533.9

LANDAU DAMPING AND ECHO IN A PLASMA

B. B. KADOMTSEV

Usp. Fiz. Nauk 95, 111–129 (May, 1968)

1. INTRODUCTION

IN 1946, L. D. Landau has shown that waves in a plasma should be damped even in the absence of collisions[1]. The effect of the Landau damping, as it was subsequently called, plays a fundamental role in plasma; it serves as the basis for the theory of collective phenomena in a rarefied plasma. However, for a long time this interesting physical effect remained outside the scope of experimental research. Only most recently were direct laboratory experiments performed on Landau damping, and have shown good agreement between the measurement results and the theory.

The Landau damping is not connected directly with dissipation, so that it cannot be regarded as a completely irreversible process, inasmuch as even a damped wave in a collisionless plasma retains the "memory" of the preceding oscillatory motion. This memory can become manifest in effect of the echo type. Echo in a plasma was observed only recently. The present article is devoted to a review of work on Landau damping and echo in a plasma and to a discussion of the connection between these two effects.

2. LANDAU DAMPING

We recall first the history of the problem. In 1938, A. A. Vlasov proposed to describe wave processes in a rarefied plasma by means of a kinetic equation with a self-consistent field[2]. For Langmuir oscillations, which represent waves propagating along the x axis, the Vlasov equation is of the form

$$\frac{\partial F}{\partial t}+v\frac{\partial F}{\partial x}+\frac{e}{m}\frac{\partial \varphi}{\partial x}\frac{\partial F}{\partial v}=0. \quad (2.1)$$

Here F is the electron distribution function with respect to the velocity component along the x axis, which we designate simply by v, m is the electron mass, $-e$ its charge, and φ the electric field potential, which, according to A. A. Vlasov, should be determined from the equation

$$\frac{\partial^2\varphi}{\partial x^2}=4\pi e\left\{\int F\,dv-n_0\right\} \quad (2.2)$$

(here n_0 is the ion density, which can be regarded constant in high-frequency Langmuir oscillations).

For oscillations of very small amplitude, Eqs. (2.1) and (2.2) can be linearized. To this end it is sufficient to put $F = f_0 + f$, where f_0—equilibrium function at f—deviation from equilibrium, and neglect the quadratic term $\frac{e}{m}\frac{\partial\varphi}{\partial z}\frac{\partial f}{\partial v}$ in (2.1). We shall assume that in the equilibrium state the plasma is homogeneous and neutral, so that $\int f_0 dv = n_0$. By virtue of the homogeneity, Eqs. (2.1) and (2.2) can be written out for each component of the expansion of f and φ in a Fourier integral series with respect to the variable x, so that it is sufficient to consider only the evolution of an individual harmonic. Assuming that the functions f and φ are of the form $f(v, t)e^{ikx}$ and $\varphi(t)e^{ikx}$, we write the linearized equations in the form

$$\frac{\partial f}{\partial t}+ikvf+ik\varphi\frac{e}{m}\frac{\partial f_0}{\partial v}=0, \quad (2.3)$$

$$k^2\varphi=-4\pi e\int f\,dv. \quad (2.4)$$

It is natural to attempt, as Vlasov did, to find the natural oscillations of a plasma with certain frequency ω. To this end we put

$$f(v,\ t)=f_1e^{-i\omega t},\qquad \varphi(t)=\varphi_1e^{-i\omega t}.$$

Then it might seem that we get from (2.3)

$$f_1=\frac{k}{\omega-kv}\frac{e}{m}\frac{\partial f_0}{\partial v}\varphi_1. \quad (2.5)$$

Substituting this expression in (2.4), we get a dispersion equation connecting the natural frequency ω with the wave number k:

$$\varepsilon(k,\ \omega)\equiv 1+\frac{4\pi e^2}{km}\int\frac{\frac{\partial f_0}{\partial v}dv}{\omega-kv}=0. \quad (2.6)$$

We have introduced here the symbol ϵ for the quantity (2.6), which represents the dielectric constant of the plasma.

We see that (2.6) has a singularity under the integral sign, so that this expression cannot be used until we determine how to remove these singularities. A. A. Vlasov[3] proposed to carry out the integration in (2.6) in the sense of the principal value, but there are not sufficient grounds for this procedure.

A correct approach to the solution of the problem of small plasma oscillations which simultaneously resolved the difficulty with the divergence in (2.6), was pointed out by L. D. Landau[1]. He called attention to the fact that in a real formulation of the problem of small oscillations of a plasma it is necessary to deal either with specified initial data or with specified boundary conditions, and he showed how to solve the two problems.

Let us consider, for example, the initial-value problem. In this case it must be assumed that when $t < 0$ there is no perturbation, and only at an instant $t = 0$ is an external action applied, and produces a certain initial perturbation of the distribution function $g(v)$. The problem consists of determining the temporal evolution of the perturbation. To this end, it is necessary to use Eqs. (2.3) and (2.4) with an external source $g(v)\delta(t)$, added to the right side of (2.3). To solve this system of equations we can use the Laplace-transformation method, putting

$$f_p(v)=\int_0^\infty f(v,\ t)\,e^{-pt}\,dt, \quad (2.7)$$

and accordingly for $\varphi(t)$.

Multiplying (2.3) with a source $g(v)\delta(t)$ in the right side by e^{-pt} and integrating with respect to t, we get

$$f_p=\frac{ig(v)}{ip-kv}+\frac{k}{ip-kv}\frac{e}{m}\frac{\partial f_0}{\partial v}\varphi_p. \quad (2.8)$$

We see that this expression differs from (2.5) in that a term with g has been added and that ω has been replaced by ip. Substitution of (2.8) in (2.4) allows us to find φ_p:

$$\varphi_p = \frac{4\pi e}{k^2 \varepsilon(k, ip)} \int \frac{g(v)\,dv}{ip - kv}. \qquad (2.9)$$

From the known value of φ_p we can now easily find $\varphi(t)$:

$$\varphi(t) = \frac{1}{2\pi i} \int\limits_{-i\infty+\sigma}^{i\infty+\sigma} \varphi_p e^{pt}\,dp. \qquad (2.10)$$

It is convenient here to use in lieu of p the variable $\omega = ip$, and since the integration with respect to the variable p takes place in the right half-plane, the integration with respect to the complex ω should be carried out in the upper half plane:

$$\varphi(t) = \frac{2e}{k^2} \int\limits_{-\infty+i\sigma}^{\infty+i\sigma} \frac{e^{-i\omega t}}{\varepsilon(k, \omega)} \int\limits_{-\infty}^{+\infty} \frac{g(v)\,dv}{\omega - kv}\,d\omega. \qquad (2.11)$$

