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D. I. Blokhintsev, V. L. Ginzburg, B. B. Kadomtsev, L. D. Keldysh,
S. T. Konobeevskii, F. L. Shapiro, V. A. Ugarov, V. I. Veksler,
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SOME PROBLEMS OF THE ELECTRON THEORY OF METALS*

III. KINETIC PROPERTIES OF ELECTRONS IN METALS

I. M. LIFSHITZ and M. I. KAGANOV

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INTRODUCTION

IN the present (third) part of this survey we consider the kinetic properties of metals mainly at low temperatures. Here the main consideration, as in the second part,^[1] is devoted to those phenomena and properties which are sensitive to the dispersion law for the conduction electrons. The mathematical apparatus used here is the Boltzmann kinetic equation.

In this part we restrict our treatment to static or quasi-static properties. This permits us, as shown in the first paragraph, to use the "gas" approximation, since the Fermi-liquid interaction drops out of the final formulas.

*This paper is the third part of a survey; the first part was published in U.F.N. 69, 419 (1959), and the second part in U.F.N. 78, 411 (1962) [Soviet Phys. Uspekhi 2, 831 (1960) and 5, 878 (1963)].

The majority of kinetic phenomena are extremely sensitive to the nature of the interaction of the conduction electrons with impurities, with phonons, and with one another. In particular, this interaction dictates the temperature behavior of the kinetic coefficients. The survey does not attempt to give a presentation of the wide range of theoretical material available at present concerning this question. Particular attention is devoted to those regularities and properties which are more or less independent of the character of the interaction (cf. Secs. 5-7, 9). At times in this summary we use the τ -approximation, i.e., the integral collision operator is replaced by the operator of multiplication with a phenomenological constant (the relaxation time τ); this is specifically stated in all cases.

In this part of the survey we restrict ourselves to problems concerned with the computation of the co-

efficients of electrical conductivity (resistance), thermal conductivity, and the thermoelectric coefficients of bulk (infinite) metals* in the absence of a magnetic field H and in a magnetic field. We do not consider effects resulting from the quantization of energy in relatively strong magnetic fields ($\mu H \gtrsim T$, μ equals the Bohr magneton). A separate section of this survey will be devoted to this class of problems.

Most of the quantum kinetic phenomena in a magnetic field are of the same nature as the de Haas-van Alphen effect (cf.^[1], Sec. 6). However, unlike the de Haas-van Alphen effect, quantum kinetic phenomena $\mu H \ll \epsilon_F$ (where ϵ_F is the Fermi energy of the conduction electrons) appear in most metals as slight oscillations of amplitude superimposed on a curve describing a relatively smooth dependence of the kinetic coefficients on magnetic fields. The latter is explained by classical theory. For this reason, we can divide the calculation of the kinetic coefficients into two parts: a classical part and a quantum part, where in the first part we need not consider the quantization of energy in the magnetic field. If we add that the condition $\mu H \ll \epsilon_F$ for most metals means that the magnetic field is much less than 10^8 oe, it becomes clear that the neglect of quantum effects is permissible in describing the kinetic properties of metals in magnetic fields.

Although high-frequency and, in particular, resonance properties of metals will be discussed in a separate part of this survey,[†] various problems from this domain (the theory of the skin effect, theory of absorption of ultrasonics) seem to us worth discussing here since they are closely connected with the other material in this part of our survey.

Certain more special, as well as very involved, questions are left for the appendices. In a first reading they may be omitted. Besides, in the appendices we discuss various questions related to further developments of the mechanics (classical and quantum) and statistical thermodynamics of electrons with a complicated dispersion law.

As in the preceding parts of this survey, the authors did not feel that it was their duty to exhaust all the work that has appeared in recent years, so that the literature references are not a bibliography on the present problem. Experimental work is only mentioned in this survey, while the discussion of the experimental results requires a special summary.

1. THE BOLTZMANN KINETIC EQUATION

In those cases where the motion of a conduction electron in the crystal lattice can be regarded as

“free”* (or as motion under the action of external forces), rarely interrupted by collisions, the Boltzmann kinetic equation is valid. The mean free path l , the mean distance between collisions, is determined both by the properties of the electrons (in particular, by their dispersion law) as well as, mainly, by the interruptions of the period structure of the crystal: the presence of chemical and physical inhomogeneity, phonons, electron-electron collisions, etc.

To develop the kinetic equation it is of fundamental importance to write down the “collision integral” term in the Boltzmann equation which describes the comparatively rare collisions of the electrons.

Obtaining the expression for the collision integral involves a solution of the scattering problem and requires knowledge of the laws of interaction of electrons with phonons, with impurities, and with one another. However, the development of the electron theory of metals during the last decade has shown that there are a large number of non-equilibrium kinetic properties of metals which depend only slightly on the detailed structure of the collision integral and are determined for the most part by the kinematics of the conduction electrons, i.e., by their dispersion law. Naturally, it is just these properties, as a rule, which are sensitive to the structure of the electron energy spectrum. It is to this problem that most of the present summary is devoted. This enables us to devote special attention to the field part of the kinetic equation without worrying about an investigation of the structure of the collision integral (cf., however, Appendix I).

As already stated, we shall not take account of quantum effects; more precisely, we shall not take account of the quantization of the electron energy (for example, in a magnetic field). The quantum character of the problem will manifest itself in the nature of the dispersion law of the conduction electrons and in their statistics. Between collisions (by assumption) the electron moves along a classical trajectory. The limitations imposed in connection with the neglect of the quantum character of the motion are described in detail in Sec. 3 of the first part of this survey.^[3] We recall only that in a classical treatment it is unnecessary to consider interband transitions caused by external fields. This effect is of a quantum nature, but the many-zone character of the electron energy spectrum does manifest itself in the summation over states and in calculating the probabilities for different collisions. The latter must be taken into account in any specific formulation for the collision integral.

The state of the electrons in a metal will be characterized by the electron distribution functions $f_{\mathbf{s}}(\mathbf{p}, \mathbf{r}, t)$ which have the significance of an electron

*A separate section of this survey will be devoted to the kinetic properties of bounded samples (films, wires).

[†]In addition, we may refer the reader to the survey by M. Ya. Azbel' and I. M. Lifshitz.^[2]

*To avoid misunderstandings we emphasize that we are speaking of electrons with a complicated dispersion law, whose mechanics was discussed in the first part of this survey.

density for the s 'th band ($2f_s/(2\pi\hbar)^3 dp dr$ is the number of electrons in the s 'th band, in the phase volume* $dp dr = dp_x dp_y dp_z dx dy dz$). Here r is the coordinate, t the time, and p the kinematic momentum. In the absence of a magnetic field, the kinematic momentum p coincides with the momentum P which is canonically conjugate to the coordinate.† If however, the magnetic field differs from zero,

$$\mathbf{P} = \mathbf{p} + \frac{e}{c} \mathbf{A}, \quad (1.1)$$

where \mathbf{A} is the vector-potential of the magnetic field \mathbf{H} ($\text{curl } \mathbf{A} = \mathbf{H}$).

All of the quantities which will be of interest to us: the current density \mathbf{j} , the energy flux density \mathbf{Q} etc., can be calculated if we know the distribution function f_s . Thus

$$\mathbf{j} = \frac{2e}{(2\pi\hbar)^3} \sum_s \int \mathbf{v}_s f_s(\mathbf{p}, \mathbf{r}; t) d\mathbf{p} \equiv e \int \mathbf{v} f d\Gamma, \quad (1.2)$$

$$\mathbf{Q} = \frac{2}{(2\pi\hbar)^3} \sum_s \int \epsilon_s \mathbf{v}_s f_s(\mathbf{p}, \mathbf{r}; t) d\mathbf{p} \equiv \int \mathbf{v} \epsilon f d\Gamma, \quad (1.3)$$

where $\mathbf{v}_s = \partial \epsilon_s / \partial \mathbf{p}$ is the velocity of an electron with energy $\epsilon_s(\mathbf{p})$, and the integration over $d\Gamma$ includes a summation over all the partially filled bands in the metal.

According to Liouville's theorem, in the absence of collisions the change in the distribution function with time is equal to zero, i.e.,

$$\frac{df_s}{dt} \equiv \frac{\partial f_s}{\partial t} + \dot{\mathbf{p}} \cdot \frac{\partial f_s}{\partial \mathbf{p}} + \dot{\mathbf{r}} \cdot \frac{\partial f_s}{\partial \mathbf{r}} = 0. \quad (1.4)$$

Equation (1.4) describes the invariance of the number of particles in an element of phase volume during the motion of the electrons along their phase trajectories. The quantities $\dot{\mathbf{p}}$ and $\dot{\mathbf{r}}$ should be taken from the equations of motion, according to which $\dot{\mathbf{p}} = \mathbf{F}$, where \mathbf{F} is the external force acting on the electron, and $\dot{\mathbf{r}} = \mathbf{v}_s$.

Since $\mathbf{F} = -\partial \epsilon_s / \partial \mathbf{r}$ (where ϵ_s is the total energy including external fields), df_s/dt becomes zero if f_s is an arbitrary function of the energy ϵ_s .

If an electric \mathbf{E} and a magnetic field \mathbf{H}^\ddagger act on the electron, then

$$\mathbf{F} = e \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}\mathbf{H}] \right). \quad (1.5)**$$

We emphasize once more the classical nature of this last expression. In particular, we have not included the interaction of the magnetic moment of the electron with the magnetic field. This is completely justified in most of the interesting cases (in a

homogeneous magnetic field, the force acting on the magnetic moment is actually equal to zero).

The external force acting on the conduction electron cannot always be expressed in terms of the macroscopic field intensities \mathbf{E} and \mathbf{H} . Thus, for example, in the passage through a metal of a sound wave, the force acting on the electron includes, in addition to the Lorentz force (1.5), an additional force caused by the deformation interaction of the electron with the lattice (cf. Sec. 10).

Collisions violate the condition (1.4). The degree of this violation is given by the "collision integral"

$$\frac{df_s}{dt} = \hat{\mathcal{L}}_{\text{coll}}(f_s). \quad (1.6)$$

The collision integral is a complicated non-linear function of the distribution functions whose structure and specific form are determined by the interaction of the electrons with impurities, with one another, or with other quasiparticles. In the latter case the system (1.6) must be extended by giving the kinetic equations for the distribution functions of the various quasiparticles (for example, for the phonons). The writing of the kinetic equation (1.6), as has been shown, is possible only in those cases where the motion of the particles can be divided into a motion along the phase trajectory and into collisions—sharp changes in the momenta of the particles without any significant change in coordinates (from the macroscopic point of view). From this it is clear that the "collision integral" is an operator associated with the dependence of the distribution function on momentum, but not on coordinates or time. Certain very general properties of the collision integral are discussed in Appendix III. Here we mention only that the collision integral vanishes for the equilibrium Fermi function for arbitrary values of the parameters—temperature* T and chemical potential ζ . T and ζ can depend only on the coordinates and the time.

The Boltzmann kinetic equation (1.6) is a system of complicated non-linear integro-differential equations which, if we give exact boundary and initial conditions, uniquely determine the state of the solid. In the general case, naturally, it is impossible to solve this system, and we require a considerable number of simplifications which are determined by the physical formulation of the problem.

Since the external electric field directly applied to the metal or arising as the result of external action (for example, by a sound wave) is as a rule very small compared with the internal interatomic electric fields, the deviation of the electron system from its equilibrium state is in most cases very small. This

*Neglecting quantum effects permits us to disregard the spin variables: each electron state is assumed to be doubly degenerate.

†We remind the reader that in a classical treatment the concepts of quasimomentum and momentum coincide.

‡Since we are not considering ferromagnetic metals, we shall not distinguish \mathbf{H} from the induction \mathbf{B} .

** $[\mathbf{v}\mathbf{H}] \equiv \mathbf{v} \times \mathbf{H}$.

*If the collision integral describes collisions with other quasiparticles (for example, with phonons), it vanishes only when we substitute all the equilibrium functions: the Fermi functions for the electrons and Bose functions for the phonons, etc.

allows us to linearize the system of equations (1.6) by replacing the distribution function f_s by the sum

$$f_s = n_F + f_1, \quad (1.7)$$

where n_F is the equilibrium Fermi function (in zero'th approximation) and the smallness of the function f_1 (first approximation) is guaranteed by the smallness of the external interaction. In other words, the function f_1 is proportional to those external forces which remove the system from its equilibrium state, (for example, the function f_1 is proportional to the electric field when a current flows through an isothermal conductor, or to the temperature gradient when a metal sample acts as a heat conductor). The choice of the zeroth approximation, or more precisely the choice of the parameters T and ζ in the Fermi function is determined by the statement of the problem. It is most natural to start from the assumption of local equilibrium, assuming that the parameters in the Fermi function are chosen so that $T = T(\mathbf{r})$ determines the temperature at the point \mathbf{r} , and $\zeta = \zeta(\mathbf{r})$ is the chemical potential. This means that the electron density at point \mathbf{r} and their average energy are given by the function of the zeroth approximation n_F and

$$\int f_1 d\Gamma = \int \varepsilon f_1 d\Gamma = 0. \quad (1.8)$$

In addition the energy of the external electric field is not included in the energy $\varepsilon_s(\mathbf{p})$.

As we have already stated, the main content of this part of our survey is the calculation of the tensors for electrical conductivity, thermal conductivity, and of the thermoelectric coefficients. It is therefore natural to treat those cases where the removal of the system from its equilibrium state is caused by an electric field \mathbf{E} and a temperature gradient ∇T , which by assumption are so small that they guarantee the validity of the linearization procedures.*

Substituting (1.7) in (1.6), taking account of the expression for the Lorentz force (1.5) and neglecting quadratic terms, we obtain

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{\partial f_1}{\partial \mathbf{p}} \frac{e}{c} [\mathbf{v}\mathbf{H}] - \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}} \\ = - \frac{\partial n_F}{\partial \varepsilon} \mathbf{v} \cdot (e\mathbf{E} - \nabla \zeta) - \frac{\partial n_F}{\partial T} \mathbf{v} \nabla T. \end{aligned} \quad (1.9)$$

Here $(\partial f_1 / \partial t)_{\text{coll}}$ is the linearized collision integral:

$$\left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}} = - \hat{W} f_1,$$

where \hat{W} is the linear collision operator, equal to

$$- \left[\frac{\delta \mathcal{L} \{f_1\}}{\delta f} \right]_{f=n_F}.$$

*In order to be able to perform the linearization it is sufficient that $l|\nabla T| \ll T$ and $eEl \ll T$. Both these conditions impose practically no restrictions on the values of the electric field and the values of the temperature gradient.

The observed field intensity in the conductor \mathbf{E}' , i.e., the force acting on unit charge, is the sum of the intensity of the field due to the external potential difference imposed on the conductor ($\mathbf{E} = -\nabla\varphi$) and the quantity $-1/e \nabla\zeta$, where ζ is the chemical potential of the electrons (cf., for example, [4], Sec. 25), i.e.

$$\mathbf{E}' = \mathbf{E} - \frac{1}{e} \nabla \zeta.$$

From now on we shall omit the prime on the electric field, but remember that the derivative of the Fermi function with respect to the temperature is taken for constant chemical potential ($\partial n_F / \partial t \equiv (\partial n_F / \partial t)_\zeta$).

Thus, the kinetic equation for the additional terms, linear in the perturbation, in the distribution function has the form

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{\partial f_1}{\partial \mathbf{p}} \frac{e}{c} [\mathbf{v}\mathbf{H}] - \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}} \\ = - \frac{\partial n_F}{\partial \varepsilon} e\mathbf{E} \cdot \mathbf{v} - \frac{\partial n_F}{\partial T} \mathbf{v} \nabla T. \end{aligned} \quad (1.10)$$

Frequently for estimates and sometimes in solving relatively complicated problems of kinetics, the linear collision operator is replaced by an operator of multiplication by a phenomenologically introduced constant τ , i.e., one assumes

$$\left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}} = - \frac{f_1}{\tau}. \quad (1.11)$$

The positive* quantity τ has the dimensions of the time and is called the relaxation time or the mean free time; the quantity $l = v\tau$ is called the mean free path, and Eq. (1.11) is called the τ -approximation. It should be remembered that for an anisotropic dispersion law the τ -approximation cannot be justified theoretically, and thus the substitution (1.11) can serve only for estimate or in those cases where the final result does not depend on the form of the collision integral (cf. Secs. 9 and 10).

Let us look at the first two terms in (1.10). The time derivative of the distribution function $\partial f_1 / \partial t$ takes account of the effects of time dispersion in the kinetic coefficients, i.e., effects associated with retardation of the reaction of the electron gas to the external force. If the characteristic frequency of the external field is ω , then $\partial f_1 / \partial t \sim \omega f_1$. This term plays an important role for frequencies of the order of or greater than $\nu = 1/\tau$. For $\omega \ll \nu$ it can be neglected. The relaxation time changes over a very wide range: from 10^{-14} sec at room temperature to 10^{-9} sec for particularly pure metal samples [5] at liquid helium temperature (4.2° K).

The derivatives with respect to the coordinates (the term $\partial f_1 / \partial \mathbf{r} \cdot \mathbf{v}$ in (1.10)) are responsible for effects of spatial dispersion of the kinetic coefficients. If the characteristic distance over which the distribution function varies is of order d , then

*The fact that τ is positive assures the increase of entropy, with time.

$\partial f_1 / \partial \mathbf{r} \cdot \mathbf{v} \sim v/d f_1$. This term is important when $v/d \gtrsim v$, i.e., $l \gtrsim d$. When $l \ll d$, we can also neglect this term. Since for the purest samples the mean free path l is no greater than $10^{-3} - 10^{-1}$ cm,^[5] in calculating the kinetic coefficients of bulk metals we may neglect the spatial derivatives. As we shall see (cf. Sec. 8-10), the calculation of the high-frequency conductivity and sound absorption under particular conditions requires one to take account of the spatial dispersion.

Inhomogeneity of the sample (presence of boundaries etc.) may result not only in an inhomogeneity of the function f_1 , but also in the distribution of the conduction electrons (for example, in an open-circuited conductor). However, in metals (in contrast to semi-conductors) this inhomogeneity, which is actually present, has no observable macroscopic effect since the Debye-Hückel radius r_D (the measure of the inhomogeneity of the distribution of the charged particles) is very small ($r_D \lesssim 10^{-8}$ cm) for the degenerate electron gas*. This fact allows one always in calculating the kinetic coefficients to regard the Fermi function appearing in (1.10) as a homogeneous function (independent of the coordinates).

From now on it will be convenient to use the following notation:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \frac{\partial}{\partial \mathbf{r}} + \frac{e}{c} [\mathbf{vH}] \frac{\partial}{\partial \mathbf{p}}, \quad (1.12)$$

$$\hat{W}_p = \left(\frac{\partial n_F}{\partial \epsilon} \right)^{-1} \hat{W} \left(\frac{\partial n_F}{\partial \epsilon} \right), \quad (1.13)$$

$$\hat{W}_\epsilon = \left(\frac{\partial n_F}{\partial T} \right)^{-1} \hat{W} \left(\frac{\partial n_F}{\partial T} \right), \quad (1.14)$$

and to replace the function f_1 (for uniform \mathbf{E} and ∇T) by two vector functions ψ_i and φ_i :

$$f_1 = -e E_i \psi_i \frac{\partial n_F}{\partial \epsilon} - \nabla_i T \varphi_i \frac{\partial n_F}{\partial T}, \quad (1.15)$$

for which we easily obtain the very compact equations

$$\frac{d\psi_i}{dt} + \hat{W}_p \psi_i = v_i, \quad (1.16)$$

$$\frac{d\varphi_i}{dt} + \hat{W}_\epsilon \varphi_i = v_i. \quad (1.17)$$

Comparing expressions (1.12) and (1.4), we see that the "time" t , with respect to which we differentiate (1.16) and (1.17), is the time for motion along the phase trajectory of the electron in the magnetic field (cf. Sec. 5).

Substituting the expression (1.15) for the distribution function in terms of the vector functions ψ_i and φ_i in formulas (1.2) and (1.3), and noting that for $f_s = n_f$ the current density \mathbf{j} and the energy flux \mathbf{Q}

are equal to zero, we find

$$j_i = -e^r \int \frac{\partial n_F}{\partial \epsilon} v_i \psi_k d\Gamma E_k - e \int \frac{\partial n_F}{\partial T} v_i \varphi_k d\Gamma \nabla_k T, \quad (1.18)$$

$$Q_i = -e \int \frac{\partial n_F}{\partial \epsilon} \epsilon v_i \psi_k d\Gamma E_k - \int \frac{\partial n_F}{\partial T} \epsilon v_i \varphi_k d\Gamma \nabla_k T. \quad (1.19)$$

In the sequel (cf. Secs. 2 and 3) we shall sometimes use the following notation:

$$- \int \frac{\partial n_F}{\partial \epsilon} \chi(\mathbf{p}) \eta(\mathbf{p}) d\Gamma \equiv \langle \chi, \eta \rangle, \quad (1.20)$$

and consider such integrals as the scalar product of functions χ and η . It is easy to show that all the properties required for a scalar product are present for such integrals.^[6]

In this notation

$$j_i = e^2 \langle v_i \psi_k \rangle E_k - \frac{e}{T} \langle (\epsilon - \zeta) v_i \varphi_k \rangle \nabla_k T, \quad (1.21)$$

$$Q_i = e \langle \epsilon v_i \psi_k \rangle E_k - \frac{1}{T} \langle \epsilon (\epsilon - \zeta) v_i \varphi_k \rangle \nabla_k T. \quad (1.22)$$

In Appendix I, we show for a number of examples (scattering by impurities and by phonons) that the operator \hat{W}_p has the following important properties. First it is hermitian

$$\langle \chi, \hat{W}_p \eta \rangle = \langle \eta, \hat{W}_p \chi \rangle, \quad (1.23)$$

and, second, it is positive:

$$\langle \chi, \hat{W}_p \chi \rangle > 0. \quad (1.24)$$

The operator \hat{W}_ϵ has similar properties.

So far we have tacitly assumed that the conduction electrons form an almost ideal gas of quasiparticles. In other words we have not considered the fact that the energy of an individual quasiparticle depends on the state of the whole system, i.e., on its distribution function.* This dependence is taken into account by the theory of the Fermi liquid developed (for the case of He³) by Landau^[7] and generalized^[8] to electrons in a metal.

We describe here what changes have to be made in writing the Boltzmann kinetic equation when we take account of the Fermi liquid interaction between electrons.

The kinetic equation for the distribution function is constructed in the theory of the Fermi-liquid in completely analogous fashion to its construction in the gas model, i.e., one makes use of formula (1.6). One need only consider that the energy of the individual quasiparticle (conduction electron) ϵ is determined not only by the dispersion law $\epsilon_0(\mathbf{p})$ but also by the distribution function $f(\mathbf{p})$, for states close to equilibrium

$$\begin{aligned} \epsilon(\mathbf{p}) &= \epsilon_0(\mathbf{p}) + \int \Phi(\mathbf{p}, \mathbf{p}') f_1(\mathbf{p}') d\Gamma' \equiv \epsilon_0(\mathbf{p}) + \eta, \\ f_1(\mathbf{p}) &= f(\mathbf{p}) - n_F(\epsilon_0), \quad |f_1(\mathbf{p})| \ll n_F(\epsilon_0). \end{aligned} \quad (1.25)$$

Here† $\epsilon_0(\mathbf{p})$ is the energy of the electron with momentum \mathbf{p} in

*By definition, $e^2/r_D = \hbar\omega_0$, where ω_0 is the plasma frequency of the electron gas ($\omega_0^2 = 4\pi n e^2/m$), n is the density of electrons, m is the electron mass; since $n \approx 1/a^3$ (where a is the interatomic distance), $r_D \approx a\sqrt{U_C/\epsilon_F}$, while $U_C = e^2/a$ is the Coulomb interaction energy of the electrons; $\epsilon_F \approx \hbar^2/a^2 m$ is the Fermi energy.

†We recall that the dispersion law for the elementary excitations (electrons) of which we are speaking in this and earlier parts of our survey naturally includes the interaction between electrons (cf. [1], beginning of Sec. 7).

‡In Sec. 7 of the second part of this survey [1] the correlation function was defined in terms of $f(\mathbf{p}, \mathbf{p}')$ and the deviation of the distribution function from its equilibrium value by $\nu(\mathbf{p})$.

equilibrium, given by the Fermi function; $\Phi(p, p')$ is the correlation function, the fundamental characteristic of the interaction between electrons in the Landau theory of the Fermi-liquid.^[7] In the microscopic theory the correlation function $\Phi(p, p')$ is related to the amplitude for electron scattering.^[8] The experimental determination of this quantity is an important problem of the physics of metals. As will be clear from what follows, the quasistatic kinetic properties are not suitable for such a determination.

With the definition of the quasiparticle energy given here, the energy $\epsilon_0(p)$ is naturally a function of temperature. Limitations on such a treatment exist, of course (the temperature must be much less than the Fermi temperature) and result from the fact that the lifetime of single particle excitations falls off rapidly as one moves away from the Fermi surface. But at high temperatures, in all phenomena, electrons participate that have energies which are significantly far from the Fermi energy.

It would be extremely important to study experimentally the temperature dependence of the dispersion law $\epsilon = \epsilon_0(p)$. The corresponding estimates made in reference^[9] show that changes in the dispersion law (from temperature zero up to the Debye temperature) apparently are detectable even though the Debye temperature is assumed to be small compared to the Fermi temperature.

Let us now turn to the derivation of the linearized kinetic equation. Using formula (1.25) it is easy to show that

$$\left. \begin{aligned} \dot{\mathbf{p}} &= e\mathbf{E} + \frac{e}{c} \left[\frac{\partial \epsilon}{\partial \mathbf{p}} \mathbf{H} \right] - \frac{\partial \eta}{\partial \mathbf{r}}, \\ \dot{\mathbf{r}} &= \mathbf{v} + \frac{\partial \eta}{\partial \mathbf{p}}, \\ \dot{\epsilon} &= e\mathbf{E} \frac{\partial \epsilon}{\partial \mathbf{p}}, \end{aligned} \right\} \quad (1.26)$$

where $\mathbf{v} = \frac{\partial \epsilon_0}{\partial \mathbf{p}}$ is the velocity of the electrons in a state of thermodynamic equilibrium. It is convenient now to introduce, in addition to the function f_1 describing the deviation of the system from its thermodynamic equilibrium state, another function f_1^* , by the following equation:

$$f = n_F(\epsilon_0) + f_1 \equiv n_F(\epsilon) + f_1^*. \quad (1.27)$$

In the linear approximation in f_1 ,

$$f_1^* = f_1 - \frac{\partial n_F}{\partial \epsilon_0} \eta. \quad (1.27')$$

Using formula (1.26) and further linearization of the left side of the kinetic equation (1.6), one can obtain *

$$\begin{aligned} \frac{df}{dt} &\approx \frac{\partial n_F(\epsilon_0)}{\partial \epsilon_0} e\mathbf{v}\mathbf{E} + \frac{\partial n_F(\epsilon_0)}{\partial T} \mathbf{v}\nabla T \\ &+ \frac{\partial f_1}{\partial t} + \frac{\partial f_1^*}{\partial \mathbf{r}} \mathbf{v} + \frac{\partial f_1^*}{\partial \mathbf{p}} \frac{e}{c} [\mathbf{v}\mathbf{H}]. \end{aligned} \quad (1.28)$$

In considering the collision integral (the right side of the kinetic equation (1.6)) we should note that the dependence of the distribution function is contained first of all in the probabilities for collision processes (because of the statistical properties of the electrons) and secondly in the δ functions describing the law of conservation of energy. The latter dependence is a consequence of Fermi-liquid effects (cf. formula (1.25)). Both dependences are

taken into account by writing

$$\mathcal{L}_{\text{coll}} = \mathcal{L}_{\text{coll}}\{f, \epsilon\}. \quad (1.29)$$

The collision integral goes to zero when we substitute the equilibrium distribution function in it (for an arbitrary dispersion law), i.e.,

$$\mathcal{L}_{\text{coll}}\{n_F(\epsilon), \epsilon(p)\} = 0. \quad (1.30)$$

We note that $\epsilon(p)$ includes the Fermi liquid correction η , i.e., it is a complicated functional of the distribution function.

Substituting the second of the expansions (1.27) in (1.29) and noting that in the approximation which is linear in the perturbation,

$$\left\{ \frac{\delta \mathcal{L}}{\delta f} \right\}_{f=n_F(\epsilon), \epsilon=\epsilon_0(p)}, \quad f_1^* \approx \left\{ \frac{\delta \mathcal{L}}{\delta f} \right\}_{f=n_F(\epsilon_0), \epsilon=\epsilon_0(p)}, f_1^*. \quad (1.31)$$

we arrive at the conclusion that in the collision integral the inclusion of liquid effects results only in replacing the function f_1 by the function f_1^* , and we can finally write the kinetic equation (1.6) in the following form:

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial f_1^*}{\partial \mathbf{r}} \mathbf{v} + \frac{\partial f_1^*}{\partial \mathbf{p}} \frac{e}{c} [\mathbf{v}\mathbf{H}] - \left(\frac{\partial f_1^*}{\partial t} \right)_{\text{coll}} \\ = - \frac{\partial n_F(\epsilon_0)}{\partial \epsilon_0} e\mathbf{v}\mathbf{E} - \frac{\partial n_F(\epsilon_0)}{\partial T} \mathbf{v}\nabla T. \end{aligned} \quad (1.32)$$

From this it is clear that in all those cases where the term $\frac{\partial f_1}{\partial t}$ can

be neglected, for instance quasistatic cases, i.e., when $\omega\tau \ll 1$, the kinetic equation can be put in "gas" form by introducing the new distribution function f_1^* (cf. (1.27)). The correlation functions $\Phi(p, p')$ drops out of the Boltzmann equation. If we add that the fluxes (current density \mathbf{j} and energy flux \mathbf{Q}) are also expressed in the linear approximation in terms of the function f_1^* :

$$\mathbf{j} = e \int \mathbf{v} f_1^* d\Gamma, \quad \mathbf{Q} = \int \epsilon \mathbf{v} f_1^* d\Gamma, \quad (1.33)$$

it becomes clear that in treating quasistatic problems we can completely ignore the Fermi-liquid interaction between electrons. The correlation between electrons manifests itself only at sufficiently high frequencies ($\omega\tau \gtrsim 1$).

2. SPECIFIC ELECTRICAL CONDUCTIVITY. OHM'S LAW

Let us consider the passage of a constant current through an isothermal metal in the absence of magnetic field.

Using the results of the preceding paragraph, we need not take account of the Fermi-liquid interaction between the electrons. The results obtained (the values of the kinetic coefficients) can be formulated in "gas" terms. We must, however, remember that the basic feature of the electron, its dispersion law $\epsilon_0(p)$, depends on the electron-electron correlation (see above). The analogous situation occurs in the de Haas-van Alphen effect (cf. ^[1], Sec. 7): the periods

*We remind the reader that the linearization is carried out with respect to the electric field \mathbf{E} (including the gradient of the chemical potential) and the temperature gradient ∇T . In the zeroth approximation in these quantities, ϵ and ϵ_0 coincide, and the

electron velocity is equal to $\mathbf{v} = \frac{\partial \epsilon_0}{\partial \mathbf{p}}$.

*Equation (1.33) follows immediately from formulas (1.2) and (1.3) if we use the fact that

$$\int \frac{\partial \epsilon}{\partial \mathbf{p}} n_F(\epsilon) d\Gamma = \int \epsilon \frac{\partial \epsilon}{\partial \mathbf{p}} n_F(\epsilon) d\Gamma = 0. \quad (2.3)$$

of oscillation are determined only by the shape of the Fermi surface $\epsilon_0(\mathbf{p}) = \epsilon_F$.

In the case we are considering the kinetic equation can be simplified in a very natural way: we can restrict (1.16) by dropping the term $d\psi_i/dt$. Thus

$$\hat{W}_p \psi_i = v_i. \quad (2.1)$$

The neglect of terms containing spatial derivatives means that the distance over which the electric field or the distribution function varies considerably is large compared to the mean free path, while the neglect of the time derivative means that the frequency of the field ω is much less than the collision frequency $\nu = 1/\tau$ (for estimates it is convenient to use the τ -approximation).

Introducing the operator \hat{W}_p^{-1} , the inverse of the collision operator, we find from (2.1)

$$\psi_i = \hat{W}_p^{-1} v_i. \quad (2.2)$$

If (2.1) requires an extension by linearized kinetic equations for the distribution functions of the phonons or other quasiparticles with which the electrons collide, the operator \hat{W}_p^{-1} should be understood as the operator which is obtained after including all the distribution functions in addition to the electron function.

From formulas (1.18), (1.21), and (2.2) we have

$$j_i = -e^2 \int \frac{\partial n_F}{\partial \epsilon} v_i \hat{W}_p^{-1} v_k d\Gamma \cdot E_k = e^2 \langle v_i, \hat{W}_p^{-1} v_k \rangle E_k. \quad (2.4)$$

Comparing expression (2.4) with Ohm's law

$$j_i = \sigma_{ik} E_k, \quad (2.5)$$

we find the formal expression for the tensor of the specific electrical conductivity:

$$\sigma_{ik} = -e^2 \int \frac{\partial n_F}{\partial \epsilon} v_i \hat{W}_p^{-1} v_k d\Gamma \equiv e^2 \langle v_i, \hat{W}_p^{-1} v_k \rangle. \quad (2.6)$$

The properties of the operator \hat{W}_p formulated above in (1.23) and (1.24), which, as can easily be shown, also hold for the operator \hat{W}_p^{-1} insure the symmetry of the tensor σ_{ik} and the positive nature of its principal values.

We note that the symmetry of the tensor σ_{ik} ($\sigma_{ik} = \sigma_{ki}$) is a manifestation of a general principle of non-equilibrium thermodynamics—the principle of the symmetry of the kinetic coefficients. The fact that the principal values of the tensor σ_{ik} are positive assures the validity of the law of increasing entropy.

The number of independent components of the tensor σ_{ik} is determined by the symmetry class of the particular crystal. Most metals have either cubic or hexagonal symmetry. In the first case the tensor degenerates to a scalar, and in the second case it has two coincident principal values. Some metals (for example, Mg) have three principal values for the electrical conductivity—they belong to the rhombic syngony.

We should point out that the absence of anisotropy of the tensor of specific conductivity by no means proves the isotropy of the dispersion law for the conduction electrons (cf., for example, Au, Ag, Cu, etc., metals which have a cubic lattice). One of the main problems of the theory of metals is to calculate the electrical conductivity tensor and in particular to obtain its temperature dependence. Although at present this problem cannot be regarded as finally solved, it is clear that the basic mechanism for the resistance is already correctly described in the papers of F. Bloch: over a wide range of temperatures the main role in the resistance is played by scattering of electrons by vibrations of the crystal lattice (by phonons). At low temperatures (of the order of a few degrees absolute), the main mechanism for resistance is scattering by static inhomogeneities (by impurity atoms, grain boundaries, dislocations, sample boundaries etc.).

