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NUCLEAR SHAPE, DEFORMABILITY, AND EXCITED STATES

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1. INTRODUCTION

DURING the past fifteen years our conceptions of the shapes of atomic nuclei have changed greatly. Prior to 1950 it was generally believed that all nuclei possess a spherically symmetric equilibrium shape and that excited nuclear states can be represented as vibrations of the nuclear surface about that shape and changes in the states of motion of individual nucleons in a self-consistent spherically symmetric field. These ideas were well supported by the very successful shell model of the nucleus that was proposed by Goeppert-Mayer and Jensen in 1949. The shell model provided a qualitative explanation of the magic numbers, spins, and magnetic moments of nuclei. It was found, however, that the observed quadrupole moments of rare earth elements and several others are almost 30 times larger than those predicted by means of the shell model. Rainwater [1] gave the first qualitative explanation of large deviations from nuclear sphericity by considering interactions between outer nucleons and the nuclear core. On the basis of these ideas A. Bohr proposed a collective model in 1952,[2] and nuclear theory entered a new stage. Bohr's theory explained the principal regularities observed for the first rotational states of nonspherical even-even nuclei. Following the work of Bohr and of Bohr and Mottelson [3] the investigation of nuclear equilibrium shapes became one of the most active branches of nuclear physics.

The shape of a nucleus, or more exactly the symmetry of the self-consistent field acting on the nucleons within the nucleus, is very important for the classification of single-nucleon and collective excited nuclear states. In the case of spherical symmetry, which undoubtedly exists in the ground states of doubly magic nuclei, a single-nucleon state is characterized by its energy, parity, and the quantum numbers j and m_j specifying, respectively, the total nucleonic angular momentum and its projection on an arbitrary quantization axis. In axially symmetric nuclei the projection Ω of nucleon angular momentum on the axial symmetry axis is conserved in addition to energy and parity. In nonaxial nuclei neither j nor Ω is an integral of the motion.

In spherically symmetric nuclei excited rotational states cannot be observed without simultaneous nuclear deformation. In nonspherical nuclei rotational states play an important role in nuclear excitations. The classification of excited nuclear states thus depends essentially on the symmetry of the intranuclear self-consistent field.

The symmetry of the self-consistent field depends on the spatial (angular) distribution of intranuclear nucleons. It has been shown experimentally that the nucleon density (with $r \leq 1.2A^{1/3} \times 10^{-13}$ cm) is nearly constant. At the nuclear boundary the nucleon density falls off rapidly to zero within about 2×10^{-13} cm. Therefore the concepts of the nuclear surface and shape are good approximations. Because of small nuclear compressibility, in the case of excitations below 10 MeV the alteration of the nucleon distribution is basically a change of nuclear shape without affecting the nuclear density. It must, of course, be remembered that even in the ground state there exist zero-point oscillations of the nuclear surface about its equilibrium shape. The character of these oscillations and their amplitudes depend on the degree of deformability and the symmetry of the equilibrium shape, which depend, in turn, on the numbers of protons and neutrons and their states of motion.

It is evident that to account for many experimental results it is sufficient to take into account the deviations from spherical shape due to quadrupole deformations, i.e., the nuclear shape can be approximated sufficiently by a triaxial ellipsoid. Then in the coordinate system fixed on the nucleus the shape of a nucleus with a given mean radius will depend on two parameters, for which it is convenient to use $\beta(\geq 0)$ and $\gamma(0 \leq \gamma \leq \pi/3)$, which were introduced by Bohr.[2]

In the adiabatic approximation the total energy of single-nucleon motions depends on the nuclear shape, i.e., on the parameters β and γ. This energy determines the potential energy $V(\beta, \gamma)$ of the surface vibrations in any given state of single-nucleon motions.

The values β_0 and γ_0 corresponding to the minimum of $V(\beta, \gamma)$ characterize the equilibrium shape in any given state of single-nucleon motions. In nuclei with a spherical equilibrium shape $\beta_0 = 0$, and for small values of β the potential energy $V(\beta, \gamma) = \frac{1}{2} C\beta^2$ is independent of γ. A nucleus with $\beta_0 \neq 0$ is not spherically symmetric and its potential energy depends on both β and γ. If then $\gamma_0 = 0$ or $\pi/3$ the nucleus is, respectively, a prolate or oblate ellipsoid of rotation (an axial nucleus).

For other values of γ_0 nuclei do not possess axial symmetry. Variations of β about an equilibrium value β_0 for a fixed value of γ_0 are called β vibrations or longitudinal vibrations. Variations of γ about γ_0 for a fixed value of $\beta_0 \neq 0$ are called γ vibrations or transverse vibrations.

In an axially symmetric nucleus ($\gamma_0 = 0$) transverse vibrations can exist in two mutually perpendicular planes (double degeneracy). Two mutually perpendicular transverse vibrations with $\pi/2$ phase difference can be regarded as a pure rotation. Figure 1 shows the scheme of longitudinal atomic vibrations along the z axis and transverse vibrations along the x and y axes in a linear triatomic molecule, which serves as the simplest model of an axial nucleus.

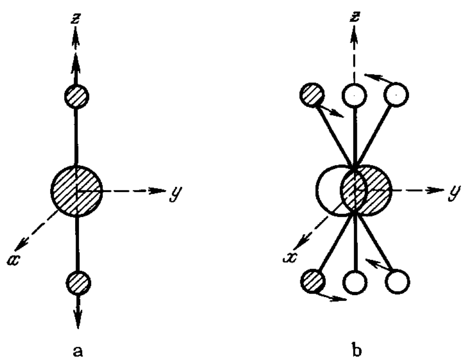

FIG. 1. Illustration of longitudinal and transverse vibrations of an axially symmetric nucleus model by a linear triatomic molecule. a) longitudinal vibrations; b) transverse vibrations.

