi'(\omega)$ determines the nonlinear addition to the phase velocity of the harmonic wave of frequency ω, and the imaginary part (χ'') the nonlinear absorption; they are connected with each other by relations of the Kramers-Kronig type [36]. The tensor χ_{ijkl} has nonzero components for all groups of point symmetry and isotropic media. The nonzero components χ_{ijkl}, determined by the symmetry properties of the medium, can be found for all crystal classes in [33,38].

Using the matrix of the components χ_{ijkl}, we can calculate the components of the vector of the nonlinear induction $D_i^{(nl)}$, which determine the effects of self-action. In two cases—isotropic and cubic medium—we can write for D_i quite general formulas.

For an isotropic medium

$$D_i^{(\mathrm{nl})} = 12\pi\chi_{1122}E_i(\mathbf{EE}). \quad (1.16)$$

For cubic crystals (in particular, for crystals such as LiF, NaCl, etc.)

$$D_i^{(\mathrm{nl})} = 12\pi\chi_{1122}E_i(\mathbf{EE}) + 4\pi(\chi_{1111} - 3\chi_{1122})E_i^3. \quad (1.17)$$

An estimate of the absolute values of the components χ must obviously be based, already on an analysis of the concrete physical mechanism.

1.2.1. Mechanisms of nonlinearity of the refractive index of a liquid.

In accordance with (1.16), we can write for a liquid

$$\varepsilon = \varepsilon_0 + \varepsilon_2|A|^2 \quad (1.18)$$

where

$$n = n_0 + n_2|A|^2,\quad n_2 = n_0\frac{\varepsilon_0}{2\varepsilon_0}. \quad (1.19)$$

The contribution to n_2 due to changes in the density of the medium in light field (these changes can be due to electrostriction and heating, if the medium has absorption) and due to changes in the polarizability of the molecules (due to the nonlinear addition to the electronic polarizability, and in the case of anisotropic molecules also due to their orientation along the field of the wave) can be estimated by using the Clausius-

Mosotti equation and the corresponding dynamic equations (see [39]). For an isotropic medium

$$\frac{n^2-1}{n^2+2}=\frac{4}{3}\pi\rho\alpha, \qquad (1.20)$$

where ρ is the density of the medium and α the polarizability of the molecule. In a strong light field, ρ and α contain terms that depend on the field; when account is taken of the lowest-order term, we can write

$$\rho=\rho_0+\Delta\rho=\rho_0+\rho_2|A|^2,\quad \alpha=\alpha_0+\Delta\alpha=\alpha_0+\alpha_2|A|^2. \qquad (1.21)$$

Substituting (1.21) in (1.20) and putting $n = n_0 + \Delta n$, where Δn is the correction to the refractive index for the action of the field, we have

$$\Delta n=(\Delta n)_\rho+(\Delta n)_\alpha=(n_0^2-1)(n_0^2+2)\left[\frac{\Delta\rho}{\rho_0}+\frac{\Delta\alpha}{\alpha_0}\right]6n_0^{-1}. \qquad (1.22)$$

Let us turn first to an analysis of the quantity $(\Delta n)\rho$. The contribution made to it by electrostriction can be determined by solving a wave equation having in its right side an additional term due to the electrostriction pressure,

$$\left(-\nabla^2+\frac{1}{u^2}\frac{\partial^2}{\partial t^2}-\frac{2\Gamma}{u^2}\frac{\partial}{\partial t}\right)\Delta\rho=-\frac{1}{8\pi}\frac{\gamma}{u^2}(\nabla^2|A|^2). \qquad (1.23)$$

Here u is the velocity of the acoustic wave, $u = (\rho\beta)^{-1/2}$, where β is the compressibility, $2\Gamma/u$ is the decrement of the damping of the acoustic wave, and $\gamma = 2n_0\rho_0\partial n/\partial\rho$. The stationary solution of (1.23) makes it possible to determine the striction addition to the refractive index in the field of an unmodulated wave of amplitude

$$(\Delta n)_\rho=K_\rho\frac{\lambda_0}{2}|A|^2\equiv n_2|A|^2\equiv\frac{n_0}{2\varepsilon_0}\varepsilon_2|A|^2, \qquad (1.24)$$

where $K\rho = \gamma^2\beta/8\pi n_0\lambda_0$ can be called the electrostriction coefficient. The result (1.24) does not depend on the type of wave polarization; in striction self-action, elliptic and plane polarizations are on par. Of great importance from the point of view of revealing the mechanism of electrostriction self-action in experiments is the temperature dependence of $K\rho$, usually $dK\rho/dT > 0$.

In the field of a modulated wave it is necessary to consider the nonstationary solution (1.23); such calculations for a specified field were made in [39,40]. In the general case it is necessary to take into account also the reaction of the medium on the field. Then, in general, it is not advantageous to use the concept of nonlinear susceptibility, and (1.23) must be solved simultaneously with the field equations (see Sec. 2). Thus, in problems of self-action, a separate analysis of the nonlinear properties of the medium and of the electrodynamic problem is, generally speaking, not always justified. For more detailed explanations see Secs. 2 and 3.

The change of density (and consequently of the refractive index) connected with the heating (due to the dissipation of energy of the light wave) leads, generally speaking, to nonstationary effects, even in the field of an unmodulated wave (see [45]). Indeed, for time intervals shorter than the time of establishment of a stationary temperature gradient ($\tau_T \cong a^2/\kappa$, where κ is the coefficient of thermal conductivity of the medium), we have

$$\varepsilon_2=\frac{1}{8\pi}\frac{\partial\varepsilon}{\partial t}\frac{\delta_0 ct}{\rho_0 c_p}; \qquad (1.25)$$

c_p is the specific heat. Usually $\tau_T \cong 0.1-1$ sec, and consequently for laser pulses it is necessary to use (1.25). For most media $\partial\epsilon/\partial T < 0$, and the thermal effect leads to defocusing. The characteristic defocusing times were estimated in [40]. At the same time, for $(\partial\epsilon/\partial T) > 0$, heating leads to self-focusing; this circumstance was discussed in detail in [45]*. The stationary problem of thermal self-action (which has a bearing on experiments with gas lasers) is discussed in [45,114].

The nonlinear addition to the refractive index in liquids with anisotropic molecules is connected with the orienting action of the light field on the molecules—their alignment in the field. When this circumstance is taken into account, we have for the distribution function of axially-symmetrical anisotropic molecules with respect to the angle, in first approximation, (see [85,53,113])

$$f(\theta)\,d\Omega=\frac{d\Omega}{4\pi}+\frac{d\Omega}{48\pi kT}(\alpha_\parallel-\alpha_\perp)(3\cos^2\theta-1)|A|^2. \qquad (1.26)$$

Here $d\Omega$ is the solid-angle element, θ is the angle between the symmetry axis of the molecule and the field direction, and $\alpha_\parallel$ and $\alpha_\perp$ are the principal polarizabilities of the molecule. In a field of a plane-polarized wave of constant amplitude, the stationary addition to the refractive index, due to the Kerr effect, was calculated in [39]:

$$(\Delta n)_\alpha=\frac{1}{3}K_\alpha\lambda_0|A|^2=n_2|A|^2. \qquad (1.27)$$

According to [39], $dK_\alpha/dT < 0$. The calculated values of the parameters K_α and K_β for different liquids are listed in Table I, which is taken from [39]. Formula (1.27) can be used also to estimate self-action of modulated waves, if the modulation period $\tau_m \gg \tau = 4\pi b^2\nu/kT$—the relaxation time, which is determined by the dimensions of the molecule b and by the viscosity ν (usually $\tau \simeq 10^{-12}$ sec). When $\tau_m \cong \tau$ it is necessary to solve the relaxation equation for the polarizability simultaneously with the field equations (see Secs. 2 and 3). An important circumstance distinguishing the Kerr nonlinearity from the striction nonlinearity is the appearance of anisotropy due to the alignment of the molecules in the field. In a field which is plane-polarized along the x axis, a medium with orienting molecules becomes birefringent, and the components of ϵ are

Table I

Liquid	$K_\rho\cdot 10^7$	$K_\alpha\cdot 10^8$	Liquid	$K_\rho\cdot 10^7$	$K_\alpha\cdot 10^8$
Carbon tetrachloride	1.21	0.67	Nitrobenzene	0.92	26.04
Carbon disulfide	2.53	32.6	Aniline	1.00	3.22
Hexane	1.06	0.45	Chloroform	1.03	1.70
Cyclohexane	1.06	0.78	Acetone	0.75	1.03
Metaxylene	1.20	7.59	Methyl alcohol	0.58	0.17
Benzene	1.33	5.73	Ethyl alcohol	0.66	0.21
Toluene	1.25	6.55	Butyl alcohol	0.64	0.41
Chlorobenzene	1.20	9.93			
Bromobenzene	1.50	14.35			

*It is noted in [41] that for $t < \tau_\perp = a/u$ heating even in a medium with $d\epsilon/dT < 0$ should lead to a noticeable self-focusing of a homogeneous light beam (see also [112]).

$$\varepsilon_{xx} = \varepsilon_0 + \varepsilon_2 |A|^2, \quad \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_0 - \frac{1}{2}\varepsilon_2 |A|^2 \tag{1.28}$$

—the self focusing component is precisely E_x. By virtue of (1.28) we have for a circularly polarized wave, in lieu of (1.27),

$$(\Delta n)_\alpha = \frac{1}{12} K_\alpha \lambda_0 |A|^2 \tag{1.29}$$

—for circularly-polarized radiation the power necessary to obtain the same addition to n increases by a factor of 4. Attention was called to this circumstance in[7] and[39]. It must be emphasized that the foregoing conclusion pertains to the case when the circularly-polarized wave becomes self-focused as a whole. A more detailed analysis shows, however, that such self-focusing has low probability; in a nonlinear medium a circularly polarized wave is unstable, and its self-focusing results in wave channels not with circular but with linear polarization. In this case the critical power is only double the critical power for linearly polarized light (see[104,106], and also the plot of Fig. 20, which is taken from[100]).

The anisotropy induced by an intense light wave in liquids with anisotropic molecules was recorded directly in[43]. The scheme of this experiment is shown in Fig. 7. Light from a xenon lamp, blocked in the absence of a laser beam by a system made up of a polarizer and analyzer, is incident on a photomultiplier whenever the laser pulse makes the liquid in the cell anisotropic*. The Kerr constant is calculated from the experimentally determined difference of the refractive indices $n_\parallel - n_\perp = \lambda_0 B A_0^2$ (A_0—laser field). The values of B_{opt} determined experimentally in[43] (optical Kerr effect) for a number of liquids are listed in Table II, which gives also the values of the static Kerr constant B_{st} ($n_2 = (2/3)\lambda_0 B_{opt}$).

It was shown in[51] that when anisotropy is induced, elliptically polarized light experiences rotation of the ellipse axes. Using this circumstance, the authors of[51] determined the nonlinear addition to the refractive index in different liquids. These measurements were repeated in[52]; the absolute values deviated greatly from the data of[51] (see also the refined data in[100]).

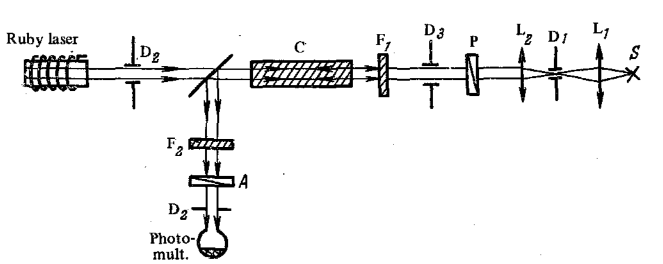

FIG. 7. Diagram of experiment in which a high-frequency Kerr effect was observed [43]. Radiation from xenon lamp (S) experiences a rotation of the plane of polarization in the cell with the investigated liquid (C), if the cell is illuminated simultaneously by a powerful coaxial beam from a ruby laser. P—polarizer; A—analyzer; F_1, F_2—filters passing the radiation of the xenon lamp ($\lambda \simeq 5000$ Å); D_1, D_2, D_3—diaphragms; L_1, L_2—lenses.

*This effect is one example of cross modulation of waves in a nonlinear medium.

Table II

Liquid	$B_{opt} \times 10^7$	$B_{st} \times 10^7$
CS_2	3.9	3.9
$C_6H_5NO_2$	3.3	362
C_6H_6	0.7	0.5
C_6H_5COCl	3.3	223

A calculation based on formulas such as (1.26) is valid for fields that are not too strong; when the field intensity increases it is necessary to take into account the higher-order terms, corresponding to inclusion of terms with $|A|^4$ and higher powers of the field in (1.2). The limiting value of the nonlinear addition to ϵ_0 is reached when all the molecules are completely aligned with the field: when $A \to \infty$ saturation takes place of the high-frequency Kerr effect (Fig. 8). The concrete law governing the saturation is discussed, for example, in[7,115] (see Sec. 3). Although orientational effects and striction are the principal mechanisms of self-focusing of light in liquids, it is necessary in many cases to take into consideration also a few other effects. Thus, Townes and his co-workers[105] explain the formation of "hyperfine" laser filaments as being due to the addition to an increment in n. connected with the growth of the polarizability of the molecules excited in stimulated Raman scattering. A general discussion of this question was published recently by Askar'yan[46], who considered the contribution made to n by excited atoms and molecules. Finally, Hellwarth[107] recently developed a theory of the nonlinear refractive index with allowance for correlations between molecules; it turned out that noticeable additions to n can occur for symmetrical molecules.

1.22. The most effective mechanism that can lead to self-focusing in solids is electrostriction. The static values of the critical powers turn out to be low in this case (see[4]) (we must remember the statements made above concerning the role of the nonstationary processes; thermal effects can also be appreciable).

The nonlinear electron polarizability is usually small; it yields usually $n_2 \simeq 10^{-14}$–10^{-15} cgs esu. In semiconductors with a narrow forbidden band, however, such as Te, we can expect appreciable values of n_2. Although no concrete measurements of n_2 have been made as yet, an approximate estimate can be made by starting from experimentally-measured values of the tensor of the quadratic polarizability (according to[102] $\chi_{ijk} \simeq 10^{-6}$ cgs esu) and the general theory of nonlinear properties of semiconductors[54]. A rough estimate yields for Te the value $n_2 \simeq 10^{-12}$ cgs esu; the latter leads us to expect self-action to be observed in Te in a beam of a CO_2 gas laser. We note, finally, that the

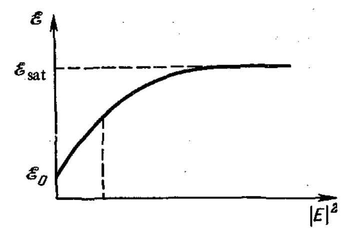

FIG. 8. Dependence of the dielectric constant of a nonlinear medium on the intensity of the light wave, with allowance for the saturation effect (for the Kerr effect).

foregoing summary does not exhaust, of course, all the possible mechanism of nonlinearity of the refractive index. For example, the possibility of self focusing in a medium with anomalous dispersion that depends on the intensity was considered in [32].

2. GEOMETRICAL OPTICS OF A NONLINEAR MEDIUM

As was already indicated in Sec. 1, the effects of self-action lead to a change in the angular spectrum of the propagating quasimonochromatic wave (these changes will be called here and henceforth self-focusing and self-trapping) and to a change in the frequency spectrum (these effects be called self-contraction of wave packets). Although for a real quasimonochromatic wave (for example, for a short laser pulse having a finite angular spectrum) both indicated phenomena occur, in general, simultaneously, in order to simplify the exposition, it is sensible to carry out their theoretical analysis separately. The results will then make it possible to discuss the entire problem as a whole. In this section we turn first to an analysis of the problem of self-focusing of a light beam in the geometrical-optics approximation ($k = 2\pi/\lambda \to \infty$). Such an analysis makes it possible to reveal the main features of the nonlinear refraction of light rays under different conditions; we note, incidentally, that inasmuch as the effects of nonlinear refraction are dominating at light-beam powers greatly exceeding the crystal value (see formula (1.6)), the geometrical-optics picture reflects correctly the main features of the phenomenon of self focusing of beams possessing "supercritical" power also at finite values of λ, particularly for three-dimensional beams (see also Sec. 3).

2.1. Stationary Self-focusing of a Bounded Beam in Cubic Lossless Medium. Fundamental Equations

In accordance with the foregoing, we turn to the investigation of the effect of self-focusing of unmodulated waves (stationary self-focusing). Stationary self-focusing is described by the wave equation

$$\operatorname{rot}\operatorname{rot}\mathbf{E}+\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}+\frac{4\pi}{c^2}\frac{\partial^2\mathbf{P}^{(l)}}{\partial t^2}+\frac{4\pi}{c^2}\frac{\partial^2\mathbf{P}^{(nl)}}{\partial t^2}=0 \tag{2.1}$$

together with material equations of the type (1.1) (the linear and nonlinear responses of the medium are assumed to be in the steady-state); for a cubic medium

$$\mathbf{D}=\hat{\varepsilon}_0\mathbf{E}+4\pi\mathbf{P}^{(nl)}=\hat{\varepsilon}_0\mathbf{E}+4\pi\hat{\chi}\mathbf{EEE}. \tag{2.2}$$

Equation (2.1) can be analyzed on the basis of the method of slowly varying amplitudes, which greatly simplifies the initial equation. The idea of the method consists in the fact that a bounded weakly converging or weakly diverging beam can be represented in the form

$$\mathbf{E}=\frac{1}{2}\mathbf{e}A\left(\mu\,(\mathbf{rp});\,\sqrt{\mu}\,[\mathbf{rp}]\right)\exp\left[i\,(\omega t-\mathbf{kr})\right]+\text{c.c.}, \tag{2.3}$$

where μ is a small parameter characterizing the difference between the beam and the plane wave $\mathbf{E} = (1/2)\,\mathbf{e}E_0\exp[i\omega t - \mathbf{k}\cdot\mathbf{r})]$, a deviation due to the nonlinearity of the medium and to diffraction. Account is taken in (2.3) of the change in the complex amplitude A both along the beam $\mathbf{p}$ and across it (we take in first approximation the same $\mathbf{p}$ as for a plane wave). The changes across the beam are faster, since a transition into the shadow region takes place.