We shall return to the problem of the temperature dependence of the resistance, but we first present various forms of writing the tensor σ_{ik} .

If $\hat{W}_p^{-1} v_k$ is a sufficiently smooth function of the energy, which occurs whenever the change in energy of the electron during a collision is small compared to its energy*, we can replace the derivative of the Fermi function in the first of formulas (2.4) by a δ -function. Integrating over the energy, we get

$$\sigma_{ik} = \frac{2e^2}{(2\pi\hbar)^3} \sum_s \oint_{\epsilon_s(\mathbf{p}) = \epsilon_F} \frac{v_i^{(s)} \hat{W}_p^{-1} v_k^{(s)}}{v^{(s)}} dS_s. \quad (2.7)$$

The integration here is taken over the Fermi surface, dS_s is an element of area in the s -th band.

The temperature variation of the components of the tensor σ_{ik} is determined by the temperature dependence of the factor $\hat{W}_p^{-1} v_k$ in the integrand.

Formula (2.7) can be given a somewhat different form by introducing the "mean free path operator" which acts along the unit normal to the Fermi surface:

$$\hat{l}_p \{n_i\} \equiv \hat{W}_p^{-1} v_i. \quad (2.8)$$

Then

$$\sigma_{ik} = \frac{2e^2}{(2\pi\hbar)^3} \sum_s \oint_{\epsilon_s(\mathbf{p}) = \epsilon_F} n_i^{(s)} \hat{l}_p n_k^{(s)} dS_s. \quad (2.9)$$

From this form of writing of the electrical conductivity tensor, it is clear that the symmetry of the tensor σ_{ik} came as a consequence of the hermiticity of the mean free path operator.

For a cubic crystal it is convenient to introduce the mean free path \bar{l}_p by the following equation:

*For collisions of phonons the condition $|\Delta\epsilon/\epsilon| \ll 1$ (where $\Delta\epsilon$ is the change in energy resulting from the collision) means that $\theta/\epsilon_F \ll 1$. For poor metals of the type of Bi or C (graphite), this condition may not be satisfied.

$$\frac{1}{3} \bar{l}_p \delta_{ik} = \frac{1}{S_F} \oint_{\epsilon(p)=\epsilon_F} n_i \hat{l}_p n_k dS,$$

where S_F is the area of the Fermi surface. For simplicity of writing of the formulas we have omitted the summation over s and we shall do this from now on. Then the electrical conductivity σ ($\sigma_{ik} = \sigma \delta_{ik}$) can be written in a very compact form:

$$\sigma = \frac{2e^2 S_F \bar{l}_p}{3 (2\pi\hbar)^3}, \quad (2.10)$$

or

$$\sigma (\text{sec}^{-1}) \approx 0.6 \cdot 10^{10} S_F \left(\frac{\text{g}^2 \text{cm}^2}{\text{sec}^2} \right) \bar{l}_p (\text{cm}). \quad (2.10')$$

The expressions introduced here for the tensor σ_{ik} are of very general character and require only a minimum number of stipulations (for example, assumptions about the degeneracy of the electron gas).

We now give various formulas whose derivations are based on simplifying assumptions. Let us replace the collision operator by the mean free time tensor:

$$\hat{W}_p^{-1} v_i = \tau_{il} v_l. \quad (2.11)$$

If we assume that the components of the tensor τ_{il} are independent of quasimomentum, it follows from formula (2.7) that

$$\sigma_{ik} = \frac{2e^2}{(2\pi\hbar)^3} \tau_{kl} \oint_{\epsilon(p)=\epsilon_F} n_i v_l dS. \quad (2.12)$$

Since $v_l = \frac{\partial \epsilon}{\partial p_l}$, using Gauss's theorem we find (cf. [1], Appendix I).

$$\begin{aligned} \sigma_{ik} &= \pm \frac{2e^2}{(2\pi\hbar)^3} \tau_{kl} \int \frac{\partial^2 \epsilon}{\partial p_l \partial p_i} dp \\ &= \pm \frac{2e^2}{(2\pi\hbar)^3} \tau_{kl} \int \left(\frac{1}{m} \right)_{li} dp. \end{aligned} \quad (2.13)$$

The sign is determined by the direction of the vector \mathbf{n} relative to the Fermi surface: if the vector \mathbf{n} is directed outward (electron band) the sign is plus; if it is directed inward (hole band) it is minus. In the first case (electron band), the integration is taken over the region of \mathbf{p} -space where $\epsilon < \epsilon_F$, and in the second case (hole band) over the region where $\epsilon > \epsilon_F$. It is natural to call the quantity

$$u_{ik} = \pm e \left(\frac{1}{m} \right)_{il} \tau_{kl} \quad (2.14)$$

the mobility tensor. From formulas (2.13) and (2.14) we have

$$\sigma_{ik} = ne \bar{u}_{ik}, \quad (2.15)$$

where the dash denotes an averaging over the band, n is the density of electrons (for an electron band) or the density of "holes" (for a hole band). The rigorous definition of the hole density is given in Sec. 5. With our definition of the mobility $e < 0$ in all cases.

There is no point in discussing all the complications which can arise in applying formulas (2.13) to bands with a complicated Fermi surface (where, in particular, the problem of the sign of the diagonal components of the effective mass tensor may arise). The initial formula (2.11) is not suitable for this purpose. But in those cases where the Fermi surface is an ellipsoid located near an external point in \mathbf{p} -space, the assumption (2.11) is natural

since it is a consequence of an approximate expansion of the collision integral in momentum components. No misunderstanding can then arise since in these cases the effective masses are constants. Formula (2.15), for the case of an ellipsoidal Fermi surface, takes on an extremely simple form:

$$\sigma_{ik} = ne u_{ik}. \quad (2.16)$$

If we assume that the mean free time tensor degenerates to a scalar,

$$\tau_{il} = \tau \delta_{il},$$

formula (2.12) can be written as follows:

$$\sigma_{ik} = \frac{2e^2}{(2\pi\hbar)^3} \oint_{\epsilon(p)=\epsilon_F} \frac{\tau v_i v_k}{v} dS. \quad (2.17)$$

Introducing the mean free path $l = \tau v$ and neglecting the anisotropy of the dispersion law (i.e., the anisotropy of the Fermi surface), we have from formula (2.17) (or (2.10))

$$\sigma_{ik} = \sigma \delta_{ik}, \quad \sigma = \frac{2e^2 l S_F}{3 (2\pi\hbar)^3}. \quad (2.18)$$

This formula, which formally coincides with formula (2.10), if we replace l by \bar{l}_p , has an approximate character and shows that large electrical conductivity for a fixed mean free path (which permits of an independent measurement) is evidence for a large area of the Fermi surface. Cases are possible where, with a relatively small volume, the Fermi surface nevertheless has a large area. This may be a result of branching of the Fermi surface.

For a spherical Fermi surface of radius p_F (for which formula (2.18) was obtained, but not formula (2.10)!))

$$S_F = 4\pi p_F^2 = \frac{3\Delta_F}{p_F},$$

while $\frac{2\Delta_F}{(2\pi\hbar)^3} = n$ (Δ_F is the volume contained within the Fermi

surface, n is the density of electrons). Under these simplifying assumptions, which we have used, one naturally obtains the usual expression for the specific electrical conductivity

$$\sigma = \frac{ne^2 l}{p_F}. \quad (2.19)$$

An expression of the type of (2.19) (or the analogous $\sigma = ne^2 \tau / m^*$ where m^* is the modulus of the effective mass) is usually used for a "proof" of the Matthiessen rule, according to which the specific resistance $\rho = 1/\sigma$ is the sum of the specific resistances caused by the various scattering mechanisms*. In fact, $1/l$ is the probability of scattering per unit path. If there are several independent causes of scattering, then, according to the theorem of superposition of probabilities,

$$\frac{1}{l} = \sum_j \frac{1}{l_j}, \quad (2.20)$$

where the subscript j enumerates the various scattering mechanisms.

From formulas (2.19) and (2.20) it follows naturally that

*From this formulation there follows the usual one: the resistance is the sum of the residual (temperature-independent) resistance and the ideal resistance which varies with the temperature.

$$\varrho = \sum_j \varrho_j. \quad (2.21)$$

The Matthiessen rule (2.21) is frequently claimed to be absolute, although it is completely obvious that even in the absence of correlations between the scattering processes, formula (2.21) can be derived only in the τ -approximation. In the general case the specific resistance depends in a more complicated way on the various scattering mechanisms.

If one of the mechanisms makes a small contribution to the scattering of the electrons as compared to the others, then we can formulate a rule analogous to the Matthiessen rule, but, as we shall see later, much weaker.

Thus, suppose that the collision operator \hat{W}_p contains a small additive correction. Let us denote it by \hat{W}_1 :

$$\hat{W}_p = \hat{W}_0 + \hat{W}_1, \quad (2.22)$$

where $|\hat{W}_1\{f\}| \ll |\hat{W}_0\{f\}|$. Then the solution of the kinetic equation (2.1) can be found by the method of successive approximations. Calculation of the specific conductivity tensor and the specific resistance tensor in the approximation linear in W_1 gives

$$\varrho_{ik} = \varrho_{ik}^0 + \Delta Q_{ik}, \quad (2.23)$$

where

$$\Delta Q_{ik} = -\varrho_{il}^0 \Delta \sigma_{lm} \varrho_{mk}^0, \quad (2.24)$$

and

$$\Delta \sigma_{ik} = \int v_i \hat{W}_0^{-2} \hat{W}_1 v_k \frac{\partial n_F}{\partial \epsilon} d\Gamma.$$

The tensor of the specific resistance ρ_{ik}^0 is related to the main collision operator \hat{W}_0 . At high temperatures ($T \gg \theta$, where θ is the Debye temperature), the main cause of scattering, as a rule, are collisions with phonons. In this temperature region $\rho_{ik}^0 \sim T$ ($\rho_{ik}^0 \sim T$ up to terms $\theta/T \ll 1$), and, as is clear from the relations (2.24), the correction to the resistance due to scattering by impurities (which is described by the operator \hat{W}_1) does not depend at all on temperature (in the first approximation in the parameter θ/T).

At low temperatures (for very pure samples, at super-low temperatures) the main mechanism for resistance is scattering by impurities and by other static inhomogeneities. Now the operator \hat{W}_0 describes the impurity scattering. The role of the small correction is played by the interaction with the phonons (the operator W_1). The temperature correction to the resistance is proportional to T^5 (see below), while the coefficient of T^5 does not depend on the total number of impurity atoms (cf. (2.24)). However, this coefficient depends on the nature of the scattering of the electrons by the impurities and thus may vary from sample to sample. Such a situation is observed in various cases (for example, in refer-

ence^[11]) and is usually regarded as a deviation from the Matthiessen rule*.

The resistance of a polycrystal can be expressed in terms of the principal values of the resistance tensor of a single crystal only when the mean free path is much less than the size of the crystallites. To calculate the average value of the conductivity of a polycrystal we may use the method developed in reference^[12] for determining averages of elastic moduli. According to reference^[13] in the case of weak anisotropy, †

$$\varrho = \frac{1}{\bar{\sigma}} \left\{ 1 + \frac{1}{3} \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}{(\sigma_1 + \sigma_2 + \sigma_3)^2} \right\},$$

$$\bar{\sigma} = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3). \quad (2.25)$$

Here $\sigma_{1,2,3}$ are the principal values of the electrical conductivity tensor for the individual crystallite. In deriving formula (2.25) it was assumed that there are no correlations whatsoever between crystallites and that the polycrystal has no texture.

In the case of sufficiently pure samples at low temperature, the cause of the resistance of a polycrystal is the scattering of electrons at the boundaries of crystallites. To calculate the specific resistance of the sample we may use a kinetic approach if we assign a probability to the scattering of an electron at the boundaries and a distribution of boundaries within the sample.

Formula (2.7) and all that follow show that the electrical conductivity depends in essential fashion on the dynamical properties of the electrons and, in particular, on the nature of their dispersion law. It can be shown that this statement, confirmed by a whole variety of experiments on the resistance of metals, is in contradiction with the classical experiments of Stewart and Tolman,^[14] according to which the value of e/m for electrons in a metal is equal to the value e/m_0 for free electrons. This apparent contradiction is most easily explained away as follows: If the electron is acted upon, in addition to the electrical field, by the inertial force, the kinetic equation should be written as follows:^[15]

$$\left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}} = - \frac{\partial n_F}{\partial \epsilon} e v (eE - m_0 a), \quad (2.26)$$

where a is the acceleration. Such an expression for the inertial force ($-m_0 a$) can be justified in two ways: either by using the principle of equivalence, where we replace the gravitational force by an acceleration, or by treating the Schrödinger equation for the electron in a periodic field in a non-inertial coordinate system (as has been done in detail in reference^[15]). In carrying out both proofs, one must take account of the fact that the number of conduction electrons (the number of quasiparticles) is equal to the number of free electrons. For this reason, the Fermi-liquid interaction does not change the expression for the inertial force. From Eq. (2.26) it is clear that the field appearing in an open-circuit con-

*Usually one observes a monotonic dependence of the coefficient of T^5 on the concentration of impurities. This fact is not explained in terms of the formulas given here.

†Smallness of the anisotropy means that

$$v = \frac{1}{3} \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}{(\sigma_1 + \sigma_2 + \sigma_3)^2} \ll 1.$$

ductor (this corresponds to one of the formulations of the experiment^[14]) is equal to $m_0 a/e$, and we see that the measured quantity is determined by the ratio m_0/e for the free electron.

According to another formulation of the Stewart-Tolman experiment, one measures the ratio of the momentum density $\bar{\mathbf{P}}$ to the current density \mathbf{j} . It is easily shown that

$$\bar{\mathbf{P}} = \frac{m_0}{e} \mathbf{j}.$$

Actually, according to the general principles of relativistic mechanics, the momentum of a particle is equal to $\epsilon v/c^2$ where ϵ is the total energy including its rest energy. Since the energy of interaction of an electron both with the lattice and with other electrons is much less than the rest energy, the momentum of an electron, to high accuracy, is equal to $m_0 \mathbf{v}$ (where m_0 is the mass of the free electron). From this there immediately follows the relation given above.

In the formulas presented above, we have tacitly assumed that the degeneracy temperature of the electron gas is much higher than the characteristic temperature of the scattering particles (for example, phonons). This permitted us to use the limiting value for the Fermi function ($-\partial n_F/\partial \epsilon = \delta(\epsilon - \epsilon_F)$). The temperature dependence of the conductivity tensor in these cases is determined by the collision integral. However, one must keep in mind that the criterion for degeneracy of the conduction electron gas is much stricter than the criterion for the degeneracy of a gas of free electrons. In reference^[1] it was shown that the gas of conduction electrons can be regarded as degenerate so long as

$$\min |\epsilon_F - \epsilon_c| \gg T, \quad (2.27)$$

where ϵ_c is a critical value of the energy (for $\epsilon = \epsilon_c$ the topology of the surfaces of constant energy changes). For various metals (for example, for the metals of group V, Bi, As, Sb, for graphite etc.) the Fermi surface passes in the neighborhood of singular points in momentum space. Thus, the value of $\min |\epsilon_F - \epsilon_c|$ is anomalously small for these metals. This means that for such metals the temperature dependence of the resistance depends significantly on the factor $-\partial n_F/\partial \epsilon$ in the integrand of (2.6) and can be calculated from a sufficiently detailed knowledge of the structure of the electron energy spectrum.

However, specifically for Bi, the energy spectrum of which has been well investigated, the construction of a quantitative theory is made difficult since one must include a specific mechanism for scattering of electrons by phonons (cf. the remarks in reference^[16]).

A complicated temperature dependence of the electrical resistance, which does not reduce to a temperature dependence of the mean free path (or more precisely, of the factor $\hat{W}^{-1}\{v_i\}$ in formula

(2.6)), but is determined by the structure of the energy spectrum, can exist in those cases where an anomalously small group of electrons (or a band; cf. Secs. 3 and 6) has a relatively large mean free path. Then, in the dependence of the resistance on the temperature, there should appear the dependence on the temperature of the number of electrons in the anomalous band (cf.^[1], end of Sec. 3).

As we have already stated, the detailed study of resistance mechanisms is not part of our present paper. This question is studied in detail, for example, in the book "Electrons and Phonons" by Ziman.^[17] We shall therefore restrict ourselves to just a few general remarks and arguments.

The principal mechanisms for resistance are the following: 1) collisions of electrons with phonons, 2) collisions of electrons with one another, and 3) collisions of electrons with impurity atoms and other static defects of the crystal lattice.* The first two mechanisms occur in an ideal crystal and cause the so-called ideal resistance, which goes to zero at absolute zero; the third mechanism is characteristic of crystals containing defects and is the cause of the residual resistance, i.e., it determines the value of the resistance of a metal at absolute zero. The value of the residual resistance varies markedly for different samples of one and the same metal.

Let us begin with the first mechanism. All the present pictures of the energy structure of a metal are based on the fact that the conduction electrons and phonons represent relatively weakly coupled subsystems. The weakness of the interaction between the conduction electrons and the phonons (lattice vibrations) results from the fact that the main interactions between the electrons and the lattice appear in the dispersion laws of the electrons and phonons. The weakness of the electron-phonon interaction permits us, in treating this interaction, to restrict ourselves as a rule to one-phonon processes—absorption and emission of phonons by electrons. Since the velocity of the electrons with energies near the Fermi energy is much greater than the sound velocity, these processes are allowed by the conservation laws.

Assuming the phonon gas is in equilibrium, i.e., neglecting disturbances of the phonons by the electrons, one can show[†] that the scattering of electrons by phonons leads to the following temperature dependence of the resistance (cf. Appendix III):

*We shall not discuss other mechanisms for scattering in metals which have very specific properties. For example, in ferro- and antiferro-magnetic materials there occur collisions of electrons with spin waves; in ordered alloys there is scattering by fluctuations of long-range order.

†A consistent derivation of the isotropic dispersion law (the so-called Bloch solution of the Boltzmann equation) is contained in any detailed course in the electron theory of metals.

$$\frac{\rho_{e,p}(T)}{\rho(\theta)} \approx \begin{cases} \left(\frac{T}{\theta}\right)^5, & T \ll \theta, \\ \frac{T}{\theta}, & T \gg \theta. \end{cases} \quad (2.28)$$

Here $\rho(\theta)$ is the resistance of the metal at a temperature equal to the Debye temperature θ , $\rho(\theta) \approx m\theta/ne^2\hbar$ (cf. Appendix III). Although these results have already become classical, the assumptions on which they are based are still not completely clear. In particular, there has been no clarification of the question of the role of processes of dragging of phonons by electrons at very low temperatures.^[18]

Collisions of electrons with one another result in a finite resistance only in those cases where they are accompanied by umklapp. Because of the fact that the Fermi surface as a rule has portions which are located arbitrarily close to the boundaries of the Brillouin zone, collisions with umklapp are not hindered. The calculation carried out by L. D. Landau and I. Ya. Pomeranchuk^[19] shows that this part of the resistance is proportional to the square of the temperature:

$$\rho_{e,e} \approx \frac{m}{ne^2} \frac{\varepsilon_F}{\hbar} \left(\frac{T}{\varepsilon_F}\right)^2. \quad (2.29)$$

Comparison of formulas (2.29) and (2.28) shows that for all metals (except transition metals) the electron-electron part of the resistance can be observed only in the region of very low temperatures and there only if it is not completely overwhelmed by the residual resistance.

We note that a comparison of experimental data on resistance at low temperatures (where the electron-electron interaction can show itself) with the theoretical computations (we are speaking not of the rough formulas like (2.28) and (2.29), but of rigorous formulas^[20]) apparently show that there is a systematic disagreement between experiment and theory. The theory gives too high values for the electron-electron resistance.

The residual resistance is completely determined by the purity of the metal and the quality of the single crystal (presence of stresses, dislocations, etc.). At present one can obtain such pure samples that the mean free path of electrons in them reaches several millimeters (for the purest tungsten, for example, $l \approx 1$ cm.).

Pomeranchuk has called attention to a very curious mechanism for the resistance.^[22] It appears that isotopic inhomogeneity of a metal, which results in inhomogeneity of the zero point oscillations of the lattice, may cause a resistance. The mean free path (l_{is}) associated with this mechanism is of order

$$l_{is} \approx 10^2 a \left| \frac{\bar{M}}{\Delta M} \right| \left(\frac{Ms}{p_F} \right)^2,$$

where a is the interatomic spacing, M is the average mass of a metal atom, ΔM its mean deviation, and s the velocity of sound.

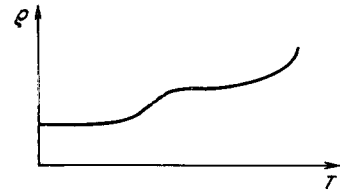


FIG. 1

According to the estimates, l_{is} can reach 0.1–1 cm. We have discussed this mechanism for resistance in particular since its cause is by no means trivial. It should be remembered that the electron clouds in isotopes are identical (a slight difference caused by hyperfine structure terms surely can be taken into account). The scattering is related, as we have already stated, only to the appearance of inhomogeneities in the zero point oscillations of the lattice.

A detailed analysis of the mechanisms of phonon and residual resistance given recently by R. N. Gurzhi^[23] showed that in the temperature variation of the resistance one should see very strange complications caused by the nature of the electron dispersion law.

The results obtained by R. N. Gurzhi, and shown in Fig. 1, are based on the assumption that the phonons are deflected by the electrons so that normal collisions of electrons with phonons (without umklapp) cannot lead to a finite resistance. The probability of such electron-phonon collisions with umklapp is extremely small. The latter is surely valid for metals for which the Fermi surface is entirely located within the first Brillouin zone (such a situation apparently occurs for K, Na, and various metals).

To this same class of problems there belongs the observed minimum of resistance which is seen in various metals (Au, etc.) whose nature apparently has been explained by Kondo,^[24] who showed that the resistance minimum is associated with the dependence of the scattering amplitude on energy (cf. also^[24]).

The weak coupling between electrons and phonons, of which we have spoken, also manifests itself in the fact that situations are possible in which each of the subsystems can be described by its own temperature. The relaxation time under these conditions coincides with the time for equilibration of the temperatures of the electrons and phonons.^[25]

The difference in temperatures of phonons and electrons is one of the reasons for the dependence of resistance on current (deviations from Ohm's law), which are observed experimentally at anomalously high current densities^[26] ($j \gtrsim 10^7$ – 10^8 A/cm²).

3. THERMAL CONDUCTIVITY. WIEDEMANN-FRANZ LAW. THERMOELECTRIC PHENOMENA

The electron participates not only in transfer of charge, but also in transport of heat, where in good

metals the electrons are the main mechanism for this thermal transport. The existence of thermoelectric phenomena is due only to the free electrons.

The kinetic coefficients describing the relaxation between the corresponding fluxes (the heat flux q and the current density j) and the forces (field intensity E and the temperature gradient ∇T) can be calculated by means of the kinetic equations formulated above.

Before giving detailed computations we consider the phenomenological description of thermal conductivity and thermoelectric phenomena.

Staying within the framework of linear relations and making use of the principle of symmetry of the kinetic coefficients (cf., for example,^[4] Sec. 25), we can write the relation between the electric field intensity E ,* the temperature gradient ∇T , the current density j , and the heat flux q :

$$\begin{aligned} E_i &= \rho_{ik} j_k + \alpha_{ik} \frac{\partial T}{\partial x_k}, \\ q_i &= T \alpha_{ki} j_k - \kappa_{ik} \frac{\partial T}{\partial x_k}. \end{aligned} \quad (3.1)$$

Here ρ_{ik} is the resistance tensor, κ_{ik} is the thermal conductivity tensor, and α_{ik} is the tensor characterizing the thermoelectric properties of the metal. We can express the Thompson and Peltier coefficients and the thermoelectric emf in terms of the tensor α_{ik} .^[4] We note that the Onsager relations (principle of symmetry of the kinetic coefficients) which require the symmetry of the tensors ρ_{ik} and κ_{ik} permit the existence of conductors with an asymmetric tensor for the thermoelectric coefficients ($\alpha_{ik} \neq \alpha_{ki}$).

We now proceed to the calculation of the tensor for the thermal conductivity and thermo-electric coefficients. In what follows it is convenient to rewrite the kinetic equation given above by directly writing an equation for the phonon distribution function χ .

We recall that collisions with phonons are almost always important. In addition, in all solids (and, of course, in metals) the phonons participate in heat transfer. If the non-equilibrium correction to the Fermi function is again denoted by f_1 and that for the equilibrium phonon function (the Bose-Einstein function $N_B = (e^{\hbar\omega/T} - 1)^{-1}$, where $\hbar\omega = \hbar\omega(p)$ is the energy of a phonon with quasimomentum p) by χ_1 , the system of Boltzmann kinetic equations can be written as follows (for the reasons mentioned above we drop the terms in the derivatives of the distribution function with respect to the coordinates and time; furthermore we drop the term which includes the action of the magnetic field; cf. Sec. 7):

$$\left. \begin{aligned} \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}}^{e,e} + \left(\frac{\partial \chi_1}{\partial t} \right)_{\text{coll}}^{e,ph} &= \frac{\partial n_F}{\partial \epsilon} e \mathbf{v} E + \frac{\partial n_F}{\partial T} \mathbf{v} \nabla T, \\ \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}}^{ph,e} + \left(\frac{\partial \chi_1}{\partial t} \right)_{\text{coll}}^{ph,ph} &= \frac{\partial N_B}{\partial T} \mathbf{u} \nabla T. \end{aligned} \right\} \quad (3.2)$$

*As we have already stated, the observed field intensity in the conductor is the sum of the field intensity caused by the potential differences imposed on the conductor and the quantity $-(1/e) \nabla \zeta$ where ζ is the chemical potential of the electrons (cf. [4]). This means that the change in the chemical potential caused by the temperature gradient is already contained in E .

Here $\mathbf{u} = \frac{\partial \hbar\omega}{\partial \mathbf{p}}$ is the velocity of a phonon with quasimomentum p , and the linearized collision integrals are defined as follows:

$$\begin{aligned} \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}}^{e,e} &= -\hat{W}_{e,e} \{f_1\}, & \left(\frac{\partial \chi_1}{\partial t} \right)_{\text{coll}}^{e,ph} &= -\hat{W}_{e,ph} \{\chi_1\}, \\ \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}}^{ph,e} &= -\hat{W}_{ph,e} \{f_1\}, & \left(\frac{\partial \chi_1}{\partial t} \right)_{\text{coll}}^{ph,ph} &= -\hat{W}_{ph,ph} \{\chi_1\}, \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} \hat{W}_{e,e} &= - \left\{ \frac{\delta \hat{\mathcal{L}}_{\text{coll}}^e}{\delta f} \right\}_{f=n_F, \chi=N_B}, & \hat{W}_{e,ph} &= - \left\{ \frac{\delta \hat{\mathcal{L}}_{\text{coll}}^e}{\delta \chi} \right\}_{f=n_F, \chi=N_B}, \\ \hat{W}_{ph,e} &= - \left\{ \frac{\delta \hat{\mathcal{L}}_{\text{coll}}^{ph}}{\delta f} \right\}_{f=n_F, \chi=N_B}, & \hat{W}_{ph,ph} &= - \left\{ \frac{\delta \hat{\mathcal{L}}_{\text{coll}}^{ph}}{\delta \chi} \right\}_{f=n_F, \chi=N_B} \end{aligned} \quad (3.4)$$

are linear operators acting on the functions f_1 and χ_1 ; $\hat{\mathcal{L}}_{\text{coll}}^e$ and $\hat{\mathcal{L}}_{\text{coll}}^{ph}$ are the electron and phonon collision operators; they, of course, include the interaction not only between electrons and phonons, but also with all static inhomogeneities (impurities, dislocations, etc.). The existence of several electron and phonon bands is assumed and is taken into account in the same way as in sec. 1 (cf. formulas (1.2) and (1.3)).

By analogy with formulas (1.13)–(1.15) we introduce the following notation for the distribution functions:

$$\begin{aligned} f_1 &= -eE_i \frac{\partial n_F}{\partial \epsilon} \psi_i - \nabla_i T \frac{\partial n_F}{\partial T} \varphi_i, \\ \chi_1 &= -eE_i \frac{\partial N_B}{\partial (\hbar\omega)} \mu_i - \nabla_i T \frac{\partial N_B}{\partial T} \nu_i, \end{aligned} \quad (3.5)$$

and for the linearized collision integrals:

$$\left. \begin{aligned} \hat{W}_p^{e,e} \dots &= \left(\frac{\partial n_F}{\partial \epsilon} \right)^{-1} \hat{W}_{e,e} \left\{ \frac{\partial n_F}{\partial \epsilon} \dots \right\}, \\ \hat{W}_p^{e,ph} \dots &= \left(\frac{\partial n_F}{\partial \epsilon} \right)^{-1} \hat{W}_{e,ph} \left\{ \frac{\partial N_B}{\partial (\hbar\omega)} \dots \right\}, \\ \hat{W}_e^{e,e} \dots &= \left(\frac{\partial n_F}{\partial T} \right)^{-1} \hat{W}_{e,e} \left\{ \frac{\partial n_F}{\partial T} \dots \right\}, \\ \hat{W}_e^{e,ph} \dots &= \left(\frac{\partial n_F}{\partial T} \right)^{-1} \hat{W}_{e,ph} \left\{ \frac{\partial N_B}{\partial (\hbar\omega)} \dots \right\}, \\ \hat{W}_p^{ph,e} \dots &= \left(\frac{\partial N_B}{\partial (\hbar\omega)} \right)^{-1} \hat{W}_{ph,e} \left\{ \frac{\partial n_F}{\partial \epsilon} \dots \right\}, \\ \hat{W}_p^{ph,ph} \dots &= \left(\frac{\partial N_B}{\partial (\hbar\omega)} \right)^{-1} \hat{W}_{ph,ph} \left\{ \frac{\partial N_B}{\partial (\hbar\omega)} \dots \right\}, \\ \hat{W}_e^{ph,e} \dots &= \left(\frac{\partial N_B}{\partial (\hbar\omega)} \right)^{-1} \hat{W}_{ph,e} \left\{ \frac{\partial n_F}{\partial \epsilon} \dots \right\}, \\ \hat{W}_e^{ph,ph} \dots &= \left(\frac{\partial N_B}{\partial (\hbar\omega)} \right)^{-1} \hat{W}_{ph,ph} \left\{ \frac{\partial N_B}{\partial (\hbar\omega)} \dots \right\}. \end{aligned} \right\} \quad (3.6)$$

Then the Boltzmann kinetic equation is written as a system of vector equations for the functions ψ_i , φ_i , μ_i , and ν_i :

$$\left. \begin{aligned} \hat{W}_p^{e,e} \psi_i + \hat{W}_p^{e,ph} \mu_i &= v_i, & \hat{W}_e^{e,e} \varphi_i + \hat{W}_e^{e,ph} \nu_i &= v_i, \\ \hat{W}_p^{ph,e} \psi_i + \hat{W}_p^{ph,ph} \mu_i &= 0, & \hat{W}_e^{ph,e} \varphi_i + \hat{W}_e^{ph,ph} \nu_i &= u_i \end{aligned} \right\} \quad (3.7)$$

Using the operators introduced here, one can easily formally construct \hat{W}_p and \hat{W}_e of Sec. 1, for example,

$$\hat{W}_p = \hat{W}_p^{e,e} - \hat{W}_p^{e,ph} (\hat{W}_p^{ph,ph})^{-1} \hat{W}_p^{ph,e} \quad (3.8)$$

In using the kinetic equation the generalized forces are the observed electric field intensity E and the temperature gradient ∇T . According to the principles of the thermodynamics of non-equilibrium processes,^[27] the fluxes corresponding to these forces

must be determined so that the time derivative of the entropy \dot{S} (the entropy production) is expressed by the following equation:

$$\dot{S} = \mathbf{j}^* \mathbf{E} + \mathbf{q}^* (-\nabla T). \quad (3.9)$$

It is easy to show that then the currents \mathbf{j}^* and \mathbf{q}^* have the following form:

$$\mathbf{j}^* = \frac{e}{T} \int \mathbf{v} f_1 d\Gamma_e, \quad (3.10)$$

$$\mathbf{q}^* = \frac{1}{T^2} \left\{ \int \mathbf{u} \hbar \omega (\mathbf{p}) \chi_1 d\Gamma_{ph} + \int \mathbf{v} (\varepsilon - \zeta) f_1 d\Gamma_e \right\}. \quad (3.11)$$

Here by the symbol $\int \dots d\Gamma_e$ we mean an integration over all electron states (cf. page 807) and by the symbol $\int \dots d\Gamma_{ph}$, an integral over all phonon states. Naturally, here we include a summation over the bands in the phonon spectrum.

Usually, in place of the fluxes \mathbf{j}^* and \mathbf{q}^* one uses the current density $\mathbf{j} = T\mathbf{j}^*$ and the density of heat flux $\mathbf{q} = T^2\mathbf{q}^*$, for which the phenomenological relations (3.1) are given.

If we assume the generalized forces $-\nabla\varphi$ and $-\nabla T$, then the corresponding fluxes will be \mathbf{j}^* and \mathbf{q}^* , which differs from \mathbf{q}^* (cf. formula (3.11) by the replacement of ζ by $w = -T^2 \partial/\partial T (\zeta/T)$ (where w is the specific heat per electron; $w \approx \zeta$ when $T \ll \epsilon_F$).

Using formulas (3.5) and going over to the fluxes \mathbf{j} and \mathbf{q} , we write expressions (3.10) and (3.11) as follows:

$$\left. \begin{aligned} j_i &= \sigma_{ik} E_k + b_{ik} \frac{\partial T}{\partial x_k}, \\ q_i &= c_{ik} E_k + d_{ik} \frac{\partial T}{\partial x_k}, \end{aligned} \right\} \quad (3.12)$$

where

$$\left. \begin{aligned} \sigma_{ik} &= e^2 \langle v_i \psi_k \rangle_e, \\ b_{ik} &= \frac{e}{T} \langle v_i \varphi_k (\varepsilon - \zeta) \rangle_e, \\ c_{ik} &= -e \{ \langle v_i \psi_k (\varepsilon - \zeta) \rangle_e + \langle u_i \mu_k \hbar \omega \rangle_{ph} \}, \\ d_{ik} &= -\frac{1}{T} \{ \langle v_i \varphi_k (\varepsilon - \zeta)^2 \rangle_e + \langle u_i v_k (\hbar \omega)^2 \rangle_{ph} \} \end{aligned} \right\} \quad (3.13)$$

and where we have introduced the notation

$$\left. \begin{aligned} - \int \frac{\partial n_F}{\partial \varepsilon} \chi(\mathbf{p}) \eta(\mathbf{p}) d\Gamma_e &= \langle \chi \eta \rangle_e, \\ - \int \frac{\partial N_B}{\partial (\hbar \omega)} \chi(\mathbf{p}) \eta(\mathbf{p}) d\Gamma_{ph} &= \langle \chi \eta \rangle_{ph}, \end{aligned} \right\} \quad (3.14)$$

where, as in Sec. 1, such integrals can be regarded as the scalar products of functions $\chi(\mathbf{p})$ and $\eta(\mathbf{p})$.

