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FERROMAGNETIC RESONANCE AND PLASMA EFFECTS IN METALS

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FERROMAGNETIC resonance (FMR) is produced when an alternating high-frequency field excites homogeneous precession of the magnetic moment in a constant magnetic field. The reaction of the system of magnetic moments to the alternating electromagnetic field can be described with the aid of the magnetic permeability $\mu(\omega)$ (ω—frequency of alternating field), whose frequency dependence reveals the resonant character. Calculation of the magnetic permeability is an important problem in the theory of FMR and can be performed in different ways, depending on the model chosen. It is usually assumed that the ferromagnetic properties of metals are connected with the magnetic moments, localized in the crystal lattice sites, of the incompletely filled d and f shells of the atoms. The system of magnetic moments is described by an equation of the Landau-Lifshitz type [1]. Another approach is to consider a single interacting-electrons system, having a nonzero total magnetic moment M[2,3]. We note immediately, however, that the frequency dependence of the magnetic permeability near the resonance frequency is in general not very sensitive to the model used for the metal in the analysis.

Ferromagnetic resonance can be observed when the length of the electromagnetic wave is large compared with the dimensions of the sample and is used, as a rule, in observations of FMR in ferromagnetic dielectrics (or ferrites). The reason is that the refractive index of a ferrite differs markedly from unity in the centimeter band, in which the FMR is observed. Consequently, the customarily employed samples are small compared with the wavelength, i.e., a quasi-stationary situation is produced.

The inverse case (skin effect) takes place in metals, owing to the high density of the conduction electrons. Therefore it is natural to observe FMR in metals by using reflection of the electromagnetic wave. It is convenient to describe the reflected waves by using the concept of surface impedance, and the FMR is manifest in the frequency dependence of this impedance.

Excitation of inhomogeneous precession of magnetic moments—spin waves—is customarily called spin-wave resonance[4]. The existence of the spin wave is due to exchange interaction between the electrons. From the macroscopic point of view, allowance for exchange interaction is manifest in spatial dispersion—the dependence of the magnetic permeability μ on the wave vector $\mathbf{k}$, i.e., $\hat{\mu} = \mu(\omega, \mathbf{k})$. It must be noted that the spin-wave dispersion law is likewise not a feature of the model used for the metal, and can be written in the case of long waves in the form

$$\omega = \omega_0(\Phi) + \frac{A}{\hbar}(ak)^2,$$

where the constant A is of the order of the Curie temperature θ_c and a is the lattice constant; the activation energy $\hbar\omega_0$ is determined by the magnitude of the real magnetic field acting inside the metal, and depends on the angle Φ between the wave vector $\mathbf{k}$ and the static moment M_S.

Attention must be called to a very curious circumstance typical of the high-frequency properties of ferromagnets. Exchange interaction—the true cause of ferromagnetism—enters directly only in that part of the high-frequency magnetic susceptibility which is connected with the spatial dispersion. The characteristic frequencies in $\mu(\omega)$ are determined by the relatively weak relativistic interaction (the anisotropy energy, the interaction with the external field, dipole-dipole interactions). Therefore the role of spatial dispersion in ferromagnets (especially in ferromagnetic metals, see below) is very appreciable.

This is formally manifest in the fact that the role of the spatial dispersion is measured, in the analysis of the electromagnetic properties of a ferromagnet, not by the ratio $(a/\lambda)^2$, where $\lambda = 2\pi/k$ is the length of the electromagnetic wave, but by the quantity $(A/\hbar\omega_0)(a/\lambda)^2$, as can be seen directly from the spin-wave dispersion law. Although this quantity should be compared not with unity but with the relative deviation from resonance $\Delta\omega/\omega_0$, the minimum value of which is determined by the line width, the condition $(A/\hbar\omega_0)(a/\lambda)^2 \gtrsim \Delta\omega/\omega_0$ can, of course, not be satisfied at radio frequencies ($\lambda \sim 1$—10 cm) even in the most perfect dielectric samples. In metals, however, the relation of interest to us is satisfied relatively easily, since the wavelength in a metal is smaller by several orders of magnitude than the wavelength in vacuum, owing to the skin effect. Of course, this condition is satisfied if the relative deviation from resonance determined by the FMR line width, is sufficiently small. FMR in very pure metals and at low temperatures has been extensively investigated of late. Such experiments reveal relatively narrow FMR lines—on the order of several dozen Oersted (cf., e.g., [5,6]). We shall show that allowance for the spatial dispersion in the metal leads to a

shift and broadening, of the same order of magnitude, of the FMR line. Thus, allowance for spatial dispersion is a vital problem in the theory of FMR in metals.

We emphasize once more that the appearance of spatial dispersion in magnets is facilitated by the unique features of the law of dispersion of the spin waves, whose dynamic part (dependence on the wave vector) is determined by the large exchange energy, and whose activation energy is determined by small relativistic interactions. In dielectrics, in which the spatial dispersion is significant in the exciton-absorption region[7], the situation is different: both the activation energy and the dependence on the wave vector are caused by the same interactions. The "facilitating" circumstances for dielectrics are the smallness of the wavelength in the optical band and the existence of very narrow lines.

The effects of spatial dispersion in a metal are significant both in the magnetic permeability and in the conductivity $\hat{\sigma}$. The conduction electrons, owing to the skin effect, not only cause the magnetic-moment inhomogeneity necessary for the appearance of the exchange effects, but can themselves be readily placed under conditions of strong spatial dispersion (anomalous skin effect[8]).

Ferromagnetic resonance is observed in relatively strong magnetic fields, in which the influence of the magnetic field on the dynamics of the conduction electrons becomes appreciable. The motion of the electrons in the magnetic field produces, in particular, galvanomagnetic effects and gives rise to helicons and other weakly-damped waves in the metal[9]. This region became accessible to experimental research relatively recently, after highly perfected ferromagnetic metals became available. Reed and Fawcett[1] measured the magneto-resistance of nickel and iron at low temperatures. Anderson and Gold[11] observed the de Haas-van Alphen effect in nickel. Grimes[12] observed a helical wave in nickel.

The most characteristic difference between FMR in metals and FMR in dielectrics is connected with the specific role of the conduction electrons. The plasma approach to a ferromagnetic metal denotes allowance for the temporal and spatial dispersion of the magnetic susceptibility in the analysis of the collective excitations in the electron gas. In other words, a ferromagnetic metal can be regarded as a plasma, whose wave properties reveal not only the characteristic branches of the oscillations of the electron gas[9,13], but also the specific magnetic branches connected with the macroscopic oscillations of the magnetization.

The complicated dependence of the magnetic permeability and of the specific conductivity on the frequency and on the wave vector lead to a highly unique pattern of propagation of the waves in such media. The high-frequency properties of ferro- and antiferromagnetic metals and semiconductors have recently been the subject of a large number of theoretical investigations[14].

1. FUNDAMENTAL EQUATIONS

In considering the electromagnetic properties of a ferromagnetic metal, it is necessary to start from Maxwell's equations

$$\operatorname{rot}\mathbf{h}=\frac{4\pi}{c}\mathbf{j},\quad \operatorname{rot}\mathbf{e}=-\frac{1}{c}\frac{\partial\mathbf{b}}{\partial t}, \tag{1}$$

where $\mathbf{b}$ is the magnetic induction, $\mathbf{j}$ the current density, $\mathbf{e}$ and $\mathbf{h}$ the intensities of the electric and magnetic alternating fields respectively, and c the speed of light.

Equations (1) must be supplemented with the material equations, which in the case of a spatially homogeneous medium are

$$\mathbf{j}(\mathbf{r},t)=\int\hat{\sigma}(\mathbf{r}-\mathbf{r}',\ t-t')\,\mathbf{e}(\mathbf{r}',\ t')\,d\mathbf{r}'\,dt',$$
$$\mathbf{b}(\mathbf{r},t)=\int\hat{\mu}(\mathbf{r}-\mathbf{r}',\ t-t')\,\mathbf{h}(\mathbf{r}',\ t')\,d\mathbf{r}'\,dt'. \tag{2}$$

The integral form of the material equations (2) reflects the nonlocal character of the connection between the current and the magnetic induction, on the one hand, and the electric and magnetic fields on the other. In the case of a spatially homogeneous medium, the integral relations (2) contain the difference kernels $\sigma(\mathbf{r}-\mathbf{r}')$ and $\mu(\mathbf{r}-\mathbf{r}')$, and this facilitates the subsequent transition to the Fourier representation. When account is taken of the boundaries the metal, of course, ceases to be homogeneous and relations (2), generally speaking, no longer hold. In many cases, however, it is possible to use the expressions obtained for $\hat{\sigma}$ and $\hat{\mu}$ in the case of an unbounded metal (see below).

In the analysis of FMR we shall confine ourselves to the linear theory. This means that $\hat{\sigma}$ and $\hat{\mu}$ do not depend on the amplitude of the high-frequency field, but depend, of course, on the external static magnetic field $\mathbf{H}_0$.

The current density can be calculated with the aid of the distribution function f of the conduction electrons. The current density is determined by the expression

$$\mathbf{j}=\frac{2e}{(2\pi\hbar)^3}\int\mathbf{v}f\,d\mathbf{p}, \tag{3}$$

where $\mathbf{v}$ is the electron velocity.

We confine ourselves to a classical treatment, although in relatively strong magnetic fields the quantization condition $\hbar\omega_H > T$ (ω_H is the cyclotron frequency and T the temperature) may turn out to be satisfied. Nonetheless, this analysis is fully justified, since allowance for the quantization of the electron motion leads to a superposition of small oscillations on the "classical" curves; the amplitude of these oscillations is determined by the relations $\hbar\omega_H/\epsilon_F$ (ϵ_F is the limiting Fermi energy), the mag-

nitude of which is comparable with unity only in fields on the order of 10^8 Oe.*

The electron distribution function f is determined from Boltzmann's kinetic equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\nabla f+\frac{2\pi}{T_H}\frac{\partial f}{\partial \varphi}+e\mathbf{e}\frac{\partial f}{\partial \mathbf{p}}+\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}=0. \tag{4}$$

Here φ and $T_H = 2\pi\omega_H^{-1}$ are the phase and the period of revolution of the electrons in their orbit, and $(\partial f/\partial t)_{coll}$ is the collision integral.

The collision integral $(\partial f/\partial t)_{coll}$ describes (along with other dissipation mechanisms) the interaction of the electrons with the magnetic-moment density oscillations—with the spin waves. However, it is not our task to describe in detail the dissipative mechanisms, and all the more to calculate the relaxation constants. Where possible, we shall omit the collision integral entirely (for example when $\omega \gg \nu$, where ν is the mean collision frequency). In some cases we shall use the τ-approximation, i.e., replace the true collision integral by the expression $f/\tau \equiv \nu f$ (τ—mean free path time). Such a substitution is in fact justified only under conditions of anomalous skin effect[16], when the "incoming" integral term in the collision operator is much smaller than the "outgoing" one. In the region of the normal skin effect the character of the collision operator should not be of any interest to us at all—in this case the final formulas contain only the macroscopic characteristics of the metal, namely the electric-conductivity tensor components.

We note also that we start from the "gas" approximation. In almost all cases, however, allowance for the Fermi-liquid interaction does not change the final results (see[17], Sec. 1).

We present the known expression for the Fourier components of the conductivity tensor $\sigma_{ik}(\omega, \mathbf{k})$ in a number of limiting cases[18,9].

A. Conductivity in the Absence of a Magnetic Field

Under the conditions of the normal skin effect ($kl \ll 1$, where $l = v_0\tau$, v_0 is the electron velocity on the Fermi boundary), the conductivity is strongly dependent on the relation between the frequency ω of the electromagnetic field and the collision frequency ν.

When $\omega \ll \nu$ the metal is characterized by a static conductivity $\sigma_{ik}^{(0)}$ (σ_0 for an isotropic metal).

When $\omega \gg \nu$ the conductivity of the metal is imaginary[17]:

$$\sigma_{ik}=i\frac{D_{ik}}{\omega}. \tag{5}$$

The matrix elements D_{ik} coincide in order of magnitude with the square of the metal plasma frequency. For an isotropic metal we have in the gas approximation

$$D_{ik}=\omega_L^2\delta_{ik},\quad \omega_L^2=\frac{4\pi Ne^2}{m^*}, \tag{6}$$

where N is the electron density and m* the modulus of the effective mass.

Under the conditions of the extremely anomalous skin effect ($kl \gg 1$) we are interested in the tensor elements σ_{ik} that are transverse to the direction of the wave vector

$$\sigma_{\alpha\beta}=\frac{3\pi}{4}\frac{B_{\alpha\beta}}{|\mathbf{k}|}, \tag{7}$$

where

$$B_{\alpha\beta}=\frac{8e^2}{3(2\pi\hbar)^3}\int_0^{2\pi}\frac{n_\alpha n_\beta}{K(\varphi)}\,d\varphi,$$

and n is a unit vector normal to the Fermi surface and $K(\varphi)$ is the Gaussian curvature of the surface at the points where $n_z = 0$. For an isotropic electron spectrum

$$\sigma_{\alpha\beta}=\frac{3\pi}{4}\frac{Ne^2}{p_0|\mathbf{k}|}\delta_{\alpha\beta}. \tag{8}$$

Here p_0 is the radius of the Fermi sphere.

B. Conductivity in a Magnetic Field

In a ferromagnetic metal, the conduction electron is acted upon by the Lorentz force $(e/c)\mathbf{v}\times\mathbf{B}$, where $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}_z$ and H is the magnetic field acting inside the ferromagnet; this field must be determined by solving the magnetostatic problem. The conductivity σ_{ik} is determined by the value of the induction B.*

In considering a metal in a magnetic field, we shall assume that the magnetic field is sufficiently strong, i.e., we shall assume the inequality $\omega_S\tau \gg 1$ to be satisfied. This means that the radius r of the electron orbit is much smaller than the mean free path l.

The parameter that determines the spatial dispersion of the electric conductivity is kR, where $R = l|(\nu - i\omega)/\omega_H|$: R coincides with the cyclotron radius r when $\omega \ll \nu$. Under conditions of weak spatial dispersion ($kR \ll 1$) the transverse part of the conductivity tensor takes the form

$$\sigma_{\alpha\beta}(\omega, 0)=\frac{Ne^2}{m^*}\frac{\omega_H}{\omega_H^2-(\omega+i\nu)^2}\begin{pmatrix}\frac{\nu-i\omega}{\omega_H} & -1\\ 1 & \frac{\nu-i\omega}{\omega_H}\end{pmatrix}. \tag{9}$$

The indices α and β take on the values x and y. The z axis is chosen along the magnetic field direction ($\mathbf{k} \parallel \mathbf{B}$).

*This statement is valid, strictly speaking, for static quantities (resistance etc.). In the high-frequency region (especially near resonances), effects are possible (for example, quantum cyclotron resonance [145], which are determined just by the quantization of the electron motion in the magnetic field.

*We do not take into account the anomalous Hall effect, since it is not significant at low temperatures [19].

If the alternating-field frequency is small compared with the cyclotron frequency, $|\omega + i\nu| \ll \omega_H$, expression (9) assumes a simpler form. The factor preceding the brackets in (9) becomes Nec/B. Expression (9) is valid for one group of carriers under the assumption that the spectrum is isotropic.