This expression solves completely the problem of the plasma oscillations produced by the initial perturbation g(v). We see therefore, that, generally speaking, there is no definite dependence of ω on k: for a specified k, the integration in (2.11) is carried out over all ω. However, if g(v) has no singularity, then the asymptotic form of the integral (2.11) will be determined at large values of t by the zeroes of $\epsilon(k, \omega)$, i.e., $\varphi(t) \sim \exp(-i\omega_k t)$, where $\epsilon(k, \omega_k) = 0$. Thus, at very large values of t there is separated from the solution (2.11) a branch of plasma oscillations with natural frequency ω_k, determined by the relation

$$\varepsilon(k, \omega_k) = 0. \qquad (2.12)$$

Since the integration in (2.11) is carried out along a horizontal line in the upper half-plane, in the calculation of ϵ by (2.6) it is necessary to assume that the frequency ω is in the upper half plane, i.e., for ω close to the real axis it is necessary to put

$$(\omega - kv)^{-1} \to (\omega + i\nu - kv)^{-1} \to \frac{P}{\omega - kv} - i\pi\delta(\omega - kv),$$

where P denotes the principal value. This rule for going around the pole is customarily called the Landau rule. With allowance for the rule for going around the pole, the dielectric constant (2.6) is complex:

$$\varepsilon = 1 + \frac{4\pi e^2}{km} \int \frac{P}{\omega - kv} \frac{\partial f_0}{\partial v}\,dv - \frac{4\pi e^2}{km} \frac{i\pi}{|k|} \frac{\partial f_0}{\partial v}\Big|_{v=\omega/k}. \qquad (2.13)$$

The presence of an imaginary part in ϵ corresponds to the Landau damping. Its magnitude is proportional to the derivative $\frac{\partial f_0}{\partial v}\Big|_{v=\omega/k}$ at the point $v = \omega/k$, where the velocity of the particles coincides with the phase velocity of the wave. It can be said that the Landau damping is connected with the absorption of the wave by the resonant particles. If the electrons have a Maxwellian distribution with temperature T, then the dielectric constant (2.3) can be represented in the form

$$\varepsilon = 1 + \frac{m\omega_0^2}{k^2 T} \left(1 + i\sqrt{\pi} z W(z)\right), \qquad (2.14)$$

where $\omega_0 = \sqrt{4\pi e^2 n_0/m}$ -- Langmuir frequency, $z = \omega/kv_e$, $v_e = \sqrt{2T/m}$ --average thermal velocity, and

$$W(z) = e^{-z^2} + \frac{2i}{\sqrt{\pi}} \int\limits_0^z e^{-z^2+\xi^2}\,d\xi. \qquad (2.15)$$

Equating (2.14) to zero, we can find the real and imaginary parts of the complex frequency. At small values of $k \ll a^{-1}$, where $a = \sqrt{T/4\pi e^2 n_0}$ is the Debye radius, the expression for the real part of the frequency is

$$\omega^2 = \omega_0^2 + \frac{3T}{m} k^2, \qquad (2.16)$$

and its imaginary part (i.e., the damping decrement γ) equals

$$\gamma = \omega_0 \sqrt{\frac{\pi}{8}} \frac{1}{(ka)^2} \exp\left(-\frac{1}{2a^2k^2}\right). \qquad (2.17)$$

We see that at small ka the damping increment is exponentially small.

At not very large values of t, a definite contribution to the integral (2.11) is made also by the pole $\omega - kv = 0$, corresponding to the free spreading of the particles of the initial perturbation, so that the dependence of the potential on the time can be quite complicated.

3. VAN KAMPEN WAVES

Thus, Langmuir oscillations excited by a certain initial perturbations $g(v)\delta(t)$ with a smooth function g(v) should attenuate in time. This does not mean, however, that it is impossible to have undamped natural oscillations of the plasma. In fact, let us return again to Eq. (2.5), which connects the disturbance of the distribution function f_1 with the potential φ_1. According to the Landau rule, it is necessary to add to the frequency ω in the denominator a small imaginary part $\omega \to \omega + i\nu$, so that we can rewrite (2.5) in the form

$$f_1 = \frac{P}{\omega - kv} \frac{ke}{m} \frac{\partial f_0}{\partial v} \varphi_1 - i\pi\delta(\omega - kv) \frac{ke}{m} \frac{\partial f_0}{\partial v} \varphi_1. \qquad (3.1)$$

Here P means that in the calculation of the different integrals containing f_1 the singularity must be integrated in the sense of the principal value. Expression (3.1) describes the perturbation of the distribution function by the potential φ_1 of the electric field of the wave. We see that the closer the velocity is to the phase velocity ω/k of the wave, the larger this perturbation. In the immediate vicinity of this point, the main contribution is made by the second term, which is a result of the imaginary part

$$\mathrm{Im}\left(\frac{1}{\omega - kv + i\nu}\right) = \frac{\nu}{(\omega - kv)^2 + \nu^2} = \pi\delta(\omega - kv)$$

as $\nu \to 0$.

We note that allowance for the damping of the perturbations due to collisions, which in the simplest variant can be carried out by introducing a small term $-\nu f$ in the left side of Eq. (2.3), will also lead to the Landau rule for going around the pole, and consequently, to exactly the same expression (3.1). This is perfectly understandable, for both effects—collisions and the finite growth time of the perturbation—lead to qualitatively the same limitation on the perturbation at the resonant point.

If we substitute (3.1) in the Poisson equation (2.4), then we obtain directly the Landau dispersion equation. In other words, the only perturbation of the distribution function (3.1) remaining in the Langmuir oscillations at large values of t is the one produced by the wave potential.

Let us assume now that at the initial instant of time there is introduced, besides the smooth perturbation

$g(v)$, also a certain modulated beam with a velocity precisely equal to the phase velocity of the wave. If we choose the magnitude and phase of this beam in suitable manner, then we can exactly cancel out the resonant-electron perturbation described by the second term of (3.1). But in this case we get expression (2.5) without the δ function, and consequently we arrive at the Vlasov dispersion equation with an integral in the sense of the principal value, describing the Langmuir waves without attenuation. Thus, the Vlasov solution also has a definite physical meaning: it describes a wave with addition of a group of resonant particles.

However, this solution is quite particular—it corresponds to a certain fully defined choice of the density of the additional particles. As shown by Van Kampen[4], Eqs. (2.3) and (2.4) have a much broader class of natural oscillations. In order to find these oscillations, it is necessary to eliminate the inaccuracy admitted in the derivation of (2.5) for the perturbation of the distribution function. In fact, if ω is a real quantity, then the homogeneous equation for $f_1(\omega - kv)f_1 = 0$ has a nontrivial solution of the form $\lambda\delta(\omega - kv)\varphi_1$, where λ is an arbitrary constant (more accurately, a function of ω and k). This solution of the homogeneous equation should be added to (2.5). Assuming for concreteness (without loss of generality) that the integral of $1/(\omega - kv)$ is taken in the sense of the principal value, we should write for the natural oscillations in place of (2.5)

$$f_1 = \frac{P}{\omega - kv}\frac{ke}{m}\frac{\partial f_0}{\partial v}\varphi_1 + \lambda\delta(\omega - kv)\varphi_1. \tag{3.2}$$

Substituting this expression in the Poisson equation, we determine the dispersion equation

$$1 + \frac{4\pi e^2}{mk}\int\frac{P}{\omega - kv}\frac{\partial f_0}{\partial v}dv + \frac{4\pi e}{k^2|k|}\lambda = 0. \tag{3.3}$$

Since this equation contains two unknown quantities λ and ω, it does not yield a unique connection between ω and the wave number k. It should be more readily regarded as an equation for the determination of λ if ω is given. This means that at a given k the frequency can be quite arbitrary. In other words, for any frequency ω we can choose a value λ, i.e., the density of the resonant particles, such that the solution has the form of an undamped wave with given frequency ω. This is the Van Kampen wave. Each of the Van Kampen waves represents a modulated beam of particles moving with a velocity equal to the phase velocity of the wave $v_{ph} = \omega/k$ (this beam is described by the second term of (3.2)), accompanied by a polarization cloud resulting from the action of the beam on the plasma electrons. The perturbation f_1 in this cloud is described by the first term of (3.2).