Substituting (2.3) in (2.1), assuming that the nonlinear polarization is of order μ, and confining ourselves to first-order terms in μ, we arrive at a simplified equation that describes the effect of self-action of a harmonic wave in a cubic medium:

$$2ik\frac{\partial A}{\partial z}=\Delta_\perp A+k^2\frac{\varepsilon_2|A|^2}{\varepsilon_0}A. \tag{2.4}$$

Here $\Delta_\perp$ is a two-dimensional Laplace operator in the plane perpendicular to the beam or to the axis $z \parallel \mathbf{p}$. When $\epsilon_2 = 0$, Eq. (2.4) goes over into the parabolic equation using the approximate theory of diffraction [55,56]. Thus, (2.4) corresponds to the so called quasioptical approximation, and can describe the self-action of a normal wave in both an isotropic and an anisotropic medium.

Introducing the eikonal s of complex amplitude:

$$A=A_0\exp(-iks), \tag{2.5}$$

which is an addition to the eikonal of the plane wave (see (2.3)), we obtain from (2.4) the system

$$2\frac{\partial s}{\partial z}+(\nabla_\perp s)^2=\frac{\varepsilon_2A_0^2}{\varepsilon_0}+\frac{\Delta_\perp A_0}{k^2A_0}, \tag{2.6}$$

$$\frac{\partial A_0}{\partial z}+\nabla_\perp s\cdot\nabla_\perp A_0+\frac{1}{2}A_0\Delta_\perp s=0. \tag{2.7}$$

for a two-dimensional beam and a three-dimensional cylindrically-symmetrical beam, (2.6) and (2.7) take the form

$$2\frac{\partial s}{\partial z}+\left(\frac{\partial s}{\partial r}\right)^2=\frac{\varepsilon_2A_0^2}{\varepsilon_0}+\frac{1}{k^2A_0}\left(\frac{\partial^2A_0}{\partial r^2}+\frac{m}{r}\frac{\partial A_0}{\partial r}\right), \tag{2.8}$$

$$\frac{\partial A_0}{\partial z}+\frac{\partial s}{\partial r}\cdot\frac{\partial A_0}{\partial r}+\frac{A_0}{2}\left(\frac{\partial^2 s}{\partial r^2}+\frac{m}{r}\frac{\partial s}{\partial r}\right)=0. \tag{2.9}$$

Here $m = 0$ corresponds to a two-dimensional beam and $m = 1$ to a three-dimensional beam. In the right side of (2.8) there are two "forces" determining the behavior of the eikonal (wave front): the "force" connected with the nonlinear refraction and the diffraction "force."

2.2. Geometrical Optics of a Cubic Medium. Self-focusing Length. Nonlinear Aberrations

A feature of the equations of nonlinear geometrical optics

$$2\frac{\partial s}{\partial z}+\left(\frac{\partial s}{\partial r}\right)^2=\frac{\varepsilon_2A_0}{\varepsilon_0}, \tag{2.10}$$

$$\frac{\partial A_0}{\partial z}+\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r}+\frac{A_0}{2}\left(\frac{\partial^2 s}{\partial r^2}+\frac{m}{r}\frac{\partial s}{\partial r}\right)=0, \tag{2.11}$$

distinguishing it from the equations of linear optics, is the fact that the equation for the eikonal (2.10) is no longer independent of the equation for the amplitude (2.11), as a result of which its solution, which determines the paths of the rays, depends essentially in general not only on the initial phase front but also on the form of the amplitude profile of the wave. It is impossible to obtain an analytic solution of Eqs. (2.10) and (2.11) in general form for arbitrarily specified initial profiles $s(0, r)$ and $A_0(0, r)$. At the same time, we can point to several particular solutions that are of practical interest.

We recall first that in a linear medium ($\epsilon_2 = 0$) Eqs. (2.10) and (2.11) are satisfied, in particular, by cylindrical and spherical waves (they are written here in a form corresponding to quasioptic approximation):

$$s=\frac{r^2}{2(z+R)},\quad A_0^2=\frac{E_0^2}{\left(\frac{z}{R}+1\right)^{1+m}}F\left\{\frac{r}{a\left(\frac{z}{R}+1\right)}\right\}, \tag{2.12}$$

where R is the radius of curvature of the initial wave front (R > 0 corresponds to a diverging wave, R < 0 to a converging wave); F is an arbitrary function which specifies the distribution of the amplitude across the beam. Since all the rays cross at a single point (at the focus $Z_f = -R$), there is no aberration.

In a nonlinear medium ($\epsilon_2 \neq 0$) a natural generalization of these solutions will be cylindrical and spherical waves with variable radius of curvature

$$s=\frac{r^2}{2}\beta(z)+\varphi(z),\quad A_0^2=\frac{E_0^2}{f^{1+m}(z)}\left[1-\frac{2r^2}{a^2f(z)}\right], \tag{2.13}$$

where $\beta^{-1}(z)$ is the radius of curvature, $\varphi(z)$ is the addition to the eikonal due to the change in the wave propagation velocity, f(z) is the dimensionless width of the beam, E_0 is the field intensity on the beam axis. At the entrance to the nonlinear medium we have for z = 0

$$\beta(0)=R^{-1},\quad \varphi(0)=0,\quad f(0)=1, \tag{2.14}$$

$$A_0^2(r,0)=E_0^2\left(1-\frac{2r^2}{a^2}\right). \tag{2.14a}$$

We note that in a cubic medium aberrationless focusing is realized not for an arbitrary initial amplitude profile of the beam (compare with (2.12) for a linear medium), but only for a parabolic profile (2.13). A solution in the form (2.13) is, in essence, one of the self-similar solutions for (2.10) and (2.11).

Substituting (2.13) in (2.10) and (2.11), we arrive at the following ordinary equations for the functions β, φ, and f:

$$\frac{d\varphi}{dz}=\frac{\varepsilon_2E_0^2}{2\varepsilon_0}f^{-(1+m)}, \tag{2.15}$$

$$\frac{d\beta}{dz}=\frac{1}{f(z)}\frac{df}{dz}, \tag{2.16}$$

$$\frac{d^2f}{dz}=-\frac{2\varepsilon_2E_0^2}{\varepsilon_0a^2f^{2+m}(z)}\ ^{*)}. \tag{2.17}$$

Taking into account the boundary conditions (2.14), the first integral of (2.17) is

$$\left(\frac{df}{dz}\right)^2=\frac{4\varepsilon_2E_0^2}{(1+m)\varepsilon_0a^2f^{1+m}}+C,\quad C=\frac{1}{R^2}-\frac{4\varepsilon_2E_0^2}{(1+m)\varepsilon_0a^2}. \tag{2.18}$$

For the case of a spherical wave (m = 1), the integration of (2.18) leads to the following dependence of the beam width on the distance covered by the wave in the nonlinear medium

$$f^2(z)=\left(\frac{1}{R^2}-\frac{2\varepsilon_2E_0^2}{a^2\varepsilon_0}\right)z^2+\frac{2}{R}z+1. \tag{2.19}$$

By virtue of (2.13), the width of the beam determines the intensity of the wave on the beam axis, $A_0^2(z,0) = E_0^2/f^2(z)$. When $f(z) \to 0$, the amplitude $A_0 \to \infty$; the corresponding point Z_f on the z axis is a focus. The quadratic equation obtained from (2.19) at f = 0 determines in the general case two focal points, $Z_{f,1}$ and $Z_{f,2}$:

$$\frac{1}{Z_{f,1}}=\frac{1}{R_{nl}}-\frac{1}{R}, \tag{2.20}$$

$$\frac{1}{Z_{f,2}}=-\frac{1}{R_{nl}}-\frac{1}{R},\quad R_{nl}=a\sqrt{\frac{\varepsilon_0}{2\varepsilon_0E_0^2}}. \tag{2.21}$$

When $\epsilon_2 > 0$ the nonlinear medium exerts a focusing action on the light beam, and when $\epsilon_2 < 0$ a defocusing action (f(z) never vanishes). A parameter characterizing the focusing properties of the nonlinear medium for a beam with an amplitude distribution described by formula (2.13) is the quantity (see [6,14,16])

$$R_{nl}=a\sqrt{\frac{\varepsilon_0}{2\varepsilon_2E_0^2}}. \tag{2.22}$$

It follows from (2.20) that the quantity R_{nl} determines the distance at which a light beam with a plane phase front ($R \to \infty$) and with amplitude distribution (2.14a) becomes self-focused in a linear medium (see Fig. 2)*. For a converging beam (R < 0) the focal distance in a nonlinear medium with $\epsilon_2 > 0$ decreases. Moreover, an initially diverging beam R > 0 becomes self-focused upon entering into the nonlinear medium if R is not too small. The critical value of the divergence is $R_{cr} = R_{nl}$.

The second focal point (2.21) exists only for a converging wave ($R < 0$, $|R| < R_{nl}$), and in this case $Z_{f,2} > Z_{f,1}$. Using (2.10) we can determine also the trajectories of the rays in the nonlinear medium:

$$\frac{dr}{dz}=r(z)\beta(z), \tag{2.23}$$

$$r(z)=r_0f(z). \tag{2.24}$$

The trajectories of all rays are similar (self-similarity of the solution).

Figure 9 shows the rays in the self-focusing of a three dimensional beam (m = 1) for different ratios of R to R_{nl}. The change of the intensity on the beam axis $A_0^2(z,0) = E_0^2/f^2(z)$ for the case $R \to \infty$ (plane initial phase front of the beam) is shown by curve 1 of Fig. 10.

For the case of self-focusing of a two-dimensional beam (m = 0) we obtain, after integrating (2.18), a more complicated expression for the width of the beam in an arbitrary cross section z

$$f=\frac{1}{\sin^2\alpha}\sin^2\left\{\alpha+\sin\alpha\left[\sqrt{f(1-f\sin^2\alpha)}-\cos\alpha+\frac{z\sin^2\alpha}{R\cos\alpha}\right]\right\}, \tag{2.25}$$

*Equation (2.17) can be used also to investigate stationary thermal defocusing of laser beams. In this case it is necessary to reverse in this equation the sign of ϵ_2 (now ϵ_2 means the addition to the dielectric constant, obtained on the basis of the stationary solution of the equation of thermal conductivity), and take into account the damping of the light wave. We then have in lieu of (2.17) $f'' = 2\epsilon_2E_0^2/\epsilon_0a^2\,[\exp(-\delta_0z)]/f^3$. For the case of greatest practical interest, $a\sqrt{\epsilon_0/\epsilon_2E_0^2} >> \delta_0^{-1}$ (which is characteristic of gas-laser beams of small and medium power) the beam divergence angle in the medium is directly proportional to the beam power P, $\theta(z) = af'/f = \theta_0 + 1/\pi \cdot dn/dT \cdot P/n\kappa a(1-\exp[-\delta_0z])$ (the notation is the same as in formula (1.25)). When $\delta_0z \to \infty$ the divergence reaches a stationary value. For more details see [114], where the results of the calculation are compared with experimental data on the propagation of an argon-laser beam in liquids.

*Thus, the results of the calculation confirm the validity of the qualitative estimates made in Sec. 1. Formula (2.22) was verified experimentally by Townes and co-workers [11]. It must be borne in mind that the character of the dimension of the beam under the square root in (2.22) has in the general case of an arbitrary beam the meaning of an effective radius of curvature of the amplitude profile (see also formulas (2.32) and (2.33)). Therefore in experimental papers (such as [18]) the formulas for R_{nl} contain in lieu of a the quantity $a/\mathfrak{F}$, where $\mathfrak{F}$ is the ratio of the radius of the beam to the characteristic radius of curvature (for a Guassian beam $\mathfrak{F} = 1$). The value of $\mathfrak{F}$ is determined from the slope of the plot of R_{nl} against P. We note, incidentally, that formula (2.20) points to an additional possibilities of verifying the theory, and consequently, of a more detailed study of the dynamics of self-focusing when the form of the phase front of the incident beam changes.

where $\cos\alpha = R_{nl}/|R| \le 1$, and

$$f=\frac{1}{\mathrm{sh}^2\alpha}\mathrm{sh}^2\left\{\alpha+\mathrm{sh}\,\alpha\left[\sqrt{f(1+f\,\mathrm{sh}^2\alpha)}-\mathrm{ch}\,\alpha-\frac{z\,\mathrm{sh}^2\alpha}{R\,\mathrm{ch}\,\alpha}\right]\right\}, \quad (2.26)$$

where $\sinh\alpha = R_{nl}/|R| \ge 1$.

Formula (2.25) describes the behavior of beams in a strongly focusing nonlinear medium ($|R| > R_{nl}$) according to (2.25), the beam is periodically focused and defocused; the number of focal points is infinite:

$$Z_f=-\frac{R\cos\alpha}{\sin^3\alpha}\left[\frac{\sin 2\alpha}{2}-\alpha+n\pi\right]. \quad (2.27)$$

In particular, a two-dimensional beam with a plane phase front ($R = \infty$) and an amplitude distribution (2.14a) becomes self-focused at a distance

$$Z_{f,1}=\frac{\pi}{2}R_{nl} \quad (2.28)$$

The corresponding plot for the intensity on the axis is shown in Fig. 10, curve 2). In the case of a weakly focusing medium, when $R_{nl} \ge |R|$, there is only one focal point

$$Z_f=-\frac{R\,\mathrm{ch}\,\alpha}{\mathrm{sh}^3\alpha}\left[\frac{\mathrm{sh}\,2\alpha}{2}-\alpha\right]. \quad (2.29)$$

If the beam has an amplitude profile different from parabolic (2.13), then it is no longer self-focused at a point as a whole, and aberration appears. In this case the focal distance for a beam with sufficiently smooth profile can be estimated in the following manner. In the first approximation, one can seek the eikonal in the same form as in the absence of aberration:

$$s=\frac{r^2}{2}\frac{1}{f}\frac{df}{dz}+\varphi(z). \quad (2.30)$$

Then the general solution of the equation for the amplitude (2.11) has the form of the arbitrary function

$$A_0^2=\frac{E_0^2}{f^{1+m}(z)}F\left[\frac{r}{f(z)}\right]. \quad (2.31)$$

Substituting A_0^2 from (2.31) into (2.10), we expand the function F in powers of the transverse coordinate r and obtain, with allowance for (2.30), the following equation for the width of the beam:

$$\frac{d^2f}{dz^2}=-\frac{2\varepsilon_2E_0^2}{\varepsilon_0f^{2+m}}F''(0)\ \left(F''=\frac{d^2F}{dr^2}\right). \quad (2.32)$$

Comparing (2.32) with (2.17) we see that the role of R_{nl} is assumed by

$$R'_{nl}=\sqrt{\frac{2\varepsilon_0}{\varepsilon_2E_0^2F''(0)}}, \quad (2.33)$$

where $1/F''(0)$ characterizes the radius of curvature of the intensity profile of the incoming beam on its axis (see (2.31)). Near the focal point, where aberration plays an important role, Eq. (2.32) and its solutions no longer hold.

A somewhat more general analysis, which makes it possible to take into account aberration phenomena, can be made for a two-dimensional beam. After introducing new independent variables

$$u=\frac{\partial s}{\partial x},\quad J=A_0^2 \quad (2.34)$$

Eqs. (2.10) and (2.11) reduce to a system of first-order equations

$$\frac{\partial u}{\partial z}+u\frac{\partial u}{\partial x}-\gamma\frac{\partial J}{\partial x}=0, \quad (2.35)$$

$$\frac{\partial J}{\partial z}+u\frac{\partial J}{\partial x}+J\frac{\partial u}{\partial x}=0, \quad (2.36)$$

where $\gamma = \epsilon_2/2\epsilon_0$. We note that (2.35) and (2.36) have the same form as the equations describing nonstationary

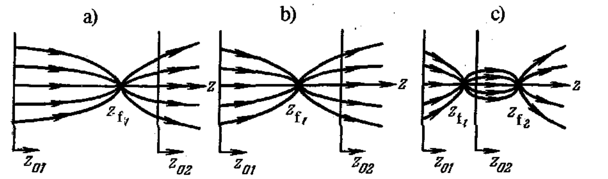

FIG. 9. Trajectories of rays of three-dimensional light beam with initial amplitude distribution $A_0^2 = E_0^2(1 + 2r^2/a^2)$. In all the Figures the cross section z_{01} corresponds to the entry into the medium of the converging wave ($R < 0$). a) $|R| > R_{nl}$—weak convergence of the beam (one focus is produced); b) $|R| = R_{nl}$—critical convergence; c) $|R| < R_{nl}$—strong convergence, two foci are produced. The cross section $z = z_{02}$ corresponds to entrance of a diverging wave ($R > 0$). a) $R < R_{nl}$ (no focus), b) $R = R_{nl}$ (no focus) c) $R > R_{nl}$ (no focus).

isentropic flow of a barotropic liquid (see [57]), and differ from the latter when $\epsilon_2 > 0$ and $\gamma > 0$ only in the sign of γ. This circumstance is very important, since a change in the sign of γ transforms the system from a hyperbolic one into an elliptical one. It is known that the system (2.35) and (2.36) can be reduced to a linear system by a hodograph transformation. The latter makes it possible to construct a sufficiently general procedure for the analysis of the equations of nonlinear geometrical optics. It can be shown, however, that the system has one particular solution of practical importance

$$u=-\frac{2\gamma zJ}{a}\,\mathrm{th}\left(\frac{x-uz}{a}\right), \quad (2.37)$$

$$J=\left(J_0+\frac{\gamma z^2J^2}{a^2}\right)\mathrm{ch}^{-2}\left(\frac{x-uz}{a}\right). \quad (2.38)$$

Formulas (2.37) and (2.38) describe, in the geometrical-optics approximation, the propagation of a wave having at the section $z = 0$ a plane phase front ($u = \partial s/\partial x = 0$) and an intensity distribution

$$J(x,0)=J_0\,\mathrm{ch}^{-2}\left(\frac{x}{a}\right). \quad (2.39)$$

in a nonlinear medium.