2. THEORETICAL CALCULATIONS OF EQUILIBRIUM NUCLEAR SHAPE

Several different nuclear models have been used in theoretical calculations of $V(\beta, \gamma)$.[2-10] In [2-4,7-8] the nucleus is viewed as a system that consists of a core containing all closed nucleon shells, plus any additional outer nucleons moving in the field of the core. The equilibrium shape of the nucleus was determined by minimizing, over the states of motion of the outer nucleons, the energy of interaction between these nucleons and the nonspherical part of the core field; the spherical part determines the states of motion of the outer nucleons.

In [2-4] an axial shape was postulated and the value of β_0 was sought with independent motion of the outer nucleons. In [8] Filippov investigated $V(\beta, \gamma)$ as a function of both variables without account of the interactions between outer nucleons. The correlation of outer nucleon motions was taken into account only in accordance with the Pauli principle. It was shown that, depending on the degree to which outer shells are filled with protons and neutrons, nuclear deformation is possible in two mutually perpendicular directions. In these cases the nucleus will not possess axial symmetry. Filippov showed that such cases should be observed near magic numbers of either neutrons or protons.

The theoretical possibility of a nonaxial shape for some nuclei was mentioned by Geĭlikman[5] and by Zaikin,[6] who investigated the dependence of nucleon energy on the parameters defining the symmetry of the self-consistent nuclear field. In these studies interparticle interactions and pairing were not considered.

Very interesting calculations of the potential energy $V(\beta, \gamma)$ have recently been performed in Canada[9,10] taking into account the residual nucleon pairing interaction. The minimum total energy was calculated as a function of β and γ (for constant nuclear volume), for a given number of neutrons and protons moving in an anharmonic three-dimensional oscillator potential, with account of spin-orbit interaction (a correction term proportional to l^2 that was introduced by Nilsson[11]) and nucleon pairing. Characterizing these interactions by means of the same set of parameters as in [12,13], Gunye, Das Gupta, and Preston [10] showed that rare earth nuclei should possess axial equilibrium symmetry while nuclei with mass numbers near 188 should be nonaxial for the same set of parameters. The potential $V(\beta_0, \gamma)$ of the latter type is relatively weakly dependent on γ; this indicates the possibility of zero-point transverse surface γ vibrations with large amplitudes.

Figure 2 illustrates the calculation[10] of $V(\beta, \gamma)$ for Hf^{180} and Os^{188} as functions of β plotted radially and γ in the main interval $0 \leq \gamma \leq \pi/3$. Equipotential energies (in MeV) are represented by solid lines (for the interaction parameters given in [12]) and by dashed lines (for the interaction parameters given in [13]). Energies are measured from the minimum, corresponding to $\gamma = \gamma_0$ and $\beta = \beta_0$.

3. DETERMINATION OF NUCLEAR SHAPE AND DEFORMABILITY FROM EXPERIMENTAL DATA

The theoretical calculations of $V(\beta, \gamma)$ can yield only qualitative results. Because of the great mathematical complexity of the problem and our incomplete knowledge of intraparticle nuclear forces, exact quantitative calculations of $V(\beta, \gamma)$ are not feasible at the present time, and would evidently be inexpedient, like, for example, exact calculations of the refractive indices of complex materials such as glass.

Methods must be developed for determining experimentally the shapes of nuclei and their deformability in connection with transitions to different excited states. Because the properties of excited states—the sequence of energy levels, spins, probabilities of transitions between excited states, mean

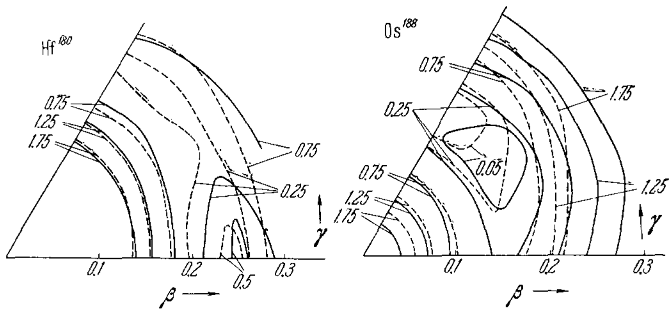

Fig. 2. Potential energy $V(\beta, \gamma)$ as a function of β and γ, calculated in [10].

electric quadrupole moments etc.—depend on nuclear shape and deformability, by measuring these quantities we can determine nuclear shapes and their changes on the basis of the existing theory.

The experimental results are interpreted most simply for even-even nuclei, although even for some of these nuclei we do not possess unique interpretations at the present time. The ground state of each of these nuclei has spin zero.

According to the theory of Bohr and Mottelson[2,3] the simplest spectrum of collective excitations should be found in spherical nuclei. These collective excitations should consist of equally spaced levels corresponding to phonon excitations with the energy E(2) and the angular momentum 2. Neglecting the degeneracy corresponding to different projections of the angular momentum, one-phonon excitations with the energy E(2) and I = 2 will correspond to a nondegenerate spin state. Two-phonon excitations with the energy 2E(2) are triply degenerate in the spin (I = 0, 2, 4), three-phonon excitations are quintuply degenerate (I = 0, 2, 3, 4, 6) etc.

Doubly magic and neighboring nuclei are spherical ($\beta_0 = 0$). However, the spectra of excited states that are predicted by the theory are not observed in these nuclei. For example, in several nuclei having magic numbers of neutrons or protons (O^{16}, Ge^{72}, Zr^{70}) the first excited state has spin zero instead of spin two. In Ge^{70}, Ce^{140}, and Ca^{42} the spin zero state lies very close to the spin-2 first excited state. According to the theory of the surface vibrations of spherical nuclei the energy of this level should be twice as high. Degenerate excited states with the energy 2E(2) and spins 0, 2, and 4 are also not observed in other nuclei (Cd^{114} and Pd^{110}) that are usually assigned to the class of spherical nuclei. The high probability of Coulomb-excited first levels of these nuclei indicates the collective nature of these excited states.