In a strong magnetic field, for a closed Fermi surface, the Hall elements of the electric conductivity ($\sigma_{xy} = -\sigma_{yx}$) are much larger than the diagonal elements σ_{xx} and σ_{yy} ($|\sigma_{xy}| \gg \sigma_{xx,yy}$). In the case of two groups of carriers we have

$$\sigma_{xy} = \frac{(N_1 - N_2)ec}{B}, \tag{10}$$

where N_1 (N_2) is the electron (hole) density. The expression for the Hall elements of the electric conductivity is independent of the electron dispersion law if the Fermi surface is closed.

Interest also attaches to the asymptotic behavior of the conductivity tensor of a metal with equal electron and hole densities ($N_1 = N_2 = N$). In this case the off-diagonal (Hall) elements of the tensor $\sigma_{\alpha\beta}$ vanish ($\omega \gg \nu$):

$$\sigma_{\alpha\beta} = \frac{Nec}{B}\begin{pmatrix} -i\frac{\omega}{\omega_H} & 0 \\ 0 & -\frac{i\omega}{\omega_H} \end{pmatrix}. \tag{11}$$

The case $kR \ll 1$ is rarely realized in metals. The foregoing formulas may be useful for ferromagnetic semiconductors (ferrites). In a metal one usually has the inverse condition

$$kR \gg 1. \tag{12}$$

Let us consider the asymptotic behavior of the conductivity tensor under condition (12) in the case when the wave propagates transversely to the magnetic field ($\mathbf{k} \perp \mathbf{H}$)[9]. The z axis, as before, is chosen along the magnetic field, and the x axis along the wave vector. The diagonal elements of the tensor σ_{ik} are asymptotically equal to

$$\left.\begin{aligned} \sigma_{xx} = \sigma_{zz} \equiv \sigma(\omega,\mathbf{k}) &= \frac{3\pi i}{4}\frac{Ne^2}{|\mathbf{k}|p_0}\operatorname{ctg}\frac{\pi(\omega+i\nu)}{\omega_H}, \\ \sigma_{yy} &= \frac{3Ne^2(\nu - i\omega)}{mk^2v_0^2}\left[1 - \frac{\pi(\omega+i\nu)}{2|\mathbf{k}|v_0}\operatorname{ctg}\frac{\pi(\omega+i\nu)}{\omega_H}\right]. \end{aligned}\right\} \tag{13}$$

The off-diagonal elements of the tensor turn out in this case to be smaller by a factor kR than the diagonal ones, and they can be neglected. Expression (13) at frequencies close to $n\omega_H$, where n is an integer, describes the cyclotron resonance[13]. Expanding $\cot[\pi(\omega + i\nu)/\omega_H]$ near $n\omega_H$, we get

$$\sigma(\omega, \mathbf{k}) = \frac{3i}{4}\frac{Ne^2}{|\mathbf{k}|p_0}\frac{\omega_H}{\omega - n\omega_H + i\nu}. \tag{14}$$

In the case of low frequencies ($|\omega + i\nu| \ll \omega_H$) we can replace $\cot[\pi(\omega + i\nu)/\omega_H]$ by $\omega_H/\pi(\omega + i\nu)$, and then $\sigma(\omega, \mathbf{k})$ takes the form

$$\sigma(\omega, \mathbf{k}) = \frac{3i}{4}\frac{Ne^2}{(\omega + i\nu)|\mathbf{k}|R}. \tag{15}$$

The foregoing asymptotic expressions for the conductivity tensor are extremely simple, and the fact that facilitates the calculations greatly is that they can be used when boundary-value problems are considered. This possibility is based on the "indifference" of the electrons to the boundary conditions when $kR \gg 1$. As shown by Azbel' and Kaner[16], the impedance calculated with the aid of the expression for the conductivity of an unbounded metal differs from that obtained by solving the exact boundary-value problem by an inessential real factor on the order of unity. The insensitivity of the electrons to the boundary conditions in the case of the anomalous skin effect (when H = 0) becomes manifest, in particular, in the fact that the values of the impedance in specular and diffuse reflection of the electrons from the metal boundary differ only by the factor $^8/_9$.[8,20] The physical cause of the insensitivity of the surface impedance to the boundary conditions for the electrons lies in the fact that the main contribution to the impedance is made by electrons gliding along the surface of the metal, which consequently do not collide with the boundary.

We shall now discuss the expression for the magnetic permeability of the metal. If we assume that the magnetic moments are localized in the lattice sites, then the magnetic permeability can be calculated with the aid of an equation of the Landau-Lifshitz type, describing the precession of the moment M in a magnetic field

$$\frac{d\mathbf{M}}{dt} = \gamma[\mathbf{M}, \mathbf{H}_{\text{eff}}] - \frac{\lambda}{M^2}[\mathbf{M}, [\mathbf{M}, \mathbf{H}_{\text{eff}}]], \tag{16}*$$

where γ is the gyromagnetic ratio, and the effective magnetic field for an isotropic metal takes the form

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + \alpha\Delta\mathbf{M}. \tag{17}$$

Here α is a volume constant equal to $Aa^2/\mu_B M_S$, where μ_B is the Bohr magneton.

The dissipation is taken into account in (16) phenomenologically with the aid of the reciprocal relaxation time λ of the magnetic moments. The question of the form of the dissipative term in the equation for the magnetic moment is the subject of a special analysis (cf., e.g.,[21]) and will not be dealt with here.

The solution of (16) in the linear approximation with respect to the alternating part of the magnetic moment m leads to the following expression for the magnetic permeability of an unbounded metal[1] in the simplest case when the wave propagates along the magnetic field:

$$\mu_{ik}(\omega, \mathbf{k}) = \begin{pmatrix} \mu_1 & i\mu_2 & 0 \\ -i\mu_2 & \mu_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

$$\mu_1 = \frac{\Omega\Omega_1 - (\omega + i\lambda)^2}{\Omega^2 - (\omega + i\lambda)^2}, \quad \mu_2 = \frac{4\pi\gamma M_s(\omega + i\lambda)}{\Omega^2 - (\omega + i\lambda)^2}, \tag{19}$$

*$[\mathbf{M}, \mathbf{H}_{\text{eff}}] \equiv \mathbf{M} \times \mathbf{H}_{\text{eff}}$.

where $\Omega = \gamma M_S(\alpha k^2 + H/M_S)$ and $\Omega_1 = \Omega + 4\pi\gamma M_S$. The z axis is chosen along the direction of the wave vector.

Neglecting dissipation ($\lambda = 0$), the elements of the tensor turn out to be singular at the point $\omega = \Omega$. This is the consequence of the resonant absorption of the energy of the electromagnetic field. In a number of cases the absorption of the energy is determined by the effective magnetic permeability [22], which represents a combination of elements of the tensor μ_{ik}. This leads to a change in the frequency of the resonance absorption.

On approaching the resonance frequency, it is obviously impossible to neglect the exchange term in (19) (this was already mentioned in the Introduction). Since the wave-vector values determined from the dispersion equation are in general complex, allowance for the exchange interaction leads both to a shift and to broadening of the resonance line, and by the same token changes the form of the resonance curve.

Formulas similar to (18) and (19) can be obtained by starting from other more realistic models, which take into account the role played by the conduction electrons in the formation of the magnetic moment of the metal [23]. The most general approach to this problem is probably the Fermi-liquid approach, in which the ferromagnetic metal is regarded as a system of Fermi particles with exchange interaction and with a magnetic moment. The magnetic permeability was calculated for this model by Kondratenko [3,24] (see the Appendix).

The formulas for the magnetic susceptibility of an unbounded metal can be used to calculate the surface impedance of a ferromagnet in those cases when the following boundary conditions are satisfied for the magnetic moment: either $\mathbf{m} = 0$ on the boundary, or else $\partial\mathbf{m}/\partial\mathbf{n} = 0$ [25]. This is possible as a result of the following: Usually in the calculation of the impedance it is convenient to continue the electric and magnetic fields to the region outside the metal. Such a continuation can be effected in two ways, either by assuming that $\mathbf{e}(z)$ is an even function and $\mathbf{h}(z)$ an odd one, or vice versa.

If the condition $\mathbf{m} = 0$ on the boundary is satisfied, then, by continuing the magnetic field in odd fashion to the region $z < 0$ we obtain for the Fourier component of the linearized equation (16)

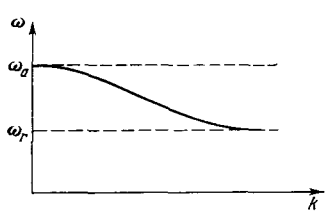


FIG. 1.

$$-\omega\mathbf{m}_k = \gamma[\mathbf{M}_s, \mathbf{h}_k - \alpha k^2\mathbf{m}_k] + \gamma[\mathbf{m}_k, \mathbf{H}], \qquad (20)$$

from which we get

$$\mathbf{m}_k = \hat{\chi}(\omega, \mathbf{k})\,\mathbf{h}_k, \qquad (21)$$

where $\hat{\chi}(\omega, \mathbf{k})$ is the susceptibility of the unbounded metal. We shall denote the impedance* in the case $\mathbf{m}(0) = 0$ by ζ_p. For an even continuation of the electric field we obtain from Maxwell's equations

$$\zeta_p = \frac{2}{\pi i}\int_0^\infty \frac{\mu(\omega, x)\,dx}{x^2 - \varepsilon(\omega, x)\,\mu(\omega, x)}. \qquad (22)$$

Here $\epsilon(\omega, x) = 4\pi i\sigma(\omega, x)/\omega$ and $\mu(\omega, x)$ are respectively the dielectric constant and the magnetic permeability of the metal, determined by the elements of the tensors σ_{ik} and μ_{ik} in accordance with the geometry of the problem. We have gone over to integration with respect to x, where $x = ck/\omega$ is the refractive index of the wave with wave vector k.

In the case when $\partial\mathbf{m}/\partial\mathbf{n}$ on the boundary, relation (21) can be obtained by means of an even continuation of the magnetic field. The surface impedance (ζ_n) is given in this case by

$$\zeta_n^{-1} = \frac{2}{\pi i}\int_0^\infty \frac{\varepsilon(\omega, x)\,dx}{x^2 - \varepsilon(\omega, x)\,\mu(\omega, x)}. \qquad (23)$$

It should be noted that the even and odd continuations of the field e correspond to very special boundary conditions for the electron distribution function, but this circumstance does not play an important role, owing to the already mentioned insensitivity of the impedance to the behavior of the electrons on the boundary. If we neglect the spatial dispersion of the magnetic susceptibility, then the resultant expressions for the impedance either merely coincide (normal skin effect) or differ by an inessential factor (limiting anomalous skin effect). If there is no spatial dispersion of either the magnetic permeability or the dielectric constant, we get from (22) and (23) the well known expression

$$\zeta_p = \zeta_n = \left(\frac{\mu(\omega)}{\varepsilon(\omega)}\right)^{1/2}. \qquad (24)$$

Given a general boundary condition for the magnetic moment, neither an even nor an odd continuation of the field allows us to use the magnetic susceptibility of the unbounded metal.

2. NORMAL SKIN EFFECT

The dielectric constant of a metal under the conditions of the normal skin effect is imaginary in the

*The surface-impedance tensor can be introduced in invariant fashion by means of the equality $e_\alpha = \zeta_{\alpha\beta}[\mathbf{h} \times \mathbf{n}]_\beta$, which holds on the boundary ($\mathbf{n}$ – unit vector normal to the surface) or with the aid of the relation $Z_{\alpha\beta} = \partial e_\alpha/\partial J_\beta$, where J_β is the component of the total current in the volume of the metal. The two definitions are equivalent, ($Z_{\alpha\beta} = 4\pi\zeta_{\alpha\beta}/c$), but we shall make use of both.

case of low frequencies and real but negative at high frequencies. In either case, the electromagnetic field does not penetrate into the metal. At low frequencies the electric field attenuates at a skin-layer depth $\delta = c/\sqrt{2\pi\sigma_0\omega}$, and at high frequencies it is totally reflected.

Analysis of the behavior of a ferromagnetic conductor under the same conditions shows that the frequency dependence of the magnetic susceptibility $\mu = \mu(\omega)$ alters the electromagnetic properties of the metal appreciably. This circumstance is most evident at high frequencies ($\omega\tau \gg 1$). The dispersion equation $k^2 = \omega^2\epsilon\mu/c^2$ takes in this case the form

$$k^2 = -\frac{\omega_L^2}{c^2}\frac{\omega_a-\omega}{\omega_r-\omega}. \qquad (25)$$

It is easy to see that in a relatively narrow region of frequencies, $\omega_r < \omega < \omega_a$, an undamped electromagnetic wave can propagate in the metal. Its dispersion is given by

$$\omega = \frac{\omega_L^2\omega_a+\omega_r c^2k^2}{\omega_L^2+c^2k^2}. \qquad (26)$$

The corresponding plot is shown in Fig. 1. This wave has anomalous dispersion. The possible existence of a wave with negative group velocity in a high-frequency plasma, in the region where $\epsilon < 0$ and $\mu < 0$, was first pointed out by Pafomov [26] (see also [26a]). A ferromagnetic metal is a natural example of a medium in which this situation can be realized.

Under FMR conditions, the low frequency case is frequently realized. Then the metal is characterized by a static conductivity σ_0, and the wave vector is complex. In the case when the wave propagates parallel to the magnetic field ($\mathbf{k} \parallel \mathbf{H} \parallel Oz$) we have

$$k_\pm^2 = \frac{4\pi i\sigma_0\omega}{c^2}(\mu_1 \pm i\mu_2) = \frac{4\pi i\sigma_0\omega}{c^2}\frac{\Omega_1 \mp \omega}{\Omega \mp \omega}, \qquad (27)$$

where the indices ± correspond to circularly polarized waves. In the case of transverse propagation ($\mathbf{k} \parallel Ox$) the dispersion equation of the resonating wave takes the form

$$k^2 = -\frac{4\pi i\sigma_0\omega}{c^2}\frac{\det\hat{\mu}}{\mu_1} = -\frac{4\pi i\sigma_0\omega}{c^2}\frac{\Omega_1^2-\omega^2}{\Omega\Omega_1-\omega^2}. \qquad (28)$$

The extraordinary wave, in which the magnetic field is parallel to the constant field H, does not interact with the magnetic moments. In the case of transverse propagation, the frequency of the homogeneous resonance is equal to γH, and in transverse propagation it is equal to $\gamma(HB)^{1/2}$.

In the general case when the wave propagates at an angle φ to the direction of the magnetic field, the frequency of the homogeneous resonance, as can be easily verified, is equal to $\gamma H\cos\varphi + \gamma(HB)^{1/2}\sin\varphi$. The impedance of a ferromagnetic metal under the conditions of the normal skin effect was calculated by Ament and Rado [27] and subsequently by a number of authors [28-31].