If the wavelength of the perturbation is sufficiently large, $ka \ll 1$, and the frequency ω is close to the plasma frequency (2.14), then the quantity λ, defined by expression (3.3), is very small, for in this case the sum of the first two terms in (3.3) is close to zero. We then deal with a plasma wave with a small addition of resonant particles. The van Kampen waves proper are best taken to be solutions that differ noticeably from the Langmuir waves when the second term of (3.2) is larger than or comparable with the first. Thus, we can state roughly that the natural oscillations of an electron plasma consist of Van Kampen waves—modulated beams—and Langmuir waves.

Van Kampen has shown that the system of functions (3.2) is complete, i.e., any initial perturbation $g(v)$ can be expanded in terms of these functions and consequently, the waves excited in the plasma can be regarded as a superposition of natural oscillations. The corresponding solution coincides exactly with the Landau solution (2.11).

4. DAMPING OF FINITE-AMPLITUDE WAVES

Landau solved completely the problem of small oscillations of a plasma and revealed a new physical effect—damping of waves even in the absence of collisions. However, the physical meaning of the Landau damping is not yet physically clear. In particular, it has not been completely explained how a fully reversible kinetic equation leads to irreversibility, under which physical conditions can the undamped waves be realized, what happens if the wave amplitude is not infinitesimally small, etc. All these questions were subsequently clarified in studies of the damping of waves of finite amplitude, made by Bohm and Gross[25], Vedenov, Velikhov, and Sagdeev[5], and later by Al'tshul and Karpman[6] and O'Neil[7]. We present here only the qualitative results of their analysis.

Following the authors of[5-7], we assume that a wave of small but finite amplitudes φ_0 has been excited in the plasma. We assume that the wavelength is sufficiently large, so that the phase velocity $v_{ph} = \omega_0/k \gg \sqrt{2T/m}$. Then the fraction of the resonant electrons will be very small, so that during the course of a long time interval the amplitude of the wave can be regarded as constant, i.e., $\varphi = \varphi_0\cos(\omega_0 t - kx)$. This means that we can first consider approximately the behavior of the resonant particles in a wave of constant amplitude, and take into account the effect of their reaction on the wave.

Since by assumption the amplitude of the wave is small, the interaction with the field of the wave is small for all particles, except the resonant ones, i.e., we can use the linear approximation. Let us consider now the resonant particles. To this end, it is convenient to change over to a coordinate system moving together with the wave. In this system of coordinates, the wave is a stationary perturbation of the electric field: $\varphi = \varphi_0\cos kx$. The electrons in such a field can be broken up into two classes: captured electrons oscillating near points with maximum potential φ_0, and transit electrons, whose energy is sufficiently large to overcome the potential barrier.

It is convenient to consider the particle motion on the phase plane (x, v). Fig. 1a shows the dependence of φ on x, and Fig. 1b shows the phase trajectories of the electrons as they move in the wave. The electrons with low velocities are captured by the waves. The electron captured, say, near $x = 0$ is acted upon by a force

$$-eE = e\frac{\partial\varphi}{\partial x} = -e\varphi_0 k\sin kx \approx -e\varphi_0 k^2 x,$$

so that it will execute oscillations with frequency

$$\Omega = k\sqrt{\frac{e\varphi_0}{m}}. \tag{4.1}$$

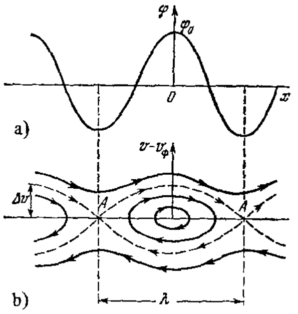

FIG. 1. Phase trajectories of electrons in a wave.

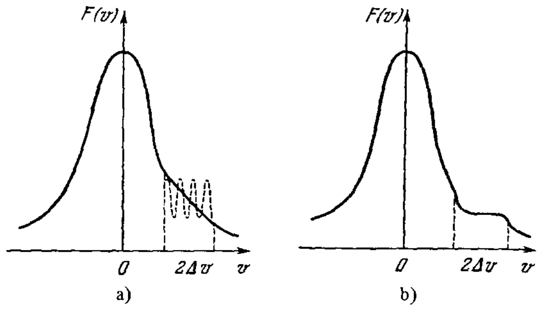

FIG. 2. Formation of a "plateau" on the distribution function.

With increasing electron-oscillation amplitude, the frequency decreases and it vanishes on the separatrix between the captured particles and the transit particles, shown dashed in Fig. 1b, since the corresponding electrons can be located for an infinitely long time on the "vertex" A of the potential energy $-e\varphi$. The average velocity of the transit particles near the separatrix is also low and increases with increasing distance from the separatrix.

Since the total electron energy is

$$\frac{m(v-v_\Phi)^2}{2}-e\varphi=\text{const},$$

and at the point A the velocity is $v - v_{ph} = 0$, the half-width of the separatrix at $x = 0$ obviously equals $\Delta v = \sqrt{e\varphi_0/m}$, i.e., it decreases with the wave amplitude much more slowly than linearly. This indicates that even if the wave amplitude is very small the number of captured particles can be relatively large.

Let us trace now the time evolution of the distribution function of the resonant particles. Since the width of the interaction region Δv is large, the small linear perturbation f can be neglected in this region, i.e., it can be assumed that the initial distribution function of the resonant particles coincides with the unperturbed function. We shall assume that it decreases with increasing v as shown by the solid line in Fig. 2a.