In writing the fluxes in the form (3.12) the Onsager principle of the symmetry of the kinetic coefficients requires that there be the following relations between the components of the tensors σ_{ik} , b_{ik} , c_{ik} , and d_{ik} :

$$\sigma_{ki} = \sigma_{ik}, \quad d_{ik} = d_{ik}, \quad c_{ik} = -T b_{ik}. \quad (3.15)$$

Naturally these relations are a consequence of definite properties of the collision integral. We shall not discuss these (but see (3.28)), but mention only that to prove the various relations (3.15) it is suffi-

cient to make use of the hermiticity of the operator \hat{W}_p of (1.23).

Comparing formulas (3.1) and (3.12), one easily sees that from (3.15) there follow the Onsager relations which we used in writing (3.1), and

$$\left. \begin{aligned} Q_{ik} &= \sigma_{ik}^{-1}, \quad \alpha_{ik} = -\sigma_{ik}^{-1} b_{ik}, \\ \kappa_{ik} &= -d_{ik} + c_{im} \sigma_{ml}^{-1} b_{lk}. \end{aligned} \right\} \quad (3.16)$$

If, using the last of relations (3.15), we replace the tensor C_{im} in the formula for the thermal conductivity tensor κ_{ik} by the equal tensor $-T b_{mi}$, then, according to formulas (3.13), the thermal conductivity splits into the sum of an electron and a phonon part:

$$\kappa_{ik} = \kappa_{ik}^e + \kappa_{ik}^{ph}, \quad (3.17)$$

where

$$\kappa_{ik}^e = -d_{ik}^e - T b_{mi} \sigma_{ml}^{-1} b_{lk}, \quad d_{ik}^e = -\frac{1}{T} \langle v_i \varphi_k (\varepsilon - \zeta)^2 \rangle_e, \quad (3.18)$$

and

$$\kappa_{ik}^{ph} = \frac{1}{T} \langle u_i v_k (\hbar \omega)^2 \rangle_{ph}. \quad (3.19)$$

This splitting, of course, does not mean the absence of mutual interactions between the electron and phonon subsystems. Over a wide range of temperatures the phonons are the main cause of scattering of electrons, which in turn at practically all temperatures are the main cause of the thermal resistance for the phonon flux. It is just because of the scattering by the electrons that the mean free path of the phonons is small, and their role in the thermal conductivity of metals is negligible. Formally this permits us in formula (3.17), and also in the expressions for C_{ik} , to neglect the second terms. The Wiedemann Franz relation is established between the electronic part of the thermal conductivity coefficient and the electronic conductivity.

Thus the electronic part of the thermal conductivity of a metal and the thermoelectric coefficients are determined by the quantities b_{ik} , d_{ik} , and σ_{ik} . As we have seen in Sec. 2, the electrical conductivity is expressed as an integral over the Fermi surface (cf. formula (2.9) et seq.). A similar form can also be given to formulas (3.16) by expanding the expressions in powers of the temperature (more precisely in powers of $T/|\epsilon_F - \epsilon_C|$); the expansion coefficients are here also functions of the temperature (because of the temperature dependence of the collision operators). All the tensors appearing here can be written explicitly separating off the integration over the energy:

$$\left. \begin{aligned} \sigma_{ik} &= -e^2 \int \frac{\partial n_F}{\partial \varepsilon} \psi_{ik}(\varepsilon) d\varepsilon, \\ b_{ik} &= \frac{e}{T} \int \frac{\partial n_F}{\partial \varepsilon} (\varepsilon - \zeta) \varphi_{ik}(\varepsilon) d\varepsilon, \\ c_{ik} &= -e \int \frac{\partial n_F}{\partial \varepsilon} (\varepsilon - \zeta) \psi_{ik}(\varepsilon) d\varepsilon, \\ d_{ik}^e &= \frac{1}{T} \int \frac{\partial n_F}{\partial \varepsilon} (\varepsilon - \zeta)^2 \varphi_{ik}(\varepsilon) d\varepsilon, \end{aligned} \right\} \quad (3.20)$$

where

$$\psi_{ih} = \frac{2}{(2\pi\hbar)^3} \oint_{\epsilon(\mathbf{p})=\epsilon} \frac{v_i \psi_{ih}}{v} dS, \quad \varphi_{ih} = \frac{2}{(2\pi\hbar)^3} \oint_{\epsilon(\mathbf{p})=\epsilon} \frac{v_i \varphi_{ih}}{v} dS, \quad (3.21)$$

and the integration is carried out over the constant-energy surface $\epsilon(\mathbf{p}) = \epsilon$.

Using the well-known properties of the Fermi function (cf., for example, formulas (3.7) of [1]), one can easily show that

$$\left. \begin{aligned} \sigma_{ih} &\approx e^2 \psi_{ih}(\epsilon_F), & b_{ih} &\approx -\frac{\pi^2}{3} eT \left(\frac{d\psi_{ih}}{d\epsilon} \right)_{\epsilon=\epsilon_F}, \\ c_{ih} &\approx \frac{\pi^2}{3} eT^2 \left(\frac{d\psi_{ih}}{d\epsilon} \right)_{\epsilon=\epsilon_F}, \\ d_{ih} &\approx -\frac{\pi^2}{3} T \varphi_{ih}(\epsilon_F). \end{aligned} \right\} (3.22)$$

We note that the Onsager relations (3.15) in this approximation mean that

$$\left(\frac{d\varphi_{ih}}{d\epsilon} \right)_{\epsilon=\epsilon_F} = \left(\frac{d\psi_{hi}}{d\epsilon} \right)_{\epsilon=\epsilon_F}. \quad (3.23)$$

Substituting the expressions (3.22) in formula (3.16) and keeping only terms of lowest order in the ratio $T/|\epsilon_F - \epsilon_c|$, we get

$$\left. \begin{aligned} \rho_{ih} &\approx \frac{1}{e^2} \psi_{ih}^{-1}(\epsilon_F), \\ \alpha_{ih} &\approx \frac{\pi^2}{3} \frac{T}{e} \psi_{ih}^{-1}(\epsilon_F) \varphi_{ih}(\epsilon_F) = \frac{\pi^2}{3} \frac{T}{e} \psi_{ih}^{-1}(\epsilon_F) \psi_{hi}(\epsilon_F), \\ \kappa_{ih} &\approx \frac{\pi^2}{3} T \varphi_{ih}(\epsilon_F). \end{aligned} \right\} (3.24)$$

By analogy with Sec. 2 it is convenient to introduce mean free path operators acting on the unit vector $\mathbf{n}_i = \mathbf{v}_i/v$. If, for the collision operators, we use the notation of Sec. 1, we can write

$$\left. \begin{aligned} \psi_i &= \hat{W}_p^{-1}(v_i) \equiv \hat{l}_p(n_i), \\ \varphi_i &= \hat{W}_e^{-1}(v_i) \equiv \hat{l}_e(n_i). \end{aligned} \right\} (3.25)$$

The subscripts \mathbf{p} and ϵ emphasize that in the first case in the collision the change in momentum is most significant, and in the second the change in energy. In other words, in order of magnitude $\psi \approx l_p$ where l_p is the length over which the momentum of the electron relaxes, and φ is equal to l_ϵ where l_ϵ is the relaxation length for the energy, (more precisely, $\epsilon - \zeta$). At low temperatures ($T \ll \theta$), when the collisions with long-wave phonons are most important, these lengths are drastically different ($l_p \approx \theta^2/T^2 l_\epsilon$). This latter fact results in an observable experimental deviation from the Wiedemann-Franz law. In those cases where inelastic collisions can be neglected* the difference between

l_ϵ and l_p disappears, which is natural, since then the relaxation is connected only with the bringing to isotropy of the motion of the electrons (in particular, the difference between l_p and l_ϵ vanishes in those rare cases where the τ -approximation is valid).

In the case of a cubic metal it is convenient to introduce average mean free paths. These are functions of the electron energy and are defined by the following formula:

$$\frac{1}{S(\epsilon)} \oint n_i \hat{l}_{p,\epsilon}(n_k) dS = \frac{1}{3} l_{p,\epsilon}(\epsilon) \delta_{ik}, \quad (3.26)$$

where the integration is extended over the surface $\epsilon(\mathbf{p}) = \epsilon$, whose surface area is $S(\epsilon)$.

The mean free path l_p introduced in Sec. 2 coincides with that used here if, in place of the energy ϵ , we substitute the Fermi energy: $l_p = l_p(\epsilon_F)$, similarly $l_\epsilon = l_\epsilon(\epsilon_F)$.

Using formulas (3.21), (3.24)–(3.26), we have

$$\left. \begin{aligned} \sigma_{ih} &= \sigma \delta_{ih}, & \sigma &= \frac{2}{(2\pi\hbar)^3} e^2 \frac{S(\epsilon_F) l_p}{3}, \\ \kappa_{ih} &= \kappa \delta_{ih}, & \kappa &= \frac{2}{(2\pi\hbar)^3} \frac{\pi^2 T}{3} \frac{S(\epsilon_F) l_e}{3}, \\ d_{ih} &= \alpha \delta_{ih}, & \alpha &= \frac{\pi^2}{3} \frac{T}{e} \frac{d(S l_p)_{\epsilon=\epsilon_F}}{S(\epsilon_F) l_p}. \end{aligned} \right\} (3.27)$$

We note that in the new notation the Onsager relations (3.15) or (3.23) are written in a particularly compact fashion:

$$\frac{d}{d\epsilon} (S l_p)_{\epsilon=\epsilon_F} = \frac{d}{d\epsilon} (S l_e)_{\epsilon=\epsilon_F}. \quad (3.28)$$

If $l_p = l_\epsilon$, then

$$\kappa = \frac{\pi^2 T}{3e^2} \sigma, \quad (3.29)$$

i.e., the familiar relation, the Wiedemann-Franz law, is satisfied. As we see from the derivation of formula (3.29), this relation is determined only by the nature of the scattering and, for example, is completely independent of the dispersion law. In the case of a non-cubic crystal, this relation (for $\varphi_i = \psi_i$) is satisfied for each of the components of the tensors σ_{ik} and κ_{ik} .

We shall not spend time on presenting the theory which explains (or, more precisely, tries to explain) the observed experimental facts concerning the thermo-emf and thermal conductivity (many paragraphs of J. Ziman's monograph "Electrons and Phonons" are devoted to this problem). We only note that the opposite sign* of the coefficient α (i.e., of the thermo-emf) for various metals can be explained by the hole character of the Fermi surface: If the Fermi surface is located near a point of \mathbf{p} -space where the energy reaches a maximum, then with increasing energy the area of the surface decreases (the surface contracts); furthermore, as we approach the extremal point, the velocity decreases and slow

*This means not only that one can set $\mu_i = v_i = 0$ (cf. (3.7)), but also, and most important, that one can neglect energy transfers from the electrons to the equilibrium phonons in each collision. We recall (cf. appendix I) that when $T \gg \theta$, collisions with the phonons are quasielastic. This is the reason why the Wiedemann-Franz law is valid at high temperatures.

*For free electrons $\alpha < 0$.

electrons are more frequently scattered and therefore with increasing energy the mean free path also clearly decreases.

4. GALVANOMAGNETIC PHENOMENA. INTRODUCTION

The influence of a magnetic field on the conduction of metals is due to its effects on the motion of the electrons (cf. [3], Secs. 3 and 5). Here we should remember that, unlike equilibrium thermodynamic properties, the kinetic characteristics (specific resistance, thermal conductivity coefficients, etc.) depend essentially on magnetic field, even in the classical approximation. In other words, the dependence on magnetic field appears even if we do not consider the quantization of the electron energy in a magnetic field. The characteristic dimensionless parameter which determines the importance of the magnetic field is the ratio r/l , where r is the radius of the electron orbit and l is the mean free path. Since the orbit radius r is inversely proportional to the magnetic field, we should regard those fields as small for which $r \gg l$ and regard fields as great for which the reverse inequality holds. The orbit radius r for a free electron is equal to pc/eH , where by p in this case we should mean the radius of the Fermi sphere, while in the general case $p = \sqrt{S}$, where S is the section of the Fermi surface that is cut out (for example, an extremal section).

The importance of quantum effects is determined by the two parameters: $\hbar\omega_H/|\epsilon_F - \epsilon_C|$ and $\hbar\omega_H/T$ (cf. [3], Sec. 3). The first parameter as a rule is always very small* which justifies the neglect of quantum effects. The parameter $\hbar\omega_H/T$ in different fields and temperatures changes over wide limits. If $\hbar\omega_H/T \gtrsim 1$, we have quantized oscillations of the galvanomagnetic characteristics—the Shubnikov-de Haas effect and other effects similar to the de Haas-van Alphen effect (cf. [1], Sec. 6). For most metals quantum oscillations appear in the form of slight bumps on the basic dependence of the galvanomagnetic characteristics on the magnetic field (Fig. 2). This allows one first to consider the classical effect and then to take account of the quantum effects later as corrections.†

The simplest variant of the electron theory of the

*If the Fermi surface is sufficiently well-branched, which is characteristic for most metals, one finds cases where $\hbar\omega_H/|\epsilon_F - \epsilon_C|$ reaches a value of the order of unity. This apparently can result in a phenomenon analogous to magnetic breakdown (cf. also sec. 6 of [1]).

†Quantum corrections always contain the factor $(\hbar\omega_H/|\epsilon_F - \epsilon_C|)^n$ where $n > 0$. For some metals, for example for Bi, quantum effects have an important role in determining the dependence of the resistance and particularly the Hall "constant" R on magnetic field. For these metals the results presented here must be applied with care.

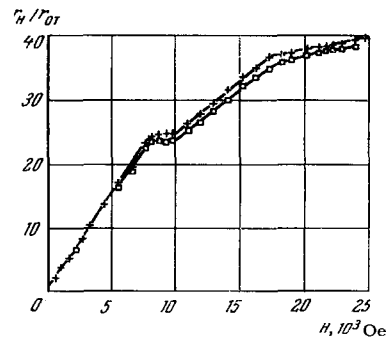


FIG. 2. Dependence of resistance of Zn on magnetic fields. Upper curve: $T = 4.29^\circ\text{K}$; lower curve $T = 1.81^\circ\text{K}$.

kinetic phenomena (τ -approximation, one group of carriers with an isotropic quadratic dispersion law) cannot even qualitatively explain the dependence of the resistance on magnetic field, although the estimate made for the Hall "constant" R is correct in many cases ($R = 1/nec$, where n is the electron concentration). Investigations of recent years have shown that galvanomagnetic characteristics in strong fields are extremely sensitive to the structure of the electron energy spectrum. They serve as one of the most reliable methods for determining the topology of the Fermi surface.

We shall not give here a survey of the experimental data (a separate summary would be needed for this), but in the course of presenting the theory the results of experiment will be given to illustrate conclusions of the theory. We note only a few well-known facts:

a) For polycrystals and for various single crystals the Kohler rule [29] is approximately satisfied, according to which the quantity

$$\Delta_H \equiv \frac{\rho(H) - \rho(0)}{\rho(0)}$$

is a function of the effective field

$$H_{\text{eff}} = H \frac{\rho_{273}}{\rho(0)}$$

($\rho(H)$ is the resistance in the magnetic field H , ρ_{273} is the resistance at 273°K for $H = 0$). This rule, which is illustrated in Fig. 3, confirms the statement made above that the parameter on which the change in resistance depends is r/l .

Recently there have been observed considerable deviations from the Kohler rule. [30] Jones and Sondheimer [31] have attempted to formulate a more general rule, which has the following form:

$$\frac{\Delta_H}{\Delta_\infty} = f\left(\frac{H}{\rho(0)\Delta_\infty}\right), \quad (4.1)$$

where $\Delta_\infty = \Delta_{H=\infty}$ and $f(\zeta)$ is a certain universal function (for the given metal) which increases from zero up to unity as the argument changes from zero to infinity.

Like the Kohler rule, the Jones and Sondheimer rule cannot be proved rigorously and must be regarded as semi-empirical. In Fig. 4 are shown ex-

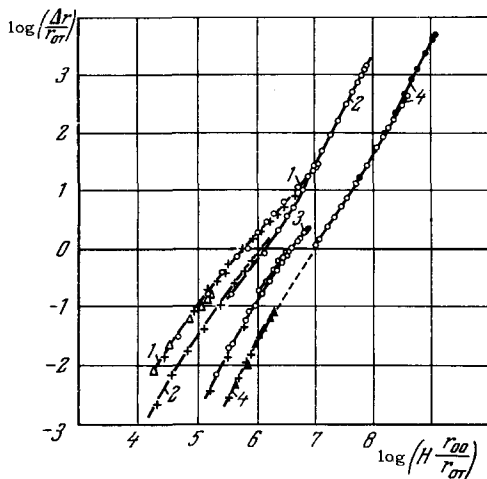


FIG. 3. Dependence of increase of resistance in a magnetic field on the effective value of the magnetic field. 1 - Mg; 2 - Cd; 3 - Cu; 4 - Pb; ● - $T = 2^\circ\text{K}$; ○ - $T = 4.22^\circ\text{K}$; □ - $T = 14^\circ\text{K}$; × - $T = 20.35^\circ\text{K}$; △ - $T = 78^\circ\text{K}$ [93].

perimental data on the resistance of Al in a magnetic field, [32] which were the basis for the formulation of the rule (5.1).

b) All metals investigated so far for the dependence of their resistance on magnetic field can be divided into three groups. In the first group are those metals for which the resistance tends to saturation as $H \rightarrow \infty$ irrespective of the direction of the magnetic field. The increase in resistance (though the resistance always increases in large fields) for these metals reaches hundreds of percent (Δ_∞ amounts to a few units). In the second group are those metals for which the resistance increases quadratically with increase in magnetic field ($r \ll l$), again for all directions of the magnetic field. The resistance here can increase by factors of a million (Bi, for example); here the metal in the strong magnetic field has the properties of a dielectric or insulator. A third group has intermediate properties: for certain directions of magnetic field the resistance

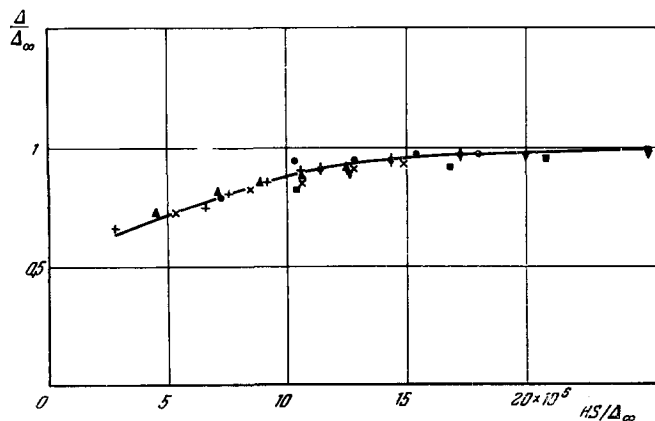


FIG. 4. Graph characterizing the Jones-Sondheimer rule (4.1) for Al of various purities (according to E. S. Borovik et al. [32]).

increases quadratically, while for others it tends to saturation.

So far we have considered transverse galvanomagnetic effects, i.e. the case where the current measured is perpendicular to the magnetic field. The longitudinal galvanomagnetic effect is relatively small: the longitudinal resistance for all metals tends to saturation. In the dependence of the resistance on magnetic field ($j \perp H$) for most metals one observes a large region in which the resistance depends linearly on the magnetic field (the Kapitza law). For certain metals this region is intermediate either between two quadratic dependences or between a quadratic dependence and saturation; for other metals (or samples) one has not succeeded in detecting a deviation from the linear law in large fields.

Most of the experimental facts find their explanation in the present-day kinetic theory of the galvanomagnetic effects. True, certain observations require one to assume that final clarity in the understanding of this area of problems has still not been achieved (for example, it is very difficult to explain [32] the behavior of extremely pure Al). For presenting the theory of galvanomagnetic effects let us start with their phenomenological description. The connection between the electric field and the current is naturally linear in character (Ohm's law):

$$j_i = \sigma_{ik} E_k, \quad E_i = \rho_{ik} j_k, \quad \partial_{ik} = \sigma_{ik}^{-1}. \quad (4.2)$$

In a magnetic field the tensors σ_{ik} and ρ_{ik} , which are functions of the magnetic field are not symmetric tensors. The principle of the symmetry of kinetic coefficients requires a somewhat more complicated relation between the components of the tensor

$$\sigma_{ik}(H) = \sigma_{ki}(-H). \quad (4.3)$$

The tensor σ_{ik} , like every second-rank tensor, can be written as the sum of a symmetric ρ_{ik} and an antisymmetric tensor a_{ik} :

$$\sigma_{ik} = s_{ik} + a_{ik}. \quad (4.4)$$

Using the Onsager relations (4.3), it is easy to show that the tensor ρ_{ik} is an even function, and the tensor a_{ik} is an odd function of the magnetic field:

$$s_{ik}(H) = s_{ik}(-H), \quad a_{ik}(H) = -a_{ik}(-H). \quad (4.5)$$

From the components of the antisymmetric tensor a_{ik} we can construct the dual vector a :

$$a_x = a_{yz}, \quad a_y = a_{zx}, \quad a_z = a_{xy},$$

or

$$a_i = \epsilon_{ikl} a_{kl},$$

where ϵ_{ikl} is the unit antisymmetric tensor of third rank.

A similar expansion can be performed with the tensor for the specific resistance ρ_{ik} , the tensor which is inverse to σ_{ik} :

$$\rho_{ik} = \rho_{ik}^s + \rho_{ik}^a,$$

$$\rho_{ik}^s(H) = \rho_{ki}^s(H) = \rho_{ik}^s(-H), \quad \rho_{ik}^a(H) = -\rho_{ki}^a(H) = -\rho_{ik}^a(-H).$$

The components of the tensors ρ_{ik}^s and ρ_{ik}^a can be expressed in terms of the components of the tensors ρ_{ik} and a_{ik} :

$$\rho_{ik}^s = \frac{|s| (s^{-1})_{ik} + a_i a_k}{|s| + (a \hat{s} a)}, \quad b_i = \frac{(\hat{s} a)_i}{|s| + (a \hat{s} a)}. \quad (4.6)$$

Here $|s|$ is a determinant made up of the components of the tensor ρ_{ik} ; $a_s a \equiv a_i s_{ik} a_k$, b_i is the vector dual to the antisymmetric tensor ρ_{ik}^a ; $b_i = \epsilon_{ikl} \rho_{lk}^a$.

Using the quantities we have introduced, Ohm's law for a conductor in a magnetic field is written as follows:

$$E = \hat{Q}^s j + [b]j. \tag{4.7}$$

The vector $\hat{Q}^s j$ is the vector with components ρ_{ijk}^s ; the second term is called the Hall field. We note that it is, first of all, perpendicular to the current, and second, it changes sign with change in sign of the magnetic field. The Joule heat liberated in the conductor is determined by the tensor ρ_{ik}^s :

$$Q = Q_{ik}^s j_i j_k; \tag{4.8}$$

so the tensor ρ_{ik}^s (i.e., the symmetric part of the resistance tensor) is frequently simply called the resistance tensor.

From formula (4.7) it is clear that a complete description of the galvanomagnetic properties is possible if one knows the three principal values of the tensor ρ_{ik}^s for all values of the magnetic field, the direction of its principal axes, and the three components of the vector b . In general case, the principal directions for the tensor ρ_{ik}^s do not coincide with the crystallographic directions, but depend on the magnitude and direction of the magnetic field.

The determination of the current density and the electric field intensity in a sample of finite dimensions is a complicated mathematical problem which is solvable only in various special cases,* so that as a rule in experiment one uses long, cylindrical conductors (wires) in which the direction of the current lines is fixed by the sample geometry.

Additional possibilities for measurement occur if one uses alternating fields of relatively low frequency (cf. Sec. 8).

The components of the tensor ρ_{ik}^s and the vector b depend in a complicated way on the magnetic field. If the magnetic field is small, they can be expanded in powers of the magnetic field. From the conditions (4.5) it follows that the expansion has the following form:

$$Q_{ik}^s = Q_{ik}^s(0) + \lambda_{iklm} H_l H_m + \dots, \\ b_i = R_{ik} H_k + \dots \tag{4.9}$$

We restrict ourselves to the first non-zero term containing the magnetic field; $\rho_{ik}(0)$ is the resistance tensor for $H = 0$.

The number of independent components of the tensors λ_{iklm} and R_{ik} is determined by the symmetry class of the metal. Thus the angular dependence on magnetic field and current direction) of the resistance and the Hall field for the case of weak field is determined completely by the crystal class. For example, in the case of a cubic crystal, as for any isotropic body, $R_{ik} = R \rho_{ik}$. The quantity R is called the Hall constant. This quantity (R) for most metals is actually comparatively weakly dependent on temperature, sample purity, etc. However, a more convenient characteristic of metals (although by no means applicable for all metals), as we shall see, is the Hall "constant" in high fields R_∞ (for the definition see Sec. 5).

In the case of an isotropic metal (a polycrystal) the only distinguished direction is that along the magnetic field. The tensor component along the field is denoted by $\rho_{||}^s$, and in the plane

*We are speaking of sufficiently large samples so that one need not consider surface phenomena and can use a macroscopic description.

The mathematical formation of the problem of determining the current density and field intensity E is the following: $\text{div } j = 0$, $\text{curl } E = 0$, where $j \cdot \nu = 0$ at all points of the boundary except those through which the current is fed; at lead points $j \cdot \nu$ is continuous (ν is the normal for the surface of the body).

perpendicular to the field by ρ_{\perp}^s . From expression (4.9) it follows that:

$$\left. \begin{aligned} Q_{ik}^s &= (Q(0) + \lambda_1 H^2) \delta_{ik} + \lambda_2 H_i H_k, \\ Q_{||}^s &= Q(0) + \lambda_{||} H^2 \quad (\lambda_{||} = \lambda_1 + \lambda_2), \\ Q_{\perp}^s &= Q(0) + \lambda_{\perp} H^2 \quad (\lambda_{\perp} = \lambda_1). \end{aligned} \right\} \tag{4.10}$$

As a rule, the constants $\lambda_{||}$ and λ_{\perp} are positive. Exceptions are ferromagnetics, in which the opposite sign for the galvanomagnetic effect (decrease of resistance in a weak magnetic field) has been explained in references^[3,13].

Formulas (4.9) and (4.10) are phenomenological in character. The problem of the microscopic theory is to calculate the tensors λ_{iklm} and R_{ik} . In the Appendix, where we establish the connection between the components of the tensors λ_{iklm} and R_{ik} and the microscopic characteristics of the conduction electron, it is shown that the Onsager relations (4.3), are a consequence of the general properties of the collision operator formulated in Sec. 1 (cf. formulas (1.23) and (1.24)); furthermore, starting from these same conditions it is shown that the principal values of the tensor ρ_{ik}^s in weak fields are greater than the principal values of the tensor $\rho_{ik}(0)$; in other words, the resistance increases with increasing magnetic field.

Although the dependence of the components of the resistance tensor for the case of weak fields ($r \gg l$) can be established from extremely general considerations (cf. formula (4.9)), their dependence in large fields (asymptotic behavior with magnetic field) cannot be determined from general arguments, but requires an analysis of the solutions of the kinetic equation (cf. the next sections).

5. GALVANOMAGNETIC PHENOMENA. STRONG FIELDS. CLOSED TRAJECTORIES

The development of a microscopic theory of galvanomagnetic phenomena, i.e., the calculation of the tensor of specific resistance ρ_{ik} as a function of magnetic field, is based on the solution of the linearized Boltzmann equation (cf. Sec. 1), which is conveniently used in a somewhat different form than that of Sec. 1.

The character of the motion of an electron in a constant homogeneous magnetic field shows (cf.^[3], Sec. 3) that to describe the location of the electron in momentum space it is convenient (when a magnetic field is present) to use not cartesian coordinates (p_x, p_y, p_z) but coordinates related to the trajectory of the electron in momentum space. The position of the electron can be specified by assigning it to a trajectory in momentum space, i.e., an energy ϵ and a projection of its momentum on the magnetic field p_z (as always, the homogeneous and constant magnetic field H is directed along the z axis), and also its position along the trajectory. To give the position of the electron along the trajectory one can use either the arc length s measured from some point, or (and this is more convenient) the time t for the motion of the electron from some fixed initial point to the given point. Thus, we shall describe the position of the electron by three quantities (coordinates): ϵ, p_z , and t . The formulas for transformation of coordinates (from p_x, p_y, p_z to ϵ, p_z, t) are given

by the dispersion law $\epsilon = \epsilon(p_x, p_y, p_z)$ and the equations of motion of the electron in the magnetic field:

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c} [\mathbf{v}\mathbf{H}], \quad \mathbf{v} = \frac{\partial \epsilon}{\partial \mathbf{p}}, \quad (5.1)$$

according to which

$$dt = \frac{c}{eH} \frac{ds}{v_{\perp}},$$

where v_{\perp} is the projection of the vector \mathbf{v} on the plane perpendicular to the magnetic field.

In the chosen variables the Boltzmann kinetic equation (1.6) has the form*

$$\frac{\partial f}{\partial t} \dot{t} + \frac{\partial f}{\partial p_z} \dot{p}_z + \frac{\partial f}{\partial \epsilon} \dot{\epsilon} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}}, \quad (5.2)$$

where the generalized velocities (\dot{t} , \dot{p}_z and $\dot{\epsilon}$) describe the change of state of the electron caused by external fields (electric field \mathbf{E} and magnetic field \mathbf{H}). They must be calculated using the equations of motion

$$\dot{\mathbf{p}} = e\mathbf{E} + \frac{e}{c} [\mathbf{v}\mathbf{H}]. \quad (5.3)$$

Using Eqs. (5.1) and (5.3) we easily find that

$$\dot{\epsilon} = e\mathbf{v}\mathbf{E}, \quad \dot{p}_z = eE_z, \quad \dot{t} = 1 - \frac{c}{v_{\perp}^2 H} [\mathbf{v}_{\perp}\mathbf{E}]_z, \quad (5.4)$$

where \mathbf{v}_{\perp} is a vector with components $v_x, v_y, 0$.

Using relations (5.4), Eq. (5.2) is easily linearized in the electric field and takes the following simple form:

$$\frac{\partial f_1}{\partial t} - \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}} = -e\mathbf{v}\mathbf{E} \frac{\partial n_F}{\partial \epsilon}. \quad (5.5)$$

As in Sec. 1, we introduce in place of the function f_1 a vector function ψ_1 by the relation

$$f_1 = -e \frac{\partial n_F}{\partial \epsilon} \psi_1 E_i. \quad (5.6)$$

Then ψ_1 satisfies the following equation (cf. (1.16)):

$$\frac{\partial \psi_1}{\partial t} + \hat{W}_p \psi_1 = v_i. \quad (5.7)$$

The operator \hat{W}_p is defined in Sec. 1 (cf. p. 808 and formula (1.13)).

Equation (5.7) coincides with (1.16) not merely formally but also in principle, since when spatial and temporal inhomogeneities are not considered, the quantity df/dt (cf. formula (1.12)) describes the change of the electron distribution caused only by their motion in the constant homogeneous magnetic field. As we see, Eq. (5.7) is a differential equation in t . We discuss the conditions which serve as boundary conditions. If for given ϵ and p_z the trajectory of the electron in the magnetic field is closed,

then one boundary condition is the periodicity condition

$$\psi_i(t+T) = \psi_i(t), \quad (5.8)$$

where $T = T(\epsilon, p_z)$ is the period of the Larmor precession. According to (3.12) of the first part of this survey^[3]

$$T = -\frac{c}{eH} \frac{\partial S}{\partial \epsilon},$$

where S is the area of the section of the constant-energy surface $\epsilon(\mathbf{p}) = \epsilon$ by the plane $p_z = \text{const}$.

If the trajectory is open, the boundary condition is the condition that ψ_k be finite for $t = \pm \infty$.

The presence of the factor $-\partial n_F / \partial \epsilon$ in (5.6) shows that the electrons with energies of the order of the Fermi energy play the main part in the problem. Because of this we can consider separately the cases of closed and open surfaces (cf.^[3], Sec. 2). The next section will be devoted to the latter case.

In this paragraph, therefore, we shall consider only the case of closed surfaces. More precisely, we shall treat only the case of closed trajectories. First of all, cases are possible where open surfaces have no open sections, and secondly, for certain definite directions of the magnetic field the open surfaces may have only closed sections (cf.^[3], Sec. 2).

Averaging (5.7) over t and using condition (5.8), we get

$$\overline{\hat{W}_p \psi_1} = \bar{v}_i, \quad (5.9)$$

where the bar denotes an average over the period T :

$$\bar{u} = \frac{1}{T(\epsilon, p_z)} \int_0^T u dt. \quad (5.10)$$

We shall regard Eq. (5.9) as a boundary condition imposed on the function ψ_1 .

A large value of the magnetic field manifests itself in the fact that in Eq. (5.7) the term $\partial \psi_1 / \partial t \approx \psi_1 / T$ is much larger than the "collision" term $\hat{W}_p \psi_1 \approx \psi_1 / \tau_p$. The condition $|\partial \psi_1 / \partial t| \gg |\hat{W}_p \psi_1|$ means that $T \ll \tau_p$ or $r \ll l$ (cf. Sec. 4), and allows us to use the method of successive approximations, which corresponds to an expansion in inverse powers of the magnetic field. One can show that the solution of Eq. (5.7) that satisfies the condition (5.9) can be represented as follows:

$$\begin{aligned} \psi_1 &= \overline{W}_p^{-1} \bar{v}_i + \hat{q} (\overline{W}_p^{-1} \bar{v}_i - \hat{W}_p^{-1} v_i) + \hat{q}^2 (\overline{W}_p^{-1} \bar{v}_i - \hat{W}_p^{-1} v_i) + \dots \\ &\equiv \overline{W}_p^{-1} \bar{v}_i + \frac{\hat{q}}{1 - \hat{q}} (\overline{W}_p^{-1} \bar{v}_i - \hat{W}_p^{-1} v_i), \end{aligned} \quad (5.11)$$

where the operator \hat{q} is given by the following equation*

$$\hat{q} \varphi = \overline{W}_p^{-1} \hat{W}_p \int_{-\infty}^t \overline{W}_p \varphi dt - \int_{-\infty}^t \hat{W}_p \varphi dt. \quad (5.12)$$

*In order for the expansion (5.12) to be meaningful, it is necessary that the function φ satisfying the condition $\hat{W}_p \varphi = 0$.