In constructing the ray trajectories for this case, we can use the analog of the isocline method, used to construct trajectories of motions on the phase plane. We note first that the angle between the wave vector p and z axis (unit vector Z_0) for near-axis rays is approximately equal to u, since

$$\mathrm{tg}\,\mathbf{p}\mathbf{z}_0 \simeq \mathbf{p}\mathbf{z}_0 \simeq \frac{\partial s}{\partial x}=u. \quad (2.40)$$

Using the implicit solution of (2.37) and (2.38), we can plot the field of the ray vectors on the (x, z) plane. The

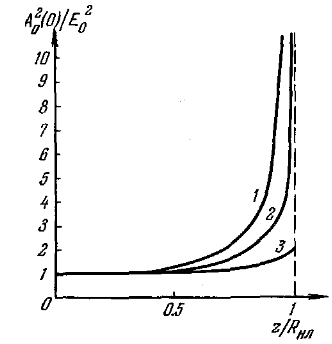

FIG. 10. Variation of intensity along the axis of the light beam: 1—Aberrationless self-focusing of three-dimensional beam; 2—aberrationless self-focusing of a two-dimensional beam; 3—aberration self-focusing of a two-dimensional beam.

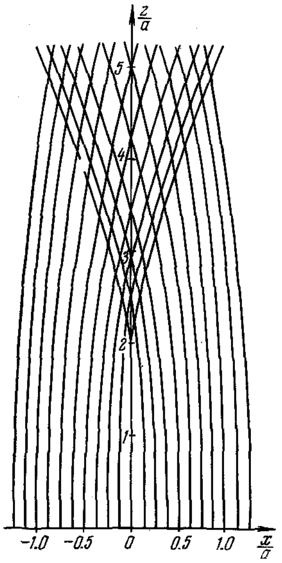

FIG. 11. Ray trajectories in a two-dimensional light beam propagating in a nonlinear medium with $\epsilon_2 > 0$. For $z = 0$ and $u = ds/dx = 0$, $I = I_0 \cosh^{-2}(x/a)$; $\epsilon_2 E_0^2/\epsilon_0 = 1/8$ is a parameter.

lines tangent to these vectors at each instant will be the ray trajectories. The rays constructed in this manner are shown in Fig. 11 for the coefficient $\epsilon_2 E_0^2/\epsilon_0 = 1/8$, while Fig. 12 shows the corresponding plots of the distribution of the intensity in different sections of the beam. The rate of growth of the intensity on the beam axis is shown by curve 3 of Fig. 10.

It follows from Fig. 11 that initially the rays approach the beam axis and self-focusing takes place; however, unlike the self-focusing of a wave with initial profile (2.14a), which was considered above, rather strong aberrations occur in the self-focused beam in the present case. The peripheral rays cross the z axis later than the near-axis rays. The light intensity reaches a (finite) maximum in a plane located at a distance Z_f from the boundary of the nonlinear medium,

$$Z_f = 2R_{nl}. \tag{2.41}$$

When $z > Z_f$ the rays begin to intersect.

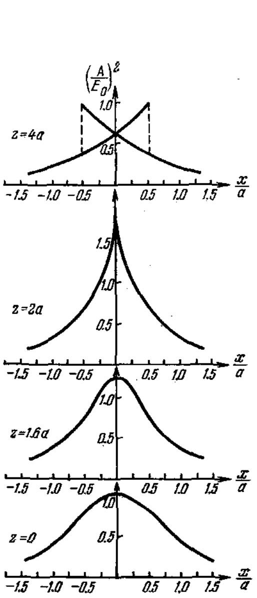

FIG. 12. Distribution of intensity over the cross section of a two-dimensional light beam propagating in a nonlinear medium, for different values of the parameter z/a. The beam parameters are the same as for Fig. 11.

Aberrational self-focusing is apparently the most typical for real beams. Although for three-dimensional beams the picture of the aberrations cannot be easily calculated analytically (in this sense, the example considered above is a fortunate exception), it is clear that the rate of growth of the field on the axis of real three-dimensional beams subject to aberration will be lower than on curve 1 of Fig. 10. The aberrations can play a certain role in the formation of the proper optical waveguide.

2.3. Stationary Self-focusing of Complicated Beams. Role of Linear and Nonlinear Absorption

The analytic results obtained above pertain to waves close to cylindrical and spherical and having a sufficiently smooth amplitude front. A complete calculation of the behavior of beams of more complicated structure is possible, apparently, only by numerical methods (see [29]). However, interesting analytic results can be obtained by the perturbation method, that is, by assuming the changes of the amplitude and phase fronts of the plane wave due to the nonlinearity to be small. In this case, an appreciable simplification of the equations makes it possible to consider the self-action problem under assumptions which are much more general than those used so far, with respect to the properties of the nonlinear susceptibility of the medium (in particular, to take into account the linear and nonlinear absorption, the spatial dispersion of the nonlinear susceptibility, the finite relaxation times, etc.).

We turn first to stationary self-focusing of complicated beams in a cubic lossless medium. Assume that a plane homogeneous wave with complex amplitude

$$A = E_0 \exp\left\{-ik\frac{\varepsilon_2}{2\varepsilon_0}E_0^2 z\right\} \tag{2.42}$$

propagates in such a medium. Let us trace the behavior of small perturbations of the square of the real amplitude α and of the eikonal ψ, that is, let us represent the real amplitude and the eikonal in the form

$$A_0^2 = E_0^2 + \alpha, \quad s = \frac{\varepsilon_2}{2\varepsilon_0}E_0^2 z + \psi, \quad \frac{\partial s}{\partial r} = \frac{\partial \psi}{\partial r} = u. \tag{2.43}$$

For small perturbations ($\alpha \sim \mu'$, $u \sim \mu'$, where μ' is a small parameter characterizing the perturbation), Eqs. (2.10) and (2.11) become linearized and reduce to a single equation of the elliptic type for a perturbation of the intensity α

$$\frac{\partial^2 \alpha}{\partial z^2} + \frac{\varepsilon_2}{\varepsilon_0}E_0^2 \Delta_\perp \alpha = 0. \tag{2.44}$$

In the two dimensional case ($\Delta_\perp = \partial^2/\partial x^2$), Eq. (2.44) has a general solution for arbitrary boundary conditions

$$\alpha(x, 0) = \Phi(x, 0) \text{ и U } u(x, 0) = u_0(x),$$

$$\alpha(x, z) = \frac{1}{2}\left[\Phi(x + i\Gamma z) + \Phi(x - i\Gamma z)\right] - \frac{E_0^2}{2i\Gamma}\left[u_0(x + i\Gamma z) + u_0(x - i\Gamma z)\right], \tag{2.45}$$

where $\Gamma^2 = \epsilon_2 E_0^2/2\epsilon_0$, and Φ and u_0 are analytic functions. The presence of the solution (2.45) thus makes it possible, in the case of a two-dimensional beam, to trace the spatial development of a perturbation of arbitrary type. For a Gaussian beam with a plane phase front

$$u_0(x, 0) = 0, \quad \Phi(x, 0) = \alpha_0 \exp\left(\frac{-2x^2}{a_1^2}\right) \tag{2.46}$$

we obtain from (2.45)

$$\alpha=\alpha_0 \exp\left[-\frac{2(x^2-\Gamma^2 z^2)}{a_1^2}\right]\cos 4\frac{\Gamma x z}{a_1^2}, \tag{2.47}$$

that is, the Gaussian beam reveals a tendency to become stratified on propagating in the nonlinear medium. The characteristic distance at which the perturbations build up appreciably is $Z_f \simeq a_1/2\Gamma \simeq R_{nl}$—the self-focusing length which figured earlier in the more exact theory (see (2.33), (2.41))*.

It is impossible to write general solutions for a three-dimensional beam; in this case the functions $\Phi(x, y)$ and $u_0(x, y)$ can be expanded in Fourier integrals and it is possible to trace the behavior of individual Fourier components:

$$\Phi(x, y)=\alpha_0 \exp\left\{-\frac{i(x+y)}{a}\right\}. \tag{2.48}$$

Substituting (2.48) in (2.44), we have

$$\alpha=\alpha_0 \exp\left\{-\frac{i(x+y)}{a}+\frac{z}{R_{nl}}\right\} \tag{2.49}$$

—in the perturbation method, the self-focusing length R_{nl} is the increment of exponentially growing perturbations. The fact that the perturbation method makes it possible not only to trace the qualitative picture of the behavior of the beam, but also to determine the self-focusing length, can be used for the calculation of the value of R_{nl} under much more general assumptions concerning the properties of the medium than was the case in Secs. 2.1–2.3.

Let us account first for the influence of the dissipative processes on self-focusing. In the presence of linear and nonlinear absorption (quadratic in the field, for example two-photon absorption, see formula (1.2a)), the equation for the amplitude in the system (2.10) and (2.11) is altered and takes the form

$$\frac{\partial A_0^2}{\partial z}+\frac{\partial s}{\partial r}\frac{\partial A_0^2}{\partial r}+A_0^2\Delta_\perp s+2\delta_0 A_0^2+2\delta_2 A_0^4=0. \tag{2.50}$$

Assume first that $\delta_0 > 0$ and $\delta_2 = 0$. We consider a linear system in which a plane damped wave propagates, $A_0^2 = E_0^2 \exp(-2\delta_2 z)$. Representing the intensity of the perturbed wave in the form

$$A_0^2=\left\{E_0^2+\alpha(z)\exp\left(-i\frac{x+y}{a}\right)\right\}\exp(-2\delta_0 z) \tag{2.51}$$

and linearizing the system (2.10), (2.50), we obtain an equation for

$$\frac{d^2\alpha}{dz^2}+\frac{2\alpha}{R_{nl}^2}\exp(-2\delta_0 z)=0. \tag{2.52}$$

Equation (2.52) has a solution that grows with increasing z and is expressed in terms of the modified Bessel functions:

$$\alpha=\alpha_0 K_0^{-1}\left(\frac{1}{\delta_0 R_{nl}}\right)K_0\left[\frac{\exp(-\delta_0 z)}{\delta_0 R_{nl}}\right]. \tag{2.53}$$

(When $\delta_0 \to 0$, formula (2.53) goes over into (2.49); $\alpha(z) \sim \exp(z/R_{nl})$.) We see from (2.53) that linear losses increase the self-focusing length. An approximate estimate of the self-focusing length Z_f can be obtained by assuming that the argument of the Bessel function changes by unity over this length:

$$R_{nl}=\frac{1-\exp(-\delta_0 Z_f)}{\delta_0}. \tag{2.54}$$

From (2.54) there follows directly also the characteristic difference between the self-focusing in a dissipative medium and self-focusing in a lossless medium. Whereas in the latter, by virtue of the relation $Z_f = R_{nl}$, the power necessary for self-focusing is the smaller the larger the length over which the self-focusing is observed*, in a lossy medium there is a minimum self-focusing threshold power; it is determined from the equation $R_{nl} = \delta_0^{-1}$. The meaning of the latter equation is obvious; the effect of self-action of the wave, which leads to self-focusing, accumulates only over distances not exceeding the photon mean free path in the medium.

It is possible to determine in similar fashion the self-focusing length in a medium with $\delta_0 < 0$ (active medium). We have

$$\alpha=\alpha_0 I_0^{-1}\left(\frac{1}{\delta_0 R_{nl}}\right)I_0\left[\frac{\exp(\delta_0 z)}{R_{nl}\delta_0}\right]. \tag{2.55}$$

in this case, obviously, $Z_f < R_{nl}$ the self-focusing of the beam is faster than in a lossless medium. At large gains ($R_{nl}\delta_0 \gg 1$) we can write $Z_f \delta_0 = \ln R_{nl}\delta_0$. One cannot disregard, apparently, the appreciable shortening of the self-focusing length as a result of amplification in experiments on stimulated Raman scattering (here $\delta_0 < 0$ for the scattered radiation), and also in the analysis of the angular and spatial structure of the radiation of certain types of lasers (particularly semiconductor and CO_2 lasers).

With the aid of calculations similar to those described above, we can calculate the value of Z_f also for a medium with nonlinear absorption. Considering the most typical case $\delta_0 = 0$ and $\delta_2 > 0$, and a perturbed wave in the form

$$A_0^2=\left\{E_0^2+\alpha(z)\exp\left(-i\frac{x+y}{a}\right)\right\}[1+2\delta_2 E_0^2 z]^{-1}, \tag{2.56}$$

we obtain the formula

$$\alpha(z)=\alpha_0\frac{K_2\left[\frac{(1+\delta_2 E_0^2 z)^{1/2}}{\delta_2 E_0^2 R_{nl}}\right]}{K_2\left[\frac{1}{\delta_2 E_0^2 R_{nl}}\right]}. \tag{2.57}$$

When $\delta_2 \to 0$ formula (2.57) goes over into (2.49). In accordance with (2.57), we can define Z_f as follows (see also [72])

$$Z_f=R_{nl}+\frac{ka^2}{2}\frac{\epsilon_2''}{\epsilon_2'}. \tag{2.58}$$

It follows from (2.58) that the presence of nonlinear absorption increases the self-focusing length. In this case Z_f does not vanish even if $P \to \infty$, and tends to a value $ka^2/2 \cdot \epsilon_2''/\epsilon_2'$, which is determined by the diffraction length of the beam $R_d = ka^2/2$ and by the nonlinear properties of the medium. The ratio ϵ_2''/ϵ_2' is determined to a considerable degree by the radiation wavelength. In liquids, where ϵ_2' is determined essentially by the high frequency Kerr effect, and ϵ_2'' by two-photon absorption at the wavelength of a ruby laser we usually have $R_d\epsilon_2''/\epsilon_2' \ll R_{nl}$. However in experiments at shorter wavelengths (in particular, in the study of self-focusing of radiation of optical harmonics of ruby

*The difference in the numerical factors involved in the value r_{nl} are connected here and throughout with differences in the definition of the effective diameter of the beam.

*It must be recalled, of course, that in this section we use the geometrical-optics approximation, that is, we are dealing with lengths $Z_f < R_d$ or powers $P > P_{cr}$ (see formulas (1.6) and (1.7) and the corresponding results of Sec. 3).

and glass lasers), situations in which $\epsilon_2''/\epsilon_2' \simeq R_{nl}/R_d$ are possible*.

The change of the field intensity of the self-focusing wave, taking into account in formula (2.54)—(2.58), is, of course, not the only consequence of dissipative processes. Heating of the medium can also be very significant. As indicated in Sec. 1, a stationary distribution of the temperature is established only after times exceeding $\tau_T \simeq a^2/\kappa$, so that the problems of the influence of thermal effects on self-focusing of laser pulses (whose durations usually do not exceed 10^{-3} sec) turns out to be essentially nonstationary.

Problems of nonstationary self-focusing do arise, however, and not only in connection with the problem of self-focusing in a dissipative medium. There is a clearly pronounced tendency in recent years towards a shortening of laser pulses (pulses of 10^{-9} sec duration are now typical; there are reports of generation of powerful pulses of duration 10^{-10}—10^{-11} sec; see [49]) makes it necessary to take relaxation processes into account also for one of the "fastest" self-focusing mechanisms, that due to the Kerr effect. We present below an analysis of the features of nonstationary self-focusing, using as an example Kerr self-focusing and self-focusing due to electrostriction.

2.4. Nonstationary Self-focusing. Nonlinear Dispersion. Chromatic Envelope Aberration. Spatial Dispersion of Nonlinearity.

To investigate nonstationary self-focusing it is necessary to introduce derivatives with respect to time in (2.10) and (2.11), and to solve these equations jointly with the material equation characterizing the behavior of the nonlinear polarization (we can no longer use the simple algebraic connection between the polarization and the field (see [62])). The time derivatives in the abbreviated equations come from expansions of the type (1.13); confining ourselves, as before, to the geometrical-optics approximation, we need include in them only terms with the first derivatives of the complex amplitudes†. The system of equations of the nonstationary self-focusing due to the Kerr effect then takes the following form:

$$\frac{2}{v}\frac{\partial s}{\partial t}+2\frac{\partial s}{\partial z}+\left(\frac{\partial s}{\partial r}\right)^2=\frac{\varepsilon_2}{\varepsilon_0}p, \qquad (2.59)$$

$$\frac{1}{v}\frac{\partial A_0}{\partial t}+\frac{\partial A_0}{\partial z}+\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r}+\frac{A_0}{2}\Delta_\perp s=0 \qquad (2.60)$$

(v—group velocity of light),

$$\tau\frac{\partial p}{\partial t}+p=A_0^2. \qquad (2.61)$$

(Here τ is the relaxation time, see Sec. 1.) To calculate the self-focusing length we shall use, as before, the perturbation method; now, however, we shall trace the behavior of one spectral component of the space-time Fourier expansion. We represent the intensity A_0^2, the transverse derivative of the eikonal, and the quantity p (proportional to the corresponding component of the anisotropic tensor; see Sec. 1) in the form (cf. formula (2.43))

$$A_0^2=E_0^2+\alpha(z)\exp\left\{i\nu t-i\frac{x+y}{a}\right\}, \qquad (2.62a)$$

$$u=u'(z)\exp\left\{i\nu t-i\frac{x+y}{a}\right\}, \qquad (2.62b)$$

$$p=E_0^2+p'(z)\exp\left\{i\nu t-i\frac{x+y}{a}\right\}. \qquad (2.62c)$$

Substituting (2.62) in (2.59)—(2.61) we arrive at an equation for α:

$$\frac{d^2\alpha}{dz^2}-\frac{\alpha}{R_{nl}(1+i\nu\tau)}=0, \qquad (2.63)$$

from which we can easily determine the self-focusing length of the specified spectral component (ν, a^{-1}):

$$Z_f(\nu,\ a^{-1})=R_{nl}\frac{\sqrt{2}\sqrt{1+(\nu\tau)^2}}{\sqrt{1+\sqrt{1+(\nu\tau)^2}}}, \qquad (2.64)$$

where, as before, Z_f is determined from the real part of the increment of (2.63). It is seen from (2.64) that in a relaxing medium the self-focusing length of a modulated wave increases with increasing modulation frequency ν. The indicated effect can be called "chromatic aberration" for the envelope. As a result of the indicated aberration, sufficiently short wave packets are not self-focused as a unit, but a space-time stratification of the packet takes place: the focal points for the different spectral components are distributed over a length which increases with increasing $\nu\tau$. The physical cause of this effect is quite obvious—the nonlinear response of the system is different for different modulation frequency, nonlinear dispersion of the phase velocity takes place of the form $[1+(\nu\tau)^2]^{-1/2}$ (deceleration or acceleration of the modulated wave as a result of nonlinearity is smaller than that of the unmodulated wave)*.

