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SYNCHROTRON RADIATION AND ITS REABSORPTION

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INTRODUCTION

SYNCHROTRON radiation has been discussed in the literature for many years, and the corresponding formulas are extensively used, especially in astrophysics (see, e.g., [1,2]). It is all the more surprising that new aspects have been clarified in recent times in this field. Moreover, it turned out that many articles contain and make partial use of incorrect or inaccurate formulas. This pertains to the radiation of particles moving in a magnetic field not along a circle, but along a helix (in other words, particles moving with a velocity v at an angle $\theta \neq \pi/2$ to the magnetic field H). To be specific, the formulas obtained in rather well known papers,[3-5] for the intensity of synchrotron radiation of one electron moving at an angle $\theta \neq \pi/2$ are incorrectly interpreted and are simply speaking untrue. At the same time, the expressions for the intensity of the synchrotron radiation from an ensemble of particles, used in radioastronomy (see [1,2]), are based precisely on these formulas for the intensity of radiation of an individual electron. Fortunately, this inaccuracy* of the initial formulas does not come into play when dealing with the radiation intensity of a system of electrons situated in a fixed region (e.g., a supernova shell). Therefore, all the applications of the formulas of the synchrotron radiation theory in [1,2] and in a few other articles turned out to be correct. However, in the case of synchrotron radiation from a cloud of relativistic electrons exploding from a certain source and moving likewise with relativistic velocity, the use of some of the formulas contained in [1,2] would lead to considerable errors.

The purpose of the present article is, first, to cast light on this question of synchrotron radiation in the case of helical motion, a question of both practical and methodological significance. Second, we shall discuss the problem of reabsorption of synchrotron radiation, which recently has been attracting much attention (especially in connection with discussions of radio emission from quasars).[8-10] At the same time, in [1,2] this question was not discussed in sufficient detail. In the present article we consider it advisable to include also a section (6) devoted to the motion of particles in a magnetic field in the presence of losses, and, simultaneously, with allowance for their interaction with the source that produces the field (e.g., with a "solenoid"). We do not obtain here any unexpected results, but from the methodological point of view the problem is of undisputed interest. It can possibly turn out to be also of practical importance when dealing with radiation from an ensemble of particles with sufficiently high energy density.

*This circumstance (the incorrectness or inaccuracy of certain expressions when $\theta \neq \pi/2$, was pointed out, in so far as we know, by others (V. A. Razin, E. G. Mychelkin, G. B. Field, and J. Skargle). In addition a number of articles contain correct general formulas for the radiation of single particles [6,7], but these formulas were not explicity employed or interpreted for the ultrarelativistic case.

1. ELEMENTARY ANALYSIS OF RADIATION OF A CHARGE MOVING ALONG A HELIX

An ultrarelativistic electron moving in vacuum* (this is essentially the only case of interest to us) emits practically only in the direction of its instantaneous velocity or, more accurately, in a cone with an apex angle

$$\psi \sim \frac{mc^2}{E}, \qquad E \gg mc^2. \tag{1.1}$$

In the qualitative analysis that follows we assume, where possible, that the radiation is needle-like, i.e., the angle ψ is arbitrarily small. When an electron moves in a constant and homogeneous magnetic field of intensity H, its trajectory is, generally speaking, helical and its velocity is $v_{\parallel} = v\cos\theta$ along the field and $v_{\perp} = v\sin\theta$ across the field (of course, the total velocity is $v = \sqrt{v_{\parallel}^2 + v_{\perp}^2}$). The revolution frequency ω_H depends only on v and is equal to

$$\omega_H = \frac{eH}{mc}\sqrt{1-\frac{v^2}{c^2}} = \frac{eH}{mc}\frac{mc^2}{E}. \tag{1.2}$$

*For concreteness, we refer to electrons. Of course, all the expressions presented below pertain to all particles with charge e and rest mass m.

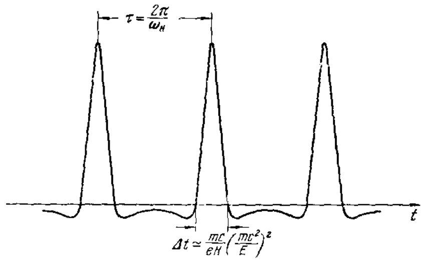

FIG. 1.

If the motion is along a circle (i.e., $v_{\parallel} = 0$, $v_{\perp} = v$) then, by virtue of the foregoing, the ultrarelativistic electron radiates only in the plane of the orbit. An observer (recording instrument) located in this plane will "see" flashes of radiation at those instants when the electron moves precisely in the direction of the observer (it is, of course, necessary to take into account the delay due to the propagation time of the radiation, which in the case of vacuum is equal to r/c, where r is the distance from the electron to the observer). Obviously, the flashes will repeat each period, or, in other words, they will follow each other at intervals

$$\tau = \frac{2\pi}{\omega_H} = \frac{2\pi mc}{eH}\frac{E}{mc^2}.$$

As shown in detail in [1, 2], under condition (1.1) the characteristic duration of each flash is of the order of

$$\Delta t \simeq \frac{mc}{eH}\left(\frac{mc^2}{E}\right)^2,$$

and the observer records the field shown schematically in Fig. 1. It is clear that an expansion of this field in a Fourier series leads to a spectrum consisting of overtones of frequency ω_H. All the corresponding expressions given in [1-5] for the field, intensity, and other quantities are valid for this case and there is no reason to discuss them here.* On the other hand, as already mentioned, the formulas are incorrect for noncircular motion when the longitudinal velocity $v_{\parallel}$

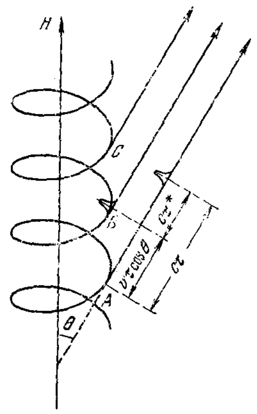

FIG. 2.

*Radiation in the case of circular motion was considered in detail in [11].

$= v\cos\theta \neq 0$, i.e., when $\theta \neq \pi/2$. The source of the error is seen particularly clearly from the initial expression (see, e.g., the article [5]) for the intensity of the radiation field, which is written in the form

$$\mathcal{E} = \mathrm{Re}\left(\sum_{n=1}^{\infty} \mathcal{E}_n e^{-in\omega_H t}\right). \tag{1.3}$$

The point is that when $v_{\parallel} \neq 0$, the radiation pulses follow each other not at intervals $\tau = 2\pi/\omega_H$, but at intervals τ^* that differ from τ as a result of the Doppler effect.

The time τ^* can be readily obtained in elementary fashion by using Fig. 2. For a selected observer, the flashes of radiation occur when the electron is situated at the points A, B, C, ... (for simplicity we assume here and below that the radiation is strictly needle-like). In other words, it is precisely at these points that the electron "looks" on the observer. The time interval between the instants when the electron passes through the points A and B is, of course, equal to the period $\tau = 2\pi/\omega_H$. The distance between the points A and B is $v_{\parallel}\tau = v\tau\cos\theta$ (θ—angle between **v** and **H**), and a pulse emitted at the point A will cover during that time the path $c\tau$. It is clear from Fig. 2 that the pulse emitted at the point B will reach the observer at a time

$$\tau^* = \tau\left(1 - \frac{v_{\parallel}\cos\theta}{c}\right) = \tau\left(1 - \frac{v}{c}\cos^2\theta\right) \simeq \tau\sin^2\theta = \frac{2\pi}{\omega_H}\sin^2\theta \tag{1.4}$$

after the first pulse, where on going over to the next-to-the-last expression we took into account the fact that the entire calculation is carried out for the limiting case $v \to c$. We note that this picture, in which the radiation reaches the observer in the form of individual pulses, is suitable only if $\theta \gg \psi \sim mc/E$. Actually, however, the expression

$$\tau^* = \tau\left(1 - \frac{v_{\parallel}\cos\theta}{c}\right)$$

is general and it is connected neither with the assumption that the radiation is "needle-like" nor with the possibility of subdividing it into discrete pulses (in this case $v_{\parallel}\cos\theta$ is replaced by $v_{\parallel}\cos\theta'$; see Sec. 3).

Thus, the radiation field of an ultrarelativistic electron consists in the wave zone of overtones of the frequency

$$\omega_H^* = \frac{2\pi}{\tau^*} = \frac{\omega_H}{\sin^2\theta}. \tag{1.5}$$

This circumstance in itself is not very significant, if it is recognized that in the cases of interest to us the overtones are not resolved and we are dealing with a continuous spectrum. On the other hand, the estimate given in [1, 2] for the pulse width

$$\Delta t \sim \frac{mc}{eH_{\perp}}\left(\frac{mc^2}{E}\right)^2,$$

meaning also for the characteristic frequency $\omega_m \sim 1/\Delta t$, is perfectly correct (here and below, $H_{\perp} = H\sin\theta$). However, the change of the interval between pulses affects not only the spectrum but also all the characteristics of the radiation field, particularly its intensity as recorded at the observation point. Indeed, let the electron lose to the radiation during each revolution (within a time $\tau = 2\pi/\omega_H$) an energy $\Delta E = P\tau$. Then, by virtue of the foregoing, it is obvious that this energy is reached by "observers" located on a certain fixed sphere at a distance r from the electron within a time τ^* and, consequently, the average observed radiation power (the total energy flux) is

$$P^* = \frac{\Delta E}{\tau^*} = \frac{P\tau}{\tau^*} = \frac{P}{\sin^2\theta}. \qquad (1.6)$$

At first glance it may seem that this contradicts somehow the energy conservation law. The electron loses an energy P per unit time (the value of P is determined by a well known formula, for example, formula (2.10) of [2] and formula (4.1) below. This entire energy goes over into radiation and it might seem that it should be equal to the total flux of radiation through the sphere under consideration. This indeed is the frequent procedure—the radiation loss experienced by the particle is calculated and is equated to the total radiation flux. In the stationary case, for a radiator whose center of gravity is fixed, it is of course possible to proceed in this manner. In general, however, as is well known, the work performed by a radiator per unit time (the power loss P) is equal to the total energy flux through a certain surface plus the change in the field energy

$$\frac{\partial}{\partial t}\int\left(\frac{\mathscr{E}^2+H^2}{8\pi}\right)dV$$

in the volume enclosed by this surface. In the case of interest to us, the region of space occupied by the radiation and located between the moving electron and a fixed surface in space, on which the observations are performed, decreases all the time. Therefore the energy contained in this region also decreases, and consequently the power P^* of the received radiation exceeds the power loss P. Yet, for example, in [4], on going over to spectral quantities, they used the loss power P. Such an approach, of course, can not lead to correct results for the radiation intensity determined at a certain fixed surface when account is taken of the displacement of the radiator. At the same time, if the radiating particles are in a fixed volume (e.g., in a supernova shell) or, more accurately, if the distribution function of the radiating particles does not vary with time, the radiation intensity of the ensemble of particles coincides with the spectral loss power. This conclusion is clear from the energy conservation law and, of course, is confirmed by direct calculation (see Sec. 4).

This is the gist of the matter. We propose that the fact that this entire essentially elementary question remained unexplained for so long a time justifies our attempts to present a sufficiently detailed exposition in the present article.

