

SIMILARITY HYPOTHESIS IN THE THEORY OF PHASE TRANSITIONS

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1. INTRODUCTION

THE critical phenomena in various substances have been investigated since the second half of the 19th century. The nature of critical phenomena varies. At the Curie point T_C matter goes over from the paramagnetic into the ferromagnetic (or antiferromagnetic) state. Critical phenomena occur when matter goes over into the ferroelectric state. In solids there occur also phase transitions in which the crystal symmetry changes. There are known cases when such transitions are in general not accompanied by an abrupt change in the specific volume and by release of heat, or when the changes are very small (for example, on going from the α to the β phase of quartz). Mention should also be made of alloy ordering. The transition of liquid helium into the superfluid state and the transition of certain metals into the superconducting state are also second-order phase transitions—there are no jumps of the specific volume and no heat is released. From this point of view, the critical point between a liquid and a gas should also be regarded as a second order phase transition, since it is precisely at this point that the jumps of the density and of the entropy vanish, and the difference between the liquid and gas disappears.

These phenomena, which at first glance seem so different, have certain general features in common.

The first and the most common of them, is the anomaly of the specific heat. Second, we note the growth of the "susceptibility" of the system to external action. We have in mind the magnetic susceptibility χ in the case of a ferromagnet, the isothermal compressibility k_T in the case of the liquid-gas critical point, the quantity $(\partial c/\partial \mu)_{p,T}$ in the case of the critical point of a two-component solution (c —concentration, μ —chemical potential), etc. In some cases (helium, quartz) the generalized susceptibility has no simple physical meaning. In almost all cases one can speak of the occurrence of a long-range (a macroscopically nonvanishing correlation between the elements of the system) below the transition point.

In some of the described cases, anomalous scattering of light (the liquid-gas critical point) or scattering of neutrons (ferromagnets) were observed near the transition point. This is evidence of an increase in the fluctuations. The cause of the anomalous growth of the fluctuations is clear: the ordered and disordered phases hardly differ from near the transition point.

We shall call all these phenomena critical. The theory of critical phenomena, which dates back to Gibbs, has predicted the thermodynamic properties near the critical point on the basis of the simplest assumption that it is possible to expand the thermodynamic potential in powers of the deviations of the temperature and of

the volume from the critical values (see, for example, ^[1]).

The main conclusions of the theory are as follows. When $V = V_C$ the specific heat C_V experiences a finite discontinuity. Near the critical point, the isotherms are approximately described by the Van der Waals equation. Consequently the liquid-gas coexistence curve is of the form

$$T_c - T = A(V - V_c)^2.$$

The compressibility becomes infinite in accordance with the law

$$k_T \sim (T - T_c)^{-1}.$$

Ornstein and Zernike took into account the fluctuations of the density within the framework of the same basic assumptions, and this led to the well known density correlation function $\langle n(\mathbf{r})n(\mathbf{r}') \rangle \sim \exp[-k(\mathbf{r} - \mathbf{r}')]/|\mathbf{r} - \mathbf{r}'|$, and made it possible to explain anomalous scattering of light (critical opalescence).

As more experimental facts were accumulated, it became clear that the premises of the classical theory are incorrect (at least sufficiently close to the critical point). Voronel' and co-workers have observed ^[2] that the specific heat C_V increases anomalously when the critical point is approached. However, deviations from the predictions of the classical theory were observed even earlier in the behavior of the compressibility and of the critical isotherm ^[3].

A similar situation arose in the theory of second-order phase transitions. A phenomenological theory was first proposed by Weiss for ferromagnets. A more general theory was constructed by Landau, who first pointed out that these transitions are connected with changes of symmetry. As in the theory of critical phenomena, the classical theory of second-order phase transitions was based on very simple mathematical assumptions, and reduced physically to the theory of a homogeneous self-consistent field, the magnitude of which is determined from the condition that the thermodynamic potential has a minimum. This theory holds with great accuracy for phase transitions in superconductors and in certain solids.

Let us see what the theory leads to in the case, for example, of a ferromagnet. The specific heat experiences a finite jump. The magnetic susceptibility behaves like $(T - T_C)^{-1}$ (the Curie law). The magnetic moment below the transition point behaves like $(T_C - T)^{1/2}$, and therefore the following equation is satisfied:

$$T_c - T = \tilde{A}M^2.$$

By extending the theory in the spirit of Ornstein and Zernike, we obtain for the correlation function of the

moments a value proportional to $\exp[-k|r - r'|]/|r - r'|$.

We purposely emphasized the analogy between the theories of Gibbs, Weiss, and Landau. This analogy is perfectly natural, since any of these theories starts out from a certain self-consistent field and presupposes that the density of the thermodynamic potential can be expanded in even powers of the field. It is usually sufficient to retain terms of fourth order. The role of the self-consistent field is played by $V - V_C$ in the Gibbs theory, by the "effective" magnetic field H_{eff} in the Weiss theory, and by the order parameter η in the Landau theory.

However, as a result of Onsager's^[4] investigations, which were devoted to a simple mathematical model (a plane Ising lattice), it became clear that the physical picture of the transition may not be so simple. In particular, it turned out that in this model the specific heat behaved like $-\ln|T - T_C|$, and the magnetic moment is $M \sim (T_C - T)^{1/8}$. Numerical investigations of the three-dimensional Ising model (see, for example,^[5]) likewise do not agree with the theory of the self-consistent field.

