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*RELATIVISTIC ASTROPHYSICS. II.**

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CONTENTS

1. INTRODUCTION

ASTRONOMY is indeed undergoing a dramatic period, a period when many of the most important fundamental problems have already been formulated and distinctly posed, but have not yet been resolved.

The stirring conflict of this drama has arisen long before the discovery of the celestial bodies of the new type, called superstars or quasistellar radio sources (quasars). The conflict concerns the final fate of ordinary stars.

The theory of the structure of stars, which are in a state of slow evolution, has been developed in detail and is in excellent agreement with observations. The distribution of the temperature and of the density in the sun and other stars containing a sufficient reserve of hydrogen (stars of the principal series of the Hertzsprung-Russell diagram) has been fully calculated. Their luminosity (total energy release), radius, spectrum, and evolution were calculated. It turned out that the relations obtained agree with the observations. This outstanding accomplishment of the last twenty years convinces us of the correctness of the main premises of the theories concerning the properties of matter at the temperature of stellar interiors and the rate of nuclear reactions under these conditions.

However, whereas the theory is correct as applied to the stationary state of the star, it becomes also necessary to regard seriously the deductions of the theory concerning the final fate of stars. The general trend of evolution consists in the consumption of the nuclear fuel, and in a gradual rise of the temperature and density at the center of the star.

The final state, however, is sought without tracing in detail the entire evolution, but by using a different approach. We shall assume that all the nuclear fuel has already been consumed (or else the reactions would continue), but the temperature has dropped to zero (or else the outward radiation of energy would continue), and we shall seek the distribution of matter satisfying the condition of mechanical equilibrium.

For stars having a mass smaller than 1.2 solar masses ($M_\odot$) the answer is well known: an equilibrium state is obtained in which the electron shells of the atoms have been crushed, but the nuclei are still at a sufficient distance from one another. The pressure of the degenerate electron gas counteracts the gravitation. This possibility was pointed out by Fowler[147] and J. I. Frenkel.[138] The stars in such a state are called white dwarfs. The observations confirm this prediction of the theory. When the stellar mass exceeds 1.2 $M_\odot$, but is smaller than the critical mass

*The first part of the review was published in UFN 84 (3), 377 (1964), [Soviet Phys. Uspekhi 7, 763 (1965)]. This will henceforth be cited as[1].

$M_{crit} \approx 2M_{\odot}$, *the equilibrium state is a neutron star.* The matter has been compressed to a density of the same order as the density of the atomic nucleus (10^{14} g/cm^3). The radius of the star is on the order of 10 km, the potential of the force of gravity is on the order of $0.2\,c^2$.

Under these conditions it is necessary to take into account those changes of the laws of gravitation which follow from Einstein's general theory of relativity. It is meaningless in principle to separate here the effects of special relativity (the weight of energy) and the general theory (curvature of space, effect of gravitation on the flow of time).

What does relativity theory contribute to the question of the fate of a star?

When the mass is smaller than M_{crit}, only quantitative changes take place. But the very existence of a maximum critical mass is the result of relativity theory. It turns out here that the critical mass exists for any conceivable equation of state compatible with relativity theory. When the mass exceeds the critical value, there is no equilibrium solution. The final stage of the evolution should be unbounded compression. At this stage, an account of the general theory of relativity leads to a deduction which is paradoxical at first glance: owing to the slowing down of the course of time, a remote observer will register an asymptotic approach of the star to a definite state (see[1]). This is not an equilibrium state, and can be called the "cooled" state.* Actually there is no paradox at all, the deduction of the theory is simply unexpected and unusual. The relativistic slowing down of the time denotes simultaneously that the frequencies of the quanta received by the observers tend to zero. A gravitational self-closing of the star takes place, the star ceases to radiate energy, and the flow of information to the external observer stops.

Thus, the theory predicts three types of celestial bodies in the final state, depending on their mass: 1) white dwarfs, 2) neutron stars, 3) "cooled" stars. The drama (and possibly also the tragedy) of astronomy consists in the fact that the latter two types of bodies have not yet been observed. It is precisely those bodies for which relativity theory plays an important or decisive role, which have not been observed.

The question of the existence of such bodies plays an important role also for cosmology, since the presence of neutron and cooled stars influences the average density of matter in the metagalaxy; the average density of all types of matter determines the curvature of space in large scales, and consequently determines whether the uniform metagalaxy is closed or infinite.

The first rough estimate[36] has led to the hypothesis that the total mass of the cooled stars can be comparable with the mass of the visible stars. However, this estimate depends strongly on the assumptions made in[36].

What are the possible ways of resolving the conflict between theory and observation? First, it is possible that we have so far not observed neutron and cooled stars only because they are difficult to observe. Consequently, it is necessary first to solve the question of their properties. What properties do they possess? How must they manifest themselves in the neighborhood of other stars, in the interstellar medium containing dust, gas, or a magnetic field? Can there be many cooled and neutron stars in the galaxies, and in our own galaxy?

Second, it is necessary to analyze the assumptions which have led to the conclusion that this indicated final state of the star is unavoidable, particularly the role of rotation of the star and its magnetic field.

During the course of evolution of a massive star, the increase in density is accompanied by an increase in temperature. At a definite instant of time the star approaches the limits of stability, beyond which catastrophic compression begins. However, up to that instant, the matter of the star still contains a reserve of nuclear energy.* The release of this energy can cause the contraction to give way to expansion and to an explosion of the star. It must be borne in mind, however, that the observed frequency of stellar explosions yields a number many times smaller than the expected number of stars whose evolution is terminating; in other words, the observational data provide evidence which speaks rather against the assumption that all stars are transformed into neutron or cooled states by explosion.

Thus, the disparity between the deductions of the theory and observations has been objectively in existence for a long time; however, the discovery of quasars has made the situation much more acute. All attempts to describe a quasar by means of traditional concepts, by transferring to a mass $\sim 10^8\, M_{\odot}$ the usual picture of a gas sphere in the state of mechanical equilibrium, in which energy is released as a result of nuclear reactions, have failed. In connection with the discovery of quasars, dozens of theoreticians have returned to the theory of equilibrium and contraction of stars, with account of general relativity; the astronomers have recalled the classical papers of Oppenheimer, Volkoff, and Snyder of the 1938—1939 period. At the same time, in accord with a natural psychological law, the assumption appeared that the two puzzles, namely the fate of ordinary stars and the nature of quasars, are interrelated and perhaps have a common answer.

The present article presents a detailed review of the above questions. The material is schematically

*In this part of the review we shall use a new term, "cooled" star, for stars during the stage of relativistic collapse. In the first part of the review such stars were called "collapsed."

*This margin depends essentially on whether different layers of the star have become intermixed by convection.

arranged in such a way that the first part of the review[1] contains the justification of the need for the stars to go over into a neutron or cooled state under very simple assumptions; this, as it were, is the statement of the problem. In part II we consider, on the one hand, the observational properties of such objects and, on the other, we analyze in detail the reasons why a star can avoid collapse or, to the contrary, will reach the state of collapse; in other words, we consider the possible variants of the answer; some problems in the theory of quasars are expounded. There is no complete theory at present. Many of the variants have already dropped out (Secs. 2—5), and ways of creating such a theory are beginning to appear (Sec. 16). The rapid development of the question hindered greatly the compilation of the review; the authors have reviewed the literature up to the end of 1964, and less completely the later papers.

2. EQUILIBRIUM OF A SUPERMASSIVE STAR

a) Energy Approach to the Theory of Equilibrium of a Star

The first attempt at explaining the nature of quasars as sources of energy sufficient for the formation of radio galaxies was the attempt to represent them as supermassive stars with $M \sim (10^5-10^9) M_\odot$,[2] while applying to them all the usual concepts and approximations of stellar theory. The theory of such hypothetical stars is, obviously, also of interest in itself. Therefore, after having considered in[1] questions concerning the equilibrium and evolution of an ordinary star, we shall stop to discuss briefly the theory of a supermassive star.

We recall that a star in the usual state is in hydrodynamic equilibrium, and the release of nuclear energy (if it takes place) proceeds slowly and does not upset the conditions of hydrodynamic equilibrium. The energy from the central regions seeps out to the surface and is radiated into the surrounding space. Initially the star consists principally of hydrogen. Such a star is situated on the "principal sequence" of the Hertzsprung-Russell diagram. As the hydrogen is burned up, the parameters of the star change gradually. The limiting case is a star which has completely exhausted its energy reserve and which consists almost entirely of iron. However, we shall show below that for stars with mass $M \gtrsim 5 \times 10^5 M_\odot$ the nuclear reactions are of no importance at all. In an equilibrium star with mass larger than 100 $M_\odot$, the entropy is so large that the pressure and the internal energy are determined essentially by the radiation, and the pressure and energy of the plasma* are relatively small, a fact noted already by Eddington.[3] This singularity is indeed the cause of the difference between the structure and evolution of a supermassive star and an ordinary star in which the decisive factor is the plasma energy, and the contribution of the radiation is relatively small.

For pure radiation, the adiabatic exponent is $\gamma = 4/3$, i.e., it has a critical value for the equilibrium of the star (see[1]). By virtue of this, the adiabatic exponent in massive stars differs little from 4/3 and much greater care is necessary in consideration of deviations of γ from 4/3. The deviation of γ from 4/3 is connected with the fact that the plasma makes its own contribution to the pressure, while at high temperatures, the contribution is made by production of e^+, e^- pairs and by the dissociation of the iron, $Fe \rightarrow 13\alpha + 4n$. It is precisely as a result of the influence of the plasma that $\gamma > 4/3$ and the star can be in stable hydrodynamic equilibrium.*

The gravitational potential near the star is small: $\varphi \ll c^2$, and one might think that the theory of the equilibrium of such stars is not connected at all with effects of general relativity (GR). This is not the case, however. After all, the difference between γ and 4/3 is small, and there are enough small effects to influence appreciably the stability of the star, so that small corrections to GR must also be taken into account.

As noted in[1], S. A. Kaplan[4] (see also[139]) was the first to emphasize the importance of small GR effects when $\gamma - (4/3) \ll 1$. Recently similar work was repeated by Chandrasekhar.[140]

He applied these considerations to the theory of white dwarfs. Recently Fowler[5] developed an analogous theory for stars of large mass.

In the first part of this review it was already noted that the equations of the hydrostatic equilibrium of the star are equivalent to the variational principle of the extremum of the stellar energy for a specified total number of nucleons and a specified entropy. This principle is equally valid in either Newtonian or in GR theory. The minimum of energy corresponds to stable equilibrium, and the maximum to unstable equilibrium. In the energy approach, the clarification of stability does not call for supplementary calculations; yet a direct solution of the differential equation of equilibrium still does not allow us to decide that stability exists, and it becomes necessary to investigate in addition the linearized equation for small perturbations. It must be particularly emphasized that entropy plays an important role in the energy approach. This special role is connected with the thermodynamic relation $P = -(\partial E/\partial \rho)_S$, where E is the internal energy per unit mass, ρ—the density, and S—the entropy per unit mass. It is precisely this relation which makes it possible to establish a connection between the energy

*By plasma energy we mean here the energy of the nuclei and the electrons. The plasma pressure is the pressure produced by these particles.

*Concerning the special type of instability in the equilibrium of large stars, connected with isothermal perturbations, see Sec. 5.

of a star, which contains E, and the equilibrium equation, which contains the pressure P. Therefore the theory contains E rather as a function of ρ and S, and not of ρ and T, in terms of which the energy is usually more conveniently expressed. The energy approach was indicated, for example, in[6,7,149,150]. A formal derivation in GR theory is given in[142]. It is possible to develop in this manner an asymptotically exact theory. The Appendix presents a comparison of our method and Fowler's method.[5]

We shall proceed by the method of successive approximations. We shall first find the equilibrium of the star, using Newtonian theory, and taking into account only the radiation in the energy, neglecting all other corrections. We then take into account successively the influence of the plasma, of pair production, and of GR. The processes of dissociation of iron and the neutronization of matter turn out to be immaterial for the equilibrium stage of a supermassive star.

b) Equilibrium of a Star with $\gamma = 4/3$

The total energy of the star $\mathscr{E}$ is written in the form

$$\mathscr{E} = \int\limits_V E\varrho\, dV - G\int\limits_V \frac{m\varrho}{r}\, dV. \tag{2.1}$$

The first term is the internal energy, the second—the gravitational energy, E is the internal energy per unit mass, m is the mass inside the radius r, $m = 4\pi \int\limits_0^r \rho r^2 dr$.

As the zeroth approximation we take into consideration only the energy of the light. The specific internal energy E per unit mass, the specific entropy per unit mass, and the pressure are written in the form

$$E = \frac{\sigma T^4}{\varrho}, \quad S = \frac{4}{3}\frac{\sigma T^3}{\varrho}, \quad P = \frac{1}{3}\sigma T^4 = b\varrho^{4/3}, \tag{2.2}$$

where

$$\sigma = 7.7\cdot 10^{-15}\,\frac{\text{erg}}{\text{erg/cm}^3\,\text{deg}^4}, \quad b = \left(\frac{3}{256\sigma}\right)^{1/3} S^{4/3}. \tag{2.2$'$}$$

Hence we express E in terms of ρ and S:

$$E = 3^{4/3}4^{-4/3}\sigma^{-1/3}S^{4/3}\varrho^{1/3} = 3b\varrho^{1/3}. \tag{2.3}$$

If we know the distribution of the matter in the star $\rho = \rho(r)$, then, substituting (2.3) in (2.1) and integrating, we get

$$\mathscr{E} = k_1 bM\varrho_c^{1/3} - k_2 GM^{5/3}\,\varrho_c^{1/3}, \tag{2.4}$$

where ρ_c is the central density, while the constants k_1 and k_2 depend on the distribution of matter in the star (see Appendix I). In our case the dependence of the pressure on the density at constant S is in the form of a polytrope $P \sim \rho^{4/3}$ with polytropic index $n = 1/(\gamma - 1) = 3$ (see[1], Sec. 3). The distribution of density in equilibrium polytropic gas spheres is given in the basic paper of Emden.[8] In an equilibrium star with $n = 3$, the density distribution is $\rho = \rho(m/M) = \rho_c\psi(m/M)$, where m is the running mass, and is shown in Fig. 1.

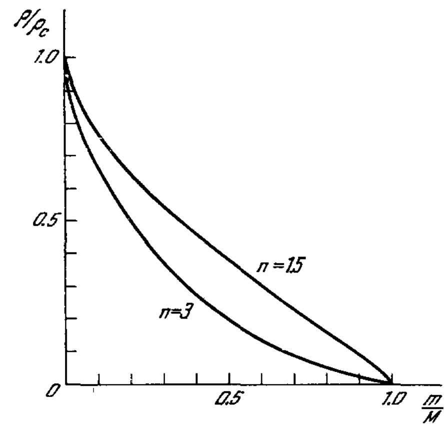

FIG. 1. Emden's function $\rho/\rho_c = \psi(m/M)$ for a polytropic index $n = 3$ ($\gamma = 4/3$) and $n = 1.5$ ($\gamma = 5/3$).

Using these data, we obtain for the constants the numerical values $k_1 = 1.75$ and $k_2 = 0.638$. The equilibrium of the star is determined by the extremum of $\mathscr{E}$ at constant mass (more accurately, at a constant number of nucleons) and constant entropy S. The only quantity which varies in (2.4) is ρ_c, but the equilibrium condition, that is, the condition for the extremum of $\mathscr{E}$, namely $d\mathscr{E}/d\rho_c = 0$, is satisfied only when

$$k_1 bM - k_2 GM^{5/3} = 0. \tag{2.5}$$

In this case the equilibrium is neutral and does not depend on ρ_c.

We emphasize that neutral equilibrium occurs only with respect to contraction and expansion of the star as a whole, that is, with respect to a similar change of the entire star, the star being stable with respect to deformation of the density distribution in it.

For a star in equilibrium we obtain from (2.5)

$$b_e = \frac{k_2}{k_1} GM^{2/3} = 0.364\, GM^{2/3}, \tag{2.6}$$

and from (2.2′) and (2.6) we obtain the corresponding unique value of the equilibrium entropy for the given mass

$$S_e = 7.85\cdot 10^7 \left(\frac{M}{M_\odot}\right)^{1/2}. \tag{2.7}$$

In this approximation the total energy of the star is identically equal to zero, and the density and temperature at any point are connected by the relation

$$T(^\circ\text{K}) = 1.97\cdot 10^7 \left(\frac{M}{M_\odot}\right)^{1/6} \varrho_c^{1/3}. \tag{2.8}$$

If the entropy is not equal to its equilibrium value, then from the general expression for the energy (2.4) with $|S - S_e| < S_e$ we obtain*

$$\mathscr{E} = 2.3\cdot 10^{40} \left(\frac{M}{M_\odot}\right)^{7/6} (S - S_e)\,\varrho_c^{1/3}.$$

*We use the CGS system of units and °K throughout.

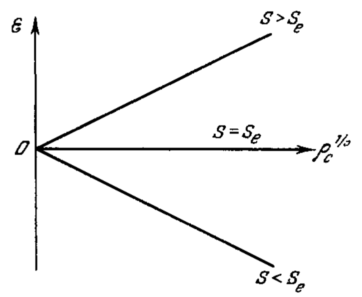

FIG. 2. Energy $\mathscr{E}$ of a star with fixed entropy and with pressure determined only by the radiation, as a function of the cubic root of the central density, $\rho_c^{1/3}$.

When $S > S_e$, the energy increases monotonically with ρ_c, and when $S < S_e$, it decreases monotonically (Fig. 2); (it is convenient to plot $\rho_c^{1/3}$ on the abscissa axis).

c) Effect of Plasma

We now take into account the change in the equation of state, connected with the energy and pressure of the nuclei and electrons of the plasma.

At a given temperature, account of the plasma increases the internal energy. In fact, now in place of (2) we should write for E

$$E = \frac{\sigma T^4}{\varrho} + \frac{3}{2}\frac{R}{\mu}T,$$

where the second term on the right is the plasma energy, and μ is the molecular weight. However, at a given entropy, an allowance for the plasma decreases the energy. Qualitatively this is clear already from the general principle the state of thermodynamic equilibrium corresponds to maximum entropy at a given energy or, what is the same, to minimum energy at a given entropy. The zeroth approximation, when the plasma energy is not taken into account, corresponds to the state in which there is radiation and a cold plasma with zero energy. The transition to a complete equilibrium state for a given entropy can, according to the general principle, only reduce the energy. Numerically, the correction to the internal energy of the matter at constant entropy, necessitated by the plasma, is given by

$$\Delta F_{\mathrm{pl}} = -3.85 \cdot 10^{12} \frac{S^{1/3}}{A} \varrho^{1/3} \left(\ln \frac{aS^{1/2}}{\varrho^{1/2}} + Z \ln \frac{bS^{1/2}}{\varrho^{1/2}} \right),$$

where

$$a = 8.63 \cdot 10^3 g A^{3/2} \qquad b = 2.18 \cdot 10^{-1} \frac{1}{Z} \tag{2.9}$$

A—atomic weight, Z—nuclear charge, g—statistical weight of the nucleus.

This expression is valid in the region when the corrections for the plasma in the expression for the energy and entropy are small, and in addition, the plasma is a nondegenerate ideal gas. Under the conditions of an equilibrium star with mass 10^4—10^8 $M/M_\odot$, these limitations are satisfied with sufficient accuracy.*

We can now calculate the energy of the entire star with account of the influence of the plasma. A correction for the equation of state of the order $\alpha = \Delta E/E \ll 1$ changes not only the energy in the given volume element, but causes also in principle a change in the distribution of the matter in the star (of the same order α), and this change must be taken into account in the calculation of the energy. However, owing to the extremal properties of the distribution function of matter, as a solution of zeroth approximation, a change in this function of order α will produce a change of order α^2 in the total energy of the star, since the first variational derivative of the total energy with respect to the distribution function of matter is equal to zero. Therefore, to calculate the correction to the total energy of the star, of order α, it is necessary to integrate ΔE over the distribution of the zeroth approximation (with respect to the Emden function), and this yields exactly the first term (of order α) of the expansion of the energy in powers of α. In this sense we can speak of an asymptotically exact theory (with error $\sim\alpha^2$) of the equilibrium of stars with $\gamma - (4/3) \sim \alpha$.

Integrating (2.9) over the zeroth-approximation distribution of the density in a star, we obtain the correction $\Delta\mathscr{E}_{\mathrm{pl}}$ to the energy of the star.

$$\Delta\mathscr{E}_{\mathrm{pl}} = -\varrho_c^{1/3}\{a_1 - b_1[1.176 \ln \varrho_c - 1.615]\},$$

$$a_1 = 4.5 \cdot 10^{45} S^{1/3} \left(\frac{M}{M_\odot}\right) A^{-1} \Big\{ \ln (8.6 \cdot 10^3 g A^{5/2} S^{1/2})$$

$$Z \ln \left(2.2 \cdot 10^{-1} S^{1/2} \frac{A}{Z} \right) \Big\},$$

$$b_1 = 1.925 \cdot 10^{45} S^{1/3} \left(\frac{M}{M_\odot}\right) A^{-1} (1 + Z) \tag{2.10}$$

This expression, of course, is valid also only under the limitations indicated above.

For different values of the entropy, we now obtain in place of Fig. 2 a series of curves of $\mathscr{E}$, shown in Fig. 3. All the curves pertain to one value of the mass, and the entropy plays the role of a parameter. The curves now have minima. These minima correspond to the equilibrium state of the star and are designated by bars. The dashed line is the geometric locus of the minima $\mathscr{E}_e(\rho_c)$. In the coordinates of Fig. 3, the

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*We present for reference, without giving the calculations, the first order correction to the entropy, due to a hydrogen plasma

$$\Delta S_{\mathrm{pl}} = 8.3 \cdot 10^7 (15.5 + 2 \ln T^{3/2} - 2 \ln \varrho)$$

In this formula the temperature should be expressed in °K and the density ρ in g/cm³. For an equilibrium hydrogen star this yields $\Delta S_{\mathrm{pl}}/S \approx 60(M/M_\odot)^{-1/2}$, whereas for the pressure we have $\Delta P_{\mathrm{pl}}/P \approx 8.6\,(M/M_\odot)^{-1/2}$. Since an additive constant does not affect the entropy, one can develop a more complicated method with an error on the order of $(\Delta P/P)^2$.

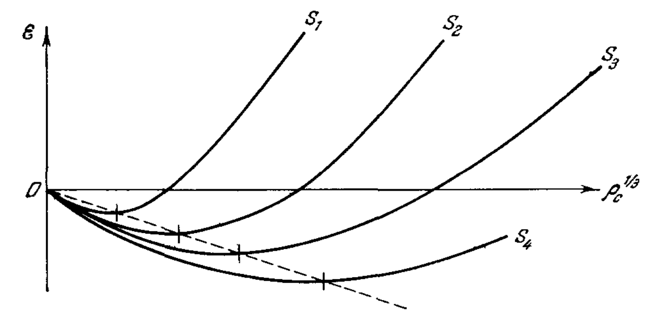

FIG. 3. Energy ℰ of a star with account of the contribution of the plasma (of electrons and nuclei) to the pressure. $S_1 > S_2 > S_3 > S_4$. The minima of the curves correspond to the equilibrium position of the star with given entropy. The dashed line is the geometric locus of the equilibrium positions.

curves $\mathscr{E}(\rho_c)$ are obtained from one another by a similarity transformation, and the dashed line is a straight line and (in accordance with the well-known deduction from the virial theorem)

$$\mathscr{E}_e = -\frac{3}{2}\frac{R\bar{T}}{\mu}M,$$

that is, the energy of the star is equal to the thermal energy of the plasma, taken with the opposite sign. Numerically, recognizing that $\bar{T}/T_c \approx 0.6$, we obtain

$$\mathscr{E}_e = -5\cdot 10^{41}\bar{T}\left(\frac{M}{M_\odot}\right) = -3\cdot 10^{41}T_c\left(\frac{M}{M_\odot}\right).$$

Substituting the expression for T_c from (2.8), we get

$$\mathscr{E}_e = -5.9\cdot 10^{48}\left(\frac{M}{M_\odot}\right)^{7/6}\varrho_c^{1/3}\frac{1}{\mu}.$$

d) Account of Production of Electron-positron pairs

At temperatures on the order of 5×10^8 °K ($kT/m_ec^2 \approx 0.1$) and above, there are electron-positron pairs in equilibrium. At a given temperature, pair production increases the energy of the matter, but, as noted above, it follows from the general principle of thermodynamics that at a given entropy pair production decreases the energy.

We shall verify below that the appearance of pairs leads to an instability of a massive star at a temperature which (in energy units) is 10—15 times smaller than the energy of one pair $2mc^2$. For reference purposes, $2mc^2 = 1.02$ MeV and $\Theta = 2mc^2/k = 11.9\times 10^9$ deg. The number of positrons in the entire star is not larger in this case than several per cent of the number of electrons, and even in the center of the star $n_+ < 0.25\,n_-$. We shall therefore use asymptotic formulas pertaining to a nonrelativistic nondegenerate gas:

$$n_+n_- = \frac{4(2\pi mkT)^3}{(2\pi\hbar)^6}e^{-\frac{2mc^2}{T}}. \qquad (2.11)$$

We assume that $n_+ \ll n_-$, $n_- = n_{-0} = \rho/\mu_e m_p$, and

$$\frac{n_+}{n_-} = \frac{n_+n_-}{n_{-0}^2},$$

where n_{-0} is the number of electrons in the matter (plasma) of given density, without account of pair production. Substituting the numbers, we write

$$\frac{n_+}{n_-} = 1.1\cdot 10^{14}\mu_e^2\frac{1}{\varrho^2}\left(\frac{T}{\Theta}\right)^3 e^{-\Theta/T};$$

the quantity Θ was defined above.

With the aid of the zeroth approximation [see (2.8)] we express also the density in terms of the mass of the star and of the temperature:

$$\frac{n_+}{n_-} = \frac{\left(\frac{M}{M_\odot}\right)\mu_e^2}{430}\left(\frac{T}{\Theta}\right)^{-3}e^{-\Theta/T}.$$

In this formula we can substitute the local temperature and obtain the local value of n_+/n_-. In particular, the formula is, of course, also valid for the center, where $T = T_c$.

To average over the star any quantity x, which varies rapidly with the temperature (or with the density $\rho \sim T^3$), there is a convenient formula

$$\bar{x} = x_c\ 3.2\left(\frac{d\ln x}{d\ln T}\right)_c^{-3/2},$$

which in the case of interest to us yields

$$\overline{\left(\frac{n_+}{n_-}\right)} = \left(\frac{n_+}{n_-}\right)_c 3.2\left(\frac{\Theta}{T_c}-3\right)^{-3/2} \approx \left(\frac{n_+}{n_-}\right)_c 3.2\left(\frac{T_c}{\Theta}\right)^{3/2}.$$

We now turn to the thermodynamic aspect of the matter.

The supplementary energy, occurring at a given temperature in connection with the pair production, is equal to $2mc^2n_+$ per unit volume or $\Delta E|_T = 2mc^2n_+/\rho$ per unit mass. We consistently neglect here terms of order T/Θ; this means neglecting the kinetic energy of the positrons compared with their rest mass, meaning also automatically that their pressure is neglected.

The change in energy for a given entropy is connected with the change in energy for a given temperature by the relation

$$\Delta E\Big|_S = -T\frac{d\Delta E|_T}{dT} = -\frac{T}{\Theta}\Delta E\Big|_T = -\frac{kTn_+}{\varrho}.$$

We write down the energy of the entire star at a given entropy. In place of $\mathscr{E}(S, \rho_c)$ it is technically more convenient to change to $\mathscr{E}(S, T_c)$, expressing ρ_c in terms of T_c by means of the zeroth-approximation formula (2.8). We obtain

$$\mathscr{E} = A(S-S_e)T + \frac{3}{2}\frac{R\bar{T}}{\mu}M(B+\ln T) - k\bar{T}N_+, \qquad (2.12)$$

where N_+ is the total number of positrons in the star.

It is convenient to rewrite this expression in the form

$$\mathfrak{E} = A'(S-S_e)T_c + DT_c\left(\ln T_c - \frac{2}{3}\frac{\mu}{\mu_e}\frac{n_+}{n_-} + B'\right)$$
$$= A'(S-S_e)T_c - DT_c\left[\ln T_c - \frac{2}{3}\mu\mu_e\frac{M/M_\odot}{430}\left(\frac{T}{\Theta}\right)^{-3/2}\right.$$
$$\left.\times\, e^{-\Theta/T}3.2\left(\frac{T}{\Theta}\right)^{3/2} + B'\right]. \qquad (2.13)$$

For iron $\mu \approx \mu_e \approx 2$, and

$$\mathcal{E} = A'(S-S_e)T_c + DT_c\left[\ln T_c - \frac{M/M_\odot}{50}\left(\frac{T_c}{\Theta}\right)^{-3/2} e^{-\Theta/T_c} + B'\right].$$

The contribution of the pairs to the energy, the sign of which is negative, increases rapidly in absolute magnitude with increasing temperature. Allowance for the pairs changes the curves of Fig. 3 into the form shown in Fig. 4.

The isentropic curves now have in addition to minima also maxima which, like the minima, are marked with vertical bars. On the curve corresponding to a certain entropy, which we shall call critical, S'' on Fig. 4, the maxima and the minima merge and produce a point of horizontal inflection (marked by two bars). When $S < S''$ (for a fixed mass) there is no extremum of $\mathcal{E}(\rho_c)$ at all, that is, there is no state of equilibrium.

The equilibrium corresponding to the maxima of the curves is unstable in accordance with the general ideas of the energy approach.

The geometrical locus of the extrema of $\mathcal{E}(\rho_c, S)$, that is, the curve of equilibrium energy $\mathcal{E}_e(\rho_c)$, is shown in Fig. 5. The minimum of this curve corresponds to the horizontal inflection on Fig. 4, the descending branch of $\mathcal{E}_e$ corresponding to stable, and the ascending to unstable equilibrium.

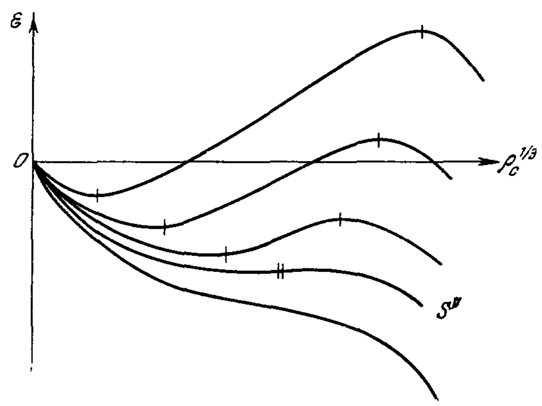

FIG. 4. Change in the energy curves of Fig. 3 when account is taken of production of e^+, e^- pairs (for $M < 10^4 M_\odot$) or GR effects (for $M > 10^4 M_\odot$).

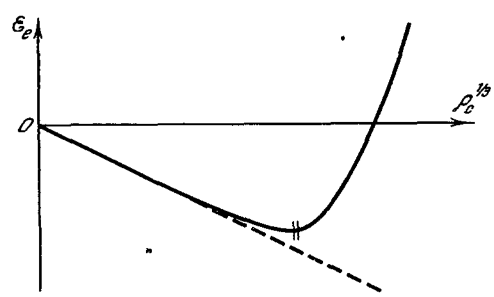

FIG. 5. Energy $\mathcal{E}_e$ of equilibrium star. The dashed line shows the energy of the equilibrium star without account of pair production and GR.

On the ascending branch there is a region where $\mathcal{E}_e > 0$. We recall that these states, which correspond to maxima of the isentropic curves, are the result of a negative correction to the energy. This correction has caused the appearance of an extremum where previously there was none, but the energy itself, of course, without correction was positive.

The critical state, as already noted, is attained when

$$\frac{\partial \mathcal{E}}{\partial \rho_c} = \frac{\partial^2 \mathcal{E}}{\partial \rho_c^2} = 0, \quad \text{i.e.,} \quad \frac{\partial \mathcal{E}}{\partial T_c} = \frac{\partial^2 \mathcal{E}}{\partial T_c^2} = 0.$$

We write immediately the condition with $\partial^2 \mathcal{E}/\partial T^2$, since the condition with the first derivative can always be satisfied by choosing a suitable value of S. From the condition with $\partial^2 \mathcal{E}/\partial T^2$ all the constants A', S_e, D, and B' will drop out. We obtain the condition

$$1 - \frac{M/M_\odot}{50}\left(\frac{T_c}{\Theta}\right)^{-7/2} e^{-\Theta/T_c} = 0.$$

Table I lists the parameters of the critical state for three typical stars (composition—iron), $T_{c9} = T_c \times 10^{-9}$.

The estimates given above confirm the correctness of the assumptions made, $T_c/\Theta \ll 1$, $n_+/n_- < 1$.

For a hydrogen star with $\mu = 1/2$, $\mu_e = 1$, we obtain Table II.

