

## HYDRODYNAMIC EFFECTS IN SOLIDS AT LOW TEMPERATURE

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## Introduction

I. Kinetic Phenomena in Dielectrics . . . . .	256
1. Statement of the Problem . . . . .	256
2. Hydrodynamic Mechanism of Thermal Conductivity . . . . .	257
3. Influence of Higher-order Anharmonicity on Transfer Processes in Solids at Low Temperatures . . . . .	259
4. Second Sound in Dielectrics . . . . .	260
II. Kinetic Phenomena in Metals	
5. Electrical Conductivity of Metals at Low Temperatures . . . . .	262
6. Electrical Conductivity of Thin Samples . . . . .	264
7. Electrical Conductivity of Bulk Samples . . . . .	266
8. High-frequency Properties . . . . .	268
References . . . . .	269

## INTRODUCTION

ONE of the most important achievements of the quantum theory of solids was the explanation of the basic regularities of the electrical conductivity of metals and of the thermal conductivity of dielectrics. As for the whole of the physics of the solid state, the main role in this was played by the fundamental papers of Bloch and Peierls. The results obtained in that period were found to be in good agreement with the available experimental data (at least qualitatively). It is probable that this was just the reason why the further more detailed development of the general aspects of the theory was relatively slow. Recently the direction of investigations has changed from the explanation of general laws to a study of the properties of actual substances, taking their particular properties into account. In the meantime, up to the present, there remain in the theory of transfer processes a whole number of unexplained or badly based problems of a general nature. The problem of longitudinal long-wavelength phonons in the consideration of the thermal conductivity of dielectrics, or the problem of the electrical conductivity of a number of pure metals in the low-temperature region, are examples of such unexplained problems.

Another problem of general character is connected with taking the normal collisions, i.e., collisions where the total quasi-momentum of the quasi-particles is conserved, into account. We are dealing here with the fact that the phenomena of thermal conductivity of dielectrics and the electrical conductivity of metals have specific properties: in both cases the total quasi-particle current turns out to be non-vanishing. (We note that, e.g., the heat transfer by molecules of a normal gas or by the electrons in a metal these peculiar circumstances do not occur.) Hence it follows that when only normal collisions occur in the system there could exist

an undamped (electrical or thermal) current in the absence of an external field which could sustain it. This situation can be made to look more obvious by noting that the normal collisions lead only to internal equilibrium in the quasi-particle system which can as a whole move in relation to the crystal lattice with an arbitrary velocity. In other words, the thermal resistance of dielectrics or the electrical resistance of metals occurs when we take into account processes where the quasi-momentum is not conserved (Umklapp processes, scattering by defects in the crystal lattice, etc.). Because of this the problem of the normal collisions is usually not taken into account for a quantitative consideration.

It is natural to expect that such a procedure leads to a qualitatively correct result for the cases when the probabilities for normal collisions and for collisions with loss of quasi-momentum are of the same order of magnitude. However, in sufficiently pure and large samples at low temperatures the normal collisions may dominate. Such a position is typical for dielectrics, since in that case the probability for Umklapp processes in collisions between quasi-particles is exponentially small. The same is also true for a number of metals—it is sufficient, for example that the Fermi surface is closed and that the number of electrons is not equal to that of holes. It is clear that under such conditions the usually applied method of discussion is unsuitable.

The present survey is devoted to the development of a theory of transfer processes for the case where normal collisions between quasi-particles have the largest probability. It turns out that although normal collisions by themselves do not lead to a finite resistance, they can all the same appreciably influence the result of the action of other scattering processes. As a result the order of magnitude of the kinetic coefficients and their dependence on temperature and other param-

eters is changed. The nature of the influence of the normal collisions depends essentially both on the kind of quasi-particles (electrons, phonons, spin waves) and on the nature of the mechanisms which lead to a loss of quasi-momentum (Umklapp processes, collisions with impurity atoms, with the boundaries of the sample, etc.).

From a formal point of view the dominance of normal collisions shows that the state of the system of quasi-particles is in first approximation characterized by an equilibrium function with drift  $f_0(\epsilon - \mathbf{p} \cdot \mathbf{u})$ . The problem thus consists in determining the velocity of the ordered motion  $\mathbf{u}$  as a function of the coordinates, time, temperature, and other parameters. In certain cases the problem then reduces to solving the hydrodynamical Navier-Stokes equations (Secs. 2, 4, 6, 8) and in other cases the quantity  $\mathbf{u}$  can be determined directly from the kinetic equations (Secs. 3 and 7).

## I. KINETIC PHENOMENA IN DIELECTRICS

### 1. Statement of the Problem

In a perfect single crystal of infinite dimensions thermal resistivity occurs only when we take into account those collisions between quasi-particles which are accompanied by Umklapp processes. In the low temperature region the corresponding mean free path  $l^U$  increases exponentially when the temperature is lowered:  $l^U \sim \exp(\gamma\Theta/T)$  where  $\Theta$  is the Debye temperature,  $T$  the absolute temperature, and  $\gamma$  a numerical constant of order unity. It is clear that when the temperature is lowered for a sufficiently pure sample the main role begins to be played by collisions with the boundaries and the effective mean free path  $l^{\text{eff}}$  will turn out to be of the order of the dimensions  $d$  of the cross section of the sample.

For a rough estimate of the thermal conductivity we can use the following formula known from the kinetic theory of gases:

$$\kappa \approx Cl^{\text{eff}} v_T, \quad (1.1)$$

where  $C$  is the specific heat and  $v_T$  the average velocity of the quasi-particles.

In the very low temperature region  $l^{\text{eff}} \approx d$ , and  $\kappa$  increases with increasing temperature. At very high temperatures (but  $T \ll \Theta$ )  $\kappa$  decreases exponentially. The temperature dependence of the thermal conductivity has thus a bell-shaped form (see Fig. 1 below). If we disregard details, the qualitative considerations given here are experimentally confirmed.

However, doubt was cast by Pomeranchuk on the validity of these statements from a theoretical point of view.<sup>[1]</sup> He noted that if we restrict ourselves to three-phonon processes it follows from the appropriate conservation laws that the longitudinal long-wavelength phonons can collide only with phonons of comparable wavelength. As a result the probability for scattering so steeply tends to zero when the momenta of the longitudinal phonons is decreased that the thermal conductivity coefficient diverges.\* However, this difficulty turns out to be connected to an appreciable extent with

\*We note that in the case of spin waves which have a quadratic dispersion law such a difficulty does not arise.

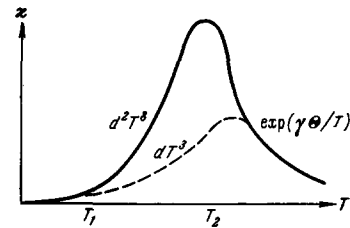


FIG. 1.

Pomeranchuk's assumption of isotropic phonon spectra. Herring<sup>[2]</sup> has shown that if we take the anisotropy and the degeneracy of the phonon spectra into account, for many crystals the divergence either is not present or leads to size effects which for real samples are of little importance. (Simons<sup>[3]</sup> has recently proposed an interesting solution of the problem.)

Another difficulty of the theory of the thermal conductivity of dielectrics is connected with the fact that the kinetic equation in this case cannot be solved. To obtain qualitative results it is usual to introduce in one way or another a relaxation time (see, e.g.,<sup>[1]</sup>). The normal collisions between quasi-particles (N-processes) which are not accompanied by Umklapp processes are as a rule not taken into account. Such a procedure is possibly reasonable at not too low temperatures when the probabilities for normal collisions and for collisions with Umklapp (U-processes) are of the same order of magnitude (i.e.,  $l^N \sim l^U$ ) since N-processes by themselves do not lead to thermal resistivity. However, in the region of rather low temperatures where  $l^N \ll l^U$  N-processes may be very important. However, just in that case it is possible to solve the kinetic equation consistently starting essentially from an expansion in the small parameter  $l^N/l^U$  (see, e.g.,<sup>[4]</sup>).

Under actual conditions the thermal resistivity is connected both with Umklapp processes and with scattering of quasi-particles by defects in the crystal lattice and by the boundaries of the sample. Calculations show that taking N-processes into account does not lead to a qualitative change in the results connected with volume collisions at low temperatures (U-processes in a system of spin waves and phonons,<sup>[5]</sup> scattering by impurities: phonons,<sup>[6]</sup> spin waves<sup>[7]</sup>).

The situation may be different in the case where the thermal conductivity is due to the scattering of quasi-particles by the sample boundaries or by some macroscopic lattice defects. The mechanism for thermal conductivity occurring then reminds us of the Poiseuille flow of a viscous liquid. As a result, the order of magnitude of the thermal conductivity coefficient changes and also its dependence on temperature and sample thickness.<sup>[8, 9]</sup> This problem is considered in Sec. 2.

The next section is devoted to the role of Umklapp processes in collisions between quasi-particles, of course, under conditions when normal collisions dominate.

One usually assumes that at low temperatures those collisions in which the smallest number of quasi-particles takes place have the largest probability. From a formal point of view this corresponds to taking into account only the first non-vanishing anharmonicity in the

corresponding interaction Hamiltonian. However, we shall show below that transfer phenomena in solids at sufficiently low temperatures can be determined by anharmonic terms of arbitrarily large order. For this it is necessary that the quasi-particle energy as function of the modulus of the quasi-momentum changes faster than linearly.<sup>[10]</sup> As an example we consider the spin-wave-connected thermal conductivity of a ferroelectric.

Under conditions when normal collisions dominate, macroscopic oscillations may propagate in the system of quasi-particles which are similar to second sound in liquid helium. We consider some features of this phenomenon in dielectrics and ferroelectrics in Sec. 4.

## 2. Hydrodynamic Mechanism of Thermal Conductivity

1. At sufficiently low temperatures, the cause of thermal conductivity of dielectrics is usually the scattering of quasi-particles at the boundaries of the sample. Indeed, when the temperature is lowered the volume collisions become less and less effective. It is clear that in the final analysis, a Knudsen situation arises, i.e., the quasi-particles will move from boundary to boundary without colliding in the volume.

The practical influence of boundaries may turn out to be important under conditions when the mean free path for Umklapp collisions  $l^U$  is comparable to the diameter of the sample  $d$ . On the other hand, at low temperatures normal collisions are appreciably more probable than Umklapp collisions. For a sufficiently large sample, therefore, a situation occurs where one can neglect Umklapp processes while normal collisions still occur more frequently than collisions with the boundaries. For this it is necessary that for some range of temperatures the following inequality hold:

$$l^N \ll d \ll l^U. \quad (2.1)$$

The result obtained when we take normal collisions into account can be easily understood from the following simple considerations. When  $d \ll l^N$  a quasi-particle moving rectilinearly traverses a path of order  $d$  between two collisions with the boundaries. Correspondingly the effective mean free path  $l^{\text{eff}} \approx d$ , and according to Eq. (1.1) the coefficient of thermal conductivity  $\kappa \sim C(T)d v_T$ .

However, when  $d \gg l^N$  a quasi-particle inside the sample will, on the other hand, undergo many normal collisions before it reaches the boundary. It is clear that as a result the path traversed between two collisions with the boundaries will be appreciably increased. The fact that for normal collisions there is an exchange of quasi-momentum with other quasi-particles is clearly unimportant.

Using the well-known formulae of Brownian motion one shows easily that the length of the trajectory between two collisions with the boundaries will be of the order of  $d^2/l^N$ . Since this quantity has the meaning of an effective mean free path, we have according to (1.1)

$$\kappa \approx \frac{C(T) v_T d^2}{l^N(T)}.$$

The order of magnitude of the thermal conductivity coefficient and its dependence on the parameters  $T$ ,  $d$ ,

and so on is thus changed appreciably thanks to the normal collisions.

The mechanism of thermal conductivity considered here is in no way different from the flow of a viscous liquid or gas (the temperature gradient plays the role of the external force). For a quantitative description of this mechanism it is thus natural to use a hydrodynamic approach. We are dealing here with the fact that the normal collisions lead only to internal equilibrium in each volume element (large compared to  $l^N$ ) which can, for instance, move as a whole with an arbitrary velocity  $\mathbf{u}$ . It is thus clear a priori that in the conditions of interest to us the distribution function will have the form

$$N_0(\varepsilon - \mathbf{p}\mathbf{u}) \equiv \left[ \exp\left(\frac{\varepsilon - \mathbf{p}\mathbf{u}}{T}\right) - 1 \right]^{-1},$$

where  $\varepsilon(\mathbf{p})$  is the quasi-particle energy and  $\mathbf{p}$  its quasi-momentum.

The velocity of the ordered motion must as a function of the coordinates,  $\mathbf{u}(\mathbf{r})$ , satisfy a hydrodynamic equation of the Navier-Stokes type. In the following such an equation is derived which afterwards will be applied to a consideration of the thermal conductivity of dielectrics and ferrites.