Further, the experiment of Buckingham, Fairbank and Kellers^[6] has shown that the specific heat of helium increases logarithmically in the most immediate vicinity of the λ point.

Different anomalies of the thermodynamic quantities were observed in ferromagnets and antiferromagnets (see^[5]). Thus, at least some of the experiments do not agree with the self-consistent field theory.

It is remarkable that different phase transitions reveal similar features. For example, the specific heat behaves like $-\ln|T - T_C|$ (or, within the limits of experimental accuracy, like $|T - T_C|$ raised to a small power) in all the "anomalous" cases. Further, in ferromagnetic transitions, the magnetic moment below the transition point behaves like $|T - T_C|^\beta$ with $\beta \approx 1/3$ (see^[5]). A similar behavior was found for the jump of the density near the critical point^[7].

The construction of a consistent macroscopic theory of phase transitions turned out to be a difficult problem. The reason for the difficulties lies in the fact that the growing fluctuations interact strongly with one another near the critical points. Even in the case of a weak "bare" interaction between the particles (spins), the effective interaction is strong near the transition point. Inasmuch as the correlation radius becomes large compared with the distance between the particles, many-particle interactions are just as important as two-particle interactions. The theoretician can no longer use his main weapon—perturbation theory. The general problem of the phase transition is one of the last unsolved problems in the physics of condensed state. It is therefore not surprisingly that it attracts the attention of the physicists. At first glance the difficulties are purely mathematical, since in principle the solution of the problem is contained in the Gibbs statistics and in the laws of classical or quantum mechanics.

It is appropriate to mention in this connection the theory of turbulence, where a similar situation existed at one time. It might seem at first glance that the entire information is contained in the equations of hydrodynamics and heat conduction. However, it was necessary to clarify first the simple physical picture of the turbulent state (transfer of energy to vortices having

nearly equal scales, etc), and this made it possible for Kolmogorov to construct a simple theory based on dimensionality considerations.

This suggests the idea of a similar approach to the theory of critical phenomena. Simple hypotheses were advanced in a number of recent papers^[9-12] concerning the structure of matter in the region of the phase transition.*

2. CELL EQUIVALENCE HYPOTHESIS

The authors use the growth of the correlation radius r_C of the system near the critical point (curve) and mentally divide the system into a number of subsystems (cells), each containing a large number of particles, but each having a dimension smaller than or of the order of r_C . The simplest premises concerning the properties of these systems lead to conclusions that a connection exists between the different thermodynamic functions near the transition point.

The first considerations of this type were advanced by Buckingham^[9]. The model of the substance is not made very specific in the subsequent considerations. This may be a "lattice gas" (see, for example,^[11]), the models of Ising or Heisenberg for magnets, an ordered alloy, etc. For concreteness we shall speak of a lattice gas. The method of "translating" from the language of this model to the language of the model of magnets is well known (see Sec. 5).

As already mentioned, Buckingham divided the system into cells of volume V . The interaction energy of such cells is small compared with the energy of the cells themselves (the surface of the cell is small compared with the volume). But it is precisely this weak surface effect which leads to singularities in the thermodynamic quantities. Indeed, by "turning off" the interaction, we arrive at an aggregate of non-interacting systems of finite volume. The thermodynamic functions of the finite system have no singularities. Therefore the phase transition in the system of volume V becomes smeared out, as it were, over a temperature interval $\Delta T(V)$. We shall henceforth measure the temperature in units of T_C and introduce the notation $t(V) = \Delta T(V)/V_C$. It is obvious that $t(\infty) = 0$. Turning on of the interaction changes the entropy of the system appreciably. According to Buckingham's simplified scheme, on the equilibrium curve, when the interaction is turned off, each cell can be filled with matter in the first phase or else in the second phase. This yields an entropy $\ln 2$ per cell or $\ln 2/V$ per particle (V is measured in units of the cell volume). Turning on the interaction leads to a complete correlation between the neighboring volumes and changes the entropy by an amount $-\ln 2/V$.

Turning on the interaction is equivalent to replacing the volume V by an infinitely large volume. We can therefore write for the per-particle entropy s_V of the volume V

$$\Delta s_V = s_\infty - s_V = -\ln 2/V. \quad (1)$$

We now vary the temperature. We introduce the nota-

*The similarity hypothesis was first advanced in the theory of phase transitions in [8].

tion $\tau = (T - T_C)/T_C$. Formula (1) is valid under the assumption that $t(V) \ll |\tau| \ll 1$ ($\tau < 0$). Increasing τ , we go over into the single-phase region, where all the cells are filled in like fashion. In this region, $\Delta s_V = 0$. Buckingham's main assumption is that the only characteristic dimension of Δs_V as a function of τ is $t(V)$, and that $\Delta s_V(\tau)$ can be written in the form

$$\Delta s_V(\tau) \equiv s_\infty(\tau) - s_V(\tau) = \frac{\ln 2}{V} f\left(\frac{\tau}{t(V)}\right), \quad (2)$$

where $f(x)$ is a universal function that takes on the values

$$f(-\infty) = -1, \quad f(+\infty) = 0.$$

A quantitative description of the transition proposed above can hardly be regarded as physically justified, since fluctuations can exist in either the two-phase or single-phase regions and can greatly change the entropy. It therefore seems natural to employ the less restrictive assumption

$$\Delta s(\tau, V) = t^{1-\alpha} f\left(\frac{\tau}{t}\right). \quad (3)$$

The quantity Δs remains finite under all conditions. Therefore $\alpha < 1$. Differentiating (3) with respect to τ for a specified $V(t)$, we obtain

$$C(\tau, 0) - C(\tau, t) = \text{const} \cdot t^{-\alpha} f'\left(\frac{\tau}{t}\right), \quad (4)$$

where $C(\tau, t)$ is the specific heat per site in a cell of volume V , situated in a thermostat with specified τ . The relation between V and $t(V)$ is not established in this theory.