Table I. Parameters of the critical state due to the influence of e^+, e^- pairs for stars of iron

$M/M_\odot$	300	3000	6000
T_{c9}	1.2	0.92	0.87
ρ_c	10 000	1600	800
Θ/T_c	10	13	14
$e^{-\Theta/T_c}$	$5\cdot10^{-5}$	$2.4\cdot10^{-6}$	$8\cdot10^{-7}$
$\overline{n_+/n_-}$	0.022	0.013	0.011
$(n_+/n_-)_c$	0.22	0.18	0.18

Table II. Parameters of the critical state due to production of e^+, e^- pairs for stars of hydrogen

$M/M_\odot$	2400	24 000
T_{c9}	1.2	0.92
ρ_c	3600	550

At equal temperature (see Table I), the remaining quantities do not depend on the composition. The extreme masses (6000 for iron, 24,000 for hydrogen, on the average $\sim 10^4 M_\odot$) constitute the limit where the GR effects, which we shall consider shortly, come materially into play.

We note also that the corrections to the equation of state, connected with dissociation of nuclei, for example[10] $Fe^{56} \rightarrow 13\alpha + 4n$, necessitate, as a rule, a higher temperature compared with pair production and in the theory of equilibrium of supermassive stars these processes are immaterial, precisely as is the case for processes of neutronization of matter.

e) Correction for General Relativity and Conclusions

The introduction of the correction for general relativity calls for careful attention to the very definition of the "correction at a given density distribution," since it is necessary to take into account both the fact that the space is not Euclidean, and the difference between the density of the rest mass and the density which includes the energy, divided by c^2.

Rather lengthy calculations (see Appendix III) lead to the following type of correction to the energy of the star, connected with the GR effects:

$$\Delta\mathscr{E}_{\rm GR} = -0.93\,\frac{G^2M^{7/3}}{c^2}\,\varrho_c^{2/3}. \tag{2.14}$$

In order of magnitude, $\Delta\mathscr{E}_{\rm GR}$ is equal to the product of the gravitational energy of the star, $\sim -GM^2/R$, and the ratio of its gravitational radius to the actual radius:

$$\Delta\mathscr{E}_{\rm GR} \approx \left(-\frac{GM^2}{R}\right)\left(\frac{r_g}{R}\right).$$

The sign of this correction and the character of its influence on the general picture is the same as for the e^+, e^- pairs. Thus, there are two causes of the transition from Fig. 3 to Fig. 4. It turns out that depending on the mass of the star, only one of the causes plays a significant role: for a mass smaller than $10^4\ M_\odot$—only pairs, for a mass larger than $10^4\ M_\odot$—only GR.

Such a sharp distinction is a consequence of the strong dependence of the equilibrium number of pairs on the temperature. In equilibrium stars with mass larger than $10^4\ M_\odot$, when the density increases the GR effects change the course of the isentropic curves long before a temperature sufficient for intense production of e^+, e^- pairs is attained.

Thus, for stars with $M > 10^4\ M_\odot$ the transition from Fig. 3 to Fig. 4 is due to GR effects. We again turn to the curve $\mathscr{E}_e$ on Fig. 5. For stars with $M > 10^4\ M_\odot$ the appearance of a minimum and of an ascending branch on the curve is connected with GR effects. The presence of the region where $\mathscr{E}_e > 0$, is here also due to GR. But we emphasize once more (see[1], Sec. 11) that positive energies in the equilibrium state are not necessarily connected with GR, and can arise also for other reasons. Initially solutions with positive energy were obtained in the problem of Oppenheimer and Volkoff[11] concerning a sphere of ideal degenerate gas. In this case their appearance was actually connected with GR. However, when the e^+, e^- pairs are taken into account, that is, for a non-ideal gas, such solutions as we have seen in the preceding subsection are obtained also without GR effects, so that they cannot be regarded at all as being specific consequences of the curvature of space and other characteristic features of GR.

The most characteristic for a given mass is the critical state corresponding to the minimum of $\mathscr{E}_e$, designated by two bars on Fig. 5. It corresponds to the last equilibrium state in a series with decreasing entropies. In this state the star has the minimum energy possible under equilibrium for a given mass, and the maximum possible temperature and density.

For $M/M_\odot > 10^4$, the critical state is determined by the effects of general relativity. In this case the following formulas are valid for the critical state (see Fowler[5])

$$\varrho_c'' = 2.43\cdot 10^{17}\,\frac{1}{\mu^3}\left(\frac{M}{M_\odot}\right)^{-7/2}\ \text{g/cm}^3, \tag{2.15}$$

$$T_c'' = 1.23\cdot 10^{13}\,\frac{1}{\mu}\left(\frac{M}{M_\odot}\right)^{-1}\ {}^\circ\text{K}, \tag{2.16}$$

$$\mathscr{E}'' = -0.93\cdot 10^{54}\,\frac{1}{\mu^2}\ \text{erg}. \tag{2.17}$$

The energy $\mathscr{E}''$ of the critical state does not depend on the mass.

3. EQUILIBRIUM OF A ROTATING STAR WITH $\gamma = 4/3$

To find the equilibrium, let us consider the energy of a rotating star and let us find its extremum. The GR effects will be assumed to be small. Therefore we shall consider first the condition for the equilibrium in Newtonian theory,* and then the corrections for GR. We shall assume that the density of matter in the rotating star on similar ellipsoids of revolution is constant.

The energy is written in the form [see (2.2)]

$$\mathscr{E} = -k_2GM^{5/3}\varrho_c^{1/3}g(\lambda) + k_1Mb\varrho_c^{1/3} + k_3K^2\lambda\varrho_c^{2/3}M^{-5/3},$$
$$k_2 = 0.64,\quad k_1 = 1.75,\quad k_3 = 1.25.$$

The first term here is the gravitational energy, the second the thermal energy, the third the energy of rotation; λ is a parameter characterizing the oblateness: λ is equal to the ratio of the diameter of the ellipsoid on the axis of revolution to the diameter of the equivalent sphere, $\lambda < 1$; the factor $g(\lambda)$ takes into account the change in the gravitational energy as the result of the oblateness of the star as it rotates; K is the moment of rotation.

Even from the very form of $\mathscr{E}$ it follows that the energy of rotation depends on the density ρ_c like the energy of a gas with adiabatic exponent $\gamma = 5/3$, that is, it contributes to stability.

We denote by $b_0 = 0.364\ GM^{2/3}$ the value of b at which there is neutral equilibrium in the absence of rotation, so that (see Sec. 2)

*The energy approach to the problem was adopted from[18].

$$k_2 GM^{5/3} \varrho_c^{1/3} = k_1 b_0 M \varrho_c^{1/3},$$
$$b_0 = \frac{k_2}{k_1} GM^{2/3},$$

and we introduce the dimensionless quantities $b/b_0 = h$ and $r = \rho_c/\rho_0$, where ρ_0 is the characteristic density, made up of G, M, and K:

$$\varrho_0 = \frac{G^3 M^{10}}{K^6}.$$

The expression for the energy is now rewritten in the form

$$\mathscr{E} = \frac{G^2 M^5}{K^2} \{r^{1/3}[-k_2 g(\lambda) + k_1 0.364h] + k_3 \lambda r^{2/3}\}.$$

We denote the curly bracket by A. The factor preceding the bracket is made up of conserved specified quantities. To find the extremum of $\mathscr{E}$ we vary λ and r. The value of h is determined by the entropy and is also specified; during the course of the evolution of the star, it varies slowly. The equilibrium conditions (the conditions for the extremum of $\mathscr{E}$) have the form $\partial A/\partial\lambda = 0$ and $\partial A/\partial r = 0$. From these two relations we express λ and r in terms of h. The solution exists only when $h < 1$, which is perfectly natural. If $h > 1$, then the entropy is so large that even without rotation the given mass is scattered apart without restraint; it is clear that rotation is incapable of changing this result. When $h < 1$ without rotation, the gas is compressed without restraint. In this case the rotation stops the compression. A decrease in h is accompanied by an increase in the density and by an increase in the oblateness, characterized by the quantity $1 - \lambda$. For arbitrary $h < 1$ there exists a formal solution of the problem, in which it was specified that the surfaces of constant density are similar ellipsoids; here $h \to 0$ gives $\lambda \to 0$, $\rho \to \infty$, that is, a solution corresponding to a flat disc. This solution is, as is well known, unstable; the disc breaks up into clusters with dimensions of the order of the thickness of the disc.

If we consider the decrease of h, then for a given h we can expect instability with respect to the transformation of the ellipsoid of revolution into a triaxial ellipsoid, as is the case for an incompressible liquid. In the expressions used by us, the dependence of the gravitational energy and the energy of rotation on the shape is separated in a factor (factorization), and therefore in the case with arbitrary adiabatic exponent the instability arises for the same value of λ as in an incompressible liquid. However, as we shall see below, the solution becomes meaningless long before this instant, when the oblateness is still small, when h is close to unity, and when accordingly λ is close to unity. In this region we have the relations

$$g(\lambda) = 1 - \frac{(1-\lambda)^2}{5}, \quad r = \left(\frac{k_3^{-1}}{1-h}\right)^3,$$
$$1 - \lambda = \frac{5}{4}(1-h),$$

and the average density is

$$\bar{\varrho} = \text{const} \cdot \frac{G^3 M^{10}}{K^6} \frac{1}{(1-h)^3}. \quad (3.1)$$

For a gas, unlike an incompressible liquid, the density distribution is characterized by a decrease from the center towards the edge. For a given mass and for a given average density, the outer radius of the gas sphere is larger than that of an incompressible liquid.

In the case of the gas, the gravitational field is smaller on the edge, and at the same time, for a given moment of rotation, the centrifugal force is larger. Therefore in the case of a gas sphere the condition for the breakaway of the matter from the equator is attained much earlier than in a liquid, that is, at relatively small deformation.

Let us find the breakaway condition for the Emden solution with n = 3. We shall use the method of successive approximations: we find the breakaway condition, specifying a definite value of $\bar{\rho}$ and neglecting deformation, that is, regarding the rotation of the sphere and determining its moment of rotation. We then find h and λ, corresponding to the assumed density and the moment.

We set up the condition for equality of the centrifugal force to the gravitational force on the equator:

$$\omega^2 R = \frac{GM}{R^2},$$

where ω is the angular velocity.

From this condition we obtain after calculation for the critical state

$$1 - \lambda = 0.05, \quad h = 0.96.$$

Since we found that λ is quite close to unity, this justifies also post factum the method of successive approximations used above.

In the critical state, when the centrifugal and attraction forces are equal, the star differs little from a sphere, that is, it is very far from a disc. In the critical state the matter on the equator ceases to fall, even if it is not supported by the pressure from the inside, but at the same time the energy of the matter is insufficient to permit it to escape to infinity.

The general considerations of dimensionality and similarity theory undoubtedly remain in force also when the specific ellipsoidal model is no longer applicable. We are therefore fully assured of the correctness of the general formula (3.1), according to which for a specified mass and entropy the density is proportional to K^{-6}, and the shape depends only on the ratio of the mass and entropy, and does not depend on the momentum. In particular, the critical condition for the start of escape is likewise independent of the momentum. There is a definite critical value of the entropy (for a given mass), at which matter begins to escape, and there is no hydrostatic solution. We emphasize once more that this unique result pertains specially to matter with exponent $\gamma = 4/3$.

Tentative estimates from above show that in the course of the decrease of the entropy the condition for escape was reached earlier than the condition for a loss of stability of shape with formation of a triaxial ellipsoid.

We must emphasize specially that rotation stabilizes the star with $\gamma = 4/3$ and leads to the appearance of a definite equilibrium density (which depends on the entropy) while the shape is still practically spherical.

We can imagine for the sake of illustration that a non-rotating star with $\gamma = 4/3$ is in neutral equilibrium (at $b = b_0$) or is compressed without limit (when $b < b_0$), retaining a spherical form. However, even for neutral equilibrium, that is, in the absence of elasticity with respect to a change in the radius, the star has a finite elasticity with respect to a change in the shape: the thermal energy Q does not depend on the shape, and the potential energy of gravitation U increases with deviation from sphericity, the dependence of U on the shape being the same for different γ.

At first glance the rotation can prevent the compression only at the equator, and does not prevent axial collapse. Actually, however, the elasticity of the shape causes indirectly the delay in the contraction of the equator also to give rise to a delay in the contraction at the pole. This is connected with the fact that upon contraction into a disc the gravitational force at the pole tends to a constant (unlike in the contraction of a sphere into a point), whereas the force due to the pressure gradient at arbitrary positive γ in the relation $P = \mathrm{const} \cdot \rho^\gamma$ tends to infinity when the thickness of the disc tends to zero. This force unavoidably balances the gravitation and stops the contraction.

We emphasize that the escape condition is reached, for small deviation from sphericity, in the case when the turbulent viscosity and the magnetic forces ensure rotation of the body as a whole. Prior to the start of the escape, the evolution proceeds at a rate that depends on the rate of change of the entropy, and the system goes through a series of equilibrium states. An analysis of this stage can be easily carried out in analogy with the preceding. By way of the first approximation, we substitute in the energy equation $\lambda = 1$ and $g(\lambda) = 1$. It is curious that the additional term depends on the density in exactly the same way as the correction for general relativity, but has an opposite sign and cancels out this correction exactly at some definite value of the momentum.

In order of magnitude, the compensation condition is of the form

$$cv_{\text{circular}} \approx v^2_{\text{parabolic}} \sim GM/R.$$

When the effects of rotation overcome the effects of general relativity, the slow evolution continues until the escape conditions are reached. An analysis of the next stage has not yet been made, and for this analysis it would be essential to clarify the kinetics of the exchange of angular momentum between the different layers of the star (see Sec. 11).

4. POSSIBILITY OF OCCURRENCE OF A SUPERMASSIVE STAR

Let us consider the process of gravitational condensation of a rarefied gas cloud in a supermassive star.* In order for the gas to start compressing spontaneously under the influence of gravitation, it is necessary that the gravitational force exceed the elasticity of the gas. This question was first considered and solved by J. H. Jeans.[13] (For a simple exposition see[14].) In a homogeneous medium, in a volume of diameter D, the gravitational energy is $GM^2/D \sim D^5$, whereas the internal energy of the gas is proportional to the volume, that is, to D^3. It is clear that in a small volume gravitation can be neglected, and any sufficiently small local fluctuation of density will not build up, but will propagate through the medium with the velocity of sound. This is a sound wave. With increasing dimension of the fluctuation D, the role of gravitation increases, and at a certain critical value D_{crit} it exceeds the role of gas pressure. Fluctuations with dimensions larger than D_{crit}, after having set in, increase with time and gravitational instability develops. The critical dimension is

$$D_{\text{crit}} = a_{\text{ac}}\sqrt{\frac{\pi}{G\varrho}} = 2\cdot 10^{-11}\sqrt{\frac{\gamma T}{\mu\varrho}}\ (nc). \tag{4.1}$$

Here a_{ac} is the speed of sound, γ is the adiabatic exponent, and μ the molecular weight. The mass of gas in a sphere of diameter D_{crit} is

$$\frac{M}{M_\odot} \approx 3\,\frac{\gamma T}{\mu}\,D_{\text{crit}}\,(nc), \tag{4.2}$$

where D_{crit} is in parsec.

Jeans suggested that an infinite homogeneous medium against the background of which the fluctuations develop, is stationary. Obviously, such a formulation of the problem is incorrect,[131,15] for a homogeneous medium cannot be in stationary equilibrium, and this assumption must be discarded. The introduction of suitable changes does not change the critical length of the perturbation, but it changes essentially the dependence of the growth of the fluctuations on the time. For details we refer the readers to[104,131,15].

The process of gravitational condensation is very complicated. Its details depend on the possible presence of external pressure of heated gases, on the mechanisms of heating and cooling of the matter, on the development of the fluctuations during the process of compression, on the presence of a magnetic field and of rotation, etc. In addition, the calculation calls

*V. A. Ambartsumyan[12] and co-workers adhere to a different point of view, according to which the stars arise from superdense bodies. For more details see Sec. 17.

for knowledge of the concrete astrophysical conditions under which gravitational condensation takes place.

In spite of numerous papers (for reviews see[16,17]) the process of formation of ordinary stars from a diffuse medium is still unclear, as is the process of formulation of galaxies[16] (concerning this topic see the papers by L. M. Ozernoĭ[34] and by Bird[35]). A fortiori, we cannot specify more concretely the conditions for possible occurrence of supermassive stars.

However, some considerations of principle can already be advanced here. In order to maintain in equilibrium the mass of gas, the entropy of the supermassive star must be sufficiently large (see Sec. 2). During the course of the compression of initially rarefied gas into a star, its entropy due to energy radiation can only decrease. The separation of nuclear energy prior to the attainment of the equilibrium state, obviously does not take place, owing to the low temperature. Consequently, if we assume that during the course of contraction there are no processes in which the entropy increases as a result of energy transfer from the macroscopic motions into heat, the entropy of the gas in the initial state should be no lower than the entropy in the final state.

The entropy of the initial state consists only of the entropy of the gas (neutral hydrogen), and the role of radiation can be neglected.* The entropy of a gas made up of neutral hydrogen, calculated per unit mass, is given by the expression

$$S_{\rm H}=8.3\cdot 10^{8}\left(-5.6+\ln\frac{T^{3/2}}{\varrho}\right). \tag{4.3}$$

Equating this value to the entropy per unit mass in the star (see Sec. 2), we obtain the critical value of the initial temperature necessary (at a given initial density) for the formation of an equilibrium star:

$$T^{\circ}_{\rm crit}=\left(\frac{\varrho}{10^{-24}}\right)^{2/3}10^{0.27(M/M_\odot)^{1/2}-14.4}. \tag{4.4}$$

Here $T^{\circ}_{\rm crit}$ is the required minimum initial temperature of the gas, expressed in degrees, and ρ is the initial density in g/cm^3. In Table III are listed the values of the critical temperature for two values of the initial density ρ_1 and ρ_2, characteristic for interstellar and metagalactic conditions, respectively.

Thus, stars with mass $M/M_\odot \gtrsim 10^4$ cannot be produced without energy transfer from the macroscopic motion of the gas into heat. Consequently, for the occurrence of such stars, an essential factor is the presence of turbulence in the process of gravitational condensation, with subsequent formation of shock waves, which increase the entropy.

Thus, if $M \lesssim 10^3\ M_\odot$, the rate of compression is determined by the decrease in entropy upon radiation of energy, and an equilibrium star with $M \geq 10^4\ M_\odot$ is produced only if the entropy increases sufficiently in the shock waves to hold the star together.

Table III. Critical temperature of the diffuse medium as a function of the mass of the formed star, °K

ρ, g/cm^3 \ $M/M_\odot$	10^3	$3\cdot10^3$	10^4
$\varrho_1=10^{-24}$	10^{-6}	~ 3	$\sim 5\cdot10^{12}$
$\varrho_2=10^{-29}$	$\sim 5\cdot10^{-10}$	$\sim 10^{-3}$	$2\cdot10^{9}$

*Even if the entropy of radiation were noticeable, during the course of compression this radiation would still leave the star, since the cloud is *transparent* during the start of contraction.

5. EVOLUTION OF A SUPERMASSIVE STAR

a) General Remarks

A stable star in equilibrium, for a given chemical composition and a specified entropy, is at the minimum of the energy curve $\mathscr{E}(\rho, S)$. The equilibrium energy is negative. If the reserves of nuclear energy of the star have not been exhausted, then this energy is incomparably larger than the absolute value of $\mathscr{E}_{\rm eq}$. In fact, the reserve of nuclear energy per gram of matter is on the order of $q = 10^{18}$ erg/g and in the entire star we have*

$$\mathscr{E}_{\rm nuc}\approx 10^{18}M=10^{51}\frac{M}{M_\odot}\ \text{erg}, \tag{5.1}$$

which for the masses in question, 10^4—$10^9\ M_\odot$, is much more than $\mathscr{E}''_{\rm e} = 3.56 \times 10^{54}$ erg.

Were the nuclear energy to be released instantaneously, then the material of the star would scatter. However, we have seen in the first part of the review that the star has a mechanism for automatically regulating its energy sources. When it is produced from rarefied matter, it contracts until the release of nuclear energy near the center balances the radiation from the surface. This determines its value of $T_{\rm crit}$ and consequently also the position of the representative point on the energy curve $\mathscr{E}_{\rm e}$ (Fig. 5). If there were no factors to cause instability of equilibrium in such a state, the star would gradually burn up all its nuclear fuel. However, stars in which the principal role is played by the radiation pressure are subject to isothermal instability (see below), which develops within a time of the order of the time of thermal relaxation of the star. Such instability causes the star to break up into clusters with $M \approx 10^2\ M_\odot$ and leads to catastrophic contraction of the entire system. Finally, if the temperature for the occurrence of nuclear reactions is low, the evolution of the star (without loss of mass) concludes in the fact that the star gradually

*To estimate $\mathscr{E}_{\rm nuc}$, we take into account the mass of the entire star, not only of the core, where the temperature is high. This is explained by the fact that in massive stars the energy is transferred to the surface by convection, thus causing mixing of the material.

decreases its entropy and energy during the course of emission of light. The representative point then moves downward along the curve $\mathcal{E}_e$. The minimum of this curve corresponds to a transition to states in which the equations of hydrostatic equilibrium have no solutions. A catastrophic contraction takes place, with a velocity determined by the equations of hydrodynamics and the representative point begins to slide downward along the curve S = const, which has no maxima or minima (the curves below S″ in Fig. 4).

Let us consider first the rate and time of equilibrium evolution due to cooling of a star without nuclear sources of energy; we shall then see what is obtained by taking into account nuclear sources of energy and isothermal instability, and in the final paragraph we shall turn to the stage of catastrophic contraction.

b) Condition of Optical Equilibrium

This condition was considered in Eddington's classical book.[3] In a supermassive star, the force of gravitation is balanced by the light pressure. Let us consider the forces acting in the surface layer of the star. In a strongly ionized plasma, the Compton scattering by the electrons is the fundamental process which causes the matter to become opaque. Let us calculate the force of light pressure acting on one free electron. This force is, obviously,

$$F_e = -\frac{1}{n_e}\frac{dP}{dr} = -\frac{1}{3n_e}\frac{dE}{dr}, \tag{5.1a}$$

where n_e is the concentration of the plasma, and E the density of radiant energy. In a medium whose optical thickness is larger than unity, the radiation flux q is equal to

$$q = -D\frac{dE}{dr} \tag{5.2}$$

and the diffusion coefficient is

$$D = \frac{1}{3}\frac{c}{n_e\sigma_e}, \tag{5.3}$$

where σ_e is the scattering cross section, $\sigma_e = 6.7 \times 10^{-25}$ cm^2. The cross section does not depend on the frequency of the quanta so long as $\hbar\omega \ll m_e c^2$.

Substituting (5.3) and (5.2) in (5.1a), we get

$$F_e = \frac{\sigma_e q}{c}. \tag{5.4}$$

We note that expression (5.4) does not depend on the assumption that the optical thickness is large. Indeed, the time-averaged force acting on one electron, for a radiation flux equal to q, is given by (5.4) independently of the angular distribution of the quanta of radiation. The same force acts on an isolated electron in the radiation field of a point source.

Since the plasma is electrically neutral, the mass per electron is μ_e/A, where $A = 6 \times 10^{23}$. Under equilibrium, the force of radiation pressure on one electron F_e is equal to the force of attraction of the mass per electron

$$\frac{GM\mu_e}{Ar^2} = \frac{\sigma_e q}{c}. \tag{5.5}$$

From this, expressing q in terms of the luminosity L, namely $q = L/4\pi r^2$, and substituting the numerical constants, we obtain finally for a hydrogen plasma

$$L = 1.3\cdot 10^{38}\frac{M}{M_\odot}\ \text{erg/sec},$$
$$\frac{L}{L_\odot} = 3\cdot 10^4\frac{M}{M_\odot}, \quad \frac{L}{M} = 3\cdot 10^4\frac{L_\odot}{M_\odot}. \tag{5.6}$$

We emphasize that formula (5.6) gives an upper limit of luminosity of any stationary star (and not necessarily a supermassive one). A larger flux will cause the surface layers to drop off.

In the stars in question, in which the pressure is determined by the radiation, expression (5.6) is not only an upper limit, but also the actual luminosity. Digressing somewhat, we note that in such stars the condition for equilibrium (5.5) should be observed not only on the surface, but also in the entire star. Consequently, the following equality should hold true

$$L_r = 1.3\cdot 10^{38}\frac{M_r}{M_\odot}\ \text{erg/sec}, \tag{5.7}$$

where M_r is the mass inside a sphere of radius r and L_r is the total flux of light through this sphere.

It is clear that the nuclear energy sources (if they exist) are located near the center and the light flux cannot increase in the entire star in proportion to the mass in accordance with (5.7). At first glance it seems that a star with a light pressure and central source cannot exist at all. But the point is that the central source causes convection in the star, for even without this source the star is on the border of convective instability. Convective energy transport ensures the necessary flow of heat in the supermassive star. No matter what causes the transport of energy inside the star, at the surface itself the transport should be realized by means of radiant thermal conductivity, for the energy flux into outer space leaves the surface in the form of light rays, and therefore relation (5.6) remains in force also for the total luminosity.

c) Time of Cooling to the Critical State

Let us return to a star without nuclear energy sources. As can be seen from the formulas of Sec. 2, the temperature in the center of a star with $M > 10^5 M_\odot$ is insufficient, even in the critical state, to cause neutrino radiation. Therefore the cooling of the star is determined by the photon luminosity (5.6). This luminosity determines the rate of evolution—the rate of motion along the curve $\mathcal{E}_e$.

When the material of the star was in the dispersed state, its energy was equal to zero. In order to reach the critical state, it is necessary to radiate an energy

$-\mathcal{E}'' = 3.56 \times 10^{54}$ erg (see the end of Sec. 2). Thus, the time of evolution is given by the formula

$$t = -\frac{\mathcal{E}''}{L} = 2.4 \cdot 10^8 \mu^{-2} \left(\frac{M}{M_\odot}\right)^{-1} \text{ years.} \tag{5.8}$$

We have already emphasized (see Sec. 2) that $\mathcal{E}''$ is much smaller than the thermal energy of the star. Therefore t_{ev} is much smaller than the time of thermal relaxation $t_{cool} = Q/L$.

It is obvious that a star can be regarded to be in quasi-equilibrium only when the time of evolution up to $\mathcal{E}''$ greatly exceeds the characteristic time of the hydrodynamic processes (see[1], Sec. 2). The latter time has an order of magnitude [see[1], formula (2.2)]

$$t_{\text{hydr}} = \sqrt{\frac{1}{6\pi G\bar{\varrho}}} \approx \frac{10^3}{\sqrt{\bar{\varrho}}} \text{ sec,} \tag{5.9}$$

Substituting $\bar{\rho} = \rho_c/54$ (an expression which is valid for a polytrope with n = 3), and substituting for ρ_c the expression (2.14) for ρ_c'', we get

$$t_h = 5 \cdot 10^{-5} \mu^{3/2} \left(\frac{M}{M_\odot}\right)^{7/4}. \tag{5.9'}$$

Comparing (5.9′) with (5.8) we see that the times t_{ev} and t_{hydr} become equal when $M \approx 10^8 M_\odot$. Certainly no equilibrium stars with such a large mass (without a magnetic field!) can exist.

d) Nuclear Sources of Energy

We have already noted above that a star with $M < 10^8 M_\odot$ will contract in quasi-equilibrium fashion until the temperature near the center rises sufficiently high for the release of nuclear energy to compensate for the radiation from the surface. Table IV gives the effective power per gram of matter of the star $\tilde{A}_{nuc} = (\int A_{nuc} dM)/M \approx 0.1 A_{nuc}$ calculated for the critical state.

Data for the carbon-nitrogen cycle are given under the usual assumption that the content of carbon and nitrogen in the material is approximately 0.5%. It is seen from Table IV that under these assumptions the proton-proton reaction can be completely neglected for the region where $\tilde{A}_{nuc}$ is comparable with the energy lost by the star to radiation, calculated per gram of matter L/M. For L we used formula (5.6).

Thus, $\tilde{A}_{nuc}$ and L/M become comparable for $M \approx 5 \times 10^5 M_\odot$. This means that when $M > 5 \times 10^5 M_\odot$ the temperature, even in the critical state, is still insufficient for the released nuclear energy to compensate for the radiation, and the star can exist for a long time in equilibrium as a result of the nuclear energy. Thus, for stars with $M > 5 \times 10^5 M_\odot$, the nuclear sources of the energy are insignificant during the entire time of equilibrium evolution. The evolution of such stars after their appearance is determined by the cooling process, as described in Sec. 5c.

For stars with $M < 5 \times 10^5 M_\odot$, the release of nuclear energy becomes appreciable even before the critical state is reached. Were it not for factors that lead to instability of such a star, the star could exist in equilibrium until it exhausted all its reserve of nuclear energy. This reserve amounts approximately to $\mathcal{E}_{nuc} \approx 10^{51} M/M_\odot$ erg [see (5.1)]. Consequently the time of nuclear evolution is on the order of $\mathcal{E}_{nuc}/L \approx 10^5$ years. However, as we shall show later, the isothermal instability of the star changes this result.

Table IV. Effective power of nuclear energy released per gram of material of the star

erg/g-sec \ $M/M_\odot$	10^5	$5 \cdot 10^5$	10^6
$\tilde{A}_{pp}$	$2 \cdot 10^2$	$6 \cdot 10^{-2}$	$5 \cdot 10^{-4}$
$\tilde{A}_{CN}$	$2 \cdot 10^{14}$	$6 \cdot 10^4$	0,1
L/M	$6 \cdot 10^4$	$6 \cdot 10^4$	$6 \cdot 10^4$

e) Stability of a Supermassive Star

The entire theory developed above is based on a consideration of the stability of the star relative to over-all adiabatic expansion or contraction. For this process, as already repeated many times, the critical factor is the adiabatic exponent $\gamma = 4/3$, which is characteristic for matter in which radiation pressure predominates.

The linearized small-perturbation problem has, obviously, an entire spectrum of solutions; this means that there exist many different perturbations, which depend exponentially on the time with different exponents

$$\delta\varrho = \sum (\delta\varrho)_i, \quad (\delta\varrho)_i = \varphi_i(\mathbf{r})\, e^{\omega_i t}, \quad \mathbf{u}_i = \mathbf{W}_i(\mathbf{r})\, e^{\omega_i t}.$$

The exponential time dependence follows from the fact that the unperturbed state does not depend on the time, and consequently the equation for the perturbations contains only dt, but not t itself, so that when a constant is added to t the solution should again go over into a solution; on the other hand in the linearized problem, when the solution is multiplied by a constant it still remains a solution; this is precisely the property possessed by the exponential:

$$e^{\omega(t+c)} = e^{\omega c} e^{\omega t} = C_1 e^{\omega t}.$$

Moreover, the equation without viscosity for $\varphi_i(\mathbf{r})$ contains only ω_i^2. An over-all expansion or contraction is the "fundamental tone," an oscillation without nodes, with $\varphi_0(\mathbf{r})$ having everywhere the same sign. All the remaining types of perturbations are orthogonal to it; this means that $\varphi_i(\mathbf{r})$ with $i \neq 0$ have opposite signs in different regions* and consequently have

*We must use Lagrangian rather than Euler coordinates, or else a trivial solution is obtained, connected with the displacement of the star as a whole, φ_1 = a grad ρ_0, with nodal surface and $\omega_1^2 = 0$.

nodal surfaces $\varphi_1(\mathbf{r}) = 0$. The corresponding $\omega_1^2 < \omega_0^2$ are indeed real. Therefore, if we prove the stability of the star relative to an over-all expansion or contraction, $\omega_0^2 < 0$, one can be assured of the stability of the star relative to all other adiabatic deformations. This result can be obtained in a different manner, as follows. We start by considering the stability of an unbounded homogeneous medium, recalling the statements already made in Sec. 3. We allow the same inaccuracy as Jeans we assume that this state can be stationary.*

In the limit, for very long waves, the perturbations depend on the time with $\omega^2 = 4\pi G\rho_0$ and $\omega = \pm[4\pi G\rho_0]^{1/2}$. For short waves the growth of the perturbations is hindered by the pressure gradient, since an increase in the density is accompanied by an increase in the pressure. Accordingly

$$\omega^2 = 4\pi G\varrho - k^2 a^2,$$

where a is the speed of sound and k is the wave vector of the perturbation $\varphi(\rho) \sim \exp(i\mathbf{k}\cdot\mathbf{r})$, in the limit when $k^2a^2 \gg 4\pi G\rho_0$, the equation describes the propagation of sound. The only perturbations which are unstable and growing are those with $k < [4\pi G\rho]^{1/2}/a$ or, in other words, with a wavelength larger than critical·

$$\lambda > \lambda_{\text{crit}} = \frac{2\pi a}{\sqrt{4\pi G\varrho}}$$

The diameter of the star in an equilibrium state is precisely of the order of the critical "Jeans" wavelength λ_{crit}. This means that within the confines of the star we cannot produce perturbations that would grow. A star which is in equilibrium is always stable against breakup into many small parts, see also[149,150].