2. We can obtain the hydrodynamic equation, starting from the kinetic equation in a manner analogous to what is done for normal gases. In the case of interest to us the linearized kinetic equation for the quasi-particle distribution function  $N(\mathbf{p}, \mathbf{r})$  can conveniently be written in the form

$$\mathbf{v} \frac{\partial}{\partial \mathbf{r}} N + (\mathbf{v}\nabla T) \frac{\partial}{\partial T} N_0(\varepsilon) = \hat{I}^N N + \hat{J}^V N,$$

where  $\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}$ ; the first term on the right-hand side describes  $N$ -processes and the second one collisions accompanied by a loss of quasi-momentum (U-processes, scattering by impurity atoms, etc.).

To solve this equation approximately it is natural to use the fact that the term  $\hat{I}^N N$  is large compared with the terms  $\mathbf{v} \partial/\partial\mathbf{r} N$  and  $\hat{J}^V N$ . More exactly, we shall write the solution as a series expansion in the small parameters  $l^N/d$  and  $l^N/l^V$ . One can easily show that the method of successive approximations leads to the following set of equations:

$$\begin{aligned} \hat{I}^N N^{(0)} &= 0, \quad \mathbf{v} \frac{\partial}{\partial \mathbf{r}} N^{(0)} = \hat{J}^V N^{(1)}, \\ \mathbf{v} \frac{\partial}{\partial \mathbf{r}} N^{(1)} + (\mathbf{v}\nabla T) \frac{\partial}{\partial T} N_0(\varepsilon) - \hat{J}^V N^{(0)} &= \hat{I}^N N^{(2)}, \dots, \\ N &= N^{(0)} + N^{(1)} + N^{(2)} + \dots \end{aligned}$$

From the first equation follows that  $N^0 = N_0(\varepsilon - \mathbf{p} \cdot \mathbf{u})$ . The second equation which now becomes

$$-v_i F_k \frac{\partial u_i}{\partial x_k} N_0'(\varepsilon) = \hat{I}^N N^{(1)},$$

can in general not be solved. One can only state that  $N^{(1)}$  contains  $\partial u_i/\partial x_k$  linearly.

Since we are interested in the very low temperature region and since in the equations considered clearly only energies  $\varepsilon \lesssim T$  are important we may assume that the dispersion law for the quasi-particles is a power-law:  $\varepsilon(\mathbf{p}) = p^{\text{df}}(n)$ ,  $n = \mathbf{p}/p$ . We can then easily prove that

$$N^{(1)} = A \psi_{ik} \left( \frac{\varepsilon}{T}, n \right) \frac{\partial u_i}{\partial x_k}.$$

The functions  $\psi_{ik}$  of dimensionless variables satisfy equations which do not contain small parameters; in the

case of an isotropic dispersion law  $\psi_{\mathbf{ik}} = n_{\mathbf{i}} n_{\mathbf{k}} \psi(\varepsilon/T)$ . The temperature-dependent constant  $A$  can easily be found in each actual case.

We multiply further the third equation by  $\mathbf{p}$  and integrate. Since the integral of the right-hand side vanishes exactly, as a result we get the equation for  $\mathbf{u}(\mathbf{r})$  we are looking for. In the isotropic case it is convenient to write this equation in the form\*

$$\alpha \nabla T = \nu \Delta \mathbf{u} - \frac{\mathbf{u}}{\tau^V}, \quad (2.2)$$

where

$$\begin{aligned} \alpha &\approx \beta T^{-1} \int p v \varepsilon N'_0(\varepsilon) dp, \quad \nu = v_T l^N \approx \beta A \int p v \psi dp, \\ \frac{\mathbf{u}}{\tau^V} &\approx \beta \int \mathbf{p} l^V [(\mathbf{u} \cdot \mathbf{p}) N'_0(\varepsilon)] dp, \quad \beta^{-1} = \int p^2 N'_0(\varepsilon) dp. \end{aligned} \quad (2.3)$$

By analogy with a normal gas the kinematic viscosity  $\nu$  is written as the product of the average quasi-particle thermal velocity  $v_T$  and the quantity  $l^N$  which has the meaning of a mean free path for N-processes. We must add to this equation the boundary condition  $\mathbf{u} = 0$  at the boundaries of the sample which corresponds to diffuse scattering.

It is clear from the derivation that for an anisotropic dispersion law the hydrodynamic equation retains the same structure. However, all coefficients will now, of course, be tensors. For instance, the term  $\nu \nabla^2 \mathbf{u}_i$  takes the form  $\nu_{\mathbf{ik}lm} \partial^2 \mathbf{u}_k / \partial x_l \partial x_m$ ; however, it is important that all  $\nu_{\mathbf{ik}lm} \sim \nu$ . We shall therefore start in the following from Eq. (2.2), claiming only correct orders of magnitude for the various quantities and their dependence on the main parameters.

The thermal current density

$$\mathbf{Q} = \frac{1}{h^3} \int v \varepsilon N_0(\varepsilon - \mathbf{p} \cdot \mathbf{u}) dp \approx -\frac{\mathbf{u}}{3h^3} \int p v \varepsilon N'_0(\varepsilon) dp$$

depends on the coordinates and to find the thermal conductivity coefficient  $\kappa$  we must average the vector  $\mathbf{Q}$  over the cross-section of the sample:  $\bar{\mathbf{Q}} = -\kappa \nabla T$ .

We give as an example the result of solving Eq. (2.2) for two simple cases: a plate and a circular cylinder.

We write the thermal conductivity coefficient in the form  $\kappa = C l^{\text{eff}} v_T$  where  $C = -h^{-3} T^{-1} \int \varepsilon^2 N'_0(\varepsilon) dp$  is the specific heat. We then get for a plate

$$l^{\text{eff}} \approx l^V (1 - z^{-1} \text{th } z)$$

and for a cylinder

$$l^{\text{eff}} \approx l^V \left[ 1 - \frac{I_1(z)}{z I_0(z)} \right].$$

Here  $l^V = \tau^V v_T$ ,  $z = d/2 \sqrt{l^N l^V}$ ,  $I_1$  and  $I_0$  are Bessel functions of an imaginary argument,  $d$  is the thickness of the plate or the diameter of the cylinder.

In the limiting cases both formulae lead apart from an unimportant numerical factor to the same result: when  $z \gg 1$ ,  $l^{\text{eff}} = l^V$ , while for  $z \ll 1$ ,  $l^{\text{eff}} \approx l^V z^2 \approx d^2/l^N$ . This is natural as  $z^2 \approx d^2/l^N l^V$  while we have seen that the quantity  $d^2/l^N$  has the meaning of an ef-

fective mean free path for collisions with the boundaries.

It is clear that the small difference between the results for a plate and for a cylinder in the hydrodynamic region is not accidental. We shall understand by  $d$  in the following a characteristic cross-sectional length of the sample (or the distance between macroscopic deformations of the lattice).

3. Let us now turn to concrete results for the phonon thermal conductivity of dielectrics. At low temperatures the phonons interact, as is well known, through ternary collisions. Starting from the appropriate collision integral (see, e.g., [4]) and Eqs. (2.3) one can show that, in order of magnitude

$$l_{pp}^N(T) \approx l_{pp}^N(\Theta) \left( \frac{\Theta}{T} \right)^5, \quad (2.4)$$

where  $l_{pp}^N(\Theta) \approx a M s^2 / \Theta$ ,  $a$  the lattice constant,  $M$  the mass of an atom,  $s$  the sound velocity, and  $\Theta \approx h s / a$  the Debye temperature.

For ternary U-processes one can show easily, by a method similar to the one used in [5], that

$$l_{pp}^U(T) = s v_{pp}^U(T) \approx l_{pp}^U(\Theta) \left( \frac{T}{\Theta} \right)^{5/2} \exp\left(\gamma \frac{\Theta}{T}\right). \quad (2.5)$$

The numerical value of the parameter  $\gamma$  depends on the behavior of the phonon spectrum at the limits of the whole Brillouin zone.\* It is interesting that otherwise the results obtained, including the multiplying factor  $(T/\Theta)^{5/2}$ , do not depend on the nature of the dispersion law.

We have given in Fig. 1 the typical temperature dependence of the thermal conductivity coefficient of a dielectric. In the very low temperature region it is clear that while  $l_{pp}^N \gg d$ ,  $l^{\text{eff}} \approx d$  and  $\kappa \approx C_p d s \sim d T^3$ . Later, starting from a temperature  $T_1$  for which  $l_{pp}^N \approx d$  up to a temperature  $T_2$  for which  $d^2/l_{pp}^N \approx l^V$ , the diffusion mechanism plays the main role and

$$\kappa \approx C_p s \frac{d^2}{l_{pp}^N} \sim d^2 T^8.$$

Further on, the usual decrease of the thermal conductivity starts either exponentially (if U-processes dominate) or as a power law (if scattering by impurities plays the main role). For comparison we have also drawn (dashed curve) in Fig. 1 the usual temperature dependence  $\kappa(T)$ .

It follows from inequality (21) that the diffusion mechanism can occur only in a sufficiently large dielectric sample. It is difficult to obtain any reliable estimate for the thickness  $d$  but, it is apparently sufficient that  $d$  is of the order of 1 cm.

We note that the behavior given above was recently confirmed experimentally in a study of the thermal conductivity of crystalline  $^4\text{He}$ .<sup>[11]</sup> For the thickest samples studied ( $d \approx 2.5$  mm) the mean free path in the maximum of the  $\kappa(T)$  curve turned out to be almost of the order of the large thickness  $d$  of the sample decreasing to the left of the maximum according to approximately a  $T^8$  law.

\*The equation obtained is valid under the additional condition  $\text{div } \mathbf{u} = 0$  which follows from the equation for  $N^{(1)}$  (one shows easily that

$$\int \varepsilon v_{\mathbf{i}p} N'_0(\varepsilon) dp \sim \delta_{\mathbf{i}k}$$

for any dispersion law) and which is equivalent to the absence of volume heat sources.

\*We denote by  $\gamma\Theta$  the smallest possible value of the sum of the energies of the colliding phonons allowed by the energy and momentum conservation laws (see Sec. 3); in the case of a linear isotropic dispersion law  $\gamma = 1$ .

4. At low temperatures the heat transfer in ferrites occurs predominantly through spin waves. This is connected with the fact that the specific heat of the spin waves decreases with a lowering of the temperature appreciably more slowly than the phonon specific heat (as to order of magnitude  $C_D \approx (1/a^3)(T/\Theta)^3$  and  $C_S \approx (1/a)^3 \times (T/\Theta_C)^{3/2}$ ,  $\Theta_C$  the Curie temperature). Moreover, normal collisions between spin waves are appreciably more probable than N-collisions of spin waves and phonons.<sup>[12]</sup> We may thus neglect the presence of phonons when considering the temperature region where the hydrodynamic mechanism of thermal conductivity is possible.

Collisions between spin waves are connected with exchange and relativistic interactions. The mean free paths for the corresponding N-processes are as to order of magnitude equal to<sup>[12]</sup>

$$l_{ss}^{(c)} \approx a \left( \frac{\Theta_C}{T} \right)^{7/2}, \quad l_{ss}^{(r)} \approx a \left( \frac{\Theta_C}{\mu_B M_0} \right)^2 \left[ 1 + \exp \left( \frac{\beta \mu_B M_0}{T} \right) \right],$$

where  $\mu_B$  is the Bohr magneton,  $M_0$  the nominal magnetization,  $\beta$  the anisotropy constant; we assume the external magnetic field to be weak ( $\mu_B H \ll T$ ).