*In this and the following sections we shall restrict ourselves to the spatially homogeneous case, i.e., we neglect the boundaries of the sample. This means that the results obtained can apply for sufficiently massive samples. The corresponding estimates are given in Sec. 1 (cf. also^[35, 35a]). In addition we consider only the static case.

Formulas (5.11) and (5.12) provide an algorithm for calculating the functions ψ_i in the form of a power series in the inverse magnetic field. To accomplish this calculation we must know the explicit form of the operators \widehat{W}_p and \widehat{W}_p^{-1} . But for various basic conclusions, in particular, to explain the dependence of the different components of the resistance tensor on the strong magnetic field, it is sufficient to know only the topology of the Fermi surface and the neighboring constant-energy surfaces.

In the case of closed trajectories it follows from the equation of motion (5.1) that

$$\bar{v}_\alpha = 0 \quad (\alpha = x, y), \quad \bar{v}_z \neq 0. \quad (5.13)$$

Thus the expansion of ψ_α ($\alpha = x, y$) begins with terms proportional to $1/H$, while that of ψ_z starts with a term independent of the magnetic field. Using formulas (5.11) and (5.12) and the equation of motion (5.1) one can easily obtain

$$\left. \begin{aligned} \psi_x &= -\frac{c}{eH} (p_y - \widehat{W}_p^{-1} \widehat{W}_p p_y) + \dots, \\ \psi_y &= \frac{c}{eH} (p_x - \widehat{W}_p^{-1} \widehat{W}_p p_x) + \dots, \\ \psi_z &= \widehat{W}_p^{-1} \bar{v}_z + \dots \end{aligned} \right\} \quad (5.14)$$

For computing the electrical conductivity tensor σ_{ik} one must use formula (1.18), transforming from integration over p_x, p_y , and p_z to integration over ϵ, p_z , and t . The Jacobian is easily computed using the equation of motion (5.1). If we write (5.1) in components as a ratio of determinants

$$\frac{\partial(p_x, \epsilon, p_z)}{\partial(t, \epsilon, p_z)} = \frac{eH}{c} \frac{\partial(\epsilon, p_x, p_z)}{\partial(p_y, p_x, p_z)}, \quad \frac{\partial(p_y, \epsilon, p_z)}{\partial(t, \epsilon, p_z)} = -\frac{eH}{c} \frac{\partial(\epsilon, p_y, p_z)}{\partial(p_x, p_y, p_z)}$$

and cancel common factors, we get

$$\frac{\partial(p_x, p_y, p_z)}{\partial(t, \epsilon, p_z)} = \left| \frac{eH}{c} \right|. \quad (5.15)$$

Thus

$$\sigma_{ik} = -\frac{2e^2}{(2\pi\hbar)^3} \left| \frac{eH}{c} \right| \int \int \int \frac{\partial n_F}{\partial \epsilon} v_i \psi_k d\epsilon dp_z dt. \quad (5.16)$$

Using this formula for the electrical conductivity tensor, one can show that the Onsager relations in a magnetic field (4.3) do not require any other assumptions about the collision integral than those formulated in Sec. 1 (cf. Appendix IV).

We now substitute the expansion (5.14) in (5.16). We start by calculating the components σ_{xy} and σ_{yx} , where in place of v_x and v_y we substitute their values from the equation of motion (5.1) ($v_x = -(c/eH)(\partial p_y / \partial t)$, $v_y = (c/eH)(\partial p_x / \partial t)$):

$$\begin{aligned} \sigma_{xy} &= -\frac{2e^2}{(2\pi\hbar)^3} \int \frac{\partial n_F}{\partial \epsilon} d\epsilon \int dp_z \\ &\times \oint \frac{\partial p_y}{\partial t} [p_x - \widehat{W}_p^{-1} \widehat{W}_p p_x] \frac{c}{eH} dt + \dots \end{aligned}$$

From this equation* it is clear that the term contain-

*We shall not give the obvious values for the integration limits. We note that, by assumption, we are dealing with closed trajectories.

ing $\widehat{W}_p^{-1} \widehat{W}_p p_x$ drops out because of the t integration, and

$$\oint \frac{\partial p_y}{\partial t} p_x dt = \int_{(\epsilon, p_z)} p_x dp_y = \pm S(\epsilon, p_z). \quad (5.17)$$

Here $S(\epsilon, p_z)$ is the area of the section of the constant-energy surface $\epsilon(\mathbf{p}) = \epsilon$ by the plane $p_z = \text{const}$. The sign is determined by the direction of circulation, i.e., by the sign of the effective mass m^* (cf. [3], Sec. 3). Thus, asymptotically in high fields

$$\sigma_{xy} = -\frac{ec}{H} \frac{2}{(2\pi\hbar)^3} \int \frac{\partial n_F}{\partial \epsilon} d\epsilon \int dp_z \{S_1(\epsilon, p_z) - S_2(\epsilon, p_z)\}, \quad (5.18)$$

where the first integral is taken over those parts of the energy bands where $m^* > 0$, and the second over those where $m^* < 0$. We also assume a summation over all partially filled bands.

If we assume that $-\partial n_F / \partial \epsilon = \delta(\epsilon - \epsilon_F)$, then

$$\sigma_{xy} = \frac{ec}{H} \frac{2}{(2\pi\hbar)^3} \int \{S_1(\epsilon_F, p_z) - S_2(\epsilon_F, p_z)\} dp_z \quad (5.19)$$

and, finally,

$$\sigma_{xy} = \frac{2ec}{(2\pi\hbar)^3 H} \{V_1(\epsilon_F) - V_2(\epsilon_F)\}. \quad (5.20)$$

Here $V_1(\epsilon_F)$ is the volume bounded by that portion of the many-sheeted Fermi surface within which the energy is less than the Fermi energy ($m^* > 0$), while $V_2(\epsilon_F)$ is the volume bounded by that part of the Fermi surface within which the energy is greater than the Fermi energy ($m^* < 0$).

If the closed trajectories are located on an open Fermi surface, $V_1(V_2)$ is the volume bounded by the Fermi surface and the boundaries of the reciprocal lattice cell (Fig. 5).

Noting that at zero temperature the electrons occupy all states with energy less than ϵ_F , we find

$$\sigma_{xy} = \frac{ec(n_1 - n_2)}{H}, \quad (5.21)$$

where n_1 is the number of occupied electron states with positive effective mass, while n_2 is the number of empty electron states with negative effective mass. It is natural to call n_1 the number of "electrons" and n_2 the number of "holes."

Thus, if the Fermi surface is closed (or, in the case of an open Fermi surface, if for all p_z there are only closed sections) the asymptotic form of σ_{xy} is determined by (5.21). We emphasize that in this case σ_{xy} does not depend on either the direction of the magnetic field or on the nature of the collisions of the conduction electrons, but is determined only by

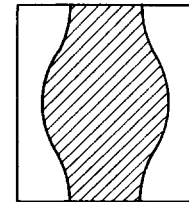


FIG. 5

the numbers of electrons and holes. The component $\sigma_{yx} = -\sigma_{xy}$. This statement is proved by direct calculation, but also follows from the Onsager relation (as we shall see, the symmetric part of the tensor is asymptotically much smaller: it is proportional to $1/H^2$).

One should treat specially the case when there are equal numbers of "electrons" and "holes" ($n_1 = n_2$). In this case the expansion of the components σ_{xy} , σ_{yx} starts with terms proportional to $1/H^2$. At the same time the expansion of the antisymmetric part starts with terms $\sim 1/H^3$. By a suitable choice of the x and y axes one can make the symmetric part of the tensor σ_{ik} (s_{ik}) diagonal; in this coordinate system the expansion of σ_{xy} and σ_{yx} starts with terms proportional to $1/H^3$.

We note that equality of the numbers of "electrons" and "holes" is nothing exceptional: all metals with an even number of electrons per unit cell have conduction electrons only because the energy bands happen to overlap. The number of freed states ("holes") in the lowest band is equal to the number of occupied states in the upper band ("electrons"). The concepts of "electron" and "hole" are completely unambiguous if the Fermi surface is closed. For an open surface one should speak of "electron" or "hole" orbits, depending on the direction of motion along them. If the overlap of the energy bands is small (Fig. 6), as a rule the constant-energy surfaces are closed. This seems to be the structure of the energy spectrum in Bi, As, and Sb.

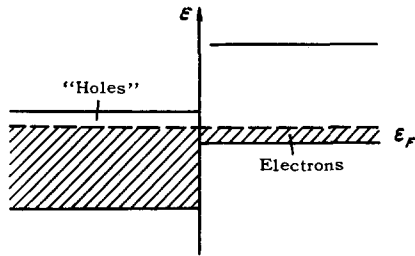


FIG. 6

Strictly speaking, complete compensation ($n_1 = n_2$) is possible only at absolute zero, and also only for an ideally pure metal. Thermal excitation leads to the participation in the galvanomagnetic effects of states with open surfaces (as is the case for the conductivity; cf. Sec. 2 and also^[37]). Thus, although the number of electrons in the upper band is equal to the number of electrons in the lower band, the equality $n_1 - n_2 = 0$ may be violated. It is obvious that at sufficiently low temperatures

$$n_1 - n_2 = \Delta n \sim ne^{-\frac{\Delta\epsilon}{T}},$$

where n is the number of electrons in the band ($\approx n_1, n_2$) while $\Delta\epsilon = |\epsilon_F - \epsilon_c|$ (ϵ_c is the critical energy at which constant-energy surfaces change their topology).

The presence of impurities increases either the number of electrons (donor impurity) or the number of holes (acceptor impurity) in proportion to the concentration, making the quantity Δn

different from zero. Concerning the dependence of the tensor components σ_{ik} on magnetic field in this case, cf. below.

We turn to the computation of the components of the tensor σ_{ik} . Proceeding just as before, we find that the expansion of σ_{xx} , σ_{yy} starts with terms $\sim 1/H^2$, and that of σ_{xz} , σ_{yz} with terms $\sim 1/H$; σ_{zz} tends to a constant as the field goes to infinity. Thus the electrical conductivity tensor in strong fields has the form ($n_1 \neq n_2$)

$$(\sigma_{ik}) = \begin{pmatrix} \frac{a_{xx}}{H^2} & \frac{ec(n_1 - n_2)}{H} & \frac{a_{xz}}{H} \\ -\frac{ec(n_1 - n_2)}{H} & \frac{a_{yy}}{H^2} & \frac{a_{yz}}{H} \\ \frac{a_{zx}}{H} & \frac{a_{zy}}{H} & a_{zz} \end{pmatrix}, \quad (5.22)$$

where the components of the matrix a_{ik} tend to constants. In especially symmetric cases, individual components of the matrix a_{ik} may go to zero. For example, in the isotropic case $a_{\alpha z} = 0$ while $a_{zz} = \sigma$; σ is the value of the electrical conductivity in the absence of the field.

From (5.22) it is evident that in those cases where the Fermi surface is closed, all components of the tensor σ_{ik} except σ_{zz} go to zero as the magnetic field goes to infinity; the way the different components approach zero will be different.

Taking account of the smearing of the Fermi levels may lead to a change in the results obtained here, but the corrections are negligibly small for actual fields. Furthermore, much earlier (at smaller fields) we must take account of the quantization of the orbits in the magnetic field (cf.^[38]), before we have to consider the temperature smearing.

For comparison we give the expression for the electrical conductivity tensor for the case of a quadratic isotropic dispersion law in the τ -approximation (magnetic field arbitrary):

$$(\sigma_{ik}) = \begin{pmatrix} \frac{\sigma}{1 + \Omega^2\tau^2} \pm \sigma \frac{\Omega\tau}{1 + \Omega^2\tau^2} & 0 \\ \mp \sigma \frac{\Omega\tau}{1 + \Omega^2\tau^2} & \frac{\sigma}{1 + \Omega^2\tau^2} \\ 0 & 0 & \sigma \end{pmatrix}. \quad (5.23)$$

Here $\Omega = eH/m^*c$ is the Larmor frequency, the "+" sign corresponds to an electron band, the "-" sign to a hole band, and $\sigma = ne^2\tau/|m^*|$. If electrons from several bands participate in the conduction, the conductivity is the sum of similar tensors.

In some cases it is convenient to have a more general expression for the electrical conductivity tensor, valid over the whole range of variation of the magnetic field. If we keep to the τ -approximation ($\hat{W}_p \{ \psi_i \} = \psi_i/\tau$), from (5.7) using the condition (5.8) it follows that

$$\psi_i = \int_0^\infty v_i(t-t') e^{-\frac{t-t'}{\tau}} dt'. \quad (5.24)$$

We note that this solution is also valid in the case of open trajectories. It is used, for example, for calculating the dependence

of the transverse resistance of noble metals on the magnetic field.^[39] In^[36] a method is also given for solving (5.7) that is valid for an arbitrary collision integral and for any value of magnetic field, but is restricted to the case of closed trajectories. The solution is obtained in the form of a series.

Substituting (5.24) in (5.16) for σ_{ik} , we find*

$$\sigma_{ik} = -e^2 \int \varphi_{ik}(\epsilon, p_z) \frac{\partial n_F}{\partial \epsilon} d\Gamma, \quad (5.25)$$

where

$$\varphi_{ik} = \int_0^\infty \overline{v_i(t) v_k(t-t')} e^{-\frac{t}{\tau}} dt'. \quad (5.26)$$

The bar denotes an average over a period in the case of closed trajectories (cf. 5.10). In the case of open trajectories the average is understood in the sense of a limit as $T \rightarrow \infty$.

Expressions (5.25) and (5.26) are convenient for investigating specific dispersion laws.

In experimental studies of galvanomagnetic phenomena, one usually transmits through the crystal a current of given magnitude and direction, measures the components of the electric field in three non-coplanar directions (possibly three orthogonal directions; cf, for example,^[41]). This means that the quantity studied is the tensor of the specific resistance in the magnetic field $\rho_{ik}(\mathbf{H})$, which is the inverse of the electrical conductivity tensor ($\rho_{ik} = \sigma_{ik}^{-1}$).

Using the expressions found earlier for the asymptotic form of the electrical conductivity tensor $\sigma_{ik}(\mathbf{H})$, one easily finds the asymptotic form of the resistance tensor in strong fields.

If the number of electrons is not equal to the number of holes[†] ($n_1 \neq n_2$), then according to (5.22)

$$\rho_{ik}(\mathbf{H}) = \begin{pmatrix} b_{xx} & \frac{H}{(n_1 - n_2)ec} b_{xz} & \\ -\frac{H}{(n_1 - n_2)ec} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{pmatrix}, \quad (5.27)$$

where the components of the matrix b_{ik} tend to constants as $H \rightarrow \infty$. They can be determined in terms of the asymptotic values of the components of the matrix a_{ik} .^[36] If the measured current j is perpendicular to the magnetic field, then the x axis can be taken along the current direction. Then the resistance ρ (ratio of electric field along the current to the current density) is ρ_{xx} (in general $\rho = \rho_{ik} j_i j_k / j^2$), and the quantity ρ_{yx}/H coincides with the Hall "constant" $R = E_y/Hj$. Thus, in this case (closed trajectories, $n_1 \neq n_2$) the resistance always tends to saturation independent of the directions of the magnetic field or the current measured. The asymptotic value of the resistance naturally depends on the directions of the magnetic field and current by virtue of the anisotropy of the matrix b_{ik} , which is related to the

anisotropy of the dispersion law. In Appendix IV it is shown that $\rho(H \rightarrow \infty)$ is always greater than $\rho(H = 0)$, and this conclusion is based only on the fact that the collision operator is independent of the magnetic field. The last statement does not hold in the quantum case.^[38]

The asymptotic value of the Hall "constant" R_∞ is an important characteristic of the electron energy spectrum of a metal. If the Fermi surface is closed (or, in the case of an open Fermi surface, if for all p_z there are only closed sections),

$$R_\infty = \frac{1}{(n_1 - n_2)ec} \quad (5.28)$$

and is independent of the direction of the magnetic field.

Investigations in recent years have shown that various metals have a closed Fermi surface with unequal numbers of "electrons" and "holes"* (In,^[42] Al,^[32] the alkaline metals K and Na^[43]). If the Fermi surface is closed and the number of electrons is equal to the number of holes ($n_1 = n_2$), then because $\sigma_{xy} \sim 1/H^2$ all the components transverse to the magnetic field of the tensor ρ_{ik} tend to infinity, where

$$\rho_{\alpha\beta} \sim H^2, \quad \rho_{\alpha z} \sim H \quad (\alpha, \beta = x, y) \quad (5.29)$$

and only the longitudinal component ρ_{zz} tends to a constant.

From this we see that in this case the resistance for any direction of the magnetic field tends to infinity, so long as the current has a component transverse to the magnetic field.

The Hall field is not the main part of the asymptotic form for the electric field perpendicular to the current. In any direction there is a term quadratic in the magnetic field which characterizes the anisotropy of the resistance (difference in principal values ρ_{xx} and ρ_{yy}).

The Hall "constant" in strong fields R_∞ (the limiting value of the quantity $[\rho_{xy}(\mathbf{H}) - \rho_{xy}(-\mathbf{H})]/2H$ for $H \rightarrow \infty$) depends on the direction of the magnetic field, and is determined not only by the energy

*Chambers^[40] gives almost the same form for the solution of the kinetic equation.

†Remember that we are considering the case of closed trajectories.

*The investigations of Al give very convincing proof that arguments about the character of the carriers in metals can be applied only to measurements of R_∞ and not to the Hall "constant"; in weak fields the Hall constant R at high temperatures has the sign corresponding to electrons, while at low temperatures (i.e., in high fields) the sign is that for holes. According to the Harrison model, the main region of the Fermi surface has hole character. The question of the bands with small numbers of carriers in Al is not completely clear. It is possible that for definite directions of the magnetic field there are open sections whose presence results in singularities in the dependence of the resistance on magnetic field. The Fermi surface of In and Al has recently been investigated carefully in the work of Gantmakher, Krylov^[103] and Vol'skii.^[104] Cf. also^[105], where the parameters of the Bi spectrum are found from the experimental data.

spectrum but also by the nature of the scattering of the conduction electrons.

We note that when $n_1 \neq n_2$ the Hall vector is very accurately parallel to the magnetic field, while when $n_1 = n_2$ all the components of the Hall vector are of the same order.

The quadratic increase in resistance for any direction of the magnetic field has been clearly established on various metals (Bi, As, Sb^[44]). This result is in beautiful agreement with our notions about the character of the energy spectra of these metals.^[16,45]

If the difference between the numbers of electrons and holes is very much less than the number of electrons ($n_1 \approx n_2$, $|n_1 - n_2| \ll n_1$), the dependence of the components of the tensor ρ_{ik} on magnetic field can be studied in somewhat more detail than was done above. According to [46], the resistance and the Hall "constant" have the following dependence on magnetic field (for $H \gg H_0$, where H_0 is the magnetic field for which the free path is equal to the electron orbit radius, $r = l$):

$$\begin{aligned} \varrho &\approx \varrho' \frac{(H/H_0)^2}{1 + \left(\frac{\Delta n}{n} \frac{H}{H_0}\right)^2}, \\ R &\approx \frac{1}{nec} \frac{a + (\Delta n/n)(H/H_0)^2}{1 + \left(\frac{\Delta n}{n} \frac{H}{H_0}\right)^2}. \end{aligned} \quad (5.30)$$

The parameters ρ' , a , H_0 depend on the angle between the direction of the magnetic field and the crystal axes; $a \approx 1$.

Formulas (5.30) are in good agreement with the experimental results of N. E. Alekseevskii, N. B. Brandt, and T. I. Kostina,^[47] who made measurements on Bi containing small amounts of impurities.

Using primitive model assumptions about the conduction electrons, one can get compact formulas describing the behavior of the resistance over a wide range of fields. Thus, if one assumes that there are two bands with quadratic anisotropic dispersion laws, with mobility tensors proportional to one another ($u_{ik}^{(2)} = ku_{ik}^{(1)}$) and with the number of electrons equal to the number of holes ($n_1 = n_2$), an elementary calculation leads to the following relations:

$$\varrho = \frac{\sigma_2 \cos^2 \alpha + \sigma_1 \sin^2 \alpha}{\sigma_1 \sigma_2} \left(1 + \frac{H^2}{H_0^2}\right), \quad R = -\frac{1}{nec} \left(\frac{1-k}{1+k}\right). \quad (5.31)$$

Here $H_0 = c/e \sqrt{u_1^{(1)} u_1^{(2)}}$, σ_1, σ_2 are the principal values of the conductivity tensor, $\sigma_{ik}^2 = ne^2 u_{ik}$ ($1+k$); the magnetic field is along one of the principal axes (the third); the current is perpendicular to the magnetic field; α is the angle between the first axis and the current.

We note that in this simple model one gets a single quadratic dependence for the resistance and no field dependence of the Hall "constant." This is particular to the two-band model with a quadratic dispersion law. The introduction of a third band drastically changes this state of affairs. One can show (with the same assumptions about the dispersion laws) that the coefficients of H^2 in weak ($r \gg l$) and strong ($r \ll l$) fields are different, where, in agreement with experiment,^[49] the coefficient of H^2 in weak fields is always greater than that for strong fields. The Hall "constants" in weak and strong fields also are different.

Although the introduction of several bands enables one to give a fair description of the experimental curves obtained using a wide range of fields,^[50] it is clear that formulas obtained using very special model assumptions cannot be used for determining quantitative parameters for metals.

The galvanomagnetic characteristics (more precisely, their asymptotic values in strong fields) are extremely sensitive to the nature of the energy spectrum and, as we shall see, are a superb instrument for determining the topology of the Fermi surface.

For metals with closed Fermi surfaces, the galvanomagnetic characteristics enable one, first of all, to establish the fact that they are closed, and second, to determine the difference in numbers of electrons and holes ($n_1 - n_2$).

The most important information about the character of the electron spectrum can be gotten by studying metals with open Fermi surfaces (cf. Sec. 6).

In concluding this section we should like to mention an interesting nonlinear galvanomagnetic effect (in the sense of a deviation from Ohm's law) which was discovered by Esaki, and which consists in the fact that at a definite value of the electric field the resistance of Bi rises sharply (cf. [51], Fig. 1). As shown by the author, this rise in resistance is explained as follows. We know that in crossed electric and magnetic fields electrons drift with a velocity $v_{dr} = cE/H$. When v_{dr} , as a result of increase of the electric field, becomes larger than the sound velocity s ($v_{dr} > s$), the electrons begin to radiate phonons ("Cerenkov" radiation of sound waves), which leads to an increase of the resistance.*

6. GALVANOMAGNETIC PHENOMENA. STRONG FIELDS. OPEN TRAJECTORIES

In the preceding section we investigated in detail the galvanomagnetic phenomena for metals having a closed Fermi surface. In this section we consider the case of open Fermi surfaces. The most significant feature of the galvanomagnetic properties of metals with open Fermi surfaces is their marked anisotropy.^[55] For some directions of the magnetic field the transverse resistance in strong fields ($r \ll l$) tends to saturation, while it increases quadratically for others. This property is used for studying the topology of the Fermi surface.^[54]

One can demonstrate the marked anisotropy of the resistance for the simplest example of an open Fermi surface. Consider a metal for which the Fermi surface (and the constant energy surfaces touching it) is a crimped cylinder (Fig. 7). We take the axis of the cylinder as the x axis. As usual the magnetic field is along the z axis.

If the magnetic field is not perpendicular to the axis of the cylinder, all the cross sections are closed. We treated this case in Sec. 5. It is true that a special treatment is required when the field is almost perpendicular to the open direction, but the electron trajectories are still closed, but very

*There is now a large literature on the Esaki effect; cf., for example,^[55] and the bibliography given there.



FIG. 7. Constant-energy surface of the "crimped cylinder" type.

elongated, so that their length is much greater than the size of the cells of the reciprocal lattice. We shall consider this situation below.

In solving the kinetic equation for the case of open surfaces we must consider various points. In the first place, the characteristic time of the motion (for determining the order of magnitude of $|\partial\psi_i/\partial t|$ in Eq. (5.7)) should be taken to be the time during which the momentum of the electron changes by an amount of order $|2\pi\hbar\mathbf{b}|$, where \mathbf{b} is the reciprocal lattice vector in the direction of opening (along the x axis in the present case); the magnetic field will be considered to be large if the relaxation time is much greater than this time (we note that from the equation of motion (5.1) it is clear that this relaxation time is inversely proportional to the magnetic field and of the order of the period of revolution of the electron T). Secondly, the boundary conditions should be made more precise. As stated in Sec. 5, the boundary conditions are provided by the requirement that the function ψ be finite for large values of the "time" t (for $t \rightarrow \pm\infty$). If we generalize the definition of the average value of a function by setting

$$\bar{u} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T u(t) dt, \quad (6.1)$$

the boundary condition for the function ψ_i is written formally the same as for the case of closed trajectories:

$$\overline{\hat{W}_p \{\psi_i\}} = \bar{v}_i. \quad (6.2)$$

Thirdly, the average value of the velocity components transverse to the magnetic field is zero. For our choice of coordinate system

$$\bar{v}_y = -\frac{c}{eH} \frac{\partial \bar{p}_x}{\partial t} \neq 0, \quad (6.3)$$

while

$$\bar{v}_x = \frac{c}{eH} \frac{\partial \bar{p}_y}{\partial t} = 0. \quad (6.4)$$

We make the last statement more precise: there are values of p_z ($|p_z| < p_1$; Fig. 8) for which the cross sections are open and $\bar{v}_y \neq 0$.

The formal solution (5.11) of Eq. (5.7) is valid also for the case of the crimped cylinder, but the expansion of ψ in inverse powers of the field now has the form (cf. (5.14)):

$$\left. \begin{aligned} \psi_x &= -\frac{c}{eH} (p_y - \overline{\hat{W}_p^{-1} \overline{\hat{W}_p p_y}}) + \dots, \quad \psi_z = \overline{\hat{W}_p^{-1} v_z} + \dots, \\ \psi_y &= \begin{cases} \overline{\hat{W}_p^{-1} v_y} + \dots & (|p_z| < p_1), \\ \frac{c}{eH} (p_x - \overline{\hat{W}_p^{-1} \overline{\hat{W}_p p_x}}) + \dots & (|p_z| > p_1). \end{cases} \end{aligned} \right\} \quad (6.5)$$

Using these asymptotic values of ψ_i , one can calculate the asymptotic values of the components of the electrical conductivity tensor

$$(\sigma_{ik}) = \begin{pmatrix} \frac{a'_{xx}}{H^2} & \frac{a'_{xy}}{H} & \frac{a'_{xz}}{H} \\ \frac{a'_{yx}}{H} & a'_{yy} & a'_{yz} \\ \frac{a'_{zx}}{H} & a'_{zy} & a'_{zz} \end{pmatrix}, \quad (6.6)$$

where the expansion of the matrix elements a_{ik} in inverse powers of the field starts with the nonzero term. Inverting this tensor, one easily finds the asymptotic value of the resistance tensor

$$(\rho_{ik}) = \begin{pmatrix} H^2 b'_{xx} & H b'_{xy} & H b'_{xz} \\ H b'_{yx} & b'_{yy} & b'_{yz} \\ H b'_{zx} & b'_{zy} & b'_{zz} \end{pmatrix}; \quad (6.7)$$

the elements of the matrix b'_{ik} are determined from the elements of the matrix a_{ik} .

Because $\bar{v}_y \neq 0$, a'_{xy} and b'_{xy} depend not only on the energy spectrum but also on the form of the collision integral (cf. the derivation of formula (5.21)). Let us analyze these results.

One of the components of the transverse resistance (ρ_{xx}) increases quadratically with increasing magnetic field, while the other (ρ_{yy}) tends to saturation. We note that the resistance in the open direction (along the x axis) increases with magnetic field. The difference in asymptotic behavior between ρ_{xx} and ρ_{yy} shows that there is a marked dependence of the resistance on the direction of the current \mathbf{j} (even when $\mathbf{j} \perp \mathbf{H}$). For an arbitrary current direction the resistance $\rho = \rho_{ijk} j_i j_k / j^2$ increases quadratically, but for a current along the y axis it tends to saturation.

If, for a metal whose Fermi surface is a crimped cylinder, we examine the dependence of the resistance on the direction of the strong magnetic field ($H \gg H_0$ or $r \ll l$) for an arbitrary direction of the current, we should observe the following picture: for an arbitrary direction of the magnetic field the resistance tends toward saturation, for particularly selected ones (magnetic field perpendicular to the

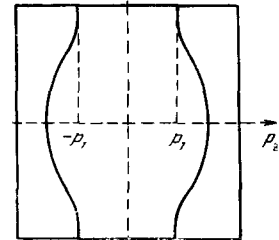


FIG. 8

open direction) it increases quadratically. Analyzing the equation for the function ψ_i , one can show^[55] that the transition from one dependence to the other occurs in a narrow range of angles $\Delta\theta$ of order $H_0/H \ll 1$. This is apparent from the fact that for elongated trajectories the period of revolution of the electron is inversely proportional to the angle between the magnetic field and the plane perpendicular to the open direction.

We note that except for the marked angular dependence described here, the resistance and other galvanomagnetic properties show a smooth dependence on the direction of the magnetic field. It can be explained only by a numerical solution of the kinetic equation and will not be discussed here.

In^[55] the analytic dependence of the resistance on the magnetic field direction was found; it had the following form:

$$\varrho = \frac{\beta H^2 \cos^2 \alpha}{\vartheta^2 H^2 + \lambda^2 H_0^2} C(\eta) + A. \quad (6.8)$$

Here β , λ and A are smooth functions of angle ($\lambda \sim 1$) $\eta = H/H_0\vartheta$, while $C(\eta)$ is a smooth function of its argument, and $C(0) = C(\infty) = 1$; α is the angle between the current direction and the x axis. We recall that in this formula ϑ is the angle between the magnetic field and the plane perpendicular to the open direction ($\vartheta \ll 1$).

Figure 9 shows schematically the dependence of the resistance on the magnetic field direction ϑ , corresponding to formula (6.8).

Practically always the direction of the current is determined by the geometry of the sample (its axis), while the magnetic field rotates in the plane perpendicular to the current. Knowing the direction of the current relative to the crystal axes, and the direction of the magnetic field for which the resistance increases quadratically, one can determine the plane in which the open direction lies. Thus, for a unique determination of the open direction one needs experiments on at least two samples with different current directions. In practice the problem of determining the open direction is more complicated than may appear from this description, since we are required not only to determine the open direction, but also to find out the type of the Fermi surface.

We also mention the following point. If the Fermi surface is almost closed, i.e., the bridges are very narrow (Fig. 10), which

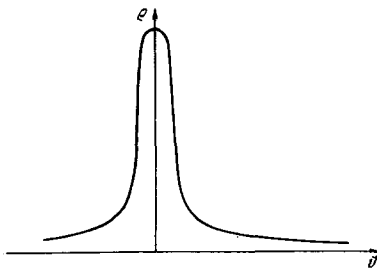


FIG. 9

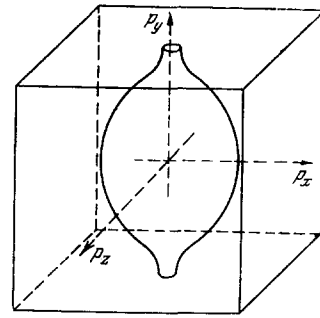


FIG. 10

in our case means that the quantity p_i is small ($\nu = p_i/2\pi\hbar b \ll 1$, where b is the size of the reciprocal lattice cell in the plane perpendicular to the open direction), the dependence of the resistance on the strong magnetic field ($H \gg H_0$) is somewhat more complicated. It is easy to show that the resistance in the open direction depends on the strong magnetic field as follows:

$$\varrho_{xx} = b'_{xx} \left(1 + \nu \frac{H^2}{H_0^2} \right), \quad (6.9)$$

i.e., up to magnetic fields of order $H_0/\sqrt{\nu} \gg H_0$ one will not be aware of the openness of the Fermi surface.

A few words about the Hall "constant." For an arbitrary direction of the magnetic field (not close to the y, z plane, perpendicular to the open direction) the Hall "constant" is isotropic and determined by the volume V' contained inside the part of the Fermi surface which is located within one cell of the reciprocal lattice (for definiteness, we are assuming that with increasing energy the crimped cylinder becomes larger):

$$R_{\infty} = \frac{1}{n'ec}, \quad n' = \frac{2V'}{(2\pi\hbar)^3}.$$

For a direction of the magnetic field close to the y, z plane, the value of the Hall "constant" may change markedly. It depends on the collision integral (this is related to the fact that $\partial \overline{p_x} / \partial t \neq 0$, so that $\widehat{W}_p^{-1} \widehat{W}_p p_y$ does not drop out of the expression for σ_{xy}). Figure 11 shows the angular dependence of the resistance and the Hall "constant" for metals for which the Fermi surface is a crimped cylinder.

The dependence of the resistance on magnetic field direction described here also occurs when the Fermi surface splits up into a "crimped cylinder" and an

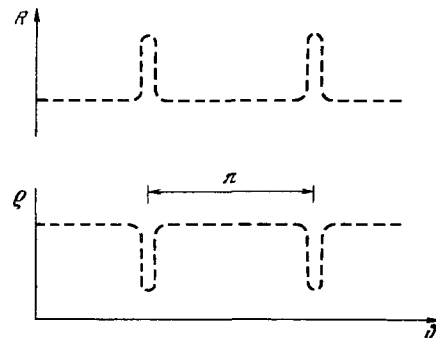


FIG. 11

arbitrary number of closed regions, so long as there is a compensation of volumes, i.e., so long as the expansion of σ_{xy} for an arbitrary direction of the magnetic field does not start with a term of order $1/H^3$. In the latter case the resistance increases quadratically for an arbitrary direction of the magnetic field (just as for a metal with the number of electrons equal to the number of holes, $n_1 = n_2$).

In treating the galvanomagnetic properties of metals for which the Fermi surface is a crimped cylinder we have shown that a marked dependence on magnetic field direction is observed for those cases where the topology of the electron trajectory changes in the magnetic field. In the case considered, the singular directions are those for which the magnetic field is perpendicular to the cylinder axis. On a stereographic projection* this corresponds to a circle of unit radius (cf. Fig. 12, where the polar axis coincides with the open direction).

