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THE GENERAL THEORY OF RELATIVITY*

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1. INTRODUCTION

DURING the past few decades there has been a considerable rise in interest in the theory of general relativity (GR). This has been due partially to lack of any essential progress in other branches of theoretical physics. Some physicists have turned to GR in the hope that by combining it with quantum theory one will find a basis for the theory of elementary particles. Others regard GR as the model of a physical theory and try to apply its geometrical concepts and methods to other branches of physics. There are also attempts to relate the symmetry of elementary particles to the asymptotic symmetry of gravitational fields.[4] On the other hand, recent advances in experimental techniques permit one to increase the accuracy of earlier experiments and also to test new predictions of GR.[5,6] Astronomical discoveries of recent years indicate that possibly it may be necessary to take account of corrections for GR in astrophysics. The development of radioastronomy and the impending use of extraterrestrial telescopes will permit us to better determine the distribution of matter in the Universe. This in turn may enable us to make a choice among the competing models of the Universe. There are indications that secular effects of gravitational radiation may play an important role for certain astronomical objects. Predictions have been made and experiments undertaken to detect gravitational waves coming to the Earth from the Cosmos.[7-9]

*Russian translation with a supplement by L. P. Grishchuk.

Beginning in 1955 there have been international conferences every two or three years on the relativistic theory of gravitation. The last such conference occurred in 1965 in London. In addition, in a few countries that are in the forefront of research on GR, there have been summer schools and local and more specialized conferences devoted to the theory of gravitation and related problems. The last such conference in the Soviet Union was in 1965 in Tbilisi.

This paper gives a survey of various results and problems of GR, with particular attention to those that were discussed at the conferences mentioned above. We omit cosmology and relativistic astrophysics, since there have been recent surveys of these in Uspekhi.[10,11] Considerable space is devoted to controversial questions: the postulate of general covariance, the equivalence principle, preferred coordinate systems, conservation laws, and gravitational radiation. The presentation is elementary, but we assume that the reader knows the basic facts about GR, as given in the book by Landau and Lifshitz "The Classical Theory of Fields." Wherever possible we shall follow their notation.

2. THE PRINCIPLE OF EQUIVALENCE

For some time after its appearance, the special theory of relativity met with criticism from nonspecialists and also from certain physicists, who welcomed the "overthrow" of the ether, but could not reconcile themselves with the relativity of time and length. At present, Einstein's interpretation of the Lorentz transformation is generally accepted. There are now no disagreements (at least, serious enough to be discussed) about the meaning of special relativity. This is not the case for general relativity: different views about the physical interpretation of GR are still expressed. In particular, physicists disagree on whether (and how) gravitational energy should be defined, on which reference frame (if such exists) is preferred in GR, and what is the meaning of the principle of general covariance. These problems are closely related to one another and to the equivalence principle, which is the basis of GR.

Einstein assigned great significance to the principle of equivalence, compared it to the relativity postulate, and regarded it as the basis of the theory of gravitation.[12] His position in this respect has been the object of criticism.[13,113,114] A partial reason for this is that sometimes the Einstein equivalence principle is formulated somewhat inexactly and in an unnecessarily crude way.

To arrive at a precise form of this principle and to explain its significance, we need only consider Newtonian physics. The Newtonian theory is based on a series of hypotheses that are not always stated explicitly. Among them are: 1) space-time is a four-dimensional manifold; 2) there is a scalar quantity t (called the absolute time) such that the hypersurfaces t = const are three-dimensional Euclidean spaces. The last assumption means that in each space t = const there are preferred (Cartesian) coordinate systems, but it says nothing about the connection between these systems for different values of t. Such a connection is established by the first law of dynamics. This law can be divided into two parts. The first part states that in the absence of gravitation there is a preferred motion of particles, called free motion. It is achieved in the idealized case where all interactions are absent. The second part of the law asserts the existence of inertial reference frames: in space-time there exist coordinate systems for which t is one of the coordinates, and if we call the others x, y, z, the free motion is characterized by the fact that

$$\frac{d^2x}{dt^2}=\frac{d^2y}{dt^2}=\frac{d^2z}{dt^2}=0.$$

It is easy to see that the first law of dynamics can be formulated geometrically. The motion of a particle corresponds to a world line in space-time, i.e., to a curve that intersects each hypersurface t = const once. Newton's first law is equivalent to the statement that there is a preferred family of world lines corresponding to free motion, that there is a symmetric affine connection Γ in space-time[14] which is integrable (flat) and such that the world lines are geodesics with respect to it. It is clear that the last formulation is sufficiently general to be applicable also to the theory of special relativity. From the fact that Γ is flat it follows that there exists a coordinate system in which the connection coefficients (Christoffel symbols) are zero. These are the inertial coordinate systems; any two such systems are related through linear (Galilean) transformations. The situation becomes more complicated if we try to explain gravitation. In this case free motion does not exist. Experiments show that for all bodies the mass is proportional to gravitational charge ("the equivalence of gravitational and inertial mass"). The best that we can do is to consider the free fall of particles, i.e., motion in a gravitational field where all other interactions are turned off. An appropriate generalization of the first law of dynamics is the following:

1. There exists a preferred motion—free fall.
2. There exists in space-time a symmetric affine connection Γ such that the world lines of freely falling particles are geodesics with respect to it.

This is equivalent to the following: freely falling particles determine the affine connection in space-time. In the Newtonian theory the statements made above must be supplemented.

3. The connection coefficients have the form

$$\Gamma^i_{kl}=\overset{0}{\Gamma}{}^i_{kl}+h^{ij}\frac{\partial\varphi}{\partial x^j}t_k t_l \qquad (i, \ldots, l=0, 1, 2, 3), \tag{1}$$

where $\overset{0}{\Gamma}$ is the flat affine connection, $t_k = \dfrac{\partial t}{\partial x^k}$, t is the absolute time and h^{ij} is the metric tensor of the Euclidean space t = const. The metric tensor is covariantly constant with respect to Γ. If (x, y, z) are cartesian coordinates on the hypersurface t = const,

$$h^{ij}=(\mathrm{grad}\, x^i)(\mathrm{grad}\, x^j),$$

where

$$\mathrm{grad}=(\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z);$$

the function φ is the Newtonian potential.

The decomposition of Γ given by (1) is not unique: if ψ is any solution of the equation

$$t_i\psi_{;\,kl}-t_k\psi_{;\,il}=0,$$

then

$$\overset{0}{\Gamma}{}^i_{kl}-h^{ij}\frac{\partial\psi}{\partial x^j}t_k t_l$$

is also a flat affine connection. It follows from (1) that t can be normalized so that it becomes an affine parameter along geodesics. If this is done, the equation of a geodesic is brought to the form of a Newtonian law of motion in the gravitational field

$$\frac{d^2x^i}{dt^2}+\overset{0}{\Gamma}{}^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}=-h^{ij}\frac{\partial\varphi}{\partial x^j}.$$

The term with $\overset{0}{\Gamma}$ corresponds to Coriolis, centrifugal and further forces of this type which arise when one uses noninertial coordinate systems. In addition, the reference system in space-time can be chosen so that $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and $\overset{0}{\Gamma}{}^i_{kl} = 0$. The equation of motion then takes the usual form

$$\frac{d^2\mathbf{r}}{dt^2}=-\mathrm{grad}\,\varphi. \tag{2}$$

But the group of transformations that preserve (2) is much wider than the Galilean group. If the vector $\mathbf{a}(t)$ depends on the time, the substitution

$$\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} + \mathbf{a}, \tag{3a}$$

$$\varphi \rightarrow \varphi' = \varphi - \ddot{\mathbf{a}}\,\mathbf{r} \tag{3b}$$

does not change (2). It is clear that formula (3b) reflects the nontensorial transformation properties of Γ and the nonuniqueness of the decomposition of Γ into $\overset{0}{\Gamma}$ and a gravitational force. One usually considers gravitational fields produced by bounded objects. One can then normalize the potential φ, using the requirement that it vanish at large distances; this excludes the possibility of transformations of the type of (3b) and establishes a privileged role for the Galilei group. But this cannot be done when the gravitational field extends over all of space as it does in cosmology. We are then faced with the choice: either we refrain from using the concept of inertial reference frames or we say that all systems in which free fall is characterized by (2) are inertial.

This analysis of the Newtonian theory can be summarized as follows. Local classical experiments with particles freely falling in space determine, in the four-dimensional space, a symmetric affine connection Γ. In general this connection is not integrable. In the case of bounded material systems, by using experiments with particles moving freely at infinity, one can determine another affine connection $\overset{0}{\Gamma}$, which is flat. It can be used for selecting the class of inertial systems, which is defined to within a Galilei transformation. In general, when such nonlocal measurements are not available, Γ is the only mechanically defined connection.

The considerations presented above included only classical mechanics and were based on the equivalence of gravitational and inertial mass. Now the following question arises: can one, by means of nonmechanical experiments, determine an inertial frame in the presence of gravitation? Or, more precisely; can one, by means of nonmechanical experiments introduce a flat connection in space-time as a supplement to the nonintegrable connection determined by moving bodies? The principle of equivalence gives a negative answer to this question. In fact, let us assume that there is a phenomenon (e.g., electromagnetism) which establishes a flat connection $\overset{0}{\Gamma}$. The difference $\Gamma^i_{kl} - \overset{0}{\Gamma}{}^i_{kl}$ is a tensor and it is meaningful to assume that the corresponding term in the equations of motion represents the gravitational force. Thus we obtain an absolute and local method by means of which we can distinguish inertial from gravitational forces. But this contradicts the principle of equivalence, in the formulation given by Einstein.

The validity of the preceding remarks is not restricted to Newtonian physics. The following two assertions summarize the role of the principle of equivalence for the theory of gravitation.

1. Equality of gravitational and inertial mass. By studying the motion of freely falling particles one can introduce a symmetric affine connection in space-time. The assumption that the geodesic lines relative to it coincide with the world lines of freely falling particles determines this connection uniquely. In the presence of gravitation the connection is nonintegrable (curved). This connection therefore cannot be used to determine a global inertial reference system.

2. The principle of equivalence extends all the preceding arguments to all other phenomena that cannot be reduced to the motion of classical particles. In accordance with this principle, there is no way of establishing the existence of a flat connection in a gravitational field by using only data from local physical experiments. All experiments lead to essentially one and the same connection Γ.

This is by no means obvious; it is not known to what types of interactions a particular geometry of space-time should give rise. As Fock has emphasized, in the formulation of the principle of equivalence the boundedness of the region of experiments is essential. As was explained above, for weak fields, produced by isolated sources, one can determine a global flat connection by studying free motion at large distances. Finally one should keep in mind the following possibility. The equation of a geodesic is not changed if Γ is replaced by $\Gamma + T$, where T is a valence tensor $(1, 2)$, antisymmetric in its covariant indices. Thus even if the physical affine connection is not symmetric, simple mechanical experiments can determine only its symmetric part. It is clear how one should generalize the principle of equivalence in order to take account of this possibility. É. Cartan was the first to understand that the affine connection of space-time could be asymmetric and that, if this was the case, its antisymmetric part is somehow related to spin.[17] Similar ideas have been proposed recently.[18,21]

It is easy to see the analogy between the principle of equivalence and the Einstein postulate of relativity: both are generalizations to all physical phenomena of certain well-established facts of classical mechanics.