When $t \neq 0$, the specific heat $C(\tau, t)$ has no singularities as a function of τ . On the other hand, $C(\tau, 0)$ has by assumption a singularity. Consequently, as $\tau \rightarrow 0$ and for finite t the difference $C(\tau, 0) - \text{const} \cdot t f(\pi/t)$ should remain finite and independent of τ . This is possible only if $f'(x)$ has the asymptotic form

$$f'(x) \sim x^{-\alpha} \quad (x \ll 1) \quad (5)$$

and

$$C(\tau, 0) \sim \tau^{-\alpha}. \quad (6)$$

In the important particular case $\alpha = 0$ we get in lieu of (4)

$$C(\tau, 0) - C(\tau, t) = \text{const} \cdot f'\left(\frac{\tau}{t}\right).$$

At small values of τ and at finite t , the quantity $C(\tau, t)$ is asymptotically independent of τ only if

$$C(\tau, 0) \sim -\ln|\tau|. \quad (7)$$

So far we have assumed that $C(\tau, \infty)$ becomes infinite when $\tau \rightarrow 0$. Of course, the case when C has a finite limit as $\tau \rightarrow 0$ does not contradict (4). Thus, the condition $\alpha = 0$ leads to a logarithmic divergence or to a finite jump of the specific heat.

Buckingham's theory does not make it possible to determine α , but leads to a certain law of corresponding states. Buckingham's idea of the equivalence of cells of different dimensions turned out to be quite fruitful. He assumed that in a two-phase system the problem reduces to the Ising model with a certain interaction energy that depends on the dimension of the cell and on the proximity to the transition point.

3. KADANOFF'S SCALE TRANSFORMATIONS

These ideas were developed most fully by Kadanoff^[11]. Let us consider an ordered ferromagnet. We break it up into cells, as already indicated. Kadanoff's main premise is that each cell possesses a certain moment that interacts with the moments of the neighboring cells, therefore following certain changes in the values of τ and of the reduced magnetic field $h = \mu_B H/T_C$, the cell problem becomes equivalent to the problem of individual spins m_r (the models of Ising, Heisenberg, etc.), where μ_B is the Bohr magneton. Let the dimension of the cell be L and let the corresponding values be $\tau(L)$ and $h(L)$. Kadanoff's second hypothesis—the similarity law—states that

$$\tau(L) \sim L^y \tau, \quad h(L) \sim L^x h. \quad (8)$$

Here x and y are certain constants.

Let the magnet be located in a nonzero external field (or at $h = 0$ below the transition point). When the field is changed by dh , the free energy changes by an amount

$$dF = \sum_r dh \langle m_r \rangle = \sum_\alpha dh \langle L \rangle \langle M_\alpha \rangle. \quad (9)$$

The brackets $\langle \rangle$ denotes averaging over the Gibbs ensemble. The first sum is taken over the lattice sites and the second over the cells. Obviously

$$\sum_{r \text{ inside the cell}} \langle m_r \rangle = \langle M \rangle L^d, \quad (10)$$

where d is the dimensionality of the space (in real space $d = 3$, but it is convenient to check the obtained relations with the exactly calculated plane Ising model). From (8)–(10) we get

$$\langle M_\alpha \rangle = L^{d-x} \langle m_r \rangle. \quad (11)$$

In accordance with the assumption made, we shall regard $\langle m \rangle$ and $\langle M \rangle$ as similar functions of the arguments τ and h and $\tau(L)$ and $h(L)$, respectively. The validity of such a hypothesis may be in question. In fact, in the Ising model, for example, $m^2 = 1$ but $M^2 \neq 1$, and one might think that the different normalization conditions will lead to different results. It is necessary to consider more consistently cells of different dimensions L_1 and L_2 , such that $1 \ll L_1$ and $L_2 \ll r_C$.

In such a formulation, the difference between m and M disappears. On the other hand, the condition $L \gg 1$ does not prevent us from extrapolating the obtained relations to cells of unit dimension.

Returning to (11), we obtain for the magnetization $\langle m \rangle$ per particle

$$\langle m \rangle(\tau, h) = L^{-d+x} \langle m \rangle(\tau L^y, h L^x). \quad (12)$$

The brackets denoting the averaging will henceforth be omitted. The general solution of the functional equation (12) takes the form

$$m(\tau, h) = \tau^{\frac{d-x}{y}} \varphi(\tau/h^{y/x}), \quad (13)$$

where $\varphi(x)$ is an arbitrary function. We can point out the asymptotic forms of $\varphi(x)$ when $x \gg 1$ and $x \ll 1$. Large x correspond to weak magnetic fields. In this case $m \approx 0$ above the transition point and m depends weakly on h below the transition point. Consequently, $\varphi(+\infty) = 0$ and $\varphi(-\infty) = \text{const} \neq 0$.