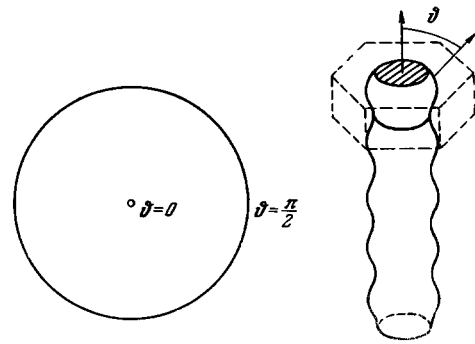


FIG. 12

Now let us consider an open surface of the type of the "space grid." In this case open trajectories occur very frequently. Figure 13 shows stereographic projections of the magnetic field direction (shaded regions and solid lines) for which open trajectories occur ($\epsilon = \epsilon_F, p_z = \text{const}$). The different pictures correspond to different types of Fermi surfaces. [55-57]

Examination of the stereographic projections (cf. Fig. 13) shows that there are different regions of magnetic field directions: a) in which there are no open trajectories (unshaded), b) in which there are layers of open trajectories with the same opening direction (shaded) and c) in which there are layers of open directions with different directions of opening (shaded twice). In addition there are at least four types of singular magnetic field directions:

1. Magnetic field directions for which there is a layer of open trajectories that form a single manifold. This occurs in particular, if there is an isolated direction of the open trajectories. Examples are the directions perpendicular to the axis for a surface of the "crimped cylinder" type, the solid lines on the stereographic projections for the case of a surface of the "space grid" type.

2. Boundaries of two-dimensional regions (solid angles) in which there are open trajectories (boundaries of the shaded regions in Fig. 13).

3. An isolated direction within the region of open trajectories, in which the layer of open trajectories degenerates into an isolated cross section (the points in Fig. 13). As a rule this direction coincides with the direction of a 3-, 4-, or 6-fold axis.

*Stereographic projection is the mapping of a hemisphere of unit radius on a plane, where the position of a point is given by the polar coordinates ρ and χ . The correspondence with the points on the hemisphere is given by the following equations:

$$\rho = 2\theta/\pi, \quad \chi = \varphi,$$

where θ and φ are the azimuthal and polar angles, respectively.

4. The boundary of the region of those magnetic field directions for which there are layers of open trajectories with different opening directions (boundaries of twice-shaded regions in Fig. 13).

Along directions of the magnetic field other than the singular ones, one should find the following situation:

a) In unshaded regions (regions of closed cross sections) the resistance tends toward a constant value (saturation), and the Hall "constant" is determined by the volume occupied by the electrons, either free electrons or "holes" (cf. Sec. 5).

b) In a once-shaded region (these are layers of open trajectories with a common opening direction) the resistance increases proportionally to the square of the magnetic field (except in the case where the electric current which is perpendicular to the magnetic field, is perpendicular to the opening direction); the Hall "constant" depends in a complicated way on the collision integral (cf. p. 828).

c) In the twice-shaded regions (where there are layers of open trajectories with different opening directions) the resistance tends toward a constant value (saturation), the Hall "constant" depends on

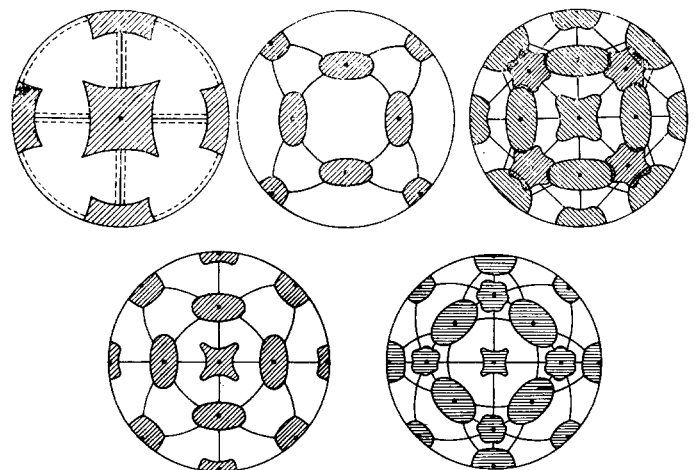


FIG. 13. Stereographic projections of magnetic field directions (shaded regions and solid lines) leading to open trajectories, for different forms of the Fermi surface of the type of the "space grid."

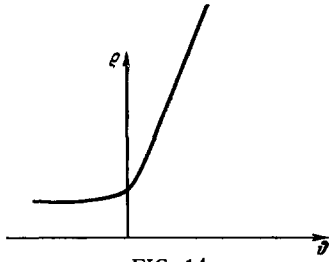


FIG. 14

the collision integral and is very small (for $H \rightarrow \infty$ it tends to zero).

Now let us consider the dependence of the resistance on the magnitude and direction of the magnetic field near the singular directions.

1. Near the directions indicated by the heavy lines in Fig. 13, the situation is completely analogous to that which occurs for a Fermi surface of "crimped cylinder" type near the plane perpendicular to the axis of the cylinder. In particular, formula (6.8) is valid. In this case the angle ϑ measures the "distance" to the heavy lines on the stereographic projections.

2. Considering the behavior of the resistance near the boundaries of the two-dimensional regions in which there are open trajectories, we must consider that in the unshaded regions there are only closed trajectories, while in the shaded regions there are open trajectories, whose number (width of layer) vanishes at the boundary. In this connection, outside the shaded region the standard dependence of resistance on magnetic field (saturation) is established out to the boundary, while if we approach the boundary from the direction of the shaded region the dependence of the resistance on the magnitude and direction of the magnetic field has the following form:

$$\rho = \vartheta\beta \left(\frac{H}{H_0} \right)^2 \cos^2 \alpha + A \quad (6.10)$$

(the notation is the same as in (6.8)). Figure 14 shows the dependence on magnetic field direction near the singular directions.

3. An analogous situation also occurs near those

isolated directions in which the layers of open trajectories degenerate to isolated sections (the points in Fig. 13). In particular, formula (6.10) retains its validity. Figure 14 shows the dependence on magnetic field in this case.

When the magnetic field is rotated in a plane, in various cases one can get a rosette shape of the resistance (dependence of resistance on direction for a fixed magnitude of the magnetic field) which has all three types of singularities.

For the dependences considered here it is characteristic that for a particular direction of the current ($\alpha = \pi/2$) the anomaly in the variation of the resistance vanishes.

4. On the boundaries of the twice-shaded regions the resistance goes over from saturation (within the region) to a quadratic dependence (outside the region). The analytic expression for the resistance near the boundary has the following form:

$$\rho = \frac{\rho'}{\vartheta A + (H_0/H)^2}, \quad (6.11)$$

where ρ' and A are smooth functions of angle, and $A \sim 1$.

As we see the study of an open Fermi surface of the "space grid" type leads to the most varied situations, and shows that the comparison of the asymptotic variation of the resistance with magnetic field for different directions with the theoretical "pictures", i.e., with stereographic projections like those in Fig. 13, allows one not only to determine the type of the Fermi surface, but also to study its form in considerable detail.

A similar analysis can also be carried out for surfaces of other types. For example ref.^[58] explains the dependence of the resistance on the magnitude and direction of the magnetic field for metals whose Fermi surfaces are "plane grids" (Fig. 15).

In all cases the characteristic properties of metals with an open Fermi surface are, first, marked anisotropy of the resistance transverse to the magnetic field, and, second, a strong dependence of the resistance on the direction of the electric current (when $j \perp H$).

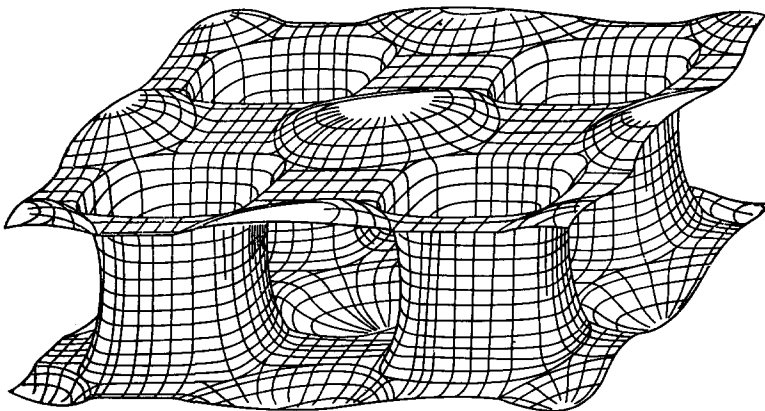


FIG. 15. Open Fermi surface of tin.^[58]

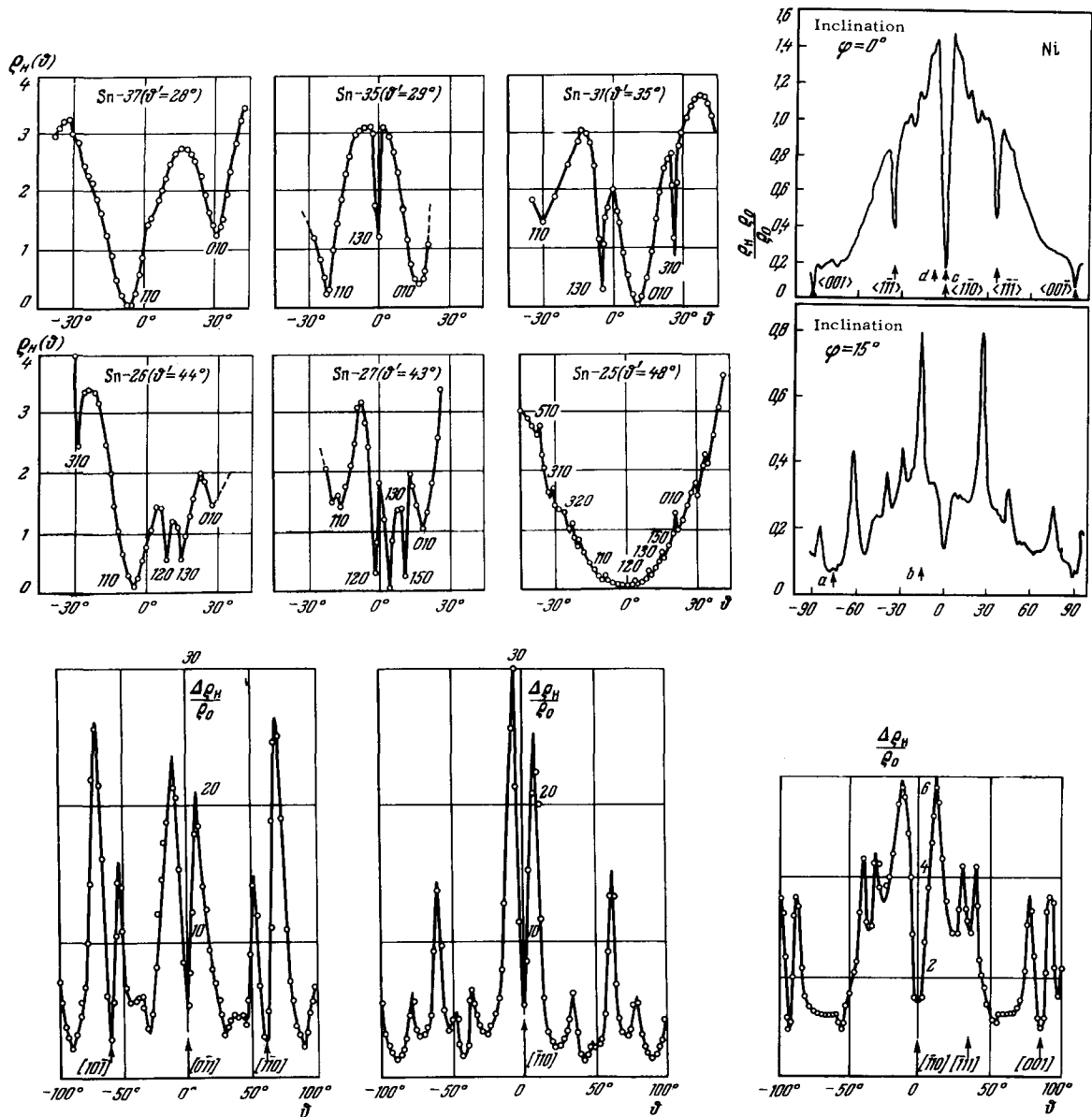


FIG. 16. Characteristic angular dependences of resistance in strong magnetic field (the lower graphs are for Au).

So far we have considered cases where in the unshaded regions, i.e., for those directions of magnetic field for which there are no open sections, the resistance tends to a constant. This means that there is no compensation of hole and electron volumes. If there is such a compensation, then in the unshaded regions one finds a quadratic increase of resistance (just as in the shaded regions). On the boundaries of these regions the resistance shows a jump. Similar jumps are also seen for all singular directions. This case has been studied in detail, in connection with studies of the galvanomagnetic properties of tin,* in

*The Fermi surface of tin consists of an open surface of "plane grid" type (cf. Fig. 15) and several closed regions, whose shape cannot be studied by the methods described here. To study the shape of these closed regions one uses resonance, magnetoacoustic, and other methods.

the work of N. E. Alekseevskii, Yu. P. Gaïdukov, I. M. Lifshitz, and V. G. Peschanskiĭ.^[58] The characteristic stereographic projections of the "plane grid" type are shown in Fig. 17. We also mention that, just as in the case of closed surfaces, the compensation of hole and electron volumes is not a random phenomenon. It should occur whenever the number of electrons per unit cell is even.

In recent years there has been a great deal of work on the comparison of experimental results on the dependence of resistance on magnetic field with different models of the Fermi surface. Figure 16 shows the results of experiments on anisotropy of resistance.

The dependence of the Hall "constant" on magnetic field direction for metals with an open Fermi surface has a distinctive character (cf. also Fig. 11), which

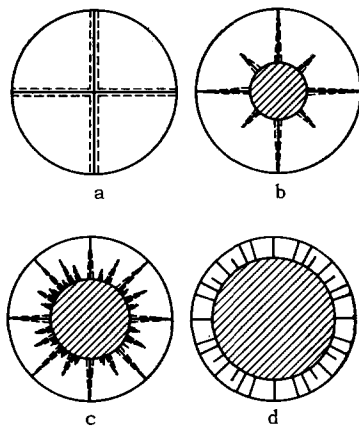


FIG. 17. Stereographic projections of singular directions of the magnetic field for constant-energy surfaces of the "plane grid" type (cf. Fig. 15).

allows one to use measurements of the Hall effect for studying the structure of electron energy spectra.^[59]

In particular, from measurements of the Hall "constant" one can say whether the resistance saturation is the consequence of closure of all cross sections, or a consequence of the presence of layers of open sections with different opening directions. In the latter case the Hall "constant" tends rapidly to zero (like $1/H^2$).^[58]

As we said above, the asymptotic value of the Hall "constant" is independent of the nature of the collisions, for all those directions of the magnetic field for which there are no open sections (cf. Sec. 5), but $n_1 \neq n_2$. A similar situation exists if, in addition to closed sections, there is an isolated open section (the third case of the singular directions). For such directions

$$R_{\infty} = \frac{2}{(2\pi\hbar)^3} \frac{1}{\Delta V \pm d_{\min} b_1 b_2}, \quad (6.12)$$

where $\Delta V = V_1 - V_2 = (2\pi\hbar)^3 (n_1 - n_2)/2$, while the definitions of the quantities d_{\min} , b_1 and b_2 are clear from Fig. 18. The sign is determined by the direction of motion of the electron along the open trajectory. It is clear that measurement of the Hall "constant" for these special directions allows one to determine not only the minimal diameter of the tube

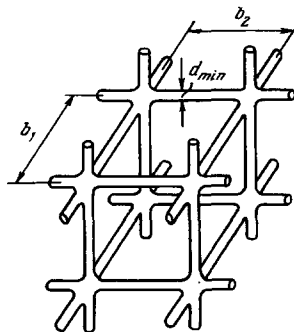


FIG. 18

containing the opening direction, but also the direction of motion of the electron. If the quantity ΔV is not zero, it must be measured for other directions of the magnetic field.

The marked dependence of the resistance on magnetic field direction allows one in certain cases to explain the linear increase of the resistance of polycrystals with increasing magnetic field (the law of P. L. Kapitza). In fact, if a quadratic growth ($\rho \sim H^2$) is observed within a narrow range of angles $\Delta\theta \sim H_0/H$ (cf. formula (6.8)), any averaging of the resistance over angles in the interval $\delta\theta \gg H_0/H$ (containing $\Delta\theta$) leads to a linear increase of resistance with field. It should be emphasized that, because of the small width of the maximum, the nature of the averaging plays no important part: in particular, averaging the conductivity and then taking the reciprocal gives the same result, a linear dependence of resistance on magnetic field.^[55] It is entirely possible that this is not the only cause of the linear rise in the resistance. As E. S. Borovik has shown,^[50] a linear dependence is frequently seen in the transition region (when $H \sim H_0$) from the quadratic dependence in small fields ($H \ll H_0$) to saturation or to a second quadratic dependence in strong fields ($H \gg H_0$). Thus in certain cases the linear dependence may mean that the measured values of the resistance should not be regarded as asymptotic. Finally, quite recently M. Ya. Azbel,^[35] and M. Ya. Azbel' and V. G. Peschanskiy^[35a] have shown that including the boundaries of the sample, even for samples of quite large size, may lead to a substantial complication of the $\rho(H)$ dependence. In particular, when $r \ll l^2/d$, where d is of the order of the wire thickness, the resistance depends linearly on the magnetic field.

So far we have used classical concepts in constructing a theory of galvanomagnetic properties. The possibility of neglecting the energy quantization was based on the smallness of the oscillating terms in the magnetic field. There is, however, a quantum effect that can change the smooth variation of the components of the resistance tensor with changing magnetic field. We are speaking of magnetic breakdown,^[60] as a result of which the electron under the action of the magnetic field can make a transition from one classical orbit to another. The probability P for such a transition, as shown by Blount,^[61] is quite high ($P = \exp(-H_B/H)$, where $H_B = K\Delta^2 m^* c / e\epsilon_F$, Δ is the energy barrier separating one classical orbit from another, and K is a constant of order unity). Thus in many cases P reaches unity in fields much smaller than H_0 (for which $l = r$). These transitions are especially important if, as a result of the magnetic breakdown, the topology of the trajectories changes (trajectories change from open to closed or vice versa). If we start from the Harrison model^[62] (i.e., from the approximation of almost

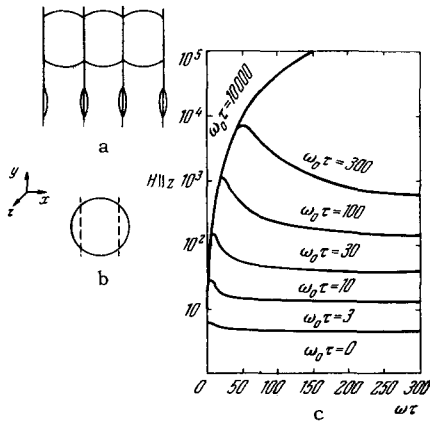


FIG. 19. Magnetic breakdown between open and closed orbits. a) Orbit with breakdown probability equal to zero; b) with breakdown probability equal to unity; c) dependence of transverse resistance on magnetic field ($\omega \tau = \frac{eH\tau}{mc}$, $\omega_0 \tau$ characterizes the probability of breakdown).

free electrons), then as a result of magnetic breakdown ($P = 1$) the electron trajectories in the magnetic field are circles for all polyvalent metals. In other words, in a sufficiently strong field (but such that one can still neglect the energy quantization) the electrons behave like an electron gas in free space. These ideas were the basis of a calculation of transverse resistance and the Hall effect in various specific cases^[63] (Fig. 19).

The experimental investigations seem to confirm the existence of magnetic breakdown for several metals (Mg, Zn^[64]).

7. THERMAL CONDUCTIVITY AND THERMOELECTRIC PHENOMENA IN A STRONG MAGNETIC FIELD

The magnetic field changes not only the resistance of the metal, but also its thermal conductivity, its Thomson and Peltier coefficients, and other characteristics of thermoelectric phenomena.^[65] This is entirely natural, since the electrons whose motions are significantly changed by the magnetic field are at the basis of all these effects.

The dependences of the thermal conductivity and thermoelectric coefficients on magnetic field are often called thermomagnetic phenomena. Like the galvanomagnetic effects, they can be divided into transverse and longitudinal, even and odd effects. The latter are analogous to the Hall effect. Starting from symmetry arguments, one can construct the dependence of the thermomagnetic coefficients on weak magnetic field as an expansion in powers of the field. The number of independent components in these tensors (proportionality coefficients) is determined by the symmetry class of the crystal. Traditionally (true, in most cases for semiconductors)

thermomagnetic investigations are used for studying the mechanisms of scattering of charge carriers. It appears that there is still no work in which these properties of metals are used for determining parameters of the electron energy spectrum.

On the other hand, as the theoretical papers show (and as we shall see below), the study of thermomagnetic properties in strong fields allows one to determine a distinctive feature of the electrons that is extremely useful for establishing the spectrum.

Without repeating the arguments for justifying the quasiclassical approach (cf. Secs. 1 and 4) we use the technique developed for studying galvanomagnetic phenomena for determining the dependence of the thermomagnetic coefficients (cf. the beginning of Sec. 3) on the magnitude and direction of a strong magnetic field ($l \gg r$).

The whole set of thermomagnetic effects is described by the dependence on magnetic field of the coefficients α_{ik} and κ_{ik} in formula (3.1). The latter are related by formulas (3.16) to the coefficients σ_{ik} , b_{ik} , d_{ik} , and c_{ik} , where, according to the principle of symmetry of the kinetic coefficients,^[67]

$$\begin{aligned} \sigma_{ik}(\mathbf{H}) &= \sigma_{hi}(-\mathbf{H}), & d_{ik}(\mathbf{H}) &= d_{hi}(-\mathbf{H}), \\ -Tb_{ik}(\mathbf{H}) &= c_{hi}(-\mathbf{H}). \end{aligned} \quad (7.1)$$

The tensors d_{ik} and b_{ik} are given by formulas (3.13), in which we must substitute the solution of the kinetic equation

$$\frac{\partial \varphi_i}{\partial t} + \hat{W}_e \varphi_i = v_i. \quad (7.2)$$

Equation (7.2) is derived similarly to Eq. (5.7) and differs from it only in the collision operator (cf. formulas (3.6) and (3.7)). In deriving it we have not improved the form of the phonon distribution function or, in other words, have not included the dragging of the phonons by the electrons (cf. below).

Since the explicit form of the collision operator does not affect the asymptotic behavior of the solution (in strong magnetic fields), the analysis given in Secs. 5 and 6 is applicable to the present case. In explaining the dependence on the strong magnetic field of the components of the thermal conductivity tensor κ_{ik} and the thermoelectric coefficients α_{ik} one can, furthermore, use formulas (3.22), keeping in mind, of course, that $\kappa_{ik} = -d_{ik}$ (cf. (3.24)).

It is completely clear that, with the possible exception of certain singular cases (cf. below) the character of the dependence of the components of the tensors b_{ik} , c_{ik} , and d_{ik} coincides with the dependence on magnetic field of the components of the electric conductivity. This allows us immediately to use the results of Secs. 5 and 6. We should consider as a special case the situation when the numbers of electrons and holes are equal. As we know, the peculiarity of this case is that, because of the compensation of volumes ($n_1 = n_2$) the expansion of the component σ_{xy} (the z axis, as always, is along the

magnetic field) begins with the quadratic term ($\sigma_{xy} \sim 1/H^2$). This in turn results in a quadratic increase in the transverse (to the magnetic field) resistance. But equality of the numbers of electrons and holes ($n_1 = n_2$) does not mean that $(dV_1/d\epsilon)_{\epsilon=\epsilon_F}$ and $(dV_2/d\epsilon)_{\epsilon=\epsilon_F}$ are equal, where $V_{1(2)}(\epsilon)$ is the volume inside (outside) the electron (hole) part of the surface $\epsilon(\mathbf{p}) = \epsilon$. Consequently, even when $n_1 = n_2$ (where $n_{1(2)} = 2V_{1(2)}(\epsilon_F)/(2\pi\hbar)^3$), the expansion of b_{xy} and c_{xy} starts with a term proportional to $1/H$ (cf. formula (3.22)).

We present some fundamental results (for more detail, cf. [66, 67]). First, for a closed Fermi surface and $n_1 \neq n_2$

$$\kappa_{xy} = \frac{\pi^2}{3} T \frac{c}{eH} (n_1 - n_2). \quad (7.3)$$

This means that measurement of the Righi-Leduc effect (like the Hall effect) allows one to measure the difference in numbers of electrons and holes. Second, under these same conditions the asymptotic form of the matrix d_{ik} is*

$$(\alpha_{ik}) = \begin{pmatrix} \nu_{xx} & \frac{1}{H} \nu_{xy} & \nu_{xz} \\ \frac{1}{H} \nu_{yx} & \nu_{yy} & \nu_{yz} \\ \frac{1}{H} \nu_{zx} & \frac{1}{H} \nu_{zy} & \nu_{zz} \end{pmatrix}. \quad (7.4)$$

The expansions of all components of the matrix ν_{ik} begin with terms of zeroth power in the inverse magnetic field, and all components (except ν_{xx} and ν_{yy}) depend on the collision integral, while

$$\nu_{xx} \approx \nu_{yy} \approx \frac{\pi^2}{3} \frac{T}{e} \left[\frac{d}{d\epsilon} \ln(n_1 - n_2) \right]_{\epsilon=\epsilon_F}, \quad (7.5)$$

and consequently, the study of the Nernst-Ettingshausen effects in strong magnetic field allow one to measure a characteristic of the electron energy spectrum $(d/d\epsilon \ln(n_1 - n_2))_{\epsilon=\epsilon_F}$, which cannot be determined from galvanomagnetic (or any other) experiments. In the case of a quadratic isotropic dispersion law ($\epsilon = p^2/2m$), when $n_2 = 0$

$$\frac{d}{d\epsilon} \ln(n_1 - n_2) \Big|_{\epsilon=\epsilon_F} = \frac{3}{2\epsilon_F}. \quad (7.6)$$

If the number of electrons n_1 is equal to the number of holes n_2 and the Fermi surface is closed, all the components of the thermal conductivity tensor κ_{ik} depend on the nature of the scattering, where, naturally, the asymptotic dependence on magnetic field is the same as the asymptotic dependence of the components of the electrical conductivity tensor. In particular, the transverse components κ_{xx} , κ_{yy} decreases sharply (κ_{xx} , $\kappa_{yy} \sim 1/H^2$). This, it appears, should enable one to separate the phonon (lattice) contribution to the heat transfer in metals. The

asymptotic form of the tensor κ_{ik} is the following:

$$(\alpha_{ik}) = \begin{pmatrix} H\nu_{xx} & H\nu_{xy} & H\nu_{xz} \\ H\nu_{yx} & H\nu_{yy} & H\nu_{yz} \\ \nu_{zx} & \nu_{zy} & \nu_{zz} \end{pmatrix}. \quad (7.7)$$

In our case all the components of the matrix ν_{ik} depend on the angles between the vector \mathbf{H} and the directions of the crystal axes, while their specific form is determined by the collision integral. We should call the reader's attention to the fact that metals with $n_1 = n_2$ should be distinguished by a relatively high thermo-emf in strong magnetic field.

Our treatment of the different cases of open Fermi surfaces shows that as in the case of the galvanomagnetic phenomena, the most characteristic feature of the case of closed surfaces is the marked anisotropy of the thermomagnetic characteristics: as we approach a selected (singular) direction, the asymptotic form of the tensor components κ_{ik} and α_{ik} changes markedly, and as is clear from our preceding remarks, the asymptotic behavior of κ_{ik} is completely analogous to the behavior of σ_{ik} (cf. Sec. 6).

L. É. Gurevich and G. M. Nedlin [68] called attention to the fact that phonon drag by the electrons in a strong magnetic field can change the asymptotic behavior of the thermomagnetic coefficients (in terms of the inverse magnetic field). The point is that the phonon drag reduces (as shown in [68]) to the appearance of "additional" forces on the right side of Eq. (7.2), while none of the components of this "force" average to zero (we recall that in many cases, $\bar{v}_x = \bar{v}_y = 0$). Because of this the expansions of all components of the function φ_i start with terms independent of the magnetic field. We shall not write out the final results of [68]. We mention only that the effect of drag is most important when the temperatures are not too low (when $T \gg T_0$, where $T_0 = \theta(\theta/\epsilon_F)^{1/2}$).

8. NORMAL SKIN EFFECT

The behavior of metals in variable electromagnetic fields is described by the Maxwell equations:

$$\left. \begin{aligned} \text{rot } \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \\ \text{rot } \mathbf{H} &= \frac{4\pi}{c} \mathbf{j}, \end{aligned} \right\} \quad (8.1)^*$$

which must be supplemented by a relation between the current \mathbf{j} and the electric field \mathbf{E} of the electromagnetic wave. Since we are not interested in nonlinear effects, we can go over to Fourier components:

$$\text{rot } \mathbf{E}_\omega = \frac{i\omega}{c} \mathbf{H}_\omega. \quad (8.2)$$

We shall drop the subscript ω from now on, remembering, however, that the time dependence of all quantities is exponential ($\mathbf{E}, \mathbf{H}, \mathbf{j} \sim e^{-i\omega t}$).

*Our notation differs somewhat from that of [67].

*rot \equiv curl.

To establish the connection between the field \mathbf{E} and the current density \mathbf{j} in the general case requires that we write the linearized kinetic equation (cf. Secs. 1 and 2) with an unknown electric field $\mathbf{E}(\mathbf{r})$, depending on the coordinates according to Eq. (8.2). The distribution function found from this equation is substituted in formula (1.2) for the current density \mathbf{j} , and the current density \mathbf{j} is put in system (8.1), in order to find the coordinate dependence of the electric and magnetic fields (cf. Sec. 9). But if the frequency of the electromagnetic field ω is much less than the collision frequency $\nu_0 = 1/\tau$, and the distance over which the field changes significantly (the skin depth δ) is much greater than the free path l , we can use for the relation between the current density and the electric field the static value of the electrical conductivity tensor or the resistance tensor (normal skin effect):

$$j_i = \sigma_{ik} E_k, \quad E_i = \rho_{ik} j_k. \quad (8.3)$$

In order to be able to take account of the effect of a constant magnetic field, we shall not assume from the start that the tensors σ_{ik} and ρ_{ik} are symmetric (cf. Sec. 4).

Since an electromagnetic field as a rule damps out over a wave length, the most convenient characteristic of the high-frequency properties of a metal* is undoubtedly its surface impedance (surface resistance)—a two-dimensional second-rank tensor, introduced through the following relation:^[4]

$$E_\alpha^{(0)} = \zeta_{\alpha\beta} [\mathbf{H}\mathbf{n}]_\beta \quad (\alpha, \beta = x, y). \quad (8.4)$$

Here the superscript (0) means that the values of the field components are taken at the boundary of the metal, \mathbf{n} is the unit vector along the normal to the surface. The coordinate axes are chosen so that the z axis is parallel to \mathbf{n} .

As shown in^[69], $\zeta_{\alpha\beta}$ is a tensor; the Onsager relations apply to it:

$$\zeta_{\alpha\beta}(\mathbf{H}) = \zeta_{\beta\alpha}(-\mathbf{H}). \quad (8.5)$$

Furthermore, one can show that there are dispersion relations for the surface resistance analogous to the Kramers-Kronig relations.^[4] In particular, for the isotropic case,

$$\zeta = \zeta' + i\zeta'', \quad \zeta''(\omega) = -\frac{2\omega}{\pi} \int_0^\infty \frac{\zeta'(x) dx}{x^2 - \omega^2}. \quad (8.6)$$

The use of the surface impedance greatly simplifies the solution of electrodynamic problems external to the metal: reflection of an electromagnetic wave, calculation of resonators, etc. This is related to the fact that for an optically dense medium (and a metal is certainly such a medium) the impedance is prac-

*In this and the following sections we consider only a metallic halfspace (more precisely, samples whose dimensions are much greater than the wave length and the damping length).

tically independent of the form of the field, i.e., of the angle of incidence of the wave, or of its polarization.* This permits us to calculate the impedance in the simplest case of normal incidence of the wave on a metallic halfspace, and to use Eq. (8.4) as an effective boundary condition (the Fock-Leontovich approximation^[72]). Without specifically saying it, we shall from now on always assume that the wave propagates normally to the surface of the metal.

Using formulas (8.2) and (8.3), one can show that the components of the tensor $\zeta_{\alpha\beta}$ are the roots of the following system of equations:^[4]

$$\zeta_{\alpha\gamma} \zeta_{\gamma\beta} = \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = -\frac{i\omega}{4\pi} \rho_{\alpha\beta}. \quad (8.7)$$

But for computations it is more convenient to use a specially selected system of coordinates. Thus, if $\rho_{\alpha\beta}$ is a symmetric tensor (no external magnetic field), then $\zeta_{\alpha\beta}$ is also a symmetric tensor, whose principal values are

$$\zeta_{1,2} = \sqrt{\frac{\omega \rho_{1,2}}{4\pi i}}, \quad (8.8)$$

while $\rho_{1,2}$ are the principal values of $\rho_{\alpha\beta}$. We note that the surface impedance (and even its principal values) depend on the direction of the normal to the surface of the metal, because the components $\rho_{\alpha\beta}$ depend on this direction. For a cubic crystal the tensor ρ_{ik} degenerates to a scalar and consequently this dependence disappears.

The coordinate system in which the tensors $\rho_{\alpha\beta}$ and $\zeta_{\alpha\beta}$ are diagonal is called the set of principal directions. If $\rho_{\alpha\beta}$ is a symmetric tensor ($\mathbf{H} = 0$) the principal directions are perpendicular to one another.

The skin depths for the different polarizations are different:

$$\delta_{1,2} = \sqrt{\frac{c^2 \rho_{1,2}}{2\pi\omega}}. \quad (8.9)$$

The propagation of electromagnetic waves in metals in a strong magnetic field ($r \ll l$) has many interesting features[†] which have been observed in recent years.

To study these features it is convenient to write the Maxwell equation (8.2) in a somewhat different form:^[74]

$$\frac{\partial E_\pm}{\partial x_3} = -\frac{i\omega}{c\beta_\pm} H_\pm, \quad \frac{\partial H_\pm}{\partial x_3} = \frac{4\pi\beta_\pm}{c\rho_\pm} E_\pm, \quad (8.10)$$

*Cf. the survey by V. L. Ginzburg and G. P. Motulevich.^[70] For the anomalous skin effect (cf. Sec. 9) the last statement is rigorously proved in^[71], in which the corrections to the surface impedance because of oblique incidence of the wave are computed.

†We remind the reader that we are considering the quasistatic case, i.e., not taking into account spatial and temporal dispersion. When $\mathbf{H} \neq 0$ we must add to the conditions $\omega \ll \nu_0$, $l \ll \sigma$ the condition $\omega \ll \omega_H = eH/m^*c$, which it is true, is always satisfied in a strong magnetic field ($\omega_H \gg \nu$ if $l \gg r$).

where

$$\left. \begin{aligned} E_{\pm} &= E_1 + \frac{1}{\beta_{\pm}} E_2, & H_{\pm} &= H_1 - \frac{1}{\beta_{\pm}} H_2, \\ \rho_{\pm} &= \rho_1 + \frac{1}{\beta_{\pm}} \rho_{21}, & \beta_{\pm} &= \frac{\rho_1 - \rho_2}{2\rho_{12}} \pm \sqrt{-1 + \left(\frac{\rho_1 - \rho_2}{2\rho_{12}}\right)^2}. \end{aligned} \right\} (8.11)$$

The significance of the parameters ρ_1 , ρ_2 and ρ_{12} is clear from the following form of the two-dimensional resistance tensor:

$$\rho_{\alpha\beta} = \begin{pmatrix} \rho_1 & \rho_{12} \\ -\rho_{12} & \rho_2 \end{pmatrix}. \quad (8.12)$$

Such a form for the tensor $\rho_{\alpha\beta}$ means that by the diagonalization of the symmetric part of the tensor $\rho_{\alpha\beta}$ the axes "1" and "2" are singled out.