A specific feature of large stars is that the pressure of the plasma constitutes a small fraction of the light pressure

$$\frac{P_{\text{pl}}}{P_{\text{rad}}} \approx 8\,6\left(\frac{M}{M_\odot}\right)^{-1/2}$$

L E Gurevich and A I Lebedinskii[18] raised the question of the stability of a large star against isothermal density perturbations. In the case of isothermal displacement of matter, only the plasma part of the pressure increases in proportion to the density, while the light part $P_r \approx \sigma T^4/3$ remains unchanged. Consequently, the restoring force which counteracts the growth of the perturbations is much smaller than in adiabatic compression. In other words, the isothermal speed of sound a_t is much smaller than the adiabatic speed of sound, and the critical Jeans wavelength for the isothermal perturbations is accordingly smaller.

By growth of isothermal perturbations, a massive star can in principle break up into an aggregate of individual stars.* The non-existence of stars with $M > 100M_\odot$ was, in particular, attributed to this.†

Actually, in the case of isothermal motion of a plasma relative to a homogeneous field of electromagnetic waves, friction is produced between the plasma and the radiation field. As a result, the isothermal perturbations, as well as the buildup of the perturbations, increase much more slowly than the adiabatic ones

Let us express the friction between the plasma and the radiation in terms of the quantities customarily used in astrophysics, the coefficient of radiant thermal conductivity and the cross section.

With the aid of (5.4) we can relate the force per gram with the energy flux

$$\mathbf{F} = \frac{\sigma_e \mathbf{q}}{cm_p}$$

We used this expression when we considered the "light equilibrium." Now, applying it to a plasma moving relative to a specified distribution of radiant energy, we write the energy flux relative to the matter in terms of the velocity of the matter **u** relative to the (stationary) radiation.

Obviously $\mathbf{q} = -\mathbf{u}E_{\text{rad}}$, where E_{rad} is the energy density of the light. Consequently, the friction law is

$$\mathbf{F} = -n\mathbf{u},\ n = \sigma_e E_{\text{rad}}/cm_p$$

The force is proportional to the velocity and the radiant-energy density, we shall denote the friction coefficient for brevity by n.

We now set up an equation for the perturbations

$$\delta\varrho = re^{ikx+\omega t},\quad u_x = ve^{ikx+\omega t},$$
$$\delta\varphi = \psi e^{ikx+\omega t},$$

where $\delta\rho$ is the density perturbation, $\delta\varphi$ the perturbation of the gravitational potential, the velocity in the unperturbed state is equal to zero, and in the perturbed state the velocity is directed along the x axis. Poisson's equation yields

$$\Delta\varphi = 4\pi G\varrho \longrightarrow -k^2\psi = 4\pi Gr$$

The continuity equation is

$$\frac{\partial\varrho}{\partial t} = -\operatorname{div}\varrho u_x \longrightarrow \omega r = -iku\varrho_0$$

*In this connection see, for example,[15]

*The total energy of a massive star is quite small in absolute magnitude, since the adiabatic exponent is close to 4/3. This has lead once to the conclusion[141] that one star with a given mass M and E ~ 0 can break up into two stars with masses M/2 and E ≡ 0, etc. Actually, with the entropy S at which the star with mass M is in equilibrium and has E ≡ 0, the equilibrium of stars of smaller mass is impossible.

†Schwarzschild and Harm[137] state that when $M > \sim 100M_\odot$, in a hydrogen-burning star, a vibrational instability develops as a result of the dependence of the rate of nuclear reactions and heat transfer on the density. We have assumed above that the thermal processes are slow and can be disregarded.

The equation of motion (with account of radiant friction!) is

$$\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}=-\operatorname{grad}\varphi+F\rightarrow\omega u=-ik\psi-nu$$

From this we obtain the characteristic equation for ω:

$$\omega^2+n\omega-4\pi G\varrho_0=0;$$

and when $n > [4\pi G\rho_0]^{1/2}$ we obtain approximately

$$\omega_1=\frac{4\pi G\varrho}{n},\quad \omega_2=-n,\ \text{where}\ n=\frac{\sigma_e}{m_pc}E_{\text{rad}}.$$

Let us compare the frequency characterizing the growth of the perturbations, ω_1, with the time τ_1 of thermal relaxation of the star. In order of magnitude (R—radius of star, l—the range of a quantum)

$$M\sim\varrho R^3,\quad E_{\text{rad}}=P=\frac{GM^2}{R^4}=G\varrho^2R^2,$$

$$\tau_1=\frac{R^2}{lc}=\frac{R^2\varrho\sigma_e}{cm_p}=\frac{R^2\varrho n}{E_{\text{rad}}}=\frac{R^2\varrho n}{G\varrho^2R^2}=\frac{n}{G\varrho}=\frac{1}{\omega_1}.$$

Thus, the time necessary for the perturbation $1/\omega_1$ to increase by a factor e is in order of magnitude equal to the time during which the light can diffuse to the outside through the plasma of the star; the entire approach, however, is valid only if the time τ_1 is considerably longer than the hydrodynamic time of contraction of the star in free fall

$$\tau_2\sim\frac{1}{\sqrt{G\varrho}}.$$

Actually the energy of the star is equal to the kinetic energy of the plasma, taken with the negative sign. Consequently, if the nuclear reactions do not maintain the temperature of the star, its contraction as a whole begins long before an appreciable fraction of the radiant energy is lost (we recall that we are considering a massive star, in which the radiant energy is much larger than the plasma energy). In this case isothermal instability does not have a chance to develop.

For stars with $M \approx 10^5 M_\odot$, the nuclear reactions, as was shown in item (d), are significant. The reserves of nuclear and thermal energy of such a star are comparable, and the isothermal instability will arise.

The breakup of a star into clusters presents a picture whose details call for a concrete analysis. So far we can state only that the breakup will occur into clusters with $M/M_\odot \approx 100$, for in an equilibrium star of such a mass the pressure of the radiation and of the plasma are of the same order. When such clusters begin to arise, the plasma can at points of reduced density, under the influence of the light pressure, acquire an acceleration and can be ejected to the outside.

Thus, the ejection of matter in such a star will occur even before the pieces into which the star breaks up after its light is exhausted begin to collide in the course of their fall.

f) Catastrophic Contraction

After the start of the catastrophic contraction, the energy can be regarded as practically constant if there are no nuclear reactions. The energy outflow does not have a chance to reduce the entropy appreciably (the neutrino energy outflow and the neutrino spectrum are considered in Sec. 6).

We note that the growth of entropy for a given energy, due to shock waves, viscosity, etc., cannot stop the contraction after the critical state is reached. Indeed, as can be seen from Fig. 6, a state with critical energy $\mathscr{E}''$ and with increased entropy S must of necessity lie on that branch of the curve $\mathscr{E}(\rho_c, S)$, on which unrestrained contraction proceeds. In exactly the same manner, the ejection of part of the mass cannot stop the contraction of the main part of the star after the critical state, if we recognize that the ejected mass should have positive energy (or else it could not overcome the gravitational field of the star and escape to infinity). This means that ejection of the mass only decreases the (negative) energy of the remaining star, so that it is impossible to go over from the critical state to a state of stable equilibrium by ejection of mass.

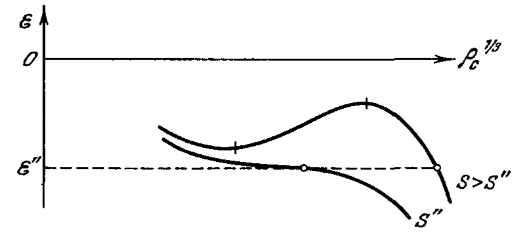

FIG. 6. Increase in the entropy of a star during the appearance of shock waves during collapse shifts the representative point to the descending branch of the isentrope S > S'', that is, it cannot stop the collapse.

The contraction can be stopped only by the following factors. First, by rapid exothermal nuclear reactions at high temperature. In this case it is easy for the heat release to be such as to cause the matter of the star to be completely scattered.* However, if the nuclear reactions have already converted during the equilibrium stage practically all the matter of the star into iron, this is certainly excluded, since all the possible nuclear energy has already been released.

Second, a contraction could be stopped in principle by the powerful baryon repulsion forces, which come into play at densities larger than nuclear and which bend the curve $\mathscr{E}(\rho, S)$ upward (see[1], Sec. 3). How-

*For this reason, Fowler's idea, according to which the oscillations of the luminosity of the quasar 3C 273 are periodic cycles of contraction and release of nuclear energy, seems little likely.

ever, when the mass is large [larger than (2—3) $M_\odot$], this is impossible, since general relativity effects turn from being merely corrections into a decisive factor long before the nuclear density is reached. Indeed, it was noted in the first part of the review that gravitational self-closure with the star going inside the Schwarzschild sphere with radius $r_g = 2GM/c^2$ is attained at an average density

$$\bar{\varrho} = 2\cdot 10^{16} (M/M_\odot)^{-2} \text{ g/cm}^3.$$

Nuclear density is reached before self-closure only when $M/M_\odot < 10$, that is, not for supermassive stars. After self-closure, no pressure can stop the contraction. We recall that a remote observer can obtain information only about the processes which occur in the star prior to self-closure; he sees only the asymptotic approach of the stellar radius to the gravitational radius. For him, the star cools, as it were, with $R \to r_g$. The observer can obtain information only on processes occurring in the star prior to self-closure, that is, up to densities $\bar{\rho}$ given by the formula indicated above.

6. EVOLUTION OF A STAR OF MEDIUM MASS

In Secs. 2, 3, and 12 of the first part of the review we considered the fundamental principles of stability and evolution of stars. We explained the role of the adiabatic index, the concept of negative specific heat of the star as a whole, and presented the concept of slow, quasistatic evolution, in which the instantaneous state at each moment is close to hydrodynamic equilibrium. In the opposite case, that of catastrophic contraction, the pressure at each instant is much smaller than the gravitational forces, so that the acceleration does not differ much from the acceleration in free fall. These concepts were used above to analyze the evolution of a supermassive star.

Referring the reader to the first part of the review for the general picture, we confine ourselves here to an analysis of particular cases, for the most part on the basis of papers published between the writing of the first and second parts. These papers are for the most part connected with the role of the neutrino in astrophysical processes.

a) Neutrino Emission Upon Contraction of Cold Matter

Let us consider, following[109,110], neutrino emission accompanying the contraction and neutronization of cold matter. In considering the equilibrium configurations of the star it is assumed that the Fermi energy of the electrons is equal, for each specified density, to the energy difference of the coexisting nuclei. For example, the coexistence of stable $_{26}Fe^{56}$ and radioactive $_{25}Mn^{56}$ occurs at a Fermi energy equal to 3.3 MeV, corresponding to matter with density 6×10^8 g/cm^3. Almost the same state is realized in slow contraction of matter. The near equality of the Fermi energy of the electrons to the reaction threshold signifies that in the case of slow contraction the neutrinos produced in the reaction $e^- + {}_{26}Fe^{56} = {}_{25}Mn^{56} + \nu$ carry away little energy.

It was mentioned in[1] that in fast contraction the neutronization can lag and produce an effective viscosity. The most interesting and most important aspect of the neutronization kinetics, however, is the following: if neutronization lags, then the Fermi energy rises above threshold, and the electron energy excess is picked up by the neutrinos. This is the mechanism whereby neutrinos of high energies (up to 30—40 MeV) are produced. It is accompanied by the formation of only neutrinos (but not antineutrinos), since it is assumed that there are no stars made up of antimatter. The yield of neutrinos is not more than one per nucleon, if the initial matter is hydrogen, and one per 2—2.5 nucleons in the case of helium or heavier elements.

Rough estimates of the neutrino energy were made in[109,110]. It was assumed that the density varies in accordance with the same law as in free fall of homogeneous matter:

$$\varrho = \frac{1}{6\pi G (t_0 - t)^2}, \quad \left|\frac{d\varrho}{dt}\right| = \frac{1}{3\pi G (t_0 - t)^3} = \varrho^{3/2} 2\sqrt{6\pi G} = \frac{\varrho^{3/2}}{450}.$$

The Fermi momentum and the Fermi energy of the electrons are expressed in terms of the density ρ and the number of nucleons per electron μ_e:

$$p_F = m_e c \left(\frac{\varrho}{\mu_e 10^6}\right)^{1/3},$$

$$E_F = m_e c^2 \sqrt{1 + \left(\frac{p_F}{m_e c}\right)^2} \approx m_e c^2 \left(\frac{\varrho}{\mu_e 10^6}\right)^{1/3}.$$

We denote by x the fraction of the nuclei which have undergone neutronization. The probability of neutronization depends on the properties of the initial and final nuclei Z_1 and Z_2, or concretely on the matrix element M_{12}. Under the usual laboratory conditions (without degenerate electrons) Z_1 is stable and Z_2 is β-active. The probability of the decay of Z_2 makes it possible to obtain M_{21}, and according to quantum mechanics $|M_{12}| \equiv |M_{21}|$. It is convenient to express the probabilities of the neutronization process of Z_1 by degenerate electrons directly in terms of τ, the half-life of the radioactive Z_2, and in terms of the well-known function f of the decay energy Q.

When $E_F \gg Q$ we get

$$\frac{dx}{dt} = -\frac{1}{5} x \frac{(E_F/m_e c^2)^5 \ln 2}{f\tau}.$$

For allowed transitions, for example, $n \to p + e^- + \bar{\nu}$, $f\tau = 800$ sec. The expressions given above are sufficient for a complete solution of the problem. The simplest example of neutronization of cold hydrogen leads to the conclusion that $x = 0.5$ will be attained for $E_F \sim 7$—8 MeV, which is much higher than the threshold (1.25 MeV, including the rest energy); this means that in this process the neutrinos carry away an energy of 5—7 MeV. The transformation of the proton into a neutron in a medium consisting of protons gives rise to a chain of nuclear reactions, which terminate with formation of He^4: $n + p = D + \gamma$, $D + p = He^3 + \gamma$, $n + He^3 = T + p$ or $He^3 + e^- = T + \nu$, $n + He^3 = He^4 + \gamma$, while $p + T = He^4 + \gamma$.

The formation of one He^4 nucleus from four protons and two electrons is accompanied by a release of 26 MeV. However, almost half of this energy is carried away by the two high-energy neutrinos. The neutronization of hydrogen during the course of free fall occurs predominantly at a density 5×10^8 g/cm^3, although the threshold density amounts to only 1.6×10^7 g/cm^3. A more difficult problem is the neutronization of helium under catastrophic contraction (free fall). In helium the threshold of the reaction $e^- + He^4 = T + n + \nu - Q$, with $Q = 22$ MeV, is very high. Furthermore, inasmuch as there is no bound state of H^4 (see the review[111]), the right side of the equation contains three particles. The probability of the reaction depends also on the energy which is carried away by the neutron.

It is understandable that experimental data on the inverse process $n + T = He^4 + e^- + \bar{\nu}$ are nonexistent, since the probability of weak interaction in flight (with a free neutron) is negligible. Therefore, to estimate the matrix element, the experimentally investigated capture of a negative muon was used in[110], namely $e^- + He^4 = T + n + \nu_\mu$. For μ^- on the lower orbit, the probability is 370 sec^{-1}.

Assuming that the matrix element does not depend on the neutron energy, they found

$$\frac{dx}{dt} = -x\, 660 y^2 (y-1)^{9/2},$$

where

$$y = \frac{E_F}{Q} = \frac{1}{45}\left(\frac{\varrho}{\mu_e 10^6}\right)^{1/3}.$$

The integration of the equation for x together with the law of free fall leads to the conclusion that the reaction takes place at $E_F \sim 45$ MeV and a density $\sim 10^{12}$ g/cm^3.

The difficult reaction of helium neutronization is followed by the much easier reaction with smaller threshold (~10 MeV)

$$e^- + T = 3n + \nu.$$

Thus, during the course of collapse, neutronization gives rise to neutrinos with energies up to 30—40 MeV. Rough estimates show that the average cosmic flux of such neutrinos can reach several per cent of the flux of high-energy solar neutrinos, resulting from the decay $B^8 \to Be^8 + e^+ + \nu$, with end-point energy 14 MeV. Since the probability of neutrino registration increases with increase in the neutrino energy, we cannot exclude the possibility of experimentally observing cosmic high-energy neutrinos, the origin of which is connected with collapse and neutronization of matter. Of particular interest in this connection are the projected experiments for the determination of the energy and direction of neutrinos.[112]

It remains for us to make two remarks. The calculations were made for the density of freely contracting matter. The pressure gradient slows down the contraction of the central core. On the other hand, when matter falls in the peripheral part, the density increases first more slowly, and then more rapidly than given by the free-fall formula. Furthermore, we are comparing $d\rho/dt$ for a given ρ. We note that comparison at the same instant of time is meaningless. Thus, the law governing the density variation, which serves as the basis for the calculations, cannot be regarded as an upper limit; deviations from it are possible in both directions.

Does gravitational self-closure of a star influence the possibility of neutrino registration? We have seen that self-closure occurs at a density $2 \times 10^{16}(M/M_\odot)^{-2}$. The maximum density at the center of the star, which can still be seen by an observer, is somewhat smaller than this quantity. For a simple example[113] the limiting central density is smaller by a factor 2.25. The neutronization of helium occurs at $\rho \sim 10^{12}$ g/cm^3. Consequently, for the overwhelming majority of stars with $M < 50M_\odot$ the neutrinos will emerge experiencing only a small red shift. The registration of the high-energy neutrinos may turn out to be a method of observing spherically symmetrical "soundless" collapse of stars.

b) Emission of Neutrinos from a Hot Plasma

The neutronization process described above is a direct consequence of the experimentally investigated interaction connected with the transformation of protons into neutrons and vice versa. Contemporary theory predicts with great probability the possibility of emission of neutrino-antineutrino pairs following any change in the electron momentum, when the electron jumps from one orbit to another, or finally, when an electron and positron annihilate:

$$e^- = (e^-)' + \nu + \bar{\nu}, \quad e^- + e^+ = \nu + \bar{\nu}.$$

The astrophysical consequences of this process were pointed out by B. M. Pontecorvo.[115] The first calculations were made by Gandel'man and Pinaev[114] for radiation by an electron moving near a nucleus, $e^- + Z = e^{-\prime} + Z + \nu + \bar{\nu}$. However, the neutrino processes become significant in astrophysics only at a temperature on the order of 5×10^8 and above,[132,128,151] when the annihilation $e^- + e^+$ begins[128] and the urca process with electrons and protons, of the type $e^- + p = n + \nu$, $e^+ + n = p + \bar{\nu}$, begins (Pinaev[116,117]). In one of the first attempts to explain quasars,[118] it was suggested that when the central part of a star collapses, ν and $\bar{\nu}$

can carry away an energy which constitutes a noticeable fraction of the rest energy Mc^2 of the core of the star. The decrease in the mass of the star gives rise to a decrease in the forces of attraction acting on the outer shell, and the latter, heretofore in equilibrium, will now expand and be partially discarded.*

In connection with this hypothesis (which incidentally turns out to be untenable, as was shown already in 1963[119]), and in connection with problems involving the evolution of stars, it is interesting to consider the question of the rate of neutrino losses. In the case of nonrelativistic electrons and positrons, the rate of energy loss is

$$\frac{du_\nu}{dt} = 2.8\cdot 10^{-40} n_+ n_- \text{ erg/cm}^3 \text{ sec},$$

where n_- and n_+ are the concentrations of electrons and positrons in cm^{-3}.

In turn, in the case of a nonrelativistic temperature and at a density such that the gas is still not degenerate, the equation of thermodynamic equilibrium yields (see Sec. 2)

$$n_+ n_- = \frac{4(2\pi mkT)^3}{(2\pi\hbar)^6} e^{-\frac{2mc^2}{kT}}, \quad \frac{p_F^2}{2m} < kT < mc^2.$$

The inequality on the right gives the region of applicability of the formula (m—electron mass). Substituting the numerical values and expressing further the temperature in units of $10^{9\circ}$ (T_9), we obtain

$$n_+ n_- = 1.7\cdot 10^{58} T^3 e^{-11.9/T},$$

$$\frac{du}{dt} = 4.8\cdot 10^{18} T^3 e^{-11.9/T} \text{ erg/cm}^3 \text{ sec},$$

$$3\left(\frac{\varrho}{\mu\cdot 10^6}\right)^{2/3} < T < 6 \quad (T = T_9 \text{ everywhere}).$$

In the region of high temperatures, $kT > mc^2$, the number of e^+, e^- pairs increases like T^3, as does also the number of quanta; it becomes much larger than the number of the initially taken electrons (which neutralize the nuclei). The product n_+n_- increases like T^6, and the annihilation cross section like T^2. The spectrum of the produced neutrinos and antineutrinos has the approximate form

$$E^6 e^{-12E/T} dE$$

(E is in MeV and T in T_9 units), so that the average neutrino energy is six times larger than the energy kT.†

*The inner part of the shell bears against the core and is also dragged inward during the collapse. We have in mind that external part which the hydrodynamic signal concerning the start of the collapse (speed of sound) has not yet reached.

†For comparison we note that the spectrum of equilibrium photon emission has a maximum at $\hbar\omega = 4$ kT. The maximum for fermions is even higher. For this reason, whenever we write below formally, for example, the condition $kT > mc^2$, etc., the less stringent condition $2kT > mc^2$ is actually sufficient. For example, for the ultrarelativistic formulas to be valid, it is sufficient to have $T = T_9 > 3$. The spectrum of the emitted neutrinos is in general not in equilibrium, since the matter remains transparent to the neutrino. The radiated spectrum is harder than the equilibrium spectrum, because the interaction increases with the increasing energy.

In this situation we have

$$n_+ = n_- = 1.6\cdot 10^{28} T^3 \frac{1}{\text{cm}^{-3}},$$

$$u(\text{pairs}) = 1.75\sigma T^4 = 1.3\cdot 10^{22} T^4 \text{ erg/cm}^3$$

$$\frac{du_\nu}{dt} = 4.3\cdot 10^{15} T^9 \text{ erg/cm}^3 \text{ sec}, \quad T > 6, \quad \frac{T}{6} > \left(\frac{\varrho}{\mu_e\cdot 10^6}\right)^{1/3}.$$

Actually, as noted by Hoyle and Fowler,[120] the last formula is satisfactory starting with T > 3 (the overestimate is 50% and decreases rapidly to 10% at T = 6).

One can recommend an intermediate interpolation formula

$$\frac{du_\nu}{dt} = 10^{14} T^{12.5} \text{ erg/cm}^3 \text{ sec}, \quad 1 < T < 3,$$

which fills satisfactorily the gap in which both the theoretical, asymptotically correct formulas are poorly satisfied.

Finally, in a relativistically degenerate gas the chemical potential of the electrons* is given by the expression $\mu_- = mc^2 (\rho/\mu_e \times 10^6)^{1/3}$, and the concentration of the positrons is

$$T < 6, \quad \frac{\mu_-}{mc^2} > 1, \qquad n_+ = 1.3\cdot 10^{29} T^{3/2} e^{-6(1+\mu_-/mc^2)/T},$$

$$T > 6, \quad \frac{\mu_-}{mc^2} > \frac{T}{6}, \quad n_+ = 1.5\cdot 10^{28} T^3 e^{-6(1+\mu_-/mc^2)/T}.$$

For the energy losses we obtain approximately

$$\frac{du_\nu}{dt} \approx 10^{-40} n_+ n_- \left(\frac{\mu_-}{mc^2}\right)^2, \quad T < 6,$$

$$\frac{du_\nu}{dt} \approx 3\cdot 10^{-40} n_+ n_- \left(\frac{\mu_-}{mc^2}\right)^2 \left(\frac{T}{6}\right), \quad T > 6,$$

so that finally in the latter case we get

$$\frac{du_\nu}{dt} = 5\cdot 10^{17} T^4 \left(\frac{\varrho}{\mu_e\cdot 10^6}\right)^{5/3} e^{-6\left[\left(\frac{\rho}{\mu_e\, 10^6}\right)^{1/3}+1\right]/T},$$

$$6 < T < 6\left(\frac{\varrho}{\mu_e\cdot 10^6}\right)^{1/3}.$$

Expressions for the energy lost by the urca process are given by Pinaev.[116,117]

Let us consider the energy loss during the course of free fall and compression of matter with specified initial value of the specific entropy S_0. Let the initial state be such that the energy of radiation and of the pairs exceed the energy of the initial plasma. We write down immediately expressions for T > 6, when the e^+, e^- pairs comprise a full-fledged term of the energy density

$$u = u_{\text{pair}} + u_{\text{rad}} = 2.75\sigma T^4 = 2.1\cdot 10^{22} T^4, \quad S = 2.8\cdot 10^{22}\frac{T^3}{\varrho}.$$

The entropy is expressed in units of erg per gram per $10^{9\circ}$. The equation for the change in entropy is

$$\frac{dS}{dt} = -\frac{1}{\varrho T}\frac{du_\nu}{dt} = -\frac{1}{\varrho} 4.3\cdot 10^{15} T^8.$$

*Not to be confused with μ_e in the parentheses, which is the molecular weight per electron.

It remains to express the temperature in terms of the entropy and the density:

$$T = \left(\frac{\varrho S}{2.8 \cdot 10^{22}} \right)^{1/3}.$$

We ultimately obtain

$$\frac{dS}{dt} = -3.6 \cdot 10^{-45} \varrho^{5/3} S^{8/3}.$$

If we specify the density as a function of the time $\rho(t)$, for example in accordance with the law of free fall, then, by expressing dt in terms of $d\rho$, we obtain directly a simple easily solved equation

$$dt = 450 \varrho^{-3/2} d\varrho, \quad \frac{dS}{d\varrho} = -1.6 \cdot 10^{-42} \varrho^{1/6} S^{8/3}.$$

Astronomers, who are not used to dealing with entropy, should notice how much simpler the derivation of the equation becomes, as compared with the usual procedure, in which it is necessary to consider the energy and work done by the pressure forces, which do not enter into the entropy equation!

We obtain ($\rho = 0$, $S = S_0$)

$$S = [S_0^{-5/3} + 1.6 \cdot 10^{-42} \varrho^{7/6}]^{-3/5}. \qquad (6.1)$$

Knowing $S(\rho)$, we can easily write an integral which gives the total energy loss Δ per gram of matter upon compression from $\rho = 0$ to a specified density ρ. To this end, we express all the quantities S, T, du_ν/dt and dt in $\int \frac{1}{\rho} \frac{du_\nu}{dt} dt$ in terms of ρ and $d\rho$.

If we were to disregard the decrease in the entropy during the course of the compression, then we would obtain

$$\Delta = k_4 \int_0^\rho \frac{T^9}{\varrho} dt = k_5 S_0^3 \int_0^\rho \frac{\varrho^3}{\varrho} \varrho^{-3/2} d\varrho \equiv \frac{2}{3} k_5 S_0^3 \varrho^{3/2},$$

that is, an integral which diverges at the upper limit $\rho \to \infty$; when $\rho \to \infty$ the loss would increase without limit. However, by substituting the expression (6.1) for S, we obtain a converging integral

$$\Delta = 4.3 \cdot 10^{15} \cdot \frac{1}{(2.8 \cdot 10^{22})^3} \cdot 450 \int_0^\infty [S_0^{-5/3} + 1.6 \cdot 10^{-42} \varrho^{7/6}]^{-9/5} \varrho^{1/2} d\varrho.$$

Simple calculations yield*

$$\Delta = \frac{2}{3} k_5 S_0^3 \varrho^{*3/2}, \quad \varrho^* = 1.64\,(1.6 \cdot 10^{-42} S_0^{5/3})^{-6/7},$$

$$\Delta = k_6 S_0^{6/7},$$

where

$$k_5 = \frac{4.3 \cdot 10^{15} 450}{(2.8 \cdot 10^{22})^3} = 10^{-49.06}, \quad k_6 = 10^{4.85}.$$

The quantity ρ^* characterizes the density at which the decrease in entropy effectively stops the energy loss; the value of Δ, calculated for compression to $\rho = \infty$ with account of the decrease in the entropy, is expressed in terms of ρ^* in the same manner as the energy loss Δ for S = const and for compression to ρ is expressed in terms of ρ.

This characteristic quantity ρ^* must be compared with the density at which gravitational self-closure occurs as a result of the approach to the Schwarzschild radius $\rho_g = 1.8 \times 10^{16} (M/M_\odot)^2$. Let us express the initial entropy S_0 in terms of the mass of the star.

The following picture of the process is obtained. The star evolves slowly, going through a sequence of equilibrium states. Finally, a critical state is reached with an adiabatic exponent equal to 4/3 (or somewhat larger, with account of corrections for general relativity). This exponent is reached in two cases: (i) as a result of relativistic degeneration of the electron gas in the case when the mass is not much larger than the Chandrasekhar limit for white dwarfs: this case will not be considered here; (ii) in the opposite case, the exponent 4/3 corresponds to predominance of the energy of radiation, while there are still no electron-positron pairs. As was shown in detail earlier, in this case,

$$T = 0.02 \varrho^{1/3} \left(\frac{M}{M_\odot} \right)^{1/6} (T \text{ in units of } T_9).$$

The corresponding expression for the entropy (in the stage without pairs!) is

$$S = \frac{4}{3} \sigma \frac{T^3}{\varrho} = 0.97 \cdot 10^{22} \frac{T^3}{\varrho} = 8 \cdot 10^{16} \left(\frac{M}{M_\odot} \right)^{1/2} \text{erg}/10^9 \text{ deg-g}.$$

Pair production is itself a cause of decreasing γ to a value smaller than 4/3, that is, the cause of the start of catastrophic fall.

This fall proceeds adiabatically, that is, with constant entropy; the situation when the pairs become full-fledged participants in the equilibrium is attained before the neutrinos have time to decrease the entropy noticeably. Thus, in the region $0.5 < T < 3$ there occurs a transition to the formulas

$$u = 2.75 \sigma T^4, \quad S = 2.8 \cdot 10^{22} \frac{T^3}{\varrho},$$

from which we get

$$T = 0.014 \left(\frac{M}{M_\odot} \right)^{1/6} \varrho^{1/3}.$$

In the case of adiabatic (lossless) contraction to a density corresponding to gravitational self-closure, $\rho_g = 1.8 \times 10^{16} (M/M_\odot)^{-2}$, we obtain an important physical conclusion: the energy of the photons and of the pairs per unit volume constitutes 0.24 of the rest energy ρc^2. In this state

$$T = 4000 \left(\frac{M}{M_\odot} \right)^{-1/2}.$$

One cannot give more than one has. As applied to neutrino radiation, this means that the total radiated energy cannot exceed $0.24\ Mc^2$. This, however, is a

*The value of the dimensionless integral is

$$\int_0^\infty (1 + Z^{7/6})^{-9/5} Z^{1/2} dZ = 1.4.$$

highly exaggerated overestimate.

Actually, the integration of the loss equation leads to the conclusion that the limit is equal to $0.1\ Mc^2$. If the mass of the star is large, $M > 10^4\ M_\odot$, then the neutrino emission, which occurs under conditions of practically constant entropy, is limited by the gravitational closure. For masses $M > 10^6$ the temperature is such that pair production becomes exponentially small.

On the other hand, when $M < 200 M_\odot$, the decrease in the entropy as a result of the neutrino radiation itself limits the total energy loss before relativistic self-closure takes place; the transition from one region to the other is quite gradual.

For the effective density and temperature, at which the loss reaches $\Delta/2$, we have in this case the following expressions

$$\varrho^* \sim 8\cdot 10^{11}\left(\frac{M}{M_\odot}\right)^{-5/7}, \quad T \sim 100\left(\frac{M}{M_\odot}\right)^{-1/14}.$$

Table V (in part taken from[119]) gives the values for different masses.

Actually the entire Table V is a very crude approximation. In the region of small masses, $M < 100 M_\odot$, it is necessary to take into account the plasma energy and the electron degeneracy. All the processes, including gravitational closure, must be considered against the background of the true hydrodynamic solution, which depends both on the time and the coordinates.[115] Neutrino scattering must also be taken into account.[128] This can reduce the given values of Δ by a factor of several times.

At the same time, the following qualitative conclusions can be regarded as reliably established:

1. The neutrino energy loss upon collapse always constitutes a small fraction of the rest energy and cannot cause stripping of the shell.

2. The over-all density of the intergalactic neutrinos and antineutrinos produced in the collapse does not exceed 5% of the density of the collapsing stars.

3. In the collapse of stars with small mass ($M < 3M_\odot$), neutronization may be accompanied by production of neutrinos with energies up to 30—40 MeV, in amounts of 0.25—0.5 per nucleon, yielding $\Delta/c^2 \sim 1\%$.

4. In the collapse of stars with $100 M_\odot > M > 3M_\odot$, neutrinos and antineutrinos are produced with a broad spectrum and with average energy on the order of 30—50 MeV; the energy loss up to $\Delta/c^2 \sim 5\%$ corresponds to the emission of up to one neutrino or antineutrino per nucleon.

The region of small masses and high temperatures calls for additional analysis. The production of a small number of muonic neutrinos is not excluded.

It is obvious at any rate that the realization of modern experimental ideas, which would make it possible to determine the energy and flight direction of the neutrino, can yield information of extreme value to astrophysics.

c) Release of Nuclear Energy, Rate and Stability of Evolution

Besides general regularities, astronomical observations disclose also large qualitative diversities in the world of stars. These diversities concern the chemical and isotopic composition of the stars. Stars are observed in which the contents of rare earths is 1000 times larger than the average, stars with the ratio $C^{13} : C^{12} \approx 1$ (as against 0.01 on earth); finally, there is one lone star with $He^3 : He^4 = 4$ (as against 10^{-7}). Some stars have anomalously large magnetic fields.