From the expressions given here it follows that the hydrodynamic mechanism must occur in relatively thin ferrite specimens. In particular, it is sufficient that

$$d \gg a \left( \frac{\Theta_C}{\mu_B M_0} \right)^2 \sim 10^{-3} - 10^{-4} \text{ cm.}$$

If this inequality is satisfied the thermal conductivity coefficient behaves as follows (Fig. 2). At very low temperatures when  $d \ll l_{SS}^{(r)}$

$$l^{eff} \approx d \text{ and } \kappa \approx C_s d v_T \approx \frac{\Theta_C}{\mu_B} \frac{d}{a} \left( \frac{T}{\Theta_C} \right)^2.$$

Later, in the range from  $T_1$  ( $l_{SS}^{(r)}(T_1) \approx d$  to  $T_2$  ( $d^2/l_{SS}^{(c)}(T_2) \sim l^V(T_2)$ ) the thermal conductivity is determined by the diffusion mechanism and

$$\kappa \approx \frac{\Theta_C}{\mu_B} \frac{d^2}{a^2} \left( \frac{T}{\Theta_C} \right)^2 \begin{cases} \left( \frac{\mu_B M_0}{\Theta_C} \right)^2 \exp \left( -\frac{\beta \mu_B M_0}{T} \right), & T_1 \ll T \ll \mu_B M_0, \\ \left( \frac{\mu_B M_0}{\Theta_C} \right)^2, & \mu_B M_0 \ll T \ll \Theta_C \left( \frac{\mu_B M_0}{\Theta_C} \right)^{4/7}, \\ \left( \frac{T}{\Theta_C} \right)^{7/2}, & \Theta_C \left( \frac{\mu_B M_0}{\Theta_C} \right)^{4/7} \ll T \ll T_2. \end{cases}$$

The way in which  $\kappa(T)$  decreases when  $T > T_2$  depends on the dimensions and purity of the ferrite. In sufficiently large and pure specimens Umklapp processes play the main role. Some higher-order anharmonics may then manifest themselves (see Sec. 3). The impurity thermal resistivity of a ferrite depends in an essential way on the magnetic properties of the impurity atoms.<sup>[7]</sup>

### 3. Influence of Higher-order Anharmonicity on Transport Processes in Solids at Low Temperatures

1. As we already noted, finite values for the thermal and electrical conductivity coefficients of an ideal infinite single crystal can be obtained only when we take Umklapp processes into account. If the quasi-particles obey Bose statistics (for instance, phonons or spin waves) the probability for collisions accompanied by Umklapp processes is exponentially small ( $\sim \exp(-\gamma\Theta/T)$  where  $\gamma$  is a numerical coefficient of order unity,  $\Theta$  the Debye temperature in the case of

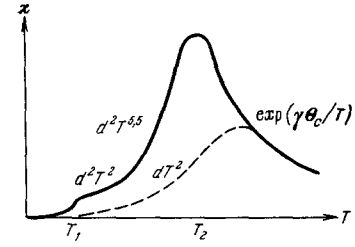


FIG. 2.

phonons and the Curie temperature in the case of spin waves,  $T \ll \Theta$ ). To elucidate the occurrence of this exponent we consider as an example the case when there are two quasi-particles both before and after the collision.

From the energy and momentum conservation laws

$$\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 = 0, \quad \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 = \mathbf{b} \quad (3.1)$$

( $\mathbf{b}$  is a reciprocal lattice vector) it is clear that at least two of the four wave vectors  $\mathbf{k}$  must be of the order of  $\mathbf{b}$ . Corresponding to those large  $\mathbf{k}$  we have energies  $\varepsilon \sim \Theta \gg T$ . The probability for the collision will be proportional to

$$\exp \left( -\frac{\varepsilon_1 + \varepsilon_2}{T} \right) = \exp \left( -\frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{2T} \right). \quad (3.2)$$

The quantity occurring in the exponent is in general a complex function of the wave vectors  $\mathbf{k}$ . Of physical interest is the probability for an Umklapp process for a well-defined quasi-particle, say, in the state  $\mathbf{k}_1$ . We must then take for the sum of energies in the exponent (3.2) the smallest value allowed by the conservation laws (3.1) for fixed  $\mathbf{k}_1$ . As a result the probability for Umklapp turns out to be appreciably different even for states with the same value of the energy (it is, for instance, clear that the largest probability corresponds to the section of the energy surface which is least far removed from the Brillouin zone boundaries). Heat transfer would then appear to be caused by those quasi-particles which are least decelerated by Umklapp processes.

However, the arguments given here have one important deficiency, viz., the role of the normal collisions, which at low temperatures proceed considerably more frequently than Umklapp collisions, has not at all been taken into account. Owing to the normal collisions the momentum acquired by one of the quasi-particles as the result of Umklapp will fast be redistributed among all other quasi-particles. The thermal resistivity is then determined by the total number of U-processes independently of whether these processes are possible for all quasi-particles. In other words, we must take in the exponent (3.2) the smallest value of the sum of energies compatible with the conservation laws (3.1).<sup>[4, 5]</sup>

The probability for the analogous process in which not necessarily two but some arbitrary number of particles  $n$  take part is proportional to  $\exp(-T^{-1} \sum \varepsilon_i)$  where we must take for  $\sum \varepsilon_i$  the smallest value allowed by the conservation laws

$$\sum_{i=1}^n \varepsilon_i - \sum_{i=1}^{2n} \varepsilon_i = 0, \quad \sum_{i=1}^n \mathbf{k}_i - \sum_{i=1}^{2n} \mathbf{k}_i = \mathbf{b}.$$

Moreover when the order of the process increases,

the power of the small parameter  $T/\Theta$  in the multiplying factor increases as  $(T/\Theta)^{\beta n}$  (the constant  $\beta$  depends on the nature of the dispersion law). One understands, however, easily that the multiplying factors which basically determine the smallness of the probability for collisions accompanied by Umklapp may increase when the order  $n$  of the process increases. For this it is necessary that the quasi-particle energy as function of the absolute magnitude of the quasi-momentum changes faster than linearly.

For concreteness, we consider spin waves in a cubic crystal. At not too low temperatures  $\varepsilon(\mathbf{k}) = \Theta_C(ak)^2$  and the smallest non-vanishing value of  $|\mathbf{b}|$  is  $|\mathbf{b}| = 2\pi/a$  and one shows easily that  $\beta = 3$ ; here  $\Theta_C$  is of the order of the Curie temperature and  $a$  the lattice constant. If we assume that the quadratic dispersion law is valid up to  $k \approx b/2n$ , the smallest value of  $\sum_i \varepsilon_i$  is reached at the point  $\mathbf{k}_i = \mathbf{b}/2n$  ( $i \leq n$ ),  $\mathbf{k}_i = -\mathbf{b}/2n$  ( $i > n$ ) and is equal to  $n\varepsilon(b/2n) = \pi^2 \Theta_C/n$ . We can thus expect that the total probability for the loss of momentum of the system of spin waves will, when we take anharmonics of all orders into account, be proportional to an expression of the form

$$\sum_n B_n \left(\frac{T}{\Theta_C}\right)^{3n} \exp\left(-\frac{\pi^2 \Theta_C}{nT}\right),$$

where the  $B_n$  are numerical coefficients.

It follows from this equation that when the temperature is lowered the kinetic processes will be determined by all higher-order anharmonics. We note that this statement is based essentially on only two assumptions: "non-linearity" of the dispersion law and the presence of different orders of anharmonicity in the Hamiltonian of the quasi-particle interaction.

2. As an illustration we give the result of a calculation of the spin-wave-connected thermal conductivity of a ferroelectric:<sup>[10]\*</sup>

$$\kappa \approx \frac{T}{ha} \left[ \sum_{n \geq 2} B_n \left(\frac{T}{\Theta_C}\right)^{3n-6} \exp\left(-\frac{\pi^2 \Theta_C}{nT}\right) \right]^{-1},$$

$$B_n \approx \pi^{11-3n} 2^{5-6n} \frac{(n!)^2}{n^3} S^{2n-2},$$

where  $S = a^3 M_0/\mu_B$  is the spin of an atom; the summation is from  $n = 2$  to  $n \sim \Theta_C/T$ .

The last limitation is important, for owing to the presence of factors  $(n!)^2$  in the coefficients  $B_n$  the terms in the series begin to increase when  $n \sim \Theta_C/T$ . This is not surprising as we assumed in the calculations that  $\Theta_C/nT \gg 1$ . The exact value of the upper limit for the summation over  $n$  is unimportant as for  $n \sim \Theta_C/T$  the terms in the series are negligibly small.

At relatively high temperatures (but  $\exp(\Theta_C/T) \gg 1$ ) the first term in the sum plays the main role and, in

\*As we noted already at low temperatures heat transfer in ferroelectrics occurs predominantly through spin waves. At the same time Umklapp processes can serve as the cause of thermal resistivity both for collisions between spin waves and for spin-phonon and phonon-phonon collisions.<sup>[12]</sup> However, since the dispersion law for the phonons is linear, the probability for multiple particle collisions in which they take part is relatively small. More exactly, this will be the case at sufficiently low temperatures when higher-order anharmonics play the main role. Up to very low temperatures collisions between spin waves are determined by exchange interactions which were the only ones to be taken into account when the calculation was made.

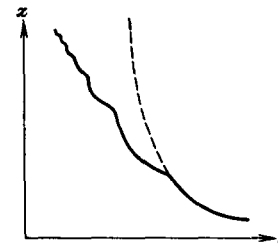


FIG. 3.

accordance with<sup>[5]</sup>,

$$\kappa \approx \frac{T}{ha} \exp\left(\frac{\pi^2 \Theta_C}{2T}\right). \quad (3.3)$$

Starting at a temperature  $T_1 \approx \Theta_C/10$  the second term in the sum ( $n = 3$ ) dominates and the temperature dependence becomes flatter:

$$\kappa \approx \frac{T}{ha} \left(\frac{\Theta_C}{T}\right)^3 \exp\left(\frac{\pi^2 \Theta_C}{3T}\right).$$

Finally at sufficiently low temperatures a small group of terms in the sum with  $n \approx (T/\Theta_C \ln(\Theta_C/T))^{-1/2}$  (we assume that  $\ln(\Theta_C/T) \gg 1$ ) plays the main role and one shows easily that

$$\kappa \approx \frac{T}{ha} \left(\frac{\Theta_C}{T}\right)^6 \exp\left(2\pi \sqrt{2 \frac{\Theta_C}{T} \ln \frac{\Theta_C}{T}}\right).$$

The temperature dependence of the thermal conductivity coefficients has thus a wavy character (see Fig. 3; the dotted curve corresponds to the usual behavior (3.3)).

To clarify the possibility of finding such a dependence, we find the effective mean free path in the region where deviations from the law (3.3) begin, i.e., at  $T \approx T_1$ . Comparing the results given above with Eq. (1.1) (in the present case  $C \approx a^{-3}(T/\Theta_C)^{3/2}$ ,  $v_T \approx a\hbar^{-1}\sqrt{T\Theta_C}$ ), we show easily that  $l^{\text{eff}}(T_1) \approx 1$  cm. It is thus necessary to have pure specimens with a transverse dimension  $d > 1$  cm. In the opposite case it would appear that for  $T \lesssim T_1$  the scattering of the spin waves by the boundaries will play the main role. However, we showed in Sec. 2 that  $l^{\text{eff}}$  must be compared not with the diameter  $d$  of the specimen but with the appreciably larger diffusion length  $d^2/l^N$  (e.g.,  $d^2/l^N \approx 1$  cm when  $d \approx 10^{-2}$  cm and  $l^N \approx 10^{-4}$  cm).

#### 4. Second Sound in Dielectrics

1. It is well known that macroscopic oscillations of essentially different nature—first and second sound—can propagate in superfluid helium.<sup>[13]</sup> In the approximation in which thermal expansion is neglected the first kind of waves are the usual sound oscillations which are possible in any liquid or gas while the second kind of waves reduces to pure temperature oscillations.

From the microscopic point of view second sound is the vibrations of the density of thermal excitations in helium and it can thus be treated as ordinary sound in a gas of quasi-particles. It is then, of course, assumed that collisions between quasi-particles occur sufficiently often (the period of the oscillations and the wavelength of the sound must be large compared with the mean free flight time and the mean free path, respectively).

From what we said it is clear that second sound is not connected with any specific properties of superfluid helium and is, generally speaking, possible in any gas of quasi-particles. For this it is, however, necessary that when quasi-particles collide the total quasi-momentum be conserved, as well as the total energy. In other words, second sound is only possible in solids under conditions when the normal collisions between quasi-particles are appreciably more probable than Umklapp collisions (in liquids U-processes are clearly impossible as a matter of principle). We saw that when the temperature is lowered such a situation unavoidably occurs and is, moreover, realized experimentally (see Sec. 2).

The problem of the possibility of second sound in solids has been often discussed in the literature (see, e.g., [14-16]). Recently this interesting phenomenon has been observed experimentally. [17]

2. We can describe the properties of second sound most naturally by starting from the hydrodynamic equations, which we shall now derive. The Navier-Stokes kind of equation obtained in Sec. 2 is in the present case unsuitable (in its derivation we assumed that the temperature was given and the process was stationary). We shall start from the kinetic equation

$$\frac{\partial}{\partial t} N + (\mathbf{v}\nabla) N = \hat{I}^N N + \hat{I}^V N.$$

In the zeroth approximation

$$\hat{I}^N N^{(0)} = 0$$

and thus  $N^{(0)} = N_0(\varepsilon - \mathbf{u} \cdot \mathbf{p} - \varepsilon\varphi + \mu)$ , where  $\mathbf{u}(\mathbf{r}, t)$  is the velocity of the ordered motion,  $\varphi$  the relative correction to the temperature:  $T(\mathbf{r}, t) = T(1 + \varphi(\mathbf{r}, t))$ ,  $\mu$  the non-equilibrium chemical potential.