The reduced probabilities of electric quadrupole transitions from the ground state to the spin-2 first level are given by the single formula

$$B(E2;\ 0 \rightarrow 2) = a^2 \langle \beta^2 \rangle \tag{1}$$

for both spherical and nonspherical even-even nuclei. Here $a = 3ZeR_0^2/4\pi$, R_0 is the mean nuclear radius, and $\langle \beta^2 \rangle = \langle 0 | \beta^2 | 0 \rangle$ is the mean square of β in the ground state.

For a spherical nucleus ($\beta_0 = 0$) the ground-state wave function is given by

$$|0\rangle = N \exp\left(-\frac{1}{2} \frac{\beta^2}{\beta_{00}^2} \right), \tag{2}$$

where $\beta_{00} = \hbar (BC)^{-1/2}$ is the amplitude of zero-point β surface vibrations, B is the mass parameter, and C is the elastic coefficient of these vibrations. From (2) we obtain

$$\langle \beta^2 \rangle = \frac{5}{2} \beta_{00}^2. \tag{3}$$

The first excited level of spherical nuclei that is quintuply degenerate in the spin projections is a vibrational level with spin 2 and the energy

$$E(2) = \hbar \sqrt{\frac{C}{B}} = \frac{\hbar^2}{B\beta_{00}^2} = \frac{\hbar^2}{B' \langle \beta^2 \rangle}, \tag{4}$$

where $B' = 2B/5$.

In the ground state of a nonspherical nucleus the surface performs zero-point longitudinal and transverse vibrations represented by the wave functions[14]

$$|0\rangle = N \exp\left\{ -\frac{1}{2} \left(\frac{\beta - \beta_0}{\mu \beta_0} \right)^2 - \left(\frac{\gamma - \gamma_0}{2\Gamma} \right)^2 \right\}, \tag{5}$$

where μ and Γ characterize the zero-point surface vibrations and are defined by

$$\mu^2 = 2 \langle 0 | (\beta - \beta_0)^2 | 0 \rangle / \beta^2, \quad \Gamma^2 = \frac{1}{2} \langle 0 | (\gamma - \gamma_0)^2 | 0 \rangle. \tag{6}$$

The parameter μ was introduced in [15], where it was called the "nonadiabaticity parameter." For $\mu = 0$ the rotational motion of the nucleus and the longitudinal vibrations are completely separated.

In accordance with (5), in the ground state of a nonspherical nucleus we have

$$\langle \beta^2 \rangle = \beta_0^2 \left(1 + \frac{1}{2} \mu^2 \right), \tag{7}$$

i.e., for small values of μ we have $\langle \beta^2 \rangle \approx \beta_0^2$. The first excited level of a nonspherical nucleus is represented by a rotation and has spin 2. Without account of nonadiabatic corrections the energy of this level is

$$E(2) = \frac{\hbar^2}{B\beta_0^2}. \tag{8}$$

By measuring the reduced probabilities of E2 transitions in even-even nuclei we can with the aid of (1) calculate the values of $\sqrt{\langle \beta^2 \rangle}$ for each of these nuclei. These values are represented by the filled circles in Fig. 3 on the basis of data in [16,17]. The experimental energies of the first excited level enable us to determine the mass parameter B for each nucleus using (4) or (8). However, neither the data in Fig. 3 nor the values of B enable us to determine whether the ground-state nucleus is spherical.

We see from Fig. 3, for example, that the values of $\sqrt{\langle \beta^2 \rangle}$ for Pd^{110} and U^{238} are almost equal (0.26 and 0.28, respectively). It is not clear whether these values resulted from static nonsphericity in the

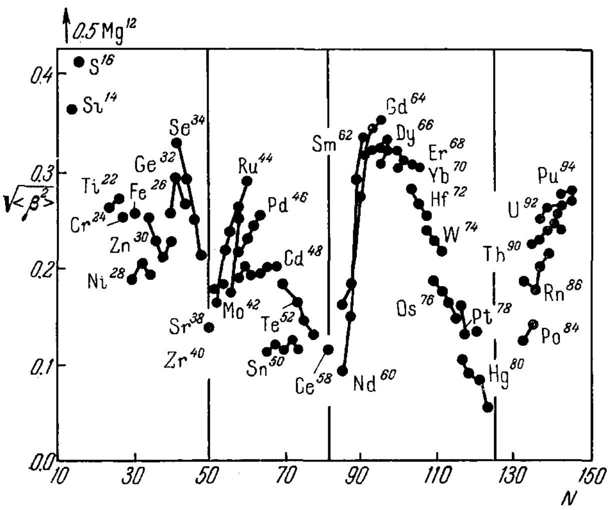

FIG. 3. Values of $\sqrt{\langle\beta^2\rangle}$ for even-even nuclei, calculated from data on the Coulomb excitation of the 2 + first excited level.

ground state or only from zero-point vibrations. From the analysis of data on higher excited levels the 2^+ first level of U^{238} is usually assigned to the rotational spectrum, while the 2^+ level of Pd^{110} is assigned to the vibrational spectrum of a spherical surface. In this case we can assume that the large value of $\langle\beta^2\rangle$ indicates easy deformability of the Pd^{110} surface.

In order to account for discrepancies between the observed spectrum of excited states in nearly spherical nuclei and the vibrational theory of spherical nuclei, different complications of the model have been proposed. For example, Scharff-Goldhaber and Weneser [18] considered a weak interaction of four $f_{7/2}$ nucleons and investigated the coupling between two interacting $f_{7/2}$ nucleons and the surface vibrations. Tamura and Komai [20] investigated the anharmonicity with respect to β and γ vibrations. Ovcharenko [21] in Kiev recently calculated the effects of small transverse and longitudinal deviations from a spherical surface. All these attempts and several others [22-25] have thus far failed to yield satisfactory results. The cause of this failure appears to lie in the fact that in each of these attempts only a single factor was being investigated—interaction with outer nucleons, anharmonicity of surface vibrations, small sphericity and nuclear "softness" etc. In actuality, all these factors appear to have comparable roles in nearly spherical nuclei and should be considered simultaneously, although this would lead to very great mathematical difficulties.