3. RELATIVITY, SYMMETRY AND INVARIANCE.

Another subject of controversy is the question of the role of the principle of general invariance. Is it a significant principle and does it really generalize the fundamental postulate of special relativity? Does the Einstein gravitational theory contain more of relativity than the special theory? According to Fock, the answer must be negative, and one should even avoid the use of the term "general theory of relativity."* The argu-

*The views of V. A. Fock about the role and the place of the principle of equivalence, and about the term "general theory of relativity" are given in detail in his talk "Principles of Galilean Mechanics and Einstein's Theory," published in Uspekhi,[114] and not listed in Trautman's bibliography. In publishing Trautman's paper, the editor points out to the reader the existence of other points of view about the theory of gravitation (Ed. note).

ment would be simply one of terminology if it were not related to the question of privileged reference systems in GR.

To clarify the points of disagreement, let us consider the concept of symmetry of a physical system. In general, a symmetry is a recipe according to which we put in correspondence with any physically admissible state of motion of a system some other state of the same system. One can formulate the notion of symmetry more precisely as follows. In any physical theory we consider the system E of all states and the subsystem $F \subset E$ of physical states, which can actually be realized. The elements of F are selected from among the elements of E by what we usually call the equations of motion. A symmetry is some one-to-one mapping of E on E which takes F into F. Since F is determined by the laws of motion, we usually say that a symmetry is a transformation that leaves the equations of motion unchanged. This can be expressed in a different way, which is convenient for our purpose.[22] All quantities that appear in any physical theory can be divided into two groups: external or absolute quantities and dynamical variables. Only the latter satisfy equations of motion and correspond to degrees of freedom of the system. A symmetry of the theory is a transformation that conserves the absolute quantities. Whether to regard some quantity as external or dynamical depends in many cases on the particular problem and the approximations that are made. In the Newtonian one-body problem the absolute quantities are the geometrical elements of space-time and the potential. If the potential is spherically symmetric and time-independent, the symmetry group consists of time displacements and spatial rotations. Now let us consider the two-body problem. The absolute quantities are those that refer to the space-time. The symmetry group becomes wider and includes, in addition to the rotations, all displacements and the Galilei transformations.

One can distinguish at least three different methods of extending the symmetry group. First, certain of the absolute elements can in general be eliminated from the theory. Second, absolute quantities may take on a dynamical character. The third method, which we shall not consider here, consists in an extension of the systems E and F.

An interesting illustration of the extension of the symmetry group resulting from the elimination of unphysical absolute quantities, is related to the special theory of relativity. In Newtonian mechanics, the quantities

$$t,\ h^{ij}\ \text{and}\ \Gamma^i_{kl}. \tag{4}$$

are absolute elements. We know that these are not enough for constructing a Newtonian theory of electromagnetism. This theory requires, in some form or other, the introduction of an ether. Generally speaking, the ether can be defined as a field of directions in space-time, intersecting the hypersurfaces t = const. The vector field u^i corresponding to these directions can be normalized so that $u^k t_k = 1$. In the presence of the ether one can introduce the auxiliary metric tensor

$$g^{ij} = h^{ij} - u^i u^j / c^2$$

and write the Maxwell equations in vacuum in the form

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{li}}{\partial x^k} + \frac{\partial F_{kl}}{\partial x^i} = 0, \quad F^{ik}_{\ ;k} = 0, \tag{5}$$

where F_{ik} is the electromagnetic field tensor and

$$F^{ik} = g^{ij} g^{kl} F_{jl}.$$

If the connection Γ is flat and the ether is at rest, $u^k_{;l} = 0$, and in the reference frame at rest relative to the ether (i.e., for $u^k = (1, 0, 0, 0)$), the system of equations (5) reduces to the usual Maxwell equations. Thus the absolute quantities in prerelativistic electrodynamics were the following:

$$t,\ u^k,\ g^{ij}\ (\text{or}\ h^{ij})\ \text{and}\ \Gamma^i_{kl}. \tag{6}$$

More rigorously, the Maxwell equations are required only for

$$g^{ij}\ \text{and}\ \Gamma^i_{kl}, \tag{7}$$

while the absolute time and the Euclidean metric are concomitants of Newtonian mechanics. The essential step taken by Einstein in 1905 consisted in denying any (absolute) physical significance to the Newtonian elements t and h^{ij}, and also in reformulating the laws of mechanics on the basis of (7). The Galilei group, which is the symmetry group of Newtonian mechanics, conserves the absolute quantities (4). After the ether is added to these elements, the group is reduced to time displacements and space rotations, but on eliminating the absolute time, the ether and h^{ij}, we get a bigger group, the group of inhomogeneous Lorentz transformations.

The transition from special to general relativity is different in nature. The absolute quantities (7) do not disappear, but rather take on a dynamical character: they now represent the gravitational field. (Actually, the metric and the affine connection are not independent. In Riemannian geometry the connection is uniquely determined by the metric. This, however, is a secondary feature, that can be generalized as pointed out in the preceding section.) In GR there are in general no absolute elements. Thus the symmetry group consists of all regular mappings of space-time into itself ("general invariance"). In this sense the Einstein theory of gravitation is more symmetric than the special theory, and the expression "general relativity" is not unjustified. It is interesting to note that the Newtonian theory of gravitation contains traces of the general invariance. Since the affine connection in that theory has a partially dynamical character, the number of absolute elements in the theory is less than in the theory without gravitation, and the symmetry group is therefore bigger. This is best seen from the appearance of the arbitrary vector a(t) in the Eqs. (3).

The principle of relativity, or invariance, can now be formulated as follows: within the framework of any theory, the physical laws can contain only dynamical variables and absolute quantities appropriate to the theory.

The requirement that the laws of physics be formulated properly, using only those elements which the theory has at its disposal, is actually fundamental. This requirement becomes nontrivial when, based on experiments, we state precisely just what these elements are. It is essential that the absolute elements (if there are any) should be related to physical phenomena. By this we mean the following. Whenever one introduces such a quantity one must give a recipe, using at least gedanken experiments, for reducing the quantity to measurements of one sort or another. If this cannot be done the absolute quantity should be discarded. This is precisely the way that Einstein disproved absolute time and the notion of an inertial frame in a gravitational field. According to our present physical knowledge, there is no phenomenon that could be used to determine a global inertial reference frame (i.e., a flat connection) in a gravitational field. This is the reason why linear theories of gravitation in a flat space-time are physically unsatisfactory.[23]

The principle of relativity, when applied to GR, is called the principle of general invariance. It asserts that the metric is a dynamical variable and that there exist no absolute quantities associated with space-time. Almost from the moment of formulation of GR, this principle has been the target of criticism. Regardless of their motivation, all these attempts to overthrow it reduce to the same idea: the introduction of an absolute geometrical structure in addition to the dynamical metric structure. The most extreme case is a theory with two metrics.[24] The proposal of Fock to single out harmonic coordinate systems also implies the possibility of introducing an auxiliary flat metric: it is sufficient to say that the components of this metric in harmonic coordinates have the values of the Minkowski metric. The Lorentz transformations considered by Fock leave this (absolute) flat metric invariant. In recent years attempts have been made to define the gravitational energy by postulating the existence of certain privileged fields of orthogonal basis vectors ("tetrads").[25-27] There is nothing wrong in considering such fields. But as soon as we select a class of fields of local bases, such that any two representatives of the same class are related to one another by a Lorentz transformation with constant coefficients, we introduce in a canonical way a flat metric in space-time. From our point of view, to justify any such attempts one must connect the second flat metric with physical phenomena (for example, by showing how one can measure the proper time corresponding to this new metric).

4. CONSERVATION LAWS

The connection between symmetry and absolute quantities can be made more explicit for the example of the conservation laws.[22,28] On the other hand, the problem of the energy is of interest in its own right in GR, and has attracted considerable attention in recent years.[29] For this reason a considerable part of this survey is devoted to the conservation laws and their connection with symmetry and the problem of gravitational energy.

We shall begin with a short derivation of Noether's theorem[30] about the existence of conservation laws and identities in classical theories whose equations of motion are derivable from an action principle. Since the energy, momentum and angular momentum are of more interest to us than "internal" conserved quantities like the charge, isospin and baryon number, we shall make suitable simplifying assumptions. A more general treatment can be found in other papers.

First we consider the classical theory of fields. We denote the dynamical variables by ψ, and use ω for the absolute quantities (each symbol can denote a system of tensor fields). The field equations are derived by varying the action

$$W = \int_\Omega L\, d\Omega$$

with respect to ψ. Here Ω is the volume of space-time, and L denotes the Lagrangian (density). In order for W to be defined correctly (in a coordinate-independent way), L must be a scalar density. From the principle of relativity, L can depend only on the functions ψ and ω and their derivatives. For simplicity we shall assume that L depends on the functions ψ and ω and their first derivatives. (This assumption is not satisfied in GR. There L depends on $\partial^2 g_{ij}/\partial x^k \partial x^l$. But the treatment of second derivatives requires only minor modifications of the formulas.) The field equations can be written symbolically as

$$\frac{\delta W}{\delta \psi} = 0,$$

where

$$\frac{\delta W}{\delta \psi} = \frac{\partial L}{\partial \psi} - \frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial \psi_i}\right) \quad \text{and} \quad \psi_i = \frac{\partial \psi}{\partial x^i}.$$

In the simple case that we are considering, the Noether identities can be gotten as follows. Let X be an arbitrary vector field in space-time, and Xf the Lie derivative of the field f* (in the physics literature the Lie derivative is often written as $\delta^* f$ or $\bar{\delta} f$). For a scalar density

$$XL - \frac{\partial}{\partial x^i}(LX^i) = 0. \tag{8}$$

If we sum over repeated indices and use

$$X\left(\frac{\partial f}{\partial x^i}\right) = \frac{\partial}{\partial x^i}(Xf),$$

the identity (8) can be written as

*Relative to the vector field X. (Ed. note)

$$\frac{\delta W}{\delta \psi} X\psi + \frac{\delta W}{\delta \omega} X\omega + \frac{\partial t^i}{\partial x^i} = 0, \tag{9}$$

where

$$t^i = \frac{\partial L}{\partial \omega_i} X\omega + \frac{\partial L}{\partial \psi_i} X\psi - LX^i.$$

Every vector field on a manifold generates a one-parameter group of transformations of the manifold. The group corresponding to X is a symmetry group if it leaves ω invariant; the necessary and sufficient condition for this is

$$X\omega = 0. \tag{10}$$

Thus we have established the connection between symmetry, field equations and conservation laws:

$$\text{if } X\omega = 0 \text{ and } \frac{\delta W}{\delta \psi} = 0, \text{ then } \frac{\partial t^i}{\partial x^i} = 0. \tag{11}$$

The conditions imposed on the symmetry of absolute elements usually restrict them so severely that only a limited number of linearly independent vector fields X satisfy Eq. (10). In these cases the symmetry is characterized by a Lie group. The solution of Eqs. (10) gives a representation of its Lie algebra. The most important example of this situation is the theory of a field in a given Riemannian space-time. In this case (which includes the theory of a field in a flat space-time) ω coincides with the metric tensor g_{ik}. For a wide class of fields ψ the Lie derivative has the form

$$X\psi = X^i \psi_i + F_i^j \psi \frac{\partial X^i}{\partial x^j},$$

where F_i^j is a constant matrix with respect to the summation indices of ψ. As a further simplification we assume that L does not depend on the derivatives of the metric tensor. The symmetric tensor of the energy-momentum density is

$$T^{ik} = -2\frac{\delta W}{\delta g_{ik}} = -2\frac{\partial L}{\partial g_{ik}}, \tag{12}$$

and (9) can be rewritten as

$$\frac{\delta W}{\delta \psi} X\psi - \frac{1}{2} T^{ik} X g_{ik} + \frac{\partial t^i}{\partial x^i} = 0. \tag{13}$$