We first of all consider the case when the number of quasi-particles is not conserved in normal collisions so that  $\mu = 0$ . We saw that in fact this equation can also be satisfied when the number of particles is conserved.

In first approximation

$$\frac{\partial}{\partial t} N^{(1)} + (\mathbf{v}\nabla) N^{(0)} = \hat{I}^N N^{(1)}, \quad (4.1)$$

or

$$-\frac{\partial N_0}{\partial \varepsilon} [\dot{\mathbf{u}} + \varepsilon \dot{\varphi} + (\mathbf{v}\nabla)(\mathbf{u}\mathbf{p}) + \varepsilon(\mathbf{v}\nabla)\dot{\varphi}] = \hat{I}^N N^{(1)}.$$

We can write the conditions that this equation can be solved for  $N^{(1)}$  (which is a consequence of the conservation of energy and momentum in normal collisions) in the form

$$\begin{aligned} \langle \varepsilon^2 \rangle \dot{\varphi} + \frac{1}{3} \langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle \operatorname{div} \mathbf{u} &= 0, \\ \alpha_{ik} \dot{u}_k + \langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle \frac{\partial}{\partial x_i} \dot{\varphi} &= 0. \end{aligned} \quad (4.2)$$

We have used here the notation

$$\langle f(\mathbf{p}) \rangle = - \int f(\mathbf{p}) \frac{\partial N_0}{\partial \varepsilon} d\mathbf{p}, \quad \alpha_{ik} = 3 \langle p_i p_k \rangle$$

and used the fact that for any dispersion law the tensor  $\langle \varepsilon p_i v_k \rangle$  is diagonal and therefore, for instance  $\langle \varepsilon(\mathbf{v}\nabla)(\mathbf{u} \cdot \mathbf{p}) \rangle = \frac{1}{3} \langle \varepsilon(\mathbf{p} \cdot \mathbf{v}) \rangle \operatorname{div} \mathbf{u}$ . Eliminating  $\mathbf{u}$  we get

$$\ddot{\varphi} = \frac{1}{3} \frac{\langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle^2}{\langle \varepsilon^2 \rangle} (\hat{\alpha}^{-1})_{ik} \frac{\partial^2 \dot{\varphi}}{\partial x_i \partial x_k}.$$

The dispersion equation has thus the form  $[\varphi \sim \exp i(\omega t$

$-\mathbf{k} \cdot \mathbf{r})]$

$$\omega^2 = \frac{1}{3} \frac{\langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle^2}{\langle \varepsilon^2 \rangle} (\hat{\alpha}^{-1})_{ik} k_i k_k.$$

Hence, the velocity of second sound is

$$V = \frac{1}{\sqrt{3}} \langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle \sqrt{\frac{(\hat{\alpha}^{-1})_{xx}}{\langle \varepsilon^2 \rangle}}, \quad (4.3)$$

where  $\mathbf{x}$  is the direction in which the wave propagates.

In the isotropic case

$$V = \frac{\langle \varepsilon p v \rangle}{\sqrt{3} \langle \varepsilon^2 \rangle \langle p^2 \rangle}. \quad (4.4)$$

When  $\varepsilon = sp$ , the well-known relation  $V = s/\sqrt{3}$  follows from this.

We can consider in a similar way the case when the number of quasi-particles is conserved in normal collisions. Starting from the set of equations

$$\left. \begin{aligned} \langle \varepsilon \rangle \dot{\varphi} + \langle 1 \rangle \dot{\mu} + \frac{1}{3} \langle \mathbf{p}\mathbf{v} \rangle \operatorname{div} \mathbf{u} &= 0, \\ \langle \varepsilon^2 \rangle \dot{\varphi} + \langle \varepsilon \rangle \dot{\mu} + \frac{1}{3} \langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle \operatorname{div} \mathbf{u} &= 0, \\ \alpha_{ik} \dot{u}_k + \langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle \frac{\partial \dot{\varphi}}{\partial x_i} + \langle \mathbf{p}\mathbf{v} \rangle \frac{\partial \dot{\mu}}{\partial x_i} &= 0. \end{aligned} \right\} \quad (4.5)$$

we find as the result of simple calculations

$$V^2 = \frac{1}{3} (\hat{\alpha}^{-1})_{xx} \frac{2 \langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle \langle \mathbf{p}\mathbf{v} \rangle \langle \varepsilon \rangle - \langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle^2 \langle 1 \rangle - \langle \mathbf{p}\mathbf{v} \rangle^2 \langle \varepsilon^2 \rangle}{\langle \varepsilon \rangle^2 - \langle \varepsilon^2 \rangle \langle 1 \rangle}. \quad (4.6)$$

The second sound velocity depends thus on whether or not the number of quasi-particles is conserved in the collisions.

We must, however, bear in mind that at low temperatures the quasi-particle dispersion law may as a rule be assumed to be a power law:  $\varepsilon(\mathbf{p}) = p^n f(\mathbf{p}/p)$ . Then  $\mathbf{p} \cdot \mathbf{v} = n\varepsilon(\mathbf{p})$  and

$$\langle \varepsilon^2 \rangle \langle \mathbf{p}\mathbf{v} \rangle - \langle \varepsilon \rangle \langle \varepsilon(\mathbf{p}\mathbf{v}) \rangle = 0. \quad (4.7)$$

Using this relation we show easily that Eqs. (4.3) and (4.6) for the second sound velocity are the same and it follows from the first two Eqs. (4.5) that  $\mu = 0$ .

To elucidate these results we note that by virtue of (4.7) the equation of continuity for the number of particles

$$\langle \varepsilon \rangle \dot{\varphi} + \frac{1}{3} \langle \mathbf{p}\mathbf{v} \rangle \operatorname{div} \mathbf{u} = 0$$

is a consequence of the energy conservation (first of Eqs. (4.2)). It is natural that then it is completely immaterial whether or not the number of particles is conserved in collisions.

We can easily generalize the considerations given to the case of a system consisting of several different kinds of interacting quasi-particles. As usual the problem is reduced to a set of equations such as (4.2) or (4.5) with the difference that now the coefficients will be sums over different kinds of quasi-particles (since the energy and momentum is only conserved for the system as a whole). Therefore

$$V = \frac{1}{\sqrt{3}} \sum_{\sigma} \langle \varepsilon_{\sigma} \langle p_{\sigma} v_{\sigma} \rangle \rangle \sqrt{\frac{(\sum_{\sigma} \hat{\alpha}_{\sigma})_{xx}^{-1}}{\sum_{\sigma} \langle \varepsilon_{\sigma}^2 \rangle}}.$$

In the case of a normal dielectric  $\sigma$  takes on three values corresponding to one longitudinal and two transverse branches of the phonon spectrum.

The damping of second sound in dielectrics is connected with two facts: 1) finiteness of the mean free flight time and of the mean free path of the quasi-particles in relation to normal collisions; 2) the presence of collisions in which the quasi-momentum is not conserved (Umklapp processes, scattering by lattice defects, etc.). Since the second sound velocity  $V$  and the average thermal velocity are of the same order of magnitude the contribution from the first mechanism will be proportional to the small parameter  $l^N/\lambda$  ( $\lambda = 2\pi/k$  is the sound wavelength) while the contribution from the second mechanism is proportional to the small parameter  $\lambda/lV$ . In other words (see [15a, 15b])

$$\frac{\text{Im } k}{k} \sim -\frac{l^N}{\lambda} - \frac{\lambda}{lV}.$$

In principle one can obtain an exact result in the framework of the approach developed above. To do this we must, in the derivation of the hydrodynamic equations, cut off the chain of coupled equations not in the second but in the third stage and look for the quantities  $u$  and  $\delta$  in the form of expansions in the small parameters  $l^N/\lambda$  and  $\lambda/lV$  (Eqs. (4.2) and (4.5) correspond to the zeroth approximation in these parameters). However, the situation is complicated by the fact that the function  $N^{(1)}$  satisfies the integral equation (4.1) which cannot be solved in its general form (cf. Sec. 2).

3. Second sound in ferroelectrics has some interesting properties.<sup>[16]</sup> Oscillations of the spin wave density are accompanied by magnetic-moment oscillations and lead thus to the appearance of a variable magnetic field. It turns out that this field has a relatively small absolute value (so that second sound remains mainly a temperature wave) but at the same time it is completely accessible to experimental observation. The propagation velocity of second sound is determined by the usual formula (4.3) but now, of course, it depends on the temperature:

$$V \approx \frac{a}{\hbar} \sqrt{T\theta_c}.$$

Relativistic interactions which do not conserve the number of spin waves may considerably affect the damping of second sound on top of the mechanisms enumerated above in subsection 2. We noted already that normal collisions between spin waves at not too low temperatures are connected with exchange interactions which conserve the number of spin waves. At the same time the spin wave dispersion law is not strictly a power law (due to the anisotropy energy or the presence of an external magnetic field). Under such conditions normal collisions connected with relativistic interactions lead to a relaxation of the chemical potential  $\mu$  (just as collisions which do not conserve quasi-momentum lead to a relaxation of the velocity of the ordered motion  $u$ ).

## II. KINETIC PHENOMENA IN METALS

### 5. Electrical Conductivity of Metals at Low Temperatures

In this section we discuss some basic mechanisms for the scattering of electrons and their influence on the low-temperature electrical conductivity of metals.

The discussion has no pretension at being complete; our aim is to make clear the conditions under which normal collisions between electrons and phonons may turn out to be important.

It is well known that at sufficiently low temperatures the electrical resistivity is as a rule constant and is determined by the scattering of the electrons by impurity atoms and other structural lattice defects and also by the boundaries of the sample. The temperature dependence is due to different interactions in the electron-phonon system. One usually employs the equation

$$\rho(T) = \rho_0 + \alpha T^2 + \beta T^5, \quad (5.1)$$

where  $\rho_0$  is the residual resistance, the second term is connected with electron-electron collisions and the third one with electron-phonon collisions. In the following we shall be dealing with sufficiently pure and large single crystal specimens for which the residual resistance occurs only at helium temperatures.

1. Electron-electron collisions. A calculation based upon the Born approximation shows that in the helium temperature region the resistivity must be determined by the second term in Eq. (5.1), i.e.,  $\rho(T) - \rho_0 \sim T^2$ .<sup>[18]</sup> From the point of view of principle the existence of such a dependence causes, apparently no doubts. However, if we exclude transition metals this behavior is not observed experimentally. The problem reduces to an estimate of the numerical factor of  $T^2$ .

We note first of all that the Born approximation gives an overestimate for the electron-electron scattering cross section.<sup>[19]</sup> We must, moreover, take into account that a direct contribution to the resistivity comes only from collisions accompanied by Umklapp (U-processes). Such collisions are clearly possible only between electrons which lie in well-defined regions which are cut from the Fermi surface by planes perpendicular to the Umklapp vector and bisecting the distance between the center of the Brillouin zone and its boundaries. It is probable that the fact that the matrix element for a U-process contains an interference factor which apparently is small compared with unity (see [20], pp. 165, 371) is more important. We note, however, that N-processes (i.e., collisions without Umklapp) can also lead to a correction proportional to  $T^2$  for any non-spherical Fermi surface (see below, Sec. 7).

From what has been said it is clear that theoretical considerations at the moment do not allow us to make a sufficiently reliable estimate of the order of magnitude of the coefficient  $\alpha$  in Eq. (5.1) and to indicate with what accuracy the quadratic dependence of the resistivity must be observed experimentally in this or that metal. In this connection it is of interest to make a comparison with experimental data on infrared absorption in metals. According to [21] the effective frequency of electron-electron collisions can be written in the following form which is useful both in the static and in the high-frequency case:

$$\tau_{ee}^{-1} = \tau_0^{-1} \left[ \left( \frac{T}{\epsilon_F} \right)^2 + \left( \frac{\hbar\omega}{2\pi\epsilon_F} \right)^2 \right] \quad (5.2)$$

( $\omega$  is the frequency of the electromagnetic field,  $\epsilon_F$  the Fermi energy). The electron-electron collisions are thus in the near infrared region ( $\lambda = 2\pi c/\omega \approx 1$  to  $20 \mu$ ,



$\hbar\omega \approx 10^3$  to  $10^4$  (K)  $10^3$  to  $10^5$  times more probable than in the static case at helium temperatures.

For a not too impure metal the infrared absorption coefficient can be written in the form

$$\Gamma = \Gamma_0 + \frac{2}{\omega_0 \tau_{ee}}, \quad (5.3)$$

where the quantity  $\Gamma_0$ , which does not depend on  $\omega$  or  $T$ , is connected with the collisions of the electrons with the phonons or with the boundaries of the specimen,  $\omega_0 = (4\pi n e^2/m)^{1/2}$  is the plasma frequency.