From (13) with $\tau < 0$ and $h = 0$ we obtain the depen-

dence of the spontaneous moment on τ :

$$m(\tau, 0) = (-\tau)^\beta, \quad \beta = \frac{d-x}{y}. \quad (14)$$

In the opposite limiting case, corresponding to large h , the moment m should not depend on τ . From this we get the asymptotic behavior as $x \rightarrow 0$, namely $\varphi(x) \sim x^{-\beta}$, and when $h \gg \tau^{x/y}$ the magnetization takes the form

$$m(0, h) \sim h^{1/\delta}, \quad \delta = \frac{x}{d-x}. \quad (15)$$

Let us turn to a calculation of the specific heat C . We denote by $F(\tau, h)$ the free energy per site. By definition, $F(\tau(L), h(L))$ is the free energy per cell of dimension L . Therefore

$$F(\tau(L), h(L)) \equiv F(\tau L^y, h L^x) = L^d F(\tau, h). \quad (16)$$

The general solution of (16) is

$$F(\tau, h) = \tau^{d/y} \tilde{f}(\tau/h^{y/x}). \quad (17)$$

Differentiating (17) twice we get

$$C(\tau, h) = \tau^{-\alpha} \tilde{\varphi}(\tau/h^{y/x}), \quad \alpha = 2 - \frac{d}{y}, \quad (18)$$

where $\tilde{\varphi}(x)$ is an arbitrary function with the following asymptotic values:

$$\begin{aligned} \tilde{\varphi}(\pm\infty) &= \varphi_{\pm}, \\ \tilde{\varphi}(x) &= a_{\pm} |x|^{\epsilon} \quad (x \ll 1), \end{aligned} \quad (19)$$

where φ_{\pm} and a_{\pm} are generally different constants. From this we get the behavior of $C(\tau, h)$:

$$C(\tau, h) \sim \begin{cases} \tau^{-\alpha} & (\tau \gg h^{y/x}), \\ h^{-\epsilon} & (\tau \ll h^{y/x}), \quad \epsilon = \alpha \frac{y}{x}. \end{cases} \quad (20)$$

We see that the four quantities α , β , δ , and ϵ , which can be determined experimentally, are expressed in terms of exponents x and y . Therefore they are connected by the following two relations

$$\alpha + \beta + \beta\delta = 2, \quad (21)$$

$$\epsilon\beta\delta = 2\alpha. \quad (22)$$

So far we have considered the states of a system with nonzero moment. Above the Curie point ($\tau > 0$) at $h = 0$ the magnetic moments of the cells have a fluctuating character. This is true also for sufficiently small nonvanishing h . In order to relate the exponents of the singularities of the physical quantities (the critical indices) in this region with x and y , it is necessary to make one more fundamental assumption: the fluctuating moments depend on the cell dimension L in the same manner as the constant ones (formula (11)). This assumption corresponds to a picture in which an "ordered" phase is established in a fluctuating manner in a cell of dimension L . It is assumed that the dependence of such a moment on L is described by formula (12). The foregoing hypothesis makes it possible to obtain information on the correlation function of the moments. We introduce

$$G(R, \tau, h) = \langle m_0, m_R \rangle. \quad (23)$$

from the equivalence of cells of different dimensions it follows that for cells of dimension L the correlation function $\langle M_0, M_R \rangle$ coincides identically with

$G(R/L, \tau L^y, h L^x)$. On the other hand, from (11) we get

$$G\left(\frac{R}{L}, \tau L^y, h L^x\right) = L^{2(d-x)} G(R, \tau, h). \quad (24)$$

The general solution of the functional equation (24) is

$$G(R, \tau, h) = R^{-2(d-x)g} g(\tau R^y, h R^x), \quad (25)$$

where $g(\xi, \eta)$ is an arbitrary function of its arguments. From general considerations it is obvious that $g(\xi, 0)$ and $g(0, \eta)$ are functions that decrease with increasing arguments ξ and η respectively.

The magnetic susceptibility can be obtained from the relation

$$\chi(\tau, h) = \sum_R G(R, \tau, h). \quad (26)$$

In limiting cases of small τ and h we get from (25) and (26)

$$\chi(\tau, 0) \sim \tau^{-\nu}, \quad \nu = -\frac{d-2x}{y}, \quad (27)$$

$$\chi(0, h) \sim h^{-\lambda}, \quad \lambda = \gamma \frac{y}{x} = 2 - \frac{d}{x}. \quad (27')$$

The exponents α and ϵ of the singularity of the specific heat are the same on both sides of the phase-transition point. This is obvious for the quantity ϵ , which corresponds to the asymptotic form for a strong magnetic field. As to the quantity α , we can present here the following reasoning. When the sign of τ changes, ϵ and δ remain unchanged. But then it follows from (21) and (22) that α and β also remain unchanged.

We introduce one more critical index ν defined by the relation

$$G(R, 0, 0) \sim R^{-\nu}. \quad (28)$$

From (26)–(28) we get

$$\gamma y = \lambda x = d - \nu.$$

A summary of the foregoing arguments is presented in two tables. Table I contains the definitions of the critical indices α , β , γ , δ , ϵ , λ , μ , and ν . Their expressions in terms of x and y and the four relations between the six quantities α , β , γ , δ , μ , and ν are shown in Table II. Some of these relations were obtained earlier by Fisher, Sykes, et al.^[13] and by Widom^[14].