Several investigators [14,15,26-28] have developed a theory of collective quadrupole excitations in nonspherical even-even nuclei with account of the coupling between the rotational motions and the longitudinal and transverse surface vibrations. This work has led to the following main results. If it is assumed that the equilibrium value of γ_0 is zero, then the energy of collective excitations, given for each nucleus in units of the first excited level energy, can be determined from the values of only the two independent parameters μ and Γ:

$$\frac{E(I, K, n_\beta, n_\gamma)}{E(2)} = \frac{n_\beta}{\mu^2} + \frac{\left(\frac{K}{4} + n_\gamma\right)}{\Gamma^2} + \frac{I(I+1)}{6} - \frac{K^2}{8} + \varepsilon(\mu, \Gamma, \ldots), \qquad (9)$$

where n_β and n_γ are the respective quantum numbers of the longitudinal and transverse vibrations and have the values 0, 1, 2, . . . ; I is the nuclear spin; K ($\approx$ 0, 2, 4, . . .) is an approximate quantum number for the spin projection on the nuclear symmetry axis; $\epsilon(\mu, \Gamma, \ldots)$ is an additional energy characterizing the coupling among different types of collective excitations. In the limit of small $\epsilon(\mu, \Gamma, \ldots)$ the collective excitations can be separated into rotational and vibrational excitations, although in the general case this separation is very arbitrary.

a) Ground-state Rotational Band

The excited states represented by (9) for $n_\beta = n_\gamma = 0$, $K \approx 0$, I = 0, 2, 4, ... form the ground-state rotational band. In very hard nuclei ($\mu \approx 0$) the excitation energies of the ground-state band satisfy the simple interval rule

$$1 : 10/3 : 7 : 12 : 55/3 : 26 : 35 \ldots \qquad (10)$$

This rule follows from the rotational energy representation

$$E(I) = AI(I+1), \qquad I = 0, 2, \ldots \qquad (11)$$

The excited states given by (11) represent nuclear rotation around an axis perpendicular to the nuclear symmetry axis. All rotating real nuclei undergo deformation, which increases with the nonadiabaticity parameter. However, the simple formula (11) does not determine excited state energies. For small values of I and $\mu \ll 1$, perturbation theory can be used to represent nuclear excitation energy in the form of the series

$$E(I) = AI(I+1) + BI^2(I+1)^2 + CI^3(I+1)^3 + \ldots, \qquad (12)$$

where B, C, ... characterize the deformability of the nuclear surface. However, for "soft" nuclei and when investigating high-spin excited states (see below) we cannot use (12), because considerable deformation can result from centrifugal forces.

A nucleus rotating with the angular momentum I becomes stretched. The ground-state equilibrium value β_0 then becomes $\beta_I = p\beta_0$, where p is the root of the equation

$$p^4 - p^3 = \frac{1}{3}\mu^4 I(I+1), \qquad I = 0, 2, \ldots$$

Figure 4 shows the ratio between the equilibrium

Table I. Ratio between excitation energies and the 2^+ level of an even-even nucleus. $R_{10}(I)$ - ground-state rotational band, $R_{11}(I)$ - rotational band of longitudinal vibrations, $R_{20}(I)$ - rotational band of transverse vibrations.

μ	Γ	$R_{10}(4)$	$R_{10}(6)$	$R_{10}(8)$	$R_{10}(10)$	$R_{10}(12)$	$R_{11}(0)$	$R_{11}(2)$	$R_{11}(4)$	$R_{20}(2)$	$R_{20}(3)$	$R_{20}(4)$	$R_{20}(5)$
0.10	8	3.327	6.978	11.90	18.09	25.51	94.77	95.80	98.20	25.02	26.01	27.34	28.98
0.20	8	3.299	6.823	11.47	17.12	23.68	22.77	23.88	26.43	23.25	24.11	25.26	26.66
0.40	8	2.962	5.513	8.448	11.65	15.06	5.394	6.676	9.069	14.85	15.28	15.85	16.54
0.60	8	2.597	4.449	6.444	8.536	10.69	3.160	4.407	6.216	10.56	10.83	11.18	11.61
0.80	8	2.417	3.994	5.651	7.359	9.100	2.620	3.866	5.350	8.993	9.210	9.494	9.838
1.00	8	2.323	3.770	5.272	6.805	8.360	2.401	3.659	4.966	8.265	8.475	8.710	9.016
0.10	15	3.271	6.678	11.07	16.36	22.52	85.99	87.02	89.36	6.844	7.841	9.250	10.82
0.20	15	3.241	6.533	10.66	15.49	20.95	20.67	21.78	24.26	6.691	7.638	8.966	10.44
0.40	15	2.898	5.263	7.867	10.61	13.48	4.925	6.206	8.511	5.369	5.990	6.832	7.732
0.60	15	2.543	4.264	6.045	7.854	9.688	2.914	4.154	5.898	4.338	4.770	5.347	5.954
0.80	15	2.373	3.844	5.331	6.817	8.308	2.432	3.664	5.103	3.907	4.270	4.751	5.256
1.00	15	2.285	3.639	4.990	6.331	7.667	2.238	3.476	4.752	3.696	4.027	4.465	4.922
0.10	25	2.834	5.335	8.533	12.43	17.02	69.63	70.66	72.55	2.406	3.405	5.505	6.397
0.20	25	2.809	5.226	8.243	11.82	15.91	16.75	17.86	19.86	2.390	3.365	5.388	6.237
0.40	25	2.526	4.281	6.225	8.319	10.53	4.050	5.324	7.167	2.194	2.952	4.391	4.956
0.60	25	2.248	3.550	4.914	6.329	7.786	2.449	3.675	5.081	1.989	2.573	3.630	4.031
0.80	25	2.120	3.248	4.403	5.584	6.788	2.073	3.276	4.453	1.892	2.405	3.316	3.658
1.00	25	2.055	3.100	4.160	5.235	6.324	1.923	3.122	4.177	1.841	2.320	3.163	3.477