Recently it has been ascertained that a quadratic frequency dependence of the infrared absorption occurs for many metals. Most trustworthy are here the results obtained for noble metals, [22-24] for which the limit of the internal photoeffect lies in the optical region of the spectrum so that the behavior of interest to us can be separated reliably. Comparison of these data with Eqs. (5.2), (5.3), and (5.1) shows that the term proportional to  $T^2$  may at helium temperatures be only a few percents of the residual resistivity even in the purest samples where the impurity mean free path  $l_{ei} \approx 0.1$  cm.

2. Interaction with phonons. The  $T^5$  law for the resistivity is well known to have been obtained by Bloch under the assumption that the phonons were in equilibrium. In other words, one assumes that there exists some mechanism for the dissipation of the momentum which flows continuously from the electrons into the phonon system. Peierls showed that phonon-phonon Umklapp collisions are such a mechanism. However, we have seen that when the temperatures is lowered the probability for these U-processes decreases exponentially as  $\exp(-\gamma\theta/T)$ . It is difficult to estimate the magnitude of the numerical factor  $\gamma$ , since it is determined by the behavior of the acoustic branches of the phonon spectrum at large values of the quasi-momentum ( $2\gamma\theta$  is equal to the least value of the sum of energies of all phonons taking part in the collision taking the appropriate energy and momentum conservation laws into account, see Sec. 3). It is in any case clear that there exists for every metal some well-defined temperature starting from which U-processes can no longer provide equilibrium of the phonons and the corresponding contribution to the resistivity must decrease exponentially. Comparison with data on the thermal conductivity of dielectrics shows that such a situation must apparently occur in a rather wide range of low temperatures, say when  $T/\theta \lesssim 1/10$ .

In principle there is yet another possibility to which Peierls also drew attention. This is electron-phonon Umklapp collisions. From the momentum conservation law  $\mathbf{p} - \mathbf{p}' - \mathbf{q} = \hbar\mathbf{b}$  it follows that at low temperatures when the phonon momentum  $\mathbf{q}$  is small compared to the electron momentum on the Fermi surface (as to order of magnitude  $q/p \approx T/\theta$ ) the difference  $\mathbf{p} - \mathbf{p}'$  is approximately equal to  $\hbar\mathbf{b}$ . This means that the initial and final states of the electron  $\mathbf{p}$  and  $\mathbf{p}'$  must be close to equivalent points on opposite boundaries of the Brillouin zone. Umklapp processes are thus only possible with an appreciable probability when the Fermi surface intersects the zone boundary or comes very close to it. We must, however, remember that the elementary cell in  $\mathbf{p}$ -space can be chosen arbitrarily and the separation of electron-phonon collisions into N- and U-processes is to a large extent arbitrary. In particular, one can al-

ways achieve that a given N-process should be considered as a U-process, and vice versa. To elucidate the physical situation in a given case it is thus convenient not to restrict ourselves to one cell but to consider the whole of  $\mathbf{p}$ -space (the so-called "repeated zone" method). When this is done all collisions are "normal" (since momentum is conserved) and we ascertain chiefly the connection between the electrical resistivity and the topology of the Fermi surface.

Let us assume that the electrical field  $\mathbf{E}$  is non-vanishing only during a short time interval  $\Delta t$  which is small compared to the relaxation time. As a result of the action of the field the electron distribution in  $\mathbf{p}$ -space is clearly shifted as a whole by the amount  $\Delta\mathbf{p} = e\mathbf{E}\Delta t$ . One can verify that the resistivity is non-vanishing if the electron system tends to equilibrium under the action of collisions with phonons.

In the case of a closed Fermi surface the non-equilibrium electrons will predominantly absorb phonons of a single direction. As a result there will appear an excess phonon current in the opposite direction. The stationary state in which the drift velocities of the electrons and of the phonons are the same will clearly be a non-equilibrium one. All we have said refers also to the hole surface—only the drift direction of the phonons will be the reverse.

Equilibrium is possible in the case when there are both electron and hole surfaces while the number of electrons must necessarily be equal to that of holes. This condition means that the total momentum obtained by the system from the external field vanishes. The role of the phonons reduces to redistributing momentum between electrons and holes.

Equilibrium can apparently be established in the case of an open Fermi surface, for then a shift in the electron distribution can be reduced by means of phonons in both directions (one can easily understand that electron transitions which in the usual terminology correspond to N- and U-processes, generate phonon currents in opposite directions).

3. Statement of the problem. The considerations given above show that in typical metals electron-electron collisions are unimportant. On the other hand, we saw that at low temperatures electron-phonon interactions can by far not always lead to an appreciable electrical resistivity. This is, for instance, the case for closed Fermi surfaces (provided the number of electrons is not equal to that of holes) or for surfaces which are open, but not in all directions. Moreover, in a number of cases the Fermi surfaces although open contain relatively narrow connecting pieces.

From what has been said it is clear that in many metals at sufficiently low temperatures only normal collisions between electrons and phonons take place, apart from the scattering of electrons by different kinds of static objects (localized defects, specimen boundaries, etc.). A study of the electrical conductivity of metals in such a situation is the topic of the further discussion.

We denote by  $l_{ep}$  and  $l_{pe}$  the mean free paths for electron-phonon and phonon-electron scattering, respectively. For a typical metal in which the number of conduction electrons is of the order of the number of atoms

$$l_{ep}(T) \approx l_{ep}(\Theta) \left(\frac{\Theta}{T}\right)^5, \quad l_{pe}(T) \approx l_{ep}(\Theta) \frac{\Theta}{T}, \quad (5.4)$$

where  $l_{ep}(\Theta) \approx a\varepsilon_F/\Theta$  (see, e.g., [4]). Thus, although the two mean free paths are connected with the same collision processes we have at low temperatures always  $l_{ep} \gg l_{pe}$ . A phonon emitted by an electron will thus after a very short trajectory and correspondingly after a very short time be absorbed by some other electron. In fact, we are thus dealing with effective electron-electron collisions which are characterized by a mean free path  $l^N \approx l_{ep}(T)$ .

Since for the collisions considered the quasi-momentum is conserved they do not by themselves lead to an electrical resistivity. It turns out, however, that these collisions may affect very considerably the process of transfer of momentum from the electron gas to static objects and thus change the resistivity. The mechanism of this influence is determined by the nature of the residual resistance. More precisely, everything depends on the fact that the mean free path  $l^N$  and the characteristic distance  $d$  between static objects are connected at those temperatures when the probabilities for "electron-electron" collisions and for collisions with static objects are the same.

If the cause of the residual resistance is macroscopic objects (boundaries of the specimen, dislocations, etc.) the inequality  $l^N \ll d$  may be satisfied. A peculiar mechanism of electrical resistivity may then occur, which reminds us of the flow of a viscous liquid (cf. Sec. 2). The resistivity depends in an unusual fashion on  $T$  and  $d$  and in particular there must be a minimum in the temperature dependence  $\rho(T)$  (see Sec. 6).

If, however, the residual resistance is connected with the scattering of electrons by microscopic lattice defects (such as impurity atoms) in the temperature region of interest to us we have always  $l^N \gg d$ . The influence of  $N$ -processes occurs then only in the case of an anisotropic dispersion law. The resistivity initially increases with increasing temperature, later reaches saturation which again changes to an increase (see Sec. 7).

In the next section we consider the high-frequency electrical conductivity of metals. We show that taking the normal collisions into account may lead to an appreciable change in the nature of the skin effect; in particular, there appears a rather wide range of frequencies in which the surface impedance of the metal depends in an unusual way on the frequency of the electromagnetic field and on the temperature (Sec. 8).

In the first two sections we follow mainly [25], and in the last one we follow [26].

## 6. Electrical Conductivity of Thin Samples

In this section we shall consider for concreteness the case when the residual resistance occurs due to the scattering of electrons at the boundaries of the sample although it should be clear from what follows that the results obtained are qualitatively valid for any macroscopic deformations. We noted already that under the conditions of interest to us the electron-phonon interaction in fact reduces to electron-electron collisions characterized by a mean free path  $l^N \approx l_{ep}(T)$ .

To explain the physical cause of the influence of the

normal collisions on the electrical conductivity we consider the limiting cases  $l^N \gg d$  and  $l^N \ll d$ . In the first case the electron travels between two collisions with the walls along a path of order  $d$  without practically any collisions inside the volume. Hence the resistivity  $\rho \sim d^{-1}$  and is practically temperature independent. If, however,  $l^N \ll d$  an electron inside the sample will, on the other hand, before reaching the wall undergo many normal collisions. It is clear that by this the path traversed by the electron between two collisions with the boundaries (and also the corresponding time) is considerably increased. All the time the electron is accelerated in the external field. The fact that as a result of collisions there is an exchange of momentum between the electrons does clearly not change the situation. Using the well-known formulas for Brownian motion we can easily show that the length of the trajectory between two collisions with the boundaries is of the order  $d^2/l^N$ . Since this quantity has the meaning of an effective mean free path it is natural to expect that  $\rho \sim l^N/d^2$ . In contrast to the Knudsen case the resistivity thus contains the square of the thickness of the sample and mainly depends on the temperature in an unusual manner: at the same time as the mean free path  $l^N \approx l_{ep}(T)$  it decreases with increasing temperature.

Starting from these intuitive considerations we can now easily establish the behavior of the resistivity qualitatively (Fig. 4). At very low temperatures while  $l^N \gg d$  the resistivity is roughly speaking independent of the temperature and proportional to  $d^{-1}$ . Then, starting with the temperature for which  $l^N \approx d$  the resistivity decreases with increasing  $T$  as  $\rho \sim T^{-5} d^{-2}$  (when  $l^N \ll d$ ). This decrease will go on until the quantity  $d^2/l^N$  is no longer comparable with one of the mean free paths characterizing the volume collisions with loss of momentum. If this is the impurity mean free path the resistivity remains further constant and equal to the resistivity of a sample of infinite dimensions (with the same concentration of impurity atoms). Later an increase of the resistivity with temperature must begin which is, for instance, connected with phonon-phonon  $U$ -processes. The corresponding mean free path  $l^U$  changes basically as  $\exp(-\gamma\Theta/T)$  and will be estimated in the following.

We formulate the conditions under which a diffusion mechanism must occur for the electrical conductivity:

a) The transverse dimensions of the sample must be small compared with the electron-impurity mean free path:  $d \ll l_{ei}$ .

b) It is necessary that the phonon-phonon  $U$ -processes still be ineffective when  $l_{ep}(T) \approx d$ . More precisely, in some temperature range the following inequality must hold:

$$l_{ep}(T) \ll d \ll l^U(T). \quad (6.1)$$

c) Electron-phonon  $U$ -processes are unimportant (see Sec. 5).

For a rough orientation we note that when  $l_{ei} \approx 0.1$  cm and  $d \approx 10^{-2}$  to  $10^{-3}$  cm condition (6.1) must be satisfied in the interval between helium and hydrogen temperatures although the situation in reality depends essentially on the properties of the metal.

For a quantitative description of the mechanism considered it is natural to use the hydrodynamic approach

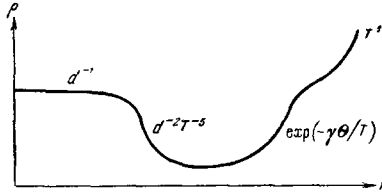


FIG. 4.

(cf. Sec. 2). This method is particularly convenient in the present case since it allows us to take into account together with the influence of the boundaries the volume collisions with loss of quasi-momentum (under conditions where they are considerably less probable than N-processes).

The hydrodynamic equation for the electron-phonon gas can be obtained in the usual way from the appropriate kinetic equations. We write these equations in the form

$$\begin{aligned} \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + e \mathbf{E} \mathbf{v} \frac{\partial f_0}{\partial \mathbf{e}} &= \hat{I}_{ep} \{f, N\} + \hat{I}_{eif}, \\ s \frac{\partial N}{\partial \mathbf{r}} &= \hat{I}_{pe} \{f, N\} + \hat{I}_{pp}^N N + \hat{I}_{pp}^U N. \end{aligned} \quad (6.2)$$

Here  $f(\mathbf{p}, \mathbf{r})$  and  $N(\mathbf{q}, \mathbf{r})$  are the electron and phonon distribution functions,  $\mathbf{v} = \partial \mathbf{e} / \partial \mathbf{p}$ ,  $\mathbf{s} = \partial \mathbf{h} \mathbf{v} \mathbf{q} / \partial \mathbf{q}$ ,

$$f_0(\mathbf{e}) = \left[ \exp \left( \frac{\mathbf{e} \cdot \mathbf{e}_F}{T} \right) + 1 \right]^{-1},$$

$\hat{I}_{ep}$  is the electron-phonon collision operator,  $\hat{I}_{ei}$  the electron-impurity operator, and so on. We neglect phonon-impurity scattering, since at low temperatures the probability of this process is roughly speaking  $(\Theta/T)^4$  times smaller than the probability for electron-impurity scattering.\* Moreover, it will become clear from what follows that the contribution from the phonon-impurity scattering is underestimated in the hydrodynamic equation by another additional factor of order  $v_F/s(T/\Theta)^4$ .