We now take into account the fact that the Vlasov equation

$$\frac{\partial F}{\partial t}+v\frac{\partial F}{\partial x}+\frac{e}{m}\frac{\partial\varphi}{\partial x}\frac{\partial F}{\partial v}=0 \tag{4.2}$$

can be regarded as the continuity equation for flow on the phase plane of a certain substance with density F. Since

$$\frac{\partial v}{\partial x}+\frac{\partial}{\partial v}\left(\frac{e}{m}\frac{\partial\varphi}{\partial x}\right)=0,$$

this flow is incompressible, i.e., F is conserved along the streamlines. Knowing these lines and the initial functions, it is easy to visualize the variation of F. Let us consider first the captured particles. We shall shade the region with smaller values of v, where the function F is larger (Fig. 3a). Since the capture particles execute oscillations with frequency $\sim\Omega$, the internal part of the phase region of the captured particles will rotate, so that after a half cycle the picture assumes the form of Fig. 3b. The particles lying on the separatrix do not rotate, therefore these lines remain in place, and for the majority of the particles a rearrangement takes place—the density of the fast electrons becomes larger than the density of the slow ones, i.e., the derivative $\partial F/\partial v$ reverses sign for the captured particles. Then this change of sign will occur through each half cycle, and with increasing t the picture will assume the form of shallower and shallower oscillations (see Fig. 3c and the dashed line in Fig. 2a). A similar "mixing" takes place also for the transit "resonant" particles that are close to the separatrix. Owing to the "mixing" effect, a state F that oscillates so rapidly on the phase plane, that Coulomb collisions can come into play, should be reached quite rapidly (compared with the average collision frequency). We recall that Coulomb collisions lead to diffusion in velocity space, and therefore the effective relaxation time of the small oscillations in phase space is much smaller than the average collision time for a smooth distribution function. Owing to the joint action of the "mixing" and the Coulomb collisions, the distribution function in the region of resonant particles should become "smoothed out" near a certain average value, i.e., a "plateau" should appear on it (Fig. 2b). During the course of the sufficiently long time interval, the damping of the wave due to collision can then still be regarded as small, so that the wave will stay purely periodic for a long time. This is the wave corresponding to the Vlasov stationary solution. In such a wave, the singularity at the point $\omega - kv = 0$ is eliminated as a result of formation of a "step" on the distribution function with $\partial f_0/\partial v = 0$ at the resonant point, leading automatically to integration of the singularity in the sense of the principal value.

Thus, the stationary wave can be realized physically as the result of evolution of a wave with finite amplitude, Of course, if we neglect the small amplitude oscillations, we can regard the wave as being practically stationary long before the "roughnesses" of the distribution function become smoothed out by the collisions. For the wave to be stationary it is sufficient that the distribution function be constant, at the assumed degree of accuracy, along the stream lines in phase space.

We can mentally visualize also the possibility of formation of a wave which is undamped from the very beginning: to this end, it is sufficient to have simultaneously with the potential perturbation produced by the nonresonant particles also such a perturbation of the distribution function of the resonant particles, that at the succeeding instants of time this function remains constant (in the coordinate system moving with the wave). For example, if the distribution function of the particles locked inside the separatrix in Fig. 3 is constant, motion of the locked particles will no longer lead to oscillations of the wave amplitude (and analogously for the transit particles near the separatrix). We see

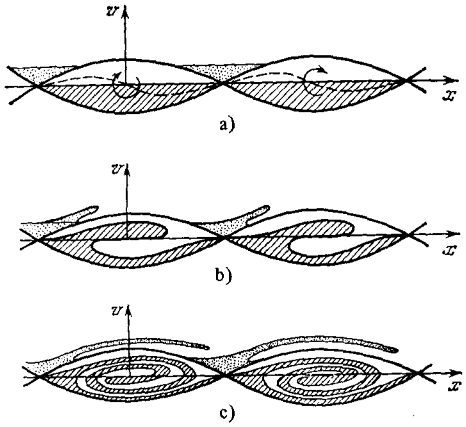

FIG. 3. Motion of resonant particles in a wave.

therefore that the stationary Langmuir waves correspond to such an addition of resonant particles, that something like a plateau is produced immediately. If the number of the additional resonant particles is increased, then we obtain, as it were, a wave with a modulated beam, corresponding to a transition to the Van Kampen wave. The Van Kampen waves proper correspond to stationary modulated beams penetrating through the plasma.

Let us turn now to the initial stage of the damping of a wave of finite amplitude, corresponding precisely to the Landau damping. If the wave amplitude tends to zero, then the first half-cycle of the damping (Fig. 4) stretches out to infinity, so that in the linear approximation there are no oscillations at all and the amplitude decreases monotonically. Here, of course, we can no longer regard the amplitude as constant, as we have done in consideration of the motion of captured particles in a wave of finite amplitude. Incidentally, even the captured particles themselves do not have time to execute even a single oscillation—they only begin their motion in the wave, as shown in Fig. 3a dashed, and their contribution to the charge density turns out to be already so appreciable, that the amplitude of the wave decreases noticeably. It can be stated that the resonant particles have too large a weight of a small-amplitude wave.

It is easy to estimate the amplitude at which the wave can be regarded as linear. To this end it is sufficient to compare the wave energy density

$$\frac{E^2}{8\pi} = \frac{k^2\varphi_0^2}{8\pi}$$

with the density of that energy which is transferred to the captured particles when the plateau is produced. The width of the plateau is of the order of $\Delta v \sim \sqrt{e\varphi_0/m}$, and the change of the electron energy due to an increase of its velocity by Δv near the phase velocity ω/k is $\sim m\omega(\Delta v)/k$. Since the distribution function changes upon formation of the plateau by an amount

$$\Delta f_0 \sim \left.\frac{\partial f_0}{\partial v}\right|_{v=\omega/k} \Delta v,$$

the change of the energy density of the captured particles is of the order of

$$\Delta\mathscr{E} \sim m\frac{\omega}{k}\Delta v(\Delta f_0 \Delta v) \sim m\left(\frac{e\varphi_0}{m}\right)^{3/2} v \left.\frac{\partial f_0}{\partial v}\right|_{v=\omega/k}. \tag{4.3}$$

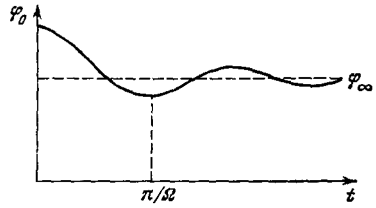

FIG. 4. Damping of waves of finite amplitude.

Comparing this value with the energy density in the wave, we find the condition for the linearity of the Langmuir wave in a plasma with a Maxwellian electron velocity distribution:

$$\sqrt{\frac{e\varphi_0}{T}} \ll \frac{1}{(ka)^4}\exp\left(-\frac{1}{2a^2k^2}\right). \tag{4.4}$$

We see therefore that at very small ka a wave of even very small amplitude cannot be regarded as linear. This is connected with the fact that for such waves the number of resonant electrons is exponentially small and accordingly their damping decreases very rapidly. Conversely, when $ka \gtrsim 1$ the wave can be regarded as linear even at moderate values of $e\varphi_0/T$.

An analysis of the damping of a wave of finite amplitude clarifies also the question of the reversibility in time. The question is as follows. On the one hand, in the absence of collisions the kinetic equation with a self-consistent field is fully reversible in time, namely, it remains its form when t is replaced by −t and when the particle velocities are reversed. Therefore all the processes described by this equation should be reversible. Yet the presence of exponential wave damping in the linear approximation would apparently indicate a patent irreversibility, and after the lapse of a sufficiently long time interval, the system of charged particles can "forget" completely that a wave propagated in it. To clarify this question, let us turn again to Fig. 3.

If there are no collisions, then the picture of the evolution of the distribution function of the resonant particles is fully reversible. When the velocities of the captured particles are reversed, the "coil" on Fig. 3c should start to unwind, after which it goes through the state $t = 0$ and begins to wind again. Accordingly, oscillations first appear on the amplitude φ_0, then φ_0 reaches a maximum value corresponding to $t = 0$, and the entire picture of damping with oscillations is repeated.