When spatial dispersion is taken into account in the Landau-Lifshitz equation, the need arises for an additional boundary condition for the magnetic moment. Ament and Rado [27] proposed that the normal derivative of the magnetic moment vanishes on the boundary: $\partial\mathbf{m}/\partial z|_{z=0} = 0$. However, a number of considerations lead to a more general boundary condition in the form

$$\frac{\partial\mathbf{m}}{\partial z} + \chi\mathbf{m} = 0. \qquad (29)$$

This condition, as shown by Pinkus [30], is due to the character of the anisotropy field on the boundary. According to [30] we have with good accuracy $\chi = -(\omega_a/\omega_e)a^{-1}$, where $\hbar\omega_a$ is the surface-anisotropy energy, which can greatly exceed the volume energy, and $\hbar\omega_e$ is of the same order of magnitude as the exchange integral. We see from (29) that the parameter χ must be compared with the magnitude of the wave vector k. Even in the case of the shortest waves we have $ka \sim 10^{-2}$, and $\chi/k \simeq 10$ when $\omega_a \sim 0.1\,\omega_e$, i.e., the condition $\mathbf{m} = 0$ is satisfied approximately on the boundary. The Ament and Rado condition $\partial\mathbf{m}/\partial\mathbf{n} = 0$ is thus valid only if there is no surface anisotropy.

Following [31], let us calculate the surface impedance under boundary condition (29). Let the magnetic field $\mathbf{H} \parallel Oz$ be parallel to the surface of the metal, and let the wave propagate normally to the surface along the y axis. For the case of a magnetic field normal to the surface of the metal, the surface impedance can be obtained from the expression derived below by simply changing the notation.

The solution of the dispersion equation of the electromagnetic waves in the metal is in this case

$$2(kd)^2 = -\frac{H}{4\pi M_s} + \left(\frac{\omega}{4\pi\gamma M_s}\right)^2 \pm \sqrt{\left(\frac{H}{4\pi M_s} - \frac{\omega^2}{(4\pi\gamma M_s)^2}\right)^2 - \frac{8id^2}{\delta^2}}, \qquad (30)$$

where $d = a(A/\mu_B M_S)^{1/2}$. In the derivation we took account of the fact that the parameters d/δ and $H/4\pi M_S$ are small compared with unity.

The conditions of continuity of e_z and h_x on the boundary, as well as the boundary condition (29) for the magnetic moment, will be written in the form

$$\left.\begin{aligned} h_{1x} + h_{2x} &= h_{0x}, \\ k_1h_{1x} + k_2h_{2x} &= \frac{4\pi i\sigma_0}{c}\zeta h_{0x}, \\ k_1^2(ik_1+\chi)h_{1x} + k_2^2(ik_2+\chi)h_{2x} &= 0. \end{aligned}\right\} \qquad (31)$$

To derive the last equation of (31) we used the fact that near resonance we have

$$m_{ix} = \frac{k_i^2\delta^2}{8\pi i}h_{ix} \qquad (i = 1,\ 2). \qquad (32)$$

From the condition for the compatibility of the system (31) we get

$$\frac{4\pi i\sigma_0}{c}\zeta = \frac{k_1k_2(k_1+k_2-i\chi)}{k_1^2+k_1k_2+k_2^2-i\chi(k_1+k_2)}. \qquad (33)$$

Substituting here the values of k_1 and k_2 from (30), we obtain an explicit expression for the impedance. For comparison with the results of Ament and Rado [27],

it is convenient to change over to their notation. Putting

$$\left(\frac{d}{\delta}\right)^2=\varkappa^2,\ \frac{\omega}{4\pi\gamma M_s}=\omega',\ \frac{H}{4\pi M_s}=\eta,\ \frac{4\pi\sigma_0\varkappa\delta}{c}\zeta=\zeta',\ \xi=\chi\varkappa\delta,$$

we get

$$\zeta'=-\frac{(\sqrt{\eta-\omega'^2-2(1+i)\varkappa}-\xi)(1+i)\varkappa}{\eta-\omega'^2-(1+i)\varkappa-\xi\sqrt{\eta-\omega'^2-2(1+i)\varkappa}}. \quad (34)$$

We introduce the effective permeability by means of the formula

$$\zeta=\left(\frac{\mu_{\text{eff}}}{\varepsilon_{\text{eff}}}\right)^{1/2},\quad \varepsilon_{\text{eff}}=\frac{4\pi i\sigma_0}{\omega}. \quad (35)$$

$\mu_{\text{eff}} = \mu' - i\mu''$ is defined, in accord with (30), (33), and (35), as follows:

$$\varkappa\mu_{\text{eff}}\equiv\varkappa(\mu'+i\mu'')=\left[\frac{\sqrt{N+1-2i}+\xi}{N-i+\xi\sqrt{N+1-2i}}\right]^2, \quad (36)$$

where $N = \kappa^{-1}(\eta - \omega'^2 + \kappa)$.

If $\xi = 0$ ($\partial m/\partial n = 0$ on the boundary), then

$$\varkappa\mu_{\text{eff}}=\frac{N+1-2i}{(N-i)^2}. \quad (37)$$

If $\xi = \infty$ ($m = 0$ on the boundary), then

$$\varkappa\mu_{\text{eff}}=\frac{1}{N+1-2i}. \quad (38)$$

Expressions (37) and (38) coincide with the results of Rado and Weertman [32]. The connection between μ' and μ'' at different values of ξ is shown in Fig. 2. The experimental points were taken from [32]. As seen from the figure, the results of the experiment can be described by assuming that ξ varies, depending on the external magnetic field and the frequency, in the range 0.4—1.0. We note that $\xi \sim 1$ means that the exchange interaction on the boundary plays the same role as surface anisotropy.

The shift and broadening of the resonance, as follows from (36)—(38), are of the same order of magnitude, regardless of the behavior of the moments on the boundary:

$$\frac{\Delta H}{H}\simeq\frac{4\pi M_s}{H}\frac{d}{\delta}. \quad (39)$$

If we neglect the spatial dispersion, then, according to (24), the impedance becomes infinite at the resonance frequency if there is no dissipation. Allowance for spatial dispersion makes the impedance finite at all frequencies. This justifies the neglect of the dissipative terms in Eq. (16), provided that

$$\lambda\ll 4\pi\gamma M_s\frac{d}{\delta}.$$

At reasonable values of the quantities involved, $4\pi\gamma M_S d/\delta \sim 10^8\ \text{sec}^{-1}$. The FMR line shape, under the conditions of the normal skin effect and with allowance for spatial dispersion of the magnetic susceptibility was calculated in [25].

3. LIMITING ANOMALOUS SKIN EFFECT

The actual calculation of the impedance of a ferromagnetic metal under the conditions of the anomalous skin effect is a very difficult task, in view of the nonlocal connection between the current and the field. If the reflection of the electrons from the metal boundary has an arbitrary character, the surface impedance is expressed in a rather complicated manner in terms of the kernel of an integral equation relating the Fourier components of the current density and of the field. In such a general formulation of the problem, the surface impedance can be calculated in a number of limiting cases [33,34] or else investigated by numerical methods [35]. However, the use of the expressions for the dielectric constant and the magnetic permeability of an unbounded metal, as already mentioned, does not give rise to an appreciable error in the surface impedance, and this greatly facilitates the calculation.

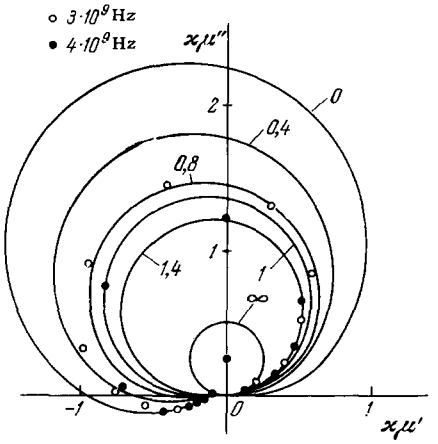


FIG. 2.

Let us rewrite the expression for the dielectric constant in accord with formula (8) in the following manner:

$$\varepsilon(\omega,x)=\frac{3\pi i}{4}\frac{\omega_L^2 c}{\omega^2 v_0 x}=i\frac{x_0}{x},\quad x_0=\frac{3\pi}{4}\frac{\omega_L^2 c}{\omega^2 v_0}. \quad (40)$$

$v_0 = p_0/m$ is the Fermi velocity and $x = ck/\omega$.

Let the magnetic field be perpendicular to the surface of the metal. For a wave propagating along the magnetic field, the permeability, in accordance with (18), is

$$\mu(\omega,x)=\frac{\gamma H+\omega_M-\omega-i\lambda+(k_0 d)^2\omega_M x^2}{\gamma H-\omega-i\lambda+(k_0 d)^2\omega_M x^2}, \quad (41)$$

where $k_0 = \omega/c$ and $\omega_M = 4\pi\gamma M_S$.

Using formula (22), we obtain for the case when $m = 0$ on the boundary

$$\zeta=\frac{2}{\pi i}\int_0^\infty\frac{\gamma H+\omega_M-\omega-i\lambda+(k_0 d)^2\omega_M x^2}{\omega_M(k_0 d)^2 x^5+(\gamma H-\omega-i\lambda)x^3-ix_0\omega_M}x\,dx. \quad (42)$$

Far from the resonant frequency $\omega_r = \gamma H$ we can neglect the spatial dispersion and put $d = 0$. In this case it is convenient to express the permeability in the form $\mu = |\mu| e^{i\vartheta}$, where

$$\operatorname{tg}\vartheta = \frac{\lambda\omega_M}{(\gamma B-\omega)(\gamma H-\omega)} \qquad (0<\vartheta<\pi). \tag{43}$$

The impedance ζ_r is in this case equal to [33]

$$\zeta_p = \frac{2}{\sqrt{3}}\left(\frac{1}{3\pi^2}\frac{\omega^2 l}{\sigma_0 c}\right)^{1/2}|\mu|^{2/3}e^{\frac{i}{3}(2\vartheta-\pi)}. \tag{44}$$

Thus, in the case of the anomalous skin effect, the singularity of the surface impedance is of the order of $|\omega_r-\omega|^{-2/3}$, and not $|\omega_r-\omega|^{-1/2}$ as in the normal skin effect.

Allowance for spatial dispersion leads to finite values of the impedance even if the dissipative mechanisms are neglected. The ferromagnetic resonance becomes manifest in this case in the presence of an extremum on the plot of Re ζ against the frequency or the external magnetic field. Exchange effects also cause a shift of the resonance frequency and a line broadening; the two are of equal order of magnitude. Considering $\gamma H-\omega$ near resonance as a small parameter, we get

$$\Delta H \sim \left(\frac{\sigma_0^2}{l^2}d^6\frac{\omega^2}{c^4}\right)^{1/5}4\pi M_s. \tag{45}$$

The impedance for $\partial m/\partial n = 0$ on the boundary differs from (45) in a numerical factor on the order of unity.

The case of a magnetic field parallel to the surface is considered in [34].

The exact form of the resonance curve can be investigated only numerically. We note in this connection the work of Hirst and Prange [35], in which the FMR curves are calculated for a number of metals by starting from the assumption that the electrons are diffusely reflected from the boundary and from the condition $\partial m/\partial n|_{z=0} = 0$. Their calculated plots of Re ζ against the external field H_e for nickel, with the field oriented parallel to the surface, at a frequency of 5 MHz, are shown in Fig. 3 ($A = \alpha M_S^2/2$). The line shape, as noted in their paper, is determined in general by three parameters—the electron effective mass, the exchange constant, and the anisotropy. Therefore a comparison of the calculated curves with the experimental ones is difficult.

4. FERROMAGNETIC METAL IN A STRONG MAGNETIC FIELD. WEAK SPATIAL DISPERSION

So far we have neglected the influence of the magnetic field on the motion of the conduction electrons. In this case the alternating field hardly penetrates into the metal and the interaction between the spin waves and the electromagnetic field occurs in a narrow layer of width δ. This situation may not occur if the metal is placed in a strong magnetic field, at which the cyclotron radius becomes smaller than the mean free path:

$$r < l. \tag{46}$$

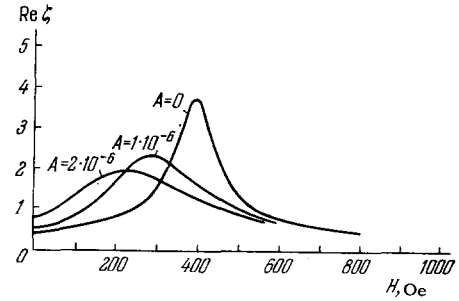


FIG. 3.

It has become clear relatively recently that weakly damped electromagnetic waves can propagate in the metal under these conditions. Helical waves were observed in many metals; magnetohydrodynamic waves were observed in bismuth, which has equal electron and hole densities. The existence of weakly damped electromagnetic waves leads to positive values of the effective dielectric constant and causes many resonance effects. The theory of electromagnetic waves in metals in a magnetic field was investigated by many (see the review [9], which refers also to the experimental papers).

Analysis of a ferromagnetic metal in a strong magnetic field under the condition (46) reveals a close analogy with the theory of the electromagnetic properties of ferrodielectrics. In the frequency region in which the effective magnetic permeability is positive, a ferromagnetic conductor has selective transparency and exhibits resonance properties similar to those observed in dielectrics. As a result of the strong interaction between the oscillations of the magnetic moment and the weakly-damped waves, a change takes place in the picture of electromagnetic wave propagation in metals.

As already mentioned, the parameter characterizing the spatial dispersion of electrons in a magnetic field is the quantity kR. When $kR \ll 1$ there can propagate in the metal helical and magnetohydrodynamic waves. The coupling between the spin and electromagnetic waves at $kR \ll 1$ was investigated by Stern and Callen [36] and by Blank [37].

Let us consider an unbounded ferromagnetic metal ($N_1 \neq N_2$) under the conditions of helical-wave propagation.

In a strong magnetic field, the dissipative elements of the conductivity tensor are proportional to $1/H^2$, and the Hall components to $1/H$, if the number of holes is not equal to the number of electrons. The metal is thus characterized under these conditions by a Hall conductivity and a magnetic permeability (18) and (19). For simplicity we shall disregard the spatial dispersion of the magnetic permeability. Allowance for this dispersion causes an insignificant

change in the overall dispersion picture. We shall discuss later (see Sec. 5) a number of effects due to allowance for the exchange interaction.

From Maxwell's equations it is easy to obtain the dispersion equation of the coupled spin and helical waves:

$$k_{\pm}^2 = \frac{4\pi\omega\sigma_H}{c^2}[\omega_r(\omega_r\cos^2\varphi + \omega_a\sin^2\varphi) - \omega^2]^{-1} \quad (47)$$
$$\times \{\omega_M \pm (\omega_r\omega_a - \omega^2)^{1/2}[\omega_a(\omega_r\cos^2\varphi + \omega_a\sin^2\varphi - \omega^2]^{1/2}|\cos\varphi|^{-1}\},$$

where $\omega_2 = \gamma H$, $\omega_a = \gamma B$, $\omega_M = 4\pi\gamma M_S$, and φ is the angle between k and H.

It follows directly from (47) that when $\sigma_H > 0$ we have $k_-^2 < 0$ at all frequencies.

The dependence of k_+^2 on ω is shown schematically in Fig. 4. The dashed lines denote the plots of k^2 against ω for helical and spin waves when their interaction is neglected. The spin wave has in this approximation the form $\omega = \omega_r'$, where $\omega_r' = [\omega_r(\omega_r\cos^2\varphi + \omega_a\sin^2\varphi)]^{1/2}$. The parameter of the coupling between the spin and helical waves is the ratio M_S/H, which in general is not small.

Equation (47) does not hold when $\varphi = \pi/2$. This direction is exceptional, for when $\varphi = \pi/2$ the helical waves are strongly damped [38]. The damping of the helical waves is small if $\cos\varphi > r/l$.