It is natural to use for an approximate solution of Eqs. (6.2) the fact that N-processes have the largest probability. The actual solution will be written as series in the small parameters  $l_{ep}/d$ ,  $l_{ep}/l_{ei}$ , and  $l_{ep}/l_U$ . The method of successive approximations leads to the following chain of equations (cf. Sec. 2):

$$\begin{aligned} \hat{I}_{ep} \{f^{(0)}, N^{(0)}\} &= 0, \quad \hat{I}_{pe} \{f^{(0)}, N^{(0)}\} + \hat{I}_{pp}^N N^{(0)} = 0, \\ \mathbf{v} \frac{\partial f^{(0)}}{\partial \mathbf{r}} &= \hat{I}_{ep} \{f^{(1)}, N^{(1)}\}, \quad s \frac{\partial N^{(0)}}{\partial \mathbf{r}} = \hat{I}_{pe} \{f^{(1)}, N^{(1)}\} + \hat{I}_{pp}^N N^{(1)}, \\ \mathbf{v} \frac{\partial f^{(1)}}{\partial \mathbf{r}} + e \mathbf{E} \mathbf{v} \frac{\partial f_0}{\partial \mathbf{e}} &= \hat{I}_{ep} \{f^{(2)}, N^{(2)}\} + \hat{I}_{ei} f^{(0)}, \end{aligned} \quad (6.3)$$

$$\begin{aligned} s \frac{\partial N^{(1)}}{\partial \mathbf{r}} &= \hat{I}_{pe} \{f^{(2)}, N^{(2)}\} + \hat{I}_{pp}^N N^{(2)} + \hat{I}_{pp}^U N^{(0)}, \\ f &= f^{(0)} + f^{(1)} + f^{(2)} + \dots, \quad N = N^{(0)} + N^{(1)} + N^{(2)} + \dots \end{aligned} \quad (6.4)$$

From the first pair of equations follows that

$$f^{(0)} = f_0(\mathbf{e} - \mathbf{p} \mathbf{u}), \quad N^{(0)} = N_0(\mathbf{h} \mathbf{v} \mathbf{q} - \mathbf{q} \mathbf{u}),$$

where  $N_0$  is the Bose distribution function and  $\mathbf{u}(\mathbf{r})$  has the meaning of a drift velocity of the electron-phonon system as a whole.

To solve the second pair of equations, which now take the form

\*A different situation may arise when the metal contains an appreciable amount of isotopes. However, the probability for phonon scattering contains in this case an additional small factor  $(\Delta M/M)^2$  ( $\Delta M$  is the difference in mass between the isotope and the main atom).

$$-v_i p_k \frac{\partial u_k}{\partial x_i} \frac{\partial f_0}{\partial \mathbf{e}} = \hat{I}_{ep} \{f^{(1)}, N^{(1)}\}, \quad -s_i q_k \frac{\partial u_i}{\partial x_k} \frac{\partial N_0}{\partial \mathbf{h} \mathbf{v} \mathbf{q}} = \hat{I}_{pe} \{f^{(1)}, N^{(1)}\} + \hat{I}_{pp}^N N^{(1)}, \quad (6.5)$$

it is necessary to use the explicit form of the collision integrals.

We write the Hamiltonian of the electron-phonon interaction in the form

$$\hat{H} = \sum_{\mathbf{p}, \mathbf{q}} A(\mathbf{p}, \mathbf{q}) a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}+\mathbf{q}} b_{\mathbf{q}}^{\dagger} + \text{c.c.}$$

where  $a_{\mathbf{p}}$  and  $b_{\mathbf{q}}$  are Fermi and Bose operators. It is important for what follows that we have  $|A(\mathbf{p}, \mathbf{q})|^2 \sim |\mathbf{q}|$  for small  $|\mathbf{q}|$ . One shows easily that in the approximation which is linear in  $f^{(1)}$  and  $N^{(1)}$

$$\begin{aligned} \hat{I}_{ep} \{f^{(1)}, N^{(1)}\} &= \int \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}+\mathbf{q}} (\varphi_{\mathbf{p}+\mathbf{q}} - \varphi_{\mathbf{p}} - \psi_{\mathbf{q}}) d\mathbf{q} + \int \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}-\mathbf{q}} (\varphi_{\mathbf{p}-\mathbf{q}} - \varphi_{\mathbf{p}} + \psi_{\mathbf{q}}) d\mathbf{q}, \\ \hat{I}_{pe} \{f^{(1)}, N^{(1)}\} &= 2 \int \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}-\mathbf{q}, \mathbf{q}} (\varphi_{\mathbf{p}-\mathbf{q}} - \varphi_{\mathbf{p}} + \psi_{\mathbf{q}}) d\mathbf{p}, \\ \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}+\mathbf{q}} &= \frac{2\pi}{\hbar} \frac{1}{\hbar^3} |A(\mathbf{p}, \mathbf{q})|^2 \delta(\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_{\mathbf{p}} - \hbar \nu_{\mathbf{q}}) f_0(\varepsilon_{\mathbf{p}}) N_0(\hbar \nu_{\mathbf{q}}) [1 - f_0(\varepsilon_{\mathbf{p}+\mathbf{q}})], \\ \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}-\mathbf{q}, \mathbf{q}} &= \frac{2\pi}{\hbar} \frac{1}{\hbar^3} |A(\mathbf{p}-\mathbf{q}, \mathbf{q})|^2 \delta(\varepsilon_{\mathbf{p}-\mathbf{q}} - \varepsilon_{\mathbf{p}} + \hbar \nu_{\mathbf{q}}) \\ &\quad \times f_0(\varepsilon_{\mathbf{p}}) [1 - f_0(\varepsilon_{\mathbf{p}-\mathbf{q}})] [1 + N_0(\hbar \nu_{\mathbf{q}})], \\ f^{(1)} &= -T \frac{\partial f_0}{\partial \mathbf{e}} \varphi_{\mathbf{p}}, \quad N^{(1)} = -T \frac{\partial N_0}{\partial \mathbf{h} \mathbf{v} \mathbf{q}} \psi_{\mathbf{q}}. \end{aligned} \quad (6.6)$$

We have not written down the phonon-phonon collision integral since we can neglect the term  $\hat{I}_{pp}^N N^{(1)}$  in comparison with  $\hat{I}_{pe} \{f^{(1)}, N^{(1)}\}$ . Analysis shows that the ratio of these terms is of the order  $(T/\Theta)^4$ , a result which could have been expected beforehand from the ratio of the mean free paths (see Eqs. (2.4) and (5.4)).

We can now solve the second of Eqs. (6.5) for  $\psi_{\mathbf{q}}$  (see (6.6)) and substitute the result into the first equation. The equation obtained for the function  $\varphi_{\mathbf{p}}$  can in principle be solved for any electron and phonon dispersion laws. We shall, however, restrict ourselves to the isotropic case as the result of a general discussion all the same contains undetermined numerical coefficients of order unity. The angular dependence then clearly can be eliminated and it is convenient to look for a solution in the form

$$f^{(1)} = -\tau_{ep}(\Theta) \frac{\partial f_0}{\partial \mathbf{e}} v_i p_k \frac{\partial u_k}{\partial x_i} \left( \frac{\Theta}{T} \right)^3 \chi \left( \frac{\mathbf{e} \cdot \mathbf{e}_F}{T} \right).$$

After rather complicated transformations the equation for the function  $\chi$  can be written in the form

$$\int_{-\infty}^{+\infty} K_{xx'} \chi(x') dx' + \left( \frac{T}{\Theta} \right)^2 \int_{-\infty}^{+\infty} Q_{xx'} \chi(x') dx' = e^x f_0^2(x), \quad (6.7)$$

where

$$\begin{aligned} K_{xx'} &= \delta(x-x') \int_{-\infty}^{+\infty} R_{xx''} dx'' - R_{xx'} + S_{xx'}^{(1)}, \\ Q_{xx'} &= (x-x')^2 R_{xx'} + S_{xx'}^{(2)}, \\ R_{xx'} &= (x-x')^2 f_0(x) f_0(x') |e^{-x} - e^{-x'}|^{-1}, \end{aligned}$$

$$\begin{aligned} S_{xx'}^{(n)} &= f_0(x) f_0(x') \exp(x+x') \int_{-\infty}^{+\infty} |z|^n f_0(x+z) [e^z f_0(x'+z) - f_0(x'-z)] dz, \\ f_0(x) &= (1+e^x)^{-1}, \quad x = (\varepsilon_{\mathbf{p}} - \varepsilon_F)/T. \end{aligned}$$

We have dropped unimportant numerical coefficients of order unity. For what follows it is important that all kernels  $K$ ,  $Q$ ,  $R$ , and  $S^{(n)}$  are symmetric and

$$\int_{-\infty}^{+\infty} K_{xx'} dx' = 0, \quad \int_{-\infty}^{+\infty} S_{xx'}^{(n)} dx' = 0. \quad (6.8)$$

Proceeding to an approximate solution of Eq. (6.7), we note that if we drop in it the term with  $(T/\Theta)^2$  the

equation obtained is insoluble. In fact, owing to the symmetry of the kernel  $K_{xx}$ , and the first of Eqs. (6.8) the corresponding homogeneous transposed equation has a solution  $\chi = \text{const}$  which is not orthogonal to the right-hand side  $e^X f_0^2(x)$ . We must thus develop the method of successive approximations as follows:

$$\int_{-\infty}^{+\infty} K_{xx'} \chi^{(0)}(x') dx' = 0, \\ \int_{-\infty}^{+\infty} K_{xx'} \chi^{(1)}(x') dx' = e^X f_0^2(x) - \left(\frac{T}{\Theta}\right)^2 \int_{-\infty}^{+\infty} Q_{xx'} \chi^{(0)}(x') dx', \dots$$

From the first equation it follows that  $\chi^{(0)} = \text{const}$ . The condition that we can solve the second equation for  $\chi^{(1)}$  gives

$$\chi^{(0)} = \alpha \left(\frac{\Theta}{T}\right)^2,$$

where

$$\alpha^{-1} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Q_{xx'} dx dx' = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x-x')^2 R_{xx'} dx dx' \approx 240\zeta(5) \approx 250.$$

Finally, we multiply Eq. (6.3) by  $\mathbf{p}$ , Eq. (6.4) by  $\mathbf{q}$ , integrate, and add. Terms containing  $f^{(2)}$  and  $N^{(2)}$  cancel one another and as a result we get the required hydrodynamic equation for  $\mathbf{u}(\mathbf{r})$ . Up to terms of higher-order in  $T/\Theta$  we get\*

$$\frac{e}{m} \mathbf{E} = -\nu \Delta \mathbf{u} + \frac{\mathbf{u}}{\tau^U}, \quad (6.9)$$

where

$$\nu \approx \frac{1}{15} v_p l_{ep}(T), \quad \frac{1}{\tau^U} = \frac{1}{\tau^V} + \frac{1}{\tau_{el}}, \\ \tau^U \approx \frac{s}{v_F} \left(\frac{\Theta}{T}\right)^4 \tau_{pp}^U, \quad \tau_{pp}^U \approx \frac{1}{s} l_{pp}(\Theta) \left(\frac{T}{\Theta}\right)^{5/2} e^{\nu\Theta/T}$$

(see Eq. (2.5)).

The quantity  $l_{ep}(T)$  coincides exactly with the mean free path occurring in Bloch's theory of electrical conductivity. As we should have expected from intuitive considerations, the kinematic viscosity  $\nu$  is in first approximation determined solely by the electron-phonon mean free path  $l_{ep}$ . One shows easily that the correction connected with the finiteness of  $l_{pe}$  is of order  $\nu(T/\Theta)^3$ . The influence of phonon-phonon U-processes is underestimated by the factor  $v_F/s(T/\Theta)^4$  which clearly refers to all collisions occurring in the phonon system.