It is possible to consider in similar fashion also the nonstationary processes occurring in striction self-focusing—to this end Eqs. (2.59) and (2.60) must be solved simultaneously with the wave equation (cf (1.23))

$$\Delta p-\frac{1}{u^2}\frac{\partial^2 p}{\partial t^2}+\frac{\Gamma}{u^2}\frac{\partial p}{\partial t}=\Delta A_0^2. \qquad (2.65)$$

Using the perturbation method (see (2.62)) we can calculate the quantity

$$Z_f(\nu,\ a^{-1})=R_{nl}\frac{\sqrt{2}\sqrt{(1+\nu^2\tau_\perp^2)^2+(2\Gamma\nu\tau_\perp^2)^2}}{\sqrt{1-\nu^2\tau_\perp^2+\sqrt{(1-\nu^2\tau_\perp^2)^2+(2\Gamma\nu\tau_\perp^2)^2}}}, \qquad (2.66)$$

where $\tau_\perp = a/u$. From (2.66) we see that chromatic aberration takes place in striction self-focusing as well as in Kerr self-focusing. However, in the case of striction the nonlocal character of the nonlinear response of the medium leads to important peculiarities. Indeed, the aberration has now not a relaxation but a resonant character; the minimum value of Z_f corresponds not to $\nu = 0$, as for the Kerr effect, but to $\nu = \tau_\perp^{-1}$ or $a = u/\nu = \lambda_{ac}$—the wavelength of sound at the frequency ν. In this case

$$Z_f^{\min}(\nu,\ a^{-1})=R_{nl}\sqrt{2\lambda_{ac}/Z_\delta},\ \text{where}\ \ Z_\delta=u/2\Gamma\ \ (\lambda_{ac}<Z_\delta).$$

Thus, the acoustic coupling transverse to the beam can greatly change the rate of self-focusing, if the pertur-

*It must be borne in mind that the Kramers-Kronig relations hold for the real and imaginary parts of ϵ_2 (see [36]).

†This corresponds to the first approximation of dispersion theory, in which account is taken only of the effect of the group delay of the wave packet. Allowance for the second derivatives (the dispersion spreading of the packet) is beyond the scope of geometrical optics, and it must be made together with an allowance for the diffraction spreading of the beam (see Sec. 3).

*The effect of nonlinear dispersion is significant, of course, also in self-action of plane wave packets; see Sec. 3.6.

bations of the light wave and the acoustic waves are matched, and $a = \lambda_{ac}$.

Summarizing the results pertaining nonstationary self-focusing, we must emphasize once more that they characterized the behavior of individual space-time Fourier components; for an analysis of the behavior of the laser pulse as a whole, it is necessary to take into consideration the energy distribution of the spectrum. At the same time, the qualitative regularities revealed above (nonlinear chromatic aberrations, effects of spatial dispersion of the nonlinearity) remain, of course, in force. We note, finally, that the foregoing results can be used directly to interpret experiments in which the envelope of laser radiation contains a sharply pronounced frequency ν_0 (for example, the frequency of the intermode beats in a giant-pulse laser or the repetition frequency of the spikes in a powerful free-running laser*). In particular, in striction self-focusing, acoustic waves of frequency ν_0 can lead to a decay of the self-focusing beam into filaments with transverse scale $a_0 \simeq u/\nu_0$. This effect should become especially strongly manifest if the acoustic waves attenuate little over the beam width $Z_\delta > a$. The latter is well satisfied in solids, especially in solids at low temperatures, where Z_δ can reach tens and hundreds of centimeters. In liquids, on the other hand, the effects of spatial dispersion of striction nonlinearity can probably determine the fine structure of a beam already self-focused as a result of the Kerr effect (see Secs. 1 and 4).

The foregoing procedure can be used also in the analysis of thermal nonstationary self-focusing [41,45] and defocusing. To this end it is necessary to add in the right side of (2.65), in accordance with [40,112], a term describing Δp_{therm}, where $p_{therm} = \gamma\delta_0 \int_0^t EE^* dt$ is the rise in the pressure of the medium at the instant, which would be produced by the heat release without a change in density, and γ is the derivative of the pressure with respect to the internal energy per unit volume, taken at a constant volume. Here, obviously, the self-focusing length will be a function of the time (temporal aberrations), and $Z_f = Z_f(t)$. Temporal aberrations, just like the chromatic aberrations considered above, will lead to a space-time stratification of the laser pulse. A number of estimates concerning this effect are contained in [40,41,112]. According to [41], when $t < \tau_\perp$ one can have $Z_f < R_{nl}$ even in a medium with $\partial\epsilon/\partial T < 0$. When $t > \tau_\perp$, Z_f increases monotonically and then becomes an imaginary quantity; by virtue of the integral character of the thermal effect at sufficiently large time segments, the thermal defocusing suppresses the self-focusing; at the same time, in media with $\partial\epsilon/\partial T > 0$ (see [45]) the situation is reversed--here the heating of the medium should lead to self-focusing.

*According to [50], the average power of such a generator can reach 1–2 MW at a pulse duration 10^{-3} sec. This power exceeds the critical value for many liquids and is close to the critical power in solids, in which the self-focusing is due to striction (for glass $P_{cr} \approx 4$ MW, for calcite $P_{cr} \approx 4$ MW; see [4]).

2.5. Self Action of Plane Wave Packets. Spectrum Broadening in a Nonlinear Medium

When we considered self-action of wave packets above, we paid principal attention to the influence of time-varying modulation on the change in the angular spectrum (spatial self-focusing) of the waves. At the same time, as indicated in Sec. 1, the presence of nonlinear additions to the refractive index leads to modification of the complex envelope of a plane wave packet.

The propagation of a modulated plane wave is described in the geometrical-optics approximation by the system (compared with (2.59) and (2.60)):

$$\frac{1}{v}\frac{\partial s}{\partial t} + \frac{\partial s}{\partial z} = \frac{\varepsilon_2 A_0^2}{2\varepsilon_0}, \qquad (2.67a)$$

$$\frac{1}{v}\frac{\partial A_0}{\partial t} + \frac{\partial A_0}{\partial z} = 0, \qquad (2.67b)$$

the solutions of which are

$$A_0^2 = E_0^2 F(t - z/v), \quad s = \frac{\varepsilon_2 E_0^2 z}{2\varepsilon_0} F(t - z/v) + s_0(t - z/v). \qquad (2.68)$$

Here F characterizes the amplitude modulation of the wave and s_0 the phase modulation. It follows from (2.68) that if the wave is modulated only in amplitude at $z = 0$ ($s_0 \equiv 0$) then, as the wave propagates in the nonlinear medium, the self-action effect leads to the appearance of phase modulation. The change in frequency is determined here by the formula

$$\Delta\omega = \frac{\varepsilon_2 E_0^2 kz}{2\varepsilon_0} \frac{\partial F(t - z/v)}{\partial t}. \qquad (2.69)$$

At sufficiently large z, the modulation frequency becomes large and it is necessary, generally speaking, to take into account the dispersion properties of the medium in the next-higher approximation (see Sec. 3.6). The phase modulation described by (2.68) can lead to a noticeable broadening of the spectrum of the light pulse.

We note, finally, that in a medium without dispersion (or with very small dispersion), Eqs. (2.67) take into account in the next higher approximation the change of the group velocity of the wave as a result of the nonlinearity of the medium:

$$\frac{\partial s}{\partial z} + \frac{\varepsilon_2 A_0^2}{v\varepsilon_0}\frac{\partial s}{\partial \xi} = \frac{\varepsilon_2 A_0^2}{2\varepsilon_0}, \qquad (2.70a)$$

$$\frac{\partial A_0}{\partial z} + \frac{\varepsilon_2 A_0^2}{v\varepsilon_0}\frac{\partial A_0}{\partial \xi} = 0, \qquad (2.70b)$$

where $\xi = t - z/v$ is the "running" coordinate. In such a medium, the amplitude is deformed like a simple wave (see Fig. 13):

$$A_0 = E_0\left(\xi - \frac{\varepsilon_2 A_0^2}{v\varepsilon_0} z\right), \qquad (2.71)$$

but the frequency increases in some sections and decreases in others [23]. It follows from (2.71) that noticeable distortions of the envelope occur at charac-

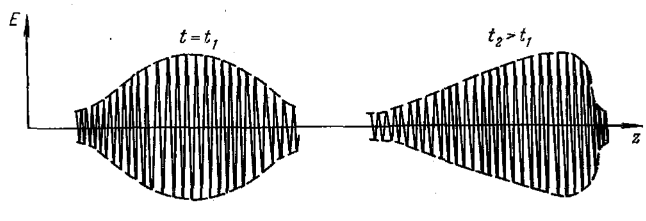

FIG. 13. Distortion of the form of the envelope of a plane wave propagating in a medium with $\epsilon = \epsilon_0 - \epsilon_2|A|^2$. As the wave propagates (along the z axis), the intially harmonic envelope becomes distorted. When $\epsilon_2 > 0$, a dip is produced on the rear end of the pulse.

teristic lengths $R \simeq \tau_p v \cdot \epsilon_0/\epsilon_2 E_0^2$ (τ_p--period of amplitude modulation). We see therefore that $R \gg R_{nl}$ under the usually realized conditions ($R/R_{nl} \simeq \sqrt{\epsilon_0/\epsilon_2 E_0^2}$ even for $\tau_p v \simeq a$).

3. WAVE OPTICS OF A NONLINEAR MEDIUM

In this section we consider effects connected with the finite wavelength; we take into account the fact that the behavior of the beam is determined not only by the effects of nonlinear refraction, which we considered in Sec. 2, but also by diffraction phenomena. As in Sec. 2, we consider primarily spherical and cylindrical waves

3.1. Stationary Aberrationless Self-focusing in a Cubic Medium with Allowance for Diffraction

We consider first the stationary processes in a medium with $\epsilon = \epsilon_0 + \epsilon_2 A_0^2$, $\epsilon_2 > 0$. The initial equations in this case are (2.8) and (2.9). We represent the eikonal in the form

$$s = \frac{r^2}{2}\beta(z) + \varphi(z), \quad \beta(z) = f^{-1}\frac{df}{dz}, \tag{3.1a}$$

and we write for the amplitude

$$A_0^2(r, z) = \frac{E_0}{f^{1+m}}\exp\left\{-\frac{2r^2}{a^2 f^2}\right\}. \tag{3.1b}$$

Just as in Sec. 2, we shall use the boundary condition (2.14). Substituting (3.1) into (2.8) and (2.9) and confining ourselves to the near-axis part of the beam (to this end it is necessary to expand in the nonlinear term in powers of r and retain only terms $\sim r^2$) we can obtain an approximate equation for the beam width f:

$$\frac{d^2 f}{dz^2} = -\frac{1}{R_{nl}^2 f^{2+m}} + \frac{1}{R_d^2 f^3}. \tag{3.2}$$

Here, as before, $R_{nl} = a\sqrt{\epsilon_0/2\epsilon_2 E_0^2}$ is the self-focusing length and $R_d = ka^2/2$ is the diffraction length of the beam. It is best to study (3.2) separately for the cases m = 0 and m = 1.

1. m = 1. We consider first the three-dimensional beam, which is of greatest practical interest. When m = 1, the first integral of (3.2) takes the form

$$\left(\frac{df}{dz}\right)^2 = \frac{1}{f^2}\left(\frac{1}{R_{nl}^2} - \frac{1}{R_d^2}\right) + C, \tag{3.3}$$

where $C = 1/R^2 - 1/R_{nl}^2 + 1/R_d$. Comparing (3.13) with (2.18), we can easily verify that, when account is taken of diffraction, the first integral, has the same form as in the geometrical-optics approximation, and only the coefficient of f^{-2} changes. The character of the behavior of the beam depends now on the relation between the quantities R_{nl} and R_d or between the total power of the beam $P = a_s^2 p_0 A_0^2/8$ and the critical power P_{cr}, determined from the equality $R_{nl} = R_d$:

$$P_{cr} = \frac{\lambda_0^2 c}{32\pi^2 n_2}. \tag{3.4}$$

When $P > P_{cr}$ ($R_{nl} < R_d$) the coefficient of f^{-2} in (3.3) is positive; the qualitative picture of the behavior of a three-dimensional beam in a cubic medium does not differ from the geometrical-optics picture investigated Sec. 2 (see the ray trajectories on Fig. 9, which apply fully also to the case considered here). In this case the diffraction changes only the spatial scale connected with the nonlinearity; therefore, at finite values of λ we can use the corresponding formulas of Sec. 2, substituting in them in lieu of R_{nl} the quantity

$$R_{nl}^{dif} = R_{nl}\left(1 - \frac{P_{cr}}{P}\right)^{-1/2} = R_d\left(\frac{P}{P_{cr}} - 1\right)^{-1/2} \tag{3.5}$$

In particular, when $P \gg P_{cr}$ a three-dimensional beam with a plane phase front and parabolic amplitude profile becomes self-focused into a point, just as in the geometrical-optics approximation, but this occurs not over a length R_{nl}, but a length $R_{nl}^{dif} > R_{nl}$.* It is important to emphasize that although the critical power does not depend on the transverse beam dimension a (see (3.4) and also (1.7)), the rate of the self focusing is determined essentially by the transverse structure of the beam. It follows from (3.5) that whereas for sufficiently broad beams ($R_{nl} \ll R_d$) we have $R_{nl}^{dif} \sim a$, just as in the geometrical-optics approximation, when $R_{nl} \to R_d$ ($P \to P_{cr}$) the dependence of R_{nl}^{dif} on a may become inverted. The latter leads to the existence of an optimal transverse spatial scale a_{opt}, determined from the condition $\partial R_{nl}^{dif}/\partial a = 0$, for which the self focusing proceeds at the fastest rate. According to (3.5) we have

$$a_{opt} = \frac{2}{k}\sqrt{\frac{\epsilon_0}{\epsilon_2 E_0^2}} \quad \text{and} \quad Z_{f,\,min} = \frac{2}{k}\frac{\epsilon_0}{\epsilon_2 E_0^2}. \tag{3.6}$$

This circumstance was pointed out by Bespalov and Talanov [16]. It is particularly important for beams with complicated amplitude profiles (inhomogeneous beams); inhomogeneities with dimensions $a \sim a_{opt}$ will become particularly strongly emphasized as a result of self-focusing in a nonlinear medium.† Estimates of a_{opt} for typical experimental conditions are as follows: if $\epsilon_2 \simeq 10^{-11}$ cgs esu (Kerr effect in CS_2), $k = 10^5\ cm^{-1}$, then a power flux of 100 MW/cm^2 yields $a_{opt} \simeq 100\ \mu$. It is interesting that the inhomogeneity of the spatial structure of ruby-laser emission from a relatively inhomogeneous crystal is of the same order of magnitude (see [68]). This means, quite probably, that experiments in which the self-focusing effect of strongly inhomogeneous laser beams is revealed by stimulated-scattering (after reaching a certain threshold light-field intensity) yield not the quantity R_{nl}^{dif}, which characterizes the entire beam as a whole, but a quantity on the order of $Z_{f.min}$ (see (3.6)). This apparently explains the discrepancies noticed in a number of experimental papers (see, for example, [10]) between the theoretically and experimentally determined self-focusing lengths.

*Formula (3.5) is, as already noted, the consequence of the relation $(1/R_{nl}^{dif})^2 = R_{nl}^{-2} - R_d^{-2}$, which follows from (3.3). A similar result is obtained also from the Talanov's calculation (see [5,16]). On the other hand, Kelley [14] presents without proof a somewhat different formula for R_{nl}^{dif}, resulting from the relation $1/R_{nl}^{dif} = 1/R_{nl} - 1/R_d$, where $R_{nl}^{dif} = n_0 a^2/4\,(c/n_2)^{1/2}(\sqrt{P} - \sqrt{P_{cr}})^{-1}$ (compared with (3.5)). The brevity of the article [14] makes it difficult to clarify in detail the causes of the discrepancy; we merely emphasize once more that all the results based on (3.3) pertain to the near-axis part of the beam. We note, finally, that in the experiment the difference between formula (3.5) and Kelley's formula is insignificant already when P exceeds P_{cr} by 2 – 3 times.

†We recall that, in accordance with the results of Sec. 2, the self-focusing of individual inhomogeneous sections proceeds independently in a medium in which there is no spatial dispersion of the nonlinearity, so that the results of the developed theory are fully applicable to each of the self-focusing filaments.

The results on the self-focusing of beams with plane phase fronts are illustrated by the diagrams of Fig. 4a; the shaded areas are those of the non-self-focusing profiles. Figure 14b shows the corresponding diagrams for beams having a finite divergence on the boundary of the nonlinear medium (R is finite). The figure shows the region of initial divergences $\theta = a/R$, for which self focusing is possible, as well as a plot of the focal distance against the initial divergence. These results follow directly from (3.3). Indeed, by means of a derivation similar to that given in Sec. 2 we obtain with allowance for diffraction (cf. (2.20)) $1/Z_f = 1/R + \sqrt{1/R^2_{nl} - 1/R^2_d}$. Letting $Z_f \to \infty$, we can obtain the critical values of the divergence angle θ_{cr} limiting the region of self-focusing beams:

$$\theta_{cr,1,2} = \frac{\varepsilon_2 E_0^2}{\varepsilon_0} \pm \sqrt{\left(\frac{\varepsilon_2 E_0^2}{\varepsilon_0}\right)^2 - \frac{4}{k^2R^2}}.$$

The angle θ_{opt} lies between θ_1 and θ_2—a beam having such a divergence becomes self-focused faster than all others. It must be borne in mind here that the power P_{div} necessary for self-focusing of a diverging beam is larger than P_{cr}. Calculation based on the foregoing formulas yields

$$P_p = P_{cr}\left(1 + \frac{k^2a^4}{R^2}\right). \tag{3.6a}$$

For inhomogeneous diverging beams, the threshold powers P_{div} can be smaller (but still larger than P_{cr}), owing to the stratification similar to that of a beam with a plane front; this circumstance was pointed out by[74], who considered the self focusing of a diverging beam in the geometrical-optics approximation and who determined the angle $\theta_{cr.2}$.

When $P = P_{cr}$ ($R_{nl} = R_d$) we get, in accordance with (3.2), df/dz = const. A beam with a plane front ($R \to \infty$) and $P = P_{cr}$ propagating in a cubic medium retains its transverse cross section ($df/dz = 0$), that is, the wave-guide propagation mode is realized (self-trapping). Finally, when $P < P_{cr}$, the behavior of the beam in the medium is determined essentially by the boundary conditions and by diffraction; nonlinear refraction leads only to quantitative corrections.