2. CERENKOV RADIATION OF A PARTICLE PASSING THROUGH A PLATE (LAYER OF MATTER)

The foregoing remark, namely that the total radiation flux of a moving or in general nonstationary source does not coincide with the energy loss per unit time, is, of course, quite general in character. By way of illustration let us stop to discuss, besides synchrotron radiation, also the flux of Cerenkov radiation pro-

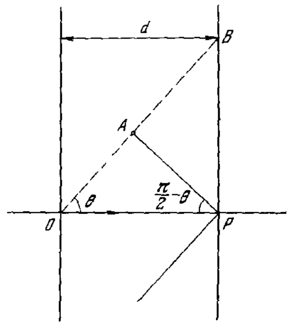

FIG. 3.

duced when a particle passes through a plate (Fig. 3). For simplicity we assume that the medium is nondispersive. In other words, we confine ourselves to the frequency region for which the dispersion is small, so that the group velocity of the light is equal to the phase velocity c/n. The energy lost to Cerenkov radiation in the frequency region under consideration per unit path, will be denoted by $(dE/dx)_c$. The particle with velocity v will pass through a plate of thickness d within a time $t = d/v$, losing thereby an energy $(dE/dx)_c d$. The loss per unit time is

$$\left(\frac{dE}{dx}\right)_c v = P.$$

The particle starts radiating on entering the plate (at the instant $t = 0$) and radiates during the entire time $t = d/v$ until it leaves the plate. The radiation in this case is on the surface of a cone with apex angle $\pi - 2\theta$, where $\cos\theta = c/nv$ (we neglect dispersion). It is clear from Fig. 3 that the light begins to emerge from the plate at an instant $t = d/v$, and the flash terminates when the point A (rear end of the cone, emitted at the instant $t = 0$) will reach the point B. Obviously,

$$OA = \frac{c}{n}t = \frac{c}{n}\frac{d}{v} = d\cos\theta$$

and

$$OB = \frac{d}{\cos\theta}.$$

Hence

$$AB = d\left(\frac{1}{\cos\theta} - \cos\theta\right) = d\frac{\sin^2\theta}{\cos\theta}$$

and the duration of the flash of Cerenkov radiation on the surface PB is*

$$t^* = \frac{d\sin^2\theta}{\cos\theta\cdot c/n} = t\,\mathrm{tg}^2\theta, \qquad (2.1)$$

since $d = vt$ and $\cos\theta = c/nv$. Consequently, the total flash radiation flux through the surface of the plate is

$$P^* = \frac{(dE/dx)_c\,d}{t\,\mathrm{tg}^2\theta} = \frac{P}{\mathrm{tg}^2\theta} = P\,\mathrm{ctg}^2\theta, \qquad (2.2)$$

where P is the energy loss per unit time, neglecting

*The question of the duration of the Cerenkov flash, particularly, as applied to Cerenkov counters, is considered in [12].

reflection from the surface of the plate (coated optics). Thus, the total flux, generally speaking, is not equal to the energy loss per unit time and in this case depends significantly on the orientation of the chosen surface.

3. SYNCHROTRON RADIATION OF AN INDIVIDUAL PARTICLE MOVING AT AN ARBITRARY ANGLE TO THE FIELD

We choose a system of coordinates in accordance with Fig. 4 such that the axis $\mathbf{e}_3$ is directed along the external magnetic field, $\mathbf{H} = H\mathbf{e}_3$. The particle with charge e moves in the field H on the trajectory

$$\mathbf{r}_0(t) = \frac{c\beta_\perp}{\omega_H}\{-\mathbf{e}_1\cos\omega_H t + \mathbf{e}_2\sin\omega_H t\} + \mathbf{e}_3 c\beta_\| t, \quad (3.1)$$
$$\boldsymbol{\beta}(t) = \beta_\perp\{\mathbf{e}_1\sin\omega_H t + \mathbf{e}_2\cos\omega_H t\} + \mathbf{e}_3\beta_\|.$$

Here β is the velocity of the particle in units of the velocity of light c, $\beta_\|$ and $\beta_\perp$ are its projections in the direction of the field and in the transverse direction, and ω_H is given by (1.2). For a negatively charged particle $\omega_H < 0$, Fig. 4 shows the trajectory of a negatively charged particle (say an electron).

At large distances from the particle, in the wave zone, the Fourier components of the vector potential and of the intensity of the electric field of the particle are respectively equal to (see [13], Sec. 66)

$$\mathbf{A}_\omega = e^{i\frac{\omega}{c}\mathbf{kr}}\frac{e}{2\pi r}\int_{-\infty}^{+\infty}\boldsymbol{\beta}(t)\, e^{i\left\{\omega t - \frac{\omega}{c}\mathbf{k}\mathbf{r}_0(t)\right\}}dt,$$
$$\boldsymbol{\mathscr{E}}_\omega = i\frac{\omega}{c}[\mathbf{k}[\mathbf{A}_\omega\mathbf{k}]] = i\frac{\omega}{c}(\mathbf{A}_\omega - \mathbf{k}(\mathbf{A}_\omega\mathbf{k})), \quad (3.2)^*$$

where k is a unit vector in the radiation direction (in the direction from the particle to the observer), $\mathbf{r} = r\mathbf{k}$, and r is the distance between the observer and the position of the electron at a certain fixed instant of time; we assume that the vector k lies in the plane $(\mathbf{e}_2, \mathbf{e}_3)$ and makes an angle θ' with the direction of the magnetic field, i.e., $\mathbf{k} = \{0, k_2, k_3\} = \{0, \sin\theta'\cos\theta'\}$. We recall that the angle between v and H is denoted θ (see Fig. 4).

As follows from (3.2), the expression for $\mathscr{E}_\omega$ contains only the velocity component transverse to the radiation direction

$$\boldsymbol{\beta}_t = \boldsymbol{\beta} - \mathbf{k}(\boldsymbol{\beta}\mathbf{k}) = \mathbf{e}_1\beta_\perp\sin\omega_H t + (\mathbf{e}_2 k_3 - \mathbf{e}_3 k_2)(\beta_\perp k_3\cos\omega_H t - \beta_\| k_2). \quad (3.3)$$

It is convenient to introduce a triad of unit vectors $\mathbf{l}_1$, $\mathbf{l}_2$, k such that

$$\mathbf{l}_2 = \mathbf{e}_3 k_2 - \mathbf{e}_2 k_3, \qquad \mathbf{l}_1 = [\mathbf{l}_2\mathbf{k}] = -\mathbf{e}_1. \quad (3.4)$$

The vector $\mathbf{l}_2$ is directed along the projection of H on the plane perpendicular to the direction of the observer (the plane of the picture), i.e., along the vector $\mathbf{e}_3 - \mathbf{k}(\mathbf{e}_3\cdot\mathbf{k})$.

From (3.2)–(3.4) we get

$$\boldsymbol{\mathscr{E}}_\omega = e^{i\frac{\omega}{c}r}\frac{ie\omega}{2\pi cr}\int_{-\infty}^{+\infty}\boldsymbol{\beta}_t(t)\, e^{i\left(\omega t - \frac{\omega}{c}\mathbf{k}\mathbf{r}_0(t)\right)}dt, \quad (3.5)$$

where

$$\boldsymbol{\beta}_t(t) = -\mathbf{l}_1\beta_\perp\sin\omega_H t - \mathbf{l}_2(\beta_\perp k_3\cos\omega_H t - \beta_\| k_2). \quad (3.6)$$

*$[\mathbf{k}[\mathbf{A}_\omega\mathbf{k}]] \equiv \mathbf{k}\times[\mathbf{A}_\omega\times\mathbf{k}]$.

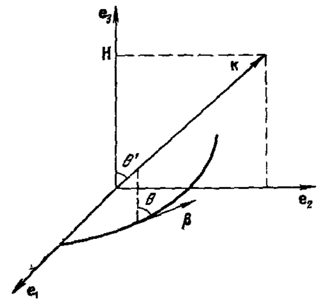

FIG. 4.

To calculate the integral (3.5), we note that the argument of the exponential in the integrand of (3.5) is equal to (see (3.1))

$$\omega t - \frac{\omega}{c}\mathbf{k}\mathbf{r}_0(t) = (1 - \beta_\| k_3)\,\omega t - z\sin\omega_H t, \quad (3.7)$$

where

$$z = \frac{\omega}{\omega_H}\beta_\perp k_2. \quad (3.8)$$

Further, we use the formula

$$e^{-iz\sin\omega_H t} = \sum_{n=-\infty}^{\infty} J_n(z)\, e^{-in\omega_H t}, \quad (3.9)$$

where $J_n(z)$ is a Bessel function of the first kind.

The integration with respect to t leads to the appearance of the functions:

$$\boldsymbol{\mathscr{E}}_\omega = e^{i\frac{\omega}{c}r}\frac{e\omega}{cr}\sum_{n=-\infty}^{\infty}\Big\{\mathbf{l}_1\beta_\perp J'_n(z) - i\mathbf{l}_2\left(\beta_\perp k_3\frac{n}{z} - \beta_\| k_2\right)J_n(z)\Big\}\,\delta\{(1-\beta_\| k_3)\,\omega - n\omega_H\}. \quad (3.10)$$

Thus, the radiation has a discrete spectrum with frequencies

$$\omega = \omega_n \equiv n\frac{\omega_H}{1-\beta_\| k_3} \equiv \frac{n\omega_H}{1-\beta\cos\theta\cos\theta'}. \quad (3.11)$$

In the ultrarelativistic case $\beta \to 1$ and the radiation is directed in practice along the instantaneous velocity of the particle (see [13], Sec. 72), i.e., the angle $\theta \simeq \theta'$ and

$$\omega_n = \frac{n\omega_H}{1 - \frac{v}{c}\cos^2\theta} \simeq \frac{n\omega_H}{\sin^2\theta} = n\omega_H^*,$$

which agrees with (1.5).

The Fourier integral of the electric field of the particle radiation reduces to the series

$$\boldsymbol{\mathscr{E}} = \int_{-\infty}^{+\infty}\boldsymbol{\mathscr{E}}_\omega e^{-i\omega t}\,d\omega = \mathrm{Re}\sum_{n=1}^{\infty}\boldsymbol{\mathscr{E}}_n e^{i\omega_n\left(\frac{r}{c}-t\right)}, \quad (3.12)$$

where the amplitude of the n-th harmonic of the field is

$$\boldsymbol{\mathscr{E}}_n = \frac{2e}{cr}\frac{n\omega_H\beta\sin\theta}{(1-\beta\cos\theta\cos\theta')^2}\left\{\mathbf{l}_1 J'_n(z_n) - i\mathbf{l}_2\frac{\cos\theta' - \beta\cos\theta}{\beta\sin\theta\sin\theta'}J_n(z_n)\right\} \quad (3.13)$$

and

$$z_n = n\frac{\beta\sin\theta\sin\theta'}{1-\beta\cos\theta\cos\theta'}. \quad (3.14)$$

Expression (3.12)–(3.14) determine completely the radiation field produced at a certain sufficiently remote point of space by a particle moving at an arbitrary angle to the magnetic field. It will be convenient in what follows to use the "radiation polarization tensor," equal by definition to

$$\tilde{p}_{\alpha\beta}(n) = \frac{c}{8\pi}\mathcal{E}_{n,\alpha}\mathcal{E}^*_{n,\beta}, \qquad (3.15)$$

where $\alpha, \beta = 1, 2$, and $\mathcal{E}_{n,\alpha}$ are the components of the electric vector (3.13). The energy flux density averaged over the period (the Poynting vector) in the n-th harmonic is

$$\tilde{p}_n = \mathrm{Sp}\,\tilde{p}_{\alpha\beta}(n) = \frac{c}{8\pi}|\mathcal{E}_n|^2. \qquad (3.16)$$

For ultrarelativistic particles

$$\xi = mc^2/E \ll 1, \qquad (3.17)$$

and the principal role is assumed by radiation at high harmonics $n \sim \xi^{-3} \gg 1$, concentrated within the small angle

$$\psi = \theta - \theta' \lesssim \xi = \frac{mc^2}{E}. \qquad (3.18)$$

The characteristic frequencies of the radiation (see (1.5) or (3.11)) at $\beta \approx 1$ and $\theta \approx \theta'$ are

$$\nu = \frac{\omega}{2\pi} = \frac{n\omega_H}{2\pi\sin^2\theta}. \qquad (3.19)$$