The solution of Eq. (6.9) must satisfy the boundary condition  $\mathbf{u}(\mathbf{r}) = 0$  on the boundaries of the specimen which corresponds to diffuse scattering of the electrons. One verifies easily that the electrical current density is  $\mathbf{j}(\mathbf{r}) = ne \mathbf{u}(\mathbf{r})$ , where  $n = (8\pi/3)(p_F/h)^3$  is the electron density. To evaluate the resistivity we must average this expression over the cross-section of the sample. It is convenient to write the result in the usual form:  $\rho^{-1} = ne^2 p_F^{-1} l^{\text{eff}}$  where  $l^{\text{eff}}$  is the effective mean free path. It is clear that the function  $l^{\text{eff}}(T, d)$  is not very sensitive to the shape of the sample (see Sec. 2) and has the form predicted earlier from intuitive considerations. We must then understand by  $d$  a characteristic transverse dimension of the sample or the distance between macroscopic deformations.

\*This equation is valid under the condition  $\text{div } \mathbf{u} = 0$ , which follows from (6.5) and is equivalent to the requirement of electrical neutrality.

## 7. Electrical Conductivity of Bulk Samples

The residual resistance of sufficiently large single crystal samples is as a rule connected with the scattering of the electrons by the microscopic defects in the crystalline lattice such as impurity atoms. We noted already that in that case the hydrodynamic situation is impossible as the mean free path  $l_{ep}$  is always appreciably larger than the average distance  $\Delta$  between atoms. Indeed, even in the purest metals where the electron-impurity mean free path  $l_{ei} \sim 1$  cm, the quantity  $\Delta \lesssim 10^{-5}$  cm ( $l_{ei} \approx a\eta^{-1}$ ,  $\Delta \approx a\eta^{-1/3}$ ,  $\eta$  is the concentration of impurity atoms). The presence of normal collisions does therefore not influence at all the probability for the collision of an electron with an impurity atom, even in the case where the first collisions occur appreciably more often than the second. However, we have seen that it does not follow at all from this that the normal collisions do not affect the resistivity although it is not possible to interpret the result of this influence as intuitively as in the case of thin specimens.

Turning to calculations, we note that in the case considered the electron dispersion law cannot be assumed to be isotropic. Indeed, when  $\varepsilon = \varepsilon(|\mathbf{p}|)$ ,  $\mathbf{p} = m\mathbf{v}$  and it follows from the momentum conservation law that normal collisions do not influence the electrical current  $\mathbf{j} = e \sum \mathbf{v}$  at all.

We write the set of kinetic equations in the form

$$eE\mathbf{v} \frac{\partial f_0}{\partial \varepsilon} = \hat{I}_{ep} \{ \varphi, \psi \} + TW \frac{\partial f_0}{\partial \varepsilon} \varphi, \quad \hat{I}_{ep} \{ \varphi, \psi \} = 0. \quad (7.1)$$

In contrast to the preceding section, we have denoted the operator for the collisions with impurities by  $\hat{W}$ . We assume as usual that these collisions are elastic. Any function of the energy can then be taken through the operator  $\hat{W}$ , in particular,

$$\hat{W} \frac{\partial f_0}{\partial \varepsilon} \varphi_p = \frac{\partial f_0}{\partial \varepsilon} \hat{W} \varphi_p.$$

Moreover, it follows from the Hermiticity of the corresponding interaction Hamiltonian that the operator  $\hat{W}$  is symmetric and real:

$$W_{pp'} = W_{p'p} = W_{pp'}^* \quad \left( \hat{W} \varphi_p = \int W_{pp'} \varphi_{p'} dp' \right).$$

Using the second of Eqs. (7.1) we eliminate  $\psi_{\mathbf{q}}$  and write the equation for  $\varphi_{\mathbf{p}}$  in the comprehensive form (see (6.6)):

$$eE\mathbf{v} \frac{\partial f_0}{\partial \varepsilon} - T \frac{\partial f_0}{\partial \varepsilon} \hat{W} \varphi_p = \\ = \int d\mathbf{q} \mathcal{L}_{p,p+q}^{p+q} \left[ \varphi_{p+q} - \varphi_p - \frac{1}{M(\mathbf{q})} \int d\mathbf{p}' \mathcal{L}_{p',p+q}^{p+q} (\varphi_{p'+q} - \varphi_{p'}) \right] + \\ + \int d\mathbf{q} \mathcal{L}_{p,p-q}^{p-q} \left[ \varphi_{p-q} - \varphi_p + \frac{1}{M(\mathbf{q})} \int d\mathbf{p}' \mathcal{L}_{p',p-q}^{p-q} (\varphi_{p'+q} - \varphi_{p'}) \right], \\ M(\mathbf{q}) = \int d\mathbf{p} \mathcal{L}_{p,p+q}^{p+q}. \quad (7.2)$$

Since this equation cannot be solved in its general form, we consider limiting cases.

1. We start with the very low temperature region, where N-processes can be considered as a small correction ( $l_{ei} \ll l_{ep}(T)$ ). In the zeroth approximation the solution of Eq. (7.2) has the form

$$\varphi_0 = \frac{e}{T} \hat{W}^{-1} E \mathbf{v}.$$

The corresponding current density

$$\mathbf{j}_0 = \frac{2e^2}{h^3} \int \mathbf{v} \frac{\partial f_0}{\partial \mathbf{e}} \hat{W}^{-1}(\mathbf{E}, \mathbf{v}) d\mathbf{p}, \quad (7.3)$$

is clearly independent of the temperature. In the next approximation

$$T \frac{\partial f_0}{\partial \mathbf{e}} \varphi_1 = -\hat{W}^{-1} \hat{I}_{ep} \{\varphi_0, \psi_0\}$$

and the corresponding current density

$$\mathbf{j}_1 = \frac{2e^2}{h^3} \int \mathbf{v} \hat{W}^{-1} \hat{I}_{ep} \{\varphi_0, \psi_0\} d\mathbf{p},$$

can be readily shown to be proportional to  $T^5$ .

To make clear the signs of the corresponding corrections to the main value of the electrical resistivity tensor we study the scalar  $\mathbf{j}_1 \cdot \mathbf{E}$ . Starting from the symmetry of the quantity  $\mathcal{L}$  ( $\mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}+\mathbf{q}} = \mathcal{L}_{\mathbf{p}+\mathbf{q}, \mathbf{q}}^{\mathbf{p}}$ ) we check easily that for any function  $g(\mathbf{p})$

$$\int g(\mathbf{p}) \hat{I}_{ep} \{\varphi, \psi\} d\mathbf{p} = \int \int [g(\mathbf{p}) - g(\mathbf{p}+\mathbf{q})] \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}+\mathbf{q}} (\varphi_{\mathbf{p}+\mathbf{q}} - \varphi_{\mathbf{p}} - \psi_{\mathbf{q}}) d\mathbf{p} d\mathbf{q}.$$

Using this relation and the Hermiticity of the operator  $\hat{W}^{-1}$  we can show that

$$\mathbf{j}_1 \mathbf{E} = -\frac{2e^2}{h^3} \int \left\{ \int F^2(\mathbf{p}, \mathbf{q}) \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}+\mathbf{q}} d\mathbf{p} - \frac{1}{M(\mathbf{q})} \left[ \int F(\mathbf{p}, \mathbf{q}) \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}+\mathbf{q}} d\mathbf{p} \right]^2 \right\} d\mathbf{q},$$

where

$$F(\mathbf{p}, \mathbf{q}) = \hat{W}^{-1} \mathbf{E} \mathbf{v}_{\mathbf{p}+\mathbf{q}} - \hat{W}^{-1} \mathbf{E} \mathbf{v}_{\mathbf{p}}.$$

Finally it is convenient to write this result in the form

$$\mathbf{j}_1 \mathbf{E} = -\frac{2e^2}{h^3} \int [(F, F) - (F, G)^2] d\mathbf{q}.$$

Here  $G^{-2} = \int \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}+\mathbf{q}} d\mathbf{p}$  and the brackets indicate the scalar product with the positive weight  $\mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}+\mathbf{q}}$ :

$$(F, G) = \int F(\mathbf{p}, \mathbf{q}) G(\mathbf{p}, \mathbf{q}) \mathcal{L}_{\mathbf{p}, \mathbf{q}}^{\mathbf{p}+\mathbf{q}} d\mathbf{p},$$

so that, for instance,  $(G, G) = 1$ . By virtue of the Schwarz inequality the quantity  $-\mathbf{j}_1 \cdot \mathbf{E}$  is non-negative and one can show that it vanishes only for an isotropic electron dispersion law. (In the latter case the action of the operator  $\hat{W}^{-1}$  reduces to multiplication by a relaxation time and the difference  $\mathbf{v}_{\mathbf{p}+\mathbf{q}} - \mathbf{v}_{\mathbf{p}}$  is independent of  $\mathbf{p}$ .) Thus the main values of the resistivity tensor change according to  $\rho = \rho_0 + \beta T^5$  where  $\beta$  is positive and non-vanishing for any anisotropic Fermi surface.

2. It is clear from general considerations that such an increase in the resistivity with temperature cannot continue arbitrarily far. Indeed, when the temperature is increased the normal collisions become more probable, but they do not lead by themselves to a finite resistivity. We find the approximate solution of Eq. (7.2) for this limiting case.

In zeroth approximation  $\varphi'_0 = \mathbf{u} \cdot \mathbf{p}/T$  where  $\mathbf{u}$  has the meaning of a drift velocity of the electron-phonon system as a whole (in this approximation  $\mathbf{f}(\mathbf{p}) = \mathbf{f}_0(\varepsilon_{\mathbf{p}} - \mathbf{u} \cdot \mathbf{p}) \approx \mathbf{f}_0(\varepsilon_{\mathbf{p}}) - \mathbf{u} \cdot \mathbf{p} \partial/\partial \varepsilon \mathbf{f}_0(\varepsilon_{\mathbf{p}})$ ). The vector  $\mathbf{u}$  can be found from the condition that the first approximation equation

$$e \int \mathbf{p}(\mathbf{E} \mathbf{v}) \frac{\partial f_0}{\partial \mathbf{e}} d\mathbf{p} = \int \mathbf{p} \frac{\partial f_0}{\partial \mathbf{e}} \hat{W}(\mathbf{u} \mathbf{p}) d\mathbf{p}$$

can be solved. If we choose the axes of the Cartesian

coordinates along the principal directions of the symmetric tensor

$$\gamma_{ik} = \int \frac{\partial f_0}{\partial \mathbf{e}} p_i \hat{W} p_k d\mathbf{p},$$

then

$$u_i = e \gamma_{ii}^{-1} \int \frac{\partial f_0}{\partial \mathbf{e}} p_i (\mathbf{E} \mathbf{v}) d\mathbf{p}.$$

The corresponding current density

$$\mathbf{j}_0 = \frac{2e^2}{h^3} \int \mathbf{v} \frac{\partial f_0}{\partial \mathbf{e}} (\mathbf{u} \mathbf{p}) d\mathbf{p}$$

does not depend on the temperature but at the same time it differs from  $\mathbf{j}_0$  (see (7.3)). We can write the difference  $\mathbf{j}_0 \cdot \mathbf{E} - \mathbf{j}'_0 \cdot \mathbf{E}$  in the form

$$\mathbf{j} \mathbf{E} - \mathbf{j}' \mathbf{E} = \frac{2e^2}{h^3} \left[ (\Phi, \Phi) - \sum_{i=1}^3 (\Phi, G_i)^2 \right], \quad (7.4)$$

where

$$\Phi(\mathbf{p}) = \mathbf{E} \mathbf{v}_{\mathbf{p}}, \quad G_i(\mathbf{p}) = (\hat{W} p_i, \hat{W} p_i)^{-1/2} \hat{W} p_i$$

and the scalar product is defined as follows:

$$(\Phi, G) = \int \left( -\frac{\partial f_0}{\partial \mathbf{e}} \right) \Phi(\mathbf{p}) \hat{W}^{-1} G(\mathbf{p}) d\mathbf{p}.$$

One checks easily that this definition satisfies all necessary properties. In particular, commutativity follows from the symmetry and "elasticity" of the operator  $\hat{W}$ . We can directly check that the scalar square of any function is positive by using the fact that

$$W_{\mathbf{p}, \mathbf{p}'} = -B_{\mathbf{p}, \mathbf{p}'} + \delta(\mathbf{p} - \mathbf{p}') \int B_{\mathbf{p}, \mathbf{p}'} d\mathbf{p}',$$

where the symmetric and positive quantities  $B_{\mathbf{p}, \mathbf{p}'}$  are proportional to the square of the matrix elements of the electron-impurity interaction. One then shows easily that

$$(\Phi, \Phi) = \int \int B_{\mathbf{p} \mathbf{p}'} [(\tilde{\Phi}(\mathbf{p}) - \tilde{\Phi}(\mathbf{p}'))^2] d\mathbf{p} d\mathbf{p}',$$

$$\tilde{\Phi}(\mathbf{p}) = \left( -\frac{\partial f_0}{\partial \mathbf{e}} \right)^{1/2} \hat{W}^{-1} \Phi(\mathbf{p}).$$

Since the functions  $G_i$  are orthonormalized, i.e.,  $(G_i, G_k) = \delta_{ik}$ , the difference (7.4) is non-negative and vanishes only for an isotropic dispersion law. In the latter case  $G_i \sim p_i$  and the function  $\Phi(\mathbf{p}) = \mathbf{E}_i p_i/m$  can be expanded in the  $G_i$ . In the temperature range considered the resistivity is thus constant in first approximation and larger than the residual resistivity. One can show that in the next approximation a correction proportional to  $T^{-5}$  occurs.