There are stars whose brilliance varies periodically (Cepheids), stars which flare up regularly, and finally stars which experience catastrophic explosions (supernovas). There is a known example of a supernova flare—the explosion which gave rise to the Crab nebula.

It can be assumed very roughly that all the recently produced young stars consisting essentially of hydrogen or 60% hydrogen and 40% helium, and 1% of heavier elements, are similar to one another. All the properties of such stars are determined entirely by their mass; these stars form a one-parameter family. On the spectrum-luminosity diagram they form the "main sequence" of Hertzsprung-Russell. The concept of youth of a star depends in turn on the rate of consumption of the fuel: a star with $M \sim M_\odot$ reaches middle age after 5×10^9 years, while a star with mass $30 M_\odot$ exhausts its hydrogen and ages in 10^6 years.

It is precisely for the period of evolution occurring after the exhaustion of hydrogen that the variety of the observed properties and behavior of the stars is characteristic. The questions which arise in connection with this stage of evolution are not fully clear and apparently are not specially connected with relativistic

Table V. Mass loss of a collapsing star due to neutrino radiation

$M/M_\odot$	10	100	10^3	10^4	10^5	10^6	10^8
Δ/c^2	0.05	0.1	0.1	0.1	$2.5\cdot 10^{-3}$	$6\ 10^{-5}$	10^{-12}
$T(T_9)$	85	70	50	36	11	3.6	0.5
$\varrho_g,\ \varrho^*$	$1.6\cdot 10^{11}$	$2.5\cdot 10^{10}$	$5\cdot 10^9$	$2\cdot 10^8$	$2\cdot 10^6$	$2\cdot 10^4$	2

effects. We shall therefore discuss them only very briefly.

What parameters can produce the variety in the properties of stars, if we assume that all the stars were condensed initially from a gas of approximately the same composition? After all, all the stars pass on the main sequence through the Lethic stage.†

In this case the star "forgets" the asymmetry, turbulence, and temperature of the initial gas, from which it was condensed. What does the star "remember"? On what does its further development depend? We already mentioned the mass, the main characteristic of the star. During the period of combustion of hydrogen on the main sequence, the loss of mass is negligibly small, the mass is conserved.

The second conserved quantity is the angular momentum of the star. Apparently, the magnetic properties must also be regarded as an innate property of the star, as an invariant; incidentally, this is far from obvious; it is still not clear to what extent the magnetic field of the star is the result of intensification of the magnetic field of the interstellar medium upon condensation of the star. Another possibility is the appearance of magnetic field as a result of convective motions in the star (dynamo effect)[121]; see Batchelor's theorem concerning the magnetic field in a turbulent conducting liquid. Finally, the fate of the star can be greatly influenced by the presence of a nearby neighboring star, with which it forms a close double system. This is frequently forgotten by the theoreticians, yet among definite classes of stars double stars are quite abundant; there is the opinion that all "novas" are double stars. We recall in this connection the considerations advanced concerning the age of the two components of Sirius (see Sec. 12 below), and also the limitations imposed on the parameters of double stars by the radiation of gravitational waves (see the article by Braginskiĭ in the present issue).

We now proceed from a listing of the parameters characterizing the star and the conditions of its evolution, to a clarification of those deep internal reasons which can cause some instability of the star. Apparently, on the whole all the listed factors, namely rotation, the magnetic field, the presence of a second component, are small perturbations compared with the most fundamental cause—the force of gravitation which depends on the total mass. These factors therefore influence strongly the evolution only during periods of low stability of the star. It is frequently—but inaccurately!—stated that rapid absorption of heat can be the cause of catastrophic contraction,† and neutrino emission is regarded on par with such energy-consuming processes as the dissociation of iron $Fe^{56}_{26} = 14\ He^4 + 4n$ or pair production $e^+ + e^-$. Actually the emission of neutrinos is a factor which causes a change in the entropy. The neutrino loss rate enters into the right side of the equation

$$\frac{dS}{dt} = -\frac{1}{\varrho T}\frac{du_\nu}{dt}.$$

In the presence of stable solutions which depend on S as a parameter, the rate of change of the entropy determines the rate of the evolution. The speed of contraction is always smaller than the speed of light. Before gravitational self-closure sets in, the neutrino has time to leave the star. The emission of the neutrinos is essentially a non-equilibrium process, and in this lies its sharp difference from production of e^+, e^- pairs. In hot matter, the time of establishment of equilibrium of these pairs is negligible in all scales; for example, when $T_9 = 6$ this time is of the order of 10^{-18} sec. Consequently, at each instant and at each point the pairs are in complete equilibrium, and their number is not determined by the rate of the process. The heat consumed in pair production has not vanished: let the matter expand and let the temperature decrease; when the number of pairs decreases in accordance with the equilibrium condition, the lost heat will be released again. The equilibrium of pair production is not a factor which changes the entropy. Pair production changes the form of $p = p(\rho, S)$, that is, it changes the dependence of the pressure on the density at a given entropy. The same pertains also to the dissociation of iron and helium. As a result, in a definite region of temperatures and densities

$$\gamma = \frac{\partial \ln P}{\partial \ln \varrho}\bigg|_S \leqslant \frac{4}{3}$$

is attained and stability is lost. The gist of the matter, of course, lies in the fact that when the loss of the rest energy of e^+ and e^- or of the energy lost to overcoming the nuclear forces is taken into account, the ratio of the additional pressure of the new particles to the density of the energy turns out to be small, less than 1/3. However, the description of this circumstance by introducing special quantities dQ/dT is the lamentable consequence of the underestimate of the thermodynamic methods and of the clarity and convenience afforded by the use of entropy.

In the ρ, T plane we can draw a line on which $\gamma = 4/3$. This line separates the region of instability from the stability region. Under the roughest assumptions concerning the structure of the star we can draw lines $p = aM^{2/3}\rho^{4/3} = \text{const}\cdot\rho^{4/3}$, corresponding to the average hydrostatic equilibrium. When such a line crosses the line $\gamma = 4/3$ this means that a star of a given mass loses stability.

For example, according to the calculations of Imshennik and Nadezhin,[130] for a star with $M = 20M_\odot$ the value $\gamma = 4/3$ is reached when $\rho \approx 6\times10^6$ g/cm^3

*This new term[35] is derived from the name of the mythological river Lethe, which separates the kingdom of the living from the kingdom of the dead. Lethe is the river of oblivion.

†For this contraction there is a special term: implosion — "inward detonation," distinguishing it from explosion, which is directed outward with ejection of matter.

and $T_9 \approx 4.8$. However, during the course of the further adiabatic contraction, beyond the region $\gamma < 4/3$, the line S = const crosses the second line $\gamma = 4/3$ and again enters into the stability region. The reason is the formation of a large number of nonrelativistic particles upon the dissociation of iron $Fe \rightarrow \alpha + n \rightarrow p + n$.

Thus, after a fast nonstationary contraction, a halt can occur in a new equilibrium state. During the course of the halt, shock waves are produced and propagate towards the surface of the atmosphere of the star, transferring their energy to an ever decreasing mass and breaking up the outermost layer. This is the mechanism of supernova flare considered in the papers of Nadezhin and Frank-Kamenetskii.[122] They investigated in great detail the hydrodynamics of the process of establishment of a new equilibrium and the ejection of the shell by the wave, without stopping to discuss the causes of the initial disruption of the old equilibrium. The outward appearances in the picture which they have calculated are in good agreement with the observations.

A second most important factor in the instability, for many years advertised by Hoyle and Fowler,[10,120,123] is connected with the inhomogeneous composition of the star. In the absence of convective mixing, at the instant when at the center of the star total thermodynamic equilibrium is attained and the matter near the center has been completely converted into iron, the adjacent layer contains oxygen and carbon, the next contains helium, and finally, in the outer layer there remains the uncombusted hydrogen.

The transmutation of hydrogen into helium is connected with weak interaction and can never become rapid. We shall therefore not take into account the energy of the hydrogen. But even without the hydrogen the energy of the transformation of heavier nuclei exceeds the negative energy of the star as a whole. This means that the reserve of nuclear energy, for example the reaction $2O^{16} \rightarrow S^{32}$, together with the thermal energy of the star, is sufficient to overcome the gravitation and to scatter the entire star to infinity. The processes $3He^4 \rightarrow C^{12}$, $2C^{12} \rightarrow Mg^{24}$, and $2O^{16} \rightarrow S^{32}$ do not need transformation of protons into neutrons, and are the result of strong interaction (nuclear forces). At sufficiently high temperatures, which weaken the action of the Coulomb repulsion of the nuclei, these processes can occur within a time shorter than the free-fall time, that is, behave explosively. Hoyle and Fowler developed a scheme for the blow up brought about by implosion: the shock wave, passing through the corresponding layers, produces in them nuclear reactions with release of heat. In other words, the shock wave is transformed into a detonation wave. All the upper layers are ejected with giant velocities. One must not think, however, that the iron core will in this case be completed: even if it is initially compressed as a result of an increase in the pressure during the start of the nuclear reaction, then afterwards, lacking the exterior pressure of the escaping shell, the core will also expand and scatter.

It should be remembered that the entropy of the core material corresponds to the equilibrium of the core only under the condition that the latter is under the pressure of the layers of the star surrounding it. This entropy, however, is much larger than the equilibrium value S_e for the smaller mass remaining after the discarding of the shell.

Thus, the star, during each instant of its evolution, almost up to the time of the total exhaustion of the fuel, ''sits on a powder keg'' and contains a fuel reserve sufficient for suicide. Is implosion the only mechanism capable of blowing up the star? To what extent is the state which is fully stable hydrodynamically also stable with respect to thermal blowup?

In the first part of the review we related the thermal stability with the negative specific heat of the star. There are two factors which cause under given conditions thermal instability.

1) The negative specific heat is characteristic of a nondegenerate plasma. At high density and not too high temperature, when appreciable degeneracy of the electrons takes place, the specific heat of the star becomes positive. An attentive reader should indeed have observed this already in the first part. During the course of the decrease of the entropy, the temperature of a star with $M = M_{\odot}$ first increases, and then decreases; a white dwarf cools down to a low temperature. The decrease of the temperature with decreasing entropy denotes positive specific heat. This circumstance has led to an abrupt halt of the nuclear reaction, to the cooling of the composition of the white dwarf. The decrease in temperature reduces the reaction rate, and the lagging of the reaction relative to the heat loss produces conditions for a decrease in entropy, which in turn, when the specific heat is positive, reduces the temperature. Under other conditions, for a different composition, the same instability can lead to a thermal explosion.

2) The second circumstance is even more important. Negative specific heat is a concept pertaining to the star as a whole, the result of the realignment of the density of the entire star with its entropy changing everywhere. Each individual small layer of matter in the star has a positive specific heat equal to c_p: each layer is under constant pressure of the matter above it. Therefore, in principle, individual layers can blow up thermally. This process is made difficult by the fact that the given layer is in thermal contact with matter lying above and below it. But if we take the layer too thick, the increase of the entropy in it will already be accompanied by a noticeable change in pressure—the specific heat increases and in the limit, passing through $c = \pm\infty$, it becomes negative when the layer is comparable with the entire star.

If the reaction occurring in the layer between the burned-up core and the shell has a sufficiently strong temperature dependence, there can exist such thermal disturbances, with respect to which the stationary mode is unstable. The process of explosion with increasing entropy in the layer is difficult in the presence of convection. The very increase of the entropy in the layer produces conditions for convective mixing.

It is possible that the thermal explosions, the development of which terminates by intensification of convection, play an important role in multiple flares (see also the third footnote on page 535).

The anomalous composition of the stellar atmosphere points to a mixing of matter which has never burned (hydrogen, helium) with matter that stayed at some time in the interior of the star and contains heavy nuclei.[123] These nuclei can be produced only by the joining of neutrons to nuclei of the middle of the periodic table, that is, they require such a high temperature that hydrogen could not survive. Finally, there are singularities in the composition ($He^3 : He^4 > 1$) which, in the opinion of several authors,[123-125] point to the very strong irradiation of the matter with particles having energies of many MeV, that is, particles such as cosmic rays, which are not in thermal equilibrium.

The foregoing exposition does not pretend to describe exhaustively the nonstationary phenomena. We hope, however, that even such a brief review will present the reader with an idea concerning the character of the possible theory and will perhaps attract new mathematicians and physicists to its development.

7. MOTION OF TRIAL PARTICLES AND LIGHT RAYS IN A SCHWARZSCHILD FIELD

a) Potential Curves of Motion

In the first part of the review we analyzed the motion of particles and of light along a radius in a Schwarzschild field. Here we shall stop to discuss the general case of non-radial trajectories. This question has been analyzed in detail long ago. A complete classification of the motions is contained, for example, in the book of A. F. Bogorodskiĭ[19]; see also[20,21]. An analysis of the principal questions involving the stability of motion on circular orbits is given by S. A. Kaplan.[22]

The equations of motion in polar coordinates are of the form (planar trajectory)

$$\left(\frac{d\tilde{l}}{d\tau}\right)^2 = \frac{E^2 - 1 + \frac{1}{r} - \frac{a^2}{r^2} + \frac{a^2}{r^3}}{E^2}, \tag{7.1a}$$

$$\left(\frac{d\varphi}{d\tau}\right)^2 = \frac{a^2}{E^2 r^4}\left(1 - \frac{1}{r}\right). \tag{7.1b}$$

For convenience, the equations were written out in terms of dimensionless quantities. Here r—Schwarzschild radial coordinate, measured in units of the gravitational radius $r_g = 2GM/c^2$; $d\tilde{l} = dr(1 - 1/r)^{-1/2}$—element of radial distance (see[1]); τ—physical time, measured by a local observer (see[1]) in units of r_g/c; a—angular momentum, measured in units of mcr_g; E—energy measured in units of mc^2; m—mass of the trial particle. The energy includes the rest mass, and therefore for a particle at rest at infinity we have E = 1. At distances which are large compared with the gravitational radius, that is, when $r \gg 1$ and when the energy of motion is small compared with unity, $|E - 1| \ll 1$, we obtain from (7.1) the equations of the Kepler problem in Newtonian gravitational theory. Indeed, under these conditions we can neglect the term a^2/r^3 in (7.1a), and we have $d\tilde{l} \approx dr$ and $E^2 - 1 \approx 2(E - 1)$. In this case $(-1/r)$ is the gravitational potential and a^2/r^2 is the potential of the centrifugal forces. The vanishing of the numerator in (7.1a) obviously yields the potential curve of radial motion.

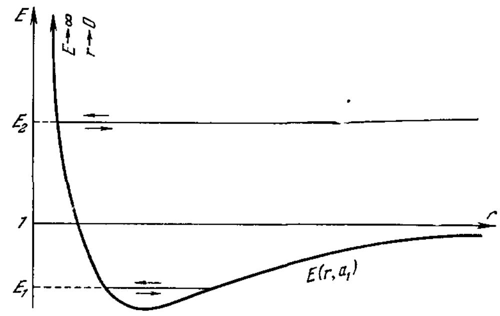

FIG. 7. Potential curve of radial motion in Newtonian theory for a fixed momentum a_1. $E_1 < 1$ – horizontal of finite (elliptical) motion; $E_2 > 1$ – horizontal of hyperbolic motion.

For Newtonian theory, such a curve $E = E(r, a_1)$ for fixed a_1 is shown in Fig. 7. For any a_1 the curve has a minimum. The qualitative singularities of the motion of the trial particle are immediately seen in this figure. The motion occurs at constant energy E_1 and is given by the horizontal line $E = E_1$. A particle with momentum a_1 moves along the horizontal to the corresponding turning curve $E = E(r, a_1)$, and then moves in the opposite direction again until it crosses the same curve, etc., carrying out finite motion in the "potential well." In accordance with the fact that we have chosen in this example $E_1 < 1$ and the energy, as in general relativity, is measured from mc^2 (from unity in our units), the particle will not go off to infinity.

If the energy of the particle is $E_2 > 1$ (Fig. 7), then it travels along the hyperbola from infinity, reaches a minimum r, corresponding to the intersection of E_2 with the curve $E = E(r, a_1)$, and again goes off to infinity. Since the potential curves tend to infinity when $r \to 0$, $E \to \infty$, for any large energy the particle, after overtaking the attraction center, will again escape to infinity, provided of course it does not collide with the surface of the attracting body. Gravitational capture is impossible in Newtonian theory of two point-like bodies.

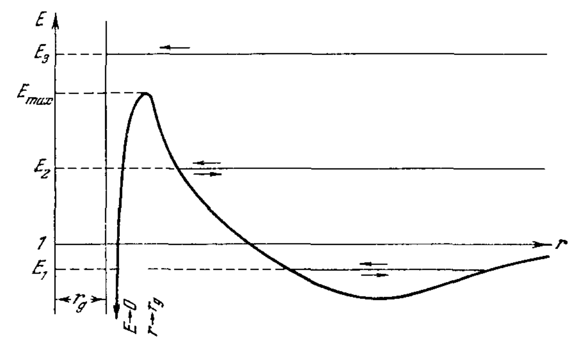

FIG 8. Potential curve of radial motion in general relativity for a fixed momentum a_1. $E_1 < 1$ – horizontal line of finite motion, $1 < E_2 < E_{max}$ – horizontal line of hyperbolic motion, $E > E_{max}$ – particle approaches gravitational radius and does not go off to infinity.

We now turn to relativistic theory, to the exact equation (7.1a). Here the potential curves have a different form (Fig. 8). Because of the term a^2/r^3, the potential curve does not rise upward without limit, as in Newtonian theory, but bends downward, tending to zero at the gravitational radius r = 1. One such curve is shown in Fig 8. The curve has both a minimum and a maximum.

The motion of a trial particle with $E_1 < 1$ in a potential well (Fig. 8) is analogous to that analyzed above. However, unlike in Newtonian theory, the orbit of the particle is not a closed curve (for details see[19]). In the Newtonian problem, the period of the radial oscillations is equal "by accident" to the time necessary for φ to change by 2π, meaning that the curve is closed. In general relativity this is not so. The famous secular shift of the perihelion of Mercury, by 42″ per century, is a manifestation of this singularity.

When $1 < E_2 < E_{max}$ the horizontal E_2 = const on the right side goes to infinity, and on the left side it bears against the turning curve In this case the particle arrives from infinity and goes off to infinity in analogy with the hyperbolic motion of the Newtonian theory.

An important singularity of the potential curve in the Schwarzschild field is the presence of a maximum. *For a frequency with $E_3 > E_{max}$ the horizontal line* $E = E_3$ does not cross the potential curve. Such a particle reaches the sphere of gravitational radius (r 1 in our units) and no longer goes off to infinity. Gravitational capture of the particle takes place. This important singularity of gravitational theory will be discussed in detail later.

We note one more curious circumstance. If a particle has an energy which is only slightly lower than E_{max}, then near the turning point a plot of the right side in (7.1a) approaches zero with an arbitrarily small slope, that is, when r is changed by a small amount dr, the particle has time to describe an arbitrarily large angle φ, meaning that near r_{min} it can make many revolutions before it goes off again to infinity. In this case the orbit near r_{min} is not at all similar to a Newtonian hyperbola. When $E = E_{max}$ the trajectory will wrap itself around the circle $r = r_{E_{max}}$.

b) Circular Orbits

If the point is located at the extremum of the curve $E(r, a_1) = 0$, this means that $dr \equiv 0$ identically and the particle moves along the circle with r = const. It is obvious that the circular motion at the minimum of E is stable in the case of a small perturbation the particle, whose values of E and a_1 experience small changes, will execute a finite motion (Fig. 9) corresponding to $E = E_{min} + \delta E_1$ and to a new turning curve $E = E(r, a_1 + \delta a_1)$. The new trajectory differs little from the previous circle.

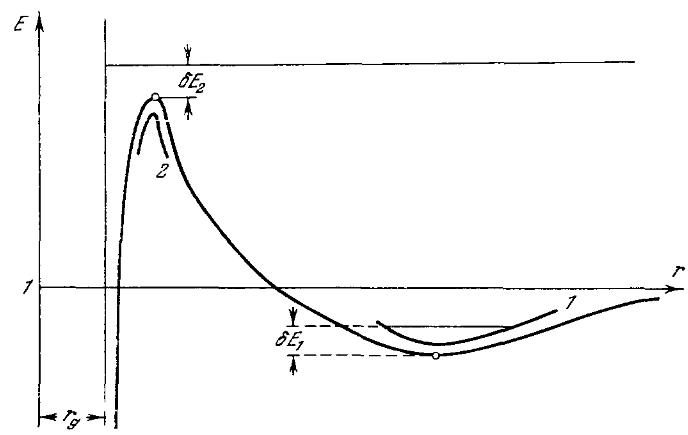

FIG. 9. Motion along a circular orbit is stable, at the minimum of the potential curve and unstable at the maximum. 1 – Potential curve $E - E(r, a_1 + \sigma a_1)$, 2 – potential curve $E - E(r, a_1 + \sigma a_2)$.

Motion along the circle $r_{E_{max}}$ at the maximum of the E curve is unstable, a small disturbance will now cause the particle either to go to infinity or to fall to the gravitational radius.

We have seen that in Newtonian theory the potential curve has a minimum for arbitrary a. Consequently, in Newtonian theory for arbitrary a there exists a stable circular orbit. The smaller a, the closer the orbit is to the center, when $a \to 0$ we have $r \to 0$. In Einstein's theory this is not the case there exists a minimum radius of the circular orbit, on which the motion is stable, and accordingly a minimum energy of circular motion. This circumstance was first pointed out by S. A. Kaplan.[22] To check on the foregoing, it is sufficient to plot E = E(r, a) for different a (Fig. 10).

We see that when $a < \sqrt{3}$ the plots have no extrema. When $a > \sqrt{3}$ each curve has two extrema—a minimum and a maximum. The minima correspond to stable orbits and have r > 3 and accordingly $\sqrt{8/9} < E_{min} < 1$. The coordinates of the maxima, when a increases from $\sqrt{3}$ to ∞, decrease monotonically from r – 3 to

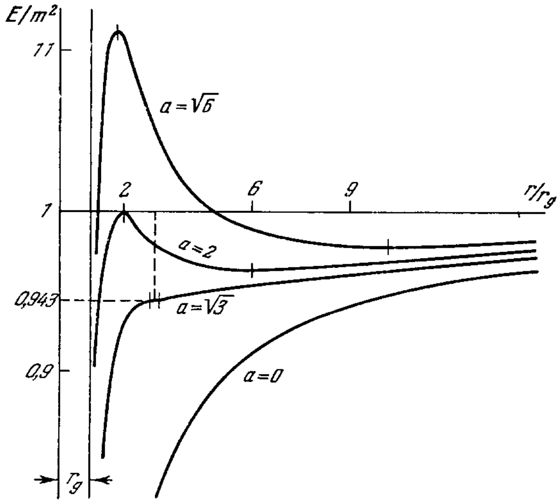

FIG. 10. Potential curves for different momenta a. The numbers near the curves denote the momenta a.

$r = 3/2$, and the energy E_{max} increases from $(8/9)^{1/2} = 0.943$ to infinity.

Thus, the circular orbit closest to the center has $r = 3$. The velocity on it $v_{circ} = c/2$, and the corresponding minimum energy $E_{crit} = 0.943\ mc^2$.

We recall that for a remote observer all the processes in the gravitational field are slowed down by a factor $\sqrt{g_{00}} = (1 - r_g/r)^{1/2}$ (see[1]). This observer will see the motion of the particle on the critical circular orbit with a period

$$T = \frac{12\pi}{(2/3)^{1/2}} \frac{r_g}{c}.$$

If the particle carries a monochromatic emitter with frequency ω_0, then the frequency of the light received by the observer is given by the formula (see[1])

$$\omega = \omega_0 \sqrt{1 - \frac{r_g}{r}} \sqrt{\frac{1 - v/c}{1 + v/c}}.$$

The first factor following ω_0 describes the slowing down of time in the gravitational field, and the second the Doppler effect. For a particle moving on r_{crit}, the plane of the orbit of which passes through the line of sight of the observer, at the instant of motion towards the observer we have $\omega = \sqrt{2}\omega_0$—violet shift, while at the instant of motion away from the observer we have $\omega = \omega_0\sqrt{2/3}$—red shift. For a source at rest having the same $r_{crit} = 3r_g$ we have $\omega = \sqrt{2/3}\omega_0$—red gravitational shift.

Unstable circular orbits are located closer to the gravitation center, in the interval $3/2 < r < 3$. The velocity of motion along the last of these (unstable) with $r = 3/2$ is equal to the velocity of light, $v = c$. This corresponds to infinite energy $E = \infty$. Closer to the gravitational radius (we recall that in our units it corresponds to $r = 1$) there are no circular orbits at all (this was noted already by Einstein).

c) Gravitational Capture of a Nonrelativistic Particle

Let us analyze a case of importance to physical applications, that of the motion of a body which has at infinity a velocity v_∞, negligibly small compared with c. And accordingly $E = 1$. We trace qualitatively the motion of such a body for different values of a. This motion, plotted in units of E and r, is represented by the horizontal line $E = 1$ (Fig. 10). If the angular momentum at infinity is smaller than $a_{crit} = 2$, then the horizontal line $E = 1$ does not cross the turning curve $E = E(r, a)$, meaning that the particle trajectory ends on the Schwarzschild sphere. When $a_{crit} = 2$, the trajectory winds around the circle $r = 2$. If $a > 2$, then it again goes off to infinity. When a differs little from $a_{crit} = 2$, the particle will execute many revolutions near $r = 2$ before going off to infinity. The asymptotic formula for the number of revolutions is[23]

$$N = -\frac{\ln(a-2)}{2^{3/2}\pi}.$$

We now turn to the question of gravitational capture. We have already emphasized that in Newtonian theory a particle arriving from infinity, it it does not strike the surface of the central body, will again go off to infinity—gravitational capture is impossible. In Einstein's theory, as we have already seen, a particle with $a \le 2$ is gravitationally captured; it will not go off to infinity. The dimensionless capture cross section is

$$\sigma_E = 4\pi\left(\frac{c}{v_\infty}\right)^2 \quad (v \ll c). \tag{7.2}$$

Let us compare this capture with "geometrical capture" of a particle by a gravitating sphere of radius R in Newtonian theory, that is, with the case when the particle encounters near the periastron the surface of the sphere. In this case the capture cross section (in the same units) is

$$\sigma_N = R\pi\left(\frac{c}{v_\infty}\right)^2, \tag{7.3}$$

where R is the radius of the sphere.

Comparing (7.2) with (7.3), we see that in the relativistic case the capture occurs effectively in the same manner as in Newtonian theory, but with a central body of radius

$$R = 4r_g.$$

We emphasize also that in Newtonian theory the capture by the sphere is accompanied by impact against its surface. In a Schwarzschild field, the captured body, after executing a finite number of revolutions on a spiral trajectory, reaches the Schwarzschild sphere, and is seen by a remote observer to slow down asymptotically. Such an approach stretches out for the remote observer over an infinite time (as was described in detail in[1] for the case of radial motion). No impact takes place here. We note also that the trajectory approaches the Schwarzschild sphere always perpendicularly, along the radius [see (7.1a) and (7.1b)]. Therefore all the formulas given in[1] for a radially-falling particle will be asymptotically correct near the Schwarzschild sphere also in the case of a non-zero momentum a of the incident particle.*

d) Motion of Ultrarelativistic Particles and of Light Rays

We now consider exactly the opposite case of motion of the particle, which is ultrarelativistic everywhere, even at infinity. Such particles, of course, are always photons and neutrinos.

The equation for a particle moving in a Schwarzschild field with fundamental velocity c is obtained from (7.1) by taking the limit as $v_\infty \to c$, corresponding to $E \to \infty$. Noting that $a/E \to l$ as $E \to \infty$, where l is the impact distance of the trajectory at infinity, we obtain in the limit as $E \to \infty$

$$\left(\frac{d\tilde{l}}{d\tau}\right)^2 = 1 - \frac{l^2}{r^2} + \frac{l^2}{r^3}, \qquad (7.4a)$$

$$\left(\frac{d\varphi}{d\tau}\right)^2 = \frac{l^2}{r^4}\left(1 - \frac{1}{r}\right). \qquad (7.4b)$$

In Newtonian theory there is no term l^2/r^3 in (7.4a), nor the factor in the round brackets in (7.4b), and $d\tilde{l} \approx dr$. This is uniform motion along a straight line.

The presence of a term l^2/r^3 causes the light ray, when passing near the gravitating mass, to be deflected from its linear motion. At large l (meaning also large r_{min}) this deflection is small. For a ray grazing the surface of the sun, it amounts to 1.75″. It was this prediction by Einstein, brilliantly confirmed during the time of the total solar eclipse of 1919, which was one of the first experimental proofs of the correctness of general relativity.

For small values of r, the trajectory of the ray can deviate greatly from a straight line. The turning curve—the dependence of r_{min} on l—is shown in Fig. 11. It is seen from the figure that the ray (or the ultrarelativistic particle) coming from infinity with impact parameter $l \leq 3\sqrt{3}/2 \approx 2.6$ (we recall that all distances are measured in units of r_g), does not meet the turning point and consequently is gravitationally captured. In this case, as in the case of a nonrelativistic particle, the trajectory approaches the Schwarzschild sphere perpendicularly. Here, too, near the limiting sphere, the asymptotic formulas derived in[1] for the case of radial motion are valid. In particular, the time that the ray approaches the Schwarzschild sphere stretches out to infinity for an external observer.

*Of course, we imply at all times that in the relativistic case of motion the central mass has already collapsed and that the particle does not strike its surface.

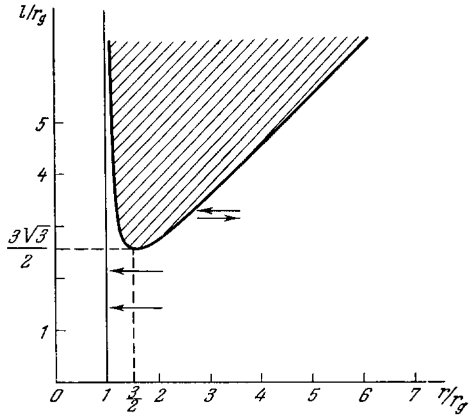

FIG. 11. Turning curve of relativistic particle. l – impact distance at infinity. Particles with $l/r_g < 3\sqrt{3/2}$ are gravitationally captured.

Thus, the cross section for gravitational capture of an ultrarelativistic particle is

$$\sigma = \frac{27}{4}\pi.$$

We note also that a light ray emitted by a source which is at rest at a radius r cannot go off to infinity for all angles of emission. In Fig. 12 the rays which emerge from the inside of the shaded cone will not go up to infinity. The formula for the angle ψ is (Fig. 12)

$$|\tan\psi| = \frac{\sqrt{1-1/r}}{\sqrt{1/r - 1 + 4/27r^2}}. \qquad (7.5)$$

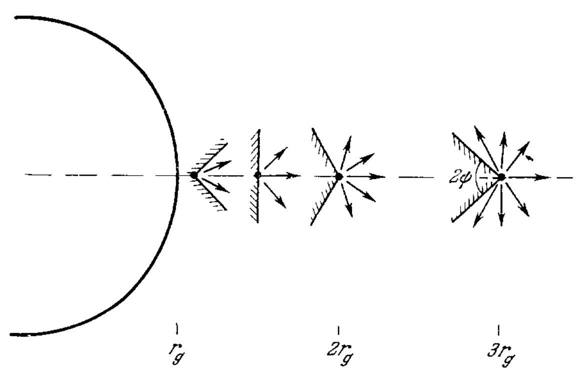

FIG. 12. Gravitational capture of radiation: the rays emerging from each point inside the shaded conical cavity are gravitationally captured.

8. RADIATION OF GRAVITATIONAL WAVES

The predicted existence of gravitational waves[24] is perhaps one of the most important and interesting predictions of general relativity. It is universally known that general relativity is mathematically very complicated, and therefore the problem of gravitational radiation could be solved so far only for a weak field. Moreover, doubts were even cast concerning the physical reality of Einstein's results with respect to the existence of gravitational waves.[25] Although the overwhelming majority of physicists believe that there are no grounds whatever for doubting the reality of gravitational waves, the question would be solved finally by direct experiments in which such waves would be recorded. Unfortunately, this has not yet been done.

The point is that in view of the smallness of the gravitational-interaction constant, the radiation power is usually exceedingly small even for astronomical masses. For example, when a planet or star of mass m moves around a star with M ($M \gg m$) at a distance r ($r \gg r_g$), the formula for the radiation power is

$$\frac{d\mathscr{E}}{dt} = 0.2\,\frac{c^5}{G}\left(\frac{m}{M}\right)^2\left(\frac{r_g}{r}\right)^5. \qquad (8.1)$$

The radiated power of the gravitational waves from the entire solar system is on the order of several hundred watts! This is approximately 10^{24} smaller than the power of the light radiated from the sun ($L_\odot \approx 4 \times 10^{33}$ erg/sec).

However, astronomers know of actually observable double stars, the gravitational radiation power of which is incomparably larger.