To go to the ultrarelativistic limit in (3.13), we can use the approximate expression for Bessel functions with large values of the index and argument (see [14], p. 979). This leads to the following expression for the amplitude of the n-th harmonic of the electric field of radiation from an ultrarelativistic electron*

$$\mathcal{E}_n = \frac{2e\omega_H}{\sqrt{3}\pi cr}\frac{n}{\sin^5\theta}\{\mathbf{l}_1(\xi^2+\psi^2)K_{2/3}(g_n) + i\mathbf{l}_2\psi(\xi^2+\psi^2)^{1/2}K_{1/3}(g_n)\}, \qquad (3.20)$$

where

$$g_n = n\frac{(\xi^2+\psi^2)^{3/2}}{3\sin^3\theta} = \frac{\nu}{\nu_c}\left(1+\frac{\psi^2}{\xi^2}\right)^{3/2}. \qquad (3.21)$$

In the second equation of (3.21) we changed over from the number of the harmonic n to the frequency ν (see (3.19)) and used the notation

$$\nu_c = \frac{3\omega_H\sin\theta}{4\pi\xi^3} = \frac{3eH_\perp}{4\pi mc}\left(\frac{E}{mc^2}\right)^2. \qquad (3.22)$$

In the region of high harmonics, the radiation spectrum is practically continuous and in place of the polarization tensor of the radiation at the n-th harmonic (3.15) it is possible to introduce the "spectral density of the polarization tensor":

$$\tilde{p}_{\alpha\beta}(\nu) = \tilde{p}_{\alpha\beta}(n)\frac{dn}{d\nu} = \frac{2\pi\sin^2\theta}{\omega_H}\tilde{p}_{\alpha\beta}(n). \qquad (3.23)$$

*Here and throughout we shall assume that $\omega_H > 0$ and, accordingly, $\omega_n >$ in (3.12). A change in the sign of the charge e will correspond to a transition to the complex-conjugate amplitude in (3.13), as in seen from (3.12) if $-\omega_n$ is replaced by ω_n. For example for a positively charged particle (positron), the amplitude is the complex conjugate of (3.20), corresponding to an opposite direction of the electric-vector rotation.

From this and from (3.15) and (3.20) we obtain the spectral density of the radiation fluxes with two principal polarization directions:

$$\tilde{p}_\nu^{(1)} \equiv \tilde{p}_{11}(\nu) = \frac{3e^2\omega_H}{4\pi^2r^2c\xi^2\sin^2\theta}\left(\frac{\nu}{\nu_c}\right)^2\left(1+\frac{\psi^2}{\xi^2}\right)^2K^2_{2/3}(g_\nu), \qquad (3.24)$$

$$\tilde{p}_\nu^{(2)} \equiv \tilde{p}_{22}(\nu) = \frac{3e^2\omega_H}{4\pi^2r^2c\xi^2\sin^2\theta}\left(\frac{\nu}{\nu_c}\right)^2\frac{\psi^2}{\xi^2}\left(1+\frac{\psi^2}{\xi^2}\right)K^2_{1/3}(g_\nu), \qquad (3.25)$$

where $g_\nu = g_n$ (see (3.21)).

We note here that formulas (3.17)–(3.25) can be easily generalized to include the case when the radiating particle is in a plasma whose refractive index can be assumed equal, with good approximation, to

$$n = 1 - \frac{\omega_0^2}{2\omega^2},$$

where $\omega_0 = \sqrt{4\pi N_0 e^2/m}$ and N_0 is the plasma electron density. This approximation is valid if $\omega \gg \omega_0$ and $\omega \gg \omega_H^0 = eH/mc$. Under these conditions, it is necessary to replace the quantity ξ in formulas (3.17)–(3.25) in those places where it enters explicitly, by

$$\eta = \sqrt{\xi^2 + \left(\frac{\omega_0}{\omega}\right)^2}. \qquad (3.26)$$

As is clear from the foregoing, it is assumed here that $\eta \ll 1$.

Expressions (3.24) and (3.25), and accordingly the Stokes parameters of the radiation of an individual electron, differ from those used in [1-5] in that a factor $\sin^2\theta$ appears in the denominator. It is precisely in this respect that the expressions given in [1-5] for the intensity and for the Stokes parameters are incorrect if one deals with radiation of an individual particle or an ensemble of particles moving in space.

On the other hand, if, as is customary, we are interested in radiation from particles in a fixed volume in space, then it is necessary to use the expressions of [1-5]. We now proceed to consider this question.

4. RADIATION OF A SYSTEM OF PARTICLES

If we calculate with the aid of (3.24) and (3.25) the total flux of the energy radiated through a fixed surface, i.e., we calculate the integral of the flux density over all frequencies and all directions, then it turns out to be larger by a factor $1/\sin^2\theta$ than the well-known expression for the energy loss of an ultrarelativistic particle

$$P = -\frac{dE}{dt} = \frac{2e^4H_\perp^2}{3m^2c^3}\left(\frac{E}{mc^2}\right)^2. \qquad (4.1)$$

As already indicated in Sec. 1, this difference is due to the nonstationary nature of the radiation field, namely, the total energy flux through a fixed surface

$$P^* \equiv \oint_S \tilde{p}\,dS = P - \frac{\partial}{\partial t}\int_V \frac{\mathcal{E}^2 + H^2}{8\pi}dV \qquad (4.2)$$

is determined not only by the work P performed by the particle, but also by the change in the energy of the field in a volume V bounded by the surface S. The change of the energy of the field is connected, obviously, with the translational motion of the particle and becomes appreciable when the velocity of the translational motion of the particle is comparable with the velocity of light.

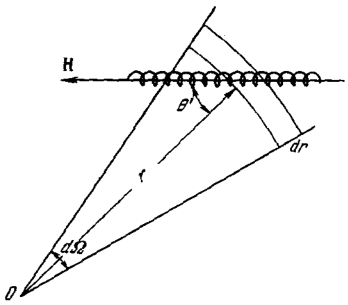

FIG. 5.

Actually, this result is the consequence of the retardation due to the finite velocity of propagation of the electromagnetic field. In fact, let us consider the radiation of an individual electron crossing a volume element $r^2 dr d\Omega$ at a distance r from the observer (Fig. 5). The electron is in the volume element under consideration during a time $dt' = dr/v_r$, where v_r is the projection of the average velocity of the translational motion of the particles $v_{\parallel}$ in the direction of the observer. Obviously, $v_r = v_{\parallel} \cos\theta' = v\cos\theta\cos\theta'$. If r(t) is the variable distance from the particle, then the instant of observation t is connected with the instant of radiation t′ by the relation $t = t' + r(t')/c$ (we are considering radiation in vacuum). Therefore the radiation emitted by the electron within a time dt′, corresponding to a displacement dr, will be received by the observer after a time

$$dt = dt'\left(1 - \frac{v_r}{c}\right) = \left(1 - \frac{v_r}{c}\right)\frac{dr}{v_r}. \tag{4.3}$$

It follows therefore that the energy radiated after a time dt′ and passing through a unit surface at the point of observation after a time dt is equal to

$$\tilde{p}_\nu\, dt = \tilde{p}_\nu\left(1 - \frac{v_r}{c}\right) dt' = p_\nu\, dt', \tag{4.4}$$

where $\tilde{p}_\nu = \tilde{p}_\nu^{(1)} + \tilde{p}_\nu^{(2)}$ (see (3.24) and (3.25)), and p_ν is given by

$$p_\nu = \tilde{p}_\nu\left(1 - \frac{v_r}{c}\right) = \tilde{p}_\nu\left(1 - \frac{v}{c}\cos\theta\cos\theta'\right). \tag{4.5}$$

As follows from (4.4), this quantity p_ν has the meaning of the energy flux density radiated by the electron per unit time. It is easy to verify that the integral of p_ν over all the frequencies and radiation directions leads to a correct expression for the energy loss (i.e., in the ultrarelativistic case, to the expression (4.1)).

Thus, relation (4.5) establishes the connection between the observed flux $\tilde{p}_\nu$ of the radiation and the "power" p_ν radiated by the electron. Obviously, a similar relation holds for all the components of the radiation polarization tensor (see (3.15) and (3.23)):

$$p_{\alpha\beta}(\nu) = \tilde{p}_{\alpha\beta}(\nu)\left(1 - \frac{v}{c}\cos\theta\cos\theta'\right). \tag{4.6}$$

In the ultrarelativistic case ($v \simeq c$, $\theta \simeq \theta'$) we get therefore

$$p_{\alpha\beta}(\nu) = \tilde{p}_{\alpha\beta}(\nu)\sin^2\theta. \tag{4.7}$$

We shall now show that if we are dealing with radiation of particles from a fixed volume, then it is necessary to use precisely the quantity $p_{\alpha\beta}(\nu)$. Actually this is clear already from (4.4), since the relation (4.4) shows precisely that the energy received by the observer from the trajectory segment dr is determined by the value of p_ν and by the time $dt' = dr/v_r$ required for the electron to traverse this segment. Let us consider this question in somewhat greater detail, in order to obtain expressions for the intensity and other Stokes parameters.

Assume that we are interested in the radiation intensity of an ensemble of particles whose distribution function is $N(E, \tau, \mathbf{r}, t)$. By definition, the quantity $N(E, \tau, \mathbf{r}, t)dE d\Omega_\tau dV$ is equal to the number of particles with energies in the interval E, E + dE and with velocity directions within the limits of the solid angle $d\Omega_\tau$, contained at the instant t in a volume element $dV = r^2 dr d\Omega$.

The number of particles falling into the volume element under consideration (see Fig. 5) is

$$v_r N\left(E, \tau, \mathbf{r}, t - \frac{r}{c}\right) dE\, d\Omega_\tau r^2\, d\Omega;$$

here t is the instant of observation and t − r/c is the instant of emission from a fixed point of space. Each particle radiates from the volume element under consideration an energy (see (4.4))

$$p_\nu\, dt' = p_\nu \frac{dr}{v_r}.$$

As a result, the total flux of the received radiation is

$$F_\nu = \int p_\nu N\left(E, \tau, \mathbf{r}, t - \frac{r}{c}\right) r^2\, dr\, d\Omega\, dE\, d\Omega_\tau, \tag{4.8}$$

and radiation intensity is

$$I_\nu = \frac{dF_\nu}{d\Omega} = \int p_\nu N\left(E, \tau, \mathbf{r}, t - \frac{r}{c}\right) r^2\, dr\, dE\, d\Omega_\tau. \tag{4.9}$$

Similar expressions hold, obviously, for all the components of the tensor

$$I_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix} \quad (\alpha, \beta = 1, 2), \tag{4.10}$$

where I, Q, U, and V are the Stokes parameters of the received radiation. Namely,

$$I_{\alpha\beta}(\nu) = \int p_{\alpha\beta}(\nu) N\left(E, \tau, \mathbf{r}, t - \frac{r}{c}\right) r^2\, dr\, dE\, d\Omega_\tau. \tag{4.11}$$

For a stationary distribution function, under conditions when $N(E, \tau, \mathbf{r}, t) \equiv N(E, \tau, \mathbf{r})$, expressions (4.8)–(4.11) coincide with those given in [1, 2] and earlier in [15].

On the other hand, if we consider an ensemble (cloud) of moving particles, then the observed intensity I_ν (or the flux F_ν) is determined essentially by the dependence of the distribution function on the time. In particular, for an individual electron $N(E, \tau, \mathbf{r}, t - r/c) \sim \delta(\mathbf{r} + \mathbf{v}_r(t - r/c))$ and as a result of integration with respect to r in (4.8) we obtain

$$F_\nu = \frac{p_\nu}{1 - (v_r/c)} = \tilde{p}_\nu = \tilde{p}_\nu^{(1)} + \tilde{p}_\nu^{(2)}, \tag{4.12}$$

as should be the case according to (3.24) and (3.25).