When $\varphi = 0$ the spectrum of the waves assumes the simple form:

$$k_+^2 = \frac{4\pi\sigma_H\omega}{c^2}\frac{\omega_a - \omega}{\omega_r - \omega}. \quad (48)$$

At low and high frequencies the spectrum approaches that of a helical wave.

The condition $kR \ll 1$ imposes a limitation an the closeness to the resonance point ($\omega = \omega_r'$). Using formula (47), we obtain

$$\frac{\omega_r' - \omega}{\omega_r'} \gg 4\pi\sigma_H\frac{v_0^2}{c^2}\cdot\frac{\omega_M}{\omega_H^2}. \quad (49)$$

In the general case of wave propagation at an arbitrary angle to the magnetic-field direction, the part of the field of the coupling wave (47) which is transverse to k is elliptically polarized. Let us introduce a coordinate system x, η, ζ with the ζ axis along k and the x axis perpendicular to the vectors k and H. The ellipticity coefficient is given by

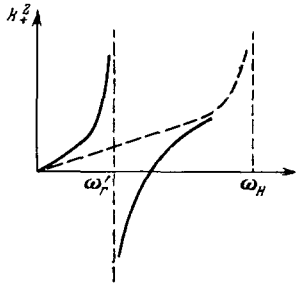


FIG. 4.

$$\frac{e_x}{e_\eta} = i\left|\frac{\omega_r\omega_a - \omega^2}{\omega_a(\omega_r\cos^2\varphi + \omega_a\sin^2\varphi) - \omega^2}\right|^{1/2} = \beta. \quad (50)$$

When $\varphi = 0$ the wave becomes circularly polarized. The wave is plane-polarized. When $\varphi \neq 0$ and in the relatively narrow frequency interval $\omega_a\omega_r(\cos^2\varphi + \omega_a/\omega_r\sin^2\varphi) \geq \omega^2 \geq \omega_a\omega_r$.

Proceeding to elliptically polarized waves $e_\pm = e_x \pm \beta e_\eta$, and eliminating the alternating field, we write the system of Maxwell's equations in the form

$$\frac{\partial^2 e_\pm}{\partial\zeta^2} + \frac{4\pi\omega}{c^2}(\tilde{\mu}_{x\eta}'' \pm \sqrt{\tilde{\mu}_{xx}\tilde{\mu}_{\eta\eta}})\,\tilde{\sigma}_{x\eta}e_\pm = 0. \quad (51)$$

Here

$$\tilde{\sigma}_{x\eta} = \sigma_H|\cos\varphi|^{-1}, \quad \tilde{\mu}_{x\eta}'' = \mathrm{Im}\,\tilde{\mu}_{x\eta}.$$

The tensor $\tilde{\mu}_{\alpha\beta}$ is given by

$$\tilde{\mu}_{\alpha\beta} = \frac{1}{\omega_r'^2 - \omega^2}\begin{pmatrix} \omega_a(\omega_r\cos^2\varphi + \omega_a\sin^2\varphi) - \omega^2 & i\omega_M\cos\varphi \\ -i\omega_M\cos\varphi & \omega_r\omega_a - \omega^2 \end{pmatrix}. \quad (52)$$

We introduce the surface impedance for elliptically polarized waves with the aid of the formulas

$$e_\pm(0) = Z_\pm J_\pm + Z' J_\mp, \quad J_\pm = J_x \pm \beta J_\eta, \quad (53)$$

where J is the total current in the volume of the metal. The elements of this impedance are connected with the elements of the tensor $Z_{\alpha\beta}$ by the following relations:

$$\left.\begin{aligned} Z_\pm &= \frac{1}{2}(Z_{xx} + Z_{\eta\eta}) \pm \frac{1-\beta^2}{2\beta}Z_{x\eta}, \\ Z' &= \frac{1}{2}(Z_{xx} - Z_{\eta\eta}) - \frac{1+\beta^2}{2\beta}Z_{x\eta}. \end{aligned}\right\} \quad (54)$$

Expressing with the aid of Maxwell's equations the electric field on the boundary in terms of the total current in the volume of the metal, and using the dispersion equation of the coupled waves, we obtain after simple calculations

$$Z_+ = \frac{1}{c}\sqrt{\frac{\pi\omega}{\tilde{\sigma}_{x\eta}}}(\tilde{\mu}_{xx} + \tilde{\mu}_{\eta\eta})\frac{[(\tilde{\mu}_{xx}\tilde{\mu}_{\eta\eta})^{1/2} + \tilde{\mu}_{x\eta}'']^{1/2}}{(\tilde{\mu}_{xx}\tilde{\mu}_{\eta\eta})^{1/2}}, \quad (55)$$

$$Z_- = -\frac{i}{c}\sqrt{\frac{\pi\omega}{\tilde{\sigma}_{x\eta}}}(\tilde{\mu}_{xx} + \tilde{\mu}_{\eta\eta})\frac{[(\tilde{\mu}_{xx}\tilde{\mu}_{\eta\eta})^{1/2} - \tilde{\mu}_{x\eta}'']^{1/2}}{(\tilde{\mu}_{xx}\tilde{\mu}_{\eta\eta})^{1/2}}, \quad (56)$$

$$Z' = \frac{i}{c}\sqrt{\frac{\pi\omega}{\tilde{\sigma}_{x\eta}}}(\tilde{\mu}_{xx} - \tilde{\mu}_{\eta\eta})\frac{[(\tilde{\mu}_{xx}\tilde{\mu}_{\eta\eta})^{1/2} - \tilde{\mu}_{x\eta}'']^{1/2}}{(\tilde{\mu}_{xx}\tilde{\mu}_{\eta\eta})^{1/2}}. \quad (57)$$

Thus, in the employed approximation, the impedance element Z_+ is real in the region of existence of the weakly damped wave (47), owing to the penetration of the wave with "+" polarization into the metal. The element Z_- is imaginary, corresponding to the reflection of the wave with "-" polarization from the surface of the ferromagnet.

The dependence of the surface impedance on the frequency, the magnetic field, and the angle φ is in general quite complicated. In the simplest case of longitudinal propagation we have

$$Z_+ = \frac{4\pi}{\omega_L c}\sqrt{\omega\omega_H}\sqrt{\frac{\omega_a - \omega}{\omega_r - \omega}}, \quad (58)$$

$$Z_- = -i\frac{4\pi}{\omega_L c}\sqrt{\omega\omega_H}\sqrt{\frac{\omega_a + \omega}{\omega_r + \omega}}. \quad (59)$$

Recently Grimes [12] reported an experiment with a nickel film, in which he observed standing electromagnetic waves. Without going into the details of the calculation of the excited coupled wave in the resonator, let us discuss Grimes' experiment.

When a standing wave is excited in the film, the wave vector assumes discrete values and, depending on the method of wave excitation, it turns out to be equal to $n\pi/d$ or $(n + \frac{1}{2})\pi/d$, where d is the thickness of the film (depending on whether the surface of the film is in a node or antinode of the electric field).

Thus, the values of k are fixed. If, in addition, the frequency is also fixed, then resonance excitation of the wave in the film should be observed at a certain value of the external magnetic field H_e. In Grimes' experiment, the standing wave was excited at several frequencies. This made it possible to obtain the dependence of the resonance frequency on the magnetic field. The electromagnetic wave was propagated parallel to the direction of the magnetic field, which in turn was perpendicular to the surface of the metal. The dependence of ω on the resonance values of the external magnetic field H_e in the region $H_e > 8$ kOe is a straight line, the extrapolation of which leads to a crossing of the H axis at the point $H_M = 6.5 \pm 0.2$ kOe. According to [39], this value of the field corresponds, within the limits of the experimental error, to the value of the saturation magnetic moment of nickel ($H_M = 4\pi M_S$). The second harmonic was excited in the experiment of [12], i.e., $k = \pi/d$.

Let us consider the dispersion equation of the coupled spin-helical wave (48). In the geometry of the experiment of [12], $H + 4\pi M_S = H_e$, where H is the field inside the film. Hence $H = H_e - 4\pi M_S$ and consequently $B = H_e$. Equation (48) thus assumes the form

$$\gamma H_e (k\delta)^2 = \omega \frac{\gamma H_e - \omega}{\gamma (H_e - 4\pi M_s) - \omega}. \qquad (60)$$

The $\omega = \omega(H_e)$ curve corresponding to Eq. (60) is a hyperbola passing through the points $\omega = 0$, $H_e = 0$, and $\omega = 0$, $H_e = 4\pi M_S$; the asymptotes of the hyperbola are the straight lines $\omega = \gamma(H_e - H_0) + \omega_0$ and $\omega = \gamma k^2\delta^2(H_e - H_0)$, where $H_0 = -8\pi M_S (k\delta)^2/[1 - (k\delta)^2]^2$ and $\omega_0 = -\omega M (k\delta)^2(1 + (k\delta)^2)/[1 - (k\delta)^2]^2$ (Fig. 5). Only the hyperbola sections sufficiently close to the asymptotes have physical meaning, since the magnetic field is bounded from below by the condition $r < l$. When $l \sim 10^{-2}$ cm this yields $H > 10^3$ Oe.

The upper branch of the hyperbola corresponds to excitation of homogeneous FMR, whereas the lower one represents the helical branch of the coupled wave and corresponds to excitation of inhomogeneous oscillations.

The solutions of (60) with respect to the frequency are

$$2\omega_\pm = \gamma H \left[1 + k^2\delta^2 \pm \sqrt{(1 - k^2\delta^2)^2 + \frac{16\pi M_s}{H} k^2\delta^2} \right]. \qquad (61)$$

Assuming that $k\delta \ll 1$, as apparently was the case in the experiment of [12], we obtain:

for the upper branch

$$\omega_+ = \gamma H \left(1 + \frac{4\pi M_s}{H} k^2\delta^2\right) \qquad (62)$$

and for the lower branch

$$\omega_- = \gamma H k^2\delta^2 \left(1 - \frac{4\pi M_s}{H}\right). \qquad (63)$$

As seen from (63), the helical branch crosses the H_e axis at the point $H_M = 4\pi M_S$.

If the dissipation is small, the system of resonance frequencies shown in Fig. 6 should be observed. If $r/l \ll 1$, this is possible for several harmonics ($2n + 1 \gg (r/l)n^2$ or $n < 2l/r$). The damping connected with the relaxation of the magnetic moment is always much smaller than the damping due to electron scattering.

The ability of a helical wave to propagate in a metal is connected, as is well known, with the character of the electron trajectories on the Fermi surface [38]. The results of a theoretical and experimental investigation of the electron spectrum of nickel [40] give grounds for assuming that at an arbitrary orientation of the magnetic field the greater part of the electron trajectories are closed curves. There is no cancellation of the electron and hole volumes in nickel. These conclusions agree with the observation of the helical wave. It should be noted that open trajectories exist in nickel in a relatively narrow angle interval, on the order of several degrees near the ⟨111⟩ directions. This should lead to

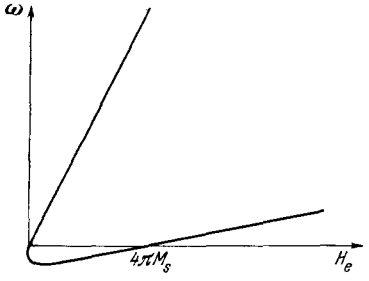


FIG. 5.

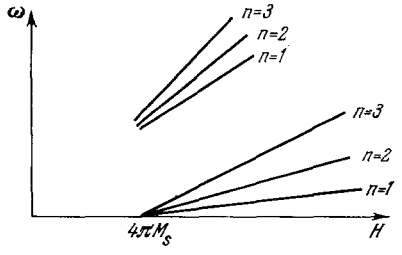


FIG. 6.

the vanishing of the spin-helical wave at a suitable orientation of the magnetic field.

The connection of the helical waves with the oscillations of the magnetic moment is also manifest in the shift of the frequency (62) and the broadening $\Delta\omega$ of the FMR line:

$$\Delta\omega = \Delta\omega_M + \omega_M (k\delta)^2 \frac{r}{l}, \tag{64}$$

where $\Delta\omega_M$ is the natural line width of the ferromagnetic resonance. As seen from formulas (62) and (64), the shift of the resonance frequency exceeds by a factor l/r the attenuation due to the scattering of the conduction electrons.

Rodbell [5] observed in his experiments ferromagnetic resonance in nickel with a line width ΔH on the order of 50 Oe. An FMR line width on the order of 40 Oe was obtained in the experiment of [6]. The resonance-frequency shift (62) can be observed if the thickness of the film is of the same order as or smaller than $\delta(4\pi M_S/\Delta H)^{1/2}$, i.e., on films $\lesssim 10^{-4}$ cm.

Observation of the coupled spin-helical wave makes it important to examine the connection between spin waves and other electromagnetic excitations in metals [27,41]. In metals with equal electron and hole densities ($N_1 = N_2 = N$), as already mentioned, propagation of magnetohydrodynamic waves is possible [9]. The character of their propagation differs with the ratio of the wave frequency to the "Doppler broadening" $k_H v_0$, where k_H is the projection of the wave vector $\mathbf{k}$ on the magnetic field $\mathbf{H}$. If

$$\nu \ll k_H v_0 \ll \omega \ll \omega_H, \tag{65}$$

then, using the asymptotic form of the conductivity tensor (11) and the usual equation for the magnetic permeability, we obtain the following dispersion equation for the coupled spin and magnetohydrodynamic waves ($\varphi \neq \pi/2$):

$$k_\pm^2 = \frac{\omega^2}{v_a^2} \frac{\cos^{-2}\varphi}{\omega_r'^2 - \omega^2} \{(\omega_a^2 - \omega_r^2)\sin^2\varphi + 2(\omega_r\omega_a - \omega^2)\cos^2\varphi \pm [(\omega_a^2 - \omega^2)^2 \sin^4\varphi + 4(\omega_M\omega)^2\cos^4\varphi]^{1/2}\}, \tag{66}$$

where $v_a = H/[4\pi N(m_1 + |m_2|)]^{1/2}$ is the Alfven velocity and m_2 is the hole mass.

The wave coupling parameter is, as before, the ratio $4\pi M_S/H$. However, whereas in the case of a helical wave only one wave propagated (corresponding to the plus sign in (47)), in a metal with equal densities of the two groups of carriers both waves can propagate under the condition (65). Neglecting the interaction, the wave corresponding to the minus sign in (66) goes over into an Alfven wave, and that corresponding to the plus sign goes over into a fast magnetosonic wave (according to (65), $\omega/k > v_0$). Calculations show that the spectrum of the coupled fast wave is analogous to the spectrum of the coupled helical wave, shown in Fig. 4. The dependence of the wave vector of the Alfven wave on the frequency has a nonresonant character. This dependence is shown in Fig. 7.

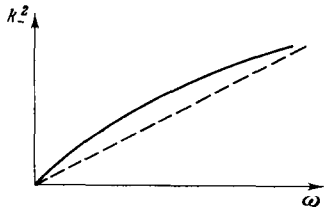


FIG. 7.

When $\varphi = 0$ the expression (66) simplifies to

$$k_\pm = \frac{\omega}{v_a}\sqrt{\frac{\omega_a \mp \omega}{\omega_r \mp \omega}}. \tag{67}$$

In the general case ($\varphi \neq 0$) coupled magnetohydrodynamic waves have elliptic polarization. When $\varphi = 0$ the field is circularly polarized, and the electric field vector rotates in opposite directions in the $\pm$ waves. In a narrow interval of angles φ close to $\pi/2$, the character of the wave propagation is appreciably altered.