deformation β_I for rotation with angular momentum I and the ground-state equilibrium value β_0 as a function of I and μ. We see that for I = 4 and $\mu = 1.05$, β_0 is replaced by $\beta_4 = 2\beta_0$. The simple character of the rotation (around an axis that is perpendicular to the symmetry axis for $\gamma_0 = 0$ and $K \approx 0$) is possible only for axially symmetric nuclei with small values of Γ, i.e., for nuclei that are sufficiently hard with respect to transverse surface vibrations. Nonaxial nuclei and axial nuclei with $\Gamma > 15°$ perform complex rotational motions. Their rotational states are described by linear superpositions of states with K = 0, 2, ... States with K = 0 will be dominant in the linear superposition for the ground-state rotational band. The contributions from states with K = 2, 4, ... will increase as Γ or γ_0 approaches 30°.

Rotational state energies with account of longitudinal nuclear deformation for all possible fixed values γ_0 of γ were investigated theoretically by Chaban and the present author in [15]. It was shown in later work [14,26,27] that the transverse surface vibrations of axial nuclei can be taken into account in the calculations of [5] formally by identifying γ with the value of Γ characterizing the amplitude of zero-point vibrations. This formal substitution is possible when we study only the first excited states pertaining to the transverse surface vibrations.* Just as in the case of a spherical nucleus the value of $\langle \beta^2 \rangle$ is equivalent to β_0^2 for some nonspherical nucleus, so in the case of an axially symmetric nucleus $\Gamma^2 = \frac{1}{2}\langle \gamma^2 \rangle$ is equivalent to γ_0^2 for some nonaxial nucleus.

*The equivalence is destroyed for higher excited states and for the probabilities of some electromagnetic transitions.[27] However, the pertinent experimental data are too sparse to permit a confident choice of a model.

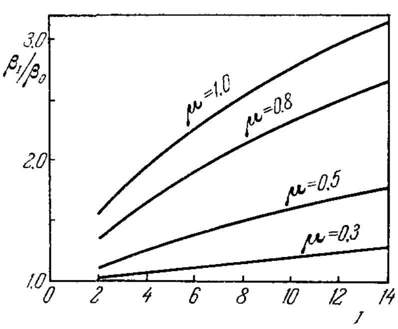

FIG. 4. Equilibrium nuclear deformation vs. rotational-state spin I and parameter μ.

Table I gives calculated [29] values of $R_{10}(I)$ based on the formulas in [15]. These are the ratios between the energies of excited states in the ground-state rotational band and the 2^+ first level as a function of μ and Γ. For soft nuclei considerable deviations should be observed from the interval rule (10) for a perfectly hard axially symmetric nucleus. For example, in nuclei with $\mu = 1$ instead of (10) the energy ratios of levels with spins 2, 4, 6, 8, 10, 12 should satisfy

$$1 : 2.32 : 3.77 : 5.27 : 6.80 : 8.34 \ldots, \text{when } \Gamma = 8°, \quad (13)$$

and

$$1 : 2.05 : 3.10 : 4.16 : 5.24 : 6.32 \ldots, \text{when } \Gamma = 25°. \quad (14)$$

Thus for $\mu \approx 1$ and $\Gamma = 25°$ the excited states of the ground-state rotational band should be almost equally spaced.

A clear proof of high deformability even for nuclei exhibiting a very clear rotational spectrum at low ex-

Table II. Comparison between experimental high-spin rotational state energies [30] and the Davydov-Chaban theory

Isotope		2^+	4^+	6^+	8^+	10^+	12^+	14^+	16^+	% deviation of theory from experiment
W^{176}	Exp.	108.7	348.5	699.4	1140	1648	2206	—	—	0.20
	Theor.	108.7	349.1	698.8	1137	1646	2213			
W^{174}	Exp.	111.9	355.0	704.2	1137	1635	2186	—	—	0.16
	Theor.	111.8	355.8	705.3	1137	1633	2181			
W^{172}	Exp.	122.9	376.9	727.2	1147	1616	2129	2677	—	0.27
	Theor.	122.6	378.7	729.3	1147	1617	2126	2668		
Hf^{172}	Exp.	94.5	307.9	627.0	1036	1519	2063	2651	—	0.31
	Theor.	94.4	307.5	625.5	1033	1517	2066	2669		
Hf^{170}	Exp.	100.0	320.6	641.1	1041	1503	2013	2564	3147	0.22
	Theor.	100.0	320.5	639.8	1038	1500	2012	2568	3160	
Hf^{168}	Exp.	123.9	385.0	756.1	1212	1734	2304	—	—	0.79
	Theor.	121.7	384.9	758.6	1216	1740	2315			
Hf^{166}	Exp.	158.7	470.7	897.6	1407	1971	2565	—	—	1.45
	Theor.	155.7	471.3	902.2	1413	1983	2600			
Yb^{166}	Exp.	101.8	329.7	667.1	1097	1604	2172	—	—	0.12
	Theor.	101.9	330.2	667.7	1097	1601	2169			
Yb^{164}	Exp.	122.5	384.0	758.0	1219	1748	—	—	—	0.52
	Theor.	121.2	384.7	760.3	1222	1752				

citations was obtained by Stephens et al. at the University of California in 1964.[30] Nuclear reactions induced by the relatively heavy ions B^{11}, N^{14}, and F^{19} indicated high-spin excited states of the nine nuclei

Yb(164, 166), Hf(166, 168, 170, 172), W(172, 174, 176).

By investigating conversion electrons and γ rays emitted by these nuclei in cascade transitions to the ground state the excitation energies of these nuclei were measured very accurately (to 0.3%). Table II, which was taken from Stephens' report at the 15th All-union Conference on Nuclear Spectroscopy compares the experimental rotational state energies with the theoretical values based on the Davydov-Chaban theory [15] for $\Gamma = 0$. The last column gives the mean percent deviations of the theoretical values from experiment. Considering that the comparison between the theory and experiment is based on only the two parameters E(2) and μ, there is good agreement.