Let us assume now that we are dealing with a stationary "cloud" of particles moving as a unit with a velocity V and a velocity projection V_r in the direction of the observer. This means that in (X.9) we have $N(E, \tau, r, t) = N_0(E, \tau, r - Vt)$.* The intensity of the radiation from such a cloud is

$$I_\nu = P_\nu \left(1 - \frac{V_r}{c}\right)^{-1},$$

where

$$P_\nu = \int p_\nu N_0(E, \tau, r) r^2 \, dr \, dE \, d\Omega_\tau$$

is the radiation power (the energy lost to radiation per unit time).

5. REABSORPTION OF RADIATION OF ULTRARELATIVISTIC PARTICLES

If the number of the radiating particles along the line of sight is sufficiently large, then absorption and stimulated emission of synchrotron radiation by the relativistic electrons themselves comes into play. This process is usually called reabsorption. Reabsorption can, in principle, greatly change the intensity under the polarization of the radiation. Moreover, under certain conditions negative reabsorption is possible, i.e., an intensification of the radiation.[9, 10] This means that the system of radiating electrons acts like a maser.

Zheleznyakov,[9] as well as earlier investigators of reabsorption (see [1, 2] and the literature cited therein) used for the radiation intensity of the individual particles expressions averaged over all the directions. The conditions for the applicability and the very feasibility of such an approach are not clear beforehand. It suffices to say that the radiation has a finite angular distribution, and the polarization properties of the radiation depend significantly on the angle $\psi = \theta - \theta'$ between the direction of the velocity and the radiation direction (see (3.24) and (3.25)). Therefore in the study of reabsorption, and specially negative reabsorption, with allowance for the polarization of the radiation, a more rigorous analysis is required of the angular and polarization properties of the cyclotron radiation. We present below the corresponding results for synchrotron radiation of ultrarelativistic particles.

The most complete characteristic of the radiation is the tensor $I_{\alpha\beta}(\nu)$ (see (4.10)). Within the limits of applicability of geometrical optics, under stationary conditions, the tensor $I_{\alpha\beta}$ can be obtained from the radiation transport equations, the form of which is

$$\frac{dI_{\alpha\beta}}{dr} = \mathscr{E}_{\alpha\beta} - R_{\alpha\beta\sigma\tau} I_{\sigma\tau}. \tag{5.1}$$

Here

$$\mathscr{E}_{\alpha\beta} = \int r^2 p_{\alpha\beta} N(E, \tau, r) \, dE \, d\Omega_\tau \tag{5.2}$$

is the emissivity per unit volume, i.e., the power of the spontaneous radiation per unit volume in a unit solid angle and a unit frequency interval, $R_{\alpha\beta\sigma\tau}$ is the reabsorption tensor, which takes into account the absorption and the stimulated emission of the relativistic electrons, and also the absorption and the change of polarization in the medium, if the latter is present. In the case when the radiating electrons are in a nonrelativistic plasma, we assume that the refractive index is $n = 1 - \omega_0^2/2\omega^2$ and the condition $\eta = \sqrt{\xi^2 + (\omega_0/\omega)^2} \ll 1$ is satisfied (see (3.26)).

Calculation of the tensor $R_{\alpha\beta\sigma\tau}$ was undertaken in [7] for a medium with refractive index $n = 1$, but with allowance for the Faraday rotation of the plane of polarization. In that paper, however, the probability of spontaneous emission per unit volume was determined incorrectly (and consequently the emissivity $\mathscr{E}_{\alpha\beta}$; see (5.2)). Namely, it was calculated using not $p_{\alpha\beta}$ but $\tilde{p}_{\alpha\beta}$ (see Sec. 4).

The tensor $R_{\alpha\beta\sigma\tau}$ was calculated for synchrotron radiation of ultrarelativistic electrons in [16]. In the case of an isotropic distribution function* $N(E, \tau, r) = N(E, r)$, the components of the tensor $R_{\alpha\beta\sigma\tau}$, referred to the axes l_1 and l_2 (see (3.4)) are equal to†

$$\left.\begin{aligned}
R_{1111} &= -\frac{c^2}{\nu^2}\int dE\, E^2 \frac{\partial}{\partial E}\left(\frac{N(E, r)}{E^2}\right)\int r^2 p_{11}(\nu)\, d\Omega,\\
R_{2222} &= -\frac{c^2}{\nu^2}\int dE\, E^2 \frac{\partial}{\partial E}\left(\frac{N(E, r)}{E^2}\right)\int r^2 p_{22}(\nu)\, d\Omega,\\
R_{1112} &= R_{2111} = R_{2212} = R_{2122} = -R_{1121} = -R_{1211} = -R_{2221} - R_{1222} =\\
&= -\frac{c^2}{2\nu^2}\int dE\, E^2 \frac{\partial}{\partial E}\left(\frac{N(E, r)}{E^2}\right)\int r^2 p_{12}(\nu)\, d\Omega,\\
R_{1212} &= R_{2121} = \frac{R_{1111} + R_{2222}}{2},\\
R_{1122} &= R_{1221} = R_{2112} = R_{2211} = 0.
\end{aligned}\right\} \tag{5.3}$$

The integrals with respect to $d\Omega = 2\pi \sin\theta \, d\theta \simeq 2\pi \sin\theta \, d\psi$, contained in (5.3), are equal to ($\psi = \theta - \theta'$; only small ψ are significant, by virtue of which it is possible to put in the final expressions $\theta' \doteq \theta$)

$$\left.\begin{aligned}
\int r^2 p_{11(22)}(\nu)\, d\Omega &= \frac{\sqrt{3}}{2}\frac{e^2\omega_H \sin\theta}{\eta}\frac{\nu}{\nu_c}\left[\int_{\nu/\nu_c}^{\infty} K_{5/3}(z)\, dz \pm K_{2/3}\left(\frac{\nu}{\nu_c}\right)\right],\\
\int r^2 p_{12}\, d\Omega &= \frac{-2i}{\sqrt{3}} e^2\omega_H \cos\theta \left[\int_{\nu/\nu_c}^{\infty} K_{1/3}(z)\, dz + \frac{\nu}{\nu_c} K_{1/3}\left(\frac{\nu}{\nu_c}\right)\right],
\end{aligned}\right\} \tag{5.4}$$

where

$$\nu_c = \frac{3\sin\theta}{4\pi}\frac{\omega_H}{\eta^3} \tag{5.5}$$

*For simplicity, we use here a velocity averaged over the period of the motion, i.e., the velocity $v_\parallel$. In this connection, $N(E, r, \tau, t)$ should be taken to mean the average over the period, by virtue of which the dependence of N on τ reduces to a dependence on the angle θ only.

*All that is actually needed is that when τ changes the distribution function $N(E, \tau, r)$ change little in directions that are close to the observation direction k. Specifically, expressions (5.3) for R_{1111} and R_{2222} remain approximately valid also for an anisotropic distribution function, if the condition $\partial N/\partial\theta \ll \eta^{-2} N$ is satisfied. Compared with R_{1111} and R_{2222}, the coefficients $R_{1112} = R_{2111} = \ldots$ etc. are of the order of magnitude of η (see (5.4) and (5.5) below), but depend significantly on the degree of anisotropy of the distribution function. Therefore, for an anisotropic distribution function, the expressions (5.3) can be used to calculate the coefficients only if $\partial N/\partial\theta \ll N$. Under the conditions, it is necessary to take $N(E, r)$ in (5.3) to mean the function $N(E, \tau, r)$ at $\tau = k$.

†The tensor $R_{\alpha\beta\sigma\tau}$ was calculated in [16] by the kinetic-equation method. This can be done also by the method of Einstein coefficients (see, in particular, [7]). It is significant here that an electron moving in a magnetic field with specified E and θ radiates a wave with definite polarization (generally elliptical) in a given direction and at a given frequency. The electron in question will absorb or intensify only waves having the polarization that it can emit (this conclusion is particularly clear from quantum theory, inasmuch as the probabilities of the spontaneous emission, absorption, and stimulated emission are proportional to the square of the modulus of the same matrix element).

and it is assumed that $\theta \gg \eta$ and $\theta \gg \pi - \eta$ (when $n = 1$, obviously $\eta = \xi$, and expression (5.5) coincides with (3.22)).

In the integration with respect to the energy E in (5.2) and (5.3), it is significant that for large values of ν/ν_c the expressions (5.4) are exponentially small. On the other hand,

$$\frac{\nu}{\nu_c(E)} = \frac{\omega_0^2}{3\pi \sin\theta} \frac{mc}{eH\nu} \left(\frac{E_0}{E}\right)^2 \left[1 + \left(\frac{E}{E_0}\right)^2\right]^{3/2}, \quad (5.6)$$

where

$$E_0 = mc^2 \frac{\omega}{\omega_0} = mc^2 \frac{2\pi\nu}{\omega_0}. \quad (5.7)$$

Therefore in the integrals of (5.2) and (5.3) the only significant energy region is $E_1 < E < E_2$, for which

$$\frac{\nu}{\nu_c(E_1)} \gg 1, \quad \frac{\nu}{\nu_c(E_2)} \gg 1, \quad E_1 \ll E_0 \ll E_2. \quad (5.8)$$

The behavior of the energy spectrum of the electrons outside the integral (5.8) has little effect on the magnitude of the integrals (5.2) and (5.3). Thus, if the electron spectrum is specified in the energy interval (E_1, E_2) and the conditions (5.8) are satisfied, then the integration with respect to dE can be extended to the entire energy interval.

We note that the first of the conditions (5.8) is in fact independent of the presence of the plasma. Indeed,

$$\frac{\nu}{\nu_c(E_1)} \simeq \frac{4\pi}{3\sin\theta} \frac{mc\nu}{eH} \left(\frac{mc^2}{E_1}\right)^2 \text{ for } E_1 \ll E_0. \quad (5.9)$$

To the contrary, the second condition of (5.8) depends significantly on the plasma density:

$$\frac{\nu}{\nu_c(E_2)} \simeq \frac{\omega_0^2}{6\pi^2 \sin\theta} \frac{mc}{eH\nu^2} \frac{E_2}{mc^2} \text{ for } E_2 \gg E_0. \quad (5.10)$$

The conditions under which it is possible to extend the integration over the entire energy interval in a total absence of plasma are indicated in [1, 2].

We shall assume that in the energy interval $E_1 < E < E_2$ the differential energy spectrum of the electrons can be approximated by the function

$$N(E, \mathbf{r}) = \frac{N_e(\mathbf{r})}{mc^2} \left(\frac{mc^2}{E}\right)^\gamma = \frac{1}{4\pi} K_e E^{-\gamma}. \quad (5.11)$$

In the isotropic case, obviously, $K_e E^{-\gamma} dE$ is the density of the particles with energy in the interval $E, E + dE$. Substituting (5.11) in (5.2) and (5.3) we obtain, for example,

$$R_{1111}(\gamma) = \frac{\gamma+2}{m\nu^2} \mathscr{E}_{11}(\gamma+1); \quad (5.12)$$

similarly

$$R_{2222}(\gamma) = \frac{\gamma+2}{m\nu^2} \mathscr{E}_{22}(\gamma+1),$$

$$R_{1112}(\gamma) = \frac{\gamma+2}{2m\nu^2} \mathscr{E}_{12}(\gamma+1). \quad (5.13)$$

Here $\mathscr{E}_{\alpha\beta}(\gamma + 1)$ is the emissivity (5.2) calculated for the emissivity spectrum (5.11) with exponent $\gamma + 1$ in lieu of γ.