In the particular case $\varphi = \pi/2$, the asymptotic form of the tensor σ_{ik} (in the x, η, ζ system) is

$$\sigma_{ik} = \sigma_H \begin{pmatrix} -i\omega\omega_H^{-1} & 0 & 0 \\ 0 & i\omega_H\omega^{-1} & 0 \\ 0 & 0 & -i\omega\omega_H^{-1} \end{pmatrix}. \tag{68}$$

With this, the fast magnetosonic wave propagates just as in an ordinary metal [9], without interacting with the oscillations of the magnetic moment. The wave corresponding to the choice of the minus sign in the solution of the dispersion equation is in this case not the analog of the Alfven wave, but has the following spectrum:

$$k_-^2 = \frac{\omega_L^2}{c^2}\frac{\omega_a^2 - \omega^2}{\omega^2 - \omega_a\omega_r}. \tag{69}$$

The wave vector is real in a relatively narrow frequency interval $\omega_r < \omega < (\omega_a\omega_r)^{1/2}$. The transverse part of the electric field in this wave is parallel to $\mathbf{H}$.

The condition for the existence of the aforementioned coupled waves $k_H v_0 \ll \omega$ or $H \gg (4\pi Nm)^{1/2} v_0$ can be satisfied in ordinary metals only in fields on the order of 10^6 Oe. In metals with low carrier density this condition is perfectly realistic.

Let us consider the propagation of coupled magnetohydrodynamic waves in metals with $N_1 = N_2$ under conditions of strong spatial dispersion, when the following inequality is satisfied:

$$\nu \ll \omega \ll k_H v_0 \ll \omega_H, \tag{70}$$

which can hold in fields that are not too strong and satisfy the condition

$$v_a \ll v_0. \tag{71}$$

It can be shown [42] that when the inequalities (71) are satisfied the tensor σ_{ik} is diagonal, and when

$\varphi \neq 0$ we have

$$|\sigma_{yy}| \ll |\sigma_{zz}|.$$

Consequently the z-component of the electric field in the coupled wave is negligibly small. Assuming e_z to be equal to zero, we obtain from Maxwell's equation the following dispersion equation for the coupled wave

$$k_-^2 = \frac{\omega^2}{v_a^2}\frac{\omega^2-\omega_a^2}{\omega^2-\omega_r\omega_a}\cos^{-2}\varphi \quad \left(\text{for } \varphi \neq \frac{\pi}{2}\right). \tag{72}$$

The slow magnetosonic wave has in this case an imaginary wave vector in the entire frequency variation interval.

When $\varphi = 0$ the spatial dispersion of the conductivity plays no role and both waves become weakly damped. Their spectrum coincides with the spectrum given by expression (67).

In the derivations of the formulas given in this section, we made use of the isotropic dependence of the electron energy on the momentum. The generalization to an arbitrary carrier spectrum is in most case unimportant and is contained in [37]. However, when the inequality (70) is satisfied, the character of the electron spectrum turns out to be important, owing to the dominant role of the collisionless damping. As shown by Kaner and Skobov [43,9], weakly damped waves in a metal having an arbitrary Fermi surface can propagate under conditions (70) only if the magnetic field is parallel to the crystal symmetry axis. In this case there is no collisionless damping (Landau mechanism). In the case of arbitrary orientation of the magnetic field relative to the symmetry axes, the collisionless damping is appreciable, as a result of which the dispersion equation has no real solutions. An exception is an electromagnetic wave linearly polarized along H (for details see [43]). The dispersion equation of this wave in a ferromagnetic metal takes the form (when $\varphi \neq 0$)

$$\frac{\hbar k^2}{2M} = \omega\left(\frac{\omega^2-\omega_a^2}{\omega^2-\omega_r^2}\right)^{1/2}|\sin 2\varphi|^{-1},$$
$$M = \frac{\hbar}{c}|4\pi e^2 \nu(\varepsilon_\mp)|^{1/2}, \tag{73}$$

where $\gamma(\epsilon_F)$ is the density of the electron states on the Fermi boundary.

Estimates given in [43] show that the wave (73) can exist in magnetic fields bounded by the inequality

$$\frac{\omega}{\omega_H}\left(\frac{v_0}{v_a}\right)^3 \gg 1. \tag{74}$$

For good metals, $N \sim 10^{22}$ and this condition implies $H \ll 10^3\omega^{1/4}$. We emphasize that this wave is brought about by the anisotropy of the Fermi surface. This wave does not exist when $\varphi = 0$. As seen from (73), the wave vector is real in the entire admissible region of frequencies, with the exception of the interval $\omega_r < \omega < \omega_a$. The impedance of the wave with polarization is

$$Z_{\eta\eta} = \frac{4\pi}{c^2}\sqrt{\frac{\hbar\omega}{2M}|\sin 2\varphi|}\sqrt{\frac{\omega^2-\omega_a^2}{\omega^2-\omega_a\omega_r}}. \tag{75}$$

Coupled magnetohydrodynamic and spin waves can probably be observed in sufficiently pure iron at low temperatures, since the concentrations of the electrons and holes in iron are equal [40].

5. FERROMAGNETIC METAL IN A STRONG MAGNETIC FIELD. STRONG SPATIAL DISPERSION [44,45,25]

Let us consider the propagation of an electromagnetic wave in a ferromagnetic metal under spatial-dispersion conditions, when the condition

$$kR \gg 1 \tag{76}$$

is satisfied. Of greatest interest from the physical point of view is in this case a geometry in which the magnetic field is perpendicular to the wave propagation direction. This circumstance is connected with the fact that weakly damped waves can propagate at this field orientation, whereas in the case of a magnetic field parallel to the wave vector the coefficients of the dispersion equation turn out to be complex and the results are essentially analogous to those in the absence of a magnetic field.

Let us direct the O_x axis along k and the O_z axis, as before, in the direction of the magnetic field H. The tensor σ_{ik} is in this case diagonal and is determined by formula (13). Owing to the diagonality of the tensor σ_{ik}, the dispersion equations of the extraordinary and ordinary waves separate:

$$k^2 = \frac{4\pi i\omega}{c^2}\sigma_{xx}(\omega, \mathbf{k}), \tag{77}$$

$$k^2 = \frac{4\pi i\omega}{c^2}\frac{\det\hat{\mu}}{\mu_{yy}}\sigma_{zz}(\omega, \mathbf{k}). \tag{78}$$

The extraordinary wave (77), in which the magnetic field is parallel to the constant field, does not interact with the oscillations of the magnetic moment. In the ordinary waves the field components e_x and e_y vanish.

Using (13) and the expression for the magnetic susceptibility (18), (19), we write the dispersion equation of the ordinary wave in the form

$$k^3 = -\pi\frac{\omega_L^2}{c^2 v_0}\omega\frac{\omega_a^2+\omega_2 a k^2-\omega^2}{\omega_r^2+\omega_1 a k^2-\omega^2}\operatorname{ctg}\pi\frac{\omega}{\omega_H}. \tag{79}$$

Here

$$\omega_2 = \gamma(HB)^{1/2},\ \omega_a = \gamma B;\ \omega_1 = \gamma(H+B) \text{ and } \omega_2 = 2\gamma B.$$

Inasmuch as the "volume" terms are significant only in the immediate vicinity of the resonance, they can be neglected in a qualitative study of the spectrum (79). The spectrum then depends essentially on the relations between the quantities ω_r, ω_a, and ω_H. For a number of cases in which the frequencies ω_r, ω_a, and ω_H are of the same order of magnitude, the

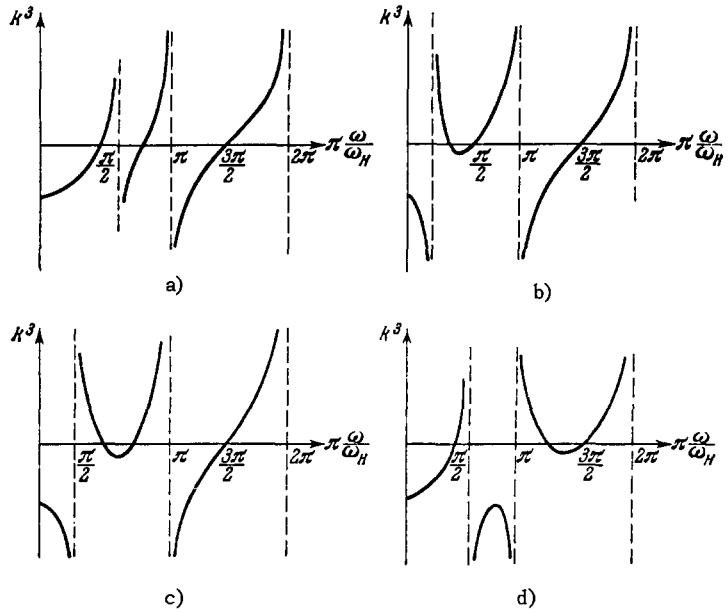


FIG. 8.

spectral dependence is shown schematically in Fig. 8. Weakly damped waves can obviously exist near the ferromagnetic-resonance frequency ω_r, near the antiresonance ω_a, and near the cyclotron frequencies $n\omega_H$. The latter case was investigated by Kaner and Skobov [9] for an ordinary metal.

Near the frequency ω_r, the wave vector in (79) becomes infinite, corresponding to spin-wave resonance. This gives rise to excitation of spin wave with a known dispersion law:

$$\omega = \omega_r + \alpha \frac{\omega_1}{2\omega_r} k^2. \tag{80}$$

When the resonance "detuning" is large compared with the exchange term, $|\omega_r - \omega| > \alpha\omega_1 k^2/2\omega_2$, there exists near the FMR frequency an electromagnetic wave with dispersion law

$$\omega = \omega_r + \frac{\pi}{2} \frac{\omega_L^2}{c^2 v_0 k^3} (\omega_a^2 - \omega_r^2) \operatorname{ctg} \pi \frac{\omega_r}{\omega_H}. \tag{81}$$

The condition for its existence is

$$k^5 \ll \frac{\omega_L^2}{c^2 v_0} \frac{\omega_M \omega_r}{\alpha}.$$

Condition (76) is in this case easily satisfied. As seen from (81), the wave has anomalous dispersion.

Near antiresonance, when $|\omega - \omega_a| \gg \alpha k^2$, there exists a wave with a dispersion law

$$\omega = \omega_a - \frac{\omega_a^2 - \omega_r^2}{2\omega_a} \frac{c^2 r}{\omega_L^2} k^3. \tag{82}$$

Inasmuch as the condition $kr > 1$ is violated at the antiresonance point itself, the condition for being close to this point is

$$\omega_a - \omega \gg \left(\frac{\delta_L}{r}\right)^2 \omega_M, \quad \left(\delta_L = \frac{c}{\omega_L}\right).$$

This wave also has anomalous dispersion.

Finally, near the cyclotron frequency, at $\nu \ll n\omega_H - \omega \ll \omega_H$, the expression for the wave spectrum becomes

$$k^3 = \frac{(1 - \omega/n\omega_H)^{-1}}{r\delta_L^2} \frac{\omega_a^2 - (n\omega_H)^2}{\omega_r^2 - (n\omega_H)^2}. \tag{83}$$

In connection with formula (83), we note that the cyclotron frequency, or a multiple thereof, can coincide with the FMR frequency when $\omega_r = n\omega_H$. This is possible, since the cyclotron frequency ω_H is determined by the effective mass ($\omega_H = eB/m^*c$). The frequencies will coincide at definite field values, equal to

$$H_n = \frac{4\pi M_s}{\left(g \frac{m^*}{2m}\right)^2 n^{-2} - 1}. \tag{84}$$

We see therefore that in order for the frequencies to coincide it is necessary to satisfy the condition $2m < gm^*$. The spectrum of a the weakly damped wave (when $H = H_n$) is written in the form

$$\omega = \omega_r - \left(\frac{\omega_a \omega_M}{k^3 r \delta_L^2}\right)^{1/2}. \tag{85}$$

We proceed to calculate the surface impedance. Neglecting exchange effects, the magnetic permeability takes the form

$$\mu(\omega) = \frac{(\omega_a^2 - \omega^2)(\omega_r^2 - \omega^2) + 2i\lambda\omega\omega_a\omega_M}{(\omega_r - \omega^2)^2 + 4\lambda^2\omega^2}. \tag{86}$$

As before, we rewrite (86) in the form

$$\mu = |\mu| e^{i\vartheta}, \quad \operatorname{tg}\vartheta = \frac{2\lambda\omega\omega_a\omega_M}{(\omega_a^2 - \omega^2)(\omega_r^2 - \omega^2)} \quad (0 < \vartheta < \pi). \tag{87}$$

Assuming for simplicity that $\omega \ll \omega_H$, we represent $\epsilon(\omega, x)$ in the form

$$\varepsilon(\omega, x) = -\frac{\omega_L^2 c}{\omega^2 r} \frac{(\omega^2 + \nu^2)^{-1/2}}{x} e^{-i\varphi}, \quad \operatorname{tg}\varphi = \frac{\nu}{\omega} \quad (0 < \varphi < \pi/2). \tag{88}$$

Using expressions (87) and (88), we find with the aid of (22)

$$\zeta_p = \frac{4}{3\sqrt{3}}\frac{\omega}{c}|\mu|^{2/3}\left(\frac{v_0 c^2(\omega^2+\nu^2)^{1/2}}{\omega_L^2\omega_H\omega}\right)^{1/3}\exp\left\{i\left(\frac{2}{3}\vartheta+\frac{\varphi}{3}-\frac{\pi}{2}\right)\right\}. \quad (89)$$

Outside the region of existence of the weakly damped waves, neglecting dissipation ($\nu = \lambda = 0$), the impedance (89) turns out to be imaginary, corresponding to total reflection of the external electromagnetic wave. We note that in the limiting case $\lambda \ll \omega \ll \nu$ expression (89) can be obtained from the expression for the surface impedance of the metal under conditions of anomalous skin effect in the absence of a magnetic field, if we introduce the effective conductivity $\sigma_{eff} = \sigma_0\omega_H\tau$. In the region of weakly-damped waves (80)—(83), the impedance becomes complex. It is easily seen from expression (87), in the region of existence of the weakly-damped waves $\vartheta = \pi$, and φ can be set equal to zero. The phase in formula (89) then becomes equal to $\pi/6$.

One might expect the impedance to be real in the region of existence of undamped waves, as for example in the case when helical waves propagate in a metal or when electromagnetic waves propagate in a dielectric. The complex impedance is in this case the result of the anomaly of the skin effect and is due to the fact that the field in the metal near the boundary is a superposition of both damped and undamped waves.

Near the cyclotron resonance frequency $\omega \ll \omega_H$ with $|\omega_H - \omega| \gg \nu$ we have

$$\zeta_p = \frac{2}{3\sqrt{3}}\frac{n\omega_H}{c}\left(\frac{(n\omega_H)^2-\omega_a^2}{(n\omega_H)^2-\omega_r^2}\right)^{2/3}\left(\frac{c^2r}{\omega_L^2}\frac{n\omega_H-\omega}{\omega_H}\right)^{1/3}(1-i\sqrt{3}) \quad (90)$$

(in deriving this formula we assumed for concreteness that $\omega_a, \omega_r < \omega_H$).