When $P > P_{cr}$ the foregoing results agree with the experimental data, as shown by the experimental papers (see, for example,[10,11,48]), only at distances z that are smaller than the self-focusing length R_{nl}. When $z > R^{dif}_{nl}$, the rays behave not in the manner shown in Fig. 9, but form quasihomogeneous wave channels, that is, in experiment the self-focusing goes over continuously when $z > R_{nl}$ into the self-trapping mode. The absence of such a transition in the theory developed above cannot be ascribed to inclusion of phenomena occurring only in the near-axis part of the beam. The results of a numerical analysis of this problem, presented in[14,29], show that although allowance for the deviation of the beam profile from parabolic does slow down the rate of growth of the field intensity on the beam axis,* it is insufficient to explain the formation of the channels. The reasons for the automatic formation of the proper wave channel and the factors determining its structure and transverse dimensions (and consequently also the limiting field intensity obtained as a result of self-focusing) are of primary interest and have not yet been explained in full. Their discussion will be presented below; however, before we proceed to this discussion, it is expedient to consider briefly the results of the solution of (3.2) for the two-dimensional case, whereas the problems listed above are peculiar only to the three-dimensional beam.

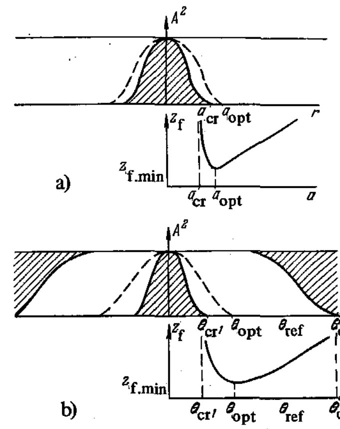

FIG. 14. Plots illustrating the conditions for optimal self-focusing of Gaussian beams with plane (Fig. 14a) and spherical (Fig. 14b) phase front. Plots of $Z_f = Z_f(a)$ for beams with plane phase front and of $Z_f = Z_f(0)$ for diverging beams characterize the retions of initial conditions, for which the self-focusing is possible; the values of $Z_{f.min}$ correspond to the optimal self-focusing. The shaded areas are those of the non-self-focusing profiles.

2. When $m = 0$ Eq. (3.2), with allowance for the boundary conditions (2.14), has as a first integral

$$\left(\frac{df}{dz}\right)^2 = \frac{2}{fR^2_{nl}} - \frac{1}{f^2R^2_d} + C, \quad C = \frac{1}{R^2} - \frac{2}{R^2_{nl}} + \frac{1}{R^2_d}. \tag{3.7}$$

The behavior of the beam is determined by the value of the parameter C. If $C < 0$, that is,

$$R^2 > \left(\frac{2}{R^2_{nl}} - \frac{1}{R^2_d}\right)^{-1} \tag{3.7a}$$

(as $R \to \infty$ we get $C < 0$ if $R_{nl} < R_d$), then self-trapping is produced, wherein the width of the wave beam oscillates within the following limits, which are obtained from the condition $df/dz = 0$:

$$a_{1,2} = \frac{a}{|C|}\left\{\frac{1}{R^2_{nl}} \pm \left[\left(\frac{1}{R^2_{nl}} - \frac{1}{R^2_d}\right)^2 + \frac{1}{R^2R^2_d}\right]^{1/2}\right\}. \tag{3.8}$$

The radius of the wave beam with a plane phase front ($R \to \infty$) remains constant when $R_{nl} = R_d$. The running power of the two-dimensional beam is equal in this case to the critical value

$$P'_{cr} = \frac{\lambda_0^2 c}{16\pi^3 n_2 a}, \tag{3.9}$$

which is inversely proportional to the dimension a (compare with the three dimensional beam, Eq. (3.4)).

Strongly focused or strongly defocused beams, which do not satisfy the condition (3.7a), do not become self-trapped. However, in this case nonlinear refraction leads to a decrease of the dimension of the focal spot; the corresponding calculations were performed in[6].

3.2. Dynamics of Formation of the Optical Waveguide. Stationary Self-focusing of Three-dimensional Beam in a Medium with Saturating Nonlinearity

One of the possible explanations of the experimentally observed formation of a optical waveguide when $z > R_{nl}$ is the decrease of the nonlinear refraction in strong fields, owing to the saturation of the nonlinear polarization (see Sec. 1.2 and Fig. 1). Indeed, the extremely high field intensities reached at the focal point

*We are dealing were, in final analysis, with allowance of the nonlinear aberrations in the waveguide theory.

make it necessary, in general, to take into account the higher-order terms in the expansion of (1.2). A decrease in the "strength" of the nonlinear refraction due to saturation, together with diffraction, ensures finite dimensions of the focal region. To verify this, let us turn to an analysis of the equation for the width of the beam in a medium with saturating nonlinearity. We shall know already specify ϵ in general form*:

$$\varepsilon = \varepsilon_0 + \varepsilon_{\rm nl}(A_0^2), \tag{3.10}$$

and by virtue of the existence of the saturation effect we have

$$\lim_{A_0\to\infty} \varepsilon_{\rm nl}(A_0^2) = \varepsilon_{\rm sat}, \quad \lim_{A_0\to\infty} \frac{\partial \varepsilon_{\rm nl}(A_0^2)}{\partial A_0^2} = 0. \tag{3.11}$$

Taking (3.10) into account and confining ourselves, as before, to the near-axis part of the beam, we obtain an equation for f (compare with (3.2))

$$\frac{d^2 f}{dz^2} = -\frac{1}{f^3}\left[\frac{2E_0^2 \varepsilon_{\rm nl}^{(1)}\left(\frac{E_0^2}{f^2}\right)}{\varepsilon_0 a^2} - \frac{4}{k^2 a^4}\right]. \tag{3.12}$$

Here $\epsilon_{nl}^{(1)}$ denotes the first derivative with respect to the argument and characterizes the slope of the nonlinear characteristic of the dielectric constant as a function of the intensity (in a cubic medium, in first approximation, $\epsilon_{nl}^{(1)} = \epsilon_2$).

The behavior of a beam in a medium with saturating nonlinearity can be traced qualitatively by analyzing the right side of (3.12). It is easy to see that the initially-negative right-hand side of (3.12) can reverse sign with decreasing normalized beam radius f; at first the rather strong nonlinear refraction decreases to such an extent, that it can already be compensated for by the diffraction divergence. The wave-beam radius corresponding to the condition of exact compensation ($d^2f/dz^2 = 0$, $f = 1$)

$$a^2 = \frac{2\varepsilon_0}{k^2 E_0^2 \varepsilon_{\rm nl}^{(1)}}, \tag{3.13}$$

now depends on the power. If the slope of $\epsilon_{nl}^{(1)}$ decreases monotonically with increasing field intensity, the quantity $E_0^2 \epsilon_{nl}^{(1)}$ in the denominator of (3.13) has a maximum, and consequently there exists a minimum dimension of the proper optical waveguide. In order to obtain more concrete relations, we shall specify the law of the saturation of the dielectric constant in the form proposed in [7]: $\epsilon_{nl} = \epsilon_2 A_0^2/1 + \epsilon_2 A_0^2/\epsilon_{sat}$; in typical cases $\epsilon_{sat} \sim \epsilon_0$ (the decrease of the gradient of the dielectric constant, and consequently of the nonlinear refraction, in strong fields and for the saturation law indicated above is illustrated in Fig. 15). Then

$$a^2 = \frac{2\varepsilon_0\left(1 + \frac{\varepsilon_2 E_0^2}{\varepsilon_{\rm sat}}\right)^2}{k^2 E_0^2 \varepsilon_2}. \tag{3.14}$$

The optimal condition corresponds to $\epsilon_2 E_0^2 = \epsilon_{sat}$, and consequently the minimum radius of the proper optical waveguide is

$$a_{\rm min}^2 = \frac{4\varepsilon_0}{k^2 \varepsilon_{\rm sat}}, \tag{3.15}$$

*Saturation was taken into account in [6] in first approximation (in the expansion of (1.2) we took into account not only the term with ϵ_2, but also with $\epsilon_4 < 0$) with the aid of equations similar to (2.8) and (2.9). Simple estimates of the self-focusing conditions in a medium with $\epsilon = \epsilon_0 + \epsilon_2 A_0^2 + \epsilon_4 A_0^4$ were presented recently in [64], where a condition of the type (1.6) served as the basis for the analysis.

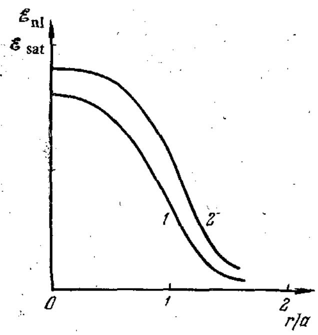

FIG. 15. Theoretical profiles of nonlinear addition to the dialectric constant $\epsilon_{nl} = \epsilon_{nl}(r/a)$ for a Gaussian beam at a different field intensities on the axis. Curve 1 corresponds to $\epsilon_2 E_0^2/\epsilon_{sat} = 4$, curve 2—$\epsilon_0 E_0^2/\epsilon_{sat} = 9$. Abscissas—normalized radius of the beam.

that is,/$a_{min} \sim \lambda_0$.* The optimal power of the self-trapping wave beam is

$$P_{\rm opt} = 2P_{\rm cr}\sqrt{\frac{1+\varepsilon_{\rm sat}}{\varepsilon_0}}, \tag{3.16}$$

that is, the optimal power coincides in order of magnitude with the critical power (it exceeds it by several times).

The behavior of a beam with arbitrary divergence at the entrance into a medium with saturating nonlinearity can be analyzed by writing down, as before, the first integral of the equation for f (in this case (3.12))

$$\left(\frac{df}{dz}\right)^2 = \left[\frac{2\varepsilon_{\rm nl}\left(\frac{E_0^2}{f^2}\right)}{\varepsilon_0 a^2} - \frac{4}{f^2 k^2 a^4}\right] + C, \tag{3.17}$$

$$C = \frac{1}{R^2} - \frac{2\varepsilon_{\rm nl}(E_0^2)}{\varepsilon_0 a^2} + \frac{4}{k^2 a^4}. \tag{3.18}$$

As before, a weakly converging (or weakly diverging) beam ($C < 0$) becomes self-trapped at z = 0; in the general case, the diameter of the waveguide channel oscillates (see also [6], where these oscillations were calculated in first approximation). Beams of supercritical power that are strongly focused † on the boundary cannot become self-trapped, just as in the case of a cubic medium; the effect of self-action leads here to a change in the focal distance and to a decrease in the dimensions of the focal spot. The minimum cross section of the beam in a medium corresponds to the condition $df/dz = 0$ and in the case of $P/P_{cr} \gg 1$ it is equal to

$$[a_f^{(\rm nl)}]^2 = \left\{[a_f^{(L)}]^{-2} + \frac{\varepsilon_{\rm sat} k^2}{2\varepsilon_0}\right\}^{-1}, \tag{3.19}$$

where $a_f^{(l)}$ is the focal cross section of the beam in a linear medium. Figure 16 shows plots of the quantity

*It should be noted that this result, strictly speaking, is already beyond the accuracy limit of the quasioptical approximation used in the calculation; therefore, for an exact estimate of a_{min}, the calculation method should be improved. At the same time, the qualitative picture established in the present section remains, of course, in force.

†Thus, strong prefocusing of a sufficiently homogeneous beam entering into a nonlinear medium can prevent self-trapping, if the latter is for some reason undesirable and it is necessary at the same time to obtain considerable light-wave field intensities. The light-field intensities obtained even in sufficiently strong self-focusing organic liquids with the aid of short-focused lenses can exceed those obtained by the self-focusing effect. The power fluxes observed in self-trapping are $p_{s.t.} \simeq P_{cr}/\pi a^2$, and for a = 30 μ and CS_2 we have $p_{s.t.} \simeq 10^9$ W/cm²; for nitrobenzene this figure is several times larger. At the same time, it must be borne in mind that for "hyperfine" filaments (a ≃ 1–3 μ, see for example [48]) the power flux can reach $10^{10} - 10^{11}$ W/cm², and the corresponding light-field intensity is ~(2 – 3) × 10^7 V/cm (see [103]).

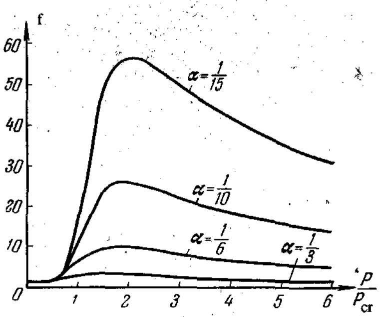

FIG. 16. Plots of the parameter $\Phi = [a^{(l)}/a^{(nl)}]$ characterizing the decrease of the area of the focal plot for a three-dimensional beam in a medium with $\epsilon = \epsilon_0 + \epsilon_2 |A|^2 - \epsilon_4 |A|^4$ vs. the ratio P/P_{cr}. The parameter of the curves is $\alpha = a/R$—half the angle of the convergence of the beam focus by a spherical lens[6].

$\Phi = [a_f^{(l)}/a_f^{(nl)}]^2$ characterizing the change of the area of the focal plot due to self-focusing, as calculated for not too large ratios P/P_{cr} (see [6]).

Thus, allowance for the saturation effect eliminates the singularity at the focus (by virtue of (3.19) the dimension of the focal region is finite). However, the process of formation of the proper waveguide at $z > R_{nl}$ still remains unexplained; apparently it is connected with the simultaneous action of saturation and losses.

3.3. Self Focusing in a Strongly Nonlinear Medium. Propagation of a Field in Self-trapping Beams. Loads of Proper Optical Waveguide

The self-focusing theory developed above on the basis of the use of the parabolic-equation method makes it possible to analyze the behavior of nearly plane waves in a weakly nonlinear and weakly absorbing medium. The change in the dielectric constant due to the self-action of the wave should be not only slow but also small ($\epsilon_{nl} \ll \epsilon_0$). However, in the process of self-focusing of powerful light beams the intensity of the field can become so large that the nonlinear and non-linear parts of the optical refractive index turn out to be quantities of the same order ($\epsilon_{nl} \simeq \epsilon_0$).* In this case the eikonal of the complex amplitude becomes comparable with the eikonal of the plane wave, which is taken as the basis of the solution, and the amplitude of the wave is no longer a slow function of the coordinates; the parabolic-equation method becomes inapplicable in this case. At the same time, if the wave remains practically plane in a medium with large nonlinearity (weakly diverging or weakly converging beam with transverse dimension larger than the wave length, $a \gg \lambda$), then it is possible to retain the quasioptical approach for the description of the diffraction of such a beam. Namely, one can again choose as the basis of the solution a plane wave, but, unlike the case of a weakly nonlinear medium, it is necessary to take directly into account the variation of the wave number (compare with (2.3) and (2.5)):

$$\mathbf{E} = \frac{1}{2}\mathbf{e}A(\mu z, \sqrt{\mu}r)\exp\left\{i\left(\omega t - \int k_{\text{eff}}\,dz\right)\right\}. \quad (3.20)$$

Substituting (3.20) into the nonlinear wave equation and taking into account the effect of self-action in the cubic medium, we obtain in the usual manner two real equations (compare with (2.8) and (2.9))

$$k_{\text{eff}}^2 + 2k_{\text{eff}}\frac{\partial s}{\partial z} + \left(\frac{\partial s}{\partial r}\right)^2 = k_0^2(\varepsilon_0 + \varepsilon_2 A_0^2) + \frac{1}{A_0}\left(\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r}\right), \quad (3.21)$$

$$k_{\text{eff}}\frac{\partial A_0}{\partial z} + \frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r} + \frac{A_0}{2}\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{\partial k_{\text{eff}}}{\partial z}\right) = 0. \quad (3.22)$$

The addition to the eikonal of the plane wave describes here only the curvature of the phase front, that is, we have in the spherical-wave approximation

$$s = \frac{k_{\text{eff}}\, r^2 f'}{2f}. \quad (3.23)$$

Equation (3.22) has as an integral the beam-power conservation law

$$P = \frac{cn_{\text{eff}}}{4}\int A_0^2 r\,dr \quad (3.24)$$

($n_{eff} = k_{eff}/k_0$ is the effective refractive index in the nonlinear medium), from which it follows, in particular, that the power necessary for self-trapping of the beam depends on the radius of the beam (this was first pointed out in [4]). A similar dependence was connected above with the saturation effect (see (2.13)).* The waveguide propagation of the beam corresponds to a plane phase front ($s = 0$); the ordinary differential equation obtained from (3.21),

$$\frac{d^2 A_0}{dr^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - (k_{\text{eff}}^2 - k^2)A_0 + \frac{\varepsilon_2 k^2}{\varepsilon_0}A_0^3 = 0 \quad (3.25)$$

describes the amplitude profiles of the self-trapping beam. We note that in the quasioptical approximation we have in (3.25) $k_{eff}^2 - k^2 \simeq 2k(k_{eff} - k)$ (see also (2.6) with $\partial s/\partial r = 0$). Equation (3.25) can be reduced to a dimensionless form

$$\frac{d^2\tilde{A}_0}{d\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{A}_0}{\partial \tilde{r}} - \tilde{A}_0 + \tilde{A}_0^3 = 0, \quad (3.26)$$

where

$$\tilde{A}_0(\tilde{r}) = \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\frac{k}{\Gamma}A_0, \quad \tilde{r} = \Gamma r, \quad \Gamma^2 = k_{\text{eff}}^2 - k^2.$$

A numerical analysis of (3.26), performed with a computer[4,67], has shown that at the principal mode the amplitude in the transverse cross section decreases monotonically with increasing distance from the beam axis. The calculation of the amplitude profile makes it possible to calculate also the critical power of the self-trapping beam

$$P_{cr} = \frac{5{,}763\lambda_0^2 c n_{\text{eff}}}{32\pi^3 n_2 n_0}, \quad (3.27)$$

which is approximately 1.8 times larger than the amplitude of the power calculated for the near-axis part of the beam (3.4); at the same time, (3.27) differs little from (1.6). The higher modes of the waveguide beam in a cubic medium, as shown in [66,67], have the character of damped oscillations of the amplitude with respect to the coordinate r (the picture of the distribution of the amplitude in the cross section of the beam is in the form of rings, the number of which depends on the number of mode). The critical power of the beam increases with the number N of the mode approximately like $2N^2 - 1$.