Attempts of observing by modern methods gravitational waves* generated under terrestrial conditions are apparently still hopeless. However, there seemingly is a real possibility of observing gravitational waves emitted by short-period double stars close to us. It can be assumed that in the not too far future these waves will be discovered. A detailed review of modern experimental possibilities in this field was presented by V. B. Braginskiĭ.[27]

We now turn to the question of the radiation of gravitational waves from bodies that move in the field of a cooled star. As already noted, the existing theory of gravitational radiation[28,29] is applicable only to processes in a weak gravitational field. However, from dimensionality considerations it is clear that the estimates given later should be of the correct order of magnitude also in the case of motion at distances that are comparable with the gravitational radius of the central body. Let us make also the following remark in this respect. Just as a charge moving uniformly along a circle with velocity $v \approx c$ emits principally higher harmonics, the emission of gravitational waves from a body in a strong gravitational field whose velocity is $v \approx c$ should have the same singularities (see[30] on this subject). However, in the problem in question $v \approx c$ is obtained only near the gravitational radius itself, where the radiation is cut off by the general relativity effects (gravitational red shift, gravitational capture of radiation). When r exceeds r_g to any appreciable degree, these effects do not change the order of magnitude estimates.

*Observation of the wave reduces in principle to a measurement of a difference in the accelerations imparted by the arriving wave to test masses which are separated in space.

An important feature of gravitational radiation is the following. When bodies come closer together under the influence of mutual gravitation, and the distance between them becomes of the order of their gravitational radii, the total amount of radiated energy should be a function of only their masses, G, and c. From dimensionality considerations it follows immediately that the small constant G cannot enter the formula, and that the total amount of radiated energy should be equal in order of magnitude to mc^2, multiplied by a function of the mass ratio m/M. If m is of the same order as M, then we can conclude immediately that the total radiation of gravitational energy is not small compared with mc^2 (m is the mass of the smaller body).[23,31,32] The formulas are given below.

Let us see how the radiation of gravitational waves influences the motion of a mass m. This radiation gives rise to a force acting on the body, that is, it leads to a unique radiant gravitational friction.[23] The friction force is due to the interaction between the mass m and the proper gravitational field, and is therefore proportional to mc^2, unlike the force of interaction with the external gravitational field, which is proportional to m. Thus, a change in the motion of the body resulting from the radiation of gravitational waves can be regarded in the case of $m/M \ll 1$ as a small correction to the motion under the influence of the force of the external field.

In the case of motion of a nonrelativistic particle m, arriving from infinity, the main fraction of the radiated energy is emitted during the time of flight at the vertex of the trajectory, that is, in the periastron. The total amount of radiated energy and the characteristic emission time are equal to, respectively[23]

$$\Delta\mathscr{E} = \frac{c^2m^2}{M}\left(\frac{r_g}{r}\right)^{3.5},$$

$$\Delta t = \frac{r^{3/2}}{(2GM)^{1/2}},$$

where r is the coordinate of the periastron. The energy loss due to radiation causes the body to become gravitationally captured by the mass M at angular-momentum values a which greatly exceed a = 2, when capture of the trial particle in the pure mechanical problem described in Sec. 7 takes place.

Taking into account the radiation, the critical values of a_{cap} and σ_{cap} depend on the parameter $x = c^2m/v_\infty^2 M$, and are determined in the following manner:

$$\text{for } x \gg 10 \quad a_{cap} = (2x)^{1/7}, \quad \sigma_{cap} = \pi\left(\frac{c}{v_\infty}\right)^2 (2x)^{2/7},$$

$$\text{for } x \ll 10 \quad a_{cap} = 2 + e^{-\frac{20}{x}}, \quad \sigma_{cap} = 4\pi\left(\frac{c}{v_\infty}\right)^2 \left(1 + e^{-\frac{20}{x}}\right).$$

For example, when $v_\infty = 10^6$ cm/sec and $m/M \approx 0.1$, we get $x \approx 10^{18}$ and hence $a_{cap} \approx 10$; the cross section σ is 25 times larger than without allowance for radiation.

As a result of the capture, the body moves away from M after going through the periastron not to infinity, but to a distance on the order of $L \approx r_g/2 \times [m/M(r_g/r)^{3.5} - (v_\infty^2/2c^2)]$. When v_∞ is small and $r = 3r_g$ we get $L = 600\ r_g$. During the next passage through the periastron, the body will radiate more energy, etc. The elongation of the orbit will decrease rapidly.*

How does the gravitational radiation affect the circular motion of the particle? This motion is represented by the minima of the curves on Fig. 10. As a result of the radiation, the point representing the motion shifts on the diagram along the minima of the curves. First, at large values of r, this evolution is very slow. The power of the radiation on the circular orbit is determined by formula (8.1). For ordinary double stars the energy lost annually amounts to $\sim 10^{-12}$ of the total energy of the star. In the case of small r, the rate of evolution is much faster. Even for real stars, which we mentioned above, the period of revolution decreases by a factor 10^{-9} annually. The circular motion continues up to the last stable orbit with $r_{crit} = 3r_g$ (see Sec. 7). This is followed by a fall to the Schwarzschild sphere. The energy in the case of motion along the critical circle amounts to 0.943 of the energy for the case of revolution at a large distance. Consequently, the total amount of radiated energy is $\Delta\mathscr{E} = 0.06\ mc^2$ and does not depend on the mass of the central body. The smaller the ratio m/M, the larger the number of revolutions executed by the body before it radiates an energy $\Delta\mathscr{E}$ and reaches r_{crit}.

During one revolution on the critical circle, the radiated energy is $\sim 0.1\ m^2c^2/M$. The body enters on a spiral-like orbit, falling to the Schwarzschild sphere. On this orbit, it goes through another $\sim(M/m)^{1/3}$ revolutions. The energy radiated per revolution is at all times of the same order as when $r = 3r_g$. Thus, after the critical orbit is reached, the body falls down on a sphere of gravitational radius, without adding practically anything to the energy already radiated before that time, if $m/M \ll 1$.

If $m/M \sim 1$, then the number of revolutions executed after reaching critical orbit is of the order of unity, and the radiated energy is of the same order as before reaching this orbit. Although the force of radiant friction is no longer a small correction to the action of the external field, from considerations of dimensionality, symmetry, and correspondence with the formula for $M \gg m$, we can immediately write down a formula for the emitted energy, valid also in the case when $m/M \sim 1$:

$$\Delta\mathscr{E}_{\text{finite}} = \alpha \frac{c^2 mM}{m+M},$$

where α is of the order of 0.06.

We present one more formula for the total amount of radiated energy when masses fall towards each other with zero angular momentum (frontal motion along a straight line):

$$\Delta\mathscr{E}_{\text{fall}} = \beta \frac{c^2 m^2 M^2}{(m+M)^3}. \tag{8.3}$$

Here $\beta \approx 0.02$, that is, of the same order as α. This formula is applicable for arbitrary m/M and is obtained from the same considerations as (8.2).

Thus, as a result of gravitational radiation the system can lose several per cent of the rest energy.

*Radiation produced in motion along an elongated orbit is analyzed by Peters and Mathews.[33]

9. COLLAPSE OF A ROTATING STAR

How will a rotating star collapse? In classical theory, as we have seen in Sec. 3, even a small momentum prevents unlimited contraction. This corresponds to the fact that in Newtonian theory a trial particle with arbitrarily small momentum can move along a stable circular orbit around a gravitating center. Therefore the particle on the equator of a rotating star will not fall to the center, as a result of the centrifugal force, for a definite value of the radius, even without support by the pressure of the material from the inside.

The position is different in relativistic theory. We have seen in Sec. 7 that for a momentum less than $a < \sqrt{3}$ there exists no finite motion and there are no circular orbits. As noted by Hoyle, Fowler and the Burbidges[6] (see also[37]), this should give rise to situations in which the small angular momentum of the star cannot prevent relativistic collapse.* However such a qualitative reasoning still does not provide the exact answer to the question and calls for further refinement.

So far we have considered only a spherical gravitational field produced by a spherical body. We recall that in Einstein's theory, unlike in Newtonian theory, the gravitational field depends not only on the distribution of the masses but also on their motion. In Newton's theory the field of a rotating sphere is perfectly identical to that of a stationary sphere. This is not the case in relativity.

When the field can be regarded everywhere as weak ($\varphi \ll c^2$), the influence of rotation of the body on its gravitational field was already established by Thirring[39] (the derivation can be found in the textbook[28]). It reduces to the fact that in vacuum, near a gravitating rotating body, the trial particle is acted upon by the Coriolis force. We can speak of an analogy with the magnetic field of a rotating charge. The local inertial reference frame rotates relative to the remote stars with angular velocity

$$|\Omega| = \frac{G|\mathbf{K}|}{c^2 r^3}(3\cos^2\theta - 1)^{1/2}, \tag{9.1}$$

where **K** is the total momentum of the body.

This means, in particular, that an inertial com-

*In a recent paper, Wagoner[38] also discusses this question.

pass (a system of gyroscopes), which points when far away from the moving masses towards the same remote stars, will turn near a rotating body at the indicated angular velocity, changing its orientation relative to the remote stars.

The rate of precession of the gyrocompass at the pole of a rotating star ($\theta = 0$) is double that at the equator ($\theta = \pi/2$). At the pole the precession is in the same direction as the rotation of the star, whereas at the equator it is in the opposite direction.

For a homogeneous sphere moving with frequency ω, formula (9.1) can be rewritten in the following convenient form

$$|\Omega| = \frac{r_g}{5r}(3\cos^2\theta - 1)^{1/2}|\omega| \tag{9.2}$$

From this we see immediately that near ordinary stars and planets the precession is negligibly small (although it is in principle measurable!) Thus, on the sun's surface $\Omega_\odot \approx 5 \times 10^{-12}$ sec^{-1}. At the earth's surface $\Omega_\oplus \approx -0.1$ sec of an angle annually on the equator, and 0.2 on the pole (we choose as positive the direction of rotation of the body). On the pole the rotation of the gyroscope proceeds in the same direction as rotation of the body, while on the equator in the opposite direction.

In a strong gravitational field, when r_g/r is no longer small, but the rotation can still be regarded as weak (K/cm $\ll r_g$), the formula for Ω, as follows from[40,129], assumes the following form

$$|\Omega| = \frac{G|\mathbf{K}|}{c^2r^3}\left(1 - \frac{r_g}{r}\right)^{-1/2}\left[4\cos^2\theta + \sin^2\theta\left(1 + \frac{r_g}{2r}\right)^2 \times \left(1 - \frac{r_g}{r}\right)^{-1}\right]^{1/2} \tag{9.3}$$

It is interesting to note that for the neutron star, calculated by Saakyan and Vartanyan[41] with $M = 1.55M_\odot$, radius $R = 9.3$ km, and momentum $K \approx 0.01cMr_g$ we obtain on the surface $\Omega \approx 50$ sec^{-1}.*

Let us now consider a contracting star. It is known that the external field of a contracting spherical nonrotating star is the same as for a stable star having the same mass. This is understandable, since the external field depends in the spherical case only on the mass, and the latter is conserved during the contraction process.

It is seen from (9.3) that the influence of rotation of the star on the external field depends only on the total momentum **K**. The momentum, just like the mass, is a conserved quantity. Therefore, the conclusion, which is rigorously proved in[40], namely that during the course of contraction a weakly rotating star (rotating like a rigid body) maintains its external field constant (accurate to terms linear in K/cM) is not surprising.

Consideration of motion of trial particles and of light rays in such a field leads to the conclusion that the properties of motion are qualitatively the same as in the case of a Schwarzschild field. For an external observer, a particle with an impact parameter smaller than the critical gravitational value is captured and, after executing a finite number of revolutions along a spiral, approaches asymptotically as $t \to \infty$ the singular Schwarzschild surface $g_{00} = 0$.

The same takes place for light rays. The preceding considerations make it possible to understand immediately how a rotating star will collapse. If we first consider a contracting weakly-rotating dust sphere without pressure, then the motion, for example, of a particle on the equator of the sphere is simply the motion of the trial particle in the field of a rotating sphere. We have already said that for an external observer this motion has qualitatively the same properties as in a Schwarzschild field, namely the particle with small initial momentum cannot execute a finite circular motion, but falls to the center and approaches asymptotically the singular surface. Consequently, the collapse of a rotating sphere proceeds for the remote observer qualitatively the same as that of a nonrotating sphere. An account of the pressure does not change the conclusion. Here, too, a characteristic feature is gravitational self-closure and the tendency to the limiting picture of a "cooled" star, as was described in detail in[1]. We emphasize that in the limit as $t \to \infty$ the observer sees the cooled star only as if it were not rotating, but in an external gravitational field the terms due to the momentum K are conserved and are invariably manifest. The apparent contradiction is resolved in the following fashion. It can be stated that the momentum is effectively produced by a mass m, which rotates with a velocity v_φ on the equator of the contracting star $K \approx mv_\varphi R$. During the course of the collapse $R \to r_g$, and the velocity v_φ seen by an external observer, as already noted, tends to zero because of the effect of the slowing down of the time in the gravitational field—the "cooling" of all the processes on the star. But the rate of contraction v_r for a local observer tends to the velocity of light as $R \to r_g$, the mass $m = m_0[1 - (v_r^2/c^2)]^{-1/2}$ tends to infinity, and the product $mv_\varphi R = K$ remains constant.

Thus, in spite of the fact that the gravitational field of the rotating star differs from a Schwarzschild field, its collapse occurs qualitatively in the same manner as for a nonrotating sphere. The star approaches asymptotically the "cooled" state[40], prior to "cool-

*The analogy noted above with the magnetic field reaches much farther. As is well known, the lines in the spectrum of an atom radiating in a magnetic field are split (Zeeman effect). It is shown in[42] that in the spectrum of an atom that radiates near a relativistic rotating star the lines are also split. The magnitude of the frequency splitting is of the order of Ω.

ing'' it has time to execute a finite number of revolutions.*

10. COLLAPSE OF NONSPHERICAL BODY

Rotation of a body makes it oblate, that is, nonspherical. Perturbations of an external field, connected with these deviations from sphericity, are quantities of second order of smallness compared with the perturbations due to the rotation itself, so that they can be neglected. But how can the body whose nonsphericity is connected not with the rotation, but, for example, with an asymmetrical mass distribution, collapse?

For small deviations from sphericity, this problem was solved in[40] We shall not present here the exact proof of the somewhat unexpected result of this investigation, referring those interested to the paper itself. We present only qualitative considerations which enable us to explain the nature of the matter.

We consider first the contraction of a homogeneous spherical dust cloud of radius R. For a remote observer, as already mentioned many times, the picture of the contraction tends to stop as $R \rightarrow r_g$ and at a dust density $\rho_g \sim 2 \times 10^{16} (M_\odot/M)^2$ g/cm^3. An observer situated on the surface of a contracting cloud will reach after a finite proper time $R = r_g$. For him, the contraction does not stop at all, and continues further, even inside the Schwarzschild field in the so-called T-region.[43,44] The density of the material of the sphere at $R = r_g$ and at a large mass is not at all remarkable, for example, for $M \sim 10^8 M_\odot$ we have $\rho_g \approx 2$ g/cm^3. After the surface of the sphere crosses the gravitational radius, the light rays move off from the surface, as can be seen in Fig. 13, to the inside from the Schwarzschild surface and never cross it, never reaching the external observer.

If initially there were in the sphere small disturbances of the density and of the velocity of matter, then these disturbances become more intense upon contraction, as investigated in detail in the papers by E. M. Lifshitz.[45,46] However, the instant when $R = r_g$ is nothing special for the dynamics of the material of the sphere, and the density is still far from infinite. Consequently, if at the beginning of the contraction of the sphere the disturbances are sufficiently small, then by the instant when $R \sim r_g$ they still do not have time to grow sufficiently. Thus, the surface of the sphere in the system of the co-moving observer crosses the sphere $R = r_g$ when the disturbances in the matter and the disturbances of the field itself around the sphere are still small.

*Kerr[148] considered the particular case of the field of a rotating body. He noted that for a pointlike mass, when the angular momentum exceeds Mr_gc, the topology of the surface $g_{00} = 0$ itself is changed. However, such a momentum will stop the contraction even during the nonrelativistic stage

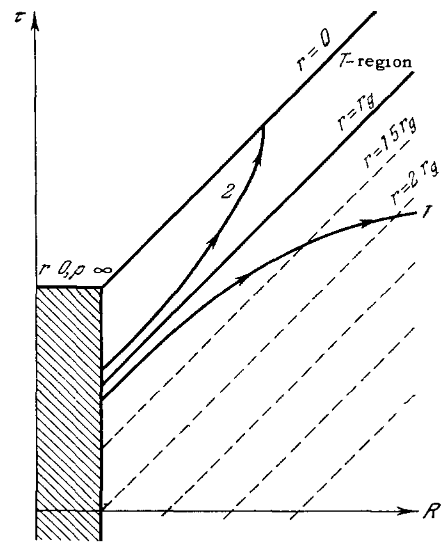

FIG. 13. Collapse of a sphere in a co-moving reference frame. τ – proper time, R – Lagrangian coordinate, r = 0 – true singularity, $r = r_g$ – Schwarzschild surface, dashed lines – world lines of points which are stationary in the Schwarzschild system, with $r = 1.5 r_g$, $2r_g$, etc., 1 and 2 – world lines of light rays. The shaded region is the one with matter. The co-moving reference frame is continued without interruption in the vacuum by the free trial particles.

The disturbances in the sphere then grow, but owing to the gravitational self-closure this is not manifest in any way in the space-time region near the Schwarzschild surface or in the exterior region of the remote observer. A reader who is inclined to believe this without explanation can omit the next paragraph.

The point is that the perturbations of the gravitational field propagate from the sphere with the velocity of light. But it is seen from Fig. 13 that the trajectories of the rays leaving the sphere in the T-region do not approach the Schwarzschild surface. Large perturbations along the characteristics-rays do not arrive in this region. This means that the perturbations in vacuum near the Schwarzschild surface are always small and the properties of this surface remain unchanged. In particular, no radiation or information ever passes through the surface to an external observer. Consequently, even in the presence of perturbations in the sphere, the external observer has access only to a finite interval of the evolution of the sphere. The observer can trace the development of the perturbations in the sphere and in the surrounding field only up to the instant when $R = r_g$.

It is now clear that the external field of the dust should tend as $t \rightarrow \infty$, for an external observer, to a stationary condition, where all $\partial/\partial t \rightarrow 0$. In fact, in his reference frame the perturbations which arise before the surface of the sphere reaches r_g should, like the gravitational waves, be scattered in space and no new perturbations can proceed from under the Schwarzschild sphere. Thus, the limiting field as

$t \to \infty$ of a contracting nonrotating body with small deviations from sphericity remains stationary.

The most curious fact is that if the corrections to the Schwarzschild field for the quadrupole and higher moments do not depend on the time and are small everywhere (up to the gravitational radius), then they vanish identically. This was noted already by Regge and Wheeler.[47] Consequently, in an external field of a contracting body, the corrections for the multipole moments must tend to zero as $t \to \infty$. The quadrupole and higher moments of the external field of a body during the relativistic stage, as shown by calculations[40], attenuate like t^{-1}.

Even if the body as a whole does not rotate, when it contracts in an external field there arise nondiagonal components g_0^α, which describe the rotation of a local inertial reference frame relative to a remote inertial system (see Sec. 9). The occurrence of these deviations is connected with the tangential components of the velocity of the contracting nonsymmetrical body. These "rotational deviations" from spherical symmetry do not attenuate as $t \to \infty$. Although the multipole moments attenuate, the state which has "cooled" for the remote observer, to which the body tends as $t \to \infty$, is far from spherically symmetrical. In spite of the fact that the mass distribution is nonspherical, this nonsphericity is not manifest in the external field.

This is easily understood by recognizing that in the Schwarzschild field any fixed local perturbation, for example a small mass at rest, is manifest far away the weaker, the closer it is located to the gravitational radius. The influence of this perturbation on the field far from the body tends to zero when the perturbation approaches the Schwarzschild surface.

From the last two sections we should draw the following conclusion the collapse of a star with small deviations from sphericity and a star which rotates slowly also leads to the state of a cooled star.

In conclusion let us stop to discuss a question which digresses somewhat from the problems considered above, but nevertheless unavoidably arises. What is the final fate of a collapsing star not for an external observer, but for an observer on its surface? What happens to the star inside the Schwarzschild surface, in the T-region?

We still have no complete answer to this question. We can only state the following. According to the derivations of Lifshitz, Sudakov, and Khalatnikov,[48] the matter cannot contract to infinite density.* As we have seen above, the star cannot expand again, even in an asymmetrical manner, so as to come out from under the Schwarzschild sphere into a region which can be accessible to an external observer. It is possible that the development of the asymmetry leads to a stronger variation of the geometry of space-time in the T region or even to a change in the topology. At any rate, no matter what occurs inside the T-region, this will never be manifest in the space-time region outside the Schwarzschild sphere and the external observer will never learn about it.

*It is stated in[49] that a singularity of the solution is unavoidably obtained in the T-region. This contradicts[48]. The question remains open.

11. DOES RAPID ROTATION INTERFERE WITH THE COLLAPSE OF A STAR?

In the preceding sections we considered stars with small momenta. The criterion of smallness of the momentum of the star is given by the condition

$$K \ll K_{\text{crit}} = kcMr_g,$$

where k is a factor on the order of 0.1, which depends on the distribution of matter in the star. Numerically

$$K_{\text{crit}} \approx 10^{48} \left(\frac{M}{M_\odot}\right)^2 \text{g-cm}^2/\text{sec}. \qquad (11.1)$$

For stars which have a similar or larger momentum, the considerations advanced above concerning the collapse are not applicable. Because of the rotational instability, matter begins to spill out from the equator, and the star cannot be compressed directly to r_g as a whole[36,37] (see below). The sun's angular momentum is $\approx 3 \times 10^{48}$ g-cm²/sec $\approx K_{\text{crit}}$ The bright stars of the main sequence usually have momenta which are appreciably larger than that of the sun. In[37] there is derived for these stars the following semi-empirical relation.

$$K \approx 10^{51} \left(\frac{M}{M_\odot}\right)^{1.75} \text{g-cm}^2/\text{sec} \qquad (11.2)$$

For ordinary stars with $10 < M/M_\odot < 100$ we have $K \gg K_{\text{crit}}$.

We do not know whether the momentum remains constant during the process of evolution of the star. Various conceivable mechanisms for loss of angular momentum were discussed in the literature. For example, when the mass of a rotating star spills out from the equator, the initial and final angular momenta of the star differ by a factor $K_1/K_2 = (M_1/M_2)^{1/k}$, where $k \approx 0.1$. Loss of half the mass reduces the angular momentum by three orders of magnitude. This estimate is valid under the assumption that the star rotates all the time like a rigid body. The tremendous size of atmospheres of red giants contributes to the outflow of matter.

Another possible mechanism whereby an appreciable loss of angular momentum can occur is magnetic braking, suggested by Hoyle.[50] If the star has a sufficiently strong magnetic field and its magnetic force lines are frozen in the surrounding plasma of interstellar matter in the HII zone, the twisting of the force lines leads to a breaking of the rotation. The rate of breaking is determined by the relation

$$\frac{dK}{dt} = -\beta H^2 R^3, \qquad (11.3)$$

where R is the radius of the star, H the field intensity, and β the efficiency of the mechanism. Babcock observed in some stars fields with intensity up to 10^3—10^4 Oe. Nonetheless, the observations apparently indicate that there is no connection between the age of a star near the stage of the main sequence and its momentum. Consequently, magnetic breaking is more likely to be negligible for ordinary stars.

At any rate, it is quite probable that massive stars can terminate their evolution while still retaining a large momentum. Of course, because of the dispersion of the values of the momenta about the mean value, even for example by formula (1.2), there undoubtedly exist stars with small momentum (for an estimate of the number see[37]), whose collapse (if not prevented by other causes) should proceed in the manner described in Secs. 9 and 10. In addition, if the momentum is concentrated principally in an extensive atmosphere of the star, which contains however a small fraction of the mass, then a core with small momentum will collapse independently of the shell. But if the momentum of the core of the star is large, what will then happen to the core? A detailed analysis of this problem is complicated, and there is still no final answer. It turns out, however, that an answer in a very crude form can be obtained without a detailed analysis of the dynamics of the process, by merely listing all the conceivable possibilities.

An important fact in the entire problem is that the total energy of the star, that is, the algebraic sum of the thermal, gravitational, and kinetic energies, is negative; therefore, if the sources of nuclear energy have been completely exhausted or are not in operation (we shall assume this at first), then the entire matter of the star cannot be scattered to infinity as a result of arbitrary processes. In Sec. 3 it was already noted that for a cooled star which has been strongly oblated by rotation there exists no stable configuration. As the contraction proceeds, matter can be ejected from the equator, forming a disc around the star, as described, for example, by Struve.[51] If at the same time a sufficiently effective viscosity is conserved which couples the escaping matter with the star (for example by means of a magnetic field), then this matter will take on the main fraction of the momentum and will allow the central condensed section to collapse.

Another alternative is that as the main mass cools down, contracting without giving up momentum, it acquires a more and more oblate form. Because of the instability of such a form, the star breaks up into two or more parts. If there are many such parts, then the system evolves like a stellar system during the later stages of evolution (see Sec. 15). The evolution is accompanied by collisions between the condensations, by ejection (evaporation) of individual bodies from the system, and leads ultimately (we are still not considering the possibility of release of nuclear energy!) either to the collapse of the entire system (see Sec. 15), or to the formation of two remaining bodies which revolve about a common center of mass.

In the presence of two bodies the motion is stable. It can be accompanied by escape of matter from the shells, but we have emphasized that the entire matter cannot be dissipated. During this stage of revolution of two nearby or almost coalescent masses, an important factor is the radiation of gravitational waves. This radiation results in loss of energy and momentum, and the stars come closer together (see Sec. 8). Dividing the reserve of gravitational energy $\mathscr{E}_{\text{grav}} = -Gm_1m_2/2r$ by the power of the gravitational radiation $d\mathscr{E}/dt$, we obtain the characteristic time of evolution of the system:

$$\tau = \frac{\mathscr{E}_{\text{grav}}}{-d\mathscr{E}/dt} = \frac{5}{8}\frac{r_{g_1}}{c}\left(\frac{r}{r_{g_1}}\right)^4\left[\frac{r_{g_1}}{r_{g_2}(1+r_{g_2}/r_{g_1})}\right]. \qquad (11.4)$$

Here r_{g_1} and r_{g_2} are the gravitational radii of the masses. The factor in the square brackets is equal to 1/2 if the masses are equal; when $r_{g_1} \gg r_{g_2}$ this factor becomes $\approx r_{g_1}/r_{g_2}$. It follows from (11.4) that for equal masses of the order of $M_\odot$, each at an initial distance $r \approx 10^5 r_g$, the time of evolution is of the order of $\sim 10^7$ years.

By losing momentum through gravitational radiation, the masses should coalesce and (if nuclear reactions induced in this process do not interfere), they should collapse; see Chiu[26], page 405.

Thus, the conclusion from the foregoing is as follows. If the matter of the rotating star, terminating its evolution, is for the most part inert with respect to nuclear reactions, that is, if during the preceding stages the nuclear fusion has led to the transformation of all the elements into elements of the iron group, then the final stage of the evolution will be a cooled star even in the presence of rapid rotation.

A similar conclusion is reached by a process in which the nuclear reactions in the matter are possible, but proceed at a much slower rate than the hydrodynamic phenomena, say upon collision of individual fragments of a massive star that is breaking up.

Of course, the possibility of a nuclear explosion (or explosions) depends on the details of the slow evolution of the star (see Sec. 6) and on the concrete processes occurring during the catastrophic stage. All this is still to be calculated. However, if the star were always to end its life by a nuclear catastrophe, then, as we shall show in the next section, the astronomers would know of it from their observations.

12. COMPARISON WITH OBSERVATIONS

Do astronomical observations give any indications of the final fate of massive stars? If the collapse of the star leads to a nuclear explosion, which destroys all or practically of the star, then such an explosion, of course, will be seen from a tremendous distance. It is natural to identify such a nuclear catastrophe with the flares of supernova stars, observed by astronomers. The energy released in such a flare is of the order of 10^{50} erg.

To be sure, it is still unclear at present what fraction of the mass of the star is ejected during flares of supernovas of types I and II (for more details see[17]). If we assume that each star with mass larger than $\sim 1.5\ M_\odot$ terminates its evolution with a supernova flare and thereby avoids relativistic collapse, then the number of supernova flares within a time interval Δt should be equal to the number of massive stars which conclude their evolution within the same interval Δt. Let us estimate the latter number[52,36,37] and let us make a comparison with observations.

We shall assume that the star does not lose appreciable mass during the process of evolution, or at any rate the loss is not large enough to make the mass of the heavy star smaller than critical. The most prolonged period of the life of the star is the stage of the "main sequence" when the hydrogen burns up in the center of the star (see[1]). We recall that the time of evolution of the star during this stage (practically the entire time of the equilibrium evolution), is

$$t = 10^{10}\frac{L_\odot M}{LM_\odot}\ \text{years}. \qquad (12.1a)$$

For bright stars of the main sequence, the approximate relation $L \sim M^3$ is satisfied, so that we can rewrite (12.1a) in the form

$$t = 10^{10}\left(\frac{M_\odot}{M}\right)^2\ \text{years}. \qquad (12.1)$$

From the observations we can determine the number dN of the main-sequence stars per unit volume of space, with masses in the interval M, M + dM. If the lifetime of the star t is smaller than the time of existence of the galaxy ($\sim 10^{10}$ years), then, dividing dN by t, we obtain the frequency, averaged over the last t years, of star production; this coincides with the frequency of "dying" of stars of a given mass. A similar calculation, carried out by Salpeter[53] yields

$$\frac{dN}{t} = 2\cdot 10^{-12}\left(\frac{M}{M_\odot}\right)^{-2.4} d\left(\frac{M}{M_\odot}\right)\ \text{stars/par sec}^3\text{ - year}. \qquad (12.2)$$

Stars with mass larger than critical $M > 1.6M_\odot$ have an evolution time shorter than the age of the galaxy. Multiplying (12.2) by the volume of the galaxy $\sim 10^{13}$ parsec^3 and integrating over a mass $M > 1.6M_\odot$, we obtain the number of stars in the galaxy with mass larger than critical, which terminate their equilibrium evolution every year:

$$F = \int_{1.6}^{\infty} 20\left(\frac{M}{M_\odot}\right)^{-2.4} d\left(\frac{M}{M_\odot}\right) \approx 7\ \text{stars/year}. \qquad (12.3)$$

It follows therefore that if each massive star were to flare up ultimately as a supernova, there should be several stars flaring up in the galaxy every year. This is three orders of magnitude larger than the observed number of flares, given by Zwicky. We see thus that the observations argue against the assumption that at the end of the evolution some nuclear explosion or some other catastrophe prevents the transformation of a massive star into a cooled star.

However, is it possible that the star still manages to get rid of the excess mass, but not by means of a catastrophic explosion, but by stationary escape of matter from the surface or in the form of small discrete ejections of mass over the duration of the equilibrium evolution? The observational data are in this case quite skimpy (a review is given in[17]) and do not make it possible to answer this question definitely.

There certainly exist stars for which intense outflow of mass from the surface is observed. These include the so-called W—R stars (Wolf-Rayet). These, however, are as a rule very massive stars ($M \sim 10M_\odot$), and as the mass decreases the discarding of surface layers, as shown by observations, becomes weaker.[17,54] It is therefore not clear whether the mass drops below the critical limit in the course of time.

Another type of stars, which lose mass intensely, are stars of the Be type, with bright lines in the spectrum. They rotate rapidly. The mass lost by them is estimated to be $(10^{-6}-10^{-10})M_\odot$ annually. These estimates are quite unreliable, and it is not clear whether this mass loss, due to escape of the shell, exceeds the loss due to photon radiation.

Strong escape of matter is observed also for the P Cygni type stars. The supergiant P Cygni itself loses $10^{-5}M_\odot$ annually as a result of mass outflow from the shell (data cited in[17]).

It must be noted that the foregoing stars are characteristic members of stellar associations and are undoubtedly young.[55] It is quite unknown what fraction of the mass is lost by them as a result of escape of matter during the course of the further evolution. In addition, astronomers are observing already non-young stars with mass larger than critical, which during the course of their further evolution apparently will not pass through the stages listed above, namely Be and P Cygni and Wolf-Rayet, and cannot lose mass in the indicated manner. Consequently, data on young stars do not have decisive significance in the solution of the problem, whether a "cooled" star is to be or not to be.

As noted already by Shaĭn,[56] conditions for the ejection of matter from the stars are most favorable during the evolution stage of red giants, when the dimensions of the shell are large and the acceleration due to the gravitational force on the surface of the star is small. Observations show that the escape of

*We recall that the terms "giant" and "supergiant" are used in astronomy to characterize the large luminosity of a star.

matter during this stage is still 1.5 orders of magnitude lower than required for an appreciable decrease in the mass of the star.[57] Finally, for the so-called flaring stars of type UV Cet, which eject matter in discrete fashion (the flares occur on the average once every 1.5 days), the observations yield quite a negligible value[58]: $dM/dt \approx 2 \times 10^{-12} M_\odot$ annually.