We indicated earlier that the FMR frequency can coincide with the cyclotron frequency ω_H. Observation of this effect in the case of an isotropic energy spectrum requires a simultaneous variation of both the frequency of the electromagnetic wave and the applied field H. In the case of an anisotropic spectrum there is a more favorable possibility of realizing this effect. When the electron energy has an anisotropic dependence on the momentum, the cyclotron frequency depends, as is well known, on the orientation of the constant field relative to the crystallographic axes. At certain field directions in a plane parallel to the surface of the metal, the cyclotron frequency may turn out to be equal to the frequency of the ferromagnetic resonance. As expected, an abrupt change should take place here in the shape of the resonance curve.

Calculation [44] shows that the surface impedance, neglecting dissipation, has in this case a root singularity at the resonance point ($\zeta \sim |\omega - \omega_r|^{-1/2}$), unlike the singularity $|\omega - \omega_r|^{-2/3}$ obtained under the conditions of the anomalous skin effect (89).

If the spatial dispersion of the magnetic permeability is neglected, the character of the behavior of the magnetic moments on the boundary plays, of course, no role, and the surface impedance ζ_s differs as before from ζ_r in all the cases considered above only by a factor on the order of unity.

Let us examine now the influence of spatial dispersion near resonance on the shape of the resonance curve. From the physical point of view, the most interesting is the frequency region in which $\cot(\pi\omega/\omega_H) > 0$. We note first that in this case, as shown by calculations, the exchange effects make no contribution to the width of the curve, but do cause a shift of the resonance. In the vicinity of the shifted resonance frequency, allowance for spatial dispersion leads to an appreciable change in the form of the resonance curve.

Let us investigate first the surface impedance in the case $m(0) = 0$. Assuming as before that $\omega \ll \omega_H$, we write down the dispersion equation (79) near resonance in the form

$$P(\eta) = x^5 + \eta x^3 + 1 = 0, \quad (91)$$

where we went over to the dimensionless variable $x = k/q$, $q^5 = \omega_L^2\beta c^2 r$, $\beta = \alpha/4\pi$. The parameter $\eta = \xi/\beta q^2$, where $\xi = (\omega_r - \omega)/\Omega$, characterizes the closeness to resonance. The function $\omega(k)$ is shown schematically in Fig. 9. As seen from the figure, the $\omega(k)$ curve has an extreme at the value $\omega = \omega_r'$, which is the end point of the spectrum. At frequencies larger than ω_r' the weakly-damped waves (80) and (81) are excited in the metal, whereas at lower frequencies the field does not penetrate into the metal. The frequency ω_r' should be identified with the shifted resonance frequency. The shift of the resonance can be readily obtained from the conditions $P(\eta_0, x_0) = 0$ and $P_x'(\eta_0, x_0) = 0$. Hence $x_0^5 = 3/2$, $\eta_0 = (-5/3)(3/2)^{2/5} \approx -2$, and the shifted frequency is

$$\omega_r' = \omega_r - \eta_0\Omega\beta q^2 \approx \omega_r + 2\Omega\beta q^2. \quad (92)$$

When $H \sim 10^4$ Oe, $\theta_c \sim 10^{13}$ erg, and $\delta_L \sim 10^{-5}$ cm, the resonance shift has the following order of magnitude:

$$\frac{\omega_r'-\omega_r}{\omega_r} \simeq \frac{\Omega}{\omega_r}\beta q^2 \sim \left(\frac{\theta_c}{\mu_B M_s}\right)^{3/5}\left(\frac{a^3}{\delta_L^2 r}\right)^{2/5} \simeq 10^{-2}. \quad (93)$$

This corresponds to a shift of the resonance field on the order of 10—100 Oe. We note that the FMR line width due to the electron collisions is small compared with the shift, by virtue of the smallness of the parameter $(kl)^{-1}$.

We shall write for the surface impedance ζ_r

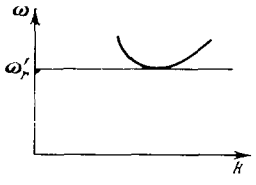


FIG. 9.

$$\zeta_r = \frac{2}{\pi i}\frac{\omega}{\beta c q^3} J_r(\eta), \quad J_r(\eta) = \int_0^\infty \frac{x\,dx}{x^5+\eta x^3+1}, \tag{94}$$

and near $\eta = \eta_0$ we get

$$J_r(\eta) \simeq \begin{cases} \frac{2\pi i}{5}|\eta-\eta_0|^{-1/2} & \eta < \eta_0, \\ \frac{2\pi}{5}(\eta-\eta_0)^{-1/2} & \eta > \eta_0. \end{cases} \tag{95}$$

Far from resonance (when $|\eta| \gg |\eta_0|$), expression (89) is valid.

Thus, allowance for the spatial dispersion of the magnetic permeability leads to a root singularity of the impedance, unlike the singularity $|\omega_r - \omega|^{-2/3}$ obtained without allowance for the exchange interaction. The frequency dependence of the impedance at $m = 0$ on the boundary is shown in Fig. 10. When the exchange constant approaches zero, the maximum value of Im ζ_r increases when $\omega > \omega_r'$, the resonance shift tends to zero, and the curve approaches its limiting form.

We investigate in similar fashion the surface impedance in the case when $\partial m/\partial z|_{z=0} = 0$:

$$\zeta_s^{-1} = \frac{2}{\pi i}\frac{c}{\omega}\beta^{2/5}\left(\frac{\omega_0^2}{c^2 r}\right)^{3/5} J_s(\eta), \quad J_s(\eta) = \int_0^\infty \frac{(x^2+\eta)\,dx}{x^5+\eta x^3+1}. \tag{96}$$

Let us consider $J_s(\eta)$ when $\eta > \eta_0$. In this case the integral does not contain any singularities on the path of integration. It is easy to see that in this region $dJ_s(\eta)/d\eta > 0$. When $\eta \gg \eta_0$ we have

$$J_s(\eta) = 2\pi\eta^{2/3}/3\sqrt{3} > 0. \tag{97}$$

On the other hand, near the end point of the spectrum the integral is equal to

$$J_s(\eta_0+\delta) \simeq -\frac{2\pi}{5}(\eta-\eta_0)^{1/2} < 0. \tag{98}$$

Thus, there exists a value $\eta > \eta_0$ at which $J_s(\eta)$ vanishes (estimates show that it lies between zero and unity). The existence of a zero of the function $J_s(\eta)$ causes the imaginary part of the impedance ζ_s to vanish.

When $\eta < \eta_0$ the denominator in (96) has real roots. Near η_0

$$J_s \simeq -\frac{2\pi i}{5}(\eta_0-\eta)^{1/2}. \tag{99}$$

When $|\eta| \gg |\eta_0|$, as already mentioned, ζ_s differs from ζ_r only in a real factor on the order of unity.

The final resonance curve at $m'(0) = 0$ is shown in Fig. 11. We note that in this case the real part of the impedance vanishes at resonance.

Comparison of the resonance curves of Figs. 10 and 11 shows that the surface impedance near the resonance is greatly different in the two limiting cases ($m = 0$ and $m'(0) = 0$ on the boundary). In this connection it is of interest to calculate the impedance under the general condition (29). Omitting the calculations, which can be found in [25], we present the final result

$$\zeta = \begin{cases} C_1\chi/[iC_2+\chi(\eta_0-\eta)^{1/2}], & \eta \leqslant \eta_0, \\ -iC_1\chi/[C_2+\chi(\eta-\eta_0)^{1/2}], & \eta \geqslant \eta_0, \end{cases} \tag{100}$$

where the constants are $C_1 = 4\omega_r/5c\beta q^3$ and $C_2 = 4q/5$. As seen from (108), when the general condition (29) is satisfied on the boundary, the impedance remains constant at resonance.

We note that when resonance is approached from the high-frequency side, the derivative $(d/d\eta)\,\mathrm{Re}\,\zeta$ has a singularity. On approaching the lower frequencies, the derivative $d/d\eta\,\mathrm{Im}\,\zeta$ becomes infinite.

The region of applicability of the obtained formulas is determined by the inequalities (76) and (46). Putting $k = qx_0$ ($x_0 \sim 1$) we obtain

$$kR \sim \left[\left(\frac{r^2}{\delta_L a}\right)^2 \frac{4\pi\mu_B M_s}{\theta_C}\right]^{1/5}, \tag{101}$$

which amounts to 10^2 in fields $H \sim 10^4$ Oe. We note that the condition $kR \gg 1$ is easily satisfied near FMR, since the effective magnetic permeability becomes anomalously large in the immediate vicinity of the resonance.

The anomalous behavior of the surface impedance as a function of the character of the boundary condition for the magnetic moment becomes understandable if we consider the distribution of the field inside the metal. For simplicity we confine ourselves to the case $m(0) = 0$. The electric field in the metal is described by the expression

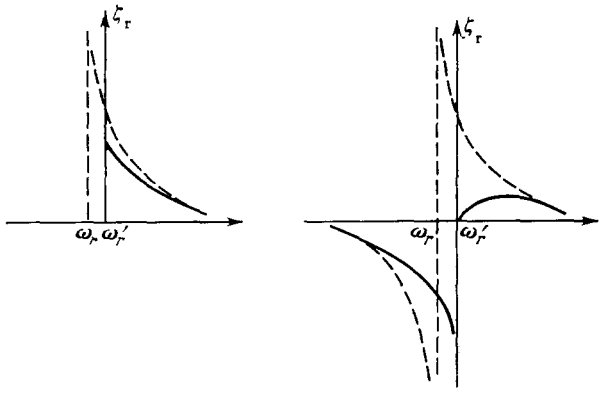


FIG. 10.

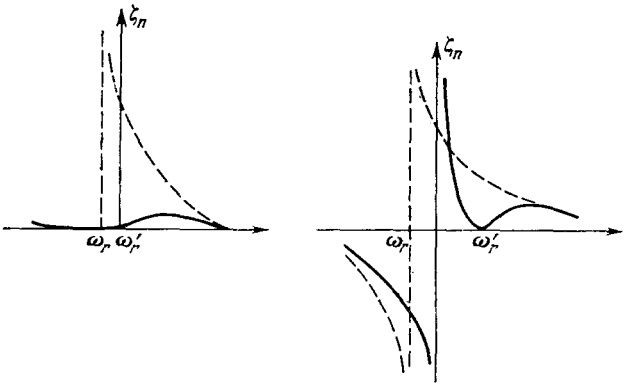


FIG. 11.

$$\mathbf{e}(z) = \frac{\mathbf{h}(0)}{\pi i}\frac{\omega}{c\beta q^3}\int_0^\infty \frac{x\,(e^{iqzx}+e^{-iqzx})\,dx}{x^5+\eta x^3+1}. \qquad (102)$$

When $\eta < \eta_0$, the denominator of the integrand in (102) has two roots located close to the real axis and lying in the upper and lower half-planes. Integrating in the upper and lower quadrants of the half-plane $x > 0$ in the first and second integrals, respectively, we obtain $|\eta| \gg |\eta_0|$ in the frequency region $\mathrm{Re}\,\eta < 0$ (i.e., in the region of the existence of weakly damped waves) for the field at infinity

$$\mathbf{e}(z) = 2\mathbf{h}(0)\frac{\omega}{c\beta q^3}\left(\frac{\exp(-i|\eta|^{1/3}qz)}{3|\eta|^{2/3}} + \frac{\exp(i|\eta|^{1/2}qz)}{2|\eta|^{3/2}}\right). \qquad (103)$$

The first term corresponds to the propagation of a wave with anomalous dispersion (81). The phase in this wave builds up towards the boundary, whereas the energy propagates towards positive z. The second term in (103) describes the weakly-damped spin wave (80).

Near the end point of the spectrum, the real positive roots of the dispersion equations coalesce. At frequencies smaller than ω_r', there are no weakly damped waves. Let us consider the distribution of the field in this case. We transform expression (102), expressing the field in terms of the amplitude E_0 of the incident wave:

$$e(z) = \frac{2}{\pi i}\frac{\omega}{c\beta q^3}\frac{E_0}{1+\zeta_p}\int_0^\infty \frac{x\,dx\cos qzx}{x^5+\eta x^3+1}. \qquad (104)$$

As indicated above, near the frequency of the shifted resonance, the impedance ζ_r has a root singularity. The same singularity is possessed also by the integral in (104). Calculating the integral with allowance for this circumstance, we get

$$e(z) = 2E_0\cos(qzx_0), \qquad (105)$$

i.e., the field has the form of a standing wave. The incident wave experiences in this case total reflection and the energy does not penetrate inside the metal.

Under the more general boundary condition (29), the field inside the metal is at resonance again a standing wave, and the parameter χ determines the phase of the field. This is the cause of the appreciable dependence of the impedance at resonance ($\omega = \omega_r'$ or $\eta = \eta_0$) on the boundary condition for the magnetic moment.

6. INTERACTION OF SPIN-HELICAL WAVES WITH SOUND WAVES. TRIPLE RESONANCE [46]

So far we have not taken into account the fact that some waves can propagate in the metal besides the electromagnetic oscillations. There are a large number of papers devoted to coupled sound and electromagnetic waves in metals [47,48]. On the other hand, the question of the coupling between the magnetic (spin) and elastic waves, which leads under certain conditions to ferroacoustic resonance, was discussed many times [49].

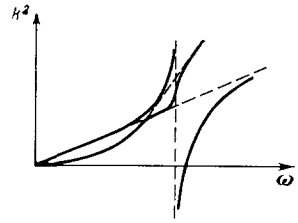


FIG. 12.

Leaving aside problems involved in the theory of sound propagation in magnets [50], we shall discuss in the present section only those features in the interaction between magnetic and sound waves, which are due to the existence of collective excitations in the system of electrons in the magnetic field. For simplicity we shall consider the interaction between a spin-helical wave and sound.

The initial system of equations to be considered consists of the Maxwell equation, the equations of the dynamic theory of elasticity, and the equations of motion of the magnetic moment. The assumption that the conditions for the propagation of helical waves are satisfied ($r \ll l$, $kR \ll 1$) denotes that we can use formula (10) for the electric conductivity (we are considering metals with an unequal number of electrons and holes). It is obvious that in the simplest case of wave propagation along a magnetic field, the equations for the longitudinal and transverse waves separate in a medium which is isotropic with respect to elastic and magnetostriction properties. The dispersion equation for a transverse circularly-polarized wave is

$$\left(1-\frac{\omega^2}{k^2s^2}\right)\left(1-\frac{4\pi\sigma_H\omega}{k^2c^2}\frac{\gamma B-\omega}{\gamma H-\omega}\right) = \frac{4\pi\gamma M_s}{\gamma H-\omega}\frac{v_M^2}{s^2}\left(1-\frac{4\pi\sigma_H\omega}{k^2c^2}\right) + \frac{4\pi\sigma_H\omega}{k^2c^2}\frac{v_a^2}{s^2} + 2\frac{4\pi\sigma_H\omega}{k^2c^2}\frac{4\pi\gamma' M_s}{B}\frac{\gamma M_s}{\gamma H-\omega}\frac{v_a^2}{s^2}, \qquad (106)$$

where s is the speed of sound, $v_M^2 = (\gamma' M_S)^2/4\pi\rho$, $v_a^2 = B^2/4\pi\rho$, ρ is the density of the medium, and γ' is the magnetostriction constant. We have confined ourselves to consideration of magnetostriction and induction interaction between the electron, spin, and lattice subsystems. The deformation reaction was disregarded. Concerning its role in a magnetic field see [48].