Thus, the reabsorption-tensor components for a system of electrons with a spectral exponent γ are expressed in terms of the components of the emissivity tensor $\mathscr{E}_{\alpha\beta}$ corresponding to the spectral exponent $\gamma + 1$.

The transport equation (5.1) can be rewritten in a more illustrative form, introducing in lieu of $I_{\alpha\beta}$ the Stokes-parameter column:*

$$S_i = \begin{pmatrix} I \\ Q \\ V \\ U \end{pmatrix} = \begin{pmatrix} I_{11} + I_{22} \\ I_{11} - I_{22} \\ i(I_{21} - I_{12}) \\ I_{12} + I_{21} \end{pmatrix}. \quad (5.14)$$

In this notation, Eq. (5.1) takes the form

$$\frac{dS_i}{dr} = \mathscr{E}_i - R_{ij} S_j \qquad (i, j = 1, 2, 3, 4), \quad (5.15)$$

where $\mathscr{E}_i$ are the Stokes parameters corresponding to the emissivity tensor:

$$\mathscr{E}_i = \begin{pmatrix} \mathscr{E}_1 \\ \mathscr{E}_2 \\ \mathscr{E}_3 \\ \mathscr{E}_4 \end{pmatrix} \equiv \begin{pmatrix} \mathscr{E}_I \\ \mathscr{E}_Q \\ \mathscr{E}_V \\ \mathscr{E}_U = 0 \end{pmatrix} = \begin{pmatrix} \mathscr{E}_{11} + \mathscr{E}_{22} \\ \mathscr{E}_{11} - \mathscr{E}_{22} \\ i(\mathscr{E}_{21} - \mathscr{E}_{12}) \\ \mathscr{E}_{21} + \mathscr{E}_{12} \end{pmatrix}, \quad (5.16)$$

and the reabsorption-coefficient matrix R_{ij} is

$$R_{ij} = \begin{pmatrix} \mu & \lambda & \rho & 0 \\ \lambda & \mu & 0 & 0 \\ \rho & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \quad (5.17)$$

where

$$\mu(\gamma) = \frac{1}{2}(R_{1111} + R_{2222}) = \frac{\gamma+2}{2m\nu^2} \mathscr{E}_I(\gamma+1),$$
$$\lambda(\gamma) = \frac{1}{2}(R_{1111} - R_{2222}) = \frac{\gamma+2}{2m\nu^2} \mathscr{E}_Q(\gamma+1),$$
$$\rho(\gamma) = 2iR_{1112} = -\frac{\gamma+2}{2m\nu^2} \mathscr{E}_V(\gamma+1). \quad (5.18)$$

The vanishing of $\mathscr{E}_U$ and of certain components of the matrix R_{ij} is connected with the choice of the coordinate system. We used formulas (5.12) and (5.13) to obtain the last expressions in (5.18).

Let us examine certain consequences of the obtained relations. A case frequently of interest is one in which the directions of the magnetic field are more or less uniformly distributed along the line of sight. For such a random field, the transport equation can be averaged over the field orientations. We note that the tensors $\mathscr{E}_{\alpha\beta}$ and $R_{\alpha\beta\sigma\tau}$ were calculated above in a special coordinate system, in which the projection of the magnetic field on the plane of the figure was directed along the $\mathbf{l}_2$ axis. To go over to a certain fixed coordinate system, in which the direction of the magnetic field is arbitrary, it is necessary to use the tensor properties of the quantities $\mathscr{E}_{\alpha\beta}$ and $R_{\alpha\beta\sigma\tau}$.

By averaging Eq. (5.1) over the field directions and changing to the form (5.15) it is easy to obtain the following relations (the bar denotes averaging over the field directions[17]):

$$\overline{\mathscr{E}}_Q = \overline{\mathscr{E}}_U = \overline{\mathscr{E}}_V = 0, \quad \overline{\lambda} = \overline{\rho} = 0, \quad (5.19)$$

$$\overline{\mathscr{E}}_I = \frac{1}{2}\int_0^\pi \mathscr{E}_I(\theta) \sin\theta \, d\theta, \quad (5.20)$$

$$\overline{\mu}(\gamma) = \frac{(\gamma+2)}{2m\nu^2} \overline{\mathscr{E}_I(\gamma+1)}. \quad (5.21)$$

*The quantities S_i do not form a vector with respect to rotations of the axes 1 and 2 in the plane of the figure. The law of transformation of the S_i is determined by their connection (4.10) with the components of the tensor $I_{\alpha\beta}$.

The conditions (5.19) denote that the intrinsic radiation is completely depolarized, as should be the case in the absence of a preferred direction. In this case

$$\mathscr{E}_i = \mathscr{E}_1\delta_{1i}, \quad R_{ij} = \mu\delta_{ij} \quad (i = 1, 2, 3, 4). \tag{5.22}$$

If we are interested in the intrinsic radiation of a certain finite volume, then Eqs. (5.15) with the parameters (5.22) must be solved with the boundary conditions $S_i = 0$ on the far boundary of the volume of the source (integration of Eq. (5.15) must be carried out along the line of sight from the far boundary to the near boundary of the source). A solution for a homogeneous source is, obviously,

$$I = \frac{\bar{\mathscr{E}}_1}{\bar{\mu}}(1 - e^{-\bar{\mu}l}), \quad Q = U = V = 0, \tag{5.23}$$

where l is the length of the source along the line of sight.

When $\gamma > -2$, the reabsorption coefficient μ is always positive, corresponding to absorption (this is clear from (5.21)). If the optical thickness of the source is large, i.e., $\bar{\mu}l \gg 1$, then $I = \bar{\mathscr{E}}_I/\mu$ and the radiation emerges only from a narrow layer of thickness on the order of $1/\bar{\mu}$. The expression for $\bar{\mathscr{E}}_I/\bar{\mu}$ determines the maximum possible intensity in the case of positive reabsorption ($\bar{\mu} < 0$).

The necessary condition for negative reabsorption ($\bar{\mu} < 0$) is $\gamma < -2$. This condition, however, is not yet sufficient. For example, negative reabsorption is in general impossible in vacuum.[9, 18] This does not contradict expression (5.21), since expression (5.21) for a power-law electron spectrum is suitable in vacuum only when $\gamma > 1/3$ (see [2]). The conditions for the realization of negative reabsorption, for which the presence of a plasma is necessary, are discussed in [9, 10]. In the case of negative reabsorption ($\bar{\mu} < 0$) at an optical thickness $|\mu|l \gg 1$ the intensity of the radiation is

$$I \simeq \frac{\bar{\mathscr{E}}_I}{\bar{\mu}} e^{|\bar{\mu}|l}$$

and can be arbitrarily large within the framework of the linear theory when $l \to \infty$.

We emphasize here that the simple form (5.15) with the coefficients (5.22) is assumed by the transport equation only in the case of a random field and a sufficiently smooth distribution of the electrons over the directions (cf. first footnote on p. 40, right column).

In other cases, the transport equation must be used in the general form (5.1) or (5.15), even when only the change of I, the total radiation intensity, is investigated. Therefore the simple concept of a single reabsorption coefficient can be used only in the isotropic case corresponding to the absence of polarization. In this case, after performing the necessary calculations of the coefficient $\bar{\mu}$ (see (5.21)), it is easy to verify that for electrons in vacuum (refractive index of the medium n = 1)

$$\bar{\mu} = \frac{1}{2}\int_0^\pi \mu(\theta)\sin\theta\, d\theta = \bar{g}(\gamma)\frac{e^3}{2\pi m}\left(\frac{3e}{2\pi m^3c^5}\right)^{\frac{\gamma}{2}} K_e H^{\frac{\gamma+2}{2}} \nu^{-\frac{\gamma+4}{2}}; \tag{5.24}$$

The values of μ and λ for the homogeneous field are

$$\mu(\theta) = \frac{\gamma+10/3}{\gamma+2}\lambda(\theta) = g(\gamma)\frac{e^3}{2\pi m}\left(\frac{3e}{2\pi m^3c^5}\right)^{\frac{\gamma}{2}} K_e H_\perp^{\frac{\gamma+2}{2}} \nu^{-\frac{\gamma+4}{2}}, \tag{5.25}$$

where $H \equiv H\sin\theta$,

$$\left.\begin{aligned} g(\gamma) &= \frac{\sqrt{3}}{4}\Gamma\left(\frac{3\gamma+2}{12}\right)\Gamma\left(\frac{3\gamma+22}{12}\right), \\ \bar{g}(\gamma) &= \frac{\sqrt{3\pi}}{8}\frac{\Gamma\left(\frac{\gamma+6}{4}\right)}{\Gamma\left(\frac{\gamma+8}{4}\right)}\Gamma\left(\frac{3\gamma+2}{12}\right)\Gamma\left(\frac{3\gamma+22}{12}\right). \end{aligned}\right\} \tag{5.26}$$

We present a table of the values of the coefficients $g(\gamma)$ and $g(\gamma)$:

γ	1	2	3	4	5
$g(\gamma)$	0.96	0.70	0.65	0.69	0.83
$\bar{g}(\gamma)$	0.69	0.47	0.40	0.44	0 46

Expression (5.25) coincides with that obtained in [2] by using the method of Einstein coefficients for the averaged radiation. The field was assumed here homogeneous, and in this case the single quantity $\mu(\theta)$ does not suffice to describe the radiation transport (see (5.17)).

If the field in the source is homogeneous, then allowance for the polarization of the radiation is important also for the total intensity. This is understandable, since, in general, all the Stokes parameters vary along the line of sight, and consequently it is necessary to consider all the components of Eqs. (5.15)–(5.17). Under the conditions of interest to us it is possible, however, to use the fact[15] that for an electron angular distribution that does not change too rapidly we have $\mathscr{E}_V/\mathscr{E}_I \sim \eta \ll 1$ and consequently $\rho/\mu \sim \eta \ll 1$. Putting in (5.15)–(5.17) $\mathscr{E}_V = 0$ and $\rho = 0$, we obtain two independent systems of equations

$$\frac{d}{dr}\begin{pmatrix} I \\ Q \end{pmatrix} = \begin{pmatrix} \mathscr{E}_I \\ \mathscr{E}_Q \end{pmatrix} - \begin{pmatrix} \mu & \lambda \\ \lambda & \mu \end{pmatrix}\begin{pmatrix} I \\ Q \end{pmatrix}, \tag{5.27}$$

$$\frac{d}{dr}\begin{pmatrix} V \\ U \end{pmatrix} = -\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} V \\ U \end{pmatrix}. \tag{5.28}$$

If we consider the intrinsic radiation of the source, then we can assume that all $S_i = 0$ on its far boundary. Integrating (5.27) with this boundary condition, we obtain at the "exit" from the source

$$\left.\begin{aligned} I &= \frac{\mathscr{E}_I + \mathscr{E}_Q}{2(\mu+\lambda)}(1 - e^{-(\mu+\lambda)l}) + \frac{\mathscr{E}_I - \mathscr{E}_Q}{2(\mu-\lambda)}(1 - e^{-(\mu-\lambda)l}), \\ Q &= \frac{\mathscr{E}_I + \mathscr{E}_Q}{2(\mu+\lambda)}(1 - e^{-(\mu+\lambda)l}) - \frac{\mathscr{E}_I - \mathscr{E}_Q}{2(\mu-\lambda)}(1 - e^{-(\mu-\lambda)l}), \\ U &= 0, \quad V = 0. \end{aligned}\right\} \tag{5.29}$$