For the sake of convenience we introduce a temperature  $T_0$  for which the probabilities for electron-phonon and electron-impurity collisions are the same (i.e.,  $l_{ep}(T_0) \approx l_{ei}$ ). The results obtained in the present section can then be written in the form

$$\rho(T) = \begin{cases} \rho_0 \left[ 1 + \gamma \left( \frac{T}{T_0} \right)^5 \right], & T \ll T_0, \\ \rho'_0 \left[ 1 - \delta \left( \frac{T_0}{T} \right)^5 \right], & T \gg T_0, \end{cases}$$

where  $\gamma, \delta > 0$  and  $\rho'_0 > \rho_0$  for any anisotropic electron dispersion law.

It is clear that when the temperature is further increased phonon-phonon Umklapp collisions are turned on and the resistivity again begins to increase (Fig. 5). It is difficult to expect that experimentally the resistivity will be constant in the intermediate temperature range. For instance, electron-electron collisions may occur here. We note that the recently performed meas-

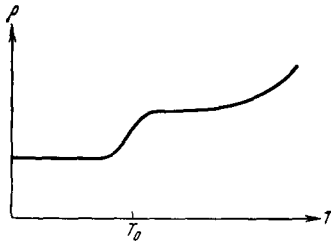


FIG. 5.

measurements of the electrical conductivity of very pure aluminum in the region of helium temperatures led to a very similar dependence  $\rho(T)$ .<sup>[27]</sup>

8. High-frequency Properties

We considered in Sec. 6 the case where the characteristic dimensions of the system did not exceed the mean free path with loss of quasi-momentum simply because of the presence of the boundaries of the sample. Such a situation can, however, arise also in an infinite sample under conditions when the electrical field and the current connected with it are non-zero only in a narrow layer. This just occurs for the skin effect in metals, which we shall treat in the present section.

The nature of the skin effect is determined by the ratio between the skin depth  $\delta$  and the mean free paths  $l^N$  and  $l^V$  (describing the normal collisions and collisions with loss of quasi-momentum). It is clear that the inequalities  $\delta \gg l^V$  and  $\delta \ll l^N$ ,  $l^V$  correspond to the well-known cases of the normal and the anomalous skin effect. There is yet another possibility,  $l^N \ll \delta \ll l^V$  which we shall now study.

We start with a formal solution of the problem, although it will become clear from what follows that the basic qualitative results could be obtained from intuitive considerations. It is natural to use, as in Sec. 6, the hydrodynamic description of the electrons in the metal. The complete set of equations has the form

$$\begin{aligned} i\omega u - \frac{e}{m} E &= v \Delta u - \frac{u}{\tau^V}, \\ \Delta E &= 4\pi i\omega e^{-2} j, \quad j = neu. \end{aligned} \tag{8.1}$$

Here  $E(r) e^{i\omega t}$  is the intensity of the electric field,  $u(r) e^{i\omega t}$  the velocity of the ordered motion of the electrons,  $v = l^N v_F$ ,  $l^N \approx l_{ep}(T)$ . For the sake of simplicity we assume the electron dispersion law to be isotropic.

The hydrodynamic approach requires apart from the condition  $l^N \ll l^V$  which we shall assume to be satisfied that the following inequality holds:

$$\omega\tau^V \ll 1, \quad l^N \ll \delta. \tag{8.2}$$

We consider as usual the problem of a normal incidence of a plane electromagnetic wave on a metal half-space. Let the  $z$  axis be directed into the metal perpendicularly to its boundary. After eliminating  $u$  from the set (8.1) we are led to the following equation for  $E(z)$ :

$$E'' - \alpha E' + i\beta E = 0,$$

where

$$\alpha^{-1} = l^N l^V (1 + i\omega\tau^V)^{-1}, \quad \beta^{-1} = \delta_0^2 l^N v_F \omega^{-1}, \quad \delta_0^2 = \frac{4\pi n e^2}{m c^2}.$$

The solution of this equation must satisfy the boundary conditions

$$E(0) = E_0, \quad E(\infty) = E'(\infty) = E''(0) = 0,$$

where  $E_0$  is the amplitude of the field at the surface; the condition on the second derivative follows from the requirement that the velocity  $u(z)$  vanish on the boundary of the specimen and at infinity.

One shows easily that

$$E(z) = \frac{E_0 (-s_2^2 e^{s_1 z} + s_1^2 e^{s_2 z})}{s_1^2 - s_2^2},$$

where  $s_1$  and  $s_2$  are the roots of the equation  $s^4 - \alpha s^2 + i\beta = 0$  with a negative real part. The surface impedance is hence equal to

$$\zeta = -i\omega c^{-1} \frac{E(0)}{E'(0)} = -i\omega c^{-1} \left( \frac{1}{s_1} + \frac{1}{s_2} \right).$$

Turning to a study of these expressions, we assume first of all that  $\omega\tau^V \ll 1$ . It will become clear from what follows that the opposite inequality is practically impossible in the frequency region considered. We note that then we need not worry about the first of conditions (8.2) as by assumption  $\tau^N \ll \tau^V$ .

We consider possible limiting cases:

a) When  $\alpha^2 \ll \beta$  one of the roots, say  $s_2$ , is much larger than the other and can thus be neglected. Then we have

$$E(z) \approx E_0 e^{s_1 z}, \quad s_1 = -\frac{1+i}{\delta_n}, \quad \delta_n = \delta_0 \left( \frac{2}{\omega\tau^V} \right)^{1/2};$$

$\delta_n$  is the skin depth for the normal skin effect. This result can easily be understood, since the inequality  $\alpha^2 \gg \beta$  when written in the form  $l^V \ll \delta_n^2 / l^N$  means that an electron diffusing within the limits of the skin layer proceeds to collide many times with loss of momentum. We note that the second of conditions (8.2) is satisfied since when  $l^N \ll l^V$  it follows from the given inequality that  $l^N \ll \delta_n$ .

b) When  $\alpha^2 \gg \beta$  one shows easily that

$$s_1 = -is_2 = -\beta^{1/4} e^{-i\pi/8}, \quad E(z) = \frac{1}{2} E_0 (e^{s_1 z} + e^{is_1 z}).$$

The electrical field thus damps into the metal, roughly speaking as  $\exp(-z/\delta^*)$  where the effective skin-depth is

$$\delta^* \approx \beta^{-1/4} = (\delta_0^2 v_F \omega^{-1} l^N)^{1/4}.$$

We can obtain this last equation by an intuitive reasoning similar to the one given by Pippard<sup>[28]</sup> for the interpretation of the results of the anomalous skin effect. We note that the skin-depth can always be written as

$$\delta \approx \frac{\delta_0}{\sqrt{\omega\tau^*}} \approx \left( \frac{m v_F}{n^* e^2 \omega l^*} \right)^{1/2},$$

as far as order of magnitude is concerned provided we understand by  $n^*$  the density of those electrons which appreciably interact with the field and by  $l^*$  the smallest of the following two quantities: path of the electrons in the skin layer and the mean free path with loss of momentum  $l^V$ . In the limits of the hydrodynamic approach  $l^N \ll \delta$  and therefore all electrons play approximately the same role, i.e.,  $n^* \approx n$ . However, it follows from the inequality  $\alpha^2 \gg \beta$  that the path traversed

by an electron in the skin layer  $(\delta^*)^2/l^N \ll l^V$  and we must thus take  $l^* \approx \delta^2/l^N$ . Ultimately we arrive at the equation

$$\delta \approx \delta_0 \left( \frac{\nu l^N}{\omega \delta^2} \right)^{1/2},$$

from which the required result  $\delta \approx \delta^*$  follows.

The inequalities  $l^N \ll \delta^*$  and  $\alpha^2 \ll \beta$  lead to the following restrictions on the magnitude of  $l^N$ :

$$\frac{\delta_0^2}{l^V} \ll l^N \ll \delta_{an},$$

where  $\delta_{an} \approx (\delta_0^2 \nu_F / \omega)^{1/3}$  is the skin depth for the anomalous skin effect.

One verifies easily that  $\delta^*/\delta_{an} \approx (l^N/\delta_{an})^{1/4}$ , i.e.,  $\delta^* < \delta_{an}$ . However, when the frequency increases  $\delta_{an}$  decreases faster than  $\delta^*$  and when  $l^N \approx \delta_{an}$  these quantities are the same. The region considered joins thus directly the anomalous skin-effect region (where the hydrodynamic description is clearly unsuitable).

We finally write down an expression for the surface impedance in different frequency ranges. We denote by  $\omega_1 = 1/\tau^V (\delta_0/l^V)^2$  the frequency where the transition from the normal to the anomalous skin effect usually takes place:  $l^V \approx \delta_n(\omega_1) \approx \delta_{an}(\omega_1)$ .

When  $\omega/\omega_1 \ll l^V/l^N$  the skin effect remains normal:

$$\zeta_n \approx \left( \frac{\omega}{\omega_0^2 \tau^V} \right)^{1/2} e^{i\frac{\pi}{4}}, \quad \omega_0 = \frac{c}{\delta_0}.$$

Further, in the region\*  $l^V/l^N \ll \omega/\omega_1 \ll (l^V/l^N)^3$  we have

$$\zeta^* \approx \left( \frac{\nu_F}{c} \right)^{1/2} \left( \frac{\omega^3 \tau^N}{\omega_0^2} \right)^{1/4} e^{3\pi i/8}. \quad (8.3)$$

Finally, when  $\omega/\omega_1 \gg (l^V/l^N)^3$  (or  $l^N \gg \delta_{an}$ ) the anomalous skin effect occurs:

$$\zeta_{an} \approx \left( \frac{\nu_F}{c} \right)^{1/3} \left( \frac{\omega}{\omega_0} \right)^{2/3} e^{i\pi/3}.$$

We have dropped in these formulae numerical coefficients of order unity since their values are connected with assumptions about the isotropy of the electron and phonon dispersion laws. It is clear that the order of magnitude of the impedance and its dependence on the basic physical parameters remain valid for any dispersion law. This statement refers also to the ratio between the real and the imaginary parts of the impedance in the cases of the normal and the anomalous skin effect. Unfortunately, in the most interesting case (8.3) the ratio of  $\text{Re } \zeta^*$  to  $\text{Im } \zeta^*$  depends in a very complicated way on the form of the electron and phonon dispersion laws and the phase factor  $\exp(3\pi i/8)$  has apparently no physical meaning. We note that in the general case the viscosity  $\nu$  is a fourth rank tensor and therefore even in the cubic lattice the surface impedance must have tensor properties.

From what has been said it is clear that normal electron-phonon collisions appreciably change the nature of the skin effect in the case when they occur more

\*We note that this region can be rather wide. For instance, when  $\delta_0 = 3 \cdot 10^{-6}$  cm,  $l^V = 0.1$  cm,  $l^N = 10^{-3}$  cm and  $\nu_F = 10^8$  cm/sec we have  $10^2 \ll \omega \ll 10^6$  sec<sup>-1</sup>. From this estimate it is incidentally clear that the inequality  $\omega \tau^V \ll 1$  is practically always satisfied.

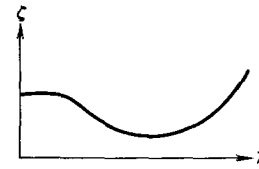


FIG. 6.

often than collisions with loss of momentum, viz:

1. The skin effect remains normal up to higher frequencies than usual. On the other hand, the region of the anomalous skin effect is considerably reduced on the low frequency side (by a factor  $(l^V/l^N)^3$ ).

2. There exists a rather wide intermediate range of frequencies in which the surface impedance depends in an unusual manner on the frequency and is mainly a function of temperature:  $\zeta^* \sim \omega^{3/4} T^{-4/5}$ .

3. The behavior of  $\zeta(T)$  has the form shown in Fig. 6. At very low temperatures  $\zeta = \zeta_{an}$  and constant. After passing into the intermediate range the impedance decreases proportional to  $T^{-5/4}$ . When the temperature increases further the skin effect becomes normal. In that region the impedance is initially constant and then, after the phonon-phonon U-processes are included, increases with temperature.

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