Since (13) is an identity for arbitrary X, the coefficients of the various derivatives of X^j can be equated to zero. This leads to the identities[32-34]

$$S_i^{lk} + S_i^{kl} = 0, \tag{14a}$$

$$T_i^k = t_i^k + S_{i;\,l}^{kl} + \frac{\delta W}{\delta \psi} F_i^k \psi, \tag{14b}$$

$$T_{i;\,k}^k = -\frac{\delta W}{\delta \psi}\psi_{;\,i} + \left(\frac{\delta W}{\delta \psi} F_i^j \psi\right)_{;\,j}, \tag{14c}$$

where

$$S_i^{kl} = \frac{\partial L}{\partial \psi_l} F_i^k \psi \quad \text{and} \quad t_i^k = \frac{\partial L}{\partial \psi_k}\psi_{;\,i} - L\delta_i^k.$$

The tensor t_i^k is called the canonical energy-momentum tensor. In the general case it is not symmetric. The divergence of the vector t^i and the vector

$$T^i = T^{ik} X_k \tag{15}$$

are equal to zero whenever X generates a symmetry transformation in space-time, i.e., whenever the Killing equation

$$X g_{ik} = X_{i;\,k} + X_{k;\,i} = 0. \tag{16}$$

is satisfied. In addition the conserved quantities corresponding to T^i and t^i are equal if the field vanishes sufficiently rapidly at large distances. This follows from the identity

$$T^i = t^i + \frac{\partial}{\partial x^j}(X^k S_k^{ij}) + \frac{\delta W}{\delta \psi} F_k^i \psi X^k,$$

which follows from (14). The physical meaning of the conserved quantities depends on the geometric properties of the Killing vector X. For example, in Minkowski space, using cartesian coordinates (x^i), we can take

$$X^i = \omega_k^i x^k, \quad \omega_{ik} + \omega_{ki} = 0$$

as the generator of a Lorentz transformation. Then the vector t^i becomes

$$t^i = \frac{1}{2} I_{kl}^i \omega^{kl},$$

where

$$I_{kl}^i = x_k t_l^i - x_l t_k^i + S_{kl}^i - S_{lk}^i$$

is the tensor density of the angular momentum. The last two terms in the expression for I are usually interpreted as the spin angular momentum. In four-dimensional Riemannian space the Killing equation (16) has at most 10 linearly independent solutions (for the case of a space of constant curvature). In the most general case of a completely asymmetric space, there are no Killing vectors.

In the general theory of relativity there are in general no absolute elements. Any vector field X generates a symmetry transformation. Equation (9) is an identity with respect to X, i.e., there are an infinite number of conservation laws.[35] In Addition, Eqs. (9) lead to a generalization of the "Bianchi identities"

$$\frac{\delta W}{\delta \psi}\psi_i - \frac{\partial}{\partial x^j}\left(\frac{\delta W}{\delta \psi} F_i^j \psi\right) = 0, \tag{17}$$

which are valid independent of the field equations. For any specific X, Eqs. (11) give a "weak" conservation law. But in the special case considered here (no absolute elements, general invariance) each of these weak laws can be transformed by using (17) into a strong law[36]

$$\frac{\partial \theta^i}{\partial x^i} = 0, \tag{18}$$

where

$$\theta^i = t^i + \frac{\delta W}{\delta \psi} F_j^i \psi X^j. \tag{19}$$

Equations (18) retain their validity irrespective of whether the field equations are satisfied or not. This implies the existence of an antisymmetric tensor density ("superpotential") U^{ij} such that

$$\theta^i = \frac{\partial U^{ij}}{\partial x^j}. \qquad (20)$$

It is clear that the differential conservation law (11) remains valid under the substitution

$$U^{ij} \to U^{ij} + V^{ij}, \qquad (21)$$

where V^{ij} is also an antisymmetric tensor. The total conserved quantity*

$$\int t^i dS_i = \frac{1}{2}\int U^{ij} df^*_{ij} \qquad (22)$$

is not affected by the transformation (21) if V^{ij} tends to zero sufficiently rapidly at large distances.

Now we consider in more detail the conservation laws in a gravitational field. We denote the purely gravitational action by W_g:

$$W_g = -\frac{1}{16\pi}\int \sqrt{-g}\, R\, d\Omega \qquad (k = c = 1).$$

Setting

$$G^{ik} = -16\pi \frac{\delta W_g}{\delta g_{ik}},$$

we get the equations of the gravitational field (we note that G^{ik} and T^{ik} are tensor densities)

$$G^{ik} = -8\pi T^{ik}. \qquad (23)$$

The fundamental identity (9) is written as

$$-\frac{1}{16\pi} G^{ik} X g_{ik} + \frac{\partial \tau^i}{\partial x^i} = 0, \qquad (24)$$

where τ^i (in this special case) denotes what we previously called t^i. The Bianchi identities (17) have the form

$$G^{ik}_{;k} = 0. \qquad (25)$$

Equation (24) can be written in the form of (18) with

$$\theta^i = \tau^i - \frac{1}{8\pi} G^{ik} X_k. \qquad (26)$$

Finally, if we use the field equation (23),

$$\theta^i = \tau^i + T^i, \qquad (27)$$

where T^i is defined by (15). An attractive interpretation of (18) and (27) is the following: in GR the total conserved "energy flux" θ^i consists of a gravitational part τ^i and a part T^i connected with matter. But in the general case of a strong gravitational field without any special symmetry, no vector fields X can be preferred above any other. The corresponding conservation laws have no clear physical meaning.

*The labelling of the integrals is the same as in the book of Landau and Lifshitz "The Classical Theory of Fields," 2nd ed., Pergamon, 1962, §6.

A possible form of superpotential, proposed by Komar,[37] was derived from a variational principle by Møller:[38]

$$U^{ij} = \frac{\sqrt{-g}}{8\pi}(X^{j;i} - X^{i;j}). \qquad (28)$$

Applying Eqs. (20), (26) and (28), we get an exact formula for τ^i.

In most cases we are interested in the gravitational field produced by isolated bounded material systems. At large distances from the system the field is weak and the metric is almost flat. In the far region one can consider the approximate symmetry of space-time and expect that it will have the same order and the same structure as for a flat space. There is justification for assuming that one will find (leaving aside the question of how this is to be done) 10 linearly independent asymptotic Killing vector fields. Since the total conserved quantities can be expressed in terms of surface integrals taken at infinity (Eqs. (22)), the values of these quantities will be uniquely determined from the known asymptotic symmetry. A formal proof of this statement was first given by Einstein[39] and Klein.[40]

All the quantities considered so far — θ^i, U^{ij}, τ^i, etc, have tensor properties. Many investigators have expended efforts to destroy the vector character of the τ^i. (One then gets quantities that are called pseudotensors or complexes of gravitational energy and momentum.) To see how this is done we need only note that the differential conservation law (18) is satisfied so long as the quantity U^{ij} is antisymmetric, and independent of its tensor properties. Correspondingly, neither V^{ij} nor X^i need necessarily be a tensor. For example, by a suitable choice of V^{ij} the Komar superpotential can be transformed to the form given by Bergmann:

$$'U^{ij} = U^{ij}_k X^k, \qquad (29)$$

where

$$U^{ij}_k = \frac{1}{16\pi\sqrt{-g}} g_{kl} \frac{\partial}{\partial x^m}\left(g\,(g^{im}g^{jl} - g^{jm}g^{il})\right).$$

The quantities θ^i and τ^i depend on g_{ij}, X^k and their derivatives, where the dependence on X is linear. We shall assume that in any coordinate system there are given 16 functions λ^{ik} (the λ^{ik} need not be the components of a tensor; the position of the indices is unimportant) and constants c_k. Let

$$X^k = c_i \lambda^{ik}. \qquad (30)$$

Then

$$\theta^k = c_i \theta^{ik},$$

where θ^{ik} is a system of 16 functions depending on λ and the metric tensor. One can apply a similar procedure to τ and to the superpotential. Since the θ^i satisfy Eq. (18) for any X, while the quantities c are arbitrary, we get

$$\frac{\partial \theta^{ik}}{\partial x^k}=0.$$

Basically, all the gravitational pseudotensors and complexes in the literature can be gotten by this method. For example the complex proposed in 1958 by Møller[41] follows from the Komar superpotential if we set $\lambda^{ik} = \delta_i^k$ (i.e., $X^i = c^i$ = const). If we start from the Bergmann superpotential, the same choice for λ leads to the canonical pseudotensor of Einstein. On the other hand, the symmetric pseudotensor of Landau-Lifshitz* can be gotten from (29) by taking $\lambda^{ik} = \sqrt{-g}\, g^{ik}$. Finally, if $(\lambda^{0k}, \lambda^{1k}, \lambda^{2k}, \lambda^{3k})$ is a field of orthogonal basis vectors, the metric tensor can be expressed as

$$g^{kl}=\lambda^{0k}\lambda^{0l}-\sum_{\alpha=1}^{3}\lambda^{\alpha k}\lambda^{\alpha l}$$

and Eqs. (30) gives a true vector X. Thus, if we start from (28), the four quantities $(\theta^{0k}, \theta^{1k}\ \theta^{2k}\ \theta^{3k})$ will be vector densities, depending on the basis vectors and their derivatives. "Strict" Lorentz transformations of the basis vectors

$$\lambda^{ik}\to L^i_j\lambda^{jk},\quad L^i_j=\text{const}$$

induce a similar transformation of the quantities θ.[25,26,42]

Although pseudotensors are not covariant objects, they lead to sensible results for the total energy and momentum if one uses a coordinate system that is cartesian at infinity. This comes about because the functions λ^{ik} are always chosen so that (at least at large distances) they become constant in a Cartesian system. Thus, for any c_i the vector (30) is asymptotically a Killing vector. Consequently, computations based on pseudotensors calculated in an asymptotically Cartesian coordinate system are equivalent to computations using asymptotic Killing vectors. Of course, the concepts of asymptotic symmetry and asymptotically rectangular coordinate systems require precise definition. It may prove necessary to revise these definitions for the particular special case. An attempt to define an asymptotically rectangular coordinate system is described in Sec. 6.

This somewhat pedantic analysis of the conservation laws shows that in GR the concepts of energy, momentum, etc, are not as well defined as they are in special relativity. In any case, the gravitational energy cannot be localized.

In classical theory the concept of energy is inseparable from the notion of force. The principle of equivalence states that (at least locally) the gravitational force cannot be defined satisfactorily.

*Cf. L. Landau and E. Lifshitz, Classical Theory of Fields, 2nd ed., Pergamon, 1962, §100. (Ed. note).