Table I. Definitions of critical indices

Specific heat C	Susceptibility χ	Magnetization m	Correlation radius r_c	Correlation function $G(R, 0)$
$ \tau ^{-\alpha}$ $ h ^{-\epsilon}$	$ \tau ^{-\nu}$ $ h ^{-\lambda}$	$ -\tau ^\beta$ $ h ^{1/\delta}$	$ \tau ^{-\mu}$	$R^{-\nu}$

Table II. Critical indices expressed in terms of the constants x and y . Relations (A)–(D) connect the indices α , β , γ , δ , μ , and ν .

α	β	γ	δ	ϵ	λ	μ	ν
$2 - \frac{d}{y}$	$\frac{d-x}{y}$	$\frac{2x-d}{y}$	$\frac{x}{d-x}$	$\alpha \frac{x}{y}$	$\gamma \frac{x}{y}$	$1 - y$	$2(d-x)$
(A) $\alpha + 2\beta + \gamma = 2$. (B) $\beta\delta = \beta + \gamma$. (C) $\mu d = 2 - \alpha$. (D) $\nu(1 + \delta) = 2d$.							

The similarity relations are satisfied in the Gibbs-Weiss-Landau theory of the homogeneous self-consistent field. More accurately speaking, relations (A) and (B) of Table II are satisfied. Inasmuch as the field is assumed beforehand to be homogeneous, the quantities μ and ν , which determine the correlation radius and the behavior of the correlation function, become meaningless. We present the values of the critical indices in the aforementioned theories:

$$\alpha = 0, \beta = 1/2, \gamma = 1, \delta = 3.$$

In the classical theories of the self-consistent field it is possible to take into account the fluctuations by the Ornstein-Zernike method. In the Landau theory with ordering parameter η , one introduces into the thermodynamic potential a term proportional to the square of the gradient of η . In this case the density of the potential $\Phi(\mathbf{r}, \tau)$ takes the form

$$\Phi(\mathbf{r}, \tau) \sim (\nabla\eta)^2 + a_0\tau\eta^2 + b_0\eta^4, \quad (29)$$

where a_0 and b_0 are constants. By definition, $\Phi(\mathbf{r}/L, \tau(L))$ is the thermodynamic potential of a cell of dimension L at temperature τ . Therefore $\Phi(\mathbf{r}/L, \tau(L)) = L^d\Phi(\mathbf{r}, \tau)$ (see (16)). However, $\Phi(\mathbf{r}, \tau)$ defined by relation (29) cannot be transformed into $L^d\Phi(\mathbf{r}, \tau)$ by the homogeneous transformations $\eta \rightarrow \eta L^\sigma$, $\tau \rightarrow \tau L^y$, and $\mathbf{r} \rightarrow \mathbf{r}/L$. In the classical theory, in which the fluctuations are taken into account by the Ornstein-Zernike method, the similarity conditions are not satisfied.

4. FREE ENERGY OF FLUCTUATIONS

It is useful to examine the results from a different point of view^[12]. The anomalies of the thermodynamic behavior of the system are due to large scale fluctuations that develop near the transition point. We visualize the physical picture of such a state as follows. We break up the entire small magnet into regions with linear dimensions on the order of r_C . Inside each such region, the local moments are correlated and the total moment M over the region differs from zero. But the moments of different regions can have, with equal probability (or with nearly equal probability below the transition point), opposite signs, so that the total moment averaged over the entire system is equal to zero. The only quantity characterizing the interaction of the fluctuations with the external magnetic field h is Mh . It is therefore natural to assume that the free energy $F(\tau, h)$ is of the form

$$F(\tau, h) = F(\tau, 0) + \frac{V}{r_C^d} f(Mh). \quad (30)$$

We have assumed that cells having the dimension r_C do not interact, so that $f(Mh)$ is the free energy of such a cell. Putting $M \sim \tau^{-\rho}$, we obtain the law of corresponding states (in Kadanoff's notation $\rho = x/y = 2 - \alpha - \beta$).

The method of scale transformations leads to a similar conclusion. Nothing prevents us from extrapolating all the relations of the preceding section and extending them up to cells of dimension $L \sim r_C$ (but not larger!). But it is then clear that the interaction with the magnetic field is determined by the moment of such a cell (since the indeterminate scale L has disappeared).

Let us show briefly how to obtain the relations of the

preceding section from (30). When $\tau \rightarrow 0$, the function $F(\tau, h)$ should tend to a finite limit. To this end it is necessary to have the function $f(\xi)$ behave like $\xi^{d/x}$ at large values of ξ . This means that $F(h) - F(0)$ behaves like $h^{d/x}$ as $\tau \rightarrow 0$. From this we get the connection between δ and x (see Table II). In a weak field when $\tau > 0$ we have $F(h) - F(0) \sim h^2$. We conclude therefore, that at small ξ we get $f(\xi) \sim \xi^2$, leading to a relation between γ , x , and y .

In order to connect the critical indices γ , δ , μ , and α , it is necessary to make an additional assumption: in a magnetic field $h \neq 0$ the phase transition becomes smeared out and $F(\tau, h)$ becomes an analytic function of τ . Indeed, when $h \neq 0$ we get a nonzero total moment, and consequently, a long-range order in the arrangement of the spins at any temperature. The phase transition vanishes, since the only feature distinguishing between the phases is the existence or absence of long-range order.

The foregoing denotes that the function $F(\tau, h)$ can be expanded at small and finite h in a series of integer powers of τ . The asymptotic form of $f(\xi)$ should contain at large ξ a term f_0 of ξ raised to the zero power. The corresponding term in (30) should compensate for the singularity in $F(\tau, 0)$. This leads to a relation between α and y .