If instead of the absolute energies we take their ratios to the energy of the 2^+ first level, the comparison between theory and experiment enables us to determine the single theoretical parameter μ for each nucleus. This comparison is shown in Fig. 5 (taken from [30]), where the nearly horizontal lines correspond to the calculated [15] ratios between the energy levels of different spins I and the energy of the 2^+ first level as functions of μ. The vertical lines connect the measured ratios for each nucleus. It can be seen that the hardest of the investigated nuclei with respect to longitudinal vibrations is Hf^{172} ($\mu \approx 0.25$) and that the softest nucleus is Hf^{166} ($\mu \approx 0.4$). Figure 6 (taken from [31]) gives the ratios A_{I+2}/A_I of successive rotational constants defined by

$$A_I = \frac{E(I) - E(I-2)}{4I-2}.$$

The dots in the figure represent the experimental values, while the solid curve represents the theory in [15]. The dashed curve represents the theoretical values (12) with the two parameters A and B determined from the energies of the 2^+ and 4^+ levels. The dot-dash curve represents the theoretical values (12) with the three parameters A, B, and C determined from data for the 2^+, 4^+, and 6^+ levels. We also see

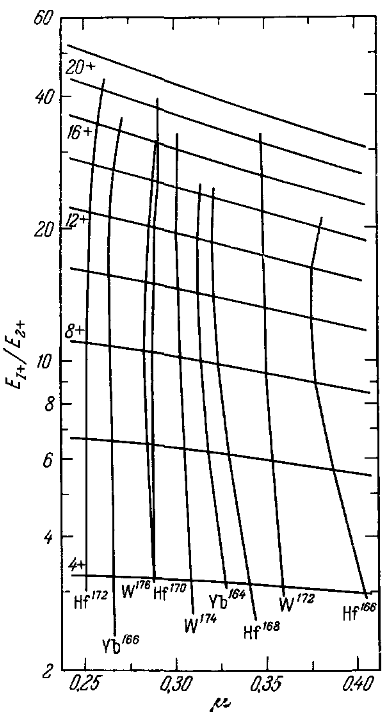

FIG. 5. Theoretical (Davydov-Chaban[15]) and experimental (Stephens et al.[30-31]) energy ratios between the rotational levels of spin I and the 2^+ first level as a function of μ.

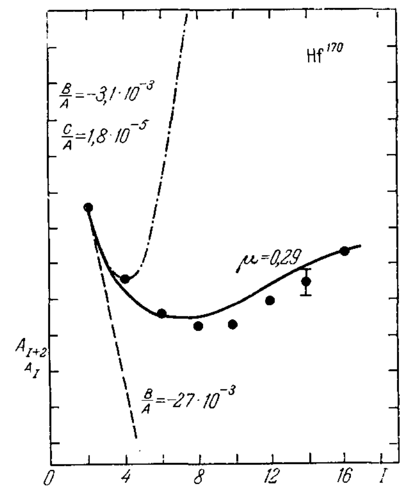

FIG. 6. Ratios of successive rotational constants A as functions of the spin I. Dots – experimental;[30-31] solid curve – Davydov-Chaban theory[15]; dot-dash and dashed curves – perturbation theory Eq. (12).

that the perturbation theory, represented by (12), cannot be used to describe rotational states with account of nuclear stretching accompanying rotation.

b) Rotational Band of Transverse Surface Vibrations

The excited states (9) for the quantum numbers $n_\beta = n_\gamma = 0$, $K \approx 2$, $I = 2, 3, 4, \ldots$ will be called the rotational band of transverse surface vibrations. In a rough approximation neglecting the correction term $\epsilon(\mu, \Gamma)$ in (9) the ratios between the energies of these excited states and the 2^+ first level are given by

$$R_{20}(I) \equiv \frac{E(I,2,0,0)}{E(2)} \approx \frac{1}{2\Gamma^2} + \frac{I(I+1)}{6} - \frac{1}{2}, \quad I = 2, 3, \ldots$$

With account of longitudinal nuclear deformation, values of $R_{20}(I)$ for different values of μ and Γ and for $I = 2, 3, 4, 5$ are given in Table I on the basis of the theory in [15] and tables in [29].

The first term of $R_{20}(2)$ for the rotational band of transverse vibrations corresponds to the transverse surface vibrations. As was pointed out at the end of the Introduction, when the ground state is axially symmetric two degenerate transverse vibrations can be regarded as a pure rotation around the z axis. The wave function of this excited state depends on the two quantum numbers K and n_γ.

These excited states are often called γ vibrations without sufficient justification. It is then assumed that the energy of these vibrations depends on the eigenvalues of the operator [32]

$$H_\gamma = -\frac{\hbar^2}{2B_\gamma}\left[\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\left(\gamma\frac{\partial}{\partial\gamma}\right) - \frac{\hat{I}_3^2}{\gamma^2}\right] + \frac{C'}{2}\gamma^2.$$

The second term in the square brackets is the rotation-operator part. Therefore in all states characterized by non-zero values of K the eigenvalues of H_γ determine rotation-vibration excitations or the energy of two transverse vibrations rather than pure γ vibrations. In truly nonaxial nuclei ($\gamma_0 \neq 0$) the states represented in $R_{20}(2)$ correspond to rotation around the z axis.