The right side of (106) is small compared with unity, since $v_M^2/s^2 \ll 1$ and $v_a^2/s^2 \ll 1$. The dependence of k^2 on ω described by this equation is shown in Fig. 12.

The condition for resonance between the sound and coupled spin-helical waves consists in equality of the wave vectors and of the frequencies:

$$\omega = \frac{4\pi\sigma_H s^2}{c^2}\frac{\gamma B-\omega}{\gamma H-\omega}. \qquad (107)$$

The two resonance frequencies determined for (107) are

$$2\omega_{1,2} = \gamma H + 4\pi\sigma_H s^2 c^{-2} \pm [(\gamma H + 4\pi\sigma_H s^2 c^{-2})^2 - 16\pi\sigma_H c^{-2} s^2 \gamma B]^{1/2}, \quad (108)$$

from which it follows that the resonance is realized only if

$$H + \frac{H_0^2}{H + 4\pi M_s} > 2H_0, \quad H_0^2 = 4\pi N m s^2. \quad (109)$$

Far from the resonance frequencies, the corrections to the unperturbed dispersion laws are small - they are proportional to the right side of the equation. Near the resonant frequencies $\omega_{1,2}$ the coupling between the oscillation branches becomes, naturally, stronger and the corrections to the amplitude dispersion laws increase:

$$\frac{\omega - \omega_{1,2}}{\omega_{1,2}} = \pm \sqrt{\frac{\pi\sigma_H}{\omega_{1,2}}} \left(\frac{4\pi\gamma M_s}{\gamma H - \omega_{1,2}} \frac{v_M}{c} + \frac{v_a}{c} \right). \quad (110)$$

If $4\pi M_S/H \ll 1$, then a weak coupling exists between the magnetic and helical waves, and we can speak of three types of quasi-independent oscillations, for which the dispersion equation is best written in the following form:

$$\left(1 - \frac{\omega^2}{k^2 s^2}\right)\left(1 - \frac{4\pi\sigma_H \omega}{k^2 c^2}\right)\left(1 - \frac{\omega}{\gamma H}\right) = \eta_e \frac{4\pi\sigma_H \omega}{k^2 c^2}\left(1 - \frac{\omega^2}{k^2 s^2}\right) + \eta_s \left(1 - \frac{4\pi\sigma_H \omega}{k^2 c^2}\right) + \eta_i \frac{4\pi\sigma_H \omega}{k^2 c^2}\left(1 - \frac{\omega}{\gamma H}\right) + 2\frac{4\pi\sigma_H \omega}{k^2 c^2}\sqrt{\eta_e \eta_i \eta_s}, \quad (111)$$

where the parameters $\eta_e = 4\pi M_S/H$, $\eta_S = 4\pi M_S v_M^2/Hs^2$, and $\eta_i = v_a^2/s^2$ are the constants of the coupling between the different waves. The existence of three types of weakly-interacting waves makes it possible to realize a "triple" resonance, i.e., one temporal agreement of the frequencies and wave vectors of the sound, magnetic, and helical waves. This equality will take place at a magnetic field $H = H_{cr}$, where

$$H_{cr}^2 = 4\pi N m s^2 \sim 10^3 \text{ э}; \quad (112)$$

Here $\omega_1 = \omega_2 = \omega_0 = \gamma H_{cr} = 4\pi\sigma_H s^2 c^{-2}$.

Let us consider the splitting of the dispersion curves in this case. The dispersion equation (111) near the frequency ω_0 of the triple resonance is

$$\left(1 - \frac{\omega}{\omega_0}\right)^3 = (\eta_e + \eta_i + \eta_s)\left(1 - \frac{\omega}{\omega_0}\right) + 2(\eta_i \eta_e \eta_s)^{1/2}. \quad (113)$$

Putting $1 - \omega/\omega_0 = x(\eta_i\eta_e\eta_S)^{1/6}$, we rewrite (113) in the form

$$x^3 = \varepsilon x + 2, \quad (114)$$

where $\epsilon = (\eta_e + \eta_i + \eta_S)(\eta_e\eta_i\eta_S)^{-1/3}$. We note that inasmuch as $\epsilon > 3$ always, all the roots of (114) are real.

Using the smallness of the two coupling constants (η_i, $\eta_S \ll \eta_e$), i.e., the fact that $\epsilon \gg 1$, we can easily find the "separation" of the dispersion curves in the region of resonance.

Putting $\epsilon \simeq \eta_e(\eta_e\eta_i\eta_S)^{-1/3}$, we find the following: for the spin-helical wave

$$\omega = \omega_0(1 \pm \eta_e^{1/2}),$$

and for the sound wave

$$\omega = \omega_0 \left[1 - \left(\frac{\eta_i \eta_s}{\eta_e}\right)^{1/2}\right].$$

The dependence of the frequency on the wave vector near the resonance frequency is shown in Fig. 13. We note that the acoustic branch of the oscillations remains practically unchanged in this case.

A similar analysis can be made for the spin-Alfven and other types of coupled electromagnetic waves.

7. CONDUCTION ELECTRONS AND THE SPECTRUM OF THE SPIN WAVES

In considering the high-frequency properties of a ferromagnetic metal, we were interested in the interaction between the collective excitations in the system of conduction electrons and the spin waves. The spectrum of the spin waves was assumed specified. There exists, in addition, a direct interaction between the electrons and the spin waves. This interaction consists in absorption and emission of spin waves by the conduction electrons and is accompanied by spin flip of the electron. It leads, in particular, to finite relaxation times for both the electrons and the magnons, causing threshold effects in the damping of the spin waves. Generally speaking, this interaction is small, thus justifying the foregoing analysis. The interaction between the spin waves and the collective excitations of the electron system, as shown before, realigns the long-wave part of the magnon spectrum. The magnons taking part in the direct interaction with the electrons have large momenta (on the order of the Fermi momentum). Thus, both interactions can be considered independently.

The direct interaction of magnons with electrons is analogous in many respects to electron-phonon interaction in metals. Just as the latter, it leads to singularities in the spin-wave spectrum, similar to the Migdal-Kohn singularities [51,52] in the phonon spectrum. A specific feature of a ferromagnetic metal is the fact that the Fermi surfaces corresponding to the two possible conduction-electron spin orientations turn out to be shifted apart as a result of the exchange interaction. As a rule, this separation is on the order of $\sqrt{\theta_C \epsilon_F}$.

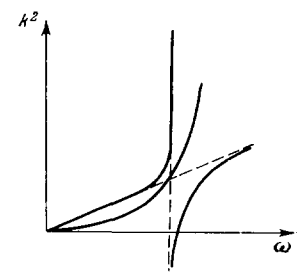


FIG. 13.

The cause of the singularities in the spectrum of the spin waves can be understood by starting from the following considerations. The singularities in the spectrum exist at those values of the magnon momenta at which the mechanism of the direct interaction with the electrons is turned on or off. These values of the momentum are threshold values, corresponding to the vanishing of the spin-wave damping. The direct interaction of the electron and of the spin wave are accompanied by the laws of conservation of momentum

$$\mathbf{p}+\mathbf{q}=\mathbf{p}' \tag{115}$$

and of energy

$$\varepsilon^{\pm}(\mathbf{p})=\varepsilon^{\mp}(\mathbf{p}'). \tag{116}$$

Here $\mathbf{p}$ and $\mathbf{p}'$ are the momenta of the electron in the initial and final states, $\mathbf{q}$ is the momentum of the magnon, $\epsilon^{\pm}(\mathbf{p})$ is the energy of an electron with spin oriented along the (+) or opposite the (-) direction of the magnetization. Equation (116) takes account of the fact that the magnon energy is small compared with the Fermi energy.

In the initial and final states, the electron is on the Fermi surfaces corresponding to opposite spin orientations. From the momentum conservation law (115), which is illustrated in Fig. 14, it follows that the threshold values of the momentum are $q = p_+ \pm p_-$, where $p_\pm$ are the limiting momenta for the electron with opposite spin directions.

In the case of an arbitrary electron dispersion law, the threshold values of the momentum are determined in similar fashion, but depend on the direction of the wave vector of the spin wave. In the case of an isotropic electron spectrum, the threshold values of the magnon momenta form in p-space two concentric spheres with radii $p = p_+ \pm p_-$ (the analog of the Kohn surface). At an arbitrary dispersion law, these surfaces are determined by the shape of the Fermi surface.

The possible existence of a singularity in the spin-wave spectrum at $q = p_+ + p_-$ was first pointed out by Kohn [52] (see also [53]). The singularity in the magnon spectrum at $q = p_+ - p_-$ was pointed out by Kondratenko [24] and independently by Kontorovich [54]. The character of the singularity in both cases is the same: $\Delta\omega \sim x \ln x$, where $x = q - (p_+ + p_-)$. One

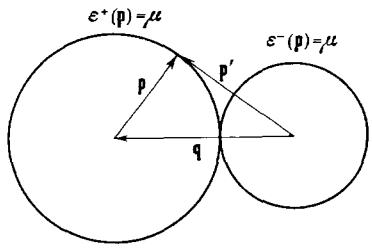


FIG. 14.

can obviously speak of changes in the spectrum if the damping is relatively small. In our case the smallness of the damping is ensured by the smallness of the parameter $\sqrt{\theta_C/\epsilon_F}$ [24]. A singularity of the type $x \ln x$ is quite weak, but the complication of the electron energy spectrum can, generally speaking, lead to an intensification of the singularity [55].

As shown in [56], a quantizing magnetic field greatly intensifies the Migdal-Kohn singularity in the phonon spectrum. Continuing the analogy between the spin waves and phonons, we might expect an intensification of the singularity at $q = p_+ - p_-$, and also the appearance of a number of resonance effects in the spectrum and damping of the spin waves, analogous to those existing in the phonon spectrum of metals in a magnetic field (oscillations of geometric resonance [57], acoustic cyclotron resonance [58], giant quantum oscillations of the phonon damping [59]). However, owing to the small value of the "separation" of the Fermi surfaces of the electrons with opposite spin directions, these effects are impossible in the spin-wave spectrum. The influence of the orbital motion of the conduction electrons on the spectrum of the spin waves in a metal was investigated in [60]. Whereas the aforementioned resonance effects can exist only in practically unattainable magnetic fields,* an effect which does not depend on the magnetic field, that of renormalization of the spin-wave velocity, occurs in not too strong fields determined by the condition $kr \ll 1$. Namely, when $kr \ll 1$ the spin-wave spectrum has the following form:

$$\omega = \gamma H - \frac{\sqrt{\theta_C \varepsilon_F}}{\hbar}(ka)^2. \tag{117}$$

Expression (125) was obtained for the case when the spin-wave vector is oriented perpendicular to the direction of the constant field. Thus, the coefficient of k^2 in the spin-wave spectrum turns out to be $\sqrt{\epsilon_F/\theta_C}$ times larger than the term Ca^2/h, which has the opposite sign at $H = 0$. This circumstance leads to the presence of a minimum on the dispersion curve shown in Fig. 15. The absolute value of the minimum is quite small, $\Delta\omega \sim \omega\sqrt{\theta_C/\epsilon_F}\,(h\omega_H/\epsilon_R)$. We note that owing to the large value of the "separation" of the Fermi surfaces, the existence of renormalization of the spin-wave spectrum does not depend on the relation between the frequency ω of the spin wave and the electron free path time τ.

The foregoing direct interaction between the spin waves and the conduction electrons can appear also when a sound wave propagates in a ferromagnetic metal. The phonon spectrum of a ferromagnetic metal was investigated by Kontorovich and Oleĭnik [61]. They have shown that the spectrum of the phonons in a ferromagnetic metal has singularities at a phonon

*For example, the condition for the existence of giant quantum oscillations in the spin-wave spectrum is the inequality $\mu_B H > \theta_C(\theta_C/\epsilon_F)$.

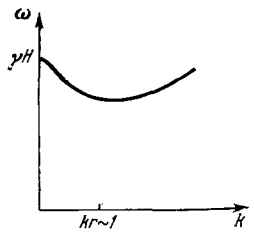


FIG. 15

momentum value equal to $p_+ \pm p_-$ as well as $2p_\pm$. Whereas the singularities of the first type arise as a result of the interaction between the phonons and the spin waves and correspond to electron transitions from one Fermi surface to the other, the singularities of the other type are connected with the decay of the phonon into an electron and hole, occurring without spin flip. Thus, in a ferromagnetic metal there takes place a splitting of the "ordinary" Migdal-Kohn singuliarity into two singularities corresponding to the two Fermi surfaces of the magnetized conduction electrons.

As noted in [61], at a certain critical value of the magnetic field, on the order of 10^4 Oe, the sound and spin-wave frequencies can coincide at the singularity point $q = p_+ - p_-$ (a unique type of "triple" resonance, see Sec. 6). The quality of the frequencies in the case of the "triple" resonance leads to a relatively large shift of the frequency of the ferroacoustic resonance, and this may be one of the methods of observing this singularity.

8. REMARKS ON ANTIFERROMAGNETIC RESONANCE IN METALS

The resonance dependence of the magnetic susceptibility on the frequency, which changes appreciably the character of the collective excitations of the metal, takes place not only in ferromagnets but also in antiferromagnets. In considering the collective excitations in antiferromagnetic metals, one must bear in mind the following circumstances.

1. The static magnetization of an antiferromagnet (compared with a ferromagnet) is quite small. This causes the resonant dependence of the magnetic susceptibility to appear in a narrower frequency region [62].

2. In uniaxial antiferromagnets with positive anisotropy constant, the antiferromagnetic-resonance frequencies are shifted (compared with ferromagnets), even in the case of not too strong fields, into the shorter wavelength region. Owing to this, a situation readily arises in which it is possible to neglect the influence of the magnetic field on the electric conductivity, but it is necessary to take into account its frequency (temporal) dispersion. In other words, in considering the wave properties of an antiferromagnetic metal one can frequently use the plasma formula for the effective dielectric constant of the electron gas. We note that in this case (which is exotic for a ferromagnetic metal), weakly damped waves can propagate in the antiferromagnetic metal, and under favorable conditions (thin antiferromagnetic-resonance lines), additional waves appear, due to exchange interaction of the spins [63].

3. The magnetic structure of the antiferromagnet is quite sensitive to the magnitude and direction of the magnetic field. This, naturally, becomes manifest in the high frequency properties of the antiferromagnetic metal, particularly in the frequency dependence of its surface impedance [63].

APPENDIX

Magnetic Susceptibility of a Ferromagnetic Metal

Following [24], let us consider a system of electrons with non-zero total magnetic moment, situated in a magnetic field H. We shall assume that the magnetic field is weak. Thus, the magnetic field plays the role of an ordering factor of the electron spins; its influence on the electron spectrum can be neglected.

We confine ourselves to the isotropic case and consider for simplicity a single-band model.

The magnetic moment of the system is determined by the expression

$$M = -2i\mu_B \lim_{\tau \to +0} \int \frac{d^4p}{(2\pi)^4} e^{i\varepsilon\tau} S_{\alpha\beta} G_{\beta\alpha}(p), \qquad (A.1)$$

where ϵ is the adiabatic parameter, $\hat{S}^i$ ($i = x, y, z$) are the Pauli spin matrices, and $G_{\alpha\beta}(p)$ are the Fourier components of the electronic Green's function.