Assuming that the electromagnetic wave is plane ($E, H \sim e^{ikx_3}$), we find the dispersion equation relating the wave vector k and the frequency ω :

$$k^2 = \frac{4\pi i \omega}{c^2 \rho_{\pm}}, \quad (8.13)$$

from which we see that two waves can propagate in the medium. In the most interesting case of strong gyrotropy $|\rho_{12}| \gg |\rho_1 + \rho_2|$, the "resistance" ρ_{\pm} is almost pure imaginary ($\rho_{\pm} \approx \mp i\rho_{21}$) and consequently one of the waves is almost undamped (helicoidal or spiral wave [73]). The absence of dissipation results from the fact that the oscillations are coupled only to the Hall current. Strong gyrotropy ($|\rho_{12}| \gg |\rho_1 + \rho_2|$) occurs, as is clear from formula (5.27), when a wave propagates in a metal, with unequal numbers of electrons and holes ($n_1 \neq n_2$), along the magnetic field.

If the anisotropy exceeds the gyrotropy $|\rho_1 - \rho_2| > 2|\rho_{12}|$, the eigenwaves are linearly polarized (β_{\pm} are real quantities), and the principal directions are not perpendicular to one another. The angle ψ between the principal directions is given by the following formula:

$$\tan \psi = \sqrt{\frac{(\rho_1 - \rho_2)^2}{4\rho_{12}^2} - 1}. \quad (8.14)$$

When $|\rho_1 - \rho_2| = 2|\rho_{12}|$ the principal directions coincide.

In a strong magnetic field ($r \ll l$) the predominance of anisotropy over gyrotropy ($|\rho_1 - \rho_2| > 2|\rho_{12}|$) should be seen for metals with equal numbers of electrons and holes ($n_1 = n_2$), and most clearly for metals with open Fermi surfaces, where for particular directions of the magnetic field the components ρ_1 and ρ_2 have different asymptotic dependences on the magnetic field (cf. Sec. 6, in particular formula (6.7)).

In our notation the impedance is conveniently defined as follows:

$$\zeta_{\pm} = \beta_{\pm} \left(\frac{E_{\pm}}{H_{\pm}} \right)_{x_3=0}. \quad (8.15)$$

This permits us to use the standard formulas relating the impedance and the complex reflection coefficient:

$$R_{\pm} = -\frac{1 + \zeta_{\pm}}{1 - \zeta_{\pm}}, \quad \zeta_{\pm} = \sqrt{\frac{\omega Q_{\pm}}{4\pi i}}. \quad (8.16)$$

In concluding this section we call attention to an effect, specific to anisotropic (or gyrotropic) conductors, which changes somewhat the value of the surface impedance, and is not related to the frequency dependence of the electrical conductivity tensor. We are referring to the inclusion of thermoelectrical forces in calculating the surface impedance.[75] If the normal to the surface of the sample does not coincide with one of the principal directions of the tensor ρ_{ik} ("skew cut"), then when an electromagnetic wave is incident there appears along the z axis an electric field E_z (which is found from the condition $j_z = 0$). It leads to the appearance of a temperature gradient proportional to the field intensity in the incident wave. A self-consistent solution of the problem, including the Maxwell equations and the heat conduction equation, in the case of a uniaxial crystal gives the following result.* If we choose the axes as shown in Fig. 20 (where "1"

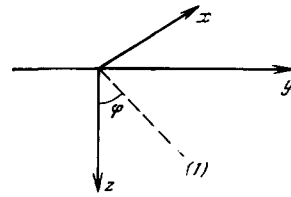


FIG. 20

is the direction of a crystal axis), then one of the principal values of the impedance is unchanged:

$$\zeta_x = \sqrt{\frac{\omega Q_{xx}}{4\pi i}}, \quad \rho_{xx} = \rho_{\perp}, \quad (8.17)$$

while the second depends strongly on the conditions of heat exchange. If the boundary is kept at constant temperature,

$$\zeta_y = \sqrt{\frac{\omega Q_{yy}}{4\pi i}} \sqrt{2} \times \frac{\sqrt{b} + \sqrt{1+a}}{(1+b + \sqrt{(1-b)^2 - 4ab})^{1/2} + (1+b - \sqrt{(1-b)^2 - 4ab})^{1/2}}, \quad (8.18)$$

while if the boundary is adiabatic (zero heat flow through the boundary)

$$\zeta_y = \sqrt{\frac{\omega Q_{yy}}{4\pi i}} \sqrt{\frac{1+a}{2}} \times \frac{(1+a + \sqrt{(1-b)^2 - 4ab})^{1/2} + (1+a - \sqrt{(1-b)^2 - 4ab})^{1/2}}{\sqrt{b} + \sqrt{1+a}}; \quad (8.19)$$

where $\rho_{yy} = \rho_{\perp} \cos^2 \varphi + \rho_{\parallel} \sin^2 \varphi$, $a = T \alpha_{xy}^2 / \rho_{yy} \kappa_{zz}$, $b = c^2 C \rho_{yy} / 4\pi \kappa_{zz}$ (C is the specific heat per unit volume; for the other notation, cf. Sec. 3). For good metals, as a rule, $a \ll 1$, $b \gg 1$, and the role of the thermoelectric forces in the skin effect is completely unimportant. But for metals like Bi (and also for semiconductors) the effect treated here may be very significant.

Thermoelectric forces may also manifest themselves in the skin effect in isotropic conductors (polycrystals with cubic symmetry) if they are placed in a magnetic field.[75]

*The principal values of the tensor ρ_{ik} are: $\rho_1 = \rho_{\parallel}$, $\rho_2 = \rho_3 = \rho_{\perp}$.

9. ANOMALOUS SKIN EFFECT

In this part of our survey (as already stated in Sec. 1) only quasistatic properties of normal metals are considered. By quasistatic properties we mean those for which the characteristic frequencies ω are much lower than the collision frequency $\nu_0 = 1/\tau$ ($\omega \ll \nu$). But the formulas that have been given in the preceding Sec. 8 will as a rule remain valid for frequencies much lower than the collision frequency ν . This is related to the fact that with increasing frequency, at low temperatures, for sufficiently pure samples, the depth of the skin layer δ is comparable with the mean free path l and can even become much smaller than the latter.

The reflection of an electromagnetic wave from a metal under such conditions ($l \gtrsim \delta$) is called the anomalous skin effect, and in those cases where $l \gg \delta$, the limiting anomalous skin effect.* It is easy to show that the condition $l \gg \delta = c/\sqrt{2\pi\sigma\omega}$ does not contradict the condition $\omega \ll \nu$. The condition for consistency of these inequalities can be written as a condition imposed on the collision frequency:

$$v \ll \frac{\omega_0}{\sqrt{2}} \frac{v}{c}, \quad (9.1)$$

where $\omega_0 = 4\pi ne^2/m$, v is the velocity of the electrons at the Fermi surface, where the simplest assumptions about the electron dispersion law have been used for the estimates (cf. formula (2.19)). For good metals, condition (9.1) means that the mean free path $l = v\tau = v/\nu$ should be much greater than 10^{-5} cm, which is easily realized at helium temperatures for pure samples of metals.†

The problem of this section is getting and discussing the formulas for the surface impedance in the case of the limiting anomalous skin effect ($l \gg \delta$). The constant magnetic field is assumed to be zero. The propagation of electromagnetic waves in metals when $H \neq 0$ has a number of interesting features and requires special detailed investigation (cf., for example, [80]).

Formally the effect of inhomogeneity of the electric field is taken into account by keeping terms in the kinetic equation that describe the change of the distribution function with the coordinates, $\mathbf{v} \cdot \partial f_1 / \partial \mathbf{r}$ (cf. Eqs. (2.1) and (1.16)). The kinetic equation is then changed into an integro-differential equa-

tion, whose solution requires formulation of boundary conditions for the distribution function f_1 of the electrons. These boundary conditions must reflect the physical picture of the collision of electrons with the surface of the sample. In a rigorous formulation the statement of the boundary conditions is a complicated problem which, as a matter of fact, goes beyond the framework of the semiphenomenological description that we are using here. But such a formulation is not necessary for the purposes of this section, since the quantity of interest to us (the surface impedance) is very insensitive to the nature of the boundary conditions. This insensitivity is explained by the following argument: when $l \gg \delta$ the most important electrons are those moving parallel to the boundary of the metal (Fig. 21). They interact only with the transverse components of the electric field which are determined by the electrodynamic boundary conditions (continuity of the tangential components of the electric and magnetic field intensities). In calculating the surface impedance when $l \gg \delta$ the normal component of the electric field can be omitted, for the following reason: it depends essentially on the boundary conditions imposed on the distribution function. We confirm this qualitative argument by the following method. Instead of the correct (physical) boundary conditions we shall use "convenient" ones (i.e., those which enable us to greatly simplify the calculation) of two types, and show that changing from one set to the other scarcely changes the value of the surface resistance. A rigorous formulation of the boundary conditions for electrons with an anisotropic dispersion law is contained in [35,35a].

So we consider the normal falloff in the electromagnetic wave in the metal halfspace ($z > 0$).

The Maxwell equations for the field in the metal in this case have the form:

$$\left. \begin{aligned} \frac{i\omega}{c} H_x &= -\frac{\partial E_y}{\partial z}, & \frac{4\pi}{c} j_y &= \frac{\partial H_x}{\partial z}, \\ \frac{i\omega}{c} H_y &= \frac{\partial E_x}{\partial z}, & \frac{4\pi}{c} j_x &= -\frac{\partial H_y}{\partial z}, \\ H_z &= 0, & j_z &= 0. \end{aligned} \right\} \quad (9.2)$$

The equation $j_z = 0$ serves to eliminate the longitudinal field E_z from the system of equations. We use the Fourier method (expansion in plane waves) and express the field in the metal ($z > 0$) in terms

*The anomalous skin effect was discovered experimentally by London.[76] The initial form of the theory of this phenomenon is contained in the papers of Reuter and Sondheimer[77], and also Pippard.[78]

† We note that condition (9.1) does not restrict the region of applicability of the expressions obtained below; rather it restricts the method used for deriving them. Since the free path and the collision frequency drop out of the final formulas, all the formulas are valid up to frequencies of the order of or greater than the collision frequency. Only when $\omega \gg \nu$ do all the formulas become invalid (cf.[77] and also[79,79]).

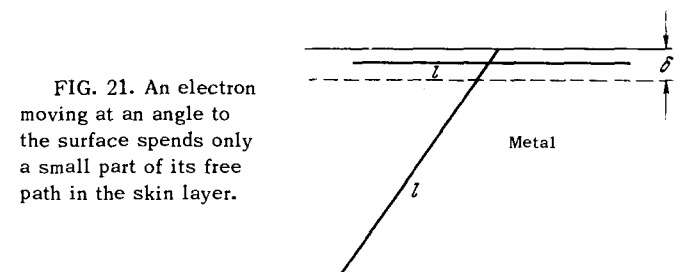


FIG. 21. An electron moving at an angle to the surface spends only a small part of its free path in the skin layer.

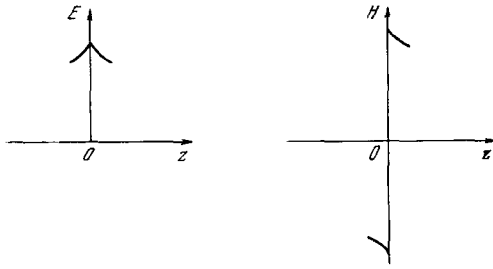


FIG. 22

of the value of the field on the boundary ($z = 0$). It is convenient to replace the actual boundary by a plane on which the field components have singularities, and consider the field to be continued into the half-space $z < 0$. We then consider two variants:

a) The electric field and the current density are continued as even functions and the magnetic field as an odd function.

b) The magnetic field is continued even, while the electric field and the current density are continued odd.

The connection of these variants with the nature of the boundary conditions will be considered below. Figure 22 shows schematically the dependence of the components of the electric and magnetic fields on the coordinates near the boundary $z = 0$ for the case of even continuation of the electric field.

Multiplying each of the equations of the system (9.2) by e^{ikz} and integrating over the coordinate from $-\infty$ to $+\infty$, we obtain in the case of variant a):

$$\left. \begin{aligned} \frac{i\omega}{c} H_{hx} &= ikE_{hy}, & \frac{4\pi}{c} j_{hy} &= -ikH_{hx} - 2H_x(0), \\ \frac{i\omega}{c} H_{hy} &= -ikE_{hx}, & \frac{4\pi}{c} j_{hx} &= ikH_{hy} + 2H_y(0), \\ H_{hz} &= 0, & j_{hz} &= 0; \end{aligned} \right\} \quad (9.3)$$

for the case of variant b):

$$\left. \begin{aligned} \frac{i\omega}{c} H_{hx} &= ikE_{hy} + 2E_y(0), & \frac{4\pi}{c} j_{hy} &= -ikH_{hx}, \\ \frac{i\omega}{c} H_{hy} &= -ikE_{hx} - 2E_x(0), & \frac{4\pi}{c} j_{hx} &= ikH_{hy}, \\ H_{hz} &= 0, & j_{hz} &= 0. \end{aligned} \right\} \quad (9.4)$$

Here E_{kx} , E_{ky} , etc. are the Fourier transforms of the corresponding field components:

$$E_{hx} = \int_{-\infty}^{\infty} E_x(z) e^{ihz} dz. \quad (9.5)$$

Hence

$$E_x(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{hx} e^{-ihz} dk. \quad (9.6)$$

So far our computations were completely exact. But to solve the system of equations (9.3) and (9.4) we must know the relation between the Fourier transforms of the current density and the electric field. If in calculating this connection we use the kinetic equa-

tion with the exact boundary conditions, this will be an integral relation. Our simplification (of which we spoke above) is that in calculating this relation we neglect the presence of the boundary. If we correct for this neglect of the boundary by introducing special boundary conditions* it turns out that variants a) and b) correspond to different boundary conditions. On the other hand, as we have seen, the results obtained are insensitive to the kind of continuation. This also serves as a confirmation of our thesis about indifference to the boundary conditions.

We shall now determine the connection between the electric field and the current. According to the first Section, the kinetic equation for this case is the following:

$$v_z \frac{\partial f_1}{\partial z} + \hat{W}_p \{f_1\} = -\frac{\partial n_p}{\partial t} e v E, \quad (9.7)$$

while the current density is

$$j = e \int v f_1 d\Gamma. \quad (9.8)$$

The main term on the left is the one containing the derivative, while the integral term (the collision integral $\hat{W}_p \{f_1\}$) is a small correction, which we could neglect if this did not result in the appearance of a singularity at momentum values for which $v_z = 0$. From this it is clear that the main contributors are the gliding electrons that move parallel to the sample boundary (for which $v_z = 0$). We recall that we are interested in the case of the limiting anomalous skin effect, i.e., the limiting value of the impedance corresponding to infinitely large mean free path. This permits us to neglect the term on the right containing the longitudinal field component E_z . Formally this means that we are interested only in the principal part of the solution of Eq. (9.7) related to the singularity at $v_z = 0$. This same argument enables us to greatly simplify the collision operator $\hat{W}_p \{f_1\}$ by dropping the integral terms in it, since the integration over momenta undoubtedly smooths out the singularity, and consequently for v_z close to zero, the integral term for the "loss" is much greater than the term for the "gain" (since it is much easier to leave a narrow region of states than to fall into it). This permits us to write, without restriction of generality:^[81]

$$\hat{W}_p \{f_1\} \cong \frac{1}{\tau_{gl}} f_1. \quad (9.9)$$

*To accomplish this program one must, following^[81], change from the distribution function f_1 to the function $\Phi(z, v) = f_1(z, v) - f_1(z_1, -v)$, in terms of which one can express the current density, and the equation for which is of second order and admits even or odd continuation. The simplifying boundary conditions are formulated for the function Φ . To the even continuation of the electric field and the use of the relation between j_k and E_k valid for an unbounded space, there corresponds the boundary condition $\frac{\partial \Phi}{\partial z} \Big|_{z=0} = 0$, and to the odd continuation, $\Phi(0, v) = 0$.

The subscript "gl" means that this form is valid only for the treatment of those effects in which gliding electrons play the main role (i.e., electrons for which $v_z = 0$). Thus, in our approximation Eq. (9.7) can be rewritten as follows:

$$v_z \frac{\partial f_1}{\partial z} + \frac{1}{\tau_{gl}} f_1 = -\frac{\partial n_F}{\partial \epsilon} e v_\alpha E_\alpha \quad (\alpha = x, y). \quad (9.10)$$

From this and Eq. (9.8) we have

$$j_{k\alpha} = \sigma_{\alpha\beta}(\omega, \mathbf{k}) E_{k\beta} \quad (\alpha, \beta = x, y), \quad (9.11)$$

and

$$\sigma_{\alpha\beta} = \frac{2e^2}{(2\pi\hbar)^3} \oint_{\epsilon(p)=\epsilon_F} \frac{l_{gl} n_{\alpha\beta} dS}{1 + (k l_{gl} n_z)^2}, \quad (9.12)$$

where $l_{gl} = \tau_{gl} v$, $\mathbf{n} = \mathbf{v}/v$; in getting the last formula we have used, first, the fact that $-\partial n_F / \partial \epsilon \approx \delta(\epsilon - \epsilon_F)$, and second, the presence of a center of inversion on the Fermi surface.

We note that in this approximation we are taking $j_z \equiv 0$, since the corresponding components of the tensor σ_{ik} ($\sigma_{\alpha z}$ and σ_{zz}) are much smaller than the components $\sigma_{\alpha\beta}$, and are set equal to zero.

The expression (9.12) allows a transition to the limit of infinitely long free path. We write $\lim_{l_{gl} \rightarrow \infty} \sigma_{\alpha\beta} = \sigma_{\alpha\beta}^{(\infty)}$. It is not difficult to show that

$$\sigma_{\alpha\beta}^{(\infty)} = \frac{3\pi}{4} \frac{B_{\alpha\beta}}{|\mathbf{k}|}, \quad (9.13)$$

where

$$B_{\alpha\beta} = \frac{8e^2}{3(2\pi\hbar)^3} \int_0^{2\pi} \frac{n_{\alpha\beta} d\varphi}{K(\varphi)}, \quad (9.14)$$

and $K(\varphi)$ is the Gauss curvature of the Fermi surface at points where $n_z = 0$. For a quadratic dispersion law $K(\varphi) = 1/p_F^2$ and $B_{\alpha\beta} = (\sigma/l) \delta_{\alpha\beta}$ (cf. formula (8.4)).

We now turn to the calculation of the surface impedance, taking x and y axes along the principal directions of the tensor $B_{\alpha\beta}$. From Eqs. (9.3), (9.4), and (9.6), and also (9.11), (9.13), and (9.14), we have, using the definition (8.4) for the impedance,

variant a):

$$\zeta_{x,y}^{(a)} = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{d\mathfrak{N}}{\mathfrak{N}^2 - \epsilon_{x,y}(\mathfrak{N})}, \quad (9.15)$$

variant b):

$$\frac{1}{\zeta_{x,y}^{(b)}} = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\epsilon_{x,y}(\mathfrak{N}) d\mathfrak{N}}{\mathfrak{N}^2 - \epsilon_{x,y}(\mathfrak{N})}, \quad (9.16)$$

where $\epsilon_{\alpha,\beta}$ is the effective dielectric constant of the metal; the principal values of the tensor $\epsilon_{\alpha\beta}$ are

$$\epsilon_{x,y}(\mathfrak{N}) = \frac{4\pi\sigma_{x,y}(\mathfrak{N})}{\omega} i, \quad (9.17)$$

or, in our case, (cf. (9.13))

$$\epsilon_{x,y}(\mathfrak{N}) = \frac{3\pi^2 c B_{x,y}}{\omega^2 |\mathfrak{N}|} i. \quad (9.18)$$

In the absence of spatial dispersion, i.e., when the

dielectric constant does not depend on the refractive index, $N = ck/\omega$ (for example, under conditions of the normal skin effect), $\zeta_{x,y}^{(a)} = \zeta_{x,y}^{(b)} = 1/\sqrt{\epsilon_{x,y}}$ (cf. formula (8.8)).

In the case of interest to us*

$$\zeta_{x,y}^{(a)} = (1 - i\sqrt{3}) \frac{2\sqrt{3}}{9} \left(\frac{\omega^2}{3\pi^2 c B_{x,y}} \right)^{1/3}, \quad (9.19)$$

$$\zeta_{x,y}^{(b)} = (1 - i\sqrt{3}) \frac{9}{8\sqrt{3}} \left(\frac{\omega^2}{3\pi^2 c B_{x,y}} \right)^{1/3}. \quad (9.20)$$

Comparing formulas (9.19) and (9.20), we see that the expressions obtained differ by a real factor of order unity ($\zeta^{(a)}/\zeta^{(b)} = 16/27$). This result confirms our statement about the relatively weak dependence of the surface impedance on the boundary conditions when we have the limiting anomalous skin effect. This statement is proved rigorously in [83], which shows that a change of the boundary conditions only changes a real factor of order unity in the impedance.

As we see from formulas (9.19), (9.20), and (9.14), anisotropy of the surface impedance, i.e., a dependence of $\zeta_{\alpha\beta}$ on the direction of the normal to the metal surface relative to the crystal axes, is closely related to the anisotropy of the dispersion law. We recall that the integration in (9.14) is taken over a line on the Fermi surface, where $v_z = 0$ (Fig. 23).

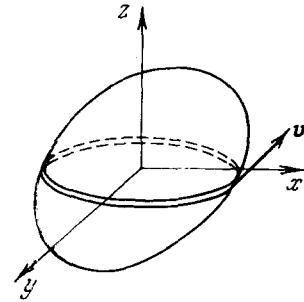


FIG. 23

When the direction of the normal is changed, this line shifts, changing the value of the integral. From the dependence of the impedance on the direction of the normal we can "sense" the shape of the Fermi surface. The shape of the Fermi surface of copper was determined in this way (Pippard [84]). Figure 24 shows the dependence of the impedance of copper on the direction of the normal, and Fig. 25 shows the Fermi surface of copper. Experiments on the De Haas-van Alphen effect [85] and on absorption of ultrasonics in a magnetic field [86] agree well with Pippard's results.

If we assume an isotropic dispersion law

*The calculation of the surface impedance under conditions for the limiting anomalous skin effect, for an arbitrary dispersion law, was done in the papers of Pippard, [82] M. Kaganov and M. Azbel'. [83]

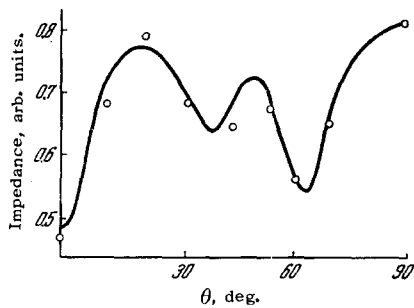


FIG. 24. Angular dependence of impedance of Cu under conditions of anomalous skin effect (after Pippard^[84]).

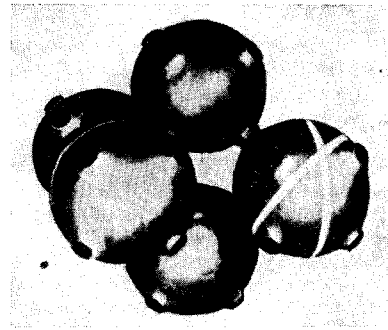


FIG. 25. Fermi surface of copper.

(potassium, sodium, etc), formulas (9.17) and (9.18) can be put in the following form:

$$\zeta^{(a)} = (1 - i\sqrt{3}) \frac{2\sqrt{3}}{9} \left(\frac{\omega^2 l}{3\pi^2 c \sigma} \right)^{1/3}, \quad (9.19')$$

$$\zeta^{(b)} = (1 - i\sqrt{3}) \frac{9}{8\sqrt{3}} \left(\frac{\omega^2 l}{3\pi^2 c \sigma} \right)^{1/3}, \quad (9.20')$$

so that they can be used for determining the free paths of electrons.^[87]

10. ABSORPTION OF ULTRASONICS BY METALS

In recent years the study of the absorption of sound in metals has been one of the most popular methods for studying electron energy spectra. This results from the fact that at low temperatures the electrons play an important part in the absorption. The latter is shown very well by the dependence of the sound absorption coefficient on temperature in the vicinity of the superconducting transition: there is a sharp reduction in the absorption coefficient at temperatures below the critical temperature, which is the best proof that the electrons are important (since we know that in the superconducting transition neither the lattice nor its vibrations undergo any significant changes).

The second fact that points to interaction of sound with the electrons is connected with the fact that at present it is comparatively easy to get metal samples in which the free path of electrons is much greater than the wave length of sound ($l \gg s/\omega$, where s is the sound velocity and ω its frequency). When $kl \gg 1$ ($k = \omega/s$) not all the electrons with energies of the order of the Fermi energy act effectively on the sound wave, but only those that move along with the wave, i.e., those for which $v = s$ (where v_n is the projection of the electron velocity along the wave vector k of the sound wave). Since the electron velocity is approximately a thousand times greater than the sound velocity, we may assume that only those electrons with $v_n = 0$ interact effectively with the sound wave. This situation is similar to that which is characteristic for the anomalous skin effect

(cf. Sec. 9). Here the gliding electrons are replaced by those that move in phase planes. In the formula for the absorption coefficient there naturally appears an integral over the band around $v_n = 0$ on the Fermi surface (cf. formula (10.8) below). But it is difficult to use this result directly to calculate the electron energy spectrum since the answer is not expressed in terms of quantities related to the dispersion law alone. More precisely, the absorption coefficient still depends on the tensor $\lambda_{ik}(\mathbf{p})$, equal to $(\partial \epsilon / \partial u_{ik})_{\epsilon = \epsilon_F}$, which plays the role of a "phonon charge," i.e., describes the interaction between electrons and phonons.*

The most interesting results concerning electron energy spectra have been obtained from measurements of absorption of sound in a strong magnetic field ($l \gg r$), naturally when $kl \gg 1$. As has been shown experimentally by Bommel^[94] and theoretically by Pippard,^[95] when $l \gg r$ and $kl \gg 1$ one observes oscillations of the absorption coefficient whose period is determined only by the extremal diameters of the Fermi surface. The observation of the Pippard oscillations is one of the main sources of information about the shape of the Fermi surface of metals.^[96]

In this Section we shall deal only with the theory of the absorption of sound by metals in the absence of a magnetic field. There is a relatively large literature on this subject at present,^[97,101] which includes a considerable amount of work about the derivation of the fundamental equations, generalizing the Maxwell equations and the equations of the dynamical theory of elasticity, and enabling one to calculate not only the absorption coefficient but also the dispersion of the sound velocity. We shall restrict our calculation of the sound absorption to two limiting cases: low frequencies ($kl \ll 1$) and high frequencies ($kl \gg 1$). It can be shown^[98] that the nature of the absorption is determined completely by

*This difficulty does not arise in the theory of the anomalous skin effect.

the ratio of the free path and the sound wave length, and is practically independent of the ratio of the sound frequency and the collision frequency.

In the first case ($kl \ll 1$) we use a macroscopic approach, while kinetic considerations enable us to establish the magnitude and temperature dependence of the coefficients appearing in the answer (viscosity, heat conductivity, etc). The collection of parameters describing the equilibrium state of a real metal results in a complicated frequency-temperature dependence of the absorption coefficient. But if these parameters are independent and we can introduce a relaxation time τ_j for each of them,* the absorption coefficient $\Gamma(\omega)$ can be written as follows:

$$\Gamma(\omega) = \sum_j A_j \frac{\omega^2 \tau_j}{1 + \omega^2 \tau_j^2}, \quad (10.1)$$

where the A_j are coefficients telling how much the sound wave shifts the various internal parameters (for example, the temperature) from their equilibrium values. A study of the frequency dependence of the absorption coefficient $\Gamma(\omega)$ allows one in certain cases to determine the relaxation times τ_j . Because of the use of a wide range of frequencies, this method (called the method of relaxation resonance) allows one to "sample" all sorts of interactions, including the very weak ones which lead to dissipation.

Very often, the absorption coefficient has a complicated dependence even when $kl \ll 1$, and is not described by (10.1), but can be expressed in macroscopic terms (cf., for example, [100]).

In calculating the absorption coefficient for high frequency sound ($kl \gtrsim 1$) the kinetic approach is necessary.

It is customary to assume that the passage of a sound wave through the metal results in a correction to the electron energy ϵ proportional to the deformation tensor and equal to

$$\delta\epsilon = \lambda_{ik}(\mathbf{p}) u_{ik}(\mathbf{r}, t), \quad (10.2)$$

where $\lambda_{ik}(\mathbf{p}) = [\partial\epsilon(\mathbf{p})/\partial u_{ik}]_{u_{ik}=0}$ is a second-rank tensor characterizing the interaction of the electrons with the acoustic phonons. A rigorous derivation of (10.2) exists. We should mention that the formula given here differs from the exact one by the omission of a term related to the Stewart-Tolman effect.^[14] As the estimates show,^[98] the term omitted is $v_F/s \approx 10^3$ times smaller than the one retained. The time dependence of the electron energy causes the system of electrons to leave the equilibrium state. Dissipative processes (collisions of the electrons with impurities, with one another and with phonons)

*In the simplest cases, τ_j can be expressed in terms of the macroscopic coefficients. For example, if the sound absorption is due to the thermal conductivity $\tau_j = \kappa/C_V s^2$, where κ is the thermal conductivity and C_V is the specific heat per unit volume.

are the cause of the sound damping, while the absorption coefficient is determined by the dissipation function per unit volume ($T\dot{S}$):

$$\Gamma = \frac{T\dot{S}}{E_{s,w.}}; \quad (10.3)$$

Here $E_{s,w.}$ is the energy of the sound wave per unit volume.

The entropy increase \dot{S} is calculated using formulas from statistical mechanics, for which one must know the nonequilibrium distribution function of the electrons (at low temperatures, when one can satisfy the condition $kl \gg 1$ given above, the role of phonons in the sound absorption is insignificant). The distribution function can be determined from the Boltzmann kinetic equation.

In this case, as shown in [98], the linearized kinetic equation can be written as follows:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \frac{\partial f_1}{\partial \mathbf{r}} + \left(\frac{\partial f_1}{\partial t} \right)_{\text{coll}} = - \frac{\partial f_F}{\partial \epsilon} e \mathbf{v} \mathbf{E} \frac{\partial n_F}{\partial \epsilon} \tilde{\lambda}_{iuvik}, \quad (10.4)$$

where $\tilde{\lambda}_{ijk} = \lambda_{ijk} - \bar{\lambda}_{ijk}$, and the dash denotes an average over all electron states, where the "weights" are the derivatives of the Fermi function:

$$\tilde{\varphi} = \left(\int \frac{\partial n_F}{\partial \epsilon} d\Gamma \right)^{-1} \int \varphi \frac{\partial n_F}{\partial \epsilon} d\Gamma. \quad (10.5)$$

The transverse part of the electric field \mathbf{E} must be found from the Maxwell equations, while the longitudinal part can be included in the renormalization of the chemical potential; it actually does not enter in the electroacoustic effects in the cases we are considering. We note that at these frequencies we cannot speak of temperature oscillations with such a frequency ($l \gg \lambda = s/\omega!$).

Since the sound frequency is as a rule much lower than the relaxation frequency ν of the electrons ($\nu = 1/\tau$), the first term on the left of (10.4) can be omitted. The second term should be written in the form $i(\mathbf{k} \cdot \mathbf{v}) f_1$, considering that the right side of the equation is proportional to $e i \mathbf{k} \cdot \mathbf{r}$. As we have already said, when $kl \gg 1$ the important electrons are those moving perpendicular to the wave vector. This (as in the case of the anomalous skin effect, Sec. 9) permits us to drop the integral (influx) term in the collision integral, keeping only the term associated with outflux of electrons from the states with $v_n = 0$.

Thus the kinetic equation (10.4) can be rewritten as follows:

$$\left(i \mathbf{k} \mathbf{v} + \frac{1}{\tau_{gl}} \right) f_1 = - \frac{\partial n_F}{\partial \epsilon} (e \mathbf{v} \mathbf{E}_\perp + \tilde{\lambda}_{ik} \dot{u}_{ik}). \quad (10.6)$$

If in place of f_1 we introduce the new function χ by the equation

$$f_1 = - \frac{\partial f_F}{\partial \epsilon} \chi,$$

one can show that, as $T \rightarrow 0$,

$$T\dot{S} = \frac{v(\epsilon_F)}{\tau_{gl}} |\chi|^2, \quad v(\epsilon_F) = - \int \frac{\partial n_F}{\partial \epsilon} d\Gamma, \quad (10.7)$$

where the dissipative function $\dot{T}S$ splits into a sum of two terms: the first is caused only by the term $\tilde{\lambda}_{ik}\dot{u}_{ik}$ on the right side of the kinetic equation (deformation interaction), and the second is caused by the Joule losses (i.e., by the term containing the electric field). When $k^2c^2/4\pi\sigma(k)\omega \gg 1$, the deformation interaction is of principal importance, while the Joule losses are most important in the opposite limiting case.^[101]

When $k^2c^2/4\pi\sigma(k)\omega \gg 1$, one can get the following expression for the absorption coefficient:^[98]

$$\Gamma(\omega) \simeq \frac{\pi\omega}{(2\pi\hbar)^3 S_0} \int_0^{2\pi} \frac{\tilde{\lambda}_{ik}\tilde{\lambda}_{lm}d\varphi}{v^2 K(\varphi)} \Phi_{iklm}; \quad (10.8)$$

where K is the Gauss curvature of the Fermi surface, and the integral is taken along the line where $\mathbf{v} \perp \mathbf{k}$ ($v_n = 0$), which, as we have already said, is connected with the predominant role of the electrons moving in the planes of equal phase of the sound wave. The components of the fourth-rank tensor Φ_{iklm} are of order unity and depend on the polarization \mathbf{q} and direction \mathbf{n} of the sound ($\Phi_{iklm} = n_i q_k n_l q_m$).

When $k^2c^2/4\pi\sigma(k)\omega \ll 1$, but $kl \gg 1$, the absorption coefficient again depends linearly on frequency, but with a different coefficient.^[101]

When $k^2c^2 \approx 4\pi\sigma(k)\omega$ ($kl \gg 1$) one should see a marked increase of the absorption coefficient, analogous to a kinematic resonance. The origin of this increase is the coincidence of wave lengths of the electromagnetic and sound waves.^[101] It is also shown in the paper of V. M. Kontorovich^[101] that there should also be marked dispersion of the sound velocity in this region of frequency.