3.4. Self-focusing of Complicated Beams in Wave Optics

Just as in Sec. 2, this problem can be treated by a perturbation method, but now the calculation should be

*According to data by Townes and Brewer [103], the nonlinear addition to n reaches 0.2 in filaments of $\sim 2\,\mu$ diameter.

*It was proposed in [74] to take saturation into account by substituting into the equation for the power the total value of n; compare with the results of Sec. 3.2.

based Eqs. (2.8) and (2.9). If we are dealing with a lossless cubic medium with stationary self-focusing, and if there is no spatial dispersion of the nonlinearity, then the need for such an approach is obviated for relatively large-scale inhomogeneities; its results (at any rate, those that can be obtained by analyzing the behavior of one Fourier component) are already contained in the formulas of Sec. 3.1. Indeed, self-focusing of an individual inhomogeneity sets in if the total power contained in it exceeds the critical value. On the basis of an analysis of the formula for R_{nl}^{dif} we can establish optimal dimensions of the most rapidly self-focusing inhomogeneity (see (3.6)). Of course, when taking into account the losses it is necessary to introduce diffraction corrections into the formulas of Sec. 2.4. The same pertains also to the formulas of Sec. 2.5 for nonstationary self-focusing. An analysis shows that now the critical power corresponding to different spectral components is different; for the Kerr effect it increases like $1 + (\nu\tau)^2$: we are dealing here with quantitative corrections. On the other hand, the most interesting qualitative effect connected with the nonstationary behavior in the self-trapping problem is the appearance of a finite rate of formation of the optical waveguide in the relaxing medium; this circumstance was discussed in[7,8]. We shall consider this question briefly in what follows.

3.5. Dynamics of Development of the Optical Waveguide in a Relaxing Medium

Let us consider the process of formation of a waveguide in a medium in which the nonlinearity is determined by the Kerr effect. Assume that at the instant of time $t = 0$ a light beam whose power is equal to the critical value enters into the nonlinear medium at $z = 0$. The nonstationary equations are of the form

$$\frac{2}{v}\frac{\partial s}{\partial t}+2\frac{\partial s}{\partial z}+\left(\frac{\partial s}{\partial r}\right)^2=\frac{\varepsilon_2}{\varepsilon_0}p+\frac{1}{k^2A_0}\left(\frac{\partial^2A_0}{\partial r^2}+\frac{1}{r}\frac{\partial A_0}{\partial r}\right), \quad (3.28)$$

$$\frac{1}{v}\frac{\partial A_0}{\partial t}+\frac{\partial A_0}{\partial z}+\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r}+\frac{A_0}{2}\left(\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}\right)=0, \quad (3.29)$$

$$\tau\frac{\partial p}{\partial t}+p=A_0^2. \quad (3.30)$$

We introduce in lieu of t a new independent variable $\xi = t - z/v$. Then we get in place of (3.28)–(3.30) the system

$$2\frac{\partial s}{\partial z}+\left(\frac{\partial s}{\partial r}\right)^2=\frac{\varepsilon_2}{\varepsilon_0}p+\frac{1}{k^2A_0}\left(\frac{\partial^2A_0}{\partial r^2}+\frac{1}{r}\frac{\partial A_0}{\partial r}\right), \quad (3.31a)$$

$$\frac{\partial A_0}{\partial z}+\frac{\partial s}{\partial r}\frac{\partial A_0}{\partial r}+\frac{A_0}{2}\left(\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}\right)=0, \quad (3.31b)$$

$$\tau\frac{\partial p}{\partial \xi}+p=A_0^2, \quad (3.31c)$$

in which the first two equations have the same form as in the case of the stationary problem, and the inertia of the nonlinear polarization is taken into account by the third equation. We shall seek the solution of the system (3.31), as before, in the form of a spherical wave with variable radius of curvature

$$s=\frac{\beta(z,\xi)r^2}{2}+\varphi(z,\xi), \qquad A_0^2=\frac{E_0^2(\xi)}{f^2(z,\xi)}\exp\left\{-\frac{2r^2}{a^2f^2(z,\xi)}\right\} \quad (3.32)$$

with boundary conditions at $z = 0$

$$\beta(0,t)=0, \quad \varphi(0,t)=1, \quad f(0,t)=1, \qquad E_0^2(t)=\begin{cases}E_0^2 \text{ for } 0\leqslant t\leqslant\tau_{\text{H}},\\ 0 \text{ for } t<0.\end{cases} \quad (3.33)$$

Confining ourselves to the near-axis part of the beam, we obtain an equation for the function $f(z, \xi)$ characterizing the variation of the beam width and its amplitude:

$$\frac{1}{f}\frac{\partial^2 f}{\partial z^2}=\frac{1}{f^4R_{\mathrm{d}}^2}-\frac{1}{R_{nl}^2\tau}\int_0^{\xi}\frac{\exp\left(\frac{\eta-\xi}{\tau}\right)}{f^4}d\eta. \quad (3.34)$$

Equation (3.34) differs from the analogous equation of the stationary theory of self trapping (see (3.2)) in the time-dependent integral in the nonlinear term; it is easy to see that the role of the non-stationary processes is determined by the relation between ξ and τ.* Let us verify first that (3.34) describes the limiting cases of a beam propagating in a linear medium and of the stationary self-trapping beam considered above.

The nonlinearity of the medium does not influence the propagation of the beam when $\xi \ll \tau$. In this case, the last term of (3.34) can be neglected, and the function

$$f^2=1+(z/R_{\mathrm{d}})^2 \quad (3.35)$$

describes the spreading of the beam due to the diffraction divergence; $A_0^2 \sim f^{-2}$. The foregoing means that the frontal part of the laser pulse, corresponding to $\xi \ll \tau$ (and the entire pulse if the pulse duration is $\tau_p \ll \tau$) does not become self-trapped in a medium with inertial nonlinearity. It is also easy to see that the stationary self-trapping mode, in which the function f does not depend on z, is attained only at sufficiently large indeed, let $\partial f/\partial z = 0$, then, recognizing that the beam power, in accordance with the conditions of the problem under consideration, is equal to the critical value ($R_{nl} = R_{\mathrm{d}}$), we arrive at the equation

$$\frac{1}{f^4}=\frac{1}{\tau}\int_0^{\xi}\frac{1}{f^4}\exp\left(\frac{\eta-\xi}{\tau}\right)d\eta, \quad (3.36)$$

which is satisfied if $f = 1$ and $\xi \to \infty$. For an analysis of the phenomena occurring in the region of formation of the optical waveguide we take into account the fact that, by virtue of the foregoing, the function f (the width of the beam) depends little on the variable ξ; therefore we can take it outside the integral sign in (3.34). Then, solving the ordinary differential equation for f, we obtain

$$f^2=1+\left(\frac{z}{R_{\mathrm{d}}}\right)^2\exp\left(-\frac{\xi}{\tau}\right), \quad 0\leqslant\xi\leqslant\tau_p \quad (3.37)$$

(we note that the solution (3.37) describes also the limiting cases indicated above.) Using (3.34), we can determine the rate of "growth" of the waveguide from the condition f = const. For simplicity we put $f^2 = 2$, corresponding to $z = R_{\mathrm{d}}$ when $\xi = 0$. By virtue of (3.37), the equality $f^2 = 2$ remains in force if the following relation is satisfied

$$t-\frac{z}{v}=2\tau\ln\frac{z}{R_{\mathrm{d}}}. \quad (3.38)$$

Relation (3.38) pertains to the region $z \geq R_{\mathrm{d}}$, and connects the length of the optical waveguide z with its development time t. For the rate of growth of the waveguide u_W we get from (3.38)

$$\frac{1}{u_W}=\frac{1}{v}+\frac{2\tau}{z}. \quad (3.39)$$

*We note that by using (3.34) we can construct for nonstationary self-focusing a more general theory than given in Sec. 2, not confined to the region in which the perturbation method is valid.

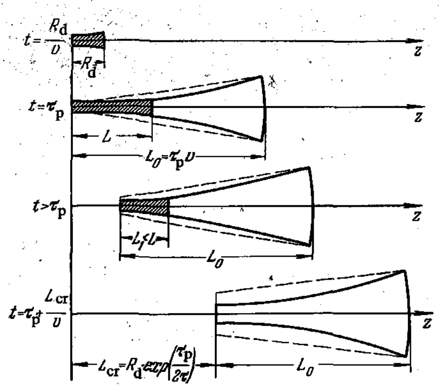

FIG. 17. Phases of development of the proper optical waveguide formed by a powerful light wave. The region occupied by the waveguide is shaded; the dashed lines show the shape of the beam in the absence of nonlinearity. The maximum length of the waveguide L is reached at the instant of termination of the pulse $t = \tau_p$. After $t > \tau_p$ the waveguide is detached from the boundary $z = 0$ and its length decreases, since $u_w < v$. Finally, when $z = L_{cr}$, the waveguide vanishes.

According to (3.39), the rate of growth of the waveguide is equal to the velocity of light when $\tau = 0$ ($u_W = v$); when $\tau \neq 0$ we have $u_W < v$, and with increasing z the growth rate decreases. The difference between the growth rate u_W and the velocity of light v causes the length of the optical waveguide produced during the time of the laser pulse to be smaller than the distance traversed by the wave, and consequently only part of the laser pulse becomes self-trapped (Fig. 17). Using (3.38) and (3.39) we obtain for the length of the optical waveguide formed during the time τ_p

$$z+2\tau v \ln\frac{z}{R_d}=\tau_p v, \tag{3.40}$$

and for the energy efficiency (η) of the self-trapping process, which is equal to the ratio of the light energy transported through the waveguide to the total energy of the light pulse, we get

$$\eta=1-2\frac{\tau}{\tau_p}\frac{\ln z}{R_d}. \tag{3.41}$$

By virtue of the difference between the velocities u_W and v, the optical waveguide vanishes at $z = L_{cr} = R_d \exp(\tau_p/2\tau)$—the wave becomes detached from the waveguide. The role of the inertia effects is determined by the values of τ, τ/τ_p, and z/R_d. Estimates show that whereas in the case of Kerr self-focusing of ordinary giant pulses the foregoing effects are not very effective ($\tau_p \simeq 10^{-9}$ sec, $\tau \simeq 10^{-12}$ sec, and L_{cr} is quite large), they cannot be neglected for lasers with synchronized modes ($\tau_p \simeq 10^{-11}-10^{-12}$ sec). Much stronger inertial effects should arise in striction self-focusing. The foregoing results are not applicable, strictly speaking, to striction self-focusing, since this phenomenon differs from the Kerr self-focusing in the nonlocal character of the nonlinear response (see Sec. 2). However, if the mean free path of the acoustic phonon $Z_\delta = u/2\Gamma$ is much smaller than the width of the beam a ($Z_\delta \ll a$), the formulas of that section can be used for approximate estimates by replacing τ by $\tau_\perp$ (see Sec. 2.5). Of course, this does not exclude the need for a rigorous theory of striction self-trapping.

3.6. Self-contraction of Wave Packets—Allowance for the Second Time Derivatives

The effect of self-action of a plane modulated wave in a dispersive medium with nonlinearity of the cubic type is described by the following abbreviated equation (we consider for simplicity a non-relaxing medium; if the relaxation time is finite it is necessary to take into account the "nonlinear dispersion")

$$2ik\frac{\partial A}{\partial z}=-kk''_{\omega\omega}\frac{\partial^2 A}{\partial \xi^2}+\frac{k^2\varepsilon_2|A|^2}{\varepsilon_0}A, \tag{3.42}$$

where $\xi = t - z/v$ and $v = \partial\omega/\partial k$; obviously $k''_{\omega\omega} = -v'_\omega/v^2$. In a linear medium ($\varepsilon_2 = 0$), Eq. (3.42) describes the dispersion spreading of a wave packet.

If we introduce the normalized quantities

$$\tilde{z}=-\frac{z}{kk''_{\omega\omega}}, \quad \tilde{\xi}=\frac{\xi}{kk''_{\omega\omega}}, \quad \tilde{\varepsilon}_2=-\varepsilon_2 kk''_{\omega\omega}, \tag{3.43}$$

then (3.42) can be written in the form

$$2ik\frac{\partial A}{\partial z}=\frac{\partial^2 A}{\partial \tilde{\xi}^2}+\frac{k^2\tilde{\varepsilon}_2|A|^2}{\varepsilon_0}A. \tag{3.44}$$

A comparison of (3.44) with (2.4), which describes the stationary spatial self-focusing, shows that there is a mathematical analogy between the behavior of a plane wave packet and spatial self-focusing of a two-dimensional beam. Therefore all the derivations and the corresponding conclusions of Sec. 3 remain in force here, provided we make the formal substitutions $z \to \tilde{z}$, $x \to \tilde{\xi}$, $\epsilon_2 \to \epsilon_2$, and $a \to \tau/k|k''_{\omega\omega}|$. The only essential difference lies in the fact that the spatial self-focusing occurs in a medium with $\epsilon_2 > 0$, and the self-contraction of the wave packets occurs when $\tilde{\epsilon}_2 > 0$, that is, when $\epsilon_2 k''_{\omega\omega} < 0$ or $\epsilon_2 v'_\omega > 0$. Using the foregoing space-time analogy, we can, for example, write down directly the values of the space scales

$$R_d^\tau=\frac{\tau_p^2}{2|k''_{\omega\omega}|}, \tag{3.45a}$$

$$R_{nl}^\omega=\tau_p\sqrt{\frac{\varepsilon_0}{2\varepsilon_2 E_0^2 k|k''_{\omega\omega}|}}, \tag{3.45b}$$

where R_d^τ is the length over which an appreciable spreading of a pulse of duration τ_p takes place in the linear medium, and R_{nl} is the length over which self-contraction of the pulse takes place. The pulse will retain its shape if $R_{nl} = R_d$, and in this case the energy density of the stationary pulse in the cross section of the wave is

$$W_{cr}=\frac{c\lambda_0^2 k|k''_{\omega\omega}|}{16\pi^3 n_2 \tau} \tag{3.46}$$

and is inversely proportional to the pulse duration. The corresponding stationary profiles were considered in [24] (compare with the results of Sec. 3.3).

If the energy and momentum are larger than critical ($W \gg W_{cr}$), then modulation of the wave takes place, as a result of which an initially modulated wave breaks up into wave packets.

4. NONLINEAR OPTICAL EFFECTS IN THE FIELD OF SELF-FOCUSING BEAMS

The appreciable increase in the intensity of the light field, caused by the self-focusing, can obviously be the cause of a strong change in the character of the behavior of other optical effects that depend on the wave intensity. The results presented in Secs. 2 and 3 of this review make it possible to present a simple quantitative criterion for estimating the contribution of self-focusing.

Indeed, self-focusing effects in experiments with unfocused or weakly focused beams can certainly be neglected if the length L over which the behavior of the beam is studied* satisfies the condition

$$L \ll R_{nl}. \quad (4.1)$$

To the contrary, the effects of self-focusing are quite appreciable if $L \simeq R_{nl}$ or $L > R_{nl}$; in the latter case self-trapping sets in, the factors determining the effective lengths of the filaments (and accordingly their lifetimes) are not yet fully clear. The contribution of self-focusing effects in strongly focused homogeneous beams, for which self-trapping is possible, can be estimated by using diagrams of the type shown in Fig. 16 (see also Fig. 14).

In accordance with the foregoing and with the data given in Sec. 1, the self-focusing effects should have the greatest influence on other nonlinear optical effects in liquids, particularly in those where the Kerr constant is large. Moreover, the registration of threshold nonlinear optical effects, such as stimulated Raman scattering (SRS) are stimulated Mandel'stham-Brillouin scattering (SMBS) in initially unfocused laser beams serves frequently as a method of indication (and frequently also as a method of quantitative study) of the self-focusing effect. Indeed, for example, since, the effect of stimulated Raman scattering takes place, only at field intensities

$$E \geqslant E_0 = \sqrt{\frac{n_0 \lambda_0 \delta_0}{8\pi^2 \sigma}} \quad (4.2)$$

(here δ_0, as before, is the damping decrement, see Sec. 2, and σ is the imaginary part of the Raman susceptibility; see, for example,[36]), it follows that observation of SRS in a beam in which the initial field intensity is $E < E_0$ offers evidence of self-focusing. The first direct experiments of this type were apparently described by l'Allemand and Bloembergen[10] †. It was established in these experiments that the critical wavelength of an SRS generator (Raman laser) can be greatly decreased by placing between its cell and the pump laser an additional cell with a strongly self-focusing liquid. Typical experimental results, obtained in[10], are illustrated by Fig. 18. The ordinates represent here the threshold length necessary for self-excitation of a nitrobenzene Raman laser, and the abscissas represent the length of the self-focusing cell with bromobenzene (which has an appreciable Kerr constant; see Sec. 1). A study of the dependence of the effect on the distance between cells[10] has shown that the effect is retained only at distances smaller than 10–15 cm; the divergence of the filament radiation in air is large.‡ In[10] they also measured the diameters of the filaments obtained as a result of self-focusing; they turned out to be ~20–80 μ; it was established at the same time that the number of filaments increases with increasing cell length (compare with data of Sec. 3), and that observation of SRS makes it possible to estimate the increase of the radiation intensity due to the self-focusing; in[10] this increase reached two orders of magnitude. Similar or nearly similar results were reported in papers[15,17] published almost simultaneously with[10]. To register the increase of the light-field intensity due to the self-focusing they used in[15] not only SRS, but also the effect of generation of the second optical harmonic in a quartz plate placed in the path of the self-focusing beam. According to the estimates given in[15], the power density in self-focusing filaments in organic liquids reaches 2×10^9 W/cm^2. The foregoing results allow us to state that the influence of self-focusing frequently plays the decisive role in the excitation of SRS in liquids with large Kerr constant. In[10,17] there is noted good correlation between the self-focusing properties of liquids and the constant of the high-frequency Kerr effect measured in[43]. From the point of view of self-focusing ability, the liquids can be arranged in the following sequence (in decreasing order of self-focusing properties): CS_2, nitrobenzene, bromobenzene, benzene, and acetone. Addition of strongly-self-focusing liquids to such liquids as CCl_4 or cyclohexane has made it possible to lower greatly the SRS threshold; the latter denotes that in experiments with mixtures, an appreciable role is played by self-action effects. It is important to emphasize that the phenomenon of self-focusing influences not only the SRS threshold (we note that by virtue of this circumstance the magnitude of the threshold is determined not so much by the cross section of the spontaneous scattering, as by the value of the Kerr constant; see Table III, which is taken from[70]), but also changes many of its important characteristics. Notice should be taken here above all of the change in

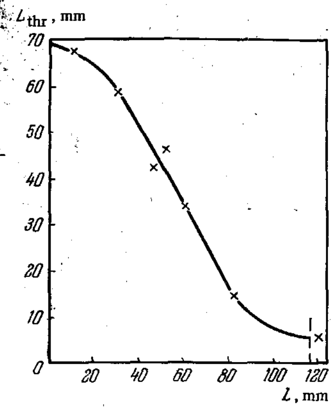

FIG. 18. Experimental dependence of the threshould length of a Raman laser using nitrobenzene, L_{thr}, as a function of the length L of the cell with the self-focusing liquid (bromobenzene) [10].