A similar picture should take place also in the case of a small-amplitude wave, i.e., in the case of a large number of resonant particles. Of course, in this case there is no capture of particles: the wave is damped long before even one oscillation of the locked particles takes place. There takes place, so to speak, a spreading out of the individual groups of the resonant particles, which is accompanied by drawing of energy from the wave, so that ultimately the entire oscillation energy is transferred to the resonant particles. But this process is also reversible. Even after the wave is damped to an extremely low amplitude, the medium can retain the "memory" of the initial perturbation for a long time (until collisions begin to assume a role), and if the particle velocities were reversed the entire process would go in the opposite direction. In practice, of course, it is impossible to reverse the velocities of

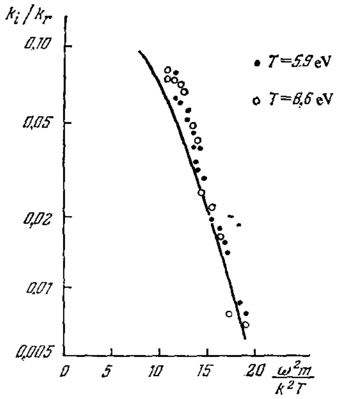

FIG. 5. Dependence of the damping decrement on the square of the ratio of the phase velocity of the average thermal velocity.

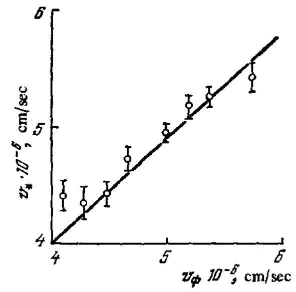

FIG. 6. Dependence of the velocity of the resonant electrons on the phase velocity of the wave.

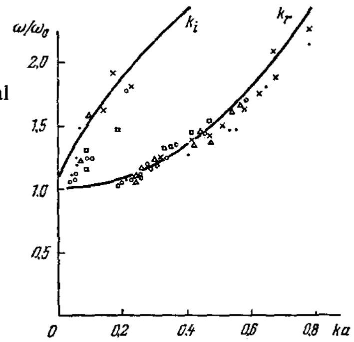

FIG. 7. Dispersion curves for Langmuir waves. The experimental points pertain to different values of the density and Langmuir frequency $f_0 = \omega_0/2$: ○ – f_0 = 75 MHz, □ – 60 MHz, Δ – 51 MHz, X – 35 MHz. The points are plotted while varying the density at a specified generator frequency f = 80 MHz.

all the particles, but the "memory" of the oscillations can appear in oscillations of the echo type.

5. EXPERIMENTS ON LANDAU DAMPING

Although recently some indications appeared that Landau damping is present in wave propagations in the ionosphere, and the experiments of Wong, d'Angelo, and Motley[8] might seem to offer quite convincing evidence of the collisionless mechanism of damping of ion-acoustic oscillations, the first most detailed and direct investigation of the Landau damping in longitudinal oscillations should still be regarded to be the experiment of Malmberg, Wharton, and Drummond[9]. In this experiment they measured the spatial damping of a longitudinal electron wave of the form $\exp(-i\omega t + ik_r x + k_i x)$, excited with the aid of high frequency oscillations applied to a Langmuir probe. A plasma of density 10^8–10^9 cm^{-3} and temperature from 5 to 20 eV was produced with the aid of a plasmatron. A second Langmuir probe was used to record to oscillations. Figure 5 shows the results of measurements of the damping (ratio of the imaginary part of the wave number k_i to the real part k_r) as a function of the ratio of the phase velocity $v_{ph} = \omega/k$ to the average thermal velocity $v_e = \sqrt{2T/m}$. Since k_i is proportional to the damping decrement, the quantity k_i/k_r should decrease exponentially with $(\omega/kv_e)^2$.

As seen from Fig. 5, such a dependence does indeed take place, and the experimental points fit well the theoretical curve.

In order to verify that this damping is actually connected with the resonant electrons, the authors of[9] changed the potential of the end electrode and by the same token "cut off the tail" of the Maxwellian distribution, i.e., they eliminated the fastest electrons. As soon as the cut off boundary reached a certain definite value v_*, a sharp decrease of the damping was observed. Figure 6 shows the dependence of the electron velocity v_* at which a jump of damping was observed, on the phase velocity of the wave $v_{ph} = \omega/k$. This plot demonstrates convincingly that the damping is indeed connected with the resonant electrons.

In subsequent experiments by Malmberg and Wharton[10], Van-Hoven[11], and Derfler and Simonen[12,13], a more detailed investigation was made of the dispersion relation for plasma waves. Figure 7 shows the results of Derfler and Simonen for the dependence of the oscillation frequency ω and of the damping (imaginary part of the wave number k_i) on k_r, which are compared with the exact dispersion relation $\epsilon = 0$ (see (2.14)) at arbitrary values of ka. (The plot in Fig. 7 should be regarded more like a dependence of k_r and k_a on ω.) We see that the experimental points fit the theoretical curves quite well. Thus, not only the magnitude of the damping, but also the complete dispersion relation $\epsilon(k, \omega) = 0$ can be regarded at present as reliably established experimental facts.

Quite recently, Malmberg and Wharton[14] investigated the damping of waves of finite amplitude. Their results are shown in Figs. 8 and 9. Figure 8 shows the experimentally measured dependence of the oscillation amplitude on the distance between the emitting probe and the receiving probe. The amplitude is given in relative units. Curve 1 pertains to a small-amplitude wave. If we disregard the region of distances smaller than 5 cm, where a plane wave has not yet been established, we see that the wave is monotonically damped in exponential fashion. With increasing wave amplitude, oscillations appear on the dependence of the amplitude

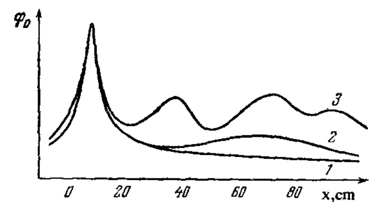

FIG. 8. Damping of waves of large amplitude. 1 – 0.9 V, 2 – 2.85 V, 3 – 9 V of the alternating field on the wave-emitting and probe unit.

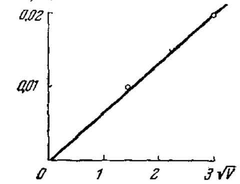

FIG. 9. Dependence of the wave number k of the wave-amplitude oscillations on the potential V on the probe-emitter unit.

on the distance, with a period that decreases with increasing amplitude. Figure 9 shows the dependence of the wave number k of the oscillations of the amplitude as a function of the potential V applied to the emitting probe.

We see that this dependence agrees with the theoretical one (see (4.1)), according to which the frequency of the oscillations Ω, and consequently also k, should increase like the square root of the amplitude.