At the present time there are no reliable observational data to offer evidence of absolute necessity for the star to get rid of an excess of mass above the critical value in a "quiet" manner, without catastrophes.

On the other hand, the observations show decisively that in clusters of stars the number of white dwarfs is appreciably smaller than expected if the massive stars, which had time to go through an evolution, were to be converted into white dwarfs. Table VI lists the data of[59].

Table VI. Expected and actual number of white dwarfs in clusters

Cluster	Expected number of white dwarfs	Observed number of white dwarfs
χ and h Persei	0	0
Pleiades	2	0
Coma Berenices	9	0
Hyades	23	7
Praesepe	20	2

These data apparently offer evidence against the statement that the star's evolution must unavoidably terminate in the white-dwarf stage.[134] Nonetheless, if our general notions concerning the evolution of the stars are true, the same white dwarfs allow us to conclude that, at any rate under certain conditions, a massive star with $M > 1.2M_\odot$ can get rid of an excess mass and be transformed into a white dwarf. As was already noted long ago in the literature, furthering this hypothesis are the double star systems, one of the components of which is a white dwarf. In two out of three cases, when such systems are investigated in detail (see Table VII), the mass of the component which is not the white dwarf is larger[55] and this component is a star of the principal sequence, i.e., one which did not evolve far. However, the star's evolution is the faster, the larger its mass [see formula (12.1)].

Since both stars were produced simultaneously (the probability of capture is negligibly small)* and the less massive one has already been transformed into a white dwarf, the principal component should terminate its evolution all the more. However, this is not the case. It is concluded therefore that the second component had previously a larger mass, evolved more rapidly, and then lost this mass.

Table VII. Double stars, one of the components of which is a white dwarf

Star	Period of revolution, years	Mass, $M/M_\odot$	Spectrum	Luminosity, $L/L_\odot$
Sirius A		2.28	A IV	38
Sirius B (white dwarf)	49.94	0 98	A 5	0.0026
Procyon A		1.76	F5 IV—V	7.24
Procyon B (white dwarf)	40.65	0.65	—	0.000705
O^2 Eridanus B (white dwarf)	247.92	0 45	B 9	0.0062
O^2 Eridanus C		0 21		0 0125

It is possible that some role can be played here by the duality of the system, although the mutual distance between the components is at present very large. Another explanation (in addition to the loss of mass) can be the fact that the components were produced at different times in the cluster in which they originated (see the footnote).

According to observations of extragalactic astronomy, the ratio of the mass of galaxies to their luminosity, M/L, is different for different types of galaxies (see, for example, the review[61]). It changes from ~100 for elliptic galaxies to 10 for spiral ones, to which our galaxy also belongs, and to ~1 for irregular ones in units of $M_\odot/L_\odot$ = 1.9 erg/g-sec. The large value of the ratio M/L for elliptic galaxies, and also data on their spectrum, offer evidence that they do not contain a noticeable number of young bright stars and that they contain a large quantity of non-luminous or weakly-luminous matter.

In these galaxies there is usually little interstellar diffused matter. In addition, if the galaxies, like our own galaxy, have relatively few white dwarfs (see Table VI), all this taken together is evidence in favor of the presence of difficult-to-observe stars, such as neutron stars and cooled stars, in elliptic galaxies which are in a very advanced stage of evolution.

In the introduction (Sec. 1) we mention that the observation of galactic x-ray sources has suggested that they might be interpreted as neutron stars. Later on, however, Friedman,[62] using the occultation of the source in the Crab nebula by the moon, measured its diameter, and found it to be $\sim 10^{18}$ cm, thus showing that it certainly is not a star. The entire continuous spectrum of electromagnetic radiation from the Crab nebula has the same nature and is attributed to synchrotron radiation of relativistic electrons in a magnetic field. Even before Friedman made his measurements, the hypothesis of the synchrotron-radiation nature of the x rays from the Crab nebula was developed by V. L. Ginzburg and S. I. Syrovat-skiĭ.[63] This might

*To be sure, there are grounds for assuming that the process of formation of stars in clusters stretched over time intervals on the order of the time of evolution of massive stars.[60]

have been thought of as the end of the astronomers' hope that they had finally found neutron stars. However, I. S. Shklovskiĭ[135] called attention to the aggregate of the following facts. As is well known, the Crab nebula is a remnant of a supernova explosion. We obtain from it x rays, visible light, and radio emission. In the locations of other x-ray sources, neither optical nor radio objects have as yet been discovered. This circumstance, along with the indication made by American observers that the x-ray spectrum of the source in the Scorpion constellation is quite steep, makes the hypothesis that their radiation is of synchrotron origin little likely. In Shklovskiĭ's opinion, the radiation is more likely to have a thermal nature and to arise from the surface of neutron stars, as was thought before. According to this hypothesis, the source in the Crab nebula is from the point of view of the nature of its radiation an annoying exception. This discovery could confuse the astronomers with respect to the nature of other sources.

Another feature of x-ray sources, noted by Shklovskiĭ on the basis of preliminary communications obtained at the Second Conference on Relativistic Astrophysics (USA, December 1964), is their apparent concentration in the galactic plane. From this fact it follows that we see them at considerable distances, on the order of the dimensions of the galaxy. In other words, we see in any case an appreciable part of all the x-ray sources which now exist in the galaxy. The astronomers know of ten such sources, and therefore their total number is not larger than 100 or 1000.

Let us compare this number with the expected number of neutron stars, which can be simultaneously seen in the galaxy. The number of neutron stars N_n, produced annually in the galaxy[36,37] (if their formation is not prevented by nuclear explosions or other causes), is determined in analogy with the determination of the expected number of supernova flares (see the beginning of this Section). Only in this case is it necessary to estimate the number of stars which terminate their evolution, in the mass interval $(1.2-1.6)M_{\odot}$. Calculation yields a value of the same order as for supernovas, namely several per year.

It was noted in[1] that a neutron star will radiate for approximately 10^3 years after its formation.* Thus, one can observe simultaneously in the galaxy, as x-ray sources, approximately 10^3 neutron stars. This approximately coincides with the estimate given above and obtained from observations.

Let us add to this hypothesis by Shklovskiĭ the following. The x-ray source in the Crab nebula is the aftereffect of a supernova flare which occurred a thousand years ago. Consequently, after the flare of the supernova, an x-ray source acts at its location for at least 10^3 years, i.e., as long as the neutron star. Since the frequency of the supernova flares is approximately 100 times smaller than the expected frequency of formation of neutron stars, approximately 1% of the visible sources should be extended sources of the Crab type, and the remainder should be neutron stars. We see thus that the observations are more likely to favor the existence of neutron stars than to oppose their existence.

A neutron star cools and stops to radiate 10^3 years after its formation, provided it is not immersed in a sufficiently dense diffuse medium, the falling of which on the surface may support the radiation. The mechanism of non-spherical accretion could also make a cooled star "visible"[84], this will be discussed later (see Sec. 13).

Both neutron and cooled stars constitute clusters of invisible stars. How can the presence of such objects be observed?

Far from a cooled or neutron star, at $r \gg r_g$, the gravitational field is exactly the same as prior to the collapse during the time of the normal evolution. Consequently, in the dynamics of stellar systems the invisible stars manifest themselves in exactly the same manner as ordinary stars. Therefore in principle invisible stars can be observed in the following manner. From the motion of the visible stars one calculates the mass of the system, for example the mass of a globular stellar cluster One then determines the mass of all the visible stars, the gas, and the dust. The difference between the first and second quantities is the mass of the invisible component of the cluster. We note that this includes not only the mass of the invisible stars, but also the mass of other difficult-to-observe forms of matter in the universe, such as neutrinos and gravitational waves. These forms of matter are not concentrated especially in the galaxies and fill the metagalaxy uniformly. Of course, for relatively small systems (stellar clusters, galaxies) the mass of the neutrinos and of the gravitons, even at the maximum density which they can have in the universe[64] is negligibly small compared with the probable mass of the invisible stars.

Let us estimate the fraction of the mass of visible stars of the galaxy that the invisible stars constitute, if their formation is not interferred with by catastrophes.[36,37] For the estimate, obviously, it is necessary to divide the total mass of the stars with $M > 1\ 2M_{\odot}$ [since the total mass of NCS (see below) is larger than the mass of the stars which do not terminate their evolution, and we neglect the mass of the latter], produced during the entire time of existence of the galaxy, by the mass of the stars with $M < 1.2M_{\odot}$. In this estimate, proceeding in analogy with the calculation of (12.3), it becomes necessary to assume that the rate of star formation remained unchanged during the entire lifetime of the galaxy. In

*At the present time calculations are being vigorously carried out of neutrino cooling of a neutron star, the authors of[106] note tentatively that the time of emission may be greatly reduced, but there are still no final conclusions.

addition, it is necessary to take into account the fact that the minimum mass of the star, which can be produced within the time of existence of the galaxy from a diffuse medium, is approximately $0.1M_\odot$. Incidentally, the result is decreased only by a factor of 1/3 if we take the minimum mass to be $0.01M_\odot$. Thus, the sought ratio, with (12.2) taken into account, is

$$\frac{M_{\text{invisible}}}{M_{\text{visible}}}=\int_{1.2}^{\infty} M^{-1.4}\,dM \Big/ \int_{0.1}^{1.2} M^{-1.4}\,dM = 0.6.$$

Thus, the invisible mass can constitute an appreciable fraction of the visible one. Unfortunately, the accuracy of observation is so far insufficient to observe the invisible stars in a similar manner.*

13. ACCRETION OF GAS BY NEUTRON AND COOLED STARS

a) General Remarks

Stars in galaxies are always surrounded by interstellar gas and dust. During certain stages of the evolution, the stars eject matter either in a continuous stream, or via catastrophic explosions (see Sec. 12). Finally, the composition may include also matter which has never been part of any star, and which has arrived in a gaseous stage during the course of expansion of the cosmological primordial, almost homogeneous matter, which had a large density, in accordance with the Friedman solution.

Such are the sources of interstellar matter in the galaxy. The observations show that in the spiral arms of our galaxy the average density of interstellar gas is on the order of 10^{-24} g/cm^3, and in the cores of the galaxy it can be much higher.

The neutron star, after it has cooled off, and also a cooled (collapsing) star, cannot eject matter; obviously, they are capable only of absorbing matter, by drawing into the sphere of their action the surrounding matter. This process is customarily called "accretion."

Accretion is of interest because the mass of the star gradually increases on account of it. In particular, a white dwarf may pass through the Chandrasekhar limit and go over in a jump into the state of a neutron star. In turn, a cold neutron star can reach the "OV" limit (see[1]) and jump over into a higher class, into a collapsing star, which soon cools down. The second aspect of accretion is connected with the change in the velocity of stars relative to the interstellar gas. The velocity of the star changes in this case not only because of the momentum of the particles which adhere to the star, but also principally because of momentum exchange with the particles traveling past the star, i.e., as a result of elastic collisions. The velocity of the star relative to the gas influences in turn the rate of accretion.

A curious new phenomenon was calculated by A. G. Doroshkevich[126] in the relativistic case. The gravitational field of a rotating body, as was shown above (see Sec. 9), differs in general relativity from the field of a body of equal mass at rest. A rotating body predominantly captures particles, the angular momentum of which has a sign opposite to the momentum of the body itself. During the course of the accretion of particles isotropically distributed in space, the momentum of the body decreases as a result of such a selectivity.

However, the main stimulus for the study of accretion lies in the energy released during accretion. Particles incident on the surface of a neutron star give up to (0.2—0.3) c^2 of energy per gram, which is much more than can be obtained from nuclear reactions. Particles incident on a cooled star accelerate to a velocity that approaches c. This raises a natural question: what fraction of their kinetic energy can be radiated to the outside? Connected with the accretion phenomenon is the very possibility of observing neutron and cooled stars; both types of stars will be designated NCS.

In quantitative estimates it is necessary to bear in mind the fact that NCS can be surrounded by a gas of much larger density than the average interstellar density: the very formation of the NCS is connected with catastrophic phenomena, in which part of the mass could be broken off the surface and form a cloud of gas around the star.

The energy released during accretion influences in turn the accretion process itself, because of the interaction between the opposing light flux and the incident matter, i.e., essentially as a result of the light pressure. In the case when the interstellar matter has sufficient density, this phenomenon leads to self-regulation of the process.[65]

From the methodological point of view it is convenient to consider problems of accretion in two limiting cases: either as the motion of individual particles (atoms, molecules, dust particles), or else as the motion of a continuous medium. Obviously, the choice of the approximation depends on the particle mean free path. The cross section of the atom is 10^{-16} cm^2, and at a density of 10^{-1} cm^{-3} this gives a mean free path of 10^{17} cm, much larger than the dimension of the neutron star.

Depending on the conditions, both cases can be realized. A small-scale magnetic field in an interstellar plasma can be regarded on the average as a term in the energy and in the pressure of the gas. The influence of the overall magnetic field of the star on the accretion will be considered in Sec. 14 which is devoted to electromagnetic phenomena.

In the nonrelativistic approximation, the main prob-

*It was noted([84, 136], Guseĭnov — private communication) that an NCS can be observed more easily when such a star is a component of a double system.

lems of accretion were considered in the late Forties. This consideration remains sufficiently well applicable not only to white dwarfs, but also to neutron stars: at a gravitational potential $(0.2-0.3)c^2$, when the radius of the star is 3–4 times larger than the Schwarzschild radius r_g, the corrections for general relativity can reach 20–30%. In the presence of other uncertainties (primarily with regard to the density of the incident gas) such corrections are insignificant. However, in connection with the question of cooling stars, a direct account of general relativity is essential for obtaining qualitatively correct deductions. To make the exposition more cohesive, we recall some universally known facts.

b) Falling of Particles on a Star

We consider particles with mass m, whose velocity far from the star is v_0 and whose density is n_0.

Here the velocity v_0 is regarded as small compared with the parabolic velocity v_p on the surface of the star:

$$v_p = \sqrt{\frac{2GM}{R}}, \quad v_p \gg v_0. \tag{13.1}$$

The velocity of the particle at the surface, which is equal to $(v_p^2 + v_0^2)^{1/2}$, can be replaced by v_p, and the maximum momentum is in this case $I = mv_pR$. Away from the star, where the motion of the particle is not perturbed by the star, the momentum is expressed in terms of the impact parameter

$$I = mv_0 b. \tag{13.2}$$

From this we obtain the maximum value of b_{max}, at which falling on the star takes place: $b_{max} = Rv_p/v_0$. The flux of particles with $b < b_{max}$ is, obviously, equal to $j = nv_0\pi b_{max}^2$. Finally we obtain an expression for the accretion rate (M—mass of the star):

$$\frac{dM}{dt} = mnv_0\pi R^2 v_0^{-2}\cdot\frac{2GM}{R} = 2\pi mn\frac{GMR}{v_0}. \tag{13.3}$$

We substitute $mn = \rho_0$, and introduce $r_g = 2GM/c^2$; we obtain

$$\frac{dM}{dt} = \varrho_0 \pi r_g^2 c \frac{c}{v_0}\frac{R}{r_g}. \tag{13.4}$$

It is curious to compare this nonrelativistic formula with the expressions for the capture of a particle by a cooled star. As shown in Sec. 7, the critical value of the momentum* is $2mcr_g$. Accordingly we obtain

$$\frac{dM}{dt} = c\varrho_0\cdot 4\pi r_g^2\frac{c}{v_0}. \tag{13.4'}$$

Consequently, when $R/r_g < 4$, the use of the nonrelativistic formula is no longer valid. Formula (13.4′) is the lower limit; this formula can be used both for cooled and for neutron stars. In convenient units, we can rewrite it in the form:

$$\frac{d(M/M_\odot)}{d(t/10^{10}\text{ years})} = 10^{-12}\left(\frac{\varrho_0}{10^{-24}\text{ g/cm}^2}\right)\left(\frac{M}{M_\odot}\right)^2\left(\frac{v_0}{1\text{ km/sec}}\right)^{-1}.$$

We note that for particles which are strictly at rest at infinity, there is no sensible answer; in fact, if in the region occupied by the resting particles we place suddenly a mass which attracts the particles, then the particles will start accelerated radial motion, and it is easy to verify that the flux on the surface will increase with time like

$$\frac{dM}{dT} = \frac{8}{3\pi}\varrho_0 GMt,$$

i.e., when $v_0 = 0$ there is no constant stationary flux.

We have considered the flux of particles, all of which move in the same direction and at the same velocity. Obviously, if for a given $|\mathbf{v}_0|$ all the velocity directions are equally probable, then dM/dt will not change; in addition, we can state that the flux is uniformly distributed over the surface of the star, which was not the case for unilateral motion of the particles at infinity. Finally, if the particles have a Maxwellian distribution far from the body, all formulas will contain a factor

$$\overline{\left(\frac{1}{v_0}\right)} = \sqrt{\frac{2}{\pi}}\left(\frac{kT}{m}\right)^{-1/2} = \frac{\sqrt{10/3\pi}}{a_0} \approx \frac{1}{a_0},$$

where a_0 is the speed of sound (the coefficient is given for a monatomic gas).

c) Hydrodynamic Solution

We shall regard the interstellar medium as a gas with a definite adiabatic exponent γ and with a definite state (ρ_0, P_0) at infinity; the velocity of the gas as a whole tends to zero with increasing distance, and the average velocity of the individual molecules at infinity is of the order of $a_0 = \sqrt{\gamma P_0/\rho_0}$.

We separate a narrow cone with solid angle $d\Omega$. We write down the continuity equation—the law of conservation of matter (dS—flux of matter in the cone $d\Omega$):

$$dS = \varrho u r^2 d\Omega = \text{const}, \quad u = \frac{A}{\varrho r^2}, \quad A = \frac{dS}{d\Omega} = \frac{1}{4\pi}\frac{dM}{dt},$$

and the Bernoulli law, which expresses the conservation of energy

$$-\frac{GM}{r} + \frac{u^2}{2} + \frac{\gamma}{\gamma-1}\frac{P}{\varrho} = \text{const} = \frac{\gamma}{\gamma-1}\frac{P_0}{\varrho_0}.$$

The right side of the formula follows from the consideration of the state as $r \to \infty$.

It is convenient to use in lieu of P and ρ, as a variable, the speed of sound a:

$$a = \sqrt{\frac{\gamma P}{\varrho}}, \quad a = a_0\left(\frac{\varrho}{\varrho_0}\right)^{(\gamma-1)/2}$$

*In[23] and in Sec. 8 we consider the influence of gravitational radiation on capture in the two-body problem. Here we do not take gravitational radiation into account, because the mass of the captured particles m is assumed to be negligibly small compared with the mass of the capturing star. The gravitational radiation is proportional to m^2, whereas the interaction energy is proportional to m; the radiation effects contain a factor m/M (see Sec. 8).

In the (a, u) plane, the Bernoulli equation for different values of r gives a family of ellipses

$$\frac{u^2}{2}+\frac{1}{\gamma-1}a^2=\frac{1}{\gamma-1}a_0^2+\frac{bM}{r},$$

and the continuity equation gives a family of hyperbolas of fractional power

$$u-\frac{A}{\varrho_0 r^2}\left(\frac{a_0}{a}\right)^{2/\gamma-1}.$$

These hyperbolas depend also on the parameter A, which is not known beforehand. This means that the analysis of the problem should itself answer the question of what the flux of matter at infinity is equal to under specified conditions at infinity.

It can be shown* that in the presence of two points of intersection of the given pair of curves (continuous and dashed), the lower point of intersection corresponds to a subsonic flow mode, and the upper point to supersonic flow (Fig. 14).

Tangency of a pair of curves must occur on the bisector u = a in the critical-flow mode. If for a given choice of A the curves do not intersect for any value of r, this means that A was chosen to be too large and such a flux is not realized. An analysis of the equations leads to the following picture: there exists a critical value A_c, and when $A > A_c$ there is no solution at all (there is a region of r without intersections).

When $A < A_c$, there are two intersections for all r, the flow at infinity is always subsonic (in the limit $r = \infty$, $a = a_0$, $u = 0$), and therefore it remains subsonic everywhere. Such a flow is possible only in the presence of sufficiently high pressure P_S at the surface of a star. In the limit with $A > 0$, we obtain a static

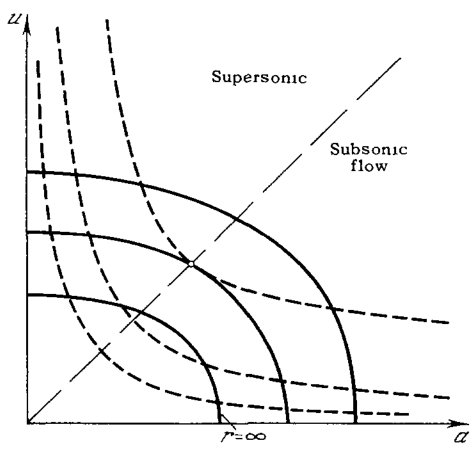

FIG. 14. Hydrodynamic accretion a – velocity of sound, b – velocity of the matter. Solid lines – family of ellipses, given by the Bernoulli equation (the parameter is the distance r). Dashed lines – family of hyperbolas of fractional power, given by the continuity equation.

*The exposition that follows is based on the assumption that $\gamma < 5/3$, which is always realized in the case of a monatomic gas because of ionization, the numbers pertain to $\gamma = 4/3$.

picture, in which the interstellar gas is the continuation of the atmosphere of the star.

When $A = A_c$ the situation arising is qualitatively new. Tangency takes place at a definite $r = r_c$, and for all other r, both $r > r_c$ and $r < r_c$, the curves intersect twice. Then far from the star, in the region $r > r_c$, subsonic flow is realized, at the point $r = r_c$ the velocity is equal to that of the sound, and closer to the star the flux is supersonic. Such a mode is realized when the pressure on the surface of the star is smaller than some value P_c, with P_c close to the static pressure which maintains the interstellar gas.

Thus, if there is no high pressure on the surface of the star, critical influx of gas with maximum possible flow is established. Without dwelling on the simple calculations, we note that at the point of transition through the speed of sound

$$u=a=a_0\sqrt{\frac{2}{5-3\gamma}}$$

the velocity of flow and the velocity of sound are of the order of their initial values (at infinity). This transition is realized at the place where the gravitational potential is of the order a_0^2:

$$\frac{GM}{r_c}=\frac{4}{5-3\gamma}a_0^2.$$

From this we can easily obtain an expression for the flux of matter:

$$\frac{dM}{dt}=4\pi r_c^2 u_c \varrho_c=\delta(\gamma)\frac{G^2M^2}{a_0^3}\varrho_0,\quad \delta=2^{\frac{\gamma+1}{2(\gamma-1)}}\pi(5-3\gamma)^{-\frac{5-3\gamma}{2(\gamma-1)}}.$$

The structure of the expression for the flux differs sharply from the case of independent particles, considered in the preceding section. The expression does not contain the radius of the star. In particular, the expression remains valid also in the case when we are dealing with a cooled star: this is perfectly natural, since the "bottleneck" of the flux is a sphere of radius r_c, where the critical velocity is attained. (Of course, the purely geometrical area of the sphere only decreases when the radius is decreased, but in the calculation of the flux it is necessary to take into account also the change in the velocity and in the density, which depend on the gravitational field.)

The expression for the flux of matter can be written in the form

$$\frac{dM}{dt}=4\pi r_g^2\varrho_0 c\frac{\delta}{16\pi}\left(\frac{c}{a_0}\right)^3.$$

It is convenient to compare this form with (13.4'): for a cooled star the gas flux exceeds the flux of the independent particles by $(c/v_0)^2$, where c is the velocity of light, v_0 the particle velocity; in order of magnitude the speed of sound a_0 does not differ from the velocity of the particles v_0. The gas differs from independent particles in that the atoms of the gas frequently collide with one another; these collisions limit the increase in the tangential velocities of the atoms, but on the other hand they increase the radial velocity which is directed towards the star.

Now a numerical estimate yields

$$\frac{d(M/M_{\odot})}{d\,(t/10^{10}\text{ years})} = \sim 0.1 \left(\frac{M}{M_{\odot}}\right)^2 \frac{\varrho_0}{10^{-24}\text{ g/cm}^3} \left(\frac{a_0}{1\text{ km/sec}}\right)^{-3}.$$

The effect is considerable even under "ordinary" conditions:

$$\frac{M}{M_{\odot}} = 1, \quad \varrho_0 = 10^{-24}\text{ g/cm}^3, \quad a_0 = 1\text{ km/sec}.$$

The condition for the applicability of the hydrodynamic analysis requires that near the critical radius, i.e., at a distance $r_c = GM/a_0^2$, the mean free path be smaller than r_c itself. To this end, at a cross section $\delta \sim 10^{-15}$ cm^2 and $a_0 \sim 1$ km/sec the density required is $\rho_0 \sim 10^{-25}\,(M/M_{\odot})^{-1}$. Consequently, such an analysis is perfectly justified.

d) Motion and Release of Energy in the Gas-dynamic Solution. Effects of the General Theory of Relativity

We have determined above the law of accretion for a gas falling in the gravitational field of a star; we found that the gas flux is determined by a sphere, on which transition takes place from subsonic to supersonic flow. How does the further flow of the gas proceed? It can be shown that in supersonic flow the pressure of the gas does not play an appreciable role. Each element of the gas moves in approximately the same way as a particle freely falling along the radius in the gravitational field. In the Newtonian region in this case the velocity is obviously equal to $u = (2GM/r)^{1/2}$. Knowing the velocity, we can easily obtain from the continuity equation the density of the gas as a function of r:

$$\varrho u r^2 = \varrho_c a_c r_c^2 \approx \varrho_0 a_0 \left(\frac{GM}{a_0^2}\right)^2,$$

$$\varrho \approx \varrho_0 \left(\frac{GM}{r a_0^2}\right)^{3/2} \sim r^{-3/2}.$$

Hence, from the law of adiabatic compression or with account of radiative cooling of the gas, we can obtain both the gas pressure and the speed of sound. For a polytropic index γ we have $a \sim \rho^{(\gamma - 1)/2} \sim r^{-3(\gamma - 1)/4} = r^{-n}$, so that when $\gamma < 5/3$ the index is $n < 1/2$, and the speed of sound actually increases more slowly than the speed of the motion.

In the case of accretion of gas by a neutron star, the gas is stopped at the surface of the star by a shock wave, so that its kinetic energy is transformed into heat and is radiated from that point to the outside.

The pressure attained in the shock wave is ρu^2. Substituting the typical values

$$\varrho_0 = 10^{-24}\text{ g/cm}^3, \quad a_0 = 1\text{ km/sec}, \gamma = 1.4, \quad M = 1.5 M_{\odot},$$

we obtain near the surface at $R = 3R_g = 15$ km the following values: $u = 2 \times 10^{10}$, $\rho = 10^{-9}$, the pressure stopping the gas is 4×10^{11} dyne/cm^2 = 0.4 million atm; such a pressure has long been exceeded in laboratory experiments.[127]

The details of this process, the radiation spectrum, its influence on the motion of the gas are subject to further discussion. It is clear, however, that the total energy released per unit mass incident on the surface of a neutron star is fixed and equal to $c^2 r_g/2R$, i.e., it amounts to 1.5×10^{20} erg/g using as an example $R = 3r_g$. This is 20 times larger than the total nuclear energy of transformation of hydrogen into iron.

However, the minimum value assumed above for the gas density yields too small a flux of matter, $\sim 10^{-11}\,M_{\odot}$/year = 10^{15} g/sec, so that the total energy released is 10^{35} erg/g-sec = 30 $L_{\odot}$; the conditions under which the energy flux is larger will be considered under heading f) below.

The situation is entirely different for accretion of a gas by a cooled star. The motion of the gas reflects the known singularities of particle motion in the Schwarzschild gravitational field, connected with general relativity. The particles reach r_g within a finite time; in the coordinate system connected to the particle, the instant of transition through r_g is in no way singled out. If the particle in question is followed by another and the distance between them was finite somewhere far from r_g, it remains finite also at the instant of crossing r_g. For a particle flux, i.e., for a gas, it follows therefore that the density remains constant in the system moving with the gas. In order of magnitude, the density is

$$\varrho_m = \varrho_0 \frac{r_c^2}{r_g^2} \frac{a_0}{c}.$$

Expressing everything in terms of a_0, we obtain

$$\varrho_m \approx \varrho_0 \left(\frac{c}{a_0}\right)^3.$$

When $a_0 = 1$ km/sec and $\rho_0 = 10^{-24}$ we get $\rho_m \sim 3 \times 10^{-8}$ g/cm^3. From the point of view of an observer at rest and situated near r_g, the closer the point of observation to r_g the closer the velocity of the gas to the velocity of light. The density of the number of particles and the flux of the particles form a four-vector. Following a Lorentz transformation from the frame moving with the gas to a stationary frame, the density of the gas, measured in the stationary frame, increases without limit, on approaching r_g, like $\rho_m[r_g/(r - r_g)]^{1/2}$ (more accurately, this is the factor by which the density of the rest mass increases, or the density defined as the number of particles per unit volume). Finally, from the point of view of a remote observer, the particle approaches r_g only asymptotically after an infinite time.

A remote observer, no matter how long he observes the stationary flux of accretion by a cooled star, assumes that none of the particles that traveled past him no matter how long ago have crossed r_g. Consequently, all are accumulated in the space adjacent to r_g.

In an exact stationary solution for the gas flux, the integral, which gives the total number of particles contained between two spheres $r = R_1$ and $r = R_2$ diverges when the lower limit tends to the gravitational radius

$$dV = 4\pi r^2 \sqrt{g_{11}}\, dr, \quad g_{11} = \left(1 - \frac{r_g}{r}\right)^{-1}$$

$$n = \text{const} \int \sqrt{\frac{r_g}{r - r_g}}\, r^2 \frac{1}{\sqrt{1 - \frac{r_g}{r}}}\, dr$$

The denominator of the integrand contains $r - r_g$, so that the integral diverges logarithmically. From the point of view of a remote observer, it is possible only to approach asymptotically the stationary flux*, and the capacity of the layer adjacent to r_g is infinite, and in the stationary solution the time necessary to fill it is infinite.

It is even more important that in the case of supersonic radial flow of matter, only a very small portion of the energy can be radiated to the outside. The kinetic energy of translational motion is conserved and is not converted into heat. In the case of ordered radial falling, the individual elements of the gas do not collide with one another and no shock waves are produced. In the layer adjacent to the gravitational radius not only matter is accumulated but also energy, or more accurately, matter accumulates possessing a tremendous kinetic energy,† which, however, is not converted into other forms of energy. Only the thermal energy of the gas, which increases during the course of the adiabatic compression, can be radiated. However, this source of energy is quite small compared with the rest energy (ρc^2) and the kinetic energy, since in supersonic flow the density (in the coordinate frame falling together with the gas) remains finite (see the expression given above for ρ_m).

Thus, spherically symmetrical accretion of gas in the gravitational field of a cooled star does not lead to effective release of energy, in exactly the same way (and essentially for the same reason) as spherically symmetrical collapse.

*The time necessary to approach the stationary flux with specified accuracy at a given point of space is the larger, the closer this point is to r_g.

†As was already noted, from the point of view of a local observer, the velocity of the particle approaches c as $r \to r_g$, so that the energy $E \to \infty$. Were a stationary body to be situated near r_g, then when struck by the particle an energy $E \gg m_0c^2$ would be released However, if this energy is diverted to the outside (for example, by means of quanta or neutrinos), the energy decreases on the path to the observer, by virtue of the red shift, and becomes close to m_0c^2 for a particle with rest mass m_0 in the case of a remote observer Actually, the energy is not released because there is no stationary body for the particle to strike.

e) Asymmetrical Accretion in the Field of a Cooled Star

The statements made above lead to the necessity of considering collisions between falling particles. By way of the simplest example we consider inelastic collision of two particles moving along hyperbolic orbits (Fig. 15). The kinetic energy of their relative motion is transformed at the instant of collision into other forms of energy (light), which is radiated in all directions. Part of the light falls on the Schwarzschild radius of the cooled star and is again captured, while the remainder, overcoming the gravitational field, goes to infinity. The colliding particles themselves, losing tangential velocity, fall down. Calculation shows that the maximum energy emitted to the outside amounts to as much as 10–20% of the rest energy (m_0c^2) of the incident particles.

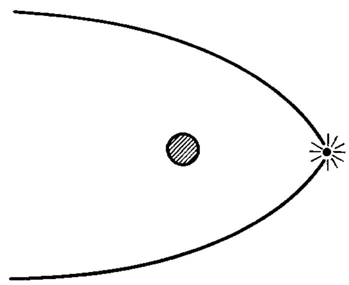

FIG 15 Collision of particles with different momenta in the field of an attracting center.