Table III. Threshold powers necessary for excitation of SRS in certain self-focusing liquids (all normalized to CS_2)[70].

Liquid	Experimental threshold	Calculated threshold from data on the Kerr constant	Calculated threshold from data on the spontaneous scattering
CS_2	1,0	1,0	1,0
Nitrobenzene	1,3	1,2	6,1
Toluene	4,9	5,0	9,5
Benzene	5,2	5,7	3,6

*We must bear in mind here, of course, the remark made in connection with formula (3.6); for complicated beams it is apparently necessary to use in (4.1) the length $Z_{f.min}$.

†It should be noted, to be sure, that the hypothesis that self-focusing can have a possible influence on SRS was already advanced earlier; it is contained, in particular, also in [4].

‡Calculation of the divergence in air, based on the diffraction formulas for a round aperture, is apparently a crude approximation; we are dealing with radiation from the open end of a waveguide.

the angular structure; calculations[13] show that it is precisely the filamentary structure of the beams which accounts for the differences between the angular structure of the scattered radiation and the structure predicted by the plane-wave theory. The shape of the SRS spectral lines is also strongly changed in self-focusing beams. In a number of cases particularly in the self focusing in ultrathin filaments, anomalous line broadening is observed[116-118]. The latter is connected with the strong changes of the complex envelope (see Secs. 2.5 and 3.6). Self-focusing can greatly influence the forward-backward asymmetry of the Raman radiation; it is not excluded that an important role can be played in this case by the self-excitation of the Stokes oscillations in individual sections of the filaments, owing to reflection from inhomogeneities. The latter can lead in general to a disintegration of the filaments. Thus, besides the influence of self-focusing on stimulated scattering, an inverse reaction can take place and can be quite noticeable.

The SRS affects particularly strongly the structure of the "hyperfine" filaments (with diameter up to 1–2 μ) observed in certain self-focusing liquids. According to Townes and co-workers[105], the excitation of molecular oscillations increases the value of n_2 and greatly influences the diameter of these filaments; the transformation of the energy of the coherent molecular oscillations into heat is attributed by the authors of[105] to the short lifetime of the "ultrathin" filaments. Finally, in self-focusing, an appreciable change can take place in the character of the competition between the different molecular oscillations in the SRS and the different types of scattering. We note that SMBS is also observed in self-focusing beams; certain information on this topic is contained in[12,69,77]. It is interesting that a definite role can be played in this case also by self-focusing of hypersound, occurring during the scattering process[96]*. It must be emphasized that the information on the self-focusing effect itself, obtained from experiments on SRS with laser beams which are not focused beforehand, it not confined to the semiquantitative conclusions described above. Comprehensive experiments performed in[18,71,72,73] make it possible to determine the critical power P_{cr} and to investigate the influence of different factors on the self-focusing length, particularly linear absorption.

The aforementioned investigations are based on the experimentally observed fact that the threshold input necessary for the appearance of SRS coincides, for many liquids, with the power at which an optical waveguide is produced in an experimental cell of length L, that is, it corresponds to the approximate satisfaction of the equality $L \simeq R_{nl}^{dif}$.† Using the foregoing circumstance and the formula for R_{nl}^{dif} in the form

$$L = R_{\mathrm{nl}}^{\mathrm{dif}} = \left(\frac{n_0}{4}\right)\left(\frac{a^2}{\mathfrak{F}}\right)\left(\frac{c}{n_2}\right)^{1/2}[\sqrt{P}-\sqrt{P_{\mathrm{cr}}}]^{-1} \tag{4.3}$$

*The latter circumstance may possibly account for the difference between the frequency shift of the SMBS components in beams of powerful lasers and the shift measured in weak light fields.

†At the same time, if the Raman scattering cross section is very small, a situation is of course possible in which the power density necessary for noticeable increase of the Stokes SRS component can greatly exceed the power density in the self-focusing filaments [70].

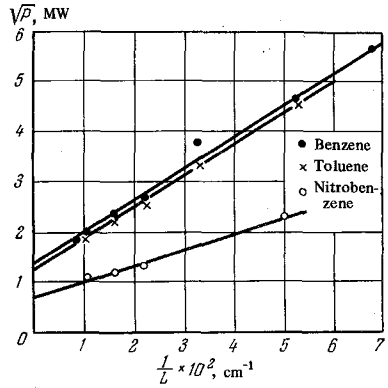

FIG. 19. Experimental dependence of the threshold power (raised to the 1/2 power) necessary for self focusing of a beam in a cell of length L, on L⁻, for benzene, toluene, and nitrobenzene. The plots were taken from [18].

(see formula (3.5) and the discussion pertaining to it), Wang[18] determined the critical power (defined as the asymptotic value of P as $L \to \infty$) and the geometric factor $\mathfrak{F}$ (we recall that $\mathfrak{F} = 1$ for Gaussian beams; see Sec. 2). Typical experimental plots, obtained in[18], are shown in Fig. 19. A study was made of the self-focusing of giant ruby-laser pulses in organic liquids. The critical powers determined from these plots were 0.064 MW for benzene, 0.019 MW for nitrobenzene, and 0.055 MW for toluene. The calculated values for benzene and nitrobenzene were respectively 0.085 and 0.021 MW; the value of the factor $\mathfrak{F}$ was also determined experimentally (from the slope of the straight lines of Fig. 19) and found to be $\mathfrak{F} \cong 2$.

Using a similar procedure, Wang[100] determined the ratio of the critical powers for circularly and plane-polarized beams (Fig. 20) in CS_2. As seen from the diagrams, the critical powers differ not by a factor of 4, but only by a factor of 2, thus confirming the assumption of the instability of the circularly polarized wave in a medium with a large Kerr constant. Detailed data in this respect for other liquids are contained in[104], where it was established that the ratio of the critical powers for two types of polarization fluctuates between 1.3 and 2.2.

Similar measurements were made in[72] with strongly absorbing liquids; carbon disulfide was used with absorbing additives (which change the absorption coefficient from $\delta_0 = 0.002\ \mathrm{cm}^{-1}$ to $\delta_0 = 0.125\ \mathrm{cm}^{-1}$). The experimental plots obtained in[72] are shown in

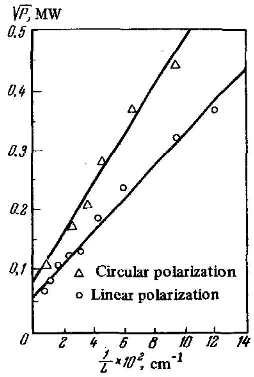

FIG. 20. Experimental dependence of the threshold power (raised to the 1/2 power) necessary for self focusing of a beam in a cell of length L with CS_2, on L^{-1}, for circularly-polarized and plane-polarized radiation.

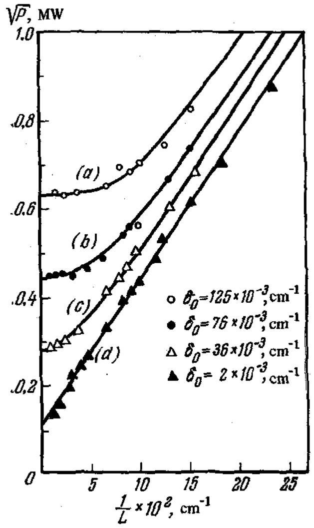

FIG. 21. Experimental plots similar to those of Fig. 19, for carbon disulfide with different absorbing additives.

Fig. 21; the solid lines are the theoretical curves based on formulas similar to (4.2) with allowance for damping (see Sec. 2). The foregoing denotes that, at any rate for giant pulses and for dampings $\delta_0 \leq 0.1\ \mathrm{cm}^{-1}$, the influence of dissipation on the self focusing is determined essentially by the damping of the field. Similar conclusions can be drawn also from the experimental results of[73] on the increase, due to damping, of the beam power necessary for self-focusing at a specified length in benzene in nitrobenzene; the media investigated there had $\delta_0 \leq 0.3\ \mathrm{cm}^{-1}$. The nonlinear optical phenomena affected greatly by self-focusing, even in the case of liquids, are of course not confined to SRS and SMBS. We must point out here also stimulated Rayleigh scattering[84,85]. The anomalously large asymmetrical broadening of the Stokes component of the SRS due to stimulated Rayleigh scattering can be attributed, according to[19,86], to their very strong self-focusing.

The self-focusing effect should also greatly influence the course of other nonlinear phenomena. The sharp increase of the light field in self-focusing beams should obviously facilitate the observation of the nonlinear scattering at the second harmonic[87,88]. Of particular interest is in this case the study of the effect of the intermolecular interaction. It is interesting that nonlinear isotropic scattering at frequencies close to the frequency of the fundamental wave was observed in self-focusing beams. Some results of its study are reported in[183]. In thin filaments, the power of the scattered radiation reaches several watts. The regions of intense nonlinear scattering turn out to be distinctly localized. Such a scattering is reported also in[48], where its appearance is connected with the scattering of the energy of coherent molecular vibrations obtained in the case of SRS.

An appreciable change of the wavelength in self-focusing beams can influence processes determined by the spatial dispersion. Among the possible effects we should mention here, in particular, the effect of nonlinear rotation of the plane of polarization in optically active liquids. In order of magnitude, the specific angle of nonlinear rotation equals $\sim\omega/cfn_2E_0^2/n_0$, where f is the optical-activity constant. Interesting features are possessed by the effect of circular dichroism in a nonlinear optically-active medium[89]. It should be noted that the spatial dispersion can influence also the self-focusing effect itself, in that it determines, alongside with diffraction, the magnitude of the critical power, but this effect is negligibly small in the optical range. Finally, in strong fields of self-focusing beams, parametric interactions are possible involving the cubic term of the expansion of the electronic polarizability in powers of the field–interactions at which $2\omega_p = \omega_1 + \omega_2$[35]. It is possible that it is precisely this effect which causes the observation of many Stokes and anti-Stokes components of SMBS in liquids under conditions when the cell with the liquid is decoupled from the laser[78]. It should be noted, to be sure, that an amplification of this type, as shown by an analysis[79,109,110], can take place only in the case of interaction between waves having nonparallel wave vectors, thereby causing the amplified signal to acquire a peculiar angular structure.

Let us note, finally, that, at any rate the initial stages of the self-focusing process can be treated as a process of frequency-degenerate (in the case of stationary self-focusing) parametric amplification of the type under consideration, or nondegenerate amplification (in the case of nonstationary self-focusing). This circumstance was pointed out in[29]. Indeed, the formulation of self-focusing problem as considered by the perturbation method is completely analogous to the parametric-amplification problem. The optimal value of the rapidly self-focusing inhomogeneity a_{opt} (see (3.6)) corresponds in "parametric" language to the synchronism conditions for the amplification of two plane waves of frequency ω (or one spatially-modulated wave) in the field of the plane pumping wave of the same frequency. An analogous "parametric" treatment can be used also for the defocusing effect—here a weak plane wave of frequency ω increases in the field of a spatially-modulated pump wave at the same frequency. The parametric approach to an explanation of the occurrence of defocusing is discussed in[79]; the same reference gives also a nonlinear theory of parametric amplification in a cubic medium; this theory can be regarded as the spectral theory of strong interactions.

5. CONCLUSION

The experimental and theoretical material presented in the review thus offers evidence that much progress has been made in the study of self-actions of powerful electromagnetic waves. The main physical effects have been predicted and observed experimentally, and a mathematical apparatus has been developed and makes it possible to trace at least qualitatively the main features of these self actions (to estimate the spatial scales of the processes, to reveal the influence of various properties of the medium, of the beam geometry, etc.). The effect most thoroughly investigated at present is spatial self-focusing. At the same time, a number of important questions must still be investigated further in this case, too.

1. It is necessary first of all to perform experimental investigations yielding reliable quantitative information on self focusing and self-trapping in different

media, and to reveal the physical self-focusing mechanisms. It is not sufficiently clear as yet whether striction self-focusing is observed in experiment. Indirect data on the observation of SRS in beams that are not focused beforehand, in such liquids as CCl_4 and alcohol [17,39], and of SMBS in CCl_4 and water (this necessitated the use of very long cells [12]) confirm, as it were, the possibility of such self-focusing; this question, however, is far from completely studied. Recently, the hypothesis was advanced that striction determines the fine structure of self-focusing beams [62,75]*. The reason for this is the nonlocal character of the striction response. This circumstance was discussed in detail in [75]; it is interesting that, as established here, the equations of striction self-focusing have much in common with the equations of superconductivity theory. A sufficiently general consideration of the instability of the light beam is given in [76]. A clarification of the contribution of striction effects is quite important for the study of the behavior of powerful light beams in crystals.†

2. The dynamics of formation of the applicable waveguide is not fully explained as yet. Worthy of particular attention are questions on the limiting power density (the diameter and the total power) carried by the waveguide (we note that, according to theoretical estimates and experimental data, it reaches 10^9-10^{11} W/cm^2) and the factors governing the limiting length of the waveguide (according to [48], the "ultrathin" filaments are very short-lived, see also [103]). Possible causes of the "break" of the waveguide are the finite growth velocities (see Sec. 3.5), inhomogeneities, strong conversion of the energy into scatter radiation (in particular, due to SRS and SMBS [77,105]), and self excitation due to inhomogeneities.

There are many unclear items in the picture of nonstationary self focusing. Experimental data are necessary to estimate the role of the thermal self-focusing and defocusing of laser pulses. It is of interest to study the effects of nonlinear dispersion, which can influence the behavior of ultrashort laser pulses.

3. The theory of self focusing (including the theory of complicated beams) has been developed to date only for spatially-coherent fields. It is of interest to generalize it to include the case of spatially-incoherent radiation. This can be done, for example, with the aid of formulas (2.45), where the field at an arbitrary point of the nonlinear medium is expressed, within the limits of applicability of the perturbation method, in terms of arbitrary boundary conditions at the input; the latter makes it possible to calculate the statistical characteristics of the field at any cross section of the medium from the statistical properties of the field at the input. It is of interest to calculate the beam spreading due to the statistical inhomogeneity of the medium.

4. Considerable interest attaches to effects of nonlinear defocusing of relatively low-power ($P < P_{cr}$) laser beams in different media—a problem directly connected with the study of the propagation of powerful radiation over large distances. The results of the corresponding experimental and theoretical investigations are given in [44,114]. Of course, the thermal effect is not the only cause of nonlinear defocusing of powerful radiation with $P < P_{cr}$; it is also necessary to take into account such effects as evaporation, ionization, etc.

5. When speaking of the wave theory of the self-focusing effects (and of the theory of self-action as a whole in general), it must admitted that, owing to the complexity of the problem, analytic results can be obtained only for a few beam models. Therefore, in order to obtain more detailed data, it is inevitably necessary to integrate the equations numerically. Many results made in this direction are contained in [14], and much material is present in [29]. However, even the latter task cannot be regarded as complete. When speaking of the theory, it should also be noted that polarization effects in self-focusing have not been extensively investigated (in particular, the effect of the instability of a circularly polarized wave, discussed in [104,106]). To take these phenomena into account, it is necessary to include in the theory an analysis of the vector character of the fields.

6. Finally, a timely problem is the unification of the theory of self-focusing with the theory of stimulated scattering. Such a general consideration makes it possible to reveal the mutual influence of the indicated effects, that is, the action of self-focusing on stimulated scattering, and the no less important reaction, which can, probably, explain many features of the behavior of the "hot" filaments. Particularly important in the construction of such a theory is allowance for nonstationary processes; owing to the latter, the influence of self-focusing on the SRS and SMBS is different.

7. Although the theory of self-action of wave packets is in general at the same level as the theory of spatial self-focusing (moreover, a "unified" theory of space-time instability of wave packets in a nonlinear medium has been developed on the basis of the perturbation method in [29]), experimental data are in this case much more skimpy. The cause of this are the larger values of the space scales than in the case of self-focusing.

However, when $E \sim 10^{-7}$ V/cm (in ultrathin filaments), these lengths amount to several centimeters and the corresponding effects are observable (they are registered experimentally by the broadening of the spectrum of the laser pulses). In the radio band (in artificial lines with nonlinear elements) the effects of self-action of time-modulated waves were originally observed by Ostrovskiĭ. Figure 22 shows oscillograms of a harmonically modulated wave (carrier frequency 300 Hz) at the input and output of a nonlinear line. One can see clearly the growth of the modulation coefficient

*At the same time, the fine structure is connected in [105] with stimulated Raman scattering.

†In crystals admitting generation of a second optical harmonic, noticeable self action can also be connected, as shown in [80], with the reaction of the second harmonic on the fundamental radiation. Estimates made in [80] show that the corresponding value of n_2 can reach $\sim 10^{-12}$ cgs esu. It must also be kept in mind that in electro optical crystals self-action can be connected with the effect of optical detection. The produced static field deforms the surfaces of the refractive indices. This circumstance was noted by Zanadvorov [81]. We note, finally, that a change in the optical parameters of the $LiNbO_3$ crystal at power levels which still do not lead to breakdown, was noted in [82] and [83], where these changes were registered in experiments on the generation of optical harmonics.