The transition to the region $\tau < 0$ calls for analytic continuation of the function $f(\xi)$. For the Ising model, Lee and Yang^[15] proved that the free energy F is an analytic function of $z = e^h$ with a cut along the circle $|z| = 1$, with the exception of a certain arc from $z = e^{-i\omega}$ to $z = e^{i\omega}$. When $T \rightarrow T_C$ from above ($\tau \rightarrow 0$, $\tau > 0$) we get $\omega \rightarrow 0$. Thus, when $\tau > 0$ the function $F(\tau, h)$ is analytic in $z - 1$ within a circle of radius $|e^{i\omega} - 1|$. At small values of τ the convergence radius is approximately equal to ω , and $z - 1 \approx h$. In the same approximation we can regard the cuts in the h plane as extending from $\pm i\omega$ to $\pm i\infty$. It follows from (30) that $\omega \sim \tau^\rho$, and $f(\xi)$ have cuts from $\pm i\xi_0$ to $\pm i\infty$.

The function $r_C^{-d}|f(Mh) - f_0|$ is an analytic function of τ at small τ and finite h . The transition from positive to negative τ denotes rotation in the ξ plane through an angle $\pm\pi\rho$ from a point on the positive ξ semiaxis ($|\xi| \gg 1$). Such a rotation inevitably leads to a crossing of the cut. Continuing the function $f(\xi)$ in the second sheet into the region of small $|\xi|$ for the same value of $\arg \xi$, we fall into the region corresponding to $\tau < 0$ and $|\tau|^\rho \ll h$. Whereas in the first sheet the expansion of $f(\xi)$ began with ξ^2 , in the second sheet the expansion contains a term proportional to ξ (i.e., h). This follows from the existence of a spontaneous moment at $\tau < 0$ and $h = 0$. The value of the moment is obtained from the well known function $F(\tau, h)$, from which we get the connection of ρ with α and β .

Widom^[10] obtained certain relations for the critical indices of a liquid by applying the thermodynamic relations to a volume of the order of r_C^d . This idea was first advanced by Pippard^[16], who proposed that in a region of linear dimensions r_C the fluctuation of the temperature $\Delta\tau$ should be of the same order as τ (the phase transition becomes smeared out in a region of this size). In conjunction with the ordinary thermodynamic formulas, this yields for the temperature fluctuation (see, for example,^[11])

$$\tau^2 \sim 1/r_C^d C_V,$$

which is equivalent to (C).

We shall apply Widom's reasoning in modified form to magnets. Like Pippard, Widom proposes the thermodynamic relations hold that in a region of dimension $L \sim r_C$ (naturally, only as far as order of magnitude goes). In this case the contribution to the free energy from a region of dimension r_C can be written in the form

$$\Delta F = \text{const} \cdot r_C^{-d} M^2 / 2\chi. \quad (31)$$

It is assumed that on the average one degree of freedom, connected with the large-scale fluctuation, is allotted to a region of this size. Then $\Delta F \sim T_C$, and from (31) we get relation (A) (see Table II). We then determine the fluctuation moment m per particle: $m^2 \sim r_C^{-d} \sim |\tau|^\mu$. In weak fields $\chi^{-1} \sim \tau\gamma \sim m^2\gamma/\mu\nu$. It is assumed that the dependence of χ on m remains the same in a strong field. From this we get

$$\delta = 1 + 2\gamma/\mu\nu. \quad (32)$$

Relation (32) can be easily obtained from Table II.

5. LIQUID-GAS CRITICAL POINT

It is possible to apply the previously obtained results to a liquid if, following Lee and Yang, we describe it by means of the lattice model (see, for example, [17]). The particles can be located only at the sites of a regular lattice. Each site is either occupied by a particle or is vacant. We introduce the site occupation numbers n_r , which take on the values 0 and 1. It is assumed that the energy of the system E can be written in the form

$$E = \sum I_{rr'} n_r n_{r'},$$

where $I_{rr'}$ decreases rapidly with increasing $|r - r'|$. The total number of particles is $N = \sum n_r$ and is conserved. We can introduce a quantity conjugate to N , namely the chemical potential ξ , and the density $\rho = \langle n_r \rangle$. By making the change of variable

$n_r = (\sigma_r + 1)/2$, the problem of the lattice gas reduces to the Ising model. The rules for the translation from the language of the three-dimensional Ising model of the magnet to the language of the lattice model of a gas-liquid system are given in Table III. Near the critical point we can introduce, just as for magnets, critical indices whose connection with the physical quantities is explained in Table IV, and use them to write down the equation of state. The latter takes the form

$$p - p_C - s_C \tau = \tau^{2-\alpha-\beta} g\left(\frac{\rho - \rho_C}{\tau^\beta}\right), \quad (33)$$

where p is the pressure, ρ the density, p_C and ρ_C their values at the critical point, s_C the specific entropy at the critical point, and $g(x)$ a certain function with the following asymptotic values:

$$g(x) \sim x^\delta \text{ as } x \rightarrow 0, \quad g(x) \sim x^2 \text{ as } x \rightarrow \infty. \quad (34)$$

In particular, the equation of the critical isotherm is

$$p - p_C \sim (\rho - \rho_C)^\delta. \quad (35)$$

The specific heat C_p can be obtained by using the well known thermodynamic relation

$$C_p = C_V - T \frac{\left(\frac{\partial p}{\partial T}\right)_V^2}{\left(\frac{\partial p}{\partial V}\right)_T}. \quad (36)$$