From the experimental ratios $R_{10}(4)$ and $R_{20}(2)$ using the tables in [29] (some of which ratios are given in Table I) we can for each nucleus determine the two parameters μ and Γ characterizing its deformability with respect to longitudinal and transverse vibrations. We thus obtain the values shown in Figs. 7 and 8 for several even-even nuclei as functions of the number of neutrons. Figure 7 shows that nuclei which are far from magic numbers are relatively hard with respect to longitudinal deformation. For example, in the regions $82 < N < 126$ and $N > 126$ we have $\mu \leq 0.2$. However, as the number of neutrons approaches magic numbers the deformability of a nucleus with respect to longitudinal vibrations increases considerably. This can be seen especially well in Gd and Xe isotopes. The given effect results from weaker coupling between the core and nucleons that begin the formation of a new shell (or holes in a nearly filled shell). A similar picture is observed in

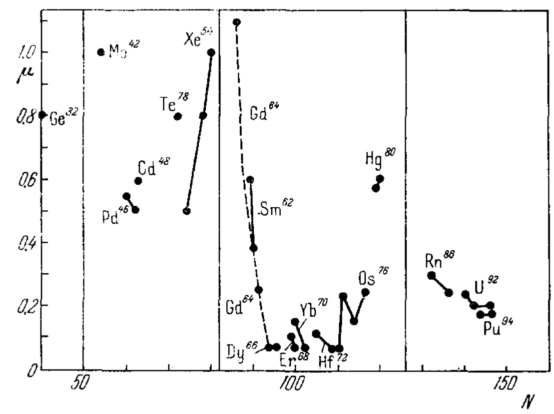

FIG. 7. The nonadiabaticity parameter μ for even-even nuclei vs. the number of neutrons. The lines connect points corresponding to nuclei with identical numbers of protons.

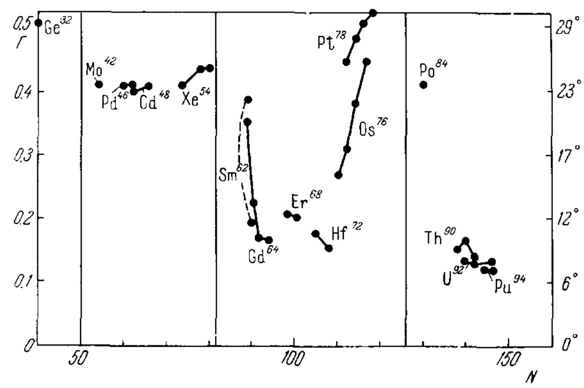

FIG. 8. The parameter Γ characterizing transverse deformability (or the value of γ_0) of even-even nuclei vs. the number of neutrons. The values of Γ are given in radians in the left-hand scale, and in degrees in the right-hand scale.

atoms when their polarizability is investigated; the effect is especially large in alkali metal atoms.

In the regions $82 < N < 126$ and $N > 126$ the deformability with respect to transverse vibrations (the value of Γ) behaves like the deformability with respect to longitudinal vibrations. However, $\Gamma \geq 0.4$ ($\Gamma > 20°$) for almost all nuclei in the region $50 < N < 82$. This indicates either that these nuclei are thus very soft with respect to transverse vibrations or that they possess no axis of symmetry in the ground state. Nuclei with large Γ do not possess the simple rotational spectrum (11) of excited states. Excited states in these nuclei are, as a rule, complex superpositions of (longitudinal and transverse) vibrational and rotational excitations. It is incorrect, although this is often done, to assign the excited states of these nuclei to vibrational excitations; it would also be incorrect to refer to them as rotational excitations.

It follows from Table I that for $\mu \approx 1$ and $\Gamma \approx 25°$ even-even nuclei have almost equally spaced excited states. This fact is sometimes regarded incorrectly as an indication of spherical symmetry, because spherical nuclei should have an equally spaced vibrational spectrum. However, soft nonspherical nuclei should also have an equally spaced spectrum.

With μ known from the experimental data for $R_{10}(4)$ and $R_{20}(2)$, and with $\langle\beta^2\rangle$ known from data on electric quadrupole transitions from the ground state to the first excited level, we can use (7) to determine the contribution of the zero-point surface vibrations of nonspherical nuclei to $\langle\beta^2\rangle$. Table III gives the values of $\langle\beta^2\rangle/\beta_0^2$ for some nuclei. It is seen here that zero-point longitudinal vibrations play a very large role in Cd^{114} but have only a small role in Os^{190}, U^{238}, and Hf^{178}.

The theory of collective quadrupole excitations [14,15,26] is important because it enables us to determine the two theoretical parameters μ and Γ from the values of $R_{10}(4)$ and $R_{20}(2)$ (or any other two energy ratios), to predict from the tables in [29] all other ratios of excited quadrupole state energies to the 2^+ first level, and to determine unambiguously the relative probabilities of transitions between excited states. Specifically, from a knowledge of μ and Γ we can determine the excitation energies of zero-spin longitudinal vibrations and of the rotational band associated with these vibrations. The theoretical and experimental ratios $R_{11}(0) = E_\beta(0)/E(2)$ are compared in Table IV.

According to (9), excited spin-0 states $E_\beta(0)$ with the quantum numbers $n_\beta = 1$, $n_\gamma = 0$, $I = K = 0$, which can be called one-phonon excitations of longitudinal surface vibrations, can be accompanied in even-even nuclei by spin-0 excitations $E_\gamma(0)$ representing one-phonon excitations of transverse vibrations (with the quantum numbers $n_\gamma = 1$, $n_\beta = 0$, $I = K = 0$). The latter excitations have been studied in [14,26,28]; their properties differ considerably from those of $E_\beta(0)$ excitations.

In axially symmetric nuclei the ratios $E_\beta(0)/E_\gamma(0)$, like other ratios of quadrupole excitation energies, depend on only the two theoretical parameters μ and Γ. It has been shown in [26] that in a rough approximation we have the equalities

$$\frac{\Gamma}{\mu} \approx \sqrt{\frac{E_\beta(0)}{E_\gamma(0)}}, \quad 2E_{20}(2) - E_{10}(2) \approx E_\gamma(0). \tag{15}$$

Two spin-0 levels were observed in Os^{188} by King and Johns,[33] who obtained $E_{10}(2) = 80.7$, $E_{20}(2) = 788$, $E_\gamma(0) = 1086$, and $E_\beta(0) = 1765$ keV. Taking $\mu = 0.25$ and $\Gamma = 0.33$ (19) for Os^{188}, we find that (15) is relatively well satisfied.