The magnetic susceptibility can be calculated in the usual manner:

$$\chi_{ik} = \frac{\partial M_i}{\partial h_k}, \qquad (A.2)$$

with the aid of (A.1), if we know the variation of the Green's function in an external alternating field.

Let the system be under the influence of a small perturbation $u_{\alpha\beta}$. The change of the Green's function in the approximation linear in u is, as is well known [64],

$$\delta G_{\alpha\beta} = G_{\alpha\beta}(p) U_{\gamma\delta}(k) G_{\delta\beta}(p+k) - iG_{\alpha\gamma_1}(p) G_{\gamma_2\beta}(p+k) \times \int \frac{dq}{(2\pi)^4} G_{\gamma_4\gamma}(q+k) U_{\gamma\delta}(k) G_{\delta\gamma_2}(q) \Gamma_{\gamma_1\gamma_2\gamma_3\gamma_4}(p, q; k). \qquad (A.3)$$

Here $\Gamma_{\alpha\beta\gamma\delta}(p, q; k) \equiv \Gamma_{\alpha\beta\gamma\delta}(p, q, p+k, q-k)$ is the vertex part (for a definition and the properties of the vertex part see [64]).

Putting $U_{\gamma\delta}(k) = 2\mu_B S_{\gamma\delta} h$ and using (A.1), (A.2), and (A.3), we obtain for the susceptibility the following expression:

$$\chi_{ik} = -4i\mu_B^2 \lim_{\tau \to +0} \int \frac{d^4p}{(2\pi)^4} e^{i\varepsilon\tau} S^i_{\alpha\beta} G_{\beta\gamma}(p) S^k_{\gamma\delta} G_{\delta\alpha}(p+k)$$

$$-4\mu_B^2 \lim_{\tau\to+0} \int \frac{d^4p}{(2\pi)^4} e^{i\varepsilon\tau} S^i_{\alpha\beta} G_{\beta\gamma_1}(p) G_{\gamma_3\alpha}(p+k) \int \frac{d^4q}{(2\pi)^4}$$

$$\times G_{\gamma_4\gamma}(q+k) S^k_{\gamma\delta} G_{\delta\gamma_2}(q) \Gamma_{\gamma_1\gamma_2\gamma_3\gamma_4}(p, q; k). \quad (A.4)$$

By virtue of the exchange character of the interelectron interaction, only the following components of the vertex part differ from zero (the indices 1 and 2 designate the orientations of the spin along and opposite the direction of the field H):

$$\begin{array}{ll} \text{longitudinal} & \Gamma_{1111},\ \Gamma_{2222},\ \Gamma_{1212},\ \Gamma_{2121} \\ \text{and transverse} & \Gamma_{1221},\ \Gamma_{2112}. \end{array} \quad (A.5)$$

The longitudinal components of the vertex part correspond to the exchange correlation between the electrons without the spin flip, and cause excitations of zero-sound type (for details see [24]). The transverse components of Γ correspond to exchange correlation between states differing in the spin direction, and are connected with excitations of the spin-wave type. It is thus obvious that the magnetic susceptibility is determined by the transverse components of the vertex part.

In fact, we direct the z axis along the magnetic field H. The Green's function of the electrons $G_{\alpha\beta}$ can be represented identically in the form

$$G_{\alpha\beta}(p) = G_+(p) P^+_{\alpha\beta} + G_-(p) P^-_{\alpha\beta}, \quad (A.6)$$

where $P^{\pm}_{\alpha\beta} = (1/2)\delta_{\alpha\beta} \pm S^z_{\alpha\beta}$ are projection operators and $G_{\pm}(p)$ is the Green's function of the electrons with spins in states 1 and 2, respectively:

$$G_{\pm}(p) = \frac{a_{\pm}}{\varepsilon - \varepsilon_{\pm}(p) - \mu + i\delta\,\mathrm{sgn}\,\varepsilon}. \quad (A.7)$$

Here ϵ is the frequency variable, δ the adiabatic parameter μ the chemical potential (which, of course, is the same for both electronic subsystems), $\epsilon_{\pm}(p)$ is the energy of electrons with spins in states along and opposite the field, and $a_{\pm}$ are normalization constants on the order of unity.

Using (A.5), (A.6), and the properties of the Pauli matrices, we obtain for the elements of the magnetic susceptibility $\chi_{\pm} = \chi_1 \pm i\chi_2$ ($\chi_1 = \chi_{xx} = \chi_{yy}$, $\chi_2 = \chi_{xy} = -\chi_{yx}$), corresponding to circularly polarized waves,

$$\chi_{\pm}(\omega, \mathbf{k}) = -2i\mu_B^2 \lim_{\tau\to+0} \int \frac{d^4p}{(2\pi)^4} e^{i\varepsilon\tau} \Big\{ G_{\pm}(p) G_{\mp}(p+k)$$

$$- iG_{\mp}(p) G_{\pm}(p+k) \int \frac{d^4q}{(2\pi)^4} G_{\pm}(q) G_{\mp}(q+k) \Gamma_{\pm}(p, q; k) \Big\}. \quad (A.8)$$

Here $\Gamma_{\pm}$ denote the components of the vertex part Γ_{2112} and Γ_{1221}, respectively. Thus, to calculate the susceptibility (A.8) it is necessary to find the explicit form of the transverse components of Γ.

As shown by Landau in the general theory of the Fermi liquid [65], the spectrum of the collective excitations of a Fermi system is determined by the poles of the vertex part, resulting from the coalescence of the singularities of the Green's functions of the electrons at small momentum transfer ($\mathbf{k} \to 0$). Let us consider the equation for the vertex part

$$\Gamma_{\alpha\beta\gamma\delta}(p, p'; k) = \Gamma^{(1)}_{\alpha\beta\gamma\delta}(p, p') \quad (A.9)$$

$$- i \int \frac{dq}{(2\pi)^4} \Gamma^{(1)}_{\alpha\varkappa_1\gamma\varkappa_2}(p, q) G_{\varkappa_2\varkappa_3}(q) G_{\varkappa_4\varkappa_1}(q+k) \Gamma_{\varkappa_3\beta\varkappa_4\delta}(q, p'; k),$$

where $\Gamma^{(1)}$ is the part of Γ which contains no singularities (and this is why we put $\mathbf{k} = 0$ in it). This equation can be written formally in the form

$$\Gamma = \Gamma^{(1)} - i\Gamma^{(1)} GG\Gamma \quad (A.10)$$

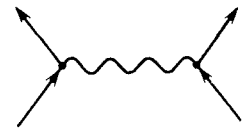
or in equivalent form

$$\Gamma = \Gamma^{(1)} - i\Gamma GG\Gamma^{(1)}. \quad (A.11)$$

We denote by Γ^{ω} the limit of the vertex part when k approaches zero. Setting k equal to zero in (A.11) and eliminating with the aid of the obtained expression for Γ^{ω} the non-singular part $\Gamma^{(1)}$ from (A.10), we obtain a formal equation relating Γ and Γ^{ω}

$$\Gamma = \Gamma^{\omega} - i\Gamma^{\omega}[GG - (GG)_{\omega}]\Gamma. \quad (A.12)$$

Inasmuch as the interelectron interaction between that leads to the spin flip consists in emission (absorption) of a spin wave, the quantity Γ^{ω} can be set in correspondence with the diagram



and accordingly, with the analytic expression

$$\Gamma^{\omega}(\mathbf{p}, \mathbf{p}'; \omega) = g(\mathbf{p}) D(\omega) g(\mathbf{p}'), \quad (A.13)$$

where $D(\omega) = (\omega - 2\mu_B H)^{-1}$ is the propagator of the spin wave (with $\mathbf{k} = 0$).

Expression (A.13) can be obtained by direct calculation. To this end, we determine in explicit manner the reaction of the system to a small disturbance and compare it with (A.3). By the same token we obtain the necessary equation for Γ^{ω}, the solution of which is (A.13).

Let the field of a circularly-polarized wave be described by the expression

$$\left.\begin{array}{l} h_x(t) = h_0 \cos\omega t, \\ h_y(t) = h_0 \sin\omega t. \end{array}\right\} \quad (A.14)$$

The Hamiltonian of the interaction between the electrons and the alternating magnetic field is written in standard form

$$H_{вз} = 2\mu_B \int d\mathbf{r}\, \psi^+_{\alpha}(x) S_{\alpha\beta} \mathbf{h}(t) \psi_B(x) \quad (A.15)$$

or in terms of circularly-polarized waves

$$H_{вз} = \mu_B \int d\mathbf{r}\, \psi^+(x)[S_- h_+ + S_+ h_-]\psi(x) = H_1 + H_1^+. \quad (A.16)$$

Here $\psi(x)$ and $\psi^+(x)$ are the quantum operators of the particles. The change of the latter, in the approximation linear in H_{int}, is determined in accordance with the equation of motion by the expression

$$\tilde{\psi}_\alpha(x) = \psi_\alpha(x) + i\int_{-\infty}^{t} dt' [V(t'), \psi_\alpha(x)] \quad (A.17)$$

and

$$\tilde{\psi}^+_\alpha(x) = \psi^+_\alpha(x) + i\int_{-\infty}^{t} dt' [V(t'), \psi^+_\alpha(x)],$$

where

$$V(t) = V_1(t) + V_1^+(t), \quad V_1(t) = e^{iH_0 t} H_1 e^{-iH_0 t}$$

(H_0 is the unperturbed Hamiltonian of the system).

Commuting the operator $e^{iH_0 t}$ with H_1 and recognizing that the only term of the unperturbed Hamiltonian which does not commute with H_1 is $2\mu_B H\int \psi^+_\alpha(x) S^z_{\alpha\beta}\psi_\beta(x)\,dr$, which describes the interaction of the electrons with a constant magnetic field, we obtain the explicit form of the operator $V_1(t)$:

$$V_1(t) = e^{i(\omega - 2\mu_B H)t}\mu_B h_0 \int d\mathbf{r}\psi^+_\alpha(x) S^-_{\alpha\beta}\psi_\beta(x). \quad (A.18)$$

Formula (A.17) thus assumes the form

$$\tilde{\psi}_\alpha(x) = S_{\alpha\beta}\psi_\beta(x) \text{ and } \tilde{\psi}^+_\alpha(x) = \tilde{\psi}^+_\gamma S^+_{\gamma\alpha}, \quad (A.19)$$

where

$$S_{\alpha\beta}(t) = \delta_{\alpha\beta} + \frac{\mu_B h S^+_{\alpha\beta} e^{-i(\omega - 2\mu_B H)t}}{\omega - 2\mu_B H + i\delta} + \frac{\mu_B h S^-_{\alpha\beta} e^{i(\omega - 2\mu_B H)t}}{\omega - 2\mu_B H - i\delta}. \quad (A.20)$$

Using (A.19), (A.20), and (A.6) we obtain the variation of the Green's function

$$\delta G_{\alpha\beta} = \frac{\mu_B h e^{i(\omega - 2\mu_B H)t}}{\omega - 2\mu_B H - i\delta}[G_- - G_+] S^-_{\alpha\beta} + \frac{\mu_B h e^{-i(\omega - 2\mu_B H)t}}{\omega - 2\mu_B H + i\delta}[G_- - G_+] S^+_{\alpha\beta}. \quad (A.21)$$

Going over in (A.21) to the Fourier representation and comparing it with (A.3), we arrive at equations for the transverse components of $\Gamma^\omega_\pm$ near resonance:

$$1 - i\int \frac{dq}{(2\pi)^4}\Gamma^\omega_+(p, q; -\omega) G_+(q) G_-(q-\omega) = \frac{G_+^{-1}(p) - G_-^{-1}(p)}{\omega - 2\mu_B H - i\delta}, \quad (A.22)$$

$$1 - i\int \frac{dq}{(2\pi)^4}\Gamma^\omega_-(p, q; \omega) G_-(q) G_+(q+\omega) = \frac{G_+^{-1}(p) - G_-^{-1}(p)}{\omega - 2\mu_B H + i\delta}. \quad (A.23)$$

Near resonance we can neglect unity in the left side of (A.22) and (A.23). We shall seek their solution in the form

$$\Gamma^\omega_\pm(p, p'; \omega) = \mp a^2 \frac{g(p) g(p')}{\omega \pm 2\mu_B H}, \quad (A.24)$$

where $q(p) = G_+^{-1} - G_-^{-1}$.

Substituting (A.24) in (A.22) and (A.23), and recognizing that

$$-i\int [G_+(p) - G_-(p)]\frac{d^4p}{(2\pi)^4} = n_+ - n_- \quad (A.25)$$

($n_\pm$ is the density of the particles in the states 1 and 2), we obtain the normalization constant a:

$$a^2 = (n_+ - n_-)^{-1} = \frac{\mu_\beta}{M},$$

where M is the total magnetic moment of the metal.

The quantities $g(p)$ are smooth functions of the argument and can be taken near the limiting momentum p_0. As seen from (A.24), the quantity $\sqrt{(\mu_B/M)}\, g(p_0)$ plays the role of the constant of interaction between the electrons and the spin waves.

The quantity $g(p_0)$, as seen from its definition, is proportional to the energy gap ϵ_0 between the Fermi surfaces of the electrons in states 1 and 2. For s-electrons, for example, $\epsilon_0 \sim \sqrt{\theta_C \epsilon_F}$ and the interaction constant is $\sim (\mu_B \theta_C \epsilon_F / M)^{1/2}$, and for d-electrons $\epsilon_0 \sim \theta_C$ and the coupling constant is $\sim (\mu_B \theta_C / M)^{1/2}$.

We can now easily obtain with the aid of (A.24) the vertex part. Putting in (A.12)

$$\Gamma_\pm = \mp\sqrt{\frac{\mu_B}{M}}\frac{g(\mathbf{p}) g(\mathbf{p}')}{\omega \pm [2\mu_B H + \Pi(k)]}, \quad (A.26)$$

we obtain for the polarization operation $\Pi(k)$

$$\Pi(k) = -i\int \frac{d^4p}{(2\pi)^4} g^2(p) G_+(p)[G_-(p-k) - G_-(\mathbf{p}, \varepsilon - \omega)]. \quad (A.27)$$

When k tends to zero, the integral (A.27), as can be verified by direct calculation, is proportional to αk^2, where α is of the same order of magnitude as the Curie temperature. Thus, the spectrum of the spin waves has the usual structure.

The expressions for the magnetic susceptibility (A.8) take on, with the aid of (A.26), the following final form:

$$\chi_\pm(\omega, \mathbf{k}) = -2i\mu_B^2 \lim_{\tau \to +0}\int \frac{d^4p}{(2\pi)^4}\left\{G_\pm(p) G_\mp(p+k) \pm ig(\mathbf{p}) G_\mp(p) G_\pm(p+k)\int \frac{d^4q}{(2\pi)^4}\frac{G_\pm(q) G_\pm(q+k) g(\mathbf{q})}{\omega \pm (2\mu_B H + \alpha k^2)}\right\}. \quad (A.28)$$

Near resonance, the first term can be omitted. In the case of an isotropic dispersion law (see (A.7)), the integrals in (A.28) can be readily calculated, and the susceptibility is written in standard fashion:

$$\chi_\pm(\omega, k) = \frac{\mu_B M_0}{\Omega \mp \omega}, \quad (A.29)$$

where $M_0 = M\theta_C/\epsilon_F$ is the saturation magnetic moment.

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