It follows from the equation of state (33) that the quantity $(dp/dT)_V$ remains finite at the critical point if $\alpha + \beta < 1$. Confining ourselves to this case, we obtain the connection between the specific heat and the compressibility near the critical point

$$C_p - C_V \simeq T s_C^2 \rho_C^{-1} k_T. \quad (37)$$

The equation of the phase coexistence curve ($h = 0$ in the case of a magnet) is

$$\rho - \rho_C \sim \pm \tau^\beta. \quad (38)$$

In the two-phase region, on approaching the critical point, the surface tension decreases. Its value can be

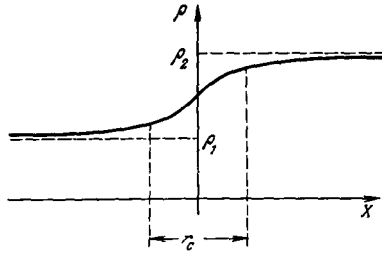
Table III. Rules for translation from the Ising-model language to the lattice-gas language

Ising model	m	$\left(\frac{1}{2} + m\right)^{-2} \chi$	h	$F - Vh$	C_m
Lattice gas	$\rho - \rho_C$	k_T	$\xi - \xi_C$	$-pV$	C_V

m - moment per site, χ - magnetic susceptibility, ρ - density, ρ_C = critical density, ξ - chemical potential, p - pressure, k_T - isothermal compressibility.

Table IV. Connection between critical indices and the quantities characterizing the liquid (lattice model)

	C_V	Compressibility k_T	Density jump $\Delta\rho = \rho_{\text{liq}} - \rho_{\text{sol}}$	Surface tension σ	Critical isotherm $\tau=0$
$ \rho - \rho_C \ll \tau^\beta$	$ \tau ^\alpha$	$ \tau ^{-\gamma}$	$ \tau ^\beta (\tau < 0)$	$ \tau ^{(d-1)\mu} (\tau < 0)$	
$ \rho - \rho_C \gg \tau^\beta$	$ \rho - \rho_C ^{\alpha/\beta}$	$ \rho - \rho_C ^{1-\delta}$			$p - p_C \sim \sim \rho - \rho_C ^\delta$



related to the critical indices. We shall therefore assume, following Widom^[10], that the dimension of the transitions region between the different phases is of the order of r_C . The dependence of the density of the coordinate in such a system is shown in the figure. In the transition region the additional free energy per unit volume is (see (31))

$$\text{const} \cdot (\rho - \rho_C)^2 / k_T.$$

Multiplying this quantity by the thickness of the transition layer, we obtained the energy per unit surface area

$$\sigma \sim r_C (\Delta\rho)^2 / k_T. \quad (39)$$

In particular, on the isochore $\rho = \rho_C$ we get

$$\sigma \sim |\tau|^{2\beta + \nu - \mu} \sim |\tau|^{(d-1)\mu}. \quad (40)$$

A real liquid, generally speaking, cannot be described by the lattice model. Nonetheless, the relations between the critical indices (Table II) can be obtained also in a more general formulation (A. Z. Patashinskiĭ^[18]).

Let us consider a real liquid described by the quantities p , ρ , and ζ (the chemical potential). We assume that fluctuations of the density ρ with large dimensions $\sim r_C$ arise near the critical point. The deviation ΔN of the number of particles from the mean value inside a region of such dimensions plays the same role as the fluctuation moment M of the magnet, and $\zeta - \zeta_C$ plays the role of the magnetic field. It must be kept in mind, however, that the role of the free energy is played in this case by the potential $\Omega = -pV$, for it is precisely this potential which changes by an amount $-N d\zeta$ when the chemical potential changes by $d\zeta$. Repeating the reasoning of Sec. 2, we get

$$\Omega = \Omega_0 + \frac{V}{r_C^d} g(\Delta N (\zeta - \zeta_C)), \quad (41)$$

where $\Omega_0 = \Omega(T, V, \zeta_C) = -Vp(T, \zeta_C)$, and $g(x)$ is a certain function. We write out the connection between the quantities p , τ , and ζ near the critical point:

$$\zeta - \zeta_C = v_C (p - p_C) - s_C \tau. \quad (42)$$

Formula (41) leads to the same relation between the critical indices.

6. BOSE CONDENSATION OF HELIUM

Josephson^[19] extended the ideas of scale transformation to include the case of a second order phase transition in liquid helium*. Josephson chose as the ordering

parameter the wave function ψ , connected with the density of the condensate n_0 by the well known relation

$$n_0 = |\psi|^2. \quad (43)$$

The superfluid part of the liquid can move with a velocity determined with the aid of the quantity

$$m_{\text{He}} v_S = \hbar \frac{\nabla \psi}{\psi}, \quad (44)$$

where m_{He} is the mass of the helium atom (it is understood that the characteristic hydrodynamic dimensions are large compared with the atomic dimensions).