5. ORDERS OF MAGNITUDE OF EFFECTS OF GENERAL RELATIVITY

A simple treatment that is based on the principle of equivalence and dimensional analysis, and that uses the weakness of the gravitational interaction, permits one to draw a variety of conclusions about the magnitude and nature of the effects of GR. For example, it is well known that the magnitude of the red shift in a gravitational field can be found in an elementary way by using the correspondence between GR and Newtonian theory. In this section we shall give in outline some similar predictions; they will be gotten without using the exact form of the Einstein equations.

First we consider the motion of a particle of mass m in the gravitational field of a rapidly rotating body, having mass M and angular momentum S. From these quantities, the universal constants k and c, and the separation vector **r** of the bodies, we can form the dimensionless quantities

$$\frac{kM}{c^2r}\ \text{and}\ \frac{kS}{c^3r^2}. \tag{31}$$

Thus, including the lowest order corrections due to gravitation and relativity, the possible form of the Lagrangian of the system is

$$\frac{L}{mc^2}=\frac{1}{2}\left(\frac{v}{c}\right)^2+\frac{kM}{c^2r}+\frac{1}{8}\left(\frac{v}{c}\right)^4 +\alpha\left(\frac{kM}{c^2r}\right)^2+\beta\frac{kM}{c^2r}\left(\frac{v}{c}\right)^2+\gamma\frac{k\,[\mathbf{Sr}]}{c^3r^3}\,\frac{\mathbf{v}}{c}. \tag{32*}$$

The first two terms are the Newtonian terms, the third describes the lowest correction from special relativity. The remaining terms enter with unknown numerical coefficients. Their exact values can be determined by a detailed treatment of the theory, but they must in any case be of order one. On the basis of (32) we consider the various relativistic corrections to the Newtonian theory for a single body. If we neglect **S**, the shift of the perihelion per revolution. obtained from (32), is of order kM/c^2r. The Einstein theory gives $6\pi kM/c^2\rho$, where ρ is the semimajor axis of the Newtonian ellipse. Another effect of general relativity that is easily gotten from (32) is the secular variation of the plane of motion of a particle in the field of a massive rotating body. If **I** is the angular momentum vector of the particle m relative to the body M, Eq. (32) leads to

$$\frac{d\mathbf{I}}{dt}=\frac{\gamma k}{c^2r^3}[\mathbf{IS}].$$

If **I** and **S** are not parallel, the vector **I** will precess about **S**; the Einstein theory gives the same formula with $\gamma = 2$.

In most cases the magnitudes of general relativity corrections are proportional to the ratio km/c^2r, formed with quantities m and r that are characteristic of the particular problem. One therefore usually

*$[\mathbf{Sr}] \equiv \mathbf{S}\times\mathbf{r}$.

says that GR is in no way connected with the structure of elementary particles. If for r we take the classical radius of the electron, then

$$\frac{km}{c^2 r} = \frac{km^2}{e^2} \sim 10^{-43}.$$

It is clear, furthermore, that on an atomic scale even Newtonian gravitation plays no part; the ratio of the gravitational interaction between a proton (mass M) and an electron (mass m) to their electric interaction is

$$\frac{kMm}{e^2} \sim 10^{-40}.$$

The insignificant role of gravitation on an atomic scale is best seen if one compares the size and energy for an ordinary atom and a "gravitational atom" (two neutrons, bound by gravitation). The radius of an electrical atom is of order

$$\frac{e^2}{mc^2}\frac{1}{\alpha^2} \sim 10^{-8}\ \text{cm},$$

whereas the radius of a gravitational atom is

$$\frac{kM}{c^2}\left(\frac{\Lambda}{l}\right)^4 \sim 10^{28}\ \text{cm}$$

where $\Lambda = \hbar/Mc$ and

$$l = \sqrt{\frac{k\hbar}{c^3}} = 1.6\cdot 10^{-33}\ \text{cm}. \qquad (33)$$

Incidentally, we note that the radius of a gravitational atom is of the same order as the "radius of the Universe," $r = cH$ (where H is Hubble's constant).

If ρ is the density of the mass M, then on the surface of the body

$$\frac{kM}{c^2 r} \sim \frac{k\varrho r^2}{c^2}. \qquad (34)$$

Consequently the effects of GR are significant for large and dense bodies. These effects are undoubtedly important in cosmology: if we set $r = cH$, and take for ρ the average density of matter in the Universe, we find a value of the order of unity for the ratio (34).

Similarly, by using simple arguments one can estimate the effects associated with gravitational radiation and discuss their main properties.[43] Elementary arguments based on the Newtonian law of conservation of mass and on the equivalence of inertial and gravitational mass show that gravitational monopole and dipole radiation do not exist.

Since gravitation exists as a classical (macroscopic) field, one would expect that it is described by a tensor field. From the fact that the gravitational force is proportional to $1/r^2$ and is attractive, it follows that this is a field of zero mass and even spin.[44]

Let φ be the component of the gravitational field that goes over into the Newtonian potential in the nonrelativistic limit. We may expect that for not too large velocities and not too strong fields φ satisfies the equation

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 4\pi k\varrho, \qquad (35)$$

where ρ is the mass density. Expansion in series of the retarded solution of (35) gives

$$\varphi = -\frac{kM}{r} - \frac{kr\mathbf{P}}{cr^2} + \text{higher multipoles}. \qquad (36)$$

Here M and **P** are the total mass and momentum. For an isolated system these quantities are constants. This would not be the case if the source of the gravitational field were not identical with the distribution of inertial mass. From field theory it follows that the flux of radiated energy is proportional to the squares of the first derivatives of the potential. Correspondingly, one may expect that the intensity of radiation of gravitational waves is of order

$$\frac{c}{k}\oint(\nabla\varphi)^2\,dS, \qquad (37)$$

where the coefficient c/k is introduced so that one gets the right dimensions; the integration is extended over a sphere surrounding the system. The derivatives of the monopole and dipole terms in (36) behave like $1/r^2$ and give no contribution to (37). Quadrupole radiation predominates; if $D_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3$) is the quadrupole moment tensor, then

$$P \sim \frac{k}{c^5}\dddot{D}_{\alpha\beta}\dddot{D}_{\alpha\beta} \qquad (38)$$

(plus the contribution of higher multipoles). Formula (38) is a simple consequence of (36) and (37) and is independent of any specific theory of gravitation.[7] Such a theory is necessary for fixing the magnitude of the numerical coefficient in (38). Sometimes the absence of dipole gravitational radiation is explained as due to the absence of negative masses. The arguments given above show that is not so and that the essential part is played by the equivalence of gravitational and inertial mass. There can be no dipole radiation even when negative masses exist, provided that the equivalence holds. One can also apply the argument for the reverse conclusion: the sources of a field with spin 2 and mass 0 must be identical with a distribution of energy and momentum. When applied to point masses this gives the equivalence of gravitational and inertial mass.[45]

The theory predicts that the energy carried by gravitational waves is extremely small. We have just seen one of the reasons for this. A more important reason is the smallness of the gravitational radius km/c^2. For a gravitating system consisting of two bodies of equal mass m, moving in circular orbits of radius r, formula (38) gives

$$P \sim \frac{mc^2}{r/c}\left(\frac{km}{c^2 r}\right)^4.$$

For a similar system in electrodynamics (two particles, of equal mass m and opposite charged e and −e, moving in circular orbits under the influence of their mutual attraction with velocity $v \ll c$), the intensity of the electromagnetic (dipole) radiation is

$$P_{\text{e.m.}} \sim \frac{mc^2}{r/c}\left(\frac{e^2}{mc^2 r}\right)^3,$$

while the intensity of the gravitational radiation is

$$P \sim \frac{km}{c^2 r} P_{\text{e.m.}}.$$

As another example we consider the nonrelativistic motion of a charge in an external constant magnetic field. If v is the velocity and r the radius of the orbit, then we find for the electromagnetic and gravitational energies, [46,47]

$$P_{\text{e.m.}} \sim \frac{mc^2}{r/c}\left(\frac{v}{c}\right)^4 \frac{e^2}{mc^2 r} \text{ and } P \sim \frac{km^2}{e^2} P_{\text{e.m.}}.$$

From these and similar examples one may conclude that the strength of the radiation depends essentially on the ratio of the gravitational (resp. electromagnetic) radius of the source to a length characterizing the size of the system. A double star may radiate a large amount of gravitational energy if its components are superdense bodies moving closely about one another. [9]

As already mentioned in the Introduction, the general theory of relativity has been applied to the problem of the last stage of stellar evolution. This problem is treated in [11,48]. But for completeness we give Chandrasekhar's formula[49] for the maximum mass of a cold star (white dwarf) maintained in equilibrium by the pressure of a degenerate electron gas. An elementary treatment[50] shows that for the stability of such a star it is necessary that its mass not exceed a magnitude of order

$$\frac{1}{M^2}\left(\frac{\hbar c}{k}\right)^{3/2}, \tag{39}$$

where M is the proton mass. The numerical value of (39) is of the same order as the mass of the Sun. This result should warn us against hasty conclusions about possible combining of different physical theories. Chandrasekhar's formula was derived on the basis of the Newtonian theory of gravitation. The corrections from GR are considerable when we deal with neutron stars—hypothetical objects that are held in equilibrium by the pressure of the degenerate gas consisting of neutrons and other baryons. In this case the maximum mass is also of order (39). The question of what happens to a star that has a mass considerably greater than (39) is known as the problem of collapse.

Undoubtedly, gravitational phenomena like all others have a quantum "background." The present "classical" theory of gravitation of Einstein is of course only an approximation to a more exact theory that takes into account the quantum nature of the microworld. Probably all physicists will agree with this statement. Many theorists have strong opinions on how one should construct a quantum theory of gravitation. The opinion is widely held that this should be done following the example of electrodynamics, i.e., by treating the metric tensor as a potential and replacing certain of its components by operators, subject to certain commutation relations, etc. Despite the difficulties that arise, because of general invariance and the nonlinearity of the equations, such a program can be carried through. One can calculate, at least to lowest order, gravitational corrections to energy levels, and estimate cross sections for processes involving gravitons. [51-53] But some physicists feel that not every physical theory can be quantized in the same way as electrodynamics.[54] Of course this procedure cannot be applied to statistical theories. On the other hand, it is difficult to think of the present theory of gravitation as in some way analogous to thermodynamics. The analogy between the theories of Einstein and Maxwell is so patent that opponents of the idea of quantizing the gravitational field are in the minority. Sticking with this majority opinion, we estimate the orders of magnitude of gravitational relativistic quantum effects. Forgetting about the difficulties of GR, we write the dimensionless action integral for the field Ψ of matter interacting with the gravitational field in the form

$$\frac{1}{\hbar c}\int\left[\frac{1}{k}(\nabla\varphi)^2 + 4\pi\varrho\varphi + (\nabla\Psi)^2\right] d\Omega.$$

Taking $\hbar = c = 1$ and introducing $\Phi = \varphi/\sqrt{k}$, we get

$$\int [(\nabla\Phi)^2 + 4\pi l\varrho\varphi + (\nabla\Psi)^2]\, d\Omega,$$

where l is given by (33) and plays the role of the coupling constant. One may expect that gravitational quantum effects will be proportional to powers of l/λ, where λ is a characteristic wave length for the particular process. In other words, these effects can be sizable only at extremely high energies. It appears that quantization of the gravitational field in the accepted way cannot lead to any significant effects, at least in the energy range attainable now or in the near future. But we must remember that GR is also a theory of space-time. The basic hypothesis of the theory is the continuity of space-time, or, more precisely, the fact that it is a differentiable manifold. This assumption would appear to be well justified in the framework of classical physics. But it is by no means obvious that it will be satisfied when quantum effects are included. In theories based on the continuity of the space-time manifold, it was assumed that arbitrarily close events could be recognized and distinguished from one another. If we consider the uncertainty relation and the finite size of elementary particles, it becomes unclear how this is to be done. One may imagine that it is actually not possible. This impossibility should be explained by the very structure of space-time, just as the local indistinguishability of gravitational and inertial mass was explained in GR. It is obvious that any change of the fundamental assumption about the continuous structure of space-time will lead to a profound reexamination of all of physics.