Equations (5.29) have a simple meaning. If we use in lieu of the Stokes parameters I and Q the parameters $I^{(1)} = I_{11} = (1/2)(I + Q)$ and $I^{(2)} = I_{22} = (1/2)(I - Q)$ and analogous parameters for the emissivity $\mathscr{E}^{(1)} = (\mathscr{E}_I + \mathscr{E}_Q)/2$ and $\mathscr{E}^{(2)} = (\mathscr{E}_I - \mathscr{E}_Q)/2$, then (5.29) reduces to

$$I^{(1)} = \frac{\mathscr{E}^{(1)}}{\mu+\lambda}[1 - e^{-(\mu+\lambda)l}], \quad I^{(2)} = \frac{\mathscr{E}^{(2)}}{\mu-\lambda}[1 - e^{-(\mu-\lambda)l}]. \tag{5.30}$$

It follows therefore that radiation intensities with two principal polarizations behave independently. For the intensity $I^{(1)}$ corresponding to polarization transverse to the field projection on the plane of the figure, the reabsorption coefficient is $\mu_\perp = \mu + \lambda$, and for radiation with polarization along the field projection (intensity $I^{(2)}$) the reabsorption coefficient is $\mu_{\parallel} = \mu - \lambda$ (see (5.18) and (5.25)). For a thin layer, when $|\mu \pm \lambda| l \ll 1$, we have

$$\frac{I^{(1)}}{I^{(2)}} \cong \frac{I_0^{(1)}}{I_0^{(2)}} = \frac{\mathscr{E}^{(1)}}{\mathscr{E}^{(2)}}. \tag{5.31}$$

If the reabsorption is positive, i.e., $\mu + \lambda > \mu - \lambda > 0$ (or, which is the same, $\mu > \lambda > 0$), we get from (5.30) at sufficiently large optical thickness $(\mu - \lambda) l \gg 1$

$$\frac{I^{(1)}}{I^{(2)}} = \frac{\mathscr{E}^{(1)}}{\mathscr{E}^{(2)}} \frac{\mu - \lambda}{\mu + \lambda} \leqslant \frac{\mathscr{E}^{(1)}}{\mathscr{E}^{(2)}} = \frac{I_0^{(1)}}{I_0^{(2)}}. \tag{5.32}$$

In particular, for electrons with a power-law spectrum in vacuum we have $I^{(1)}/I^{(2)} = (3\gamma + 5)/(3\gamma + 8) < 1$ (see (5.18) and (5.25)), i.e., the radiation becomes polarized along the magnetic field. In this case the degree of polarization decreases:*

$$\Pi \equiv \frac{\sqrt{Q^2 + U^2 + V^2}}{I} = \frac{I^{(2)} - I^{(1)}}{I^{(1)} + I^{(2)}} = \frac{3}{6\gamma + 13} < \Pi_0 = \frac{\gamma + 1}{\gamma + 7/3}.$$

Such a result is connected with the fact that for synchrotron radiation the emissivity $\mathscr{E}^{(1)}$ (polarization transverse to the field) is larger than $\mathscr{E}^{(2)}$, meaning that the inverse process, absorption, is stronger for waves with polarization (electric vector) directed across the field. In the case of negative reabsorption the picture is different. Thus, if $\mu + \nu < 0$ and $\lambda \neq 0$, it follows from (5.29) that at sufficiently large l

$$|Q| = |I^{(1)} - I^{(2)}| \to I = I^{(1)} + I^{(2)}, \quad \Pi \to 1, \tag{5.33}$$

i.e., the radiation is completely linearly polarized perpendicular to the projection of the magnetic field on the picture plane. This is the result of the large intensification of the polarization that predominates in the spontaneous emission. A strong linear polarization ($\gtrsim 75\%$) can serve as an indication of an appreciable role of negative reabsorption for the given source.

Another factor of importance in negative reabsorption may also be the dependence of the coefficients (5.18) on the angle between the line of sight and the magnetic field. If the field in the sources is homogeneous (but not fully random), then the emission will be predominantly intensified in those directions in which μ is maximal, i.e., in the directions transverse to the field. Therefore, at a large optical thickness, individual regions of the inhomogeneous source will have an anomalously bright appearance. In other words, the spatial fluctuations of the intensity in the source should increase strongly.

Let us discuss, finally, the possibility of appearance of a noticeable elliptical or circular polarization. If we disregard the highly anisotropic electron velocity distribution or the possible role of Faraday rotation of the source, then there is no elliptic polarization, accurate to terms of order η.[15] Under cosmic conditions, the existence of sharply anisotropic electron-velocity distributions for a long time, such as would take place for a particle beam, has low probability.[19] Nevertheless, it is still possible as a sporadic phenomenon, and can also be quasistationary in a sufficiently strong magnetic field (not too dense a particle beam moving along the magnetic field). Even easier to realize are apparently conditions in which it is necessary to take into account the Faraday rotation of the plane of polarization or, more accurately, regard the plasma as magnetoactive (the plasma was hitherto regarded as isotropic, by virtue of which it was characterized by a single refractive index n). The propagation and emission of waves in a magnetoactive plasma differs, as is well known, in having a large variety of different possibilities (see [20] and the literature cited there). An analysis of the positive and negative reabsorption of synchrotron radiation in the case of highly anisotropic velocity distribution functions of the radiating electrons, and with allowance for the anisotropy (magnetoactivity) relative to the cold "parent" plasma in which the radiating relativistic particles move, is of great interest and should serve as a subject of further research (see [17, 21]).

*We recently learned that this result is contained in [23].

6. MAGNETIC-FIELD VARIATION CONNECTED WITH SYNCHROTRON RADIATION (LOSSES)

In considering the radiation of a charged particle moving in a magnetic field, and also when taking into account the loss or gain of energy by this particle as the result of any other mechanism, the magnetic field is usually assumed specified. It is quite obvious that such a formulation of the problem has a limited applicability. Indeed, a particle moving in a magnetic field produces its own magnetic field $\mathbf{H}_1$, which weakens the external field $\mathbf{H}_0$ (diamagnetic effect). The field $\mathbf{H}_1$ depends on the particle energy E and, specifically, decreases with decrease of this energy, $E = mc^2/\sqrt{1 - v^2/c^2}$.

When the energy loss is taken into account, the field $\mathbf{H}_1$ decreases, but it can lead to a change not only of the total field $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1$, but also of the field $\mathbf{H}_0$ (we have in mind mutual induction; see below). As a result of the change of the magnetic field, an induced electric field $\mathscr{E}$ is produced, and can in turn change the particle energy. In this connection, the question was even raised recently whether the particle can "draw" energy from the field and by the same token lose not only its kinetic energy $E_k = E - mc^2$, but also a larger energy.[22] As will be shown below, such a conclusion is incorrect, but nevertheless the energy relations of a particle moving in a magnetic field, with allowance for the loss (or gain) of energy, is of undisputed interest, and will therefore be discussed now. A particle with charge e and mass m moving in a homogeneous magnetic field* $\mathbf{H}_0$ has a magnetic moment

$$\boldsymbol{\mu} = -\frac{mv_\perp^2 \mathbf{H}_0}{2H_0^2 \sqrt{1 - v^2/c^2}} = -\frac{|e| r_H v_\perp}{2c} \left(\frac{\mathbf{H}}{H_0}\right) = -\frac{v_\perp^2 E}{2c^2 H_0} \left(\frac{\mathbf{H}}{H_0}\right). \tag{6.1}$$

In fact, the frequency of revolution of the particle in the magnetic field is

$$\omega_H = \frac{|e| H_0}{mc} \frac{mc^2}{E} = \frac{|e| H_0}{mc} \sqrt{1 - v^2/c^2}$$

*Actually, we mean the magnetic induction $\mathbf{B}_0$. However, with the exception of the last part of the present section, we deal only with vacuum, and will not distinguish, where permissible, between H and B.

and the radius of the projection of the orbit on the plane perpendicular to $\mathbf{H}_0$ is

$$r_H = \frac{v_\perp}{\omega_H} = \frac{mv_\perp}{|e| H_0 \sqrt{1 - v^2/c^2}}.$$

Finally, the magnetic moment is

$$\boldsymbol{\mu} = \frac{e}{2c}[\mathbf{rv}],$$

whence we arrive at (6.1) where $v_\perp$ is the projection of the velocity $\mathbf{v}$ on the plane perpendicular to the field $\mathbf{H}_0$. The sign in (6.1) can be chosen already from general considerations, since it is known that a gas of charged particles is diamagnetic (if the spin is disregarded). If the number of particles is large and they move independently, then their moments are simply summed. The proper field of all the particles is in this case small compared with the external field $\mathbf{H}_0$ (this field is produced by sources located outside the region under consideration), provided $4\pi N\mu \ll H_0$, where $\mu = |\boldsymbol{\mu}|$ and N is the concentration of the particles (moments). More accurately, if we are dealing with particles having different values of μ, then the role of $N\mu$ is assumed by the total moment per unit volume, i.e., by the magnetization M. The aforementioned inequality $4\pi M \ll H_0$, in terms of the theory of magnets obviously denotes that $B = H_0 + 4\pi M \simeq H_0$ (which explains also the appearance of the factor 4π in this inequality).

Taking (6.1) into account, we arrive consequently at the condition for the weakness of the diamagnetic effect, in the form

$$\frac{\overline{v_\perp^2 EN}}{4c^2} \ll \frac{H_0^2}{8\pi}, \qquad (6.2)$$

where the bar denotes averaging over the energy spectrum. In an isotropic distribution of the ultrarelativistic particles (for concreteness we have in mind cosmic rays) with respect to their velocity directions

$$\frac{\overline{v_\perp^2}}{c^2} = \frac{2}{3}$$

condition (6.2) can be written in the form

$$w_{\text{c.r.}} \ll 6w_M, \quad w_{\text{c.r.}} = \overline{E}N, \quad w_M = \frac{H_0^2}{8\pi}, \quad E \gg mc^2, \qquad (6.3)$$

where the value of the particle mass m plays no role.

Thus, in order for the influence of the relativistic particles themselves on the magnetic field to be weak, their energy density should be small compared with the density of the magnetic energy. Yet in cosmic rays, rays, in a number of cases,[1, 19] we have

$$w_{\text{c.r.}} \sim w_M. \qquad (6.4)$$

Under these conditions the relativistic particles, obviously, already influence the field, but the field can still, generally speaking, remain sufficiently strong (in the sense that the field in the medium is of the same order as the external field H_0). On the other hand, if

$$w_{\text{c.r.}} \gg w_M = \frac{H_0^2}{8\pi}, \qquad (6.5)$$

then the dynamic effect could lead to complete screening of the field, to instability, etc. Further development of these concepts will make it possible, one might think, to obtain additional arguments in favor of relation (6.4) or, more accurately, the relation $w_{\text{c.r.}} < w_M$, against the possibility of realizing the condition (6.5). Without stopping to discuss this interesting problem in greater detail (see [19] in this connection), we consider below another case of one particle, the properties and state of which are described by the values

$$e, \ m, \ E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \text{ and } v_\perp;$$

for concreteness we shall assume that the external field $\mathbf{H}_0$ is homogeneous and is produced in a sufficiently long solenoid (Fig. 6). The current flowing in the solenoid winding, per unit length of solenoid, is

$$i = \int j\, dr = \frac{c}{4\pi} H_0,$$

where j is the current density in the "winding" (if screening is disregarded, i = jd, where d is the thickness of the "winding"). We assume that the particle trajectory is located entirely in the solenoid, and is located sufficiently far from its walls. The volume of the solenoid is $V = \pi r_0^2 L$, $L \gg r_0$.