The Landau damping, of course, not only pertains to the Langmuir waves, but also plays an essential role in many other collective processes in the plasma. In particular, Landau damping by ions determines the possibility or impossibility of propagation of sound, more accurately ion sound, in a plasma. If the ion temperature T_i is small compared with the electron temperature T_e, then there can propagate in the plasma sound waves with phase velocity $v_{ph} = c_s = \sqrt{T_e/M}$, where M is the ion mass. With increasing temperature T_i, the number of resonant ions increases, leading to a damping of the waves, and when $T_i \sim T_e$, when the mean thermal velocity of the ions $v_i = \sqrt{2T_i/M}$ becomes of the order of c_s, the propagation of ion sound waves becomes completely impossible. This conclusion was confirmed experimentally[8].

It pertains to a certain degree also to nonlinear phenomena. In[15], for example, a study was made of the question of propagation of nonlinear waves in a nonisothermal plasma. The experiment was carried out in an alkali plasma, which was produced by ionizing cesium vapor on a heated plate. By varying the pressure of the neutral gas in the chamber, the authors were able to vary the ion temperature T_i in a sufficiently wide range, so that it was possible to go over continuously from a strongly non-isothermal plasma ($T_i \ll T_e$) to an isothermal one ($T_i = T_e$).

By placing a charged negative grid absorbing an appreciable fraction of the ions incident on it in front of the heated plate on which the plasma was produced, the authors were able to produce an initial density drop of sufficiently large magnitude. When the grid potential was removed, a wave of increased density began to propagate. As seen from Fig. 10, in an isothermal plasma this wave smears out rapidly as a result of the "spreading" of the hot ions, connected with the Landau damping. To the contrary, in a nonisothermal plasma (Fig. 11), the front becomes steeper and a discontinuity in the density takes place, as it should in the propagation of a nonlinear acoustic (simple) wave. This experiment shows that the Landau damping by ions at $T_i = T_e$ is effective for waves of not only small but also finite amplitudes.

Thus, the Landau-damping effect can be regarded by now as sufficiently well confirmed experimentally. There is still the question whether it is possible to check experimentally that the Landau damping does not lead directly to irreversibility, and that a collisionless plasma retains the "memory" of the damped oscillations. Such a verification can be made by experiments with echoes.

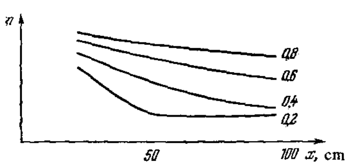

FIG. 10. Splitting of the inhomogeneities of the density in an isothermal plasma. The numbers next to the curves denote the time in microseconds.

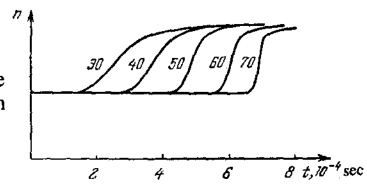

FIG. 11. Formation of a shock wave in a non-isothermal plasma. The numbers next to the curves indicate the distance from the grid producing the initial density drop.

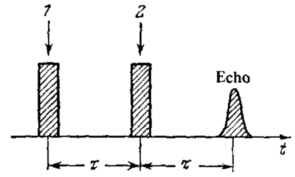

FIG. 12. Spin echo.

6. SPIN ECHO

Let us recall first what an echo is. The echo effect was observed by Hahn in nuclear-magnetic resonance experiments. It has the following outward appearance. If two short pulses, separated by an interval τ (Fig. 12) are applied at a frequency close to resonance, then the second pulse is followed, after a time τ by a pulse corresponding to spontaneous emission of nuclear spins at the resonant frequency. This effect, which is called spin echo, was explained by Hahn himself[16].

Let us consider a system of nuclear spins in a strong magnetic field and let us assume that at the initial instant all the magnetic moments are directed along the field (z axis in Fig. 13). As is well known, application of resonant high frequency oscillations causes the spins to begin to deviate from the z axis. Let us assume for simplicity that the first pulse is a 90° pulse, i.e., its amplitude is chosen such that the spins are deflected 90° and go over to the x axis. After the termination of the pulse, precession of the magnetic moment should give rise to radiation at the resonant frequency. However, by virtue of the small inhomogeneity of the external magnetic field, this radiation stops rapidly. Since the precession frequency ω changes little from point to point, soon the phases of the different spins "diverge" and on the average they can be regarded as uniformly distributed over the sample (see Fig. 13a).

Let us assume now that at the instant $t = \tau$ there is applied a second identical high frequency pulse, which can be called a 180° pulse. This pulse turns the entire "fan" of spins through 90° (see Fig. 13b), and now the initial angle $\varphi = 0$ corresponds to an angle $\vartheta = \omega\tau$.

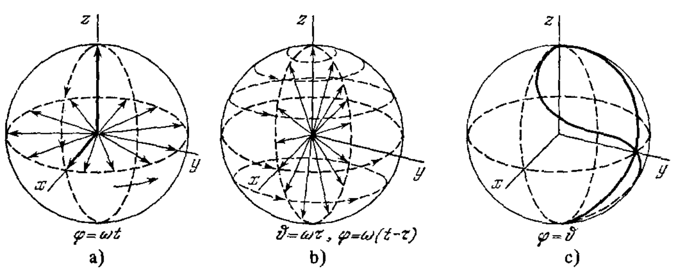

FIG. 13. Rotation and precession of spins in the presence of echo.

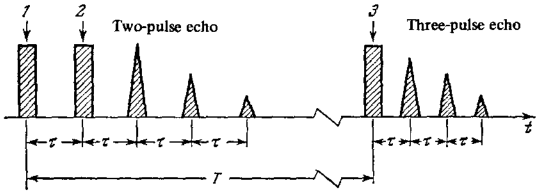

FIG. 14. Cyclotron echo.

During the subsequent instants of time the spins will process with respect to φ, each with its own frequency ω, and very soon they become distributed over the sphere of Fig. 13b. However, at a time interval τ following the second pulse, the spin will again produce a nonzero macroscopic magnetic moment. In fact, at that instant of time $\vartheta = \omega\tau$, and the phase for each spin is also equal to $\varphi = \omega\tau$, i.e., $\varphi = \vartheta$ for all spins, and consequently the ends of the spins will be uniformly distributed over the $\varphi = \vartheta$ curve, which has the form of a figure-eight (see Fig. 13c), where the angle ϑ should be regarded as varying from zero to 2π). The radiation at that instant of time is the echo. The echo effect was later observed in many physical objects (see the review[17]).

It is easy to see that the effect of vanishing of the macroscopic moment due to the scatter of the precession frequencies, connected with the inhomogeneity of the magnetic field, has much in common with Landau damping. In the former case, the magnetic moment vanishes as a result of the scatter of the phases in space, so that the resultant distribution of the magnetic moment over the volume is rapidly oscillating, while in the latter case the average electric field vanishes, owing to the velocity "smearing" of the resonant-particle beams. Naturally, one can expect an echo-type effect in plasma oscillations[18]. But let us first become acquainted with cyclotron echo in a plasma.

7. CYCLOTRON ECHO IN A PLASMA

The cyclotron echo in a plasma was observed by Hill and Kaplan[19]. The experiment was performed in the following manner[19-21]. Two high frequency-oscillation pulses separated by an interval τ were passed transversely to a column of a decaying plasma produced by a high frequency discharge at a time when there was practically no current in the plasma and the plasma was sufficiently quiescent. The plasma was located in a magnetic field of approximately 3 kG, and the generator frequency was close to the cyclotron frequency of the electrons. The amplitude of the high frequency pulses was sufficiently high, so that the electrons could acquire at cyclotron resonance an energy much higher than thermal. After the passage of the two pulses from the external generator, pulses from the plasma were observed at the cyclotron frequency, with interval τ between them (Fig. 14).