Table III. The role of zero-point longitudinal vibrations in $\langle\beta^2\rangle$ for nonspherical nuclei

Nucleus	$\sqrt{\langle\beta^2\rangle}$	μ	Γ, rad	$\langle\beta^2\rangle/\beta_0^2$	Nucleus	$\sqrt{\langle\beta^2\rangle}$	μ	Γ, rad	$\langle\beta^2\rangle/\beta_0^2$
Gd^{114}	0.2	1.0	0.40	2	U^{238}	0.28	0.2	0.1	1.02
Pd^{104}	0.21	0.5	0.38	1.12	Hf^{178}	0.30	0.15	0.15	1.01
Os^{190}	0.14	0.25	0.45	1.03					

Table IV. Theoretical and experimental ratios $R_{11}(0) = E_\beta(0)/E(2)$

Nucleus	μ	Γ	$R_{11}(0)$		Nucleus	μ	Γ	$R_{11}(0)$	
			Theory	Experiment				Theory	Experiment
Pu^{240}	0.22	8.0	19.3	20.0	Os^{188}	0.25	19.0	11.6	11.4
Pu^{238}	0.21	7.8	21.1	21.38	Er^{166}	0.20	12.7	21.1	18.12
U^{238}	0.2	8.0	22.8	22.2	Gd^{156}	0.25	10.5	13.8	11.7
U^{234}	0.2	8.7	22.5	18.66	Gd^{154}	0.37	13	5.2	5.53
U^{232}	0.25	8.8	14.1	14.6	Sm^{152}	0.37	11.5	6.1	5.62
Th^{232}	0.25	8.9	14.1	14.7	Cd^{114}	0.54	22.5	2.6	2.03
Th^{230}	0.25	9.5	13.9	11.9					

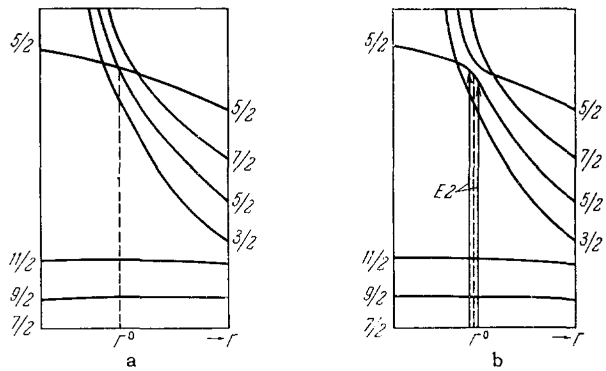

FIG. 9. Scheme of some excited states of an odd nucleus, with (a) and without (b) account of interaction between collective and and single-particle excitations.

For nonaxial nuclei, the energy ratios depend on a third theoretical parameter γ_0 in addition to μ and Γ: (15) should not then be satisfied. It must be mentioned, that at the present time we do not possess a sufficiently rigorous theory of collective excitations for nonaxial nuclei that can take into account rotations along with the longitudinal and transverse surface vibrations.

Among the experimental methods used to observe nuclear deformation occurring at the instant of a quantum transition, special importance attaches to the investigation of electric monopole (E0) transitions between states of identical spins ($0 \rightarrow 0$, $2 \rightarrow 2$ etc.). The probabilities of these transitions depend entirely on the change in the radial distribution of electric charge within the nucleus at the instant of the quantum transition. A theory of this effect has been developed by Rostovskiĭ and the present author [28] for axial nuclei with account of the coupling between longitudinal and transverse vibrations. We showed that the nuclear matrix elements of E0 transitions are expressible in terms of β_0, Γ, and μ.

In addition to the foregoing collective excitations corresponding to quadrupole deformations of the nuclear surface and always possessing positive parity, even-even nuclei are observed to possess collective excited states of negative parity (3^-).[34-36] These evidently correspond to octupole vibrations of the surface, and their energies are in the range 1–3 MeV for medium- and large-A nuclei. The theory of octupole excited states in even-even nuclei is still in its initial stage of development.[37]

Excited states of odd nuclei have not been considered in the present review. In odd nuclei the energies of single-particle excitations are of the same order of magnitude as the energies of collective excitations; therefore the division of excitations into collective and single-particle types is often unjustified. For odd nuclei Coriolis interactions are important; these lead to mixing of states with different values of K and Ω. The theory of the excited states of odd nuclei has been developed by several authors on the basis of simple nuclear models.[38-42] We mention here only the importance of the mixing of single-particle and collective excitations in odd nuclei; this mixing can play a large part when we calculate the probabilities of transitions between the ground state and excited states of such nuclei. Figure 9a shows the dependence of odd-nucleus excitation energies on Γ (characterizing surface deformability with respect to transverse vibrations), without taking into account the interaction between single-particle and collective motions. The spin $^9/_2$ and $^{11}/_2$ levels correspond to collective excitations in the ground-state rotational band, with energies that are only slightly dependent on Γ. Three excited states with spins $^3/_2$, $^5/_2$, and $^7/_2$ and energies that are strongly dependent on Γ, pertain to collective excitations including transverse surface vibrations. Finally, a spin $^5/_2$ level that is slightly dependent on Γ belongs to the single-particle excitations. Figure 9b shows the same scheme with account of the interaction between single-particle and collective excitations. We see that in the region $\Gamma \approx \Gamma_0$ the spin $^5/_2$ states are completely "entangled." Thus the probabilities of two E2 transitions, indicated by two arrows in Fig. 9b, will differ considerably. Outside of the region of "entanglement" the $^5/_2$ and $^7/_2$ levels lying above the $^3/_2$ level can be regarded as the second and third members of a rotational band based on the spin $^3/_2$ level. In the region of "entanglement" ($\Gamma \approx \Gamma_0$) this concept is meaningless for the spin $^5/_2$ level, which then "drops out" of the rotational band by virtue of all its properties (position, transition probability). It would be extremely interesting to obtain direct experimental evidence of this effect.

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