The equation of motion of the particle is

$$\frac{d}{dt}\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} = e\left\{\boldsymbol{\mathscr{E}} + \frac{1}{c}[\mathbf{vH}]\right\} - \mathbf{f}, \qquad (6.6)$$

where f is a certain "friction force," which leads to energy loss. After multiplying by the velocity $\mathbf{v}$ we get

$$\frac{dE}{dt} = e\boldsymbol{\mathscr{E}}\mathbf{v} - P, \quad E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \quad P = \mathbf{fv}. \qquad (6.7)$$

Of course, if acceleration takes place and not losses, then $P < 0$; the radiative friction force, obviously, is included in the expression for $\mathbf{f}$.

We denote the density of the current producing the field by $\mathbf{j}$; the current connected with the particle under consideration is not included here, and its density is $e\mathbf{v}\delta(\mathbf{r} - \mathbf{r}_e(t))$. Then the Poynting theorem, which follows from the field equations, should be written in the form

$$\frac{\partial}{\partial t}\left\{\frac{\mathscr{E}^2 + H^2}{8\pi}\right\} + e\boldsymbol{\mathscr{E}}\mathbf{v}\delta(\mathbf{r} - \mathbf{r}_e) = -\frac{c}{4\pi}\operatorname{div}[\boldsymbol{\mathscr{E}}\mathbf{H}] - \mathbf{j}\boldsymbol{\mathscr{E}} \qquad (6.8)$$

or, after integrating over a certain volume V and taken as (6.7) into account, in the form

$$\frac{d}{dt}\left\{\int \frac{\mathscr{E}^2 + H^2}{8\pi} dv + E\right\} = -\oint S_n\, d\sigma - \int \mathbf{j}\boldsymbol{\mathscr{E}}\, dv - P, \qquad (6.9)$$

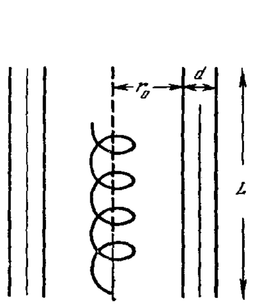

FIG. 6.

where the Poynting vector

$$\mathbf{S}=\frac{c}{4\pi}[\mathcal{E}\mathbf{H}]$$

is integrated over the surface bounding the volume V (obviously $\mathbf{n}$ is the outward normal to this surface).

Expressions (6.6)–(6.9) have, of course, a general character, but we shall apply them to the case of a field in a solenoid (in the absence the particle $\mathbf{H} = \mathbf{H}_0$ = const). Inside the solenoid the total field is $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1$, where $\mathbf{H}_1$ is the field produced by the particle in question itself. For simplicity we assume the particle to move along a circle. At a sufficiently large distance $r \gg r_H$ from the particle trajectory, its field averaged over the period is equivalent to the field of the magnetic moment (6.1) with $v_\perp = v$. Consequently, far from the particle we have

$$\mathbf{H}_1=\operatorname{rot}\operatorname{rot}\frac{\boldsymbol{\mu}}{r},\quad \mathcal{E}_1=-\frac{1}{c}\operatorname{rot}\frac{\dot{\boldsymbol{\mu}}}{r}, \tag{6.10}$$

where the value of $\boldsymbol{\mu}(t')$ must be taken at the instant $t' = t - r/c$ (see [13], Sec. 72).

We now assume that the distance r from the winding of the solenoid to the particle is much smaller than the "wavelength" $\lambda = c/\tau$, where τ is the characteristic time of variation of the moment due to the losses ($d\mu/dt \equiv \dot{\mu} \sim \mu/\tau$). In this case, i.e., neglecting retardation, $\boldsymbol{\mu}(t') = \boldsymbol{\mu}(t)$ and

$$\mathbf{H}_1=\operatorname{rot}\mathbf{A}_1=\frac{3\mathbf{r}(\mathbf{r}\boldsymbol{\mu})-\boldsymbol{\mu}r^2}{r^5},\quad \mathbf{A}_1=\left[\nabla\frac{1}{r}\boldsymbol{\mu}\right],$$

$$\mathcal{E}_1=-\frac{1}{c}\dot{\mathbf{A}}=-\frac{1}{c}\left[\nabla\frac{1}{r}\dot{\boldsymbol{\mu}}\right]=-\frac{1}{c}\left[\dot{\boldsymbol{\mu}}\frac{\mathbf{r}}{r^3}\right]. \tag{6.11}$$

We now use relation (6.9), choosing as the integration surface the internal surface of the cylindrical "winding." The fields $\mathbf{H}_1$ and $\mathcal{E}_1$ are small quantities compared with $\mathbf{H}_0$, and we can therefore put

$$\int\frac{\mathcal{E}^2+H^2}{8\pi}dV\simeq\frac{H_0^2}{8\pi}V+\frac{\mathbf{H}_0}{4\pi}\int\mathbf{H}_1\,dv,\quad \oint\mathbf{Sn}\,d\sigma\simeq\frac{c}{4\pi}\int[\mathcal{E}_1\mathbf{H}_0]_n\,d\sigma. \tag{6.12}$$

By virtue of (6.11), and also recognizing that in this case $\mathbf{n}\boldsymbol{\mu} = \mathbf{n}\dot{\boldsymbol{\mu}} = 0$, we get

$$\mathbf{S}\simeq\frac{c}{4\pi}[\mathcal{E}\mathbf{H}_0]=\frac{1}{4\pi r^3}\{\dot{\boldsymbol{\mu}}(\mathbf{H}_0\mathbf{r})-\mathbf{r}(\dot{\boldsymbol{\mu}}\mathbf{H}_0)\}, \tag{6.13}$$

since

$$\oint\mathbf{Sn}\,d\sigma=-\frac{\dot{\boldsymbol{\mu}}\mathbf{H}_0}{4\pi}\oint\frac{\mathbf{rn}}{r^3}d\sigma=-\dot{\boldsymbol{\mu}}\mathbf{H}_0,$$

$$\oint\frac{\mathbf{rn}}{r^3}d\sigma=-\oint\mathbf{n}\nabla\frac{1}{r}d\sigma=-\int\Delta\frac{1}{r}dv=4\pi\int\delta(\mathbf{r})\,dV=4\pi.$$

Further, also by virtue of (6.11),

$$\frac{\mathbf{H}_0}{4\pi}\int\mathbf{H}_1\,dv=\frac{\mathbf{H}_0}{4\pi}\int\operatorname{rot}\mathbf{A}_1\,dv=\frac{\mathbf{H}_0}{4\pi}\oint\left[\mathbf{n}\left[\nabla\frac{1}{r},\ \boldsymbol{\mu}\right]\right]d\sigma$$
$$=\frac{\boldsymbol{\mu}\mathbf{H}_0}{4\pi}\oint\frac{\mathbf{rn}}{r^3}d\sigma=\boldsymbol{\mu}\mathbf{H}_0. \tag{6.14}$$

Let us assume that the field $\mathbf{H}_0$ is maintained constant, regardless of the changes produced in the moment μ of the particle by the losses. This can be done, of course, only at the expense of work done by external emf sources ("batteries") connected in the circuit of the winding. Under such conditions, taking (6.12)–(6.14) into account and assuming that $\mathbf{H}_0$ = const, Eq. (6.9) takes the form*

$$\frac{dE}{dt}=-p(E),\quad E=\frac{mc^2}{\sqrt{1-v^2/c^2}}, \tag{6.15}$$

i.e., we obtain the customarily employed equation for the particle energy in the presence of losses. Of course, this result is directly clear from (6.7), inasmuch as when $\mathbf{H}_0$ = const the electric field $\mathcal{E} = 0$. However, the foregoing analysis shows what happens to the magnetic field and to the magnetic energy. The total energy of the field in the volume V (in the solenoid) is, according to (6.1), (6.12), and (6.14)

$$\int\frac{H^2}{8\pi}dv\simeq\frac{H_0^2}{8\pi}V+\frac{\mathbf{H}_0}{4\pi}\int\mathbf{H}_1\,dv=\frac{H_0^2}{8\pi}V+\boldsymbol{\mu}\mathbf{H}_0=\frac{H_0}{8\pi}V-\frac{v^2E}{2c^2}, \tag{6.16}$$

where we put $v = v_\perp$ (motion along a circle). As the particle loses energy, the moment $\mu = |\boldsymbol{\mu}|$ decreases, and the total magnetic energy increases, since $\boldsymbol{\mu}\cdot\mathbf{H}_0 < 0$. This increase is at the expense of the energy flux entering into the solenoid. At the end of the process (the particle has lost its energy and its moment is $\mu = 0$) the field $\mathbf{H}_0$, by assumption, remains unchanged, and the "batteries" lost an energy

$$-\boldsymbol{\mu}(0)\,\mathbf{H}_0=\frac{v^2(0)\,E(0)}{2c^2}, \tag{6.17}$$

where the argument $t = 0$ indicates the initial values of μ, v, and E. More interesting is a somewhat different formulation of the problem, in which the field $\mathbf{H}_0$ is not assumed specified,[22] but the "winding" of the solenoid is closed and is made up of a current of electrons which experience no resistance (i.e., the electrons describe circles with radius r_0, filling a thin layer of thickness d; see Fig. 6). Since in a cosmic plasma the conductivity of the medium is very large, such a case has certain features close to reality. However, one must not overestimate the degree of this closeness, since under cosmic conditions it would be necessary to assume that the entire medium inside the solenoid is also conducting. In addition, we assume for simplicity that the "winding" does not distort the field of the particle, i.e., the field of the moment $\boldsymbol{\mu}$. This means that the "winding" should be sufficiently thin ($d \ll \delta$, where $\delta = \sqrt{mc^2/4\pi e^2 N_0}$ is the depth of penetration of the field into the "winding"†). Under the conditions discussed, we place the surface bounding the volume considered in (6.9) outside the winding. Here $H_0 = 0$ and in the approximation (6.12) the energy flux $S = 0$. On the other hand, expression (6.14) remains practically unchanged if we neglect screening. As a net result Eq. (6.9) takes the form

*The integration surface is chosen such that the sources of the field $\mathbf{H}_0$ are outside the surface, by virtue of which $\int \mathbf{j}\mathcal{E}dv = 0$ (see (6.9)). In order to clarify the conditions under which the energy $\int H_1^2/8\pi dv$ can be neglected, we present the appropriate estimate. It is easy to see that

$$\int\frac{H_1^2}{8\pi}dv\sim\frac{\mu}{r_H}\sim\mu\bar{H}_1,$$

where $\bar{H}_1 \sim \mu/r_H^3$ is a certain average field of the moment μ on the particle orbit. Obviously, the condition

$$\frac{\mathbf{H}_0}{4\pi}\int\mathbf{H}_1\,dv=\mu\bar{H}_0\gg\int\frac{H_1^2}{8\pi}dv$$

takes the form $\bar{H}_1 \ll H_0$ or

$$H_0\ll\frac{vE^2}{e^3c}\leqslant\frac{m^2c^3v}{e^3}\sim10^{16}\frac{v}{c}.$$

†For a free electron gas $\epsilon = 1 - 4\pi e^2 N_0/m\omega^2$, and when $\epsilon < 0$ and $|\epsilon| \gg 1$ the field attenuates like

$$e^{-\frac{\omega}{c}\sqrt{|\varepsilon|}z}=e^{-z/\delta},\quad \delta^2=\frac{mc^2}{4\pi e^2N_0}.$$

$$\frac{d}{dt}\left\{\frac{H_0^2}{8\pi}V+\mu H_0+E\right\}=-P-\int \mathbf{j}\mathscr{E}\,dv, \qquad (6.18)$$

where $\int \mathbf{j}\,\mathscr{E}dv$ is taken over the volume of the "winding." Obviously

$$\int \mathbf{j}\mathscr{E}\,dv=\frac{d}{dt}K,$$

where

$$K=\frac{mu^2}{2}\cdot 2\pi r_0\,dLN_0$$

is the kinetic energy of the ordered motion of the electrons in the "winding," responsible for the creation of the field H_0 ($2\pi r_0 dL$ is the volume of the "winding," and N_0 is the concentration of the electrons, which are assumed to be nonrelativistic). As already indicated, the current density is

$$\frac{i}{d}=\frac{cH_0}{4\pi d}$$

and, on the other hand, $j = eN_0u$. Hence

$$u=\frac{cH_0}{4\pi eN_0d}$$

and

$$K=\frac{mc^2}{e^2N_0(2\pi r_0d)}\frac{H_0^2}{8\pi}V, \quad V=\pi r_0^2L. \qquad (6.19)$$