In addition of such a two-pulse echo, the authors observed a three-pulse echo when a third pulse was applied with a large time delay T, greatly exceeding the time of collision of the electrons with the neutral gas atoms, but shorter than the electron-energy relaxation time. As shown schematically in Fig. 14, a series of pulses from a plasma was again observed following the third pulse.

The cyclotron-echo mechanism and its main features was explained by Crawford and Houp[22] on the basis of an idea by Gould[23]. Cyclotron echo is fundamentally close to spin echo. Whereas in nuclear magnetic resonance the effect of a high frequency field leads to a rotation of the spins away from the z axis, in cyclotron resonance the energy is acquired by the electrons. If the duration of the pulse is not too large, then it contains a sufficiently large number of harmonics, so that even in the presence of a certain inhomogeneity with a magnetic field it is possible to assume that all the electrons (initially cold) acquire the same energy. Assume that in velocity space (v_x, v_y), i.e., in the plane perpendicular to the magnetic field, the position of the electrons immediately after the first pulse corresponds to the point $(v_p, 0)$ (Fig. 15a). In the succeeding instants of time the electrons will rotate with cyclotron frequency, and in the presence of a small magnetic-field inhomogeneity they can rapidly reach a homogeneous distribution with respect to the phases on the average over the sample.

Let us consider now a certain group of electrons A having the same phase for the second pulse (Fig. 15b). More accurately, this group includes all the electrons with phases $\xi + 2\pi n$, where n is an integer. During the time of the second pulse, the electrons acquire or lose energy, depending on their phase, and the action of the pulse can be taken into account by simply shifting the entire distribution function by the same amount v_p as in the first pulse. The group A of the electrons singled out by us will have in this case a velocity (in absolute magnitude) $2v_p \cos(\xi/2)$, as can be seen from Fig. 15c. Since this group includes electrons from different points of space, corresponding to different cyclotron frequency, the particles of this group will start to diverge in velocity space in the course of time. However, inasmuch as the electrons rotate with the same angular velocity as before, they will be again grouped at time intervals τ. For example, after a time $t = \tau$ following the second pulse, i.e., at the first echo signal, the particles of this group will gather at point A of Fig. 15d, with phase $3\xi/2$, and the entire electron velocity distribution will assume the form of the curve shown in Fig. 15d.

This means that the electrons executing cyclotron oscillations contain potentially the echo effect. Actually, however, the distribution of Fig. 15d will pro-

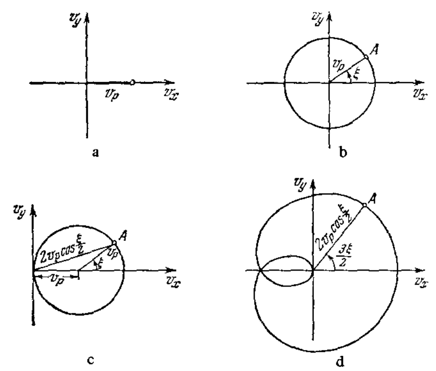

FIG. 15. Electron velocity distribution in the presence of a two-pulse echo. a) Immediately after pulse 1. b) Before pulse 2. c) Immediately after pulse 2. d) At the instant of the first echo.

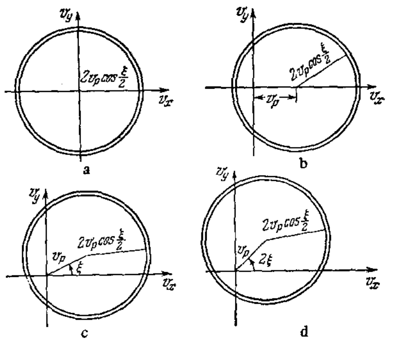

FIG. 16. Electron velocity distribution in the presence of a three-pulse echo. a) Ahead of pulse 3. b) Immediately after pulse 3; c) At the instant of the first echo; d) at the instant of the second echo.

duce no echo whatever, for it is easy to verify that at such a distribution the total current, which is expressed in terms of the mean values $\langle v_x \rangle$ and $\langle v_y \rangle$ vanishes. This is not surprising, after all, the system considered by us is linear—we have simply added up effects from two pulses. The echo, on the other hand, is an essentially nonlinear effect: in this case the total signal is not a superposition of responses to each pulse, but is determined by two pulses simultaneously.

In order for cyclotron echo actually to appear, there should come into play some nonlinear mechanism violating somehow the picture of Fig. 15d, and leading to the absence of exact cancellation of all the currents. Such nonlinear mechanism may be, for example, the dependence of the mass on the velocity (relativistic effect), the nonlinearity of the wave, etc. However, the simplest and natural mechanism is simply a velocity-dependent collision frequency $\nu(v)$. The collisions between the electrons and the atoms of the neutral gas, or between the electrons themselves causes the fraction of the electrons to be knocked out from the coherent motion. If the frequency of the collisions depends on the velocity, then the number of the electrons knocked out from the curve of Fig. 15d will differ at different points, and consequently, a macroscopic current at the cyclotron frequency will appear, i.e., echo is produced. The experimental data are in good agreement with the mechanism[21,24]. This mechanism explains also the three-pulse echo.

As already noted above, for a three-pulse echo, the electrons have time to lose completely the directional pulse by the instant of the third pulse but still do not have time to lose the energy (in elastic collisions with the atoms of a neutral gas, the energy relaxation time is larger by a factor M/m than the momentum relaxation time). This means that the electrons of group A of Fig. 15 will be distributed before the third pulse uniformly on a sphere of radius $2v_p\cos(\xi/2)$ (Fig. 16a). Immediately after the third pulse, this sphere turns out to be shifted by an amount v_p (Fig. 16b), and then the electrons of this group "spread out" in phase. However, bunching will take place in velocity space at time intervals τ, with a phase shift ξ from pulse to pulse (Figs. 16c and d). But then again, in order for the total current to be different to zero, there should be an effective mechanism causing the collision frequency to depend on the velocity. The effect itself, as can be readily seen, also reaches a maximum when the collision frequency becomes of the order of the reciprocal time between the pulses τ^{-1}. These conclusions are also in satisfactory agreement with the experimental data.

Cyclotron echo is of great interest in itself, being a new nonlinear effect in a plasma, and furthermore it can be used for diagnostic purposes, especially for the investigation of relaxation processes in a plasma.