The same velocity can be determined from the hydrodynamic relation

$$j_S = \rho_S v_S, \quad (45)$$

where j_S is the flux and ρ_S the density of the superfluid part of the liquid. The free energy per unit volume of the liquid helium, with allowance for the motion of the superfluid component, takes the form

$$F(p, T, \psi) = F_0(p, T, |\psi|^2) + \frac{1}{2} \rho_S v_S^2. \quad (46)$$

We assume that the liquid as a whole is stationary, and the flux has a fluctuating character. We perform a scale transformation, putting $p = p_C$. By definition, $F(\tau(L), \psi(L))$ is the free energy of a cell of dimension L at a temperature τ . We therefore obtain

$$F(p_C, \tau(L), \psi(L)) = L^d F(p_C, \tau, \psi). \quad (47)$$

It is quite likely that F_0 itself satisfies the scale-transformation condition

$$\rho_S(L) v_S^2(L) = L^d \rho_S v_S^2. \quad (48)$$

But according to (44), v_S goes over into $L v_S$ under the scale transformation. Consequently (see (48))

$$\rho_S(L) \equiv \rho_S(\tau L^y) = L^{d-\alpha} \rho_S(\tau). \quad (49)$$

From (49) we get

$$\rho_S \sim \tau^{-\frac{d-2}{y}}. \quad (50)$$

We recall that y is connected with the critical index of the specific heat by the relation $\alpha = 2 - d/y$. In the case of interest to us $d = 3$ and

$$\rho_S \sim \tau^{-\frac{2-\alpha}{3}}. \quad (51)$$

For the λ transition in He, the quantity α was measured by Buckingham, Fairbank, and Kellers^[6] and found to be equal to zero. Recently Clow and Reppy^[21], and also Tyson and Douglass^[22], measured ρ_S near T_C and found that $\rho_S \sim (-\tau)^{2/3 \pm 0.008}$.

Unfortunately, it is still impossible to obtain from the available experimental data any information concerning any other critical index. We therefore know nothing about the behavior of n_0 . It is obvious that n_0 changes in a different manner than ρ_S . Otherwise the correlation function $\langle \psi(r) \psi(r') \rangle$ would be the same as in the Landau theory, something impossible in a small vicinity of the λ point.

At first glance both n_0 and ρ_S can be used with equal success as the ordering parameters. However, the real long-range order is characterized by the quantity ψ (or $\sqrt{n_0}$) corresponding to the critical index β . In the theory of scale transformations, as shown, the meas-

*We present here Kadanoff's derivation of the Josephson relation [20].

Table V. Certain calculated and experimental data

	Calculations [4,5,23]			Experiment			
	2-Ising	3-Ising	Heisenberg Model $d=3$	Iron	Nickel	Argon	Xenon
α	0	$\begin{matrix} (\tau < 0) \\ 0-0.25 \\ \hline (\tau > 0) \\ -0.09-0.16 \end{matrix}$	$\approx 0(\tau > 0)$	$\approx 0^{24}$		$< 0.25^{25}$	$< 0.2^{26}$
β	1/8	$\begin{matrix} 5/16 & 0.312 \\ 0.303-0.308 \end{matrix}$				0.4 ± 0.2^{27}	0.35^3
γ	7/4	1.23-1.32	1.32-1.38	1.33 ± 0.4^{28}	1.37^{29}	0.6 ± 0.2^{30}	1.3 ± 0.2^3
δ	15	5.20 ± 0.15			4.2^{28}		4.4 ± 0.4^3
ν	1/4	1.059 ± 0.006	1.08 ± 0.04	$\sim 1^{31}$			
μ	1	0.644 ± 0.002					

ured quantity ρ_S is determined uniquely by the behavior of the specific heat.

7. CALCULATED AND EXPERIMENTAL DATA

Table V gives a certain idea of the critical indices obtained by numerical calculation for mathematical models (with the exception of the plane Ising model) and the data obtained from experiment for a number of substances. The data are far from complete. It can be verified that the relations between the critical indices are satisfied sufficiently well.

8. CONCLUSION

It is natural to raise the following question: What is the relation between the classical theory of the self-consistent field and the theory of similarity of phase transitions? A full answer to this question can be obtained only after constructing a microscopic theory. It is possible, however, to find a criterion for the applicability of the theory of the self-consistent field. V. L. Ginzburg^[32] proposed the following criterion: The theory of the self-consistent field is applicable if the fluctuations of the ordering parameter in a region having dimensions of the order of the correlation radius are much smaller than its mean value. The parameters of the theory are expressed in terms of the characteristics of the given substance. If all the coefficients (of $(\nabla\eta)^2$, $\tau\eta^2$ and η^4) are of the order of the atomic constants, then the Ginzburg condition can only be $\tau \gg 1$. It is clear that the existence of a region in which $\tau \ll 1$ and the Ginzburg condition is satisfied nonetheless imposes certain conditions on the parameters of the substance. Therefore for certain substances there exists a region of applicability of the self-consistent field theory, and for others it does not exist. The former include superconductors (see also^[33,34]), quartz, and ferroelectrics, and the latter include helium and iron.

Vaks, Larkin, and Pikin^[35] constructed a microscopic theory of ferroelectrics, in which there is a region of applicability of the Landau theory. It turned out to be sufficient to have the radius of the interaction forces exceed the atomic constant. Another possibility, first indicated by Anderson^[36] is that the long waves of the optical branch in crystals become unstable at a certain temperature, and a unique Bose condensation of the phonons occurs. The region of instability is assumed small in this case (see^[37] for details).

In all cases when the region of applicability of the

classical theory exists, there is another region, sufficiently close to T_C , where it becomes inapplicable. It can be assumed that the growth of the fluctuations in this region leads to the similarity picture described in the present review.

The question of whether all phase transitions can be described by the same critical indices (in other words, the role of the interaction in microscopic scales), has not been resolved.

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