As is clear from the preceding, the characteristic features of the sound absorption coefficient when $kl \gg 1$ are its linear dependence on frequency and independence of temperature. At low frequencies, $\Gamma \sim \omega^2\tau$ also increases with decreasing temperature, along with the relaxation time τ .

Since the components of the tensor λ_{ik} are of the order of the Fermi energy, while $K \approx 1/p_F^2$, one can use the following expression for $\Gamma(\omega)$ in making estimates:

$$\Gamma(\omega) \sim \frac{N\varepsilon_F\omega}{v_F q_s}. \quad (10.9)$$

The linear dependence of the absorption coefficient on frequency has been repeatedly observed in experiments on different metals. It serves as a reliable criterion for the fulfilment of the condition $kl \gg 1$.

It was shown in^[98] that the validity of formulas (10.8) and (10.9) is not restricted by the condition $\omega\tau \ll 1$, subject to which they were formally introduced. The linear increase in $\Gamma(\omega)$ also occurs at ultrahigh frequencies ($\omega\tau \gg 1$).

APPENDICES

I. CHANGE IN ELECTRON QUASIMOMENTUM IN SCATTERING OF AN ELECTRON BY A FORCE CENTER

In various physical problems the problem arises of the change in the quasimomentum of the electron in the individual act of scattering by a force center. In the quantum mechanical treatment one may be dealing either with the calculation of the transition probability from a state with quasimomentum \mathbf{p} to a state with quasimomentum \mathbf{p}' (such probabilities, denoted by $W_{\mathbf{p}, \mathbf{p}'}$, are usually used in studying the kinetic properties of conductors), or with the calculation of the probability that the electron has gone from the state \mathbf{p} to the state \mathbf{p}' , where the transition is accompanied by an umklapp of amount $2\pi\hbar\mathbf{b}$ where \mathbf{b} is a vector of the reciprocal lattice. The probability $W_{\mathbf{p}, \mathbf{p}'}$ is proportional to the square of the matrix element $\langle \mathbf{p} | V | \mathbf{p}' \rangle$,

$$\langle \mathbf{p} | V | \mathbf{p}' \rangle \sim \int \psi_{\mathbf{p}'}^* V(\mathbf{r}) \psi_{\mathbf{p}} d\mathbf{v}, \quad (I.1)$$

while $\psi_{\mathbf{p}}$ is a Bloch wave ($\psi_{\mathbf{p}} = e^{i\mathbf{p}\mathbf{r}/\hbar} u_{\mathbf{p}}(\mathbf{r})$), $u_{\mathbf{p}}(\mathbf{r})$ is a periodic function, $u_{\mathbf{p}}(\mathbf{r} + \mathbf{a}) = u_{\mathbf{p}}(\mathbf{r})$, \mathbf{a} is an arbitrary period of the crystal lattice, $V(\mathbf{r})$ is the potential for the interaction of the electron with the impurity center.

The probability of the second type (we denote it by $W_{\mathbf{p}', \mathbf{b}}^{\mathbf{p}}$), as shown in^[99] is equal to

$$W_{\mathbf{p}', \mathbf{b}}^{\mathbf{p}} = \frac{2\pi}{\hbar} \delta(\varepsilon_i - \varepsilon_f) |\langle \mathbf{p} | V | \mathbf{p}', \mathbf{b} \rangle|^2, \quad (I.2)$$

where

$$\langle \mathbf{p} | V | \mathbf{p}', \mathbf{b} \rangle = \int V(\mathbf{q}) e^{\frac{i\Delta\mathbf{p} \cdot \mathbf{p}}{\hbar}} d\mathbf{v}_p \sum_{\mathbf{b}'} (A_{\mathbf{b}'}^{\mathbf{p}})^* A_{\mathbf{b}'+\mathbf{b}}^{\mathbf{p}'}, \quad (I.3)$$

$$\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}' + 2\pi\hbar\mathbf{b},$$

$A_{\mathbf{b}}^{\mathbf{p}}$ are the Fourier components of the function $u_{\mathbf{p}}(\mathbf{r})$. The quantity $W_{\mathbf{p}', \mathbf{b}}^{\mathbf{p}}$ characterizes the process of transfer of momentum from the electron to the impurity center.

In the classical (more precisely, quasiclassical) approach, one can treat the motion of the electron in the field of the impurity, starting from its dispersion law, and state exactly into which final state the electron goes as a result of the scattering. Of course, in general (for an arbitrary dispersion law and for an arbitrary dependence of the interaction potential on distance) this problem can be solved only by a numerical solution of the equations of motion $\dot{\mathbf{p}} = -\frac{\partial V}{\partial \mathbf{r}}$, $\dot{\mathbf{r}} = \frac{\partial \varepsilon}{\partial \mathbf{p}}$. But some qualitative arguments about the character of the motion of the electron in the field of the impurity can be given by considering the head-on collision of the electron with the impurity. One can, for example, show that in the process of scattering the electron can go from one band on the constant energy surface to another. If the electron has the dispersion law $\varepsilon = p^2/2m$, (electron in free space) two things can occur as a result of the head-on collision: if there is a turning point in coordinate space, it returns to the opposite point on the constant-energy sphere (Fig. 26); if there is no turning point, the electron comes back to the same point on the constant-energy sphere from which it began the process of scattering (Fig. 27).

For an electron whose dispersion law is a periodic function, the situation is complicated, since the electron can be reflected not only from a point at which the energy is a minimum and the velocity vanishes, but also from a point where the energy is a maximum, since the velocity vanishes there also.

Figures 28-30 show various cases of the motion of an electron in the field of an impurity. For clarity we have taken a very

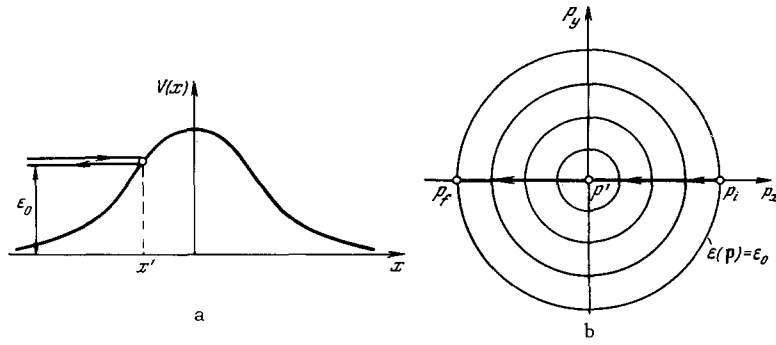


FIG. 26. Scattering of an electron with a quadratic isotropic dispersion law from a repulsive potential. ϵ_0 is the kinetic energy of the electron at infinity; x' is the turning point in x space; to it there corresponds the point p' in momentum space. The electron went over to the opposite side of the constant-energy surface.

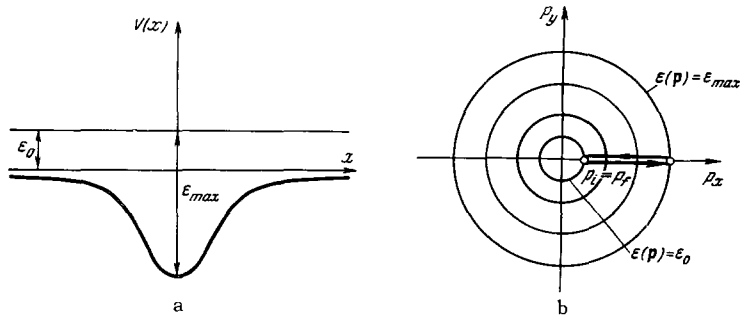


FIG. 27. Scattering of an electron with a quadratic dispersion law from an attractive potential. The notation is clear from the figure. The electron returns to the same point in p -space.

simple dispersion law. In analyzing the motion in momentum space, one must remember that in those cases where there is a turning point (Figs. 29 and 30) the force acting on the electron always has the same sign (there is no turning point in momentum space), while in those cases where there are no turning points the force changes sign, so that \dot{p} vanishes, i.e., there is a turning point in p -space.

As we see from Fig. 29, very frequently the particle in the process of scattering emerges into the neighboring cell in the reciprocal lattice.

In the case of small angle scattering (large impact parameters) it is difficult to find an analytic expression for the change of momentum in the scattering process.

If the force acting on the electron is central, the change in quasimomentum can be written as follows:

$$p' - p = n\delta_{\parallel} + m\delta_{\perp}, \quad (1.4)$$

where δ_{\parallel} and δ_{\perp} are the changes in momentum along the direction of the velocity $n = v/v$ and in the perpendicular direction $m = \rho/\rho$. Since the collision is elastic, from the energy conservation law we

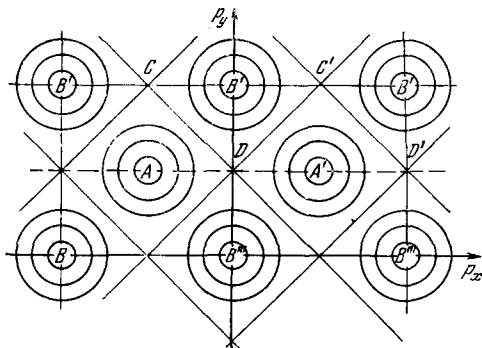


FIG. 28. Diagram of constant-energy surfaces: At A and the equivalent points (A' , A'' , etc) the energy is a minimum, at point B (B' , B'' , etc) the energy is a maximum. A head-on collision corresponds to motion only along the line AA' (and equivalents) and along BB' (and equivalents), since only on these lines is the velocity directed along the x axis. The points C and D (and the equivalent C' , D' , ...) are saddle points. During motion along the line AA' , the energy is maximum at point D; during motion along $B'B'$ the energy is minimum at point C.

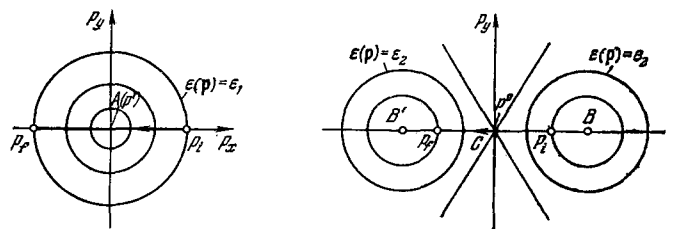
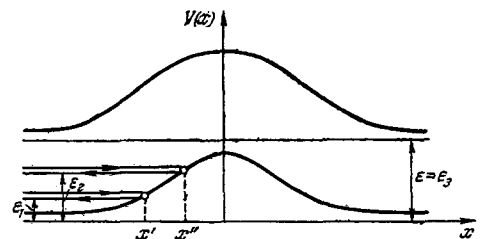


FIG. 29. Scattering of the electron with dispersion law, $\epsilon(p) = \epsilon$, shown in Fig. 28, in the field of a repulsive potential. If the electron energy far from the impurity is close to minimum ($\epsilon = \epsilon_1$), the electron remains within one cell of the reciprocal lattice; if the energy is close to maximum ($\epsilon = \epsilon_2$) the electron goes over to the neighboring cell of the reciprocal lattice; if the energy is such that there is no turning point ($\epsilon = \epsilon_3$) the electron will naturally always return to the point in p -space where the process began (cf. Fig. 27).

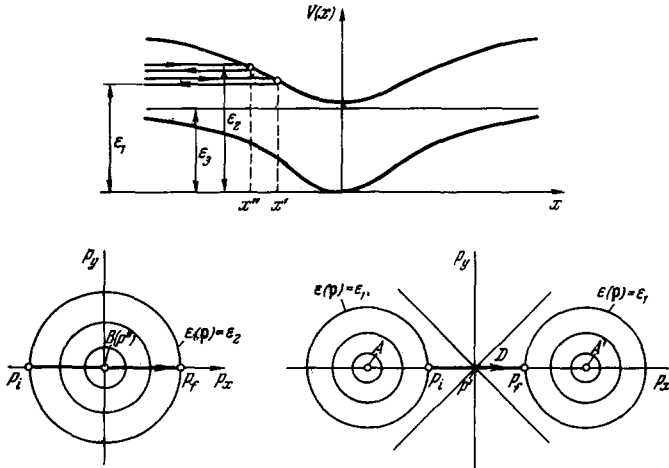


FIG. 30. Scattering of an electron with dispersion law $\epsilon(\mathbf{p}) = \epsilon$ (cf. Fig. 28) in the field of an attracting impurity (scattering from the top of the dispersion law). If the energy is close to minimum ($\epsilon = \epsilon_1$) the electron goes out into another cell; if the energy is close to maximum ($\epsilon = \epsilon_2$) the electron remains in the same cell. At sufficiently small energies (for example, $\epsilon = \epsilon_3$) there are no turning points of the electron in x -space (cf. Fig. 27).

have

$$\epsilon(\mathbf{p} + n\delta_{\parallel} + m\delta_{\perp}) = \epsilon(\mathbf{p}). \quad (\text{I.5})$$

Expanding in series in δ_{\parallel} and δ_{\perp} up to terms of second order, we get

$$0 = v(n\delta_{\parallel} + m\delta_{\perp}) + \frac{1}{2}(M^{-1})_{ik}(n_i\delta_{\parallel} + m_i\delta_{\perp})(n_k\delta_{\parallel} + m_k\delta_{\perp}), \quad (\text{I.6})$$

where $(M^{-1})_{ik} = \frac{\partial^2 \epsilon}{\partial p_i \partial p_k}$ is the inverse effective mass tensor (cf. [1],

App. I). The scalar product of the vectors \mathbf{v} and \mathbf{m} is zero, $\mathbf{v} \cdot \mathbf{n} = v$, the terms containing δ_{\parallel}^2 and δ_{\perp}^2 can be dropped, since they are small compared to the term linear in δ_{\parallel} ; thus

$$\mathbf{v}\delta_{\parallel} = -\frac{1}{2}(M^{-1})_{ik}m_i m_k \delta_{\perp}^2. \quad (\text{I.7})$$

In considering the flux of electrons we most often need the average change of momentum. Averaging (I.7) over the azimuth angle ϵ , and noting that $m_i m_k = \frac{1}{2}\delta_{ik}$ ($i, k = z, y$, the x axis is along the electron trajectory), we get

$$\delta_{\parallel} = -\frac{1}{4v} \left(\frac{1}{M_{zz}} + \frac{1}{M_{yy}} \right) \delta_{\perp}^2(\rho). \quad (\text{I.8})$$

We note that $\delta_{\parallel} v$ is the change of the average momentum of a unit current of electrons with impact parameter ρ per unit time.

The value of δ_{\perp} can be determined immediately from the equations of motion:

$$\delta_{\perp} = \int_{-\infty}^{\infty} F_{\rho} dt = \frac{1}{v} \int_{-\infty}^{\infty} F_{\rho} dx, \quad (\text{I.9})$$

or, to the same accuracy,

$$\delta_{\perp} = -\frac{q}{v} \int_{-\infty}^{\infty} \frac{\partial V}{\partial x} \frac{dx}{r}, \quad r = \sqrt{\rho^2 + x^2}. \quad (\text{I.10})$$

II. MAGNETIC MOMENT OF A METAL IN A STRONG MAGNETIC FIELD

As shown in Part II of this survey, [1] only the oscillations of the magnetic susceptibility (de Haas - van Alphen effect) can be expressed in terms of the dispersion law. The paramagnetic and dia-

magnetic susceptibility of a metal depend not only on states of the electron with energies significantly different from the Fermi energy, but also on the correlation function. Besides, experimentally one cannot distinguish the contribution of the conduction electrons from the contribution of the ion cores of the metal.

Azbel' and Skrotskaya [2] called attention to the fact that measurement of the magnetic moment of a metal in the limit of high fields ($\mu H \gg \epsilon_F$, T ; for metals like Bi this condition is satisfied in reasonable fields $H \gg 10^4$ oe) can give information about the structure of the electron ground state in high field. As shown in [2], the magnetic moment of a metal M in strong field is given by the simple formula:

$$\mathbf{M} = -N \frac{\partial \epsilon_0}{\partial \mathbf{H}} + NT \frac{\partial \ln v_0}{\partial \mathbf{H}}, \quad (\text{II.1})$$

where N is the density of electrons, $\epsilon_0(\mathbf{H})$ and $v_0(\mathbf{H})$ are the energy and density of states for the lowest energy level of the conduction electrons (with the lowest quantum numbers). Thus the measurement of the magnetic moment and its temperature behavior in principle gives the possibility of determining $\frac{\partial \epsilon_0}{\partial \mathbf{H}}$ and $\frac{\partial \ln v_0}{\partial \mathbf{H}}$.

It is interesting to note that the dependence of the magnetic moment on strong magnetic field for a free electron gas [2] is essentially different from the dependence for electrons in a metal. This is related, as shown in [2], to the difference between the ordinary and the effective mass.

III. PROPERTIES OF THE LINEARIZED COLLISION OPERATOR $\hat{W}_{\mathbf{p}}$

Let us first consider the interaction of electrons with impurities. The interaction Hamiltonian in second quantized form is written as follows:

$$H_{\text{int}}^{e, \text{imp}} = \sum_{\mathbf{p}\mathbf{p}'} B_{\mathbf{p}\mathbf{p}'}^p a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}'} + \text{c.c.}, \quad (\text{III.1})$$

where $a_{\mathbf{p}}^{\dagger}$ ($a_{\mathbf{p}}$) are the creation (annihilation) operators for an electron with momentum \mathbf{p} , while $B_{\mathbf{p}\mathbf{p}'}^p$ is the matrix element for the transition of an electron from state \mathbf{p} to state \mathbf{p}' . The presence of the hermitian conjugate terms (h.c.) guarantees the hermiticity of the interaction operator ($B_{\mathbf{p}\mathbf{p}'}^p = (B_{\mathbf{p}'\mathbf{p}}^p)^*$).

Constructing the collision operator according to the standard rules, we have

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}}^{e, \text{imp}} = \frac{2\pi}{\hbar} \int |B_{\mathbf{p}\mathbf{p}'}^p|^2 \{ f'(1-f) - f(1-f') \} \delta(\epsilon - \epsilon') d\Gamma'. \quad (\text{III.2})$$

Here $f(\mathbf{p}) \equiv f$, $f(\mathbf{p}') \equiv f'$, $\epsilon(\mathbf{p}) \equiv \epsilon$, $\epsilon(\mathbf{p}') \equiv \epsilon'$, while the δ function guarantees energy conservation (the collision is assumed to be elastic). Introducing a new function φ in place of f by the equation

$$f = n_F - \frac{\partial n_F}{\partial \epsilon} \varphi, \quad (\text{III.3})$$

we get

$$\left(\frac{\partial \varphi}{\partial t} \right)_{\text{coll}}^{e, \text{imp}} = \int P_{\mathbf{p}\mathbf{p}'} (\varphi' - \varphi) d\Gamma', \quad (\text{III.4})$$

where

$$P_{\mathbf{p}\mathbf{p}'} = -\frac{2\pi}{\hbar} |B_{\mathbf{p}\mathbf{p}'}^p|^2 \delta(\epsilon - \epsilon') \frac{\partial n_F}{\partial \epsilon}. \quad (\text{III.5})$$

According to the definition (1.13), the action of the operator $\hat{W}_{\mathbf{p}}$ is defined as follows:

$$\hat{W}_{\mathbf{p}} \varphi = -\frac{2\pi}{\hbar} \int |B_{\mathbf{p}\mathbf{p}'}^p|^2 \delta(\epsilon - \epsilon') (\varphi' - \varphi) d\Gamma'.$$

We note that for the case of an anisotropic dispersion law the operator \hat{W}_p does not reduce to a multiplication, i.e., the τ -approximation is not justified even in this simplest case of elastic collisions. Now let us construct a scalar product of the type (1.20):

$$\langle \chi, \hat{W}_p \varphi \rangle = -\frac{2\pi}{\hbar} \int \int \frac{\partial n_F}{\partial \epsilon} \chi |B_p^{p'}|^2 \delta(\epsilon - \epsilon') (\varphi' - \varphi) d\Gamma' d\Gamma.$$

Changing $p \leftrightarrow p'$ in the integral and using the hermiticity of the matrix $B_p^{p'}$, (cf. above) it is easy to show that

$$\langle \chi, \hat{W}_p \varphi \rangle = -\frac{2\pi}{\hbar} \int \int \frac{1}{2} \left(\frac{\partial n_F}{\partial \epsilon} + \frac{\partial n_F}{\partial \epsilon'} \right) \times (\chi' - \chi) |B_p^{p'}|^2 \delta(\epsilon - \epsilon') (\varphi' - \varphi) d\Gamma' d\Gamma. \quad (\text{III.6})$$

The properties (1.23) and (1.24) then follow quickly.

Now let us consider collisions with phonons. The Hamiltonian for the electron-phonon interaction is conveniently written in the following form:

$$H_{\text{int}}^{e, \text{ph}} = \sum_{pp'q} \{ A_{pp'q}^{p'q} a_p^+ a_{p'} b_q^+ + A_{pp'q}^{p'q} a_p a_{p'}^+ b_q \}, \quad (\text{III.7})$$

where $A_{pp'q}$ are the transition matrix elements, having the following symmetry: $A_{pp'q}^{p'q} \equiv (A_{pp'q}^{p'q})^*$, guaranteeing the hermiticity of the interaction Hamiltonian $H_{\text{int}}^{e, \text{ph}}$, the operators b_q^+ and b_q are the operators for creation and annihilation of a phonon with wave vector q ; we have omitted indices labelling the electron and phonon bands.

In formula (III.7) the summation goes over all momenta that satisfy the conservation condition

$$p + q - p' = 2\pi\hbar b, \quad (\text{III.8})$$

where b is an arbitrary vector of the reciprocal lattice.

Constructing the collision integral in standard form, and again introducing the function φ by (III.3), in the linear approximation in φ we have

$$\left(\frac{\partial \varphi}{\partial t} \right)_{\text{coll}}^{e, \text{ph}} = \int \int (L_{pp'}^{p'q} + L_{pp'}^q) (\varphi' - \varphi) d\Gamma'_e d\Gamma_{\text{ph}}; \quad (\text{III.9})$$

here

$$L_{pp'}^{p'q} = \frac{2\pi}{\hbar T} \sum_b \delta(p' - p - q - 2\pi\hbar b) \delta(\epsilon' - \epsilon - \hbar\omega_q) \times |A_{pp'}^{p'q}|^2 n_F(1 - n_F) N_B(\hbar\omega_q). \quad (\text{III.10})$$

For simplicity we have assumed that the phonon gas is in equilibrium*, and have used the symmetry properties of the matrix $A_{pp'}^{p'q}$. The symmetry and the positive character of the kernel of the collision integral $\int (L_{pp'}^{p'q} + L_{pp'}^q) d\Gamma_{\text{ph}}$ guarantees the properties (1.23) and (1.24). We note that in writing the action of the operator \hat{W}_p , it should be divided by $\frac{\partial n_F}{\partial \epsilon}$, and in constructing scalar products of the type (1.20) it should be multiplied by the same function.

It is also not difficult to prove similar relations for the case of electron-electron collisions.

If we use the explicit form (III.9) for the electron-phonon collision integral, one can show that the limiting temperature laws for the resistance (expression (2.28)) hold for an arbitrary dispersion law for the electrons, and that their derivation uses only extremely general properties of the collision operator.

At high temperatures a linear law is guaranteed by the fact

*One can show that including deflection of phonons and inter-phonon collisions does not change the properties of \hat{W}_p (cf., for example, [17]).

that for $T \gg \theta$ the collisions of electrons with phonons are quasi-elastic and one can, in the first place, neglect the phonon energy in the δ functions describing the energy conservation law, and secondly expand the Bose function $(\exp \frac{\hbar\omega}{T} - 1)^{-1} \approx T/\hbar\omega$. In

other words, a linear law for the resistance is determined simply by the fact that the number of phonons when $T \gg \theta$ is proportional to the temperature.

At low temperatures the situation is somewhat more complicated. From the structure of the collision integral we see that in place of the cartesian momentum components (p_x, p_y, p_z) it is more convenient to take as arguments of the function φ the energy $\epsilon = \epsilon(p)$, or more precisely the quantity $(\epsilon - \zeta)/T$, and a vector p_ϵ characterizing the position of the electron on the constant energy surface. Since at low temperatures the momentum of the phonon is much lower than the momentum of the electron, the change of the function φ in the sense of its dependence on p_ϵ should be regarded as slow. This allows us to go over for this variable from an integral to a differential operator (in the sense of the Fokker-Planck equation). The quantity $x = (\epsilon - \zeta)/T$ itself changes considerably in each collision, since $\hbar\omega \approx T$. Thus in the sense of an operation on φ as a function of x , the collision operator remains an integral operator, even after the simplifications associated with the fact that $T \ll \theta \ll \epsilon_F$ (but $T \gg \theta^2/\epsilon_F$). As the result of quite involved transformations the kinetic equation for the function $\varphi = \varphi_{p_\epsilon}(x)$ (cf. Eq. III.3) takes the form

$$T^{-1} e E v n_F'(x) = \left(\frac{T}{\theta} \right)^3 \int_{-\infty}^{\infty} dx K_{xx}^{p_\epsilon} [\varphi_{p_\epsilon}(x) - \varphi_{p_\epsilon}(x')] + \left(\frac{T}{\theta} \right)^5 \hat{L} \{\varphi_{p_\epsilon}(x)\}; \quad (\text{III.11})$$

here

$$K_{xx'}^{p_\epsilon} = \alpha \frac{(x - x')^2 n_F(x) n_F(x')}{|e^{-x} - e^{-x'}|},$$

$\alpha = \alpha(p_\epsilon)$ is a quantity of order θ/\hbar , depending on p_ϵ in the same way as the integro-differential operator \hat{L} (integral in x and second-order differential in p_ϵ). In getting (III.11) we have used the fact that when $q \rightarrow 0$, $A_{pp'}^{p'q}/\sqrt{\hbar\omega_q}$ tends to a limit that is independent of the modulus of the wave vector of the phonon.

The first term on the right, proportional to $(T/\theta)^3$, is much greater than the second, but the second term cannot be neglected: it guarantees the existence of a solution. In fact, without the second term the equation has no solution, since the integral of the left side with respect to x from $-\infty$ to ∞ is different from zero, while that of the first term is equal to zero.

We look for a solution of (III.11) by the method of successive approximations: we choose φ as a function of p_ϵ alone ($\varphi = \varphi_{p_\epsilon}$) in order to make the first term on the left of (III.11) vanish. We find the function φ_{p_ϵ} from the equation which is gotten from (III.11) by integrating with respect to x . It is clear that $\varphi_{p_\epsilon} \approx 1/T^6$. This also leads to the temperature dependence of the resistance given in (2.28) ($\rho_{e, \text{ph}} \approx T^5$ for $T \ll \theta$).

IV. GENERAL PROPERTIES OF THE DEPENDENCE OF THE ELECTRICAL CONDUCTIVITY TENSOR ON MAGNETIC FIELD.

For this discussion it is convenient to write the electrical conductivity tensor σ_{ik} as follows:

$$\sigma_{ik} = e^2 \langle v_i \Psi_k \rangle \equiv -e^2 \int \frac{\partial n_F}{\partial \epsilon} v_i \Psi_k d\Gamma, \quad (\text{IV.1})$$

where the function ψ_k must be found from the kinetic equation (5.7):

$$\frac{\partial \psi_k}{\partial t} + \hat{W}_p \psi_k = v_k, \quad (5.7)$$

while the operator \hat{W}_p (and its inverse \hat{W}_p^{-1}) satisfies conditions (1.23) and (1.24). The operator $\partial/\partial t$ changes sign when we change the sign of the magnetic field; furthermore it is easy to show that the operator $\frac{\partial}{\partial t}$ is an antihermitian operator, i.e.,

$$\langle \varphi, \frac{\partial \chi}{\partial t} \rangle = - \langle \frac{\partial \varphi}{\partial t}, \chi \rangle. \quad (IV.2)$$

Let us write φ_k as the sum of an even and an odd function (of the magnetic field):

$$\psi_k = \psi_k^s + \psi_k^a, \quad \psi_k^s(\mathbf{H}) = \psi_k^s(-\mathbf{H}), \quad \psi_k^a(\mathbf{H}) = -\psi_k^a(-\mathbf{H});$$

then

$$\begin{aligned} \frac{\partial \psi_k^s}{\partial t} + \hat{W}_p \psi_k^s &= v_k, \\ \frac{\partial \psi_k^a}{\partial t} + \hat{W}_p \psi_k^a &= 0. \end{aligned} \quad (IV.3)$$

From the last equations we easily obtain equations for each of the functions:

$$\begin{aligned} \hat{W}_p \psi_k^s - \frac{\partial}{\partial t} \left(\hat{W}_p^{-1} \frac{\partial \psi_k^s}{\partial t} \right) &= v_k, \\ \hat{W}_p \psi_k^a - \frac{\partial}{\partial t} \left(\hat{W}_p^{-1} \frac{\partial \psi_k^a}{\partial t} \right) &= - \frac{\partial}{\partial t} \left(\hat{W}_p^{-1} v_k \right). \end{aligned} \quad (IV.4)$$

The functions ψ_k^s and ψ_k^a can be used to calculate the symmetric and antisymmetric parts of the electrical conductivity tensor:

$$s_{ik} = e^2 \langle v_i \psi_k^s \rangle, \quad a_{ik} = e^2 \langle v_i \psi_k^a \rangle, \quad \sigma_{ik} = s_{ik} + a_{ik}. \quad (IV.5)$$

We shall show that the properties (1.23) and (1.24) of the collision operator guarantee the validity of the Onsager relations for the tensor σ_{ik} .

If we use formulas (IV.1) and (IV.5) and also (IV.4), we can show that

$$\left. \begin{aligned} e^2 s_{ik} &= \langle \psi_i^s \hat{W}_p \psi_k^s \rangle + \left\langle \frac{\partial \psi_i^s}{\partial t} \hat{W}_p^{-1} \frac{\partial \psi_k^s}{\partial t} \right\rangle \\ &= \langle v_i \hat{W}_p^{-1} v_k \rangle - \left\langle \frac{\partial \psi_i^s}{\partial t} \hat{W}_p^{-1} \frac{\partial \psi_k^s}{\partial t} \right\rangle \\ &\quad - \left\langle \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \psi_k^s}{\partial t} \hat{W}_p^{-1} \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \psi_i^s}{\partial t} \right\rangle, \\ e^2 a_{ik} &= \left\langle \psi_i^a \frac{\partial \psi_k^a}{\partial t} \right\rangle - \left\langle \hat{W}_p^{-1} \frac{\partial \psi_k^a}{\partial t} \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \psi_i^a}{\partial t} \right\rangle \\ &\quad - \left\langle \hat{W}_p^{-1} v_i \frac{\partial}{\partial t} \hat{W}_p^{-1} v_k \right\rangle. \end{aligned} \right\} \quad (IV.6)$$

From these expressions the following are clear:

- symmetry of the tensor s_{ik} and antisymmetry of the tensor a_{ik} , i.e., the validity of the Onsager relations;
- positiveness of the eigenvalues of the tensor s_{ik} ; and finally
- decrease of the diagonal elements of the electrical conductivity tensor with field, i.e.,

$$0 \leq \sigma_{11}(\mathbf{H}) \leq \sigma_{11}(0) \text{ etc.}$$

The last statement is obvious from the first equality (IV.6), since $e^2 \langle v_i \hat{W}_p^{-1} v_k \rangle \equiv \sigma_{ik}(0)$.

For calculating the tensors R_{ik} and λ_{iklm} we can use approximate values of the functions ψ_k^s and ψ_k^a :

$$\begin{aligned} \psi_k^s &\approx \hat{W}_p^{-1} v_k + \hat{W}_p^{-1} \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial}{\partial t} \hat{W}_p^{-1} v_k, \\ \psi_k^a &\approx -\hat{W}_p^{-1} \frac{\partial}{\partial t} \hat{W}_p^{-1} v_k. \end{aligned} \quad (IV.7)$$

If we recall that $\hat{W}_p^{-1} v_i = l_i$ (cf. (2.8)), we easily find from (IV.5) and (IV.7)

$$a_{ik} \equiv e^2 \left\langle \frac{\partial l_i}{\partial t} l_k \right\rangle, \quad s_{ik} = \sigma_{ik}(0) - e^2 \left\langle \frac{\partial l_i}{\partial t} \hat{W}_p^{-1} \frac{\partial l_k}{\partial t} \right\rangle. \quad (IV.8)$$

If, on the other hand, we write

$$a_i = a_{ik} H_k, \quad s_{ik} = \sigma_{ik}(0) + \mu_{iklm} H_l H_m, \quad (IV.9)$$

then by simple but tedious transformations we can get

$$\begin{aligned} \alpha_{ik} &= \frac{e}{2c} \left\{ \left\langle v_r \frac{\partial l_r}{\partial p_q} l_q \right\rangle - \left\langle v_r \frac{\partial l_q}{\partial p_q} l_r \right\rangle \right\} \delta_{ik} \\ &\quad + \left\langle v_i \frac{\partial l_r}{\partial p_r} l_k \right\rangle + \left\langle v_r l_r \frac{\partial l_k}{\partial p_i} \right\rangle - \left\langle v_i \frac{\partial l_k}{\partial p_r} l_r \right\rangle \\ &\quad - \left\langle v_r \frac{\partial l_r}{\partial p_i} l_k \right\rangle \}, \end{aligned} \quad (IV.10)$$

$$\mu_{iknr} = - \frac{e^2}{c^2} \{ \langle u_{mil} a_{mkl} \rangle \delta_{nr} + \langle u_{ril} a_{ikn} \rangle + \langle u_{mtr} a_{nkm} \rangle \}, \quad (IV.11)$$

where

$$u_{mil} = l_m \frac{\partial l_i}{\partial p_l}, \quad a_{mkl} = v_m \frac{\partial l_k}{\partial p_l} - v_l \frac{\partial l_k}{\partial p_m}.$$

In the case of cubic symmetry of the crystal, the second-rank symmetric tensor a_{ik} degenerates to a scalar ($a_{ik} = \alpha \delta_{ik}$), and

$$\alpha = \frac{e}{6c} \left\{ \left\langle l_i v_h \frac{\partial l_i}{\partial p_h} \right\rangle - \left\langle l_i v_i \frac{\partial l_h}{\partial p_h} \right\rangle \right\}. \quad (IV.12)$$

As we see from formula (IV.11), the tensor μ_{iklm} has an extremely complicated structure. For a cubic crystal there are six independent components.