It would be possible to develop a detailed statistical theory of the motion of the particles in the gravitational field. Let us consider the case of motion without collisions. The particles, which at infinity have an isotropic distribution of velocity directions and have an energy between E and E + dE, constitute a microcanonical ensemble. According to the Liouville theorem, their density is constant everywhere in phase space. But the volume of a layer in phase space can be expressed (using the relations $E = p^2/2m$, $p\,dp = m\,dE$) in the form $4\pi p m\,dE$ Since dE is everywhere the same, the particle density is at each point proportional to the momentum p, i.e.,

$$\rho = \rho_0 \frac{p}{p_0} = \rho_0 \sqrt{1 + \frac{GM_m}{rE}}$$

If we neglect the particle loss by accretion, the velocity distribution of the particles remains isotropic at each point in space. Thus, we construct a first-approximation particle distribution (without account of collisions and accretion), which is used to calculate the number and the energy of the collisions and to obtain the next order approximation. However, here we shall not develop this picture further, and turn directly to the opposite limiting case, to a consideration of gas flow.

Thus, let us imagine that a cooled star is immersed in a gas, which is regarded as a continuous medium. Let us assume further—and in this lies the difference from the preceding section—that at infinity the gas moves relative to the star with velocity u_0. We consider a limiting case $u_0 \gg a_0$: in the presence of ordered motion (velocity u_0), an account of the motion of the gas molecules themselves is no longer significant. Neglecting the speed of sound compared with the velocity u_0, we simultaneously also neglect the pressure. But in the absence of pressure the motion of a continuous medium does not differ essentially from the motion of individual particles: the equations of hydrodynamics constitute merely a different form of the equations of particle mechanics. In the stationary problem, the flow lines are merely particle trajectories. Salpeter[66] gives the following general picture of the motion (Fig. 16): the gas flows from left to right, the flow lines bend in the gravitational field, and the gas velocity increases in accordance with the law of energy conservation, which in this case is named after Bernoulli.

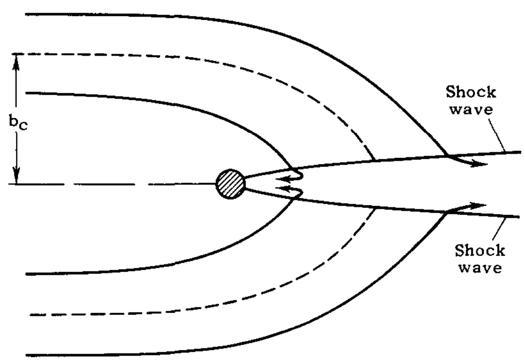

FIG. 16. Hydrodynamic picture of accretion. The particles lose in the shock wave the potential component of the velocity. At an impact distance smaller than b_c, the radial component of the velocity is smaller than parabolic, and the particle ultimately falls on the attracting center.

In the tail of the stream, adjacent to the surface of the star (or the Schwarzschild sphere), is located the elongated surface of a shock wave. The shock wave replaces the collisions of particle pairs on the axis, which we described above. Crossing the front of the shock wave, the gas loses the velocity component perpendicular to the front. The velocity component parallel to the front, i.e., directed along the radius, remains unchanged.

Using Kepler's laws, we can find the critical trajectory (dashed) and the critical impact parameter b_c. When $b > b_c$, the velocity after compression by the shock wave remains larger than parabolic, and the matter flows off to infinity; when $b < b_c$, the matter falls on the star after compression. It turns out that b_c, and with it the accretion rate, depend on the velocity of the gas u_0 approximately in the same manner as they depended on the velocity of sound a_0 in the spherical problem (see above). An important difference from the spherical problem lies in the fact that an effective conversion of kinetic energy into heat and light takes place in the shock wave.

Salpeter presents the following estimates: a body moving with supersonic velocity relative to the gas slows down after a time in which the accretion changes its mass little. Consequently, it is necessary to consider the motion of the body relative to the gas with a velocity of the order of the speed of sound, i.e., of the order of the random velocity of molecules and the gas clouds:

$$\frac{dM}{dt} \sim \alpha \frac{M^2 n}{u^3 \cdot 3 \cdot 10^{11}},$$

where u is the velocity in km/sec, M the mass in $M_\odot$ units, n the density of the interstellar gas (H atoms per cm^3), t is in years, and α is a dimensionless number ($0.1 < \alpha < 1$). The mass increases, becoming infinite in less than 10^{10} years, if $M_0 > (u/25)(0.25/\alpha n) \times 2 \times 10^6$.

Thus, bodies with mass $10^6 M_\odot$, i.e., heavier than ordinary globular clusters, should produce in our galaxy a catastrophic accretion process, accompanied by a large release of energy. The energy release is limited by the light pressure, a fact which will be discussed later. The asymmetrical picture of the motion causes the particles to collide in the gravitational field of the cooled star and can release energy of the same order as in collisions with the surface of a neutron star. Special notice must be taken of the important limitation on the Salpeter model: in this model the energy is released in the form of heat and light, and there is no mechanism capable of transforming the gravitational energy of the falling matter into the kinetic energy of an ejected jet. In fact, according to the Bernoulli theorem, in a stationary stream the law of energy conservation is applicable in each individual elementary jet bounded by the flow lines: in a stream with low influx velocity at infinity, no jets that move away from the body with large velocity can ever arise. A possible way out is to consider a nonstationary situation, wherein a cooled star, approaching the boundary of a gas cloud, causes motion of the gas which in turn, closing in on the other side of the star, yields a cumulative ejection of a jet. However, it is more probable, that to obtain a real picture of the structure of quasars it is essential to take the magnetic fields into account (see Sec. 13).

f) Regulation of Accretion by Light Pressure

We have seen in Sec. 5 above that at a definite value of the light flux, corresponding to $L/M = 3 \times 10^4\ L_\odot/M_\odot$, the light pressure balances the force of gravitation. Consequently, in the case when the release of energy during accretion exceeds this limit, the accretion will stop. These considerations should be applied to accretion on a neutron star. Assuming its mass to be $M_\odot$, we obtain the limiting luminosity $L \sim 10^{38}$ erg/sec. For a luminous surface corresponding to a radius of 10 km, we obtain a flux of 10^{25} erg/sec-cm^2, corresponding to a black body temperature 2×10^7 °K = 1.7 keV. If the flux is half as large, the temperature is 1.3 keV. In either case the temperature is sufficient for emission of x rays with energy up to 10—15 keV (wavelength longer than 1 Å), which can be registered by apparatus placed outside the atmosphere. In order to produce such a flux of light it is necessary to have an accretion rate of 2×10^{17} g/sec $= 3 \times 10^{-9} M_\odot$ annually. Near the surface, the velocity of the falling gas is of the order of 0.5 c and its density is $\rho = 10^{-6}$ g/cm^3, while the effective thickness of the absorbing layer is $\sim\rho R \sim 1$ g/cm^2. A calculation based on Thomson scattering shows that such a layer transmits more than half of the primary x rays from the surface. It is essential here that the matter is at a temperature on the order of 1 keV under the influence of the radiation, and therefore the oxygen, nitrogen, and carbon are practically fully ionized; therefore the absorption of x rays by the matter is much lower compared with their absorption by cold matter.

Finally, it remains to estimate the density of the gas (atomic hydrogen) at infinity, which is at a temperature on the order of 100°K and which ensures the required flux of matter. The speed of sound is on the order of 1 km/sec, the critical radius of the order of 10^{16} cm, and the density is of the order of 10^{-21} g/cm^3.

In this picture, which could explain the x radiation from a neutron star after the exhaustion of the thermal energy of the internal layers, there are still some unanswered questions: is the spherically symmetrical flow of matter opposite the radiation flux stable? At what distance does the ionization of the gas take place? Is it necessary to consider accretion from a gas with constant density at infinity or from a gas which is produced by catastrophic transformation of an ordinary star into a neutron star? In the latter case, the very initial distribution and the initial temperature of the gas are not arbitrary, and are determined by the preceding stage.

The main fact, however, is that the condition for light equilibrium regulates the energy release at precisely the level corresponding to the x radiation; this result depends on the relation between the mass and the radius of the neutron star, and is not sensitive to other circumstances. We note that the velocity of free fall on the surface of a neutron star is quite large. The proton energy is on the order of 200—300 MeV! Therefore the radiation of the shock wave can differ noticeably from the equilibrium radiation (the black-body spectrum) of equal power. This problem still remains to be solved, by methods analogous to the analysis of the structure of strong shock waves.[107]

The situation may become complicated by the magnetic fields frozen into the gas that is to be accreted. As noted correctly by Parker,[143] any strong motion of a plasma produces conditions for the generation of cosmic rays. If the magnetic field were to change only as a result of a similar contraction like $\rho^{2/3}$, then its role would remain small. However, entanglement may intensify the field. Thus, accretion of a magnetized gas may supply energy to relativistic particles that produce synchrotron radiation. This group of questions is still in the initial stage of study.

14. MAGNETIC AND MAGNETOHYDRODYNAMIC PHENOMENA

V. L. Ginzburg[67] was the first to call attention to the fact that the collapse of a star should be accompanied by a strong increase of its magnetic field. Later on this question was dealt with in a large number of papers.[68-71]

In considering the collapse of stars and of ultra-large masses of gas, a distinction must be made between the essentially different topology of the external magnetic fields of these objects (Kardashev[70]). According to observational data and modern theories of the origin of stars, the magnetic field of an ordinary star (say the sun) has a quasi-dipole character. The topology of such a field is shown in Fig. 17a. The magnetic force lines are closed, and in the main they do not extend far from the star. The picture of the external field is different for galaxies and metagalactic formations in general. According to contemporary notions,[72] the magnetic force lines are not closed and go practically to infinity, coupling the body with the surrounding medium and with other objects (Fig. 17b).

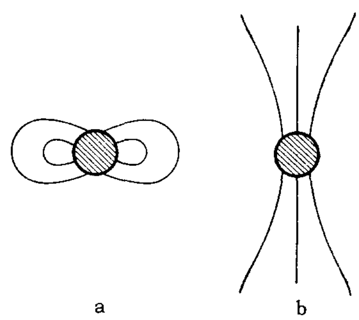

FIG. 17. Topology of the magnetic field. a) Star; b) metagalactic object.

Recently the point of view gaining in prevalence has been the one according to which the field is the result not of self-excitation during the course of formation and evolution of the galaxies, but of contraction of the initial metagalactic field, existing prior to the occurrence of the galaxies. The first to mention this point of view (but disagreeing with it) was Hoyle[73]; the cosmological theory with "originally" existing homogeneous magnetic field is developed in[74,75].

The magnetic processes occurring during collapse, with account of the phenomena occurring in the surrounding shell, are very complicated and have probably been studied even less than the effects of rotation. We therefore confine ourselves here only to the most general examination of the problem. We begin with the collapse of ordinary stars. It is known from the observations that the fields on the surfaces of the stars can reach intensities of $1-10^4$ G. In all cases, the magnetic energy is much smaller than the gravitational energy of the star. The conductivity of stellar matter is quite large and for stars of the type of the sun it amounts to $\sigma_\odot = 10^{16}$ sec^{-1}. Therefore the decay time of the field[14] is

$$t_\odot \approx \frac{4\pi\sigma_\odot R_\odot^2}{c^2} \approx 7\cdot 10^{17} \text{ sec} \approx 2\cdot 10^{10} \text{ years},$$

which is much larger than the age of the sun ($\sim 5 \times 10^9$ years). During the course of contraction of a star of any mass, this time is always much larger than the characteristic contraction time.[67] If the star is transformed into a neutron star, then its conductivity increases by many times and becomes approximately four orders of magnitude larger than the conductivity of copper under ordinary conditions.[76] In this case the time of attenuation of the field for $R \approx 10^6$ cm is of the order of a million years.

Thus, we can always assume that the field is "frozen in" into the matter of the star. Under these conditions, upon contraction $H \sim R^{-2}$ and the magnetic energy is $E_{mag} \sim H^2R^2 \sim R^{-1}$, i.e., it varies like the gravitational energy upon contraction. But in ordinary stars, as stated above, $E_{mag} \ll E_{grav}$; consequently, the dynamics of the collapse of the star is not influenced at all by the magnetic field.

How does the magnetic field vary during the time of relativistic collapse, when the star becomes cooled? This question was investigated by V. L. Ginzburg and L. M. Ozernoĭ.[67,69]

It was emphasized above that the field is frozen into the matter of the star and upon contraction to dimensions $\sim r_g$ it should reach the colossal magnitude $\sim 10^{10}$ G for an observer co-moving with the matter. A different field will be seen as $R \to r_g$ by a non-moving observer. To find this field, the authors of[69] consider first the static problem.

We mentally decrease the dimensions of the gravitating magnetized sphere and investigate its external dipole magnetic field. The dipole moment of a sphere d is proportional to R in classical theory: $d = d_0R/R_0$, where d_0 and R_0 are the initial dipole moment and the radius, respectively. The moment d tends to zero if $R \to 0$. Ginzburg has shown[67] that in relativistic theory, as $R \to r_g$, the variation of d is given by

$$d = \frac{d_0 r_g}{R_0 \cdot 3 \ln \frac{r_g}{R - r_g}}.$$

Thus, $d \to 0$ when $R \to r_g$.

We now find the variation of the magnetic moment of a collapsing star with time. On the surface, the magnetic field is calculated in the same manner as in the stationary case. Knowing the dependence of R of the star on the time (see[1], Sec. 15), we obtain ultimately as $t \to \infty$

$$d = \frac{d_0 r_g^2}{3R_0 ct}.$$

Thus, the magnetic moment for an external observer attenuates in accordance with a power law.

How can we explain the difference in the behavior of the angular momentum and the magnetic moment in the case of collapse? The former, as shown in Sec. 9, remains constant, while the latter attenuates. After all, both are produced effectively by the rotational motion: the mechanical momentum by the rotation of the mass, and the magnetic moment by the circular motion of the current. The difference lies in the following. As $R \to r_g$ the local collapse velocity $v_r \to c$. Owing to the attenuation of all the processes, the rotational velocity v_φ tends to zero, but the effective mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \to \infty,$$

and the momentum remain unchanged, $K = mv_\varphi R = \text{const}$. Unlike the mass, the charge $\tilde{e}$, which produces the current $I = \tilde{e}v_\varphi$, does not change; therefore as $v_\varphi \to 0$ the current attenuates: $I \to 0$. The attenuation of the current does indeed lead to the attenuation of the external magnetic field of the collapsing star for a Schwarzschild observer.

The change in the magnetic field upon contraction causes the appearance of a vortical electric field. In the near (non-wave) zone, this can lead to the occurrence of a current-carrying shell in the plasma surrounding the star or (and) to the occurrence of magnetohydrodynamic waves.[69] These processes have as yet not been investigated thoroughly, and we confine ourselves only to some remarks concerning the far-wave zone.[68] The characteristic time and the scales of the phenomenon are respectively r_g/c and r_g. Therefore the wave zone begins with $R > r_g$. Let the external magnetic field of the star contain a dipole magnetic moment $d = \Phi R$, where $\Phi = \text{const}$. We estimate the radiation of the external magnetic field. Since for $R \sim r_g$ the rate of contraction is on the order of c, the radiated energy will be of the order of the energy of the magnetic field, since there is no smallness parameter for the amount of radiated energy. We shall carry out a more accurate calculation. During the process of collapse, the matter falls almost freely; therefore

$$\ddot{R}=-\frac{GM}{R^2},\ \dot{R}=\sqrt{\frac{2GM}{R}}. \qquad (14.1)$$

The total flux of the radiated energy is

$$I=\frac{2}{3c^3}\ddot{d}^2. \qquad (14.2)$$

Substituting (14.1) in (14.2), we obtain

$$I=\frac{\Phi^2 c}{6r_g^2}\left(\frac{r_g}{R}\right)^4, \qquad (14.3)$$

and the total amount of radiated energy, upon contraction to dimensions R, is

$$E=\frac{\Phi^2}{15r_g}\left(\frac{r_g}{R}\right)^{2.5}. \qquad (14.4)$$

If $\Phi \approx 3\times 10^{21}$, as occurs on the sun, and $r_g \approx 3 \times 10^5$ cm, then

$$I \approx 5\cdot 10^{41}\left(\frac{r_g}{R}\right)^{4.5} \text{ erg/sec}, \qquad (14.5)$$

$$E \approx 3\cdot 10^{36}\left(\frac{r_g}{R}\right)^{2.5} \text{ erg}. \qquad (14.6)$$

The radiation is in the form of a single pulse with duration $\approx r_g/c$.* The calculation was made for the nonrelativistic theory, and the formulas cease to apply near the Schwarzschild sphere, but they give the correct order of magnitude of the estimated quantities.

The total amount of radiated energy is small in this case. Hoyle, Narlikar, and Wheeler[71] have proposed that intense electromagnetic radiation can continue for a long time if the rotating star with the magnetic field, on cooling and contracting, reaches a quasi-equilibrium state of the type, say, of a flat disc. Such a configuration with nonspherical mass distribution, according to the hypothesis of Hoyle and others, should oscillate dynamically about an equilibrium position. We have already noted, however, in Sec. 3 that such a configuration is unstable and breaks up into clusters. Another imaginable variant of an oscillating system may be, for example, the rotation of a body such as a double star which is almost merged. According to the suggestion of the authors of[71], the dimensions of this system for $M \sim 5 M_\odot$ are of the order of 10^6 cm, i.e., $\approx r_g$, and the characteristic velocity is obviously somewhat smaller than c ($\sim c/3$). The dimensions of the star, compared with the initial ones, have decreased by five orders of magnitude; consequently, the field has increased by ten orders and reaches $\sim 10^{10}$ G, while the alternating component of the field reaches, say, 10^9 G. Therefore the flux of electromagnetic radiation is $\approx R^2H^2c \approx 3\times 10^{40}$ erg/sec. The authors of[71] believe that the energy flux can last $\sim 10^3$ years $= 3\times 10^{10}$ sec, and find for the total radiated energy a value 10^{51} erg.

However, we cannot assume so long an existence of an oscillating system. The point is that along with the electromagnetic waves the system will radiate also gravitational waves both in the case of an isotropic contraction of the body upon oscillation, as well as in the case of rotation of the type of a double star. For an order of magnitude estimate of the gravitational radiation we can use formula (11.4) in which we put $r_{g_1} \approx r_{g_2}$. At the indicated parameters of the system, we find for the duration of the existence of the system

$$\tau=\frac{5}{16}\frac{r_g}{c}\left(\frac{r}{r_g}\right)^4 \approx 10^{-5} \text{ sec}. \qquad (14.7)$$

Obviously, no sensible account of the small parameter, due to some particular distribution of the masses in the system, will change this value significantly. In order for the system to exist $\sim 10^3$ years, it is necessary, as follows from (14.7), to have $r/r_g \approx 10^4$. But for such a choice of system dimensions, the intensity of the magnetic field will not exceed 10 G and the electromagnetic *radiation is negligible*. Thus, *the total* amount of radiated electromagnetic energy is small in order of magnitude, and is determined by (14.6). We shall show later that upon contraction of masses of the order of $(10^5-10^8)\ M_\odot$, the radiated energy increases sharply. So far we have tacitly assumed that the plasma surrounding the star does not interfere with the occurrence of radiation. The radiation frequencies $\omega \sim c/r_g$ are low, particularly for large masses. Even at negligible plasma density, the proper oscillation frequency $\omega_0 = [4\pi e^2 N_e/m^2]^{1/2} \approx 2\times 10^{18}\sqrt{N_e}$ sec^{-1} is much larger than ω, which it might appear should lead to the conclusion that no radiation should occur at all.

However, in order for radiation not to occur, it is necessary, in addition to the foregoing condition, also that the maximum possible current $I_{max} = N_e ec$ (N_e—electron concentration, e—electron charge), occurring in the plasma when the magnetic field varies in the non-wave zone, be in a position to compensate this variation. Let us find the critical value of the density of the surrounding plasma, at which the radiation no longer arises.[68] From Maxwell's equations we obtain

$$|\operatorname{rot} \mathbf{H}|=\frac{4\pi I_{max}}{c}=4\pi N_e e. \qquad (14.8)$$

Let us make some order of magnitude estimates. We consider a characteristic instant of contraction when $(R-r_g)\sim r_g$. Recognizing that $|\text{curl } \mathbf{H}| \sim H/r_g$ and $H \sim \Phi/r_g^2$, we obtain from (14.8)

$$\Phi_{\text{crit}}=4\pi N_e r_g^3 e.$$

Consequently, if the inequality

$$N_e < \frac{\Phi}{4\pi e r_g^3} \qquad (14.9)$$

is satisfied, then a wave zone is produced (starting with distances $L > r_g$) even when $c/r_g \ll \omega_0$.

Substituting in (14.9) $\Phi \approx 3\times 10^{21}$ and $r_{g_\odot} = 3 \times 10^5$ cm, we get

$$N_e < 10^{15} \text{ cm}^{-3}. \qquad (14.10)$$

For comparison we indicate that the electron con-

*The fall-off of the burst is due to the relativistic effect of the cooling of all the processes as $R \to r_g$ (see above).

centration in the solar corona near the sun's surface is $\sim 10^8$ cm^{-3}.

Thus, if (14.10) is satisfied, the radiation moves out of the star in the form of a single pulse, becoming absorbed by the plasma far from the star in the wave zone. We recall once more that the total radiation energy is relatively small in the case of ordinary stars.

Kardashev[70] called attention to the fact that under certain conditions the rotational energy of the star can be "pumped over" for a long time into the energy of the magnetic field. The star, having contracted to the dimensions of a white dwarf or to neutron density, should in this case still not rotate fast enough to permit a strong oblation of the main mass to cause instability in the form of the formation of individual clusters or formation of a double star.* The process of "pumping" can be realized, for example, in the following fashion. In the case of explosion of a supernova of the first type, the external layers of the star which form around it a nebula of the Crab type are ejected into space.† This shell is coupled with the star by magnetic flux lines. The rotation of the contracting star is accelerated and twists the field lines, giving up rotational energy to the magnetic field and to the shell, and transferring also momentum to the shell (the Hoyle mechanism, see Sec. 11). Let us make, following[70], some rough estimates. Let the initial parameters of the star be

$$R_0 \approx 10^{11}\ \text{cm},\ M = M_\odot = 2\cdot 10^{33}\ \text{g},\ v_0 \approx 10^6\ \text{cm/sec}$$

and $H_0 = 1$ G. Owing to the condition for the freezing-in, the flux of the field through the surface of the star remains unchanged. Therefore, if the intensity of the external field were not to increase by the twisting, the intensity H in a shell of average radius l would be $H = H_0R_0^2/l^2$. If the star executed n revolutions after the separation of the shell, then

$$H = \left(\frac{H_0R_0^2}{l^2}\right)n \tag{14.11}$$

The dimension of the star at the time of the onset of the escape of matter from the equator is, as a result of the action of the centrifugal force, $R \approx 10^8$ cm $\approx 3\times 10^2\ r_g$. Consequently, one revolution of the star lasts $\tau = (2\pi R/v_0)(R/R_0) \approx 0.6$ sec. Let us apply these estimates to a hypothetical star in the Crab nebula, produced by a supernova explosion $t = 10^3$ years ago. In this case the number of revolutions of the star is $n \approx t/\tau - 5\times 10^{10}$. The dimension of the nebula is $l \approx 2.5\times 10^{18}$ cm, and therefore $H = (H_0R_0^2/l^2)n = 10^{-4}$ G, in good agreement with the estimates obtained for the field from the observed synchrotron radiation of the nebula electrons.

*For this it is necessary that the rotational energy be noticeably smaller than the gravitational energy.

†The shell can also be produced in a different way, for example by stationary outflow of matter from the surface, etc.

The growth of the field will continue until the magnetic energy becomes of the order of the rotational energy, or until the magnetic coupling between the star and the shell is destroyed. The latter can occur either when the process becomes unstable, or else as a result of attenuation of the field. This time is estimated to be[70] several thousand years.

We now proceed to collapse of larger masses of gas. The gravitational contraction of such clouds with magnetic fields, has apparently led in the past to the formation of different forms of galaxies and radio galaxies, a fact especially emphasized recently in the papers of Piddington[77] and S. B. Pikel'ner.[72] If the spherical symmetry of the gas cloud is for some reason sufficiently good (for example in the cores of galaxies), then the contraction will continue until r_g is reached.

Unlike in ordinary stars, the topology of the field is different here, as indicated in the beginning of the section (Fig. 17). The force lines couple the contracting cloud with the surrounding medium. The magnetic energy of the cloud is probably of the order of the rotational energy even at the onset of the contraction, and the contraction is continuously accompanied by a loss of angular momentum. The contraction gives rise to the appearance of "necks" in the magnetic force lines.* The processes which arise in this case are likewise not very well known, and we present only the roughest estimates. In interstellar clouds there exist magnetic fields of intensity 10^{-5} G. Assume that the initial cloud with $M = 10^8\ M_\odot$ has $n \sim 10^4$ field-homogeneity cells with $H_1 = 10^{-5}$ G. Then the external regular field of the cloud is $H_0 \approx H_1n^{-1/2} \approx 10^{-7}$ G. Upon contraction to r_g, the density of the cloud changes by ~ 24 orders of magnitude (from $\sim 10^{-24}$ g/cm^3 to 1 g/cm^3), the field increases by 16 orders of magnitudes and reaches $H \approx 10^9$ G The increase in the field energy is accompanied by an increase in the energy of the particles frozen into the field.

Will field radiation arise during the contraction process? From the criterion for the occurrence of radiation (14.10), using the already given initial value $H_0 \approx 10^{-7}$ G on the surface of the cloud, we obtain

$$N_e < 10^4\left(\frac{10^8}{M/M_\odot}\right)^{7/3}\ \text{cm}^{-3}. \tag{14.12}$$

If in addition we take into account the fact that more likely $H_0 \sim n^{1/2} \sim M^{1/2}$, the exponent of the bracket in (14.12) rises to 3. Thus, the plasma concentration that is critical for the occurrence of radiation varies from 10^4 cm^{-3} for $M \approx 10^8\ M_\odot$ to $\sim 10^{11}$–10^{13} cm^{-3} for $M \approx 10^5\ M_\odot$.

If the plasma density satisfies the criterion (14.12), then radiation is produced. Let us estimate by means

*See the papers of Mestel[78] on this subject.

of formulas (14.5) and (14.6) the total flux and the amount of radiated energy for $M \approx 10^8 M_\odot$[68]

$$I \approx 10^{54} \left(\frac{r_g}{R}\right)^4 \text{ erg/sec},$$
$$E \approx 5 \cdot 10^{56} \left(\frac{r_g}{R}\right)^{2.5} \text{ erg}.$$

Assuming that $H_0 \sim M^{-1/2}$, we obtain $I \sim M^{-5/3}$, $E \sim M^{-2/3}$. When $H_0 = \text{const}$ we have $I \sim M^{-2/3}$, $E \sim M^{1/3}$. Thus, the electromagnetic radiation of the external field in the collapse of ultralarge masses, unlike in the collapse of ordinary stars, has a very imposing appearance and perhaps plays a noticeable role in the energetics of quasars.

15. QUASARS

The discovery of quasars was the main factor which has aroused in recent years the tremendous interest in problems of relativistic astrophysics. The history of the discovery of these objects, their study, the theories of their structure has been the subject of many original and review papers. Bearing this in mind, we shall not dwell in detail here on a description of all the works and on a listing of all the proposed quasar theories, all the more since even a simple listing of these theories would occupy more than a page. Those interested are referred to[26], to the review by Greenstein[79] which has been translated into Russian and supplemented by L. M. Ozernoĭ, to the article by Sandage[80], and also to a collection of translations.[81] Among the latest papers we point to[82,83], where a bibliography can be found.

There is no generally accepted or even merely sufficiently convincing theory of quasars at present. We shall recall briefly the experimental data on quasars and dwell on some of the attempts to explain their nature, which, in the author's opinion, are the most promising.

Quasars are observed in the optical band as point-like objects of low brightness (the brightest is 3C 273, $\sim 13^m$), sometimes with a diffuse aureole. Identified with these optical objects are radio sources of small angular dimension (a second of arc or less). The number of discovered quasars now exceeds 20.

The spectra of the quasars contain forbidden lines of highly ionized elements. The lines in the spectra show a strong red shift: in the case of 3C 273 we have $z = \Delta\lambda/\lambda = 0.158$, and in other quasars z reaches 0.4—0.8. At the May 1965 conference in Denver, Colorado, it was reported that for the object CTA 102 $\Delta\lambda/\lambda \approx 1.03$ and for C9 the shift is[144] $\Delta\lambda/\lambda = 2$ (!). This shift cannot be gravitational, if for no other reason than that the maximum gravitational shift on the surface of a stationary star is $z \approx 0.4$ (see[1], Sec. 8). Bondi[144] has shown in very general form that the gravitational red shift emitted by the surface of a static body does not exceed $\Delta\lambda/\lambda = 0.6$. Consequently, the shift is due to the fact that the objects are moving away with tremendous velocities. Objects inside the galaxy cannot just move away from us and still display a statistical relation between the visible magnitude and the red shift. It is clear that the shift is due to the cosmological expansion of the metagalaxy and that the quasars are located a tremendous distance away from us, a distance which can be determined from the red shift. Knowing the distance and the observed flux on earth, we can determine the total energy flux from the quasar. It turns out to be of the order of 10^{45}—10^{46} erg/sec, which is one or two orders of magnitude larger than the energy flux from the brightest galaxy. Taking into account the recently measured infrared radiation from 3C 273, its luminosity reaches $L \approx 3 \times 10^{47}$ erg/sec.

Having made the assumption of the approximate stationarity of the plasma of the object under the action of the radiation gravitational forces in the region where the continuous spectrum is formed,* and using the considerations advanced with respect to optical equilibrium (see Sec. 5b), we obtain an estimate of the lower limit of the quasar mass $M = (10^7$—$10^8)M_\odot$. For a flux $L \approx 3 \times 10^{47}$ erg/sec from 3C 273 we have accordingly $M = 3 \times 10^9 M_\odot$[65]. An analysis of the physical conditions in the shell of a quasar was given by I. S. Shklovskiĭ[92] and by Greenstein and Schmidt.[133]

The most amazing is the variability of the optical brightness of quasars, discovered simultaneously by Soviet[93] and American[94] astronomers. It has been reliably established that the brightness of the quasars investigated in this respect varies more or less periodically during several years, and sometimes abrupt changes in brightness are observed within a week. This means that the linear dimensions of the radiating surface do not exceed more than one light week (!).†

Finally, an analysis of the continuous spectrum (and also the presence of radio emission) argue in favor of the hypothesis that the radiation is more likely to have a synchrotron nature.

To understand the nature of quasars it is necessary first to find a source of energy with tremendous power. Astronomers hope that the same or an analogous source provides the total reserve of energy in the powerful radio galaxies of the type A-Cygni, on the order of 10^{60} erg, and also causes the explosions of the cores of several galaxies, this is especially pointed out by V. A. Ambartsumyan.[85]

*One must not confuse this region, which can provisionally be called the photosphere of the quasar with dimensions $\approx 10^{16}$ cm, with the outer shell, in which the emission lines are produced and which has apparently a dimension of $\sim 10^{19}$ cm and is expanding.[92]

†Recent observations by Soviet radio astronomers indicate that the quasar CTA 102 is variable in the radio band ($\lambda \approx 30$ cm) with a period of ~ 100 days.[152]

At the beginning of the article we indicated that a quasar cannot be a supermassive star with a nuclear energy source, as was initially suggested by Hoyle and Fowler.[2] Equally unproductive were other attempts to use nuclear power (see[86]). It was already noted above that the maximum possible yield of nuclear energy is $\approx 8 \times 10^{-3}$ mc^2, whereas the yield of gravitational energy can in principle be mc^2. In 1961, V. L. Ginzburg[87] proposed as a source of the energy of radio galaxies the gravitational contraction of the gas. It is necessary only to find a suitable mechanism for the conversion of the kinetic energy of the contracting mass into other forms of energy. In the preceding sections we considered several mechanisms of this kind. In the case of a spherical or nearly spherical contraction, the energy yield, as we have already shown, cannot be appreciable, owing to the self-closing effect. The mechanism of gravitational radiation,[5,88,23] cannot be sufficient, because the gravitational waves practically do not interact with matter; therefore gravitational waves can be regarded only as a channel for the diversion of energy from the system. Sufficient energy can be released upon accretion of matter. However, it is probable that an appreciable role in the quasar phenomenon is played by magnetic and magnetohydrodynamic processes. We shall consider the appropriate hypothesis below.

Finally, there is still another possible point of view: quasars may be galaxies in the course of their birth (see, for example, the paper by Field[89]), or, conversely, in the process of their death—in the stage of collapse.[90]

The evolution of a stellar system is accompanied by "evaporation" of stars, which carry energy away from the system, radiation of gravitational waves, formation of double systems, and direct collisions of stars. All this leads to a loss of kinetic energy of the stars of the system, to a gradual condensation of the system, which leads to a critical state and then to collapse within a time on the order of one revolution of the peripheral stars.[90] A common shortcoming of these hypotheses is that they do not explain directly the magnetic fields and the relativistic particles in the quasars, or the periodic oscillations in the light yield.

In connection with these considerations, it is interesting to note the following: a preliminary analysis of the spatial arrangement of five quasars indicates that in the remote past the formation of quasars was no more probable than at present.[91] We now proceed to hypotheses which are of particular interest at present.

16. MAGNETOTURBULENT THEORY OF QUASARS

The emission of radio waves from quasars is the most convincing proof of the presence of magnetic fields and high-energy electrons, which rotate in these fields. The unique form of the glowing jets ejected from the core also offers evidence of the decisive role of the magnetic field.