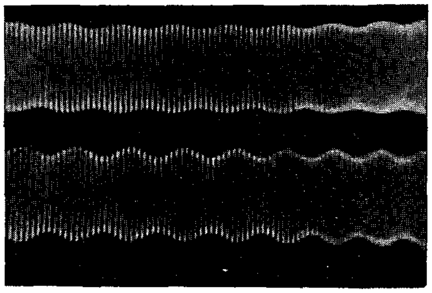

FIG. 22. Oscillograms of a harmonically-modulated wave at the input (top) and output (bottom) of a nonlinear line (carrier frequency 300 kHz). The increase in the depth of the modulation connected with the self-action effect is clearly seen.

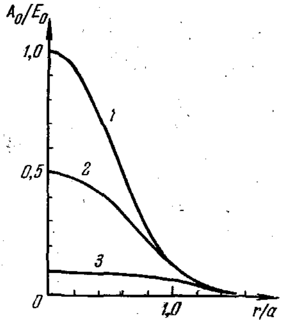

FIG. 23. Transformation of the amplitude profile of a Gaussian beam in a medium with nonlinear (two-photon) absorption. Curve 1 corresponds to the section $z = 0$, $2 - z = (2\delta_2 E_0^2)^{-1}$; $3 - z = 10\,(2\delta_2 E_0^2)^{-1}$.

(compare with the data of Sec. 3.7). In the same experiments there was observed also the transformation of amplitude modulation into a phase modulation in the nonlinear line, which was discussed in [27].

In conclusion we must emphasize that the problem of spatial self-focusing of bounded beams can be regarded as one of the divisions of the presently developing nonlinear diffraction theory. Work on the creation of theoretical methods and the development of physical concepts is far from complete in this case. To be sure, it can already be stated that one of the most effective methods in this field is apparently the method of parabolic equation, which is similar in spirit to the method of slowly varying amplitudes, the latter being in essence the theoretical basis of nonlinear optics.*

Using the procedure developed in Secs. 2 and 3, we can consider the effects of self-action of a bounded beam, due to nonlinear absorption. This problem is of independent interest without regard to self-focusing, since it is precisely on the basis of its solution that one can correctly interpret experiments on the determination of the cross section of nonlinear absorption. By way of an example we can present the results of [90] for a medium with two-photon absorption ($\delta_0 = 0$, $\epsilon_2' = 0$, $\epsilon_2'' \neq 0$); Fig. 23 shows a modification of the amplitude profile of a Gaussian beam in a medium with nonlinear absorption.

It is interesting that in the case of focusing of a wave in a nonlinear medium it is possible to connect the input and output powers by the simple relation

$$\frac{1}{P_{\text{out}}} = \frac{1}{P_{\text{in}}} + \frac{1}{P_{\text{lim}}}, \quad (5.1)$$

where $P_{\text{lim}} = cn\lambda_0/32\pi\delta_2$ is the limiting power emerging from the focus and does not depend on P_{in} (when $P_{\text{in}} \to \infty$ and $P_{\text{out}} \to P_{\text{lim}}$), and is determined only by the wavelength and by the cross section for two-photon absorption (see also [34]).

Diffraction effects can play an important role in the generation of optical harmonics (of course, also in the absence of self-focusing). This problem in the theory of diffraction in a nonlinear medium was considered in [90-94]. Similar problems are encountered also in the theory of parametric amplifications; allowance for the finite aperture can yield important corrections in this case. Finally, an important role can be played by the spatial limitation of the beam also in the dynamics of laser generation. An analysis of the generation of a giant pulse, based on the analysis of the parabolic equation, is given in [95].

*At the same time, as already indicated, there are important problems which are at the limits of the applicability of the quasioptical approximation.

[1] G. A. Askar'yan, Zh. Eksp. Teor. Fiz. 42, 1567 (1962) [Sov. Phys.-JETP 15, 1088 (1962)].
[2] V. I. Talanov, Izv. vuzov (Radiofizika) 7, 564 (1964).
[3] L. V. Keldysh, Paper at Session of Division of General and Applied Physics, USSR Academy of Sciences, 1964.
[4] R. Chiao, E. Garmire, and C. Townes, Phys. Rev. Letts 13, 479 (1964); Phys. Rev. Letts 14, 1056 (1965).
[5] V. I. Talanov ZhETF Pis. Red. 2, 218 (1965) [JETP Lett. 2, 138 (1965)].
[6] S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Zh. Eksp. Teor. Fiz. 50, 1537 (1966) [Sov. Phys.-JETP 23, 1025 (1966)].
[7] Ya. B. Zel'dovich and Yu. P. Raĭzer, ZhETF Pis. Red. 3, 137 (1966)[JETP Lett. 3, 86 (1966)].
[8] S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Zh. Eksp. Teor. Fiz. 51, 296 (1966) [Sov. Phys.-JETP 24, 198 (1967).
[9] A. P. Sukhorukov, Paper at Seminar of First All-union Summer School on Wave Diffraction, Palanga, May 1965 (Abstracts).
[10] P. Lallemand and N. Bloembergen, Phys. Rev. Letts 15, 1010 (1965).
[11] E. Garmire, R. Chiao, and C. Townes, Phys. Rev. Letts 16, 347 (1966).
[12] N. Bloembergen, P. Lallemand, and A. Pine, J. Quantum Electronics QE-2, 246 (1956).
[13] K. Shimoda, Japanese J. Appl. Phys. 5, 86 (1966).
[14] P. Kelley, Phys. Rev. Letts 15, 1005 (1965).
[15] G. Hauchecorne and G. Mayer, Compt. Rend. 261, 4014 (1965).
[16] V. I. Bespalov and V. I. Talanov, ZhETF Pis. Red. 3, 471 (1966) [JETP Lett. 3, 307 (1966)].
[17] Y. Shen and Y. Shaham, Phys. Rev. Letts 15, 1008 (1965).
[18] C. Wang, Phys. Rev. Letts 16, 344 (1966).
[19] Yu. S. Chilingaryan, Paper at II All-union Symposium on nonlinear Optics, Novosibirak, 1966.
[20] N. F. Pilipetskiĭ and A. R. Rustamov, ZhETF Pis. Red. 2, 88 (1965) [JETP Lett. 2, 55 (1965)].
[21] V. I. Bespalov and A. M. Kubarev, op. cit. [19].
[22] Ya. B. Zel'dovich, Usp. Fiz. Nauk 87, 189 (1965) [Sov. Phys.-Usp. 8, 729 (1966)].

[23] L. A. Ostrovskiĭ, Zh. Tekh. Fiz. 33, 905 (1963) [Sov. Phys.-Tech. Phys. 8, 679 (1964)].
[24] L. A. Ostrovskiĭ, Zh. Eksp. Teor. Fiz. 51, 1189 (1966) [Sov. Phys.-JETP **24**, 797 (1967)].
[25] L. Frantz and J. Nodvik, J. Appl. Phys. **34**, 2346 (1963).
[26] N. G. Basov and V. S. Letokhov, Dokl. Akad. Nauk SSSR 167, 73 (1966) [Sov. Phys.-Doklady 11, 222 (1966)].
[27] N. G. Basov and V. S. Letokhov, op. cit.[19].
[28] N. G. Basov, R. V. Ambartsumyan, V. S. Zuev, P. G. Kryukov, and V. S. Letokhov, Zh. Eksp. Teor. Fiz. 50, 23 (1966) [Sov. Phys.-JETP 23, 16 (1966)].
[29] V. I. Bespalov, A. G. Litvak, and V. I. Talanov, op. cit.[19].
[30] S. A. Akhmanov, A. I. Kovrigin, A. S. Chirkin, O. N. Chunaev, Zh. Eksp. Teor. Fiz. 50, 829 (1966) [Sov. Phys. JETP **23**, 549 (1966)]; see also S. A. Akhmanov, A. P. Sukhorukov, and A. S. Chirkin, Izv. vuzov (Radiofizika) 10, No. 7 (1967).
[31] S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, op. cit.[19].
[32] A. Javan and P. Kelley, IV Quantum Electronics Conference, Digest of Technical papers, Phoenix, 1966; IEEE Quant. Electr. 2, 470 (1966).
[33] R. Terhune, P. Maker, and C. Savage, Quantum Electronics, Proceedings of Third International Congress, Paris, 1964.
[34] P. Maker and R. Terhune, Phys. Rev. **137**, A801 (1965).
[35] S. A. Akhmanov and R. V. Khokhlov, Problemy nelineĭnoĭ optiki (Problems of Nonlinear Optics), VINITI AN SSSR, 1964.
[36] N. Bloembergen, Nonlinear Optics, Benjamin, 1965.
[37] V. M. Agranovich and V. L. Ginzburg, Kristalloop-tika s uchetom prostranstvennoi dispersii i teoriya eksitonov, Nauka, 1965 [Spatial Dispersion in Crystal Optics and Exciton Theory, Interscience, 1966].
[38] P. Butcher, Nonlinear Optical Phenomena, Ohio State Univ., Ohio, USA, 1965.
[39] Y. Shen, Phys. Letts **20**, 378 (1966).
[40] Yu. P. Raĭzer, ZhETF Pis. Red. 4, 286 (1966) [JETP Lett. 4, 193 (1966)].
[41] Yu. P. Raĭzer, ibid 4, 124 (1966) [4, 85 (1966)].
[42] R. Mink, R. Terhune, and C. Wang, Appl. Optics 5, 1595 (1966).
[43] G. Mayer and F. Gires, Compt. Rend. **258**, 2039 (1964).
[44] K. Rieckhoff, Appl. Phys. Letts 9, 87 (1966).
[45] A. G. Litvak, ZhETF Pis. Red. 4, 341 (1966) [JETP Lett. 4, 230 (1966)].
[46] G. A. Askar'yan, ibid. 4, 400 (1966) [4, 270 (1966)].
[47] J. Budin and J. Raffy, Appl. Phys. Letts 9, 291 (1966).
[48] R. Brewer and J. Lifsitz, Phys. Letts **23**, 79 (1966).
[49] D. Stetser and A. DeMaria, Appl. Phys. Letts 9, 118 (1966).
[50] M. P. Vanyukov, V. I. Isaenko, V. V. Lyubimov, V. A. Serebryakov, and O. A. Shorokhov, ZhETF Pis. Red. 3, 316 (1966) [JETP Lett. 3, 205 (1966)].
[51] R. Terhune, P. Maker, and C. Savage, Phys. Rev. Letts 12, 507 (1964).
[52] P. McWane, and D. Sealer, Appl. Phys. Letts 8, 278 (1966).
[53] M. A. Novikov, op. cit.[19].
[54] L. V. Keldysh, op. cit.[19].
[55] G. D. Malyuzhinets, Usp. Fiz. Nauk 69, 321 (1959) [Sov. Phys.-Usp. 2, 749 (1960)].
[56] G. D. Malyuzhinets, and L. A. Vainshteĭn, Radio-tekhnika i élektronika 6, 1247 and 1489 (1961).
[57] R. von Mises, Mathematical Theory of Compressible Fluid Flow, Academic Press, 1959.
[58] A. G. Litvak, Izv. vuzov (Radiofizika) 8, 1148 (1965).
[59] A. G. Litvak, ibid. 9, 629 (1966).
[60] A. G. Litvak, ibid. 9, 675 (1966).
[61] A. G. Litvak, ibid. 9, 900 (1966).
[62] S. A. Akhmanov and A. P. Sukhorukov, ZhETF Pis. Red. 5, 108 (1967) [JETP Lett. 5, 87 (1967)].
[63] S. V. Krivokhizha, D. I. Mash, V. V. Morozov, V. S. Starunov, and I. L. Fabelinskiĭ, ibid. 3, 378 (1966) [3, 245 (1966)].
[64] A. Piekara, IEEE, J. Quantum Electronics **QE-2**, 249 (1966).
[65] V. I. Talanov, Izv. vuzov (Radiofizika) 9, 410 (1966)].
[66] Z. K. Yankauskas, ibid. 9, 412 (1966).
[67] H. A. Haus, Appl. Phys. Letts 8, 128 (1966).
[68] T. M. Barkhudarova, G. S. Voronov, V. M. Gorbunkov, and N. G. Delone, ZhETF 49, 386 (1965) [Sov. Phys.-JETP 22, 269 (1966).
[69] A. Pine, Phys. Rev. **149**, 114 (1966).
[70] M. Maier and W. Kaiser, Phys. Letts **21**, 529 (1966).
[71] C. Wang, J. Appl. Phys. **37**, 1948 (1966).
[72] C. Wang and G. Racette, Appl. Phys. Letts 8, 256 (1966).
[73] W. Kaiser, A. Lauberau, M. Maier, and G. Giordamine, Phys. Letts **22**, 60 (1966).
[74] Yu. P. Raizer, ZhETF Pis. Red. 4, 3 (1966) [JETP Lett. 4, 1 (1966)].
[75] K. Grob and M. Wagner, Phys. Rev. Letts **17**, 819 (1966).
[76] K. Brueckner and S. Jorna, Phys. Rev. Letts **17**, 78 (1966).
[77] W. Kaiser, W. Rother, and M. Maier, Phys. Letts **23**, 83 (1966).
[78] R. Wick, D. Rank, and T. Wiggins, Phys. Rev. Letts **17**, 466 (1966).
[79] S. A. Akhmanov, V. G. Dmitriev, and A. P. Sukhorukov, Paper at Symposium on Self-focusing, Gor'kiĭ, April 1967.
[80] L. A. Ostrovskii, ZhETF Pis. Red. 5, 331 (1967) [JETP Lett. 5, 272 (1967)].
[81] P. I. Zanadvorov, Paper at I All-union Symposium on Nonlinear Optics, Minsk, 1965.
[82] A. Ashkin and G. Boyd, Appl. Phys. Letts 9, 72 (1966).
[83] A. G. Ershov, V. V. Fadeev, O. N. Chunaev, op. cit.[19].
[84] D. I. Mash, V. V. Morozov, V. S. Starunov, and I. L. Fabelinskiĭ, ZhETF Pis. Red. 2, 41 (1965) JETP Lett. 2, 25 (1965)].
[85] N. Bloembergen and P. Lallemand, Phys. Rev. Letts 16, 81 (1966).
[86] P. Lallemand, Appl. Phys. Letts 8, 276 (1966).
[87] R. Terhune, P. Maker, and C. Savage, Phys. Rev. Letts **14**, 681 (1965).
[88] S. A. Akhmanov and D. N. Klyshko, ZhETF Pis.

Red. 2, 171 (1965) [JETP Lett. 2, 108 (1965)].
[89] S. A. Akhmanov and V. I. Zharikov, ibid. 6, 644 (1967) [6, 137 (1967)].
[90] A. P. Sukhorukov, Candidate's Dissertation, Moscow, 1967.
[91] S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Zh. Eksp. Teor. Fiz. 50, 474 (1966) [Sov. Phys.-JETP 23, 316 (1966)].
[92] A. P. Sukhorukov, Izv. vuzov (Radiofizika) 9, 765 (1966).
[93] A. Ashkin, D. Kleinman, and G. Boyd, Phys. Rev. 145, 338 (1966).
[94] G. I. Freidman, Izv. vuzov (Radiofizika) 9, 550 (1966).
[95] V. S. Letokhov and A. F. Suchkov, Zh. Eksp. Teor. Fiz. 50, 1148 (1966) [Sov. Phys.-JETP 23, 763 (1966)].
[96] G. A. Askar'yan, ZhETF Pis. Red. 4, 144 (1966) [JETP Lett. 4, 99 (1966).
[97] M. S. Brodin, V. M. Vatulev, and O. M. Kamuz, Ukr. Fiz. Zhurn. 10, 1151 (1966)].
[98] T. M. Il'inova and R. V. Khokhlov, Izv. vuzov (Radiofizika) 8, 899 (1965).
[99] L. A. Rivlin, Zh. Eksp. Teor. Fiz. 47, 24 (1964) [Sov. Phys.-JETP 20, 16 (1965)].
[100] C. Wang, Phys. Rev. 152, 149 (1966).
[101] S. A. Akhmanov, A. P. Sukhorukov, and A. S. Chirkin, Zh. Eksp. Teor. Fiz. 52, 12 (1967) [Sov. Phys. -JETP 25, 6 (1967)].
[102] C. K. N. Patel, Phys. Rev. Letts 15, 1027 (1965).
[103] R. Brewer and C. Townes, Phys. Rev. Letts 18, 196 (1967).
[104] D. H. Close, C. R. Giuliano, R. W. Hellwarth, L. D. Hess, F. J. McClung, W. G. Wagner, IEEE J. Quantum Electronics QE-2, 1964 (1966).
[105] R. Y. Chiao, M. A. Johnson, S. Krinsky, H. A. Smith, C. H. Townes, E. Garmire, IEEE J. Quantum Electronics QE-2, 467 (1966).
[106] A. A. Chaban, ZhETF Pis. Red. 5, 57 (1967) [JETP Lett. 5, 45 (1967)].
[107] R. W. Hellwarth, Phys. Rev. 152, 157 (1966).
[108] F. Kaczmarek, Acta Phys. Polonica 30, 87 (1966).
[109] R. Chiao, E. Garmire, and P. Kelley, Phys. Rev. Letts (1966).
[110] L. R. Carman, R. Chiao, and P. Kelley, Phys. Rev. Letts 17, 1281 (1966).
[111] D. I. Abakarov, A. A. Akopyan, and S. I. Pekar, Zh. Eksp. Teor. Fiz. 52, 463 (1967) [Sov. Phys.-JETP 25, 303 (1967).
[112] Yu. P. Raizer, ibid. 52, 470 (1967) [25, 308 (1967)].
[113] S. Kielich, Acta Physica Polonica 30, 683 (1966).
[114] S. A. Akhmanov, D. P. Krindach, A. P. Sukhorukov, and R. V. Khokhlov, ZhETF Pis. Red. 6, 509 (1967) [JETP Lett. 6, 38 (1967)].
[115] R. Chiao, J. Dodson, D. Irwin, T. Gustafson, Bull. Amer. Phys. Soc. 12, 686 (1967).
[116] T. Gustafson, F. De Martini, C. Townes, P. Kelley, Bull. Amer. Phys. Soc. 12, 687 (1967).
[117] S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Paper at Session of the Division of General and Applied Physics, USSR Academy of Sciences, Moscow, June 1967.
[118] H. Grieneisen, C. Sacchi, Bull. Amer. Phys. Soc. 12, 686 (1967).

Translated by J. G. Adashko