Recently Zel'dovich[55] has predicted a new general relativistic effect, which has been called the gravitational Zeeman effect. In the gravitational field of a rotating body the local inertial system of coordinates rotates with respect to an inertial system located at infinity. For the reader who thinks mathematically, we formulate this as follows: in a stationary, but not static, space-time the Fermi displacement differs from the Lie displacement, defined by the group of timelike isometries.[56] In accordance with our earlier remarks, at a distance r from a body with angular momentum S, the angular velocity of rotation of the local inertial coordinate system is of order kS/c^2r^3. If a body of mass M has angular velocity ω, on the surface of the body

$$kS/c^2r^3 \sim \omega kM/c^2r. \tag{40}$$

Thus a spectral line radiated by a source on the surface of the rotating body will be split for a distant observer into two components with frequencies differing by an amount of the order of (40).

6. GRAVITATIONAL WAVES AND RADIATION

Some of the effects predicted by GR (for example, the shift of the perihelion) give corrections to Newtonian phenomena, others (for example, the bending of light rays) can be derived from the principle of equivalence. But there are also effects which depend essentially on the new degrees of freedom of the gravitational field. For example, the rotational effects mentioned in the preceding section depend on the "vector part" of the gravitational potential. The most interesting prediction of this latter type is the possibility of existence of gravitational waves. Some time ago this problem had the opposite character. For example, the point of view was developed that the field equations of GR do not admit any solutions of the type of plane waves, or that gravitational radiation was forbidden by the equations of motion. The problem of existence of gravitational waves is of intrinsic interest and is important for the quantization program. It makes no sense to talk of gravitons if gravitational waves are excluded by the classical equations. A large number of varied and detailed papers written in the last decade show that within the framework of the theory there can be no doubt about the existence of gravitational waves. Moreover, it has become clear from these investigations that the quantitative estimates and the basic properties of the radiation are in agreement with the predictions of the linear approximation.[47] This section, which is based on [43], gives a summary of some of the recent theoretical work on gravitational waves.

By applying the theory of gravitation, one can give a more precise estimate of the amount of energy carried off from a bounded material system by gravitational waves. This is quite easily done in the linear theory of a field with mass 0 and spin 2. To determine the energy flux we use the canonical energy-momentum tensor. We then get the formula

$$P = \frac{1}{45}\frac{k}{c^5}\dddot{D}_{\alpha\beta}\dddot{D}_{\alpha\beta}. \tag{41}$$

We have neglected higher multipoles.

The situation is not so simple in GR, where the field equations are so complicated that exact, physically meaningful, wave solutions have not been found. Moreover, as we established in Sec. 4, in this theory the very notion of gravitational energy is confused to some extent. To estimate the energy radiated one must set up some approximate method of solving the field equations and give a recipe for calculating the change in the total energy of the system. The different approaches to the study of gravitational radiation can be classified according to their methods for solving these two problems.

Einstein[57] proposed a method for obtaining approximate solutions of the field equations for the case of weak fields. He restricted himself to those coordinate systems that are now called harmonic. We set $g_{ik} = \eta_{ik} + \underset{1}{h}_{ik}$ and neglect all terms in G_{ik} that are not linear in $\underset{1}{h}$ (from now on η_{ik} denotes the Minkowski metric tensor of flat space-time). We take the retarded solution of the linearized equations, substitute it into the canonical energy-momentum pseudotensor, and on integrating the resulting Poynting vector over a large sphere, we arrive once more at (41). This approach has been criticized from several points of view: the weak field approximation neglects the main feature of the Einstein equations—their nonlinearity; it is essentially equivalent to replacing GR by a linear theory of gravitation in a flat space. Such an approach admits strictly periodic radiation fields, whereas it is clear that radiation in GR must be accompanied by secular effects. In a different context, Synge[58] proposes the following interpretation of the approximate solutions: they may be regarded as exact solutions for another matter distribution, which is determined from the Einstein equations. If we apply this view to the wave solution in the weak field approximation, we find that the corresponding flux of matter precisely balances the flux of gravitational radiation calculated from the pseudotensor.

A healthy point of view seems to be that in which the "weak field" solution is regarded as the first step in a scheme of successive approximations (the method of "fast motion approximations"). A basis of such a scheme has been developed by many authors.[59,60] In [61] radiative corrections to the motion of point masses were found in the second approximation. But it is in general by no means clear that this method "converges," and even whether it can be carried past the first step. As a basis for these doubts we may give the following argument. A typical component $\underset{1}{h}_{ik}$ has the form of a diverging wave $\underset{1}{a}(t-r)/r$ (where we assume that $\underset{1}{a}(t)$ vanishes outside the interval (t_0, t_1)). In harmonic coordinates, the equation for the second-order corrections $\underset{2}{h}_{ik}$ has the symbolic form

$$\Box \underset{2}{h} = Q(\underset{1}{h}), \tag{42}$$

where the expression $Q(\underset{1}{h})$ is quadratic in the functions $\underset{1}{h}$ and their derivatives, and $\Box$ is the wave operator in flat space. If we set $\underset{2}{h} = \psi/r$, introduce the null coordinates $t-r$ and $t+r$ and neglect terms of order $O(1/r^3)$ in Q, we can write (42) as

$$\frac{\partial^2\psi}{\partial u\,\partial v}=\frac{f(u)}{v-u},$$

where $f \sim \mathring{a}^2$. Integration of this equation gives

$$\psi(u,\ v)=\underset{2}{a}(u)+\underset{2}{b}(v)+\int_{t_0}^{u} f(t)\log(v-t)\,dt,$$

where $\underset{2}{a}$ and $\underset{2}{b}$ are arbitrary functions. We can eliminate $\underset{2}{b}$ on the basis that it corresponds to an incoming wave. For $u = \text{const} > t_0$ and $u > v$, $v \gg t_1 - t_0$,[13]

$$\psi\sim\log(v-t_0)\int_{t_0}^{u} f(t)\,dt.$$

In other words, for large r and $t-r = \text{const}$, the second order corrections may behave like $(\log r)/r$. If the exact metric behaved in this way, it would contradict the Sommerfeld radiation condition, and one could not calculate the flux of radiation. It is highly probable that this difficulty can be solved by choosing coordinate systems other than harmonic.

The method of "fast motion approximations" is not at all suited to systems consisting of freely gravitating bodies, such as the planetary system. In first order the equations of motion obtained in this way are trivial (there is no interaction). Einstein, Infeld and Hoffmann[62] and Fock[63] proposed a new method which, in first approximation, gives the Newtonian equations of motion. This is accomplished by regarding terms such as

$$\left(\frac{v}{c}\right)^2 \text{ and } \frac{km}{c^2r}, \tag{43}$$

as quantities of the same order (second). Formally this approach consists in expanding all functions in series in powers of $1/c$. This method is satisfactory (i.e., converges rapidly) for those cases where both quantities (43) are small. This means in particular that we must have

$$\frac{km}{c^2}\ll r\ll\lambda,$$

where $T = \lambda/c$ is a characteristic time interval for the system. Consequently this method is unsuited for estimates of the radiation made by using a surface integral of the type of (37): the integration must be taken over the surface of a sphere in the wave zone, i.e., for $r \gg \lambda$. From simple heuristic arguments it follows that the first terms of the series for the components of the metric tensor have the form

$$g_{00}=-1+\underset{2}{g_{00}}+\underset{4}{g_{00}}+\ldots,$$
$$g_{0\alpha}=\underset{3}{g_{0\alpha}}+\underset{5}{g_{0\alpha}}+\ldots,$$
$$g_{\alpha\beta}=\delta_{\alpha\beta}+\underset{2}{g_{\alpha\beta}}+\underset{4}{g_{\alpha\beta}}+\ldots$$

and that the first terms that can correspond to gravitational radiation are $\underset{5}{g_{\alpha\beta}}$, $\underset{6}{g_{0\alpha}}$, $\underset{7}{g_{00}}$.[64-67] Whether these terms are real or trivial (i.e., can be made to vanish by suitable coordinate transformations) depends on whether or not the linearized curvature tensor $\underset{7}{R}_{\alpha00\beta}$ vanishes. There is a great deal of arbitrariness in the choice of wave fields; this corresponds to our freedom in the choice of boundary conditions. As soon as the field is given up to some order, one can introduce the corresponding equations of motion. The wave fields $(\underset{5}{g_{\alpha\beta}}, \underset{6}{g_{0\alpha}}, \underset{7}{g_{00}})$ lead to a damping force in fifth order (where the Newtonian equations are regarded as being of first order). A suitable choice of the wave terms for a system of two bodies gives a damping force whose magnitude agrees with the decrease in energy calculated using (41). This result, which is due to Peres,[68] confirms the applicability of the "weak field" method for studying gravitational radiation.

In order to give a satisfactory and convincing theoretical answer to the question of gravitational radiation, one must find exact, or to some degree exact, solutions of the Einstein equations for a physically acceptable distribution of matter. In addition one must show that the corresponding material system undergoes secular changes, the cause of which may be assigned to the gravitational waves radiated by the system. This is still a difficult problem. In particular, it is difficult to find interesting and physically understandable nonstatic solutions of the internal problem and to connect them with the appropriate external field. But many of the important global properties of material systems such as their mass or total angular momentum can be found by studying the field at large distances. Thus, in order to calculate the radiated energy by means of the Poynting vector, it is sufficient to know the field at large distances. The clarification of these facts is the aim of a whole series of investigations of the asymptotic behavior of gravitational fields.