A large cycle of investigations of the equilibrium and evolution of massive stars without account of the magnetic field has essentially led to negative results. Let us summarize the theory developed above. A spherically symmetrical massive star becomes unstable at very low density and moderate temperature at an instant when the gravitational potential is still small and the nuclear energy has not yet had time to be released. The loss of stability is followed by collapse, in which the gravitational energy is released, becoming converted into thermal and kinetic energy of the matter, but all these forms of energy are not apparent to the outside and are buried in the gravitational field of the star after its self-closing.

An analysis of non-spherically symmetrical problems leads to the conclusion that in similar processes there can occur a release of a sufficient amount of energy in the form of thermal radiation and kinetic energy of the jets. However, to construct the true picture it is necessary to take into account magnetohydrodynamic effects, since we cannot ignore the direct observational data concerning the magnetic fields.

V. L. Ginzburg and L. M. Ozernoĭ[69] analyzed the magnetic field of a collapsing star; the field becomes stronger during the course of contraction, in accordance with the freezing-in condition ("stickiness," as is sometimes stated) of the force lines. The relation between the magnetic and gravitational energies does not change here:

$$H \sim \frac{1}{R^2}, \qquad W_m \sim H^2R^3 \sim \frac{1}{R}.$$

The authors emphasize that the magnetic energy constitutes a small fraction of the total energy of the star and does not influence the dynamics of the contraction of the main mass of the gas. Account of general-relativity effects shows that the gravitational self-closing is accompanied by a drawing-in of the magnetic field, which becomes compressed against the Schwarzschild surface. The external magnetic field disappears.

N. S. Kardashev[70] considered the mechanism of intensification of the magnetic field, connected with the contraction of a rotating plasma cloud. The relative motion of parts of the cloud is accompanied by the tangling up of the force lines of the magnetic field and by an intensification of the field. Kardashev suggests that the field energy can become equal in order of magnitude to the gravitational energy of the cloud. He considers further the formation of magnetohydrodynamic waves in the plasma when the body contracts rapidly.

Questions of magnetohydrodynamic phenomena, and particularly the explanation of the periodic changes of the brightness within the framework of the synchrotron-radiation theory, were considered by L. M. Ozernoĭ.[96,34]

A logical continuation of this line of development is an idea, most precisely formulated by D. Layzer.[95] A quasar is regarded as a body in which the gravitational field is balanced essentially by a randomly turbulent magnetic field.

It is well known that the Maxwellian magnetic-field tensor corresponds to repulsion in two directions perpendicular to the field and to contraction along the field. Consequently, a random field, in which all directions are equally represented, produces on the average a repulsion corresponding to the average pressure, equal to 1/3 the energy density. The characteristic relation $\overline{p} = \mathscr{E}/3$ holds true for a magnetic field which is stationary in the mean in all cases—also for an aggregate of electromagnetic waves in vacuum, i.e., for a photon gas, and for a random magnetic field in the plasma, maintained by currents flowing in the plasma.

Before we proceed to make such a model more concrete, let us clarify the general relations between the mass of the star and the strength of the current producing the magnetic field H necessary for the equilibrium. In order of magnitude we have

$$\frac{GM^2}{R} = H^2R^3, \qquad H = \frac{M\sqrt{G}}{R^2}.$$

The current is obtained from the equation

$$\operatorname{curl} \mathbf{H} = 4\pi \mathbf{j} = 4\pi ne\,\frac{v}{c},$$

where e is the charge of the electron in electrostatic units, n the electron concentration, v—the average velocity of the electrons producing the current. We substitute $|\operatorname{curl} \mathbf{H}| \sim H/r$, where r is the characteristic scale, and write $M = Nm_p = nR^3m_p$, where N is the total number of nucleons in the star and m_p is the proton mass. We obtain

$$v = c\sqrt{\frac{R}{r}}\sqrt{\frac{Gm_p^2}{e^2}}.$$

Thus, the expression for v contains the characteristic ratio under the second square root, namely the ratio of the gravitational interaction of two protons to their electrostatic interaction:

$$\frac{Gm_p^2}{e^2} \sim 10^{-36}, \qquad v \sim 10^{-18}\, c\sqrt{\frac{R}{r}}.$$

It is precisely because of the fact that the gravitational interaction is negligibly small compared with the electrostatic interaction, that it is sufficient to have a negligibly ordered motion of particles of the same polarity relative to particles of the other polarity to produce a repulsion that balances the gravitation.

The model of the object is incomplete in two respects. 1) With the aid of the magnetic field it is difficult to balance gravitation not only in the mean, but also locally, at each point. The magnetic force acting on a volume element of the plasma, which is equal to $\mathbf{H} \times \mathbf{j} \sim \mathbf{H} \times \operatorname{curl} \mathbf{H}$, is not potential. The gravitational force $-\rho \operatorname{grad} \varphi$ is, strictly speaking, potential only in the case when ρ is constant or $\rho = \rho(\varphi)$. It is not clear whether exact equality of the two forces can be attained at each point for an arbitrary density distribution. 2) The body on the whole is in a state of neutral equilibrium, if there are no factors other than gravitation and the magnetic field, since the magnetic field has an adiabatic exponent $\gamma = 4/3$ (with respect to a similar contraction of the body).

These difficulties are resolved if we assume, following Layzer, that along with the magnetic fields there is a macroscopic turbulence-type motion of matter, and that the kinetic energy of the matter is of the order of the magnetic energy. The turbulent motion occurs with nonrelativistic velocity, and its adiabatic exponent is 5/3. Therefore the body as a whole has an adiabatic exponent between 4/3 and 5/3 and is in stable equilibrium.

As the energy becomes dissipated, the body contracts slowly. Were there no dissipation, then the kinetic energy would increase more strongly upon contraction than the magnetic energy. The disturbed equilibrium is restored by the conversion of kinetic energy into magnetic energy at the expense of further entanglement and stretching of the force lines of the magnetic field. The volume currents which produce the magnetic field may turn out to be unstable against the pinch effect, i.e., the plasma may become compressed in individual sections as a result of attraction of parallel currents. This gives rise to electric fields which accelerate individual groups of charged particles: such phenomena were observed also experimentally in discharges in a rarefied plasma. The body in question turns out to be a powerful source of cosmic rays, and the energy acquired by the particle turns out to be proportional to its charge, if the particles move with equal velocity in the given electric field.

Finally, when the effects of general relativity are included during the course of the contraction, collapse takes place which, according to Layzer, is accompanied by ejection of part of the mass. During the ejection, the magnetic lines straighten out, the ejected matter consists of individual jets or filaments, in which the plasma is solidly connected to the magnetic field which is frozen in it and which is stretched along the filament.

Layzer's paper is for the most part descriptive, and has few quantitative estimates which are furthermore unreliable. This does not detract from its significance: its main statements—the large role played by the magnetic field in the overall energy balance,* the slow evolution of an almost-equilibrium state, and the creation of relativistic particles—are in good

*This point, as already mentioned, is contained also in Kardashev's paper.

agreement with the general picture of the phenomena occurring in quasars.

What remains unclear is the question of the rate of the evolution and the space scale of the magnetic turbulence. We recall that Batchelor's fundamental theorem concerning the equality of the magnetic and kinetic energies in a turbulent highly conducting liquid has not yet been proved.

It is possible that the greatest part of the energy is contained in the turbulence having the largest space scale. In such a case another approach is also possible, namely the analysis of stationary ordered fields and ordered motion in a gravitational field. One variant corresponds to convection with a rise at the equator and a descent at the poles. The observed data, and especially the organized ejection of one or two jets and the regular period of oscillations of the brightness, favor more readily such an approximation. Another possible variant is an axially-symmetrical solution with toroidal magnetic field and rotation of matter about the axis; the equilibrium state corresponds to the minimum energy at a given distribution of the specific angular momentum (per unit mass) and magnetic flux through the medium. In the state of minimum energy, when the conditions for the freezing-in of the magnetic field and the conservation of the momentum are satisfied, the angular velocity of different jets may be different. However, the fact that the energy is minimal denotes the suppression of turbulence by the magnetic field in this situation, which leads to an increase in the duration of such a state.

17. THE ANTI-COLLAPSE HYPOTHESIS

To conclude the article we consider a phenomenon which is the direct opposite of collapse—anti-collapse. According to the hypothesis developed in[97], anti-collapse may be a source of gigantic energy released in the processes indicated in Sec. 15.

During collapse, the observer on the surface of a contracting star will cross after a finite proper time the Schwarzschild sphere and will reach the central singularity. We shall consider this phenomenon in a reversed time sequence. Then the surface of the star, starting to expand from a point, will cross the Schwarzschild sphere after a finite proper time and will continue to expand further. Since, just as in the case of collapse, the time of reaching the Schwarzschild sphere is infinite for an external observer, it might appear that he will also see the inverse process—the expansion from the Schwarzschild sphere—as an infinitely long process and will, of course, not be able to see what occurred prior to the emergence from within the critical sphere.

Actually this is not so. The expansion picture is not the time reversal of the contraction picture, but follows in principle a different course.[98] The reason for this, roughly speaking, consists in the following. The phenomenon of attenuation of processes during the collapse is explained by the joint action of two effects: the slowing down of the flow of time in the strong field and the (generalized) Doppler effect when the surface of the contracting star moves away from the observer. Both effects are so directed as to slow down the processes. When the surface expands, the Doppler effect acts in the acceleration direction for the external observer of processes on the star. This effect turns out to be stronger than the slowing down of the processes in a gravitational field. The external observer will see the evolution starting not from the cooled picture at $R = r_g$, but will see the entire expansion process, starting from the point-like dimensions. A more detailed description of the expansion can be found in[98]. Some of the deductions of this work were later repeated in[99].

The indicated singularities of collapse and anti-collapse are connected with the following most important property of the spherical gravitational field. The physical continuation of space-time "inside" the Schwarzschild sphere (T-region) is double-valued. This was noted for empty space by Finkelstein,[100] and inside a medium in papers[43,44]. In the case of one continuation, the motion of arbitrary trial particles and light rays is directed towards the inside from the Schwarzschild sphere. In the case of the other continuation, all motions are directed outward. As shown in[43,44], the choice between the two continuations of the Schwarzschild solution in the T-region is not arbitrary, and is determined physically by the conditions for the occurrence of this region. If it occurs when the sphere contracts to dimensions smaller than the gravitational radius, then all motions in it will be directed inward. If we specify velocities of matter with dimensions smaller than r_g directed outward from the very beginning, then in the T-region all the motions will be directed outward, the light rays will emerge from within the Schwarzschild sphere and will reach an external observer. Simultaneous realization in the T-region of particles moving inward and outward is impossible. In the T-region in the spherical problem, the replacement of contraction by expansion is impossible. Therefore it is necessary to assume in the anti-collapse model that the expansion begins from point-like dimensions.

Let us see, how, in accordance with the anti-collapse hypothesis, we can attempt to explain the quasar phenomena and other giant explosions. Let us consider a homogeneous isotropic cosmological Friedman model. We assume that at the instant of infinite density (in a solution which does not take quantum effects into account) not all matter began to expand. Some regions (cores) were retained and in the world time of the model do not expand for a certain period. This time delay can be of arbitrary duration and can be different for different cores.

This is followed by expansion of these cores and the material emerges from within the gravitational radius, and its energy, by interacting with the matter which falls from the outside, goes over into energy of cosmic rays and radiation. It is possible that not all of the retained matter will expand immediately, but expansion, of individual shells will take place, i.e., repeated explosions and a continuous outflow of matter are possible. The matter falling from the outside can be ejected by the core earlier, with a velocity smaller than parabolic. In addition, in the mechanism for the conversion of energy of the expanding matter into other forms of energy, an important role may be played by the magnetic field, as was pointed out by B. V. Komberg. In[97], a mathematical model is constructed, which realizes the picture described above.

We emphasize that the "anti-collapse" hypothesis is based on ordinary physical laws, without assuming that they are violated, and to some degree is a development of the ideas of V. A. Ambartsumyan concerning the possibility of a prolonged existence of massive D-bodies and their subsequent explosion.[12] Some considerations in favor of this are advanced in[108].

On the other hand, the hypothesis proposed has nothing in common with the concepts of Hoyle[36,101] concerning the continuous creation of matter (increase of the baryon charge upon expansion). The gravitational action of the cores, whose expansion is delayed, remains all the time unchanged.

Some considerations concerning the possible causes of the delay of the expansion of cores and what happened to them prior to the onset of the expansion, as well as other details, are advanced in[97]. Solutions of the cosmological Friedman type for a limited mass were used by N. S. Kalitsyn[102] in connection with the runaway of multiple galaxies; general-relativity effects during this stage are negligible.

In connection with the anti-collapse hypothesis, it is advantageous to discuss the relation between Ambartsumyan's conception and the viewpoint of many other astronomers.

Ambartsumyan has maintained for many years that the general trend of cosmological evolution concludes in the expansion of matter from a certain superdense state. He presents many observational facts confirming his theory. The opposite point of view, which is widely prevalent at the present time, consists in the fact that the stars of the galaxy are produced by condensation from a rarefied gas. Bearing in mind the well known fact of the overall expansion of the metagalaxy (Friedman or Hubble; the choice of the name depends on what is implied, theory or observations), the usual point of view must be formulated more accurately: the orthodox astronomer believes that matter was actually 10^{10} years ago in a superdense state, but at that stage the density was with tremendous accuracy uniform in all of space. During the course of the expansion, matter reached a very low density 10^{-10}—10^{-20} g/cm^3. During this period the inhomogeneities of the density increased, but still remained small.* Only at the later stage, which occupies the lion's share of the 10^{10} years, did the matter condense into stars and galaxies. Thus, each particle of matter was originally in a superdense state, then went through a minimum of density indicated above, and then again arrived at a state of relatively large density in the stars. From the point of view of Ambartsumyan, even stars which are created at the present time are produced of matter which is in a superdense state. This matter did not go through a low-density state prior to conversion into a star.

The anti-collapse theory presented above shows that this point of view does not contradict in principle the physical laws of general relativity. The question of what occurred and what occurs actually and, in particular, the choice of these two possibilities, should be solved ultimately by observational data.

The indicated hypothesis can find application also in cosmology, as a possible variant of the development of gigantic systems of the type of a finite metagalaxy (in the limit of an unbounded metagalaxy, cosmological anti-collapse is identical with the Friedman model). As regards the application of the hypothesis for an explanation of quasars and explosions of galactic cores, although it is capable of explaining the giant release of energy and does not contradict either the observational data or the laws of physics, nevertheless its initial conditions are unusual (superdense state). The authors believe at present that it is more probable that the puzzle of quasars will be solved without using such unusual assumptions concerning the delay of the expansion. We repeat once more that the last word belongs to the observational data.

APPENDIX

GENERAL RELATIONS OF THE ENERGY APPROACH AND COMPARISON WITH THE FOWLER METHOD[5]

I. Newtonian Theory

Emden's solution with n = 3 is written with the aid of a dimensionless function of the ratio of the "running mass" m to the total mass M:

$$\varrho = \varrho_c \psi\left(\frac{m}{M}\right) = \varrho_c \psi(z), \quad m = 4\pi \int_0^r \varrho r^2\, dr, \, M = 4\pi \int_0^R \varrho r^2\, dr,$$
$$z = \frac{m}{M}.$$

We consider first formulas for a fixed entropy, for example for S = 0. In this section we shall write

*The first fundamental paper discussing the entire process of the incipience and development of an inhomogeneity, starting from the superdense state when quantum fluctuations are important, is that of A. D. Sakharov.[103] This work was preceded by[104, 105].

briefly $E(\rho)$ in place of $E(\rho, S)$ and $d/d\rho$ in place of $(\partial/\partial\rho)_S$.

By specifying $E(\rho)$ we determine the pressure:

$$P=-\frac{dE}{dV}=\varrho^2\frac{dE}{d\varrho};$$

$\gamma = 1/3$ denotes that $P = \text{const} \cdot \rho^{4/3}$. We write

$$E=A\varrho^{1/3}, \quad P=\frac{1}{3}A\varrho^{4/3}. \tag{I.1}$$

We substitute this value of E in the general expression for the energy of the star:

$$\mathscr{E}=\int E\,dm-G\int\frac{m\,dm}{r} \tag{I.2}$$

and specify the Emden distribution.

We obtain

$$\mathscr{E}=\int A(\varrho_c\psi)^{1/3}dm-G\int\frac{m\,dm}{\left(\frac{M}{\varrho_c}\xi\right)^{1/3}}, \tag{I.3}$$

$$\xi=\frac{3}{4\pi}\int\limits_0^z\psi^{-1}(z)\,dz \tag{I.4}$$

and, going over to the dimensionless integration variable z,

$$\mathscr{E}=\alpha A\varrho_c^{1/3}M-\beta GM^{5/3}\varrho_c^{1/3}, \tag{I.5}$$

where α and β are dimensionless numbers,

$$\alpha=\int\limits_0^1\psi^{1/3}(z)\,dz, \quad \beta=\int\limits_0^1\xi^{-1/3}(z)\,z\,dz. \tag{I.6}$$

We obtain the equilibrium mass M_{e0} (e for equilibrium, 0—without corrections):

$$\frac{d\mathscr{E}}{d\varrho_c^{1/3}}=\alpha AM-\beta GM^{5/3}=0, \quad M_{e0}=\left(\frac{\alpha A}{\beta G}\right)^{3/2}. \tag{I.7}$$

For a mass close to equilibrium $M = M_e + \mu$, we get, expanding in a series,

$$\mathscr{E}=-\frac{2}{3}\alpha A\mu\varrho_c^{1/3}=-\frac{2}{3}\beta GM_e^{2/3}\mu\varrho_c^{1/3}=-k\mu\varrho_c^{1/3}, \tag{I.8}$$

where k denotes $(2/3)\beta GM_e^{2/3}$. We introduce an arbitrary small correction to the equation of state

$$E=A\varrho^{1/3}+f(\varrho), \quad P=\frac{1}{3}A\varrho^{4/3}+\varrho^2f'(\varrho),$$
$$\Delta E=f(\varrho), \quad \Delta P=\varrho^2f'(\varrho). \tag{I.9}$$

The corresponding correction to the energy of the star is

$$\Delta\mathscr{E}=\int\limits_0^M f(\varrho)\,dm=\int\limits_0^M f(\varrho_c\psi(z))\,dm. \tag{I.10}$$

We assume the correction to be small, and also $\mu = M - M_e$ to be small, calculate the correction for $M = M_{e0}$, and add it to the change in energy due to the deviation of M from M_e; we obtain [the constant k is defined by formula (I.8)]

$$\mathscr{E}=-k\mu\varrho_c^{1/3}+\int\limits_0^{M_{e0}}f\left[\varrho_c\psi\left(\frac{m}{M_e}\right)\right]dm. \tag{I.11}$$

We obtain the equilibrium condition:

$$\frac{d\mathscr{E}}{d\varrho_c}=0=-\frac{1}{3}k\mu_{eq}\varrho_c^{-2/3}+\int\limits_0^{M_{eq_0}}f'\left[\varrho_c\psi\left(\frac{m}{M_e}\right)\right]\psi\left(\frac{m}{M_e}\right)dm. \tag{I.12}$$

The integral in the right side of (I.12) can be written in the following manner (see the definitions at the start of the appendix):

$$\int f'\psi\,dm=\frac{1}{\varrho_c}\int f'\varrho_c\psi\,dm=\frac{1}{\varrho_c}\int f'\varrho\,dm=\frac{1}{\varrho_c}\int\frac{\Delta P}{\varrho}\,dm. \tag{I.13}$$

From (12.13) we get

$$\mu_e=\frac{3}{k}\varrho_c^{-1/3}\int\frac{\Delta P}{\varrho}\,dm, \quad M_e=M_{e0}+\mu_e \tag{I.14}$$

and the expression for the equilibrium energy, substituting (I.14) in (I.11) and putting $1/\rho = v$, is

$$\mathscr{E}_e=-3\int\frac{\Delta P}{\varrho}\,dm+\int f\,dm=\int\limits_0^{M_{e0}}(\Delta E-3v\Delta P)\,dm. \tag{I.15}$$

The first-approximation energy and pressure satisfy the relation

$$E_0-3vP_0=0. \tag{I.16}$$

Consequently, we can add (I.16) to the expression under the integral sign; we obtain

$$\mathscr{E}_e=\int\limits_0^{M_{e0}}(E-3vP)\,dm. \tag{I.17}$$

We finally introduce, following Fowler, the dimensionless quantity

$$\varepsilon=\frac{E}{3Pv}=\frac{E\varrho}{3P}, \quad E-3vP=\frac{\varepsilon-1}{\varepsilon}E=(\varepsilon-1)E; \tag{I.18}$$

the latter equality follows from the fact that we assume that $\epsilon - 1 \ll 1$ and the entire theory is constructed in first order in $\epsilon - 1$:

$$\mathscr{E}_e=\int(\varepsilon-1)E\,dm=(\bar{\varepsilon}-1)\int E\,dm$$
$$=(\bar{\varepsilon}-1)\int E_0\,dm=(\bar{\varepsilon}-1)\int\frac{Gm\,dm}{r}. \tag{I.19}$$

This expression coincides with the expression of Fowler. We present its derivation:

$$\mathscr{E}=\int E\,dm-\int\frac{Gm\,dm}{r}. \tag{I.20}$$

We transform the first integral by parts

$$\int E\,dm=\int E\varrho\,dv=-\int v\,d(E\varrho)=-3\int v\,d(\varepsilon P)$$
$$=4\pi\int r^3\varepsilon\,dP-4\pi\int r^3P\,d\varepsilon. \tag{I.21}$$

In the right side of the first integral, Fowler substitutes the expression for dP from the equilibrium equation

$$dP=-\varrho\frac{Gm}{r^2}dr=-\frac{Gm\,dm}{4\pi r^4}. \tag{I.22}$$

The second integral is neglected by Fowler, and the final expression is

$$\mathscr{E}=\int\varepsilon\frac{Gm\,dm}{r}-\int\frac{Gm\,dm}{r}=\int(\varepsilon-1)\frac{Gm\,dm}{r}=(\bar{\varepsilon}-1)\int\frac{Gm\,dm}{r}. \tag{I.23}$$

When ϵ = const, the expression coincides with ours, but when $\epsilon = \epsilon(m)$ neglecting $4\pi\int r^3 P d\epsilon$ results in an error of the same order as the effect in question; as a result, the averaging of ϵ in Fowler's answer leads to an incorrect law, with a weight m dm/r in place of the weight E dm ~ P dv. It is more important, however, that Fowler confines himself to an examination of $\mathscr{E}_e(\rho_c)$, whereas the complete understanding of the equilibrium, of its stability, and of other properties calls for knowledge of the entire function, which is of two or three variables $\mathscr{E}(M, \rho_c)$ when S = 0 or $\mathscr{E}(S, M, \rho_c)$ near equilibrium.

II. GENERAL RELATIONS BETWEEN $\mathscr{E}_e$ and $\mathscr{E}$.

A general property of the problem is the fact that in the zeroth approximation the energy is proportional to $\rho_c^{1/3}$, with the coefficient vanishing for the state of neutral equilibrium. Thus, the general formulation of the problem is as follows: for a cold star when S = 0 and $\mu = M - M_e$ we have

$$\mathscr{E} = -k\mu\varrho_c^{1/3} + \varphi(\varrho_c), \tag{II.1}$$

and for a hot star when M = const and $s = S - S_e$ we have

$$\mathscr{E} = bs\varrho_c^{1/3} + \varphi(\varrho_c). \tag{II.2}$$

The function φ, if account is taken of only the corrections for the equation of state, is given by the expression

$$\varphi(\varrho_c) = \int (E - 3Pv)\, dm = \int (\varepsilon - 1) E dm, \tag{II.3}$$

and the integration is carried out along the Emden curve n = 3 of the zeroth approximation for $M = M_e$ or $S = S_e$, with a single free parameter ρ_c, on which it indeed depends. Taking into account the correction for general relativity (see III below):

$$\varphi(\varrho_c) = \int (E - 3Pv)\, dm - 0.93\frac{G^2M^{7/3}}{c^2}\varrho_c^{2/3} = \varphi_0(\varrho_c) - \text{const}\ \varrho_c^{2/3}, \tag{II.4}$$

the form of the function φ changes, but the character of the Eqs. (II.2) does not change. In formula (II.4), φ_0 is the quantity calculated with the aid of (II.3) without account of general relativity, and the meaning of φ is seen from formula (II.4) itself. The formulation of the problem consists in the following: we are given an equation, for concreteness (II.2), with $\mathscr{E}(s, \rho_c)$; we are required to find $\mathscr{E}_e(\rho_c)$ from the equilibrium condition and to find the limit of the existence of equilibrium configurations. It is convenient to introduce $x = \rho_c^{1/3}$ as the variable. We obtain*

$$\mathscr{E} = bsx + \varphi(x), \quad \frac{\partial\mathscr{E}}{\partial x} = 0 = bs + \varphi'(x), \quad s_e = -\frac{\varphi'}{b},$$

$$\mathscr{E}_e(x) = -x\varphi'(x) + \varphi(x) = -x^2\frac{d}{dx}\left(\frac{\varphi}{x}\right) \tag{II.5}$$

The condition for the limit of existence of the solutions is a horizontal inflection:

$$\frac{\partial\mathscr{E}}{\partial\varrho} = \frac{\partial^2\mathscr{E}}{\partial\varrho^2} = 0 \rightarrow \frac{\partial\mathscr{E}}{\partial x} = \frac{\partial^2\mathscr{E}}{\partial x^2} = 0, \tag{II.6}$$

$$bs_{\text{cr}} + \varphi'(x_{\text{cr}}) = 0, \tag{II.7}$$

$$\varphi''(x_{\text{cr}}) = 0 \tag{II.8}$$

In this case (II.8) makes it possible to obtain x_{cr}, i.e., the critical density, and after this (II.7) gives the value of the entropy on the critical curve (or the critical value of the mass μ in the analogous problem with fixed entropy and variable mass). We note that

$$\frac{d\mathscr{E}_e}{dx} = -x\varphi''(x) - \varphi'(x) + \varphi'(x) = -x\varphi''(x) = -x\left.\frac{\partial^2\mathscr{E}}{\partial x^2}\right|_S$$

Consequently, the condition for a horizontal inflection on the curve $\mathscr{E}(x, s)$ when s = const coincides with the condition for the minimum of the curve in Fig. 5

$$\mathscr{E}_e(x) = \mathscr{E}(x, s_e(x)).$$

The stability of the given equilibrium state depends, obviously, on the sign of $\partial^2\mathscr{E}/\partial x^2$ at the point where $\partial\mathscr{E}/\partial x = 0$. It follows from the formula that on the descending branch of the $\mathscr{E}_e(x)$ curve the solutions are stable: $\partial\mathscr{E}_e/\partial x < 0$, $\partial^2\mathscr{E}/\partial x^2 > 0$; this proves formally the statement made in Sec. 2.

III. CORRECTIONS FOR GENERAL RELATIVITY

We present first an expression for the correction for the energy in a given arbitrary configuration of matter, which does not correspond, generally speaking, to equilibrium. We assume that the matter is at rest at any given instant, i.e., the instantaneous velocity is equal to zero, but the instantaneous acceleration, generally speaking, is not equal to zero, since there is no equilibrium.

We must take into account the dependence of the density of the mass on the energy. The density of the rest mass will be denoted by n, and the density which includes the energy by ρ; then $\rho = n(1 + E/c^2)$, where E is the specific energy (in excess of the rest mass) per unit rest mass.

We must take into account the fact that the space is non-Euclidean:

$$dV = e^{\lambda/2}4\pi r^2\, dr, \quad V = 4\pi\int_0^r e^{\lambda/2}r^2\, dr > \frac{4}{3}\pi r^3,$$

where r stands for the "coordinate" radius, such that the length of the great circle is $2\pi r$, and the surface of the sphere is $4\pi r^2$. An invariant characteristic of the configuration occupied by the given total number

*It is curious that the correction for general relativity is proportional to $\rho_c^{2/3} \sim x^2$ and is negative for the given configuration: $\Delta\mathscr{E} = -nx^2$, therefore $\Delta\mathscr{E}_e = -x\varphi' + \varphi = +nx^2$. The correction for general relativity in the equilibrium energy $\Delta\mathscr{E}_e$ is positive and is equal in magnitude and opposite in sign to the correction to the energy for the given configuration $\Delta\mathscr{E}$:

$$\Delta\mathscr{E} = -\Delta\mathscr{E}_e.$$

of baryons is the function n(V), where V is the running volume. Equilibrium corresponds to the extremum of the observed mass of the star

$$M=4\pi\int_0^R \varrho r^2\,dr,$$

at a given rest mass

$$N=\int_0^R n(V)\,dV$$

and at a given entropy, which determines the relation

$$E=E(S,\ n).$$

Reckoning the energy from the rest mass of the star, we obtain

$$\mathscr{E}=c^2(M-N)=c^2\int_0^R(\varrho e^{-\lambda/2}-n)\,dV.$$

This expression must be compared with the Newtonian expression

$$\mathscr{E}_n=\int En\,dV-G\int\frac{m'\,dm'}{r'},$$

where m′ is the running "Newtonian" mass (calculated without corrections for the energy dependence of the mass), r′—"Newtonian or Euclidean" radius, and

$$dm'=n\,dV,\quad m'=\int_0^V n\,dV,\quad r'=\sqrt[3]{\frac{3}{4\pi}V}.$$

The correction for general relativity is defined as the difference $\Delta\mathscr{E}=\mathscr{E}-\mathscr{E}_n$, and we calculate the first nonvanishing term in the expansion in powers of G. Obviously, the dimensionless parameter is

$$\frac{r_g}{R}\sim\frac{GM}{Rc^2}\sim GN^{2/3}\varrho_c^{1/3}c^{-2}.$$

The ratio $E/c^2\sim P/\rho c^2$ is of the same order as r_g/R. The first-order terms in G have already been included in the Newtonian approximation.

We use the only relation which is valid in the absence of equilibrium:

$$e^{-\lambda}=1-\frac{2Gm}{rc^2},$$

$$e^{-\frac{\lambda}{2}}=\left(1-\frac{2Gm}{rc^2}\right)^{1/2}=1-\frac{Gm}{rc^2}-\frac{1}{2}\left(\frac{Gm}{rc^2}\right)^2.$$

With the required accuracy, we obtain

$$\Delta\mathscr{E}=\int dV\left[-En\frac{Gm}{c^2r}-\frac{1}{2}n\frac{G^2m^2}{c^2r^2}+nG\left(\frac{m'}{r'}-\frac{m}{r}\right)\right],\qquad\text{(III.1)}$$

$$\frac{m'}{r'}-\frac{m}{r}=\frac{m'-m}{r'}-\frac{m(r'-r)}{rr'},\qquad\text{(III.2)}$$

$$m'-m=-\frac{1}{c^2}\int En\,dV+\frac{G}{c^2}\int\frac{nm}{r}\,dV,\qquad\text{(III.3)}$$

$$r'-r=\frac{G}{r^2c^2}\int mr\,dr.\qquad\text{(III.4)}$$

Using these relations, we obtain ultimately the correction in the form of a sum of five integrals, in which we can always identify the density, volume, and radius with the corresponding Newtonian-Euclidean quantities, and the error due to this will be of higher order of smallness:

$$\Delta\mathscr{E}=I_1+I_2+I_3+I_4+I_5,\qquad I_1=-\frac{G}{c^2}\int E\frac{m\,dm}{r},$$

$$I_2=-\frac{1}{2}\frac{G^2}{c^2}\int\frac{m^2\,dm}{r^2},\qquad I_3=-\frac{G}{c^2}\int\left(\int E\,dm\right)\frac{1}{r}\,dm,$$

$$I_4=+\frac{G^2}{c^2}\int\left(\int\frac{m\,dm}{r}\right)\frac{dm}{r},\qquad I_5=-\frac{G^2}{c^2}\int\left(\int mr\,dr\right)\frac{m\,dm}{r^4}.$$

The integrals are taken over the entire mass of the star, while the inner integrals in I_3, I_4, and I_5 are taken from the center to the running m or r. They are arranged in the order that follows naturally from formulas (III.1)—(III.4). This expression for $\Delta\mathscr{E}$ greatly simplifies when applied to the equilibrium distribution of the gas with adiabatic exponent $\gamma = 4/3$, i.e., when taking into account

$$E=\frac{3p}{\varrho},\quad\frac{dp}{dr}=-\frac{Gm\varrho}{r^2}.$$

In this case, after making several integrations by parts, we obtain

$$I_3+I_4=-\frac{2}{3}I_1+2I_2,\quad I_5=\frac{1}{3}I_1,$$

and ultimately $\Delta\mathscr{E}=(2/3)I_1+3I_2$. This expression, coincides exactly with Fowler's correction[5] taken with the opposite sign $\Delta\mathscr{E}=-\Delta\mathscr{E}_e$.

Using now Emden's function with n = 3 for the calculation of the integrals, we obtain ultimately

$$\Delta\mathscr{E}_{GR}=-\frac{0.93G^2M^{7/3}\varrho_c^{2/3}}{c^2}.$$

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