Knowledge of the tensors $\hat{\alpha}$ and $\hat{\mu}$ enables us to determine the components of the tensors \hat{R} and $\hat{\lambda}$, by using approximate inversion formulas:

$$\left. \begin{aligned} Q_{ik}^a(\mathbf{H}) &= -Q_{il} \alpha_{lm} Q_{mk}, \\ Q_{ik}^s(\mathbf{H}) &= Q_{ik} + Q_{il} \alpha_{lm} Q_{mn} \alpha_{np} Q_{pk} - Q_{il} s'_{lm} Q_{mk}, \end{aligned} \right\} \quad (IV.13)$$

where $\hat{s}' = \hat{s} - \hat{\sigma}(0)$. Then

$$R_{pn} = \frac{1}{2} (Q_{ik}^2 - Q_{ik}^2) \alpha_{pn} + (Q_{il} Q_{pk} - Q_{pl} Q_{ik}) \alpha_{kn}. \quad (IV.14)$$

For a cubic crystal

$$R = -Q^2 \alpha. \quad (IV.15)$$

(In formulas (IV.13)-(IV.15) the resistance values are taken at $H = 0$.)

We shall not give the messy formulas for the components of the tensor λ_{iklm} . We show only that from conditions (1.23) and (1.24) there follows the positive character of the diagonal elements of the tensor $\Delta \rho_{ik}^s = \rho_{ik}^s - \rho_{ik}^a$.

According to formulas (IV.13), (IV.5) and (IV.8), we have

$$\frac{\Delta Q_{11}^s}{Q_{11}^2} = \left\langle \frac{\partial l_1}{\partial t} \hat{W}_p^{-1} \frac{\partial l_1}{\partial t} \right\rangle - \left\langle \frac{\partial l_1}{\partial t} \hat{W}_p^{-1} \xi_2 \right\rangle^2 - \left\langle \frac{\partial l_1}{\partial t} \hat{W}_p^{-1} \xi_3 \right\rangle^2, \quad (IV.16)$$

where

$$\xi_{2,3} = \frac{v_{2,3}}{\langle v_{2,3} \hat{W}_p^{-1} v_{2,3} \rangle^{1/2}}$$

are axes in a space with the norm $\langle \varphi \hat{W}_p^{-1} \varphi \rangle$, φ is an arbitrary function and the orthogonality of ξ_2 and ξ_3 is guaranteed by the

fact that the 1, 2 and 3 axes are chosen along the principal directions of the tensor σ_{ik} (and consequently of ρ_{ik}). It is easily verified that in our approximation the principal directions of the tensors ρ_{ik}^s and ρ_{ik} coincide.

It is clear that the right side of (IV.16) is greater than or equal to zero, since the square of the "vector" $\frac{\partial l_1}{\partial t}$ is greater than or equal to the sum of the squares of its projections along any two axes (of the infinite number). Equality holds only when the "vector" $\frac{\partial l_1}{\partial t}$ lies in the "plane" of ξ_2 and ξ_3 .

V. CALCULATION OF THE COMPONENTS OF THE TENSOR ρ_{ik}^s AND THE VECTOR b IN HIGH FIELDS ($l \gg r$)

According to Appendix II

$$\sigma_{ik}(\mathbf{H}) = s_{ik} + a_{ik}, \tag{V.1}$$

where

$$s_{ik} = e^2 \langle v_i \psi_k^s \rangle, \quad a_{ik} = e^2 \langle v_i \psi_k^a \rangle, \tag{V.2}$$

and $\psi_i^{s(a)}$ are subject to Eqs. (IV.4). Inverting the tensor σ_{ik} , we easily find

$$Q_{ik} = Q_{ik}^s + Q_{ik}^a, \quad Q_{ik} = \sigma_{ik}^{-1},$$

where

$$Q_{ik}^s = \frac{|s| s_{ik}^{-1} + a_{ik}}{|s| + (a\hat{s}a)}, \quad b_i = \frac{(\hat{s}a)_i}{|s| + (a\hat{s}a)}. \tag{V.3}$$

Here a_i is the vector dual to the tensor a_{ik} , and b_i to the tensor ρ_{ik}^a .

Now let us calculate the functions ψ_i^s and ψ_i^a as series in inverse powers of the field. To do this it is convenient to rewrite Eqs. (IV.4) as follows:

$$\frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \psi_i^s}{\partial t} = \gamma^2 (\hat{W}_p \psi_i^s - v_i), \tag{V.4}$$

$$\frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \psi_i^a}{\partial t} = \gamma^2 \hat{W}_p \psi_i^a + \gamma \frac{\partial}{\partial t} (W_p^{-1} v_i), \tag{V.5}$$

look for a solution in the form of a series in powers of γ , and set $\gamma = 1$ in the answer. Averaging Eqs. (V.4) and (V.5) over the period of revolution of the electron, we find the conditions which serve as boundary conditions (cf. (5.9) and (5.10)):

$$\overline{W_p \psi_i^s} = -v_i, \tag{V.6}$$

$$\overline{W_p \psi_i^a} = 0. \tag{V.7}$$

But since Eqs. (V.4) and (V.5) contain second derivatives with respect to t , we need more conditions for a unique determination of the solution (cf. below).

Thus,

$$\psi_i^s = \sum_{n=0}^{\infty} \gamma^{2n} \varphi_n^i, \quad \psi_i^a = \gamma \sum_{n=0}^{\infty} \gamma^{2n} \chi_n^i. \tag{V.8}$$

Substituting formulas (V.8) in Eqs. (V.4) and (V.5) and equating coefficients of like powers of γ^2 , we get

$$\left. \begin{aligned} \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \varphi_0^i}{\partial t} &= 0, & \overline{W_p \varphi_0} &= -v_i, \\ \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \varphi_1^i}{\partial t} &= \hat{W}_p \varphi_0^i - v_i, & \overline{W_p \varphi_1} &= 0, \\ \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \varphi_2^i}{\partial t} &= \hat{W}_p \varphi_1^i, & & \dots \dots \dots \\ & & \hat{W}_p \varphi_n^i &= 0 \quad (n \geq 1), \\ \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \varphi_n^i}{\partial t} &= \hat{W}_p \varphi_{n-1}^i \quad (n > 1) & & \dots \dots \dots \end{aligned} \right\} \tag{V.9}$$

$$\left. \begin{aligned} \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \chi_0^i}{\partial t} &= \frac{\partial}{\partial t} \hat{W}_p^{-1} v_i, \\ \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \chi_1^i}{\partial t} &= \hat{W}_p \chi_0^i, & \overline{W_p \chi_n^i} &= 0 \quad (n \geq 0), \\ & & & \dots \dots \dots \\ \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \chi_n^i}{\partial t} &= \hat{W}_p \chi_{n-1}^i \quad (n \geq 1). & & \dots \dots \dots \end{aligned} \right\} \tag{V.10}$$

The bar denotes an average over a period (cf. formula (5.10)). We first consider the system (V.9).

In place of the function φ_0^i it is convenient to introduce another function $\varphi^i = \varphi_0^i - \hat{W}_p^{-1} v_i$. Then the system (V.9) takes the form

$$\left. \begin{aligned} \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \varphi^i}{\partial t} &= -\frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial}{\partial t} \hat{W}_p^{-1} v_i, \\ \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \varphi^i}{\partial t} &= \hat{W}_p \varphi^i, & \overline{W_p \varphi_n^i} &= \overline{W_p \varphi_n} = 0, \\ & & & \dots \dots \dots \\ \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial \varphi_n^i}{\partial t} &= \hat{W}_p \varphi_{n-1}^i \quad (n \geq 1). & & \dots \dots \dots \end{aligned} \right\} \tag{V.11}$$

From (V.11) it is clear that

$$\varphi_n^i = \hat{q} \varphi_{n-1}^i, \tag{V.12}$$

where the operator \hat{q} is given by the following equation:

$$\hat{q}\chi = \int_{-\infty}^t \hat{W}_p \left\{ \int_{-\infty}^{t'} \hat{W}_p \chi dt'' - \overline{W_p^{-1} \hat{W}_p} \int_{-\infty}^{t'} \hat{W}_p \chi dt'' \right\} dt'$$

$$- \overline{W_p^{-1} \hat{W}_p} \int_{-\infty}^t \hat{W}_p \left\{ \int_{-\infty}^{t'} \hat{W}_p \chi dt'' - \overline{W_p^{-1} \hat{W}_p} \int_{-\infty}^{t'} \hat{W}_p \chi dt'' \right\} dt';$$

consequently

$$\varphi_n^i = \hat{q}^n \varphi^i. \tag{V.13}$$

Now let us calculate the function φ^i . From the first equation of the system (V.11) we have

$$\varphi^i = \overline{W_p^{-1} v_i} - \hat{W}_p^{-1} v_i. \tag{V.14}$$

We now use formula (V.8), setting $\gamma = 1$

$$\begin{aligned} \psi_i^s &= \hat{W}_p^{-1} v_i + \varphi^i + \hat{q} \varphi^i + \hat{q}^2 \varphi^i + \dots \\ &= \hat{W}_p^{-1} v_i + \frac{1}{1-\hat{q}} (\overline{W_p^{-1} v_i} - \hat{W}_p^{-1} v_i). \end{aligned} \tag{V.15}$$

Formula (V.15) corresponds to an expression of ψ_i in powers of

the inverse square of the field. From (V.15) and (V.2) it follows that

$$s_{ik} = e^2 \langle v_i \hat{W}_p^{-1} v_k \rangle + e^2 \left\langle v_i \frac{1}{1-\hat{q}} (\hat{W}_p^{-1} \bar{v}_k - \hat{W}_p^{-1} v_k) \right\rangle. \quad (V.16)$$

In completely analogous fashion, we obtain from the system (V.10)

$$\psi_i^a = \chi_0^i + \hat{q} \chi_0^i + \hat{q}^2 \chi_0^i + \dots \equiv \frac{1}{1-\hat{q}} \chi_0^i, \quad (V.17)$$

where

$$\chi_0^i = \int_{-\infty}^t (v_i - \hat{W}_p \hat{W}_p^{-1} \bar{v}_i) dt' - \overline{\hat{W}_p^{-1} \hat{W}_p} \int_{-\infty}^t (v_i - \hat{W}_p \hat{W}_p^{-1} \bar{v}_i) dt'. \quad (V.18)$$

Then

$$a_{ik} = e^2 \left\langle v_i \frac{1}{1-\hat{q}} \chi_0^k \right\rangle. \quad (V.19)$$

In the τ -approximation ($\hat{W}_p \equiv 1/\tau$) we can get even more compact formulas (analogous to formula (5.26) or (48) of [36]):

$$\psi_i^s = \int_0^\infty v_i^s(t, t') e^{-\frac{t'}{\tau}} dt', \quad (V.20)$$

$$\psi_i^a = - \int_0^\infty v_i^a(t, t') e^{-\frac{t'}{\tau}} dt', \quad (V.21)$$

where

$$v_i^{s,a}(t, t') = \frac{1}{2} [v_i(t+t') \pm v_i(t-t')]. \quad (V.22)$$

Then

$$s_{ik} = e^2 \langle \psi_{ik}^s \rangle, \quad a_{ik} = -e^2 \langle \psi_{ik}^a \rangle, \quad (V.23)$$

and

$$\psi_{ik}^{s,a} = \int_0^\infty v_i(t) v_k^{s,a}(t, t') e^{-\frac{t'}{\tau}} dt'. \quad (V.24)$$

We also note that formulas (V.16) and (V.19) can be used for calculating the tensors s_{ik} and a_{ik} in weak fields. For this purpose it is more convenient to write them as follows:

$$s_{ik} = \langle v_i \hat{W}_p^{-1} v_k \rangle + \left\langle v_i \frac{1}{1-\hat{q}} \frac{\hat{1}}{q} \hat{W}_p^{-1} v_k \right\rangle, \quad (V.25)$$

$$a_{ik} = - \left\langle v_i \frac{1}{1-\hat{q}} \hat{W}_p^{-1} \frac{\partial}{\partial t} \hat{W}_p^{-1} v_k \right\rangle, \quad (V.26)$$

where $\frac{\hat{1}}{q} = \hat{W}_p^{-1} \frac{\partial}{\partial t} \hat{W}_p^{-1} \frac{\partial}{\partial t}$; terms independent of t standing to the right of the operator $\hat{1}/q$ have been dropped. The first terms of the expansion obtained from formulas (V.25) and (V.26) coincide with formulas (IV.8).

Now we shall give the asymptotic values of the various components of the tensors s_{ik} and a_{ik} . We note that the coordinate system we use is naturally "linked" only to the magnetic field. This means that we are at liberty to choose the x and y axes in the plane perpendicular to the magnetic field ($\mathbf{H} \parallel \text{Oz}$). We choose them so that the tensor* $s_{\alpha\beta}$ ($\alpha, \beta = x, y$) has no nondiagonal com-

ponents; then

$$\left. \begin{aligned} s_{xx} &= \left(\frac{c}{eH}\right)^2 \langle p_y \hat{W}_p (p_y - \hat{W}_p^{-1} \hat{W}_p p_y) \rangle + \dots, \\ s_{yy} &= \left(\frac{c}{eH}\right)^2 \langle p_x \hat{W}_p (p_x - \hat{W}_p^{-1} \hat{W}_p p_x) \rangle + \dots, \\ s_{zz} &= \langle v_z \hat{W}_p v_z \rangle + \langle v_z \hat{q} (\hat{W}_p^{-1} v_z - \hat{W}_p^{-1} v_z) \rangle + \dots, \\ s_{\alpha z} = s_{z\alpha} &= - \langle v_z \hat{q} \hat{W}_p^{-1} v_\alpha \rangle + \dots, \end{aligned} \right\} \quad (V.27)$$

i.e., the tensor s_{ik} has the following structure:

$$s_{ik} = \begin{pmatrix} \frac{\gamma_{xx}}{H^2} & 0 & \frac{\gamma_{xz}}{H^2} \\ 0 & \frac{\gamma_{yy}}{H^2} & \frac{\gamma_{yz}}{H^2} \\ \frac{\gamma_{xz}}{H^2} & \frac{\gamma_{yz}}{H^2} & \gamma_{zz} \end{pmatrix}. \quad (V.28)$$

The components of the matrix γ_{ik} tend to constants for $H \rightarrow \infty$.

For the first terms in the expansion of the antisymmetric tensor a_{ik} we easily find the following formulas:

$$\left. \begin{aligned} a_z = a_{xy} = -a_{yx} &= \frac{c}{eH} \langle p_y v_x \rangle + \dots, \\ a_y = a_{zx} = -a_{xz} &= -\frac{c}{eH} \langle p_y u_z \rangle + \dots, \\ a_x = -a_{zy} = a_{yz} &= -\frac{c}{eH} \langle p_x u_z \rangle + \dots, \end{aligned} \right\} \quad (V.29)$$

where

$$u_z = v_z - \hat{W}_p \hat{W}_p^{-1} \bar{v}_z.$$

Since the quantity $\langle p_y v_x \rangle$ is equal to $e^2(n_1 - n_2)$ for closed constant-energy surfaces (cf. formulas (5.17) - (5.21)),

$$a_z = a_{xy} = \frac{ec}{H} (n_1 - n_2). \quad (V.30)$$

Using the values found here for the components of the tensors s_{ik} and a_{ik} we get the asymptotic values of $|s|$ and $(a \hat{s} a)$:

$$\begin{aligned} |s| &= \frac{\gamma_{xx} \gamma_{yy} \gamma_{zz}}{H^4} + \dots, \\ (a \hat{s} a) &= \gamma_{zz} a_{xy}^2 + \dots \end{aligned}$$

The dots denote terms containing higher powers of the reciprocal of the magnetic field. Starting from these last formulas, we see that the denominators in formulas (V.3) when $n_1 \neq n_2$ are of order $\gamma_{zz} a_{xy}^2$, while when $n_1 = n_2$ they contain all the components of the matrices s_{ik} and a_{ik} .

Calculating the various components of the tensor ρ_{ik}^s and the vector b_i we see that the matrix ρ_{ik}^s has the following structure: when $n_1 \neq n_2$ all the components of ρ_{ik}^s tend to saturation*, while $b_z = H/(n_1 - n_2)ec$, b_x and b_y are proportional to $1/H$, i.e., they tend to zero as $H \rightarrow \infty$. This means that, to high accuracy, in strong fields the Hall vector \mathbf{b} is parallel to the magnetic field (for any symmetry of the crystal).

When $n_1 = n_2$ all components of the tensor ρ_{ik}^s except ρ_{zz}^s increase quadratically in strong field, ρ_{zz}^s tends to saturation (but $\rho_{zz}^s \neq s_{zz}^{-1}$); all the components of the Hall vector \mathbf{b} increase linearly with increasing magnetic field (the Hall vector is not parallel to the magnetic field).

* $\rho_{zz}^s = s_{zz}^{-1}$, while the other components are given by rather complicated expressions in terms of the components of the tensor s_{ik} and the vector \mathbf{a} .

*Using the property (1.23) of the operator \hat{W}_p , one can show that $s_{ik} = s_{ki}$ (cf. Appendix IV).

In conclusion we prove the following theorem: if the magnetic field is perpendicular to one of the principal directions in the crystal, then when $n_1 \neq n_2$ the transverse resistance ρ_{xx} in strong magnetic fields ($H \rightarrow \infty$) is greater than or equal to the resistance in the absence of the field* $\rho_{xx}(0) = 1/\sigma_{xx}(0)$. The x axis coincides with one of the principal directions of the tensor $\sigma_{ik}(0)$.

Using the asymptotic behavior of the components of the tensor σ_{ik} (cf. formula (V.28)), we have

$$\rho_{xx}(\infty) \cong \frac{\sigma_{yy}\sigma_{zz} + (\sigma_{yz})^2}{(\sigma_{yx})^2\sigma_{zz}} \geq \frac{\sigma_{yy}}{(\sigma_{yx})^2};$$

so that

$$\frac{\rho_{xx}(\infty)}{\rho_{xx}(0)} \geq \frac{\sigma_{yy}(\infty)\sigma_{xx}(0)}{\sigma_{yx}^2(\infty)}.$$

Using the notation of this paper, the last inequality can be written as follows:

$$\frac{\rho_{xx}(\infty)}{\rho_{xx}(0)} \geq \frac{\langle v_y \psi_y \rangle \langle v_x \hat{W}_p^{-1} v_x \rangle}{\langle v_x \psi_y \rangle^2}.$$

Taking v_y from (5.7), we get

$$\frac{\rho_{xx}(\infty)}{\rho_{xx}(0)} \geq \frac{\langle \psi_y \hat{W}_p \psi_y \rangle \langle v_x \hat{W}_p^{-1} v_x \rangle}{\langle \psi_y v_x \rangle^2} = \frac{\langle \psi_y \hat{W}_p \psi_y \rangle \langle \hat{W}_p^{-1} v_x \hat{W}_p \hat{W}_p^{-1} v_x \rangle}{\langle \psi_y \hat{W}_p \hat{W}_p^{-1} v_x \rangle^2}.$$

In the last transformation we have used the property (1.23) of the operator \hat{W}_p . The relation obtained also proves our assertion since the numerator of the fraction is the product of the "squares" of the vectors ψ_y and $\hat{W}_p^{-1} v_x$, while the denominator is the "square" of the scalar product of the same vectors (for more detail, cf. [86]).

¹I. M. Lifshitz and M. I. Kaganov, UFN 78, 411 (1962), Soviet Phys. Uspekhi 5, 878 (1963).

²M. Ya. Azbel' and I. M. Lifshitz, Progr. Low Temperatures (Amsterdam) 3, 288 (1961).

³I. M. Lifshitz and M. I. Kaganov, UFN 69, 419 (1959), Soviet Phys. Uspekhi 2, 831 (1960).

⁴L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media) Moscow, Gostekhizdat, 1957, translation, Pergamon.

⁵B. N. Aleksandrov, Doctoral Dissertation (Khar'kov, FTINT, Academy of Sciences, Ukrainian SSR, 1964).

⁶V. I. Smirnov, Kurs vyssheĭ matematiki (Course of Higher Mathematics) vol. IV, Moscow, Gostekhizdat, 1951.

⁷L. D. Landau, JETP 30, 1058 (1956), 32, 59 (1957); Soviet Phys. JETP 3, 920 (1957), 5, 101 (1957).

⁸V. P. Silin, JETP 33, 495 (1957), Soviet Phys. JETP 6, 387 (1958).

⁹L. D. Landau, JETP 35, 97 (1958), Soviet Phys. JETP 8, 70 (1959).

¹⁰A. V. Migdal, JETP 34, 1438 (1958), Soviet Phys. JETP 7, 996 (1958).

*When $n_1 = n_2$ this statement is obvious, since $\rho_{xx} \sim H^2(\rho_{xx} \rightarrow \infty)$ for $H \rightarrow \infty$.

¹¹R. Reich and Q. Kinh, Compt. rend. 256, 156 (1963).

¹²I. M. Lifshitz and L. N. Rozentsveĭg, JETP 16, 967 (1946); I. M. Lifshitz and G. P. Parkhomovskii, Uch. zap. KhGU (Scientific Papers of Khar'kov State University) 27, 28 (1948).

¹³M. I. Kaganov, in the collection "Magnetic Structure of Ferromagnets," Moscow, AN SSSR, 1960, p. 79.

¹⁴J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941.

¹⁵V. L. Ginzburg, in the collection "Memorial to A. A. Andronov," Moscow, AN SSSR, 1955, p. 622.

¹⁶A. A. Abrikosov, JETP 44, 2039 (1963), Soviet Phys. JETP 17, 1372 (1963).

¹⁷J. Ziman, "Electrons and Phonons," Oxford, Clarendon Press, 1960.

¹⁸R. Peierls, "Quantum Theory of Solids," Oxford, Clarendon Press, 1955.

¹⁹L. D. Landau and I. Ya. Pomeranchuk, JETP 7, 379 (1937).

²⁰V. L. Ginzburg and V. P. Silin, JETP 29, 64 (1955), Soviet Phys. JETP 2, 46 (1956).

²¹B. W. Roberts, Phys. Rev. Letters 6, 453 (1961); J. F. Cochran and M. Yaqub, Phys. Letters 5, 307 (1963).

²²I. Ya. Pomeranchuk, JETP 35, 992 (1958), Soviet Phys. JETP 8, 693 (1959).

²³R. N. Gurzhi, JETP 47, 1415 (1964), Soviet Phys. JETP 20, 953 (1965).

²⁴A. D. Brailsford and A. W. Overhauser, Phys. Rev. Letters 3, 331 (1959); J. Phys. Chem. Solids 15, 140 (1960), 21, 127 (1961).

²⁵Kaganov, Lifshitz, and Tanatarov, JETP 31, 232 (1956), Soviet Phys. JETP 4, 173 (1957).

²⁶E. S. Borovik, DAN SSSR 91, 771 (1953); V. L. Ginzburg and V. P. Shabanskiĭ, DAN SSSR 100, 445 (1955); M. I. Kaganov and V. G. Peschanskiĭ, JETP 33, 1231 (1957), Soviet Phys. JETP 6, 970 (1958).

²⁷S. R. deGroot and P. Maxur, Thermodynamics of Nonequilibrium Processes.

²⁸M. H. Cohen and L. M. Falicov, Phys. Rev. Letters 7, 231 (1961).

²⁹M. Kohler, Ann. Physik 32, 211 (1938).

³⁰J. L. Olsen, Helv. Phys. Acta 31, 713 (1958).

³¹M. C. Jones and E. H. Sondheimer, Phys. Letters 11, 122 (1964).

³²Borovik, Volotskaya, and Fogel', JETP 45, 46 (1963), Soviet Phys. JETP 18, 34 (1964).

³³R. J. Balcolme, Proc. Roy. Soc. (London) 275, 113 (1963).

³⁴N. S. Akulov, Ferromagnetism, Moscow, Leningrad, ONTI, 1939; S. V. Vonsovskii and Ya. S. Shur, Ferromagnetism, Moscow-Leningrad, OGIz, 1948.

³⁵M. Ya. Azbel', JETP 44, 983 (1963), Soviet Phys. JETP 17, 667 (1963).

^{35a}M. Ya. Azbel' and V. G. Peschanskiĭ, JETP 49, 572 (1965).

- ³⁶ Lifshitz, Azbel', and Kaganov, JETP 31, 63 (1956), Soviet Phys. JETP 4, 41 (1957).
- ³⁷ F. G. Bass and M. I. Kaganov, JETP 32, 1233 (1957), Soviet Phys. JETP 5, 1002 (1957).
- ³⁸ I. M. Lifshitz, JETP 32, 1509 (1957), Soviet Phys. JETP 5, 1227 (1957); I. M. Lifshitz and A. M. Kosevich, JETP 33, 88 (1957), Soviet Phys. JETP 6, 67 (1958); E. W. Adams and T. D. Holstein, J. Phys. Chem. Solids 10, 254 (1959).
- ³⁹ Coleman, Funes, Plaskett and Tapp, Phys. Rev. 133, 521 (1964).
- ⁴⁰ R. G. Chambers, Proc. Phys. Soc. (London) 81, 877 (1963).
- ⁴¹ E. S. Borovik, JETP 30, 262 (1956), Soviet Phys. JETP 3, 243 (1956).
- ⁴² V. G. Volotskaya, JETP 45, 49 (1963), Soviet Phys. JETP 18, 36 (1964).
- ⁴³ C. C. Grimers and A. F. Kip, Phys. Rev. 132, 1991 (1963); A. C. Thorsen and T. G. Berlincourt, Phys. Rev. Letters 6, 617 (1961); R. G. Chambers and B. K. Jones, Proc. Roy. Soc. (London) A270, 417 (1962); K. Okumura and I. M. Templeton, Phil. Mag. 7, 1239; D. Shoenberg and P. J. Stiles, Proc. Roy. Soc. (London) A281, 62 (1964).
- ⁴⁴ P. B. Alers and R. T. Webber, Phys. Rev. 91, 1060 (1953); Rao, Zebouni, Grenier, and Reynolds, Phys. Rev. 133, 141 (1964).
- ⁴⁵ A. A. Abrikosov and L. A. Fal'kovskii, 43, 1089 (1962), Soviet Phys. JETP 16, 769 (1963).
- ⁴⁶ M. I. Kaganov and V. G. Peschanskiĭ, JETP 35, 1052 (1958), Soviet Phys. JETP 8, 734 (1959).
- ⁴⁷ Alekseevskii, Brandt, and Kostina, DAN SSSR 105, 46 (1955).
- ⁴⁸ Bass, Kaganov, and Slezov, FMM 5, 407 (1957).
- ⁴⁹ E. S. Borovik, JETP 23, 91 (1952).
- ⁵⁰ E. S. Borovik, Dissertation (FTIAN Ukr. SSR, Khar'kov, 1954).
- ⁵¹ L. Esaki, Phys. Rev. Letters 8, 4 (1962).
- ⁵² S. G. Kalashnikov, FTT 6, 2435 (1964).
- ⁵³ E. Fawcett, Advances in Phys. 13, 139 (1964).
- ⁵⁴ The Fermi Surface (W. A. Harrison, and M. B. Webb, Editors), J. Wiley and Son, N. Y., 1960.
- ⁵⁵ I. M. Lifshitz and V. G. Peschanskiĭ, JETP 35, 1251 (1958), Soviet Phys. JETP 8, 875 (1959).
- ⁵⁶ I. M. Lifshitz and V. G. Peschanskiĭ, JETP 38, 188 (1960), Soviet Phys. JETP 11, 137 (1960).
- ⁵⁷ V. G. Peschanskiĭ, Dissertation (Khar'kov, 1959).
- ⁵⁸ Alekseevskii, Gaĭdukov, Lifshitz, and Peschanskiĭ, JETP 39, 1201 (1960), Soviet Phys. JETP 12, 837 (1961).
- ⁵⁹ V. N. Kachinskiĭ, DAN SSSR 135, 818 (1960), Soviet Phys. Doklady 5, 1260 (1961); JETP 41, 665 (1961), 43, 1158 (1962), Soviet Phys. JETP 14, 476 (1962), 16, 818 (1963).
- ⁶⁰ M. H. Cohen and L. M. Falicov, Phys. Rev. Letters 7, 231 (1961).
- ⁶¹ E. I. Blount, Phys. Rev. 126, 1636 (1962).
- ⁶² W. A. Harrison, Phys. Rev. 118, 1182 (1960).
- ⁶³ L. M. Falicov and P. R. Sievert, Phys. Rev. Letters 12, 558 (1964).
- ⁶⁴ Stark, Eck and Gordon, Phys. Rev. A133, 441 (1964); R. W. Stark, Phys. Rev. A135, 1698 (1964).
- ⁶⁵ Boyle, Hsu, and Kunzler, Phys. Rev. Letters 4, 278 (1960); LePage, Garber, and Blatt, Phys. Letters 11, 102 (1964).
- ⁶⁶ Azbel', Kaganov, and Lifshitz, JETP 32, 1188 (1957), Soviet Phys. JETP 5, 967 (1957).
- ⁶⁷ Bychkov, Gurevich, and Nedlin, JETP 37, 534 (1959), Soviet Phys. JETP 10, 377 (1960).
- ⁶⁸ L. É. Gurevich and G. M. Nedlin, JETP 37, 765 (1959), Soviet Phys. JETP 10, 546 (1960).
- ⁶⁹ V. L. Ginzburg, JETP 29, 748 (1955), Soviet Phys. JETP 2, 589 (1956).
- ⁷⁰ V. L. Ginzburg and G. P. Motulevich, UFN 55, 469 (1955).
- ⁷¹ É. A. Kaner and M. I. Kaganov, JETP 31, 459 (1956), Soviet Phys. JETP 4, 393 (1957).
- ⁷² M. A. Leontovich, in the collection "Investigations of Propagation of Radio Waves", Moscow, AN SSSR, 1948.
- ⁷³ Bowers, Legendy, and Rose, Phys. Rev. Letters 7, 339 (1961); A. Libchaber and R. Veilex, Phys. Rev. 127, 774 (1962).
- ⁷⁴ Bass, Blank, and Kaganov, JETP 45, 1081 (1963), Soviet Phys. JETP 18, 747 (1964).
- ⁷⁵ M. I. Kaganov and V. M. Tsukernik, JETP 35, 474 (1958), Soviet Phys. JETP 8, 327 (1959).
- ⁷⁶ H. London, Proc. Roy. Soc. (London) A176, 522 (1940).
- ⁷⁷ G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) A195, 336 (1949).
- ⁷⁸ A. B. Pippard, Proc. Roy. Soc. (London) A191, 385 (1947).
- ⁷⁹ M. I. Kaganov and V. V. Slezov, JETP 32, 1496 (1957), Soviet Phys. JETP 5, 1216 (1957); Gurzhi, Azbel' and Hao, FTT 5, 759 (1963), Soviet Phys. Solid State 5, 554 (1963).
- ⁸⁰ É. A. Kaner, Dissertation, Moscow, 1964.
- ⁸¹ M. Ya. Azbel' and É. A. Kaner, JETP 32, 896 (1957), Soviet Phys. JETP 5, 730 (1957).
- ⁸² A. B. Pippard, Proc. Roy. Soc. (London) A224, 273 (1954).
- ⁸³ M. I. Kaganov and M. Ya. Azbel', DAN SSSR 102, 49 (1955).
- ⁸⁴ A. B. Pippard, Phil. Trans. Roy. Soc. A250, 325 (1957).
- ⁸⁵ D. Shoenberg, Phil. Mag. 5, 105 (1960); A. S. Joseph and A. C. Thorsen, Phys. Rev. A134, 979 (1964).
- ⁸⁶ H. V. Bohm and V. J. Easterling, Phys. Rev. 128, 1021 (1962).
- ⁸⁷ R. G. Chambers, Proc. Roy. Soc. (London) A215, 481 (1952).
- ⁸⁸ M. I. Kaganov and V. G. Peschanskiĭ, (in press).
- ⁸⁹ Kaganov, Lifshitz, and Fiks, FTT 6, 2723 (1964), Soviet Phys. Solid State 6, 2167 (1964).

- ⁹⁰ V. B. Fiks, FTT 1, 16, 1321 (1959), Soviet Phys. Solid State 1, 14, 972 (1959-60); G. E. Pikus and V. B. Fiks, FTT 1, 1062 (1959), Soviet Phys. Solid State 1, 1212 (1960).
- ⁹¹ M. Ya. Azbel' and E. G. Skrotskaya, JETP 47, 1958 (1964), Soviet Phys. JETP 20, 1315 (1965).
- ⁹² Yu. B. Rumer, JETP 18, 1081 (1948).
- ⁹³ E. S. Borovik, JETP 27, 355 (1954).
- ⁹⁴ H. E. Bömmel, Phys. Rev. 100, 758 (1955).
- ⁹⁵ A. B. Pippard, Phil. Mag. 2, 1147 (1957).
- ⁹⁶ A. A. Galkin and A. P. Korolyuk, JETP 37, 310 (1959), Soviet Phys. JETP 10, 219 (1960); V. J. Easterling and H. V. Bohm, Phys. Rev. 125, 812 (1962); K. Fossheim and T. Olsen, Physica Status Solidi 6, 867 (1964); Bezuglyĭ, Galkin, and Zhevago, JETP 47, 825 (1964), Soviet Phys. JETP 20, 552 (1965).
- ⁹⁷ V. P. Silin, JETP 38, 977 (1960), Soviet Phys. JETP 11, 703 (1960); K. V. Vlasova and B. N. Filippov, JETP 44, 922 (1963), Soviet Phys. JETP 17, 628 (1963); V. G. Skobov and É. A. Kaner, JETP 46, 273 (1964), Soviet Phys. JETP 19, 189 (1964).
- ⁹⁸ Akhiezer, Kaganov, and Lyubarskiĭ, JETP 32, 837 (1957), Soviet Phys. JETP 5, 685 (1957).
- ⁹⁹ A. I. Akhiezer, 8, 1330 (1938).
- ¹⁰⁰ M. I. Kaganov and V. G. Peschanskiĭ, FTT 5, 3215 (1963), Soviet Phys. Solid State 5, 2353 (1964).
- ¹⁰¹ V. M. Kontorovich, JETP 45, 1638 (1963), Soviet Phys. JETP 18, 1125 (1964).
- ¹⁰² J. Kondo, Progr. Theor. Phys. (Kyoto) 32, 37 (1964); Cf. also A. A. Abrikosov, JETP 48, 990 (1965), Soviet Phys. JETP 21, 660 (1965); JETP Letters 1, No. 1, 53 (1965), transl. 1, 33 (1965); S. H. Liu, Phys. Rev. A137, 1209 (1965).
- ¹⁰³ V. F. Gantmakher and I. P. Krylov, JETP 47, 2111 (1964), Soviet Phys. JETP 20, 1418 (1965).
- ¹⁰⁴ E. P. Vol'skiĭ, JETP 46, 123 (1964), Soviet Phys. JETP 19, 89 (1964).
- ¹⁰⁵ L. A. Fal'kovskiĭ and G. S. Razina, JETP 49, 265 (1965), Soviet Phys. JETP 22, 187 (1966).

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