As a first step we give a nonrigorous formulation of the Sommerfeld radiation condition for the gravitational field.[13,69] By analogy with electrodynamics we demand that Riemannian space-time admit a coordinate system (x^i), an isotropic vector field with nonzero divergence k^i, and a parameter r along the trajectories of the field such that

$$g_{kl}=\eta_{kl}+h_{kl},\qquad h_{kl}=O(1/r), \tag{44}$$

$$\frac{\partial g_{kl}}{\partial x^m}=i_{kl}k_m+O(1/r^2),\ \ i_{kl}=O(1/r), \tag{45}$$

$$k_i=O(1),\ \ \frac{\partial k_i}{\partial x^j}=O(1/r) \tag{46}$$

and

$$\left(i_{kl}-\frac{1}{2}\eta_{kl}\eta^{mn}i_{mn}\right)k^{l}=O(1/r^{2}). \qquad (47)$$

Equation (47) means that the coordinates are asymptotically harmonic; Eq. (45) contains the Sommerfeld condition:

$$k^{m}\frac{\partial g_{kl}}{\partial x^{m}}=O(1/r^{2}).$$

The coordinate transformations

$$x^{i}\rightarrow x'^{i}=x^{i}+a^{i} \qquad (48)$$

with

$$\frac{\partial a_{i}}{\partial x^{j}}=b_{i}k_{j}+O(1/r^{2}), \quad b_{i}=O(1/r) \qquad (49)$$

conserve (44), (45) and (47). In fact, it follows from (49) that

$$\frac{\partial^{2}a_{k}}{\partial x^{l}\,\partial x^{m}}=c_{k}k_{l}k_{m}+O(1/r^{2}), \quad c_{k}=O(1/r)$$

and (48) induces the transformations

$$i_{kl}\rightarrow i'_{kl}=i_{kl}+c_{k}k_{l}+c_{l}k_{k}. \qquad (50)$$

In the asymptotic region the canonical energy-momentum pseudotensor is

$$\tau_{i}^{j}=\tau k_{i}k^{j}+O(1/r^{3}),$$

where

$$\tau=\frac{1}{32\pi k}i^{mn}\left(i_{mn}-\frac{1}{2}\eta_{mn}\eta^{pq}i_{pq}\right).$$

As a consequence of (47), τ cannot be negative, and is invariant under the coordinate transformations (50). Thus we can obtain the total radiated energy and momentum by calculating the appropriate integral of the τ_i^j. Recently Cornish[70] has shown that for a wide class of energy-momentum pseudotensors the integral is independent of the form of τ_i^j if the boundary conditions (44)–(47) are satisfied. In a recent paper Komar[71] has proposed a more satisfactory formulation of the boundary conditions than that given here. He has expressed these conditions in terms of the asymptotic Killing fields.

From our boundary conditions we can also obtain the asymptotic form of the curvature tensor. We set

$$\frac{\partial i_{kl}}{\partial x^{m}}=j_{kl}k_{m}+O(1/r^{2}), \quad j_{kl}=O(1/r),$$

$$\left(j_{kl}-\frac{1}{2}\eta_{kl}\eta^{mn}j_{mn}\right)k^{l}=O(1/r^{2})$$

and

$$R_{klmn}=\frac{1}{2}k_{[k}j_{l][m}k_{n]}+O(1/r^{2}), \qquad (51)$$

where the square brackets denote the antisymmetric part. The part of the Riemann tensor that behaves like 1/r is algebraically of the same type as the Riemann tensor for a plane wave (i.e., is of zero Petrov type*).

*It is also called the second degenerate Petrov type (cf. the Supplement).

Since boundary conditions are imposed on the metric, this means that the 1/r part of the field equations is satisfied asymptotically. To get more detailed information about physics and the geometry of wave space, one must solve the field equations to high accuracy. The formulation given here is not well suited for this purpose.

Bondi was the first to give a systematic treatment of quite general metrics describing the radiation from a bounded source.[72,73] He limited himself to axially symmetric fields and postulated the form of the line element

$$ds^{2}=r^{2}\left[e^{\alpha}(d\theta-A\,du)^{2}+e^{-\alpha}\sin^{2}\theta\,d\varphi^{2}\right]-C\,du^{2}-2D\,du\,dr \qquad (52)$$

so that the area of the surface u = const, r = const, $0\le\varphi\le2\pi$, $0\le\theta\le\pi$ was $4\pi r^2$. Bondi assumed that for sufficiently large values of r, the functions α, A, C and D have the form

$$\left.\begin{aligned}\alpha&=\frac{n}{r}+O(1/r^{2}),\\ A&=\frac{a}{r}+O(1/r^{2}),\\ C&=1-\frac{2m}{r}+O(1/r^{2}),\\ D&=1+\frac{d}{r}+O(1/r^{2}),\end{aligned}\right\} \qquad (53)$$

where a, m, n, and d depend only on u and θ. These assumptions mean that (52) can be transformed to a coordinate system that is cartesian at infinity and in which the radiation conditions (44)–(47) will be satisfied. Bondi showed that the expansions (53) are compatible with the field equations in empty space and, solving certain of these equations, he found that a = d = 0, and obtained a relation between m and n

$$\frac{\partial m}{\partial u}=-\left(\frac{\partial n}{\partial u}\right)^{2}+\frac{1}{2}\frac{\partial}{\partial u}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}n\sin^{2}\theta. \qquad (54)$$

There are also other equations relating m and n with terms of higher order, but we shall not consider them. Their analysis makes it clear that m can be a completely arbitrary function of its arguments; its derivative with respect to u is called the information (news) function. The function m is closely related to the total energy of the system. In the static case $\partial m/\partial u = 0$, it follows from the other equations that $\partial m/\partial\theta = 0$; consequently m can be interpreted as the mass. In the general case Bondi defines the mass as the average of m over angle:

$$M(u)=\frac{1}{2}\int_{0}^{\pi}m(u,\theta)\sin\theta\,d\theta.$$

Equation (54) means that the mass decreases:

$$\frac{dM}{du}=-\frac{1}{2}\int_{0}^{\pi}\left(\frac{\partial n}{\partial u}\right)^{2}\sin\theta\,d\theta,$$

if and only if there is an information function. It is clear that $\partial n/\partial u$ plays the role of the quantity which we earlier denoted by i_{kl}. The general form of the Riemann tensor is:[76]

$$R = \frac{N}{r} + \frac{\mathrm{III}}{r^2} + \frac{D}{r^3} + \ldots, \qquad (55)$$

where the indices are not given, and N, III and D denote the curvature tensors of the zeroth, third and degenerate* Petrov types respectively (cf. Sec. 7). These tensors have k_i as a null eigenvector†, are covariantly constant along rays, and furthermore are proportional to quantities that can be given a physical meaning:

$$N \sim \frac{\partial^2 n}{\partial u^2}, \quad \mathrm{III} \sim \frac{\partial^2}{\partial u\, \partial\theta} n \sin^2\theta, \quad D \sim 2m + \frac{\partial n^2}{\partial\theta}.$$

Only the first of these results can be gotten from (51).

Another technique for solving the field equations has been developed by Newman and Penrose.[74] They introduce a field of null basis vectors (tetrad), associated with a congruence of rays having nonzero divergence and orthogonal to a hypersurface in V_4. If k_i is a vector tangent to the ray, the null tetrad is (k_i, l_i, m_i, m_i^*). The vectors are normalized so that $k_i l^i = 1 = -m_i m^{*i}$, while the other scalar products are zero. With such a tetrad the Riemann tensor of empty space-time can be written in the form

$$R = N(k) + \mathrm{III}(k) + D(k, l) + \mathrm{III}(l) + N(l); \qquad (56)$$

here N(k) denotes the tensor of zeroth Petrov type, admitting k_i as a wave vector; $D(k, l)$ is a degenerate tensor, admitting two null eigenvectors k_i and l_i (cf. Sec. 7). Newman and Penrose introduce as one of the coordinates the affine parameter r along the ray and replace the Einstein equations by a system of first order equations for the Ricci rotation coefficients corresponding to a null orthogonal basis. In general terms, their result about the asymptotic behavior of the Riemann tensor in empty space is the following: if $N(l) = O(1/r^5)$, then $\mathrm{III}(l) = O(1/r^4)$, $D(k, l) = O(1/r^3)$, $\mathrm{III}(k) = O(1/r^2)$, $N(k) = O(1/r)$. This is in agreement with (55) and with earlier exact results on the behavior of the curvature tensor for algebraically special metrics.[75,76]

Another important result on the asymptotic behavior of gravitational waves is the "wave-front theorem," which was proved independently in different forms by Papapetrou,[77] Peres and Rosen,[78] Infeld and Plebanski[79] and Misner.[80] Papapetrou showed that there are no periodic, nonstatic and asymptotically euclidean metrics; only pulsating waves are possible, with an excitation amplitude that falls off sufficiently rapidly for $t \to \pm\infty$. Infeld and Plebanski showed that the assumption

$$g_{ik} = \eta_{ik} + O(1/r), \qquad (57)$$

$$\frac{\partial g_{ik}}{\partial x^m} = O(1/r), \text{ but not } O(1/r^2) \quad \text{for} \quad t = \text{const} \qquad (58)$$

leads to a contradiction with the field equations.

In the most general terms their argument is the following. Let us assumed that, for a particular choice of coordinates, on a certain spacelike hypersurface t = const Eqs. (57) and (58) are satisfied. For certain of the components g_{ik} for this problem, the field equations can be written symbolically as

$$\Delta\varphi = \text{const}\,(\nabla\varphi)^2.$$

If the right side actually goes like $1/r^2$, φ contains a term of the form log r, which contradicts (58).

Arnowitt, Deser, and Misner[80] have formulated and given a rigorous proof of the wave-front theorem in a stronger form: if (57) is satisfied, then for each t = const one can introduce additional restrictions on the coordinates

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = O\left(\frac{1}{r^{3/2+\varepsilon}}\right),$$

$$K_{\alpha\beta} = O\left(\frac{1}{r^{3/2+\varepsilon}}\right),$$

where $\epsilon > 0$ and $K_{\alpha\beta}$ is the second fundamental form for the hypersurface t = const.

7. GEOMETRY OF NULL ELEMENTS

It is easy to see that there is a close connection between waves and isotropic elements of space-time. There are obvious physical reasons for such a connection. Electromagnetic and gravitational waves propagate with the velocity of light. In four-dimensional space-time the world line of the light is a null geodesic; the electromagnetic tensor for a plane propagating wave is $k_{[i}m_{j]}$ with isotropic k^i; the Cauchy problem cannot be formulated locally on a null hypersurface.

The possibility of transmission of information by means of waves is also related to the isotropic character of the corresponding geometric structure. In studying the asymptotic properties of wave fields we have already seen the role that isotropic elements play. In recent years a great many papers have been written about Riemannian spaces with characteristic isotropic structures. There are descriptive papers on this subject with a simple presentation.[81-84] There is thus no need to go into detail here. We shall only briefly enumerate some of the important results.

The paper of Pirani[85] on the physical meaning of the classification made by Petrov of curvature tensors[86,87] may be regarded as the starting point of the present day tendency to use geometrical methods in gravitational wave theory. The Petrov classification itself is the object of numerous investigations and presentations. A description of various directions of development and an extensive bibliography are contained in the survey by Pirani.[81] For our purposes it is

*This type of curvature tensor is also called the first degenerate Petrov type (cf. the Supplement).

†In place of the terminology used by the author: "null vector," "null surface," "null hypersurface," one also uses the terms "isotropic vector," "isotropic surface," "isotropic hypersurface." (Russ. transl. remark).

sufficient to mention the spinor approach.[88] There is a one-to-one correspondence between directions in spinor space (i.e., in the two-dimensional vector space) and the null directions in Minkowski vector space; every real tensor belonging to the irreducible representation $D(s, 0) \oplus D(0, s)$ of the homogeneous Lorentz group can be represented by a symmetric spinor with 2s indices $\varphi_{AB\ldots K}$; every such spinor can be factorized (the parentheses denote symmetrization):

$$\varphi_{AB\ldots K} = \xi_{(A}\eta_B \ldots \chi_{K)}.$$

Thus any nonnull tensor of the type $D(s, 0) \oplus D(0, s)$ determines 2s null directions (principal directions), some of which may coincide. A multiple principal null direction is called a direction of propagation. The Petrov classification in Penrose's formulation consists in enumerating all possible coincidences between principal null directions. The Riemann tensor in the case of empty space and the Weyl conform curvature tensor for the arbitrary case correspond to $s = 2$. They determine 4 null directions. The conform tensor belongs to type I, if all the directions are different; when two, three or four directions coincide, we get, respectively, types II, III, or N (null). Type D occurs when there are two different pairs of coincident principal null directions. A space is said to be algebraically special if its Weyl tensor does not belong to type I. For the electromagnetic field, $s = 1$, and there are only two types of tensors. Plane waves and similar simple forms of radiation belong to the null type, the Schwarzschild metric belongs to type D; it is completely clear that physically real space-time belongs to type I.