8. PLASMA-WAVE ECHO

We now turn to plasma waves. As shown by Malmberg and Wharton[14], an echo effect can appear in them, too. Let us consider first the simplest case of modulated Van Kampen beams, propagating in the form of plane waves along the x axis. Such waves can be excited in a plasma with the aid of a grid, on which a periodic signal is applied. Let the frequency of such a signal be much larger than the plasma frequency ω_0. Then the dielectric constant of the plasma can be regarded as equal to unity, i.e., the polarization of the medium in the waves can be neglected. Under such conditions, the perturbation of the distribution function of the electrons passing through the grid is simply equal to

$$f_1(x,\ v,\ t) = f_1(v)\exp\left(-i\omega t + i\omega\frac{x}{v}\right), \tag{8.1}$$

where $f_1(v)$ is the amplitude of the perturbation of the distribution function near the grid. The form of the perturbation (8.1) follows from the fact that, far from the grid, $f_1(v)$ should satisfy the equation of free motion of the particles

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} = 0. \tag{8.2}$$

The distribution function (8.1) can be regarded as a set of modulated beams. Near the grid, all these beams oscillate in phase, but as the distance from the grid increases, the phases of the beams with different v begin to differ strongly, and therefore the charge density

$$\rho = -e\int f_1\,dv \tag{8.3}$$

should decrease rapidly with decreasing x (f_1 becomes a rapidly-oscillating function of v). This means that the oscillations of the electric potential should decrease rapidly with increasing distance from the grid, and this effect is fully analogous to dephasing of the magnetic moments before the second pulse when spin-echo is observed.

Let us assume now that at a distance d from the first grid there is located a second grid, to which an alternating potential with frequency ω', also larger than ω_0, is applied. Van Kampen waves resulting from the perturbation of f_0 also travel from the second grid. But in addition, the second grid will modulate the function (8.1), so that a nonlinear response f_2 will appear at the combination frequency:

$$f_2(x,\ v,\ t) = f_2(v)\exp\left\{-i\omega\left(t-\frac{x}{v}\right) \pm i\omega'\left(t-\frac{x-d}{v}\right)\right\}. \tag{8.4}$$

When $x = \omega' d/(\omega' - \omega)$ the exponential corresponding to the frequency $\omega'' = \omega - \omega'$ ceases to depend on v. This means that at the point

$$x = \frac{\omega'}{\omega'-\omega}d = d + \frac{\omega}{\omega'-\omega}d \tag{8.5}$$

there should be observed noticeable fluctuations of the charge density at the frequency $\omega'' = \omega - \omega'$. In other words, an electric probe placed at this point should detect an echo at combination frequency.

A similar analysis can be presented also for the more general case of plasma oscillations[14]. It would be most interesting, of course, to observe echo in the case when one of the frequencies, ω, ω', or ω'' is close to the plasma frequency. Experiments with echo were carried out most recently in both electronic and ionic plasma waves[26,27]; in particular, relation (8.5) for the position of the echo-signal maximum was verified.

CONCLUSION

Twenty years have elapsed since the time when L. D. Landau has shown how to solve correctly the kinetic equation for small plasma oscillations at specified initial and boundary conditions. The new effect of collisionless damping of waves, observed by him, became an undisputed accomplishment of plasma physics. But the deep physical meaning and the value of the Landau damping for collective processes in a plasma have become clear only in related investigations, and mainly in recent years.

An analysis of the damping of nonlinear waves has shown that the linear stage of absorption of wave energy by the resonant electrons continues for only a limited time interval, and if the number of resonant particles is small, the amplitude of the wave decreases insignificantly. This investigation has demonstrated the limitations of the linear approach. A changeover to the investigation of nonlinear phenomena in the plasma has made it possible, in addition, to establish a new approach to plasma physics, which makes it possible to clarify the question of the reversibility of Landau damping. This approach is connected with the echo phenomena. The cyclotron echo, which was observed not so recently, was the first step towards an investigation of this interesting field of nonlinear phenomena. It should be assumed that in the nearest time experiments will be performed on the echo effect and on the associated collective phenomena for a wider class of plasma oscillations.

Summarizing, we can state that at the present time Landau damping is a fundamental effect that has been reliably confirmed experimentally. Landau damping and the associated set of physical phenomena produce the ground work for the understanding of many collective processes in a plasma.

[1] L. D. Landau, Zh. Eksp. Teor. Fiz. 16, 574 (1946) or Usp. Fiz. Nauk 93, 527 (1967).

[2] A. A. Vlasov, Zh. Eksp. Teor. Fiz. 8, 291 (1938) or Usp. Fiz. Nauk 93, 444 (1967) [Sov. Phys. Usp. 10, 721 (1968)].

[3] A. A. Vlasov, J. Phys. (USSR) 9, 25 (1945).

[4] N. S. Van Kampen, Physica 21, 949 (1955).

[5] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nuclear Fusion, Supplement, Part 3, 1049 (1962).

[6] L. M. Al'tshul' and V. I. Karpman, Zh. Eksp. Teor. Fiz. 49, 515 (1965) [Sov. Phys.-JETP 22, 361 (1966)].

[7] T. M. O'Neil, Phys. Fluids 8, 2255 (1965).

[8] A. Y. Wong, N. D'Angelo, and R. W. Motley, Phys. Rev. A133, 436 (1964).

[9] J. H. Malmberg, C. B. Wharton, and W. E. Drummond, Plasma Physics and Controlled Nuclear Fusion Research, vol. 1, Vienna, 1966, p. 485.

[10] J. H. Malmberg, and C. B. Wharton, Phys. Rev. Lett. 17, 175 (1966).

[11] G. Van-Hoven, Phys. Rev. Lett. 17, 169 (1966).

[12] H. Derfler, and T. C. Simonen, Phys. Rev. Lett. 17, 172 (1966).

[13] H. Derfler, and T. C. Simonen, 8th Intern. Conf. on Phenomena in Ionized Gases, Contributed Papers, Vienna, 1967, p. 335.

[14] J. H. Malmberg, and C. B. Wharton, Phys. Rev. Lett. 19, 775 (1967).

[15] H. K. Andersen, N. D'Angelo, P. Michelson, and P. Nielsen, Phys. Rev. Lett. 19, 149 (1967).

[16] E. L. Hahn, Phys. Rev. 80, 580 (1950).

[17] A. N. Oraevskii, Usp. Fiz. Nauk 91, 181 (1967) [Sov. Phys.-Usp. 10, 45 (1967)].

[18] R. W. Gould, T. M. O'Neil, and J. H. Malmberg, Phys. Rev. Lett. 19, 219 (1967).

[19] R. M. Hill, and D. E. Kaplan, Phys. Rev. Lett. 14, 1061 (1965).

[20] D. E. Kaplan, R. M. Hill, and A. Y. Wong, Phys. Lett. 22, 585 (1966).

[21] D. E. Kaplan, R. M. Hill, and G. F. Herrmann, cm[13], p. 424, 1967.

[22] R. F. Crowford, and R. S. Houp, J. Appl. Phys. 37, 4405 (1966).

[23] R. W. Gould, Phys. Lett. 19, 477 (1965).

[24] R. M. Hill and D. E. Kaplan, Phys. Rev. 156, 118 (1967).

[25] D. Bohm, and E. P. Gross, Phys. Rev. 75, 1851, 1864 (1949).

[26] J. H. Malmberg, C. B. Wharton, R. W. Gould, and T. M. O'Neil, Phys. Rev. Lett. 20, 95 (1968).

[27] H. Ikezi, and N. Takahashi, Phys. Rev. Lett. 20, 140 (1968).

Translated by J. G. Adashko