The factor

$$\frac{mc^2}{e^2N_0(2\pi r_0d)}=\frac{2\delta^2}{r_0d}$$

can be made much smaller than unity, if the assumed condition

$$\delta=\sqrt{\frac{mc^2}{4\pi e^2N_0}}\gg d$$

is satisfied, owing to the increase of the radius r_0 of the solenoid. Under similar conditions, i.e., when $2\delta^2/r_0d \ll 1$, to which we confine ourselves, the term $\int \mathbf{j}\mathscr{E}dv$ in (6.18) would mean introduction of a small correction to the term $\frac{d}{dt}\left(\frac{H_0^2}{8\pi}V\right)$. As a result we arrive at the equation

$$\frac{d}{dt}\left\{\frac{H_0^2}{8\pi}V+\mu H_0+E\right\}=-P, \qquad (6.20)$$

which was employed without commentary in [22]. Besides Eq. (6.20), it is necessary, in order to solve the problem, to make use of (6.7), expressing the field $\mathscr{E}$ in terms of dH_0/dt. The latter can be readily done, since the field $\mathscr{E}$ is simply the induction field and consequently

$$2\pi r\mathscr{E}=-\frac{1}{c}\pi r^2\frac{dH_0}{dt},$$

or on the particle trajectory (where $r = r_H = mv_\perp c/|e|H_0\sqrt{1-v^2/c^2}$) we have

$$\mathscr{E}=-\frac{r_H}{2c}\frac{dH_0}{dt}.$$

Finally,

$$\mathscr{E}=\frac{mv_\perp\frac{dH_0}{dt}}{2e\sqrt{1-v^2/c^2}\,H_0}, \qquad (6.21)$$

where the sign of $\mathscr{E}$ is determined, for example, by the Lenz rule (to conserve the flux through the contour when $dH_0/dt > 0$, the force $e\mathbf{E}$ increases the velocity $v_\perp$). Substituting the obtained field $\mathscr{E}$ in (6.7) we get

$$\frac{dE}{dt}=\frac{mv^2}{2\sqrt{1-v^2/c^2}}\frac{d}{dt}\ln H_0-P, \qquad (6.22)$$

where we put $v_\perp = v$ by virtue of the fact that we are considering only the motion of a particle along a circle. Subtracting (6.22) from (6.20) we have

$$\frac{d}{dt}\{W-F\}=-\frac{1}{2}F\frac{d}{dt}\ln W, \quad W=\frac{H_0^2V}{8\pi},$$
$$F=\frac{mv^2}{2\sqrt{1-v^2/c^2}}=\frac{v^2E}{2c^2}. \qquad (6.23)$$

The integral of this equation is

$$F=2W_0\left\{\frac{W}{W_0}-\left(1-\frac{F_0}{2W_0}\right)\sqrt{\frac{W}{W_0}}\right\}, \qquad (6.24)$$

Where W_0 and F_0 are the values of W and F at $t = 0$. After the particle has lost energy, $F = F(\infty) = 0$ and according to (6.24) $W(\infty) - W(0) \simeq -F_0$, since $F_0/W_0 \ll 1$; thus, the change of energy of the external field is

$$\frac{H_0^2(0)}{8\pi}V-\frac{H_0^2(\infty)}{8\pi}V=\frac{v^2(0)E(0)}{2c^2}=-\mu(0)H_0(0). \qquad (6.25)$$

Integrating (6.20), which can be written in the form

$$\frac{d}{dt}(W-F+E)=-P,$$

and taking (6.25) into account, we can readily see that the total loss is

$$\int_0^\infty P\,dt=E(0),$$

since $E(\infty) = F(\infty) = 0$. Thus, in this case, too, the particle loses only its own energy $E(0)$; the same, of course, follows from (6.22), (6.24), and (6.25), if the calculation is performed only accurate to terms of order F_0, but not F_0^2/W_0 (this is precisely the accuracy employed in (6.25)).

The meaning of the relation (6.25), where the equality

$$\frac{v^2(0)E(0)}{2c^2}=-\mu(0)H_0(0)$$

follows from (6.1) with $v_\perp = v$, is perfectly clear if the statements made in connection with formulas (6.16) and (6.17) are recalled. Namely, the total energy of the magnetic field in the solenoid is

$$\int\frac{H^2}{8\pi}dv\simeq\frac{H_0}{8\pi}V+\mu H_0$$

(see (6.16)). Further, by assumption, $\mu(\infty)H_0(\infty) = F(\infty) = 0$, and relation (6.25) is therefore simply the condition for the conservation of the total magnetic energy. Here, however, the field $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1$ itself changes and becomes redistributed, namely, the field $\mathbf{H}_1$ decreases with decreasing absolute value of the moment μ; in order for the total magnetic energy to be conserved it is therefore necessary that the homogeneous field $\mathbf{H}_0$ also decrease, since the field $\mathbf{H}_1$ is directed opposite to the field $\mathbf{H}_0$ (diamagnetic effect).

Thus, the situation turns out to be quite trivial in final analysis: everything reduces to a calculation of the diamagnetic effect that takes place when the charged particles move in the magnetic field, and to the use of the energy conservation law (the Poynting theorem). In both problems under consideration (constant field $\mathbf{H}_0$ and solenoid with "short-circuited winding") the particle loses only its own energy and cannot "draw" energy from the magnetic field!

To complete the picture and, principally, bearing in mind the possibility of generalization to the more complicated case of an ensemble of radiating particles, let us consider the question considered in the present section from the microscopic point of view. To this end, we introduce, besides the field intensity, the magnetic moment per unit volume $\mathbf{M}$ and the magnetic induction $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$

If N is the particle concentration, then the magnetic moment per unit volume is $\mathbf{M} = N\mu$, where the magnetic moment of one particle (6.1) is written in the form

$$\mu = -\frac{p_\perp^2}{2B}\frac{c^2}{E}\frac{\mathbf{B}}{B}, \quad E = \frac{mc^2}{\sqrt{1-v^2/c^2}}, \quad p_\perp = \frac{mv_\perp}{\sqrt{1-v^2/c^2}}. \quad (6.1a)$$

This expression differs from (6.1) only in that the field $\mathbf{H}_0$ is replaced by the average macroscopic field, i.e., by the induction $\mathbf{B}$ (for simplicity we do not distinguish between $\mathbf{B}$ and $\mathbf{B}_0$). We now take into account the constancy of the quantity $k = p_\perp^2/B$ (the adiabatic invariant) under slow variations of the field, and confine ourselves to the case of ultrarelativistic particle ($E \gg mc^2$), executing circular motion, when $E^2 \simeq c^2p_\perp^2 = kc^2B$ and

$$\mu = |\mu| = \frac{E}{2B} = \frac{1}{2}\sqrt{\frac{kc^2}{B}}. \quad (6.26)$$

Let us calculate the energy (just as in (6.28)) of a system consisting of a field and particles, assuming that the external field and the distribution of the particles N are homogeneous in this case. Then

$$dU = \frac{1}{4\pi}\mathbf{H}\,d\mathbf{B} = \frac{1}{4\pi}(\mathbf{B} - 4\pi\mu)\,d\mathbf{B} = d\left(\frac{B^2}{8\pi}\right) - \mathbf{M}\,d\mathbf{B}. \quad (6.27)$$

To calculate the last term in (6.27) we use the expression

$$\mathbf{M} = N\mu = \frac{1}{2}N\sqrt{\frac{kc^2}{B}}\frac{\mathbf{B}}{B},$$

hence

$$\mathbf{M}\,d\mathbf{B} = -d\left(N\sqrt{kc^2B}\right) = d\,(2\mathbf{MB}).$$

As a result we get

$$U = \frac{B^2}{8\pi} - 2\mathbf{MB}. \quad (6.28)$$

The energy of the particles per unit volume is, by virtue of (6.26), $NE = 2N\mu B = -2\mathbf{M}\cdot\mathbf{B}$. Thus, the energy density (6.28) consists simply of the energy density of the "true" average field $B^2/8\pi$ and the particle energy density NE.

With the aid of the relation $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, expression (6.28) can be also rewritten in the form

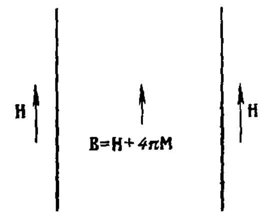

FIG. 7.

$$U = \frac{H^2}{8\pi} - \mathbf{MH} - 6\pi M^2. \quad (6.29)$$

The last term in (6.29) is proportional to the density of the magnetic energy of the currents connected with the particles, and if $6\pi M \ll H$ or $NE \ll B^2/8\pi$ (see (6.3)), this term can be omitted. Thus, in the weak-magnetization approximation used in [22] and above, we obtain

$$U = \frac{H^2}{8\pi} - \mathbf{MH} = \frac{H^2}{8\pi} + \mathbf{MH} + NE, \qquad NE = -2\mathbf{MH}. \quad (6.30)$$

The energy density of the system consisting of the field and particles is expressed in precisely the form $U = H^2/8\pi = \mathbf{M}\cdot\mathbf{H} + NE$ in [22] and (6.20).

Let us consider now the change of the energy of the system as a result of the deceleration of the particles due to radiation or other losses under two somewhat different conditions.

Let the external field be maintained constant (first formulation of the problem). Such a situation can be realized, for example, if a layer of relativistic plasma is contained by an external field (Fig. 7). In this case, by virtue of the continuity of the tangential components of $\mathbf{H}$, we have $\mathbf{H}_e = \mathbf{H}_i = \mathbf{H}$, where the indices e and i pertain to the quantities on the external and internal sides of the separation boundary.

The energy of the system in the initial state is equal to

$$U(0) = \frac{H_0^2}{8\pi} - \mathbf{M}(0)\,\mathbf{H}(0).$$

Assume that as a result of the losses the kinetic energy of the particles in the final state is zero, and accordingly $\mathbf{M}(\infty) = 0$. Then in the final state we have

$$U(\infty) = \frac{H^2(\infty)}{8\pi}$$

and the change of energy is

$$\Delta U = U(\infty) - U(0) = \mathbf{M}(0)\,\mathbf{H}(0) = -\frac{1}{2}NE(0). \quad (6.31)$$

In this case the total change of the energy of the system consists of the energy NE(0) lost by the particle, and the work done by the external currents $NE(0)/2 = -\mathbf{M}(0)\cdot\mathbf{H}(0)$, which maintain the field constant in the volume under consideration.

Let us consider now the second case, when the system is contained in a superconducting solenoid, so that the electric field on the boundaries of the volume is $\mathscr{E} = 0$. This case is closer to the problem considered in [22] and above for a short-circuited solenoid "winding." Under these conditions the total magnetic flux is $\int B_n d\sigma$ = const, and consequently B = const if the cross section is constant and the field is homogeneous.

Therefore, by virtue of (6.28), the total change of the energy of the system after the particles lose their energy is (obviously, $M(\infty) = 0$)

$$U(\infty) - U(0) = 2M(0)\,B = -NE(0), \tag{6.32}$$

i.e., it is exactly equal to the energy lost by the particles.

Obviously, the results (6.31) and (6.32) are in full agreement with the deductions made above (see, in particular, (6.17) and (6.25)).

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