An algebraically special, conformally nonflat metric determines a preferred field of null directions—the directions of propagation (a D-metric determines two such fields). This field of directions in turn determines a congruence of rays in space-time. One might think that these rays correspond to a special model of light propagation.[89] In special cases, similar to the case of plane waves, repeated principal null directions can be interpreted as directions of propagation of gravitational radiation. These remarks serve to illustrate the interest that attaches to the properties of rays, particularly to those that are related to algebraically special metrics.

Suppose we are given a congruence of rays, i.e., null geodesics, which need not correspond to principal null directions. The tangent vectors can be normalized so that $k_{i;j}k^j = 0$. Then from the first derivatives of the k_i we can form three and only three scalars: the rotation coefficient

$$\omega = \sqrt{\frac{1}{2} k_{[i;j]} k^{i;j}},$$

the expansion

$$\theta = \frac{1}{2} k^i{}_{;i}$$

and the shear

$$\sigma = \sqrt{\frac{1}{2} k_{(i;j)} k^{i;j} - \theta^2}.$$

These quantities can be given a simple optical interpretation[75] by considering the null geodesics as rays of light. Consider a small plane opaque object and a flat screen placed at some distance from the object. Let us assume that object and screen are oriented so that in their rest system they are orthogonal to the light rays, and their arrangement is such that the shadow cast by the object can be observed on the screen. By a parallel displacement along the rays the object can be put in the position occupied by the screen and compared to its shadow. Then the magnification of the shadow is proportional to θ, its rotation is proportional to ω, and σ characterizes the shear (deformation).

The following theorem is due to Goldberg and Sachs:[90] the metric in vacuum is algebraically special if and only if it contains a congruence of rays without shear; a vector tangent to a ray of the congruence belongs to a direction of propagation of the conform curvature tensor.

Sachs interpreted certain of the field equations in vacuum to find the exact behavior of algebraically special tensors along rays.[75] We mention only one of his results: for an algebraically special empty space-time with $\omega = 0 \neq \theta$, the Riemann tensor has the form

$$R = \frac{N}{r} + \frac{III}{r^2} + \frac{II}{r^3}. \qquad (59)$$

Here N, III, II are tensors (covariantly constant along the rays) of the same type as their designation; r is the affine parameter. This result can be strengthened by proving that the II tensor must belong to type D, and finding the exact form for the line element.[91,92]

It is interesting to compare the exact result (59) with the analogous formula obtained by approximation methods (cf. Sec. 6). The agreement of the first three terms in (56) and (59) can be regarded as an indication that certain algebraically special fields are good approximations to real fields of radiation at large distances from the source. But all these fields are too special to be realistic. An interesting connection between the wave equations of GR and null curvature tensors was discovered by Zakharov.[93]

A superb method for treating problems of asymptotic behavior for fields of zero mass has been developed recently by Penrose.[94] Without going into details, the basic idea of this method can be summarized as follows. When we say that a topological space is "infinite," we mean that although the space is not compact it is locally compact and can be made compact by the addition of certain ideal (infinite) elements. There are many ways of making a locally compact space compact; some of these may be preferable if the space possesses other structures in addition to its topology. For example, Minkowski space can be made compact so that its conformal geometry can be extended continuously to infinite elements. Furthermore the equations of motion of a field of mass zero are conformally invariant in the sense that if we are given two conformally related Riemannian space-times and the solution of the equations for a zero mass field in one of them, there is a natural way of mapping this solution into the solution of the same equation in the other space. According to Penrose's conclusions, instead of treating the asymptotic behavior of a field of mass zero in a noncompact space, one can study its properties in the neighborhoods of certain elements in a compact space that is conformal to it.

The construction of the Penrose manifold P in the case of Minkowski space M was simple, but this is not so simply done in the case of a Riemannian manifold, even if its topology is euclidean. Sometimes even in simple cases the conformal geometry cannot be extended beyond P. Such singularities occur for example, in the Schwarzschild space-time. When the cosmological constant is not equal to zero, the hypersurfaces cease to be null at infinity. By means of the conformal technique, Penrose gave a very simple proof of the general theorem about the asymptotic form of fields of zero mass (the peeling-off theorem). His method enables one to apply the group of asymptotic symmetries and gives rise to a new approach to the problem of gravitational energy.

8. CONCLUDING REMARKS

For obvious reasons, such as lack of space and lack of knowledge of the author, the present survey does not exhaust the problems of the general theory of relativity. For the convenience of readers wishing to obtain more complete information about the latest achievements in GR, we enumerate the most important questions not considered here, along with the main references.

Relativistic astrophysics was already mentioned in the Introduction. Aside from the surveys of Zel'dovich and Novikov, this has also been covered by the Texas symposium[95] and a monograph.[96]

There has been no essential progress in cosmology recently. Practically all the information on this subject can be found in the book of Bondi[97] and the surveys by Heckmann and Schücking[23] and Zel'dovich.[10,98] An interesting paper on the comparison of observational data with the conclusions of cosmological theory has been written recently by Kristian and Sachs.[99]

The problem of the motion of massive bodies, mentioned in Secs. 5 and 6, has been treated in detail in the monographs of Fock[13] and Infeld and Plebanski.[79] An interesting paper on the problem of stability of planetary orbits within the framework of GR was presented by Abdil'din at the Tbilisi conference.[100]

Fundamental aspects of the quantization of the gravitational field have been treated by Bergmann and Komar,[101] Arnowitt, Deser and Misner,[102] and De Witt.[103,104] Among the most important papers on the quantum theory of gravitation are those of Gupta,[105] Dirac,[106] Anderson,[107] Komar[108] and Feynman.[53]

Finally, various observational and experimental consequences and tests of GR have been considered by Ginzburg,[5,109] Adam,[110] and Bertotti, Brill and Krotkov.[111] Dicke[112] has greatly increased the accuracy of the Eötvös experiment. Shapiro[6] has proposed a new experiment in GR, using the dependence of the velocity of light on the gravitational potential. The work of Braginskiĭ and Weber has already been mentioned in the Introduction.

* * *

Many physicists are very emotional about the Einstein theory. Most of them admit that the general theory of relativity is a beautiful theory, but then they add that because of the weakness of gravitational forces the theory of gravitation is on the sidelines relative to the rest of physics. Some physicists go so far as to say that GR should not be regarded as a physical theory.

A less extreme position consists in the assertion that those who are at present developing the Einstein theory are mathematicians rather than physicists. In certain circles relativists are regarded as "socially undesirable elements." Most discussions begin or end with the reproach that GR has been subjected to too few tests—as if this were a fault of the relativists. There are many reasons for these misunderstandings. On the one hand, certain relativists regard Einstein's theory as standing in some higher relation to other theories, and believe that in the last analysis it will combine all the others (hence the search for "unified theories"); some of them are inclined to ignore quantum physics. Such a position had its beginnings with Einstein who regarded the quantum theory, with its statistical interpretation, with mistrust. On the other hand, many physicists working in other fields are not inclined to study the fundamentals of Riemannian geometry that are necessary for understanding GR, and blame the theory because it is not included among the Lorentz-invariant quantum fields.

At the present time the general theory of relativity is the best of the existing classical theories of space, time, and gravitation. It is an exceptional example of a theory having a good logical foundation, but confirmed by a small number of experimental data. Under such conditions it is natural to develop the theory as far as possible. Of course one can restrict the meaning of the word "physics" by excluding many of the papers on GR from the list of physical investigations. We doubt whether such a restriction is useful. It would undoubtedly be in contradiction with the general tendency of present-day science, in which the boundaries between different spheres of investigation become less and less sharp. Instead of continuing this discussion of the character of investigations of questions in GR, we should make the following remark: if there are *mathematicians who want to study the equations of a* physical theory, they should be welcomed, and not subjected to ostracism by a refusal to regard their work as part of physics.

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SUPPLEMENT*

TYPES OF GRAVITATIONAL FIELDS IN THE CLASSIFICATION OF A. Z. PETROV

(by L. P. Grishchuk)

It makes sense to classify gravitational fields if this can be done in an invariant way. The classification can be developed from examination of various characteristics, for example, according to the algebraic structure of the curvature tensor or according to the group of motions admitted by some field. We shall consider the algebraic classification. We shall show that the analysis of the algebraic properties of the curvature tensor

*In the survey by A Trautman the classification of gravitational fields given by A. Z. Petrov is assumed to be known to the reader. It is presented in full in the book of A. Z. Petrov "Einstein Spaces." In order for the readers of Uspekhi to read Trautman's article without having to go to the original literature, the editor has persuaded L. P. Grishchuck (State Astron. Inst.) to write this "Supplement" (Ed. note).

can be related to a study of the algebraic properties of matrices. Since we are interested in invariant properties of the matrices, we shall be dealing with the elementary divisors of λ-matrices (cf. below), which are left invariant by four-dimensional coordinate transformations. Before proceeding to present the Petrov classification, we recall some facts from the theory of elementary divisors of matrices.

Let the elements of a matrix $A(\lambda)$ of rank r be polynomials in a variable λ. We say that $A(\lambda)$ is a λ-matrix. We denote by $D_i(\lambda)$ the greatest common divisor of the i-th order minors $(i \leq r)$; we shall assume that the coefficient of the leading term in $D_i(\lambda)$ is set equal to 1. Since each i-th order minor can be expressed linearly in terms of the minors of order $(i-1)$, the polynomial $D_i(\lambda)$ is exactly divisible by the polynomial $D_{i-1}(\lambda)$. The quotient from the division of $D_i(\lambda)$ by $D_{i-1}(\lambda)$ we denote by $E_i(\lambda)$ $(i=1, 2, \ldots, r$; we take $D_0(\lambda) = 1)$.

Suppose that $\lambda_1, \lambda_2, \lambda_3, \ldots$ (which can be real or complex) are the roots of the polynomial $D_r(\lambda)$. Since $D_r(\lambda)$ is divisible by all the $D_i(\lambda)$, $D_i(\lambda)$ and consequently $E_i(\lambda)$ will have as their roots quantities from the sequence $\lambda_1, \lambda_2, \lambda_3, \ldots$. We set

$$E_i(\lambda)=(\lambda-\lambda_1)^{m_i}(\lambda-\lambda_2)^{m_i'}(\lambda-\lambda_3)^{m_i''}\ldots$$

Those factors

$$\begin{array}{lll}(\lambda-\lambda_1)^{m_1}, & (\lambda-\lambda_1)^{m_2}, \ldots, & (\lambda-\lambda_1)^{m_r}, \\ (\lambda-\lambda_2)^{m_1'}, & (\lambda-\lambda_2)^{m_2'}, \ldots, & (\lambda-\lambda_2)^{m_r'}, \\ (\lambda-\lambda_3)^{m_1''}, & (\lambda-\lambda_3)^{m_2''}, \ldots, & (\lambda-\lambda_3)^{m_r''},\end{array}$$

which do not reduce to constants are called the elementary divisors of the λ-matrix.

We shall say that the numbers $m_i, m_i', m_i'', \ldots$ determine the type of the matrix, for which we use the symbol

$$[(m_1, m_2, \ldots, m_r)(m_1', m_2', \ldots, m_r')(m_1'', m_2'', \ldots, m_r'')\ldots],$$

where the parentheses contain numbers corresponding to a particular root (the basis of the elementary divisor). This symbol is called the characteristic.

We note that the elementary divisors are invariant under the following operations on the λ-matrix: permutation of rows (or columns) of the matrix; multiplication of the rows (or columns) by a constant different from zero; addition to the elements of any row (or column) of the corresponding elements of another row (or column), all multiplied by the same polynomial in λ. An application of these operations is called an elementary transformation of the λ-matrix.

The classification of gravitational fields, based on a study of the algebraic structure of the curvature tensor $R_{\alpha\beta\gamma\delta}$* was developed for Einstein spaces, i.e., for Riemannian manifolds, in which the field equations

$$R_{\alpha\beta}=\varkappa g_{\alpha\beta}. \tag{1}$$

are satisfied. In physical investigations one deals mainly with a special case of Einstein spaces: free (or empty) space; its dimensionality n is 4, the signature of the metric is $(-\,-\,-\,+)$, and the field equations have the form

$$R_{\alpha\beta}=R^{\sigma}_{\alpha\sigma\beta}=0. \tag{1'}$$

Free space is denoted by the symbol T. Let us consider the classification for this case.

The covariant components of the curvature tensor satisfy the identities

$$R_{\alpha\beta\gamma\delta}=-R_{\beta\alpha\gamma\delta}=-R_{\alpha\beta\delta\gamma}=R_{\gamma\delta\alpha\beta}, \tag{2}$$

$$R_{\alpha[\beta\gamma\delta]}=0, \tag{2'}$$

which determine all the algebraic conditions on the covariant components of $R_{\alpha\beta\gamma\delta}$ in the sense that any other identity is a consequence of these.

Since we are interested only in the algebraic structure of the curvature tensor, our investigation will be based on relations (1′), (2) and (2′), and the treatment will be carried out at some point P of the space T.

We associate with each skew-symmetric pair of indices $\mu\nu$ a single collective index (the meaning of this operation is explained later). Of two components of the curvature tensor differing in sign and having indices $\mu\nu$ and $\nu\mu$, we fix only one. It is obvious that the number of collective indices is 6. We renumber all the collective indices, for example, as follows: $14 \to 1$, $24 \to 2$, $34 \to 3$, $23 \to 4$, $31 \to 5$, $12 \to 6$. Thus, to the curvature tensor $R_{\alpha\beta\gamma\delta}$ of the space T, given at the point P, there will correspond a tensor R_{ab} $(a, b = 1, 2, \ldots, 6)$, given in a certain six-dimensional space and associated with this point. We denote this space by the symbol R_6. It is clear that the symmetry of the tensor $R_{\alpha\beta\gamma\delta}$ with respect to interchange of skew-symmetric pairs of indices leads to symmetry of the tensor R_{ab}.

To arbitrary four-dimensional transformations in the space T there correspond not all admissible transformations in R_6, but only a definite class of transformations.

We now must introduce a metric in R_6. This is naturally and conveniently done by introducing a metric tensor g_{ab}, defined as follows:

$$g_{ab} \longrightarrow g_{\alpha\beta\gamma\delta} \equiv g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma} \tag{3}$$

(with this choice, for example, raising and lowering of indices in R_6 corresponds to raising and lowering pairs of indices in T). The tensor g_{ab} is symmetric and nondegenerate $(|g_{ab}| \neq 0)$.

If we now associate with the tensor R_{ab} a λ-matrix

$$(R_{ab}-\lambda g_{ab}) \tag{4}$$

and find its elementary divisors, the type of space will

*Greek symbols take on the values 1, 2, 3, 4; Latin symbols 1, 2, 3.

be determined by the characteristic of the λ-matrix and will be conserved in the region, containing the point P, where the characteristic does not change.

Thus, by introducing the six-dimensional space R_6 we have reduced the problem of classification to a study of the λ-matrix (4).

We construct at each point an orthogonal axis frame $\underset{6}{\xi}^{\alpha}$, with respect to which

$$(g_{\alpha\beta})_P=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \tag{5}$$

then $(g_{ab})_P$ can be written in the form

$$(g_{ab})_P=\begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \tag{6}$$

where ϵ is the three-dimensional unit matrix.

An extremely important point is that the matrix (R_{ab}) relative to the axes (5) is dual-symmetric, i.e., can be written in the form

$$(R_{ab})=\begin{pmatrix} M & N \\ N & -M \end{pmatrix}, \tag{7}$$

where M and N are symmetric square matrices of third order. This fact enables us to prove the theorem that there exist three and only three types of gravitational fields.

Let us show that formula (7) holds. Equation (1′) relative to the axes (5)

$$\sum_{\sigma} e_{\sigma} R_{\alpha\sigma\beta\sigma}=0$$

(where $e_i = -1$, $e_4 = 1$) is now rewritten in terms of collective indices. Then we get

$$\begin{aligned} &R_{12}+R_{45}=R_{13}+R_{46}=R_{23}+R_{56}=R_{15}-R_{24}=R_{16}-R_{34} \\ &=R_{26}-R_{35}=R_{11}+R_{44}=R_{22}+R_{55}=R_{33}+R_{66} \\ &=R_{11}+R_{22}+R_{33}=0. \end{aligned} \tag{8}$$

The identity (2′) gives

$$R_{14}+R_{25}+R_{36}=0. \tag{9}$$

If we use the notation

$$R_{ab}=m_{ab}, \qquad R_{a,\,b+3}=n_{ab} \qquad (a,\ b\leqslant 3),$$

then from (8) and (9), we can write the matrix (R_{ab}) in the form (7), where $M = (m_{ab})$, $N = (n_{ab})$ $(a,b = 1, 2, 3)$, where the following relations hold:

$$\sum_{s=1}^{3} m_{ss}=0, \qquad \sum_{s=1}^{3} n_{ss}=0.$$

We now prove the theorem. In accordance with the equations (6) and (7), the λ-matrix (4) has the following form:

$$(R_{ab}-\lambda g_{ab})=\begin{pmatrix} M+\lambda\varepsilon & N \\ N & -M-\lambda\varepsilon \end{pmatrix}.$$

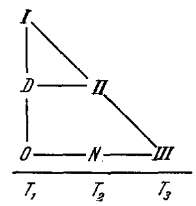
By applying elementary transformations we can bring it to the form

$$\begin{pmatrix} M+iN+\lambda\varepsilon & 0 \\ 0 & M-iN+\lambda\varepsilon \end{pmatrix}\equiv\begin{pmatrix} Q(\lambda) & 0 \\ 0 & \bar{Q}(\lambda) \end{pmatrix}.$$

The corresponding elements of the three-dimensional λ-matrices $Q(\lambda)$ and $\bar{Q}(\lambda)$ are complex conjugate, consequently their elementary divisors are also conjugate, and their characteristics coincide. This means that the characteristic of the λ-matrix (4) splits into two identical parts.

All the possible types of characteristics of the matrix $Q(\lambda)$ are included in the following: 1) [111], 2) [21], 3) [3], which proves the theorem.

We shall attach a label to the space T to indicate its type: T_i $(i = 1, 2, 3)$. The spaces T_1 and T_2 can be divided respectively into three and two subtypes, by considering the possible coincidences of bases of the elementary divisors. We picture the spaces T_i together with the subtypes by means of a Penrose diagram:



Here I, D, and O are the subtypes of the space T_1. Subtype I corresponds to the case when all three bases are different. If two of the three coincide, we have subtype D. Coincidence of all the bases corresponds to subtype O (for free space it includes only flat space-time). The "nondegenerate second type" II occurs when the basis of the multiple elementary divisor (i.e., the one having degree greater than one; in the present case the multiplicity is 2) differs from the basis of the simple elementary divisor. In the opposite case we have the "degenerate second type" N. The space III (T_3) has a single elementary divisor (its multiplicity is 3). For the spaces N and III the bases of the elementary divisors must necessarily be equal to zero.

The derivation given above for the three types of gravitational fields was obtained for the case of free space. Is there an analogous result in the general case, when Eqs. (1), and consequently, Eqs. (1′), are not satisfied?

When the space considered is not an Einstein space, the classification according to the algebraic properties of the curvature tensor is practically impossible, since equations like (8) are not satisfied, and the matrix (R_{ab}) cannot be brought to dual-symmetric form. But in this case one can construct a new tensor (called the Petrov tensor) which has all of the algebraic properties of the curvature tensor and satisfies equations analogous to (1). In fact, let the field equations have the form

$$R_{\alpha\beta}-\frac{1}{2}\,g_{\alpha\beta}=\lambda T_{\alpha\beta}, \tag{10}$$

where λ is a constant, and $T_{\alpha\beta}$ is the energy-momentum tensor of the matter. We construct the Petrov tensor:

$$P_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}-S_{\alpha\beta\gamma\delta}+\sigma\,(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}),$$

where

$$S_{\alpha\beta\gamma\delta}=\frac{\lambda}{2}\,(g_{\alpha\gamma}T_{\beta\delta}-g_{\alpha\delta}T_{\beta\gamma}+g_{\beta\delta}-g_{\beta\gamma}T_{\alpha\delta}),$$

and σ is a scalar. It is easy to see that the tensor $P_{\alpha\beta\gamma\delta}$ satisfies the identities (2) and (2′), while $P_{\alpha\beta}$ (because of (10)) satisfies the field equation $P_{\alpha\beta} = (R + 3\sigma)g_{\alpha\beta}$.

Thus, in the sense of the algebraic structure of the tensor $P_{\alpha\beta}$, we automatically arrive at a proof of the existence of three types of gravitational fields.

For the classification of gravitational fields of general type it is convenient to use the Weyl conformal curvature tensor

$$C_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}-\frac{1}{2}\,(g_{\alpha\beta}R_{\gamma\delta}-g_{\beta\delta}R_{\alpha\gamma}+g_{\gamma\delta}R_{\alpha\beta}-g_{\alpha\gamma}R_{\beta\delta})$$
$$+\frac{1}{6}\,R\,(g_{\alpha\delta}g_{\gamma\delta}-g_{\beta\delta}g_{\alpha\gamma}).$$

The Weyl tensor satisfies the identities (2) and (2′) and in addition, $C_{\alpha\beta} = 0$, i.e., algebraically it behaves like the curvature tensor for free space. It is clear that the Penrose diagram retains its form in the general case.

The physical interpretation of the different types of gravitational fields is far from complete. The overwhelming majority of solutions of the Einstein equations that are now known belong to the first type in the classification. Possibly this is related to the fact that people have mostly solved planetary problems, where one assumes the existence at infinity of a metric for a flat Minkowski space, which is typical only for spaces of the first type. A characteristic feature of spaces T_2 and T_3 is that their curvature tensor is always different from zero, i.e., they cannot be flat. This and other properties permit us to regard them as fields of wave type. But there are still no generally accepted invariant criteria for gravitational waves. According to the definition of gravitational waves, the fields of a particular class may or may not be waves. Recently more and more people in the field have expressed their preference for a definition which labels as gravitational waves the fields of type N in the Penrose diagram.

Translated